ON THE LEFSCHETZ FIXED POINT THEOREM AND SOME OF ITS EXTENSIONS

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Abstract of the Dissertation

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In this dissertation we prove several Lefschetz fixed point type theorems. In the first chapter the manifold under consideration is a compact Lipschitz manifold. This manifold is denoted X. We assume that ξ is a flat vector bundle over X. It is then possible to define L^2 forms on X with values in ξ; this space is denoted by $\Omega_d^*(X,\xi)$. An exterior derivative, denoted d, is defined on this space. This exterior derivative satisfies the property $d^2 = 0$. A corresponding de Rham cohomology complex can then be constructed. We assume

that $f:X \to X$ is a Lipschitz map. Further, we assume that f has at most a finite number of fixed points. A Lefschetz fixed point theorem is proved in this context.

Also we use Hilbert space techniques to prove two Lefschetz fixed point theorems. In the first case the manifold in question is topological. Associated to this manifold are two Hilbert spaces, denoted H_0 and H_1. We suppose $F:H_0 \to H_1$ is Fredholm. A map from the manifold into itself is assumed to induce operators on the kernel and cokernel of F.

Finally, a new proof is given for a special case of the well known Lefschetz fixed point formula for elliptic complexes proved by M. F. Atiyah and R. Bott.

To my daughter

Elizabeth Ann Slome

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Introduction

Given a compact manifold X, de Rham and various types of cochain complexes can be constructed. Cohomology spaces are then obtained. It is of interest to consider maps f, from X into itself, which commute with the operator of a complex. Such an f induces maps on the cohomology level. A map $f:X \rightarrow X$ is said to have $x \in X$ as a fixed point if $f(x) = x$. Lefschetz fixed point theorems relate the action of f on the cohomology and the behavior of f in a small neighborhood of each of its fixed points. That is, Lefschetz fixed point theorems relate global phenomena to local phenomena.

The focus of this paper is on Lefschetz fixed point theorems where the manifold in question is non-smooth. In particular, manifolds with a Lipschitz structure are studied. The importance of Lipschitz manifolds is made evident by the fact that every topological manifold of dimension not equal to four admits a Lipshitz structure; this structure is unique up to a Lipshitz homeomorphism. [9]

In the first chapter the space under consideration is a triangulated compact manifold X. Then X may be thought of as Lipschitz manifold. The work of D. Sullivan [9] shows that on a Lipschitz manifold L^2 forms and exterior derivatives can be defined. Also, given a Lipschitz vector bundle ξ over X, one can define L^2 forms with values in ξ. (This space is denoted by $\Omega_d^*(X,\xi)$). Supposing that ξ is endowed with a flat structure, an exterior derivative d can be defined on $\Omega_d^*(X,\xi)$ and satisfies $d^2=0$. Thus, one can form the corresponding de Rham complex. Then, a Lipschitz map $f:X \to X$ satisfying certain properties will give rise to a map on the cohomology spaces of this complex. Further, it is assumed that the fixed points of f are isolated. The main result of the first chapter is to prove a Lefschetz fixed point theorem for such an f and $\Omega_d^*(x,\xi)$.

In the second chapter a Lefschetz fixed point theorem is proved in an abstract setting. The underlying space is a compact topological manifold. The "complex" consists of two Hilbert spaces, H_0 and H_1, and the role of the exterior derivative is now played by a Fredholm operator $F:H_0 \to H_1$. The cohomology spaces consist of the kernel of F and the cokernel of F. A map f on the manifold, with no fixed points, which commutes with F induces operators on the kernel and cokernel. A Lefschetz fixed point theorem is proved in this setting.

The last chapter also contains a proof of a Lefschetz fixed point theorem. In this case the underlying manifold X is smooth and E and F are smooth vector bundles over X. The operator in question is an elliptic pseudodifferential operator D on the smooth sections of E into the smooth sections of F. Furthermore, D is assumed to be of order zero, so that D extends to a bounded operator on the L^2 sections of E into the L^2 sections of F. The cohomology classes are the kernel of D and the cokernel of D.

The result of the last chapter is not new [1], but the method of proof is. This method of proof, suggested by N. Teleman, is of interest because it can be extended to include the analogous Lipschitz case. Also it can be used in the case of signature-type operators introduced by S. Donaldson-D. Sullivan [10] and presently investigated by D. Sullivan and N. Teleman in the Lipschitz and quasi-conformal setting.

Chapter 1.

§1 Lipschitz Manifolds

A topological manifold X is said to be a **Lipschitz manifold of dimension n**, if it is provided with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ where the U_α are open sets of X and $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ are homeomorphisms satisfying the requirement that $\phi_\alpha \circ \phi_\beta^{-1}$ is a Lipschitz function. That is, there exist constants $c_{\alpha\beta}$ such that

$$|\phi_\alpha \circ \phi_\beta^{-1}(x) - \phi_\alpha \circ \phi_\beta^{-1}(y)| \leq c_{\alpha\beta}|x - y|$$

for all $x, y \in \phi_\beta^{-1}(U_\alpha \cap U_\beta)$.

1.1 **Theorem:** (Rademacher) Suppose U is open in $\mathbb{R}^n$ and let $g: U \to \mathbb{R}^n$. Then g is a Lipschitz function if and only if the partial derivatives of g exist almost everywhere and are bounded measurable functions.

Denote by $L_2^r(V)$, L^2- differentiable forms of degree r on $V \subseteq \mathbb{R}^n$. Suppose $g: V_\alpha \subseteq \mathbb{R} \to V_\beta \subseteq \mathbb{R}^n$ is a Lipschitz map. Then g induces a map $g^*: L_2^r(V_\beta) \to L_2^r(V_\alpha)$ defined as follows:

Let $\omega = a(y)dy_{i1} \wedge dy_{i2} \wedge \cdots \wedge dy_{ip} \in L_2^r(V_\beta)$. Then

$(g^*\omega)(x) = a(g(x))\Sigma\partial g_{i1}/\partial x_{i1}dx_{i1}\wedge\cdots\wedge\Sigma\partial g_{ip}/\partial x_{ip}dx_{ip}$.
(Note this definition of g* makes sense in view of theorem 1.1.) In light of this one can define L^2-differential forms of degree r on a Lipschitz manifold X, denoted L_2^r, as follows: Set $\omega = \{\omega_\alpha\}_{\alpha\in I}$, where each $\omega_\alpha \in L_2^r(\phi_\alpha(U_\alpha))$ satisfy the compatibility condition $(\phi_\alpha \circ \phi_\beta{}^{-1})^*\omega_\alpha = \omega_\beta$.

Let $\omega \in L_2^r(V \subseteq \mathbf{R})$. Then ω is said to have a distributional derivative $d\omega \in L_2^{r+1}(V)$ if there exists $\eta \in L_2^{r+1}(V)$ with the property that for any smooth n-r-1 form γ with compact support in V

$$\int\omega\wedge d\gamma = (-1)^{r+1}\int\eta\wedge\gamma$$

In this case set $d\omega = \eta$.

Hence, one says that $\omega = \{\omega_\alpha\}_{\alpha\in I} \in L_2^r(X)$ has distributional exterior derivative denoted $d\omega \in L_2^{r+1}(X)$ if each $d\omega_\alpha$ exits. That is, $d\omega = \{d\omega_\alpha\}_{\alpha\in I}$. In order for this definition to make sense one needs:

1.2 Proposition:[14] If $g:V_\alpha \to V_\beta$ is a Lipschitz map and $\omega \in L_2^r(V_\beta)$, $d\omega \in L_2^{r+1}(V_\beta)$ then $d(g^*\omega) = g^*(d\omega)$.

Set $\Omega_d^r(X) = \{\omega: \omega \in L_2^r(X),\ d\omega \in L_2^{r+1}(X)\}$. On this set d satisfies $d^2 = 0$, and one can form the DeRham cohomology complex

$$0 \to \Omega_d^0(X) \xrightarrow{d^1} \Omega_d^1 \xrightarrow{d^2} \cdots \xrightarrow{d^n} \Omega_d^n(X) \to 0$$

and therefore also the corresponding cohomolgy spaces $H_d^r(X) = \mathrm{Ker}\ d^r/\mathrm{Im}\ d^{r-1}$.

Let X and Y be Lipschitz manifolds with respective atlases $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$, $\{V_\beta, \psi_\beta\}_{\beta \in J}$. Then a continuous map $f:X \to Y$ is said to be a **Lipschitz map** if $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a Lipschitz map between Euclidean spaces, for all α and β. A Lipschitz homeomorphism f induces a map $f^*:L_2^r(Y) \to L_2^r(X)$ as follows: For $\omega = \{\omega_\beta\}_{\beta \in J} \in L_2^r(Y)$ set

$$f^*\omega = \{(\psi_\beta \circ f \circ \phi_\alpha^{-1})^*\omega_\beta\}_{\beta \in J, \alpha \in I}.$$

It is easily checked that $f^*\omega$ satisfies the compatibility condition

$$(\phi_\alpha \circ \phi_{\alpha'}^{-1})^*(\psi_\beta \circ f \circ \phi_\alpha^{-1})^*\omega_\beta = (\psi_{\beta'} \circ f \circ \phi_\alpha^{-1})^*\omega_{\beta'}.$$

A **Riemannian metric** on X is a collection $\Gamma = \{\Gamma_\alpha\}_{\alpha \in I}$, where Γ_α is a Riemannian metric on V_α, with measurable components, which satisfy the compatibility conditions $(\phi_\beta \circ \phi_\alpha^{-1})^*\Gamma_\beta = \Gamma_\alpha$. A **Lipschitz isometry** is a Lipschitz map $f:X \to X$ which is also an isometry.

§2 Lipschitz Vector Bundles

A vector bundle ξ over X is said to be a **Lipschitz vector bundle** if the total space $E(\xi)$ is a Lipschitz manifold, the projection map $\pi: E \to X$ is a Lipschitz map and for each $x \in X$ there exists a local coordinate system (U, h) with $x \in U$ such that h is a Lipschitz homeomorphism. Denote by $\Gamma(\xi)$ the Lipschitz sections of the bundle ξ.

A vector bundle ξ over a manifold X is called a **flat vector bundle** if there exists a family of local trivializations $\{U_\alpha, \phi_\alpha\}$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ are locally constant. On a flat vector bundle one can define locally constant sections.

Suppose now that ξ is a real or complex, flat Lipschitz vector bundle over X. Denote by $\Omega_d^r(X,\xi)$ the space $\Gamma(\xi) \otimes \Omega_d^r(X)$ (here and elsewhere the tensor product is over the Lipschitz functions on X). Locally the elements of $\Omega_d^r(X,\xi)$ can be expressed as linear combinations of elements of the form $s \otimes \omega$ where $s \in \Gamma(\xi|_{U_\alpha})$ and $\omega \in \Omega_d^r(U_\alpha)$. Further, s restricted to U_α, is given by $s = \Sigma a_i s_i$ where the s_i are constant sections on U_α and the a_i are L^2-functions. In light of this, it is possible to define the exterior derivative $d\eta$ of $\eta \in \Omega_d^r(X,\xi)$. It is given by $d\eta = d(\Sigma a_i s_i \otimes \omega) = d(\Sigma s_i \otimes a_i\omega) = \Sigma s_i \otimes d(a_i\omega)$. It is clear that $d^2 = 0$. This gives rise to the cohomology complex

$$2.1 \quad 0 \to \Omega_d^0(X,\xi) \xrightarrow{d} \Omega_d^1(X,\xi) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_d^n(X,\xi) \to 0$$

and the corresponding cohomology spaces

$$H_d^r(X,\xi) = \mathrm{Ker}\, d^r/\mathrm{Im}\, d^{r-1}.$$

§3 Statement of Theorem

Let X be a triangulated compact topological manifold of dimension n. Then X can be given a Lipschitz structure and so can be viewed as a Lipschitz manifold. Suppose that ξ is a real or complex Lipschitz vector bundle of dimension k over X. Introduce a flat affine metric on each simplex of the triangulation of X. Assume $f:X \to X$ is a Lipschitz isometry with a finite number of fixed points $\{x_i\}_{i=1}^{l}$ and call K the triangulation of X; assume also that each x_i lies in the interior of a distinct simplex of maximun dimension n. Suppose further that there exists a Lipschitz flat bundle map $\phi:f^*\xi \to \xi$ which covers the identity on X, (i.e. $\phi : f^*\xi|_x \to \xi|_x$ and carries locally constant sections to locally constant sections).

In this case it is possible to define a map $F^r : \Omega_d^r(X, \xi) \to \Omega_d^r(X, \xi)$ by setting

$$[F^r(s \otimes \omega)](x) = \phi(s(f(x)) \otimes f^*\omega \qquad \text{and}$$

extending linearly. Note two important properties of F^r:

3.1 $F^r : \Gamma(\xi|_U) \otimes \Omega_d^r(U) \to \Gamma(\xi|_{f^{-1}(U)}) \otimes \Omega_d^r(f^{-1}(U))$

for U any open subset of X.

3.2 $d^{r-1}F^{r-1} = F^r d^{r-1}$.

The second property follows from the assumption that ϕ carries locally constant sections to locally constant sections and the fact that d commutes with f^*.

From (3.2) it follows that F^r induces a map

$$(F^r)^* : H_d^r(X, \xi) \to H_d^r(X, \xi).$$

It will be shown that $H_d^r(X, \xi)$ is a finite dimensional vector space for all r so it makes sense to speak of the trace of $(F^r)^*$, denoted $tr(F^r)^*$.

The following Lefschetz fixed point formula will be proved:

3.3 **Theorem:** $\Sigma_r(-1)^r tr(F^r)^* = \Sigma_{i=1}^{l}(-1)^n a_i \, tr\phi|_{X_i}$

where the a_i are constants which depend only on the action of f on the simplex of maximum dimension containing x_i.

Actually, it will be shown that

$$\Sigma_{i=1}^{l}(-1)^{n}a_i = \Sigma_r(-1)^r tr(f^r)^*$$

where $(f^r)^*:H_d^r(X) \to H_d^r(X)$ is the map induced by f on the L^2-de Rham cohomology. Notice that the expression $tr\phi|_{x_i}$ makes sense since

$\phi:f^*\xi|_{x_i} = \xi|_{f(x_i)} = \xi|_{x_i} \to \xi|_{x_i}$. The number $\Sigma_r(-1)^r tr(F^r)^*$ is called the **Lefschetz number** of f relating to the complex $\Omega_d^*(X,\xi)$ and will be denoted L(f).

The proof of the theorem will proceed in the following order:

First, singular cochains on X with values in the bundle ξ will be defined. These will be shown to form a complex and the corresponging cohomology spaces will be denoted $H_\Delta^*(X,\xi)$. The map f together with the bundle homomorphism ϕ will induce a map also denoted

$$F^*:H_\Delta^*(X,\xi) \to H_\Delta^*(X,\xi).$$

The second step will be to show that $H_\Delta^*(X,\xi)$ and $H_d^*(X,\xi)$ are naturally isomorphic. This will be accomplished by showing

(i) $H^*_\Delta(X,\xi) \cong H^*(X,E)$ and (ii) $H^*_d(X,\xi) \cong H^*(X,E)$ where E is the sheaf of germs of locally constant sections of the bundle ξ and $H^*(X,E)$ are the sheaf cohomology spaces of X with coefficients in the sheaf E. Next it will be shown that the diagram

3.4
$$\begin{array}{ccc} H^*_\Delta(X,\xi) & \xrightarrow{F^*} & H^*_\Delta(X,\xi) \\ \wr\| & & \wr\| \\ H^*_d(X,\xi) & \xrightarrow{F^*} & H^*_d(X,\xi) \end{array}$$

is commutative.

Then simplicial cochains on X with values in ξ will be defined, giving rise to the cohomology spaces $H^*(K,\xi)$ where K is a triangulation of X. Then it will be shown $H^*_\Delta(X,\xi) \cong H^*(K,\xi)$ and again f and ϕ will induce a map, still denoted F^* so that the diagram

3.5
$$\begin{array}{ccc} H^*_\Delta(X,\xi) & \xrightarrow{F^*} & H^*_\Delta(X,\xi) \\ \wr\| & & \wr\| \\ H^*(K,\xi) & \xrightarrow{F^*} & H^*(K,\xi) \end{array}$$

is commutative.

Finally the theorem will be proved for $H^*(K,\xi)$ and so it will follow that it is true for $H_d^*(X,\xi)$.

§4 Singular Cochains on X with Values in ξ.

In order to define singular cochains on X with values in ξ, the following well known result is needed. It is stated here without proof.

4.1 **Proposition:** Any flat bundle over a connected, simply connected topological space is a product bundle compatible with the flat structure. That is, for E a flat bundle over base space B which satisfies the hypotheses, there exists an isomorphism $h:E \to B \times \mathbb{R}^n$ (or $\mathbb{C}^n$), such that h carries locally constant sections to locally constant sections.

Define $C^p(X,\xi)$ the space of **singular p-cochains on X with values in ξ** as follows: Let Δ^p be the standard Euclidean p-simplex. Let $\sigma:\Delta^p \to X$ be a map which extends to a Lipschitz map on a neighborhood of Δ^p in $\mathbb{R}^p$ into X. Denote by b_0 the barycenter of σ and let $C_p(x,Z)$ denote the space of chains, with integer coefficients, of all such σ. Then an element $\gamma \in C^p(X,\xi)$ is a map defined on $C_p(X,Z)$, such that $\gamma(\sigma) \in \xi|_{b_\sigma}$ and γ extends linearly to chains. Notice that if σ_1 and σ_2 are singular simplexes on X, then $\gamma(\sigma_1) \in \xi|_{b_{\sigma_1}}$ and $\gamma(\sigma_2) \in \xi|_{b_{\sigma_2}}$ so in general $\gamma(a\sigma_1 + b\sigma_2) = a\gamma(\sigma_1) + b\gamma(\sigma_2)$ is a formal sum.

Suppose $\gamma \in C^{p-1}(X,\xi)$, define $d\gamma$ by $d\gamma(\sigma) = \gamma(\partial\sigma) = \gamma(\Sigma(-1)^i\sigma_i) = \Sigma(-1)^i\gamma(\sigma_i)$ (the σ_i are the faces of σ). Now $(d\gamma)(\sigma) \in \xi|_{b_\sigma}$ and $\gamma(\sigma_i) \in \xi|_{b_{\sigma_i}}$, however this definition of the coboundary operator makes sense because all of the fibers over σ are identified via constant sections by proposition 4.1. The coboundary operator thus defined satisfies $d^2 = 0$ and the $C^p(X,\xi)$ together with d form a cochain complex. The associated cohomology spaces are denoted $H^*_\Delta(X,\xi)$.

Returning now to f and ϕ, these maps induce a map denoted $F:C^p(X,\xi) \to C^p(X,\xi)$ defined by

$$(F\gamma)(\sigma) = \phi(\gamma(f\circ\sigma)).$$

To see this definition makes sense, observe $(F\gamma)(\sigma) \in \xi|_{b_\sigma}$ and $\gamma(f\circ\sigma) \in \xi|_{b_{f\circ\sigma}}$. Now $b_{f\circ\sigma} = f(x)$ for some $x \in \sigma$, so $\xi|_{b_{f\circ\sigma}} = \xi|_{f(x)} = f^*\xi|_x$; however, $f^*\xi|_x$ can be identified with $f^*\xi|_{b_\sigma}$, because the pull-back bundle of a flat bundle is flat. Thus $\gamma(f\circ\sigma) \in f^*\xi|_{b_\sigma}$. From this it follows that $\phi(\gamma(f\circ\sigma)) \in \xi|_{b_\sigma}$. The map F commutes with the coboundary operator d for,

$$[d(f\gamma)](\sigma) = (F\gamma)(\partial\sigma)$$

$$= \phi(\gamma(\partial(f\circ\sigma))) = \phi(d\gamma(f\circ\sigma)) = [F(d\gamma)](\sigma).$$

That is, dF = Fd, hence F induces a map

$$F^*:H^p_\Delta(X,\xi) \to H^p_\Delta(X,\xi).$$

The proofs of propositions in the next two sections follow closely the proofs for the analogous statements for smooth manifolds and cohomology with values in a vector space as given in [13].

§5 Showing $H^*_d(X,\xi) \simeq H^*(X,E)$

To show that $H^*_d(X,\xi)$ and $H^*(X,\xi)$ are naturally isomorphic the following definitions and results from sheaf theory are needed [13].

5.1 **Definition:** A presheaf $\{S_U;\ \rho_{U,V}\}$ on a manifold M is said to be complete if whenever the open set U is expressed as a union $\cup_\alpha U_\alpha$ of open sets in M, the following two conditons are satisfied

(i) Whenever h and g in S_U are such that $\rho_{U\alpha,U}h = \rho_{U\alpha,U}g$ for all α, then h = g.

(ii) Whenever there is an element $g_\alpha \in S_{U\alpha}$ for each α such that $\rho_{U\alpha\cap U\beta,U\alpha}g_\alpha = \rho_{U\alpha\cap U\beta,U\beta}g_\beta$ for all α and β, then there exists $g \in S_U$ such that $g_\alpha = \rho_{U\alpha,U}g$ for each α.

5.2 **Definition:** Suppose $\mathcal{S}$ is a sheaf over M. The support of sheaf endomorphism $l:\mathcal{S} \to \mathcal{S}$ is the closure of the set of all points in M for which l restricted to the stalk of $\mathcal{S}$ over m is not zero. Suppose $\{U_i\}$ is a locally finite open cover of M. Then a collection of endomorphisms $\{l_i\}$ of $\mathcal{S}$ is called a partition of unity for $\mathcal{S}$ subordinate to the cover $\{U_i\}$, if

(a) support of $l_i \subset U_i$

(b) $\Sigma_i l_i$ = identity.

5.3 **Definition:** A sheaf $\mathcal{S}$ over M is said to be fine if for each locally finite open cover $\{U_i\}$ of M, there exists a partition of unity $\{l_i\}$ subordinate to this cover.

5.4 **Definition:** A sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{S}_0 \to \mathcal{S}_1 \to \cdots$$

is called a fine resolution of $\mathcal{A}$ if the sequence is exact and each of the $\mathcal{S}_i$ are fine.

If P is a presheaf on a manifold M, P canonically gives rise to a sheaf on M denoted $\beta(P)$ called the associated sheaf. Conversely, if $\mathcal{S}$ is a sheaf on M, $\mathcal{S}$ gives rise to a presheaf and also the space of sections of the sheaf $\mathcal{S}$ and both are denoted $\Gamma(\mathcal{S})$, with some abuse of notation.

5.5 **Theorem:** If P is a complete presheaf, then $\Gamma(\beta(P))$ is canonically isomorphic to P.

5.6 **Theorem:** Let $0 \to \mathcal{S} \to \mathcal{C}_0 \to \mathcal{C}_1 \to \cdots$ be a fine resolution of the sheaf $\mathcal{S}$ on M. Then

$$0 \to \Gamma(\mathcal{C}_0) \to \Gamma(\mathcal{C}_1) \to \cdots$$

is a cochain complex and there are canonical isomorphisms $H^q(M,\mathcal{S}) \cong H^q(\Gamma(\mathcal{C}^*))$ for all q.

5.7 **Proposition:** Suppose $\{S_U, \rho_{U,V}\}$ is a presheaf with associated sheaf $\mathcal{S}$ over M and suppose $\{S_U, \rho_{U,V}\}$ satisfies only condition (ii) of definiton 5.1. Then the sequence $0 \to (S_M)_0 \to S_M \xrightarrow{L} \Gamma(S) \to 0$ is exact. $((S_M)_0 =$

$\{s \in S_M\colon \rho_{m,M}(s) = 0$ for all $m \in M\}$ where $\rho_{m,U}\colon S_U \to \mathcal{S}_m$ is the natural projection which assigns each element of S_U to its equivalence class in $\mathcal{S}_m$, and τ is the homomorphism which sends $s \in S_M$ to the global section $m \mapsto \rho_{m,M}(s)$ of $\mathcal{S}$.)

Returning to the manifold X, suppose U is an open set in X. Recall, $\Omega_d^r(U) = \{\omega\colon \omega \in L_2^r(U),\ d\omega \in L_2^{r+1}(U)\}$ and set $S^r(U) = \Gamma(\xi|_U) \otimes \Omega_d^r(U)$. For any $V \supset U$ define $\rho_{U,V}\colon S^r(V) \to S^r(U)$ to be the restriction map, that is , for any $\eta \in S^r(V)$ $\rho_{U,V}(\eta)$ is the restriction of η to U. Then $\{S^r(U), \rho_{U,V}\}$ is a complete presheaf with corresponding cochain complex

5.8 $$0 \to S^0(U) \xrightarrow{d} S^1(U) \xrightarrow{d} S^2(U) \xrightarrow{d} \cdots$$

This d is the same as defined for 2.1. To each $\{S^r(U), \rho_{U,V}\}$ there is the associated sheaf of germs, denoted $\mathcal{S}^r(X)$.

Consider the following sequence of sheaves

5.9 $$0 \to \mathcal{E} \xrightarrow{i} \mathcal{S}^0(X) \xrightarrow{d} \mathcal{S}^1(X) \xrightarrow{d} \cdots$$

In this sequence d is the sheaf homomorphism induced by the presheaf homorphism d of 5.7. Recall $\mathcal{E}$ is the sheaf of germs of locally constant sections and $\mathcal{S}^0(X)$

is the sheaf of germs of Lipschitz sections, so the map i is inclusion.

5.10 Proposition: The sequence 5.8 is a fine resolution of E.

Proof: Let $\{U_i\}$ be any finite covering of X and suppose ψ_i is a partition of unity subordinate to this covering. Define presheaf endomorphisms l_i of $\{S^r(U), \rho_{U,V}\}$ by $l_i(\eta) = \psi|_U \cdot \eta$ for $\eta \in S^r(U)$. The l_i induce sheaf endomorphisms l_i which satisfy

(i) supp $l_i \subset U_i$

(ii) $\Sigma_i l_i$ = identity.

This shows the $\mathcal{S}^r(X)$ are fine.

The sequence 5.9 is clearly exact at E and $\mathcal{S}^0(X)$, also $d^2 = 0$ because 5.8 is a complex. That the sequence is exact follows from the Poincaré lemma for $\Omega_d^r(X)$ as given in [11].

It then follows from 5.10 and 5.6 that $H^q(X,E) \cong H^q(\Gamma(\mathcal{S}^r(X))$. Since $\Gamma(\beta(\{S^r(U);\rho_{U,V}\})) = \{\Gamma(\mathcal{S}^q|_U; \rho_{U,V}\}$, by 5.5 it follows that $S^r(X)$ and $\Gamma(\mathcal{S}^r(X))$ are naturally isomorphic. Hence

$$H^q(\Gamma(\mathcal{S}^*(X))) \cong H^q(S^*(X)) = H_d^q(X,\xi)$$

and so $H^q(X,E) \cong H_d^q(X,\xi)$.

§6 Showing $H^q(X,E) \approx H_\Delta^q(X,\xi)$

Let U be any open set in X, then $C^p(U,\xi)$ is the space of singular cochains on U with values in ξ. For $U \subset V$ define $\rho_{U,V}:C^p(V,\xi) \to C^p(U,\xi)$ to be the restriction of p-cochains acting on singular simplexes in V, to p-cochains acting on singular simplexes in U. Then the $\{C^p(U,\xi), \rho_{U,V}\}$ form a presheaf of vector spaces on X. Denote the associated sheaf of germs of p-cochains on X with values in ξ by $\mathcal{C}^p(X,\xi)$.

The coboundary operator d on $C^p(X,\xi)$ gives rise to a sequence of sheaves

6.1 $$0 \to E \to \mathcal{C}^0(X,\xi) \xrightarrow{d} \mathcal{C}^1(X,\xi) \xrightarrow{d} \cdots .$$

6.2 **Proposition:** The sequence 6.1 is a fine resolution of E.

Proof: Take any finite covering $\{U_i\}$ of X and a partition of unity $\{\psi_i\}$ subordinate to the cover $\{U_i\}$ in which the functions take only the values 0 or 1. For each i define $l_i:C^p(U,\xi) \to C^p(U,\xi)$ by setting $[l_i(\gamma)](\sigma) = \psi_i(\sigma(0))\gamma(\sigma)$. The l_i can be seen to be presheaf endomorphisms and the corresponding sheaf endomorphisms form a partition of unity. This shows that the sheaves in 6.1 are fine.

To see the sequence is a resolution note that $\mathcal{C}^0(X,\xi)$ is just the sheaf of germs of sections of ξ so E naturally injects into $\mathcal{C}^0(X,\xi)$. Also the sequence is exact at $\mathcal{C}^0(X,\xi)$ since $d[\gamma]_m = [d\gamma]_m$ ($[\gamma]_m$ = germ of γ at m) if and only if γ is constant on a neighborhood of m. To show the sequence is exact elsewhere it is possible to use the same homotopy operator used in the analogous proof for singular cohomology with values in a vector space.

It then follows by theorem 5.2 that $H^q(X,\xi) \approx H^q(\Gamma(\mathcal{C}^*(X,\xi))$ for all q. It remains to be seen that $H^q(\Gamma(\mathcal{C}^*(X,\xi)) \approx H^q_\Delta(X,\xi)$.

$\{C^p(U,\xi),\ \rho_{U,V}\}$ satisfy the hypotheses of 5.7. So the sequence

$$0 \to C^*_\emptyset(X,\xi) \to C^*(X,\xi) \to \Gamma(\mathcal{C}^*(X,\xi)) \to 0$$

is exact. To this short exact sequence there corresponds the long exact sequence

$$\cdots \to H^q(C^*_\emptyset(X,\xi)) \to H^q(C^*(X,\xi)) \to H^q(\Gamma(\mathcal{C}^*(X,\xi))) \to H^{q+1}(C^*_\emptyset(X,\xi)) \to \cdots$$

Thus if it can be shown that $H^q(C^*_\emptyset(X,\xi)) = 0$ for all q, then it will follow that

$$H^q(\Gamma(C^*(X,\xi))) \approx H^q(C^*(X,\xi)) = H^q_\Delta(X,\xi).$$

The proof that $H^q(C^*_0(X,\xi)) = 0$ proceeds just as in the case of cohomology with values in a vector space case. Thus

$$H^q_\Delta(X,\xi) \approx H^q(X,E).$$

§7 Showing Diagram 3.4 is Commutative

In order to show that diagram 3.4 is commutative, the following definitions and theorems from sheaf theory are required [3].

7.1 **Definition:** Let $f:X \to Y$ be a continuous map. If A and B are sheaves on X and Y respectively, an f-cohomomorphism $k:B \to A$ is a collection of homomorphisms $k_x:B_{f(x)} \to A_x$ for all $x \in X$, such that for any section $s \in \Gamma(B|_U)$, the map $x \mapsto k_x(s(f(x))$ is a section of A over $f^{-1}(U)$.

7.2 **Definition:** Let $f:X \to Y$ be continuous. If A and B are prsheaves on X and Y respectively an f-cohomomorphism $k:B \to A$ is a collection of homomorphisms $k_U:B(U) \to A(f^{-1}(U))$ for U open in Y compatible with restrictions.

Notice that an f-cohomomorphism of sheaves induces an f-cohomomorphism of presheaves by putting

$$[k_U(s)](x) = k_x(s(f(x))).$$

7.3 Theorem: If $k:\mathcal{B} \to \mathcal{A}$ is an f-cohomomorphism, then k induces a natural homomorphism $k^*:H^*(Y,\mathcal{B}) \to H^*(X,\mathcal{A})$ and futhermore if $\mathcal{B}^*$ and $\mathcal{A}^*$ are any acyclic resolutions of $\mathcal{B}$ and $\mathcal{A}$ respectively and $g^*:\mathcal{B}^* \to \mathcal{A}^*$ is an f-cohomomorphism extending k (i.e. commuting with differentials and augmentations), then the following diagram is commutative:

$$\begin{array}{ccc} H^*(\Gamma(\mathcal{B}^*)) & \xrightarrow{g^*} & H^*(\Gamma(\mathcal{A}^*)) \\ \wr\| & & \wr\| \\ H^*(Y,\mathcal{B}) & \xrightarrow{k^*} & H^*(X,\mathcal{A}) \end{array}$$

7.4 Theorem: A fine sheaf on a compact manifold is acyclic.

Return now to the manifold X, bundle ξ, map f and homomorphism ϕ. Notice ϕ induces an f-cohomorphism ϕ^* of E into itself, that is, $\phi^*: E_{f(x)} \to E_x$ is defined by

7.5 $$\phi^*[s]_{f(x)} = [\phi(s)]_x.$$

This makes sense because $\phi: f^*\xi|_x = \xi|_{f(x)} \to \xi|_x$.
In sections 3 and 4 maps

$$F: \Omega_d^*(X,\xi) \to \Omega_d^*(X,\xi)$$

and $$F: C^*(X,\xi) \to C^*(X,\xi)$$

were defined, but both these maps can be viewed as being induced by f-cohomomorphisms

$$F^*: S^*(X,\xi) \to S^*(X,\xi)$$

$$F^*: C^*(X,\xi) \to C^*(X,\xi)$$

respectively defined by

$$F^*[s\otimes\omega]_{f(x)} = [\phi(s)\otimes f^*\omega]_x$$

and $$F^*[\gamma]_{f(x)} = [\phi(\gamma)]_x.$$

Also it was shown that S^* and C^* are fine resolutions of E so by 7.4 they are acyclic. The F*'s are f-cohomomorphisms extending ϕ^*. Hence by theorem 7.3, the following diagram is commutative

$$
\begin{array}{ccc}
H^*(\Gamma(\mathcal{S}^*)) & \xrightarrow{f^*} & H^*(\Gamma(\mathcal{S}^*)) \\
\wr\Vert & & \wr\Vert \\
H^*(X,\mathcal{E}) & \xrightarrow{f^*} & H^*(X,\mathcal{E}) \\
\wr\Vert & & \wr\Vert \\
H^*(\Gamma(\mathcal{C}^*)) & \xrightarrow{f^*} & H^*(\Gamma(\mathcal{C}^*))
\end{array}
$$

which, in light of sections 5 and 6 gives the commutativity of

$$
\begin{array}{ccc}
H^*_d(X,\xi) & \xrightarrow{f^*} & H^*_d(X,\xi) \\
\wr\Vert & & \wr\Vert \\
H^*_\Delta(X,\xi) & \xrightarrow{f^*} & H^*_\Delta(X,\xi)
\end{array}
$$

§8 Showing $H^*(K,\xi) \cong H^*_\Delta(X,\xi)$

Let K be a triangulation of X. Denote the geometric realization of K by $|K|$ (that is, $|K|$ is homeomorphic to X). Denote by $C^*(K,\xi)$ the space of simplicial cochains on K with values in the flat bundle ξ over $|K|$. So if s is a p-simplex of K and $\gamma \in C^p(K,\xi)$, then $\gamma(s) \in \xi|_{|s|}$. The coboundary operator d is defined in the usual way. It is well defined because $|s|$ is connected and simply connected, so the fibers over s are naturally identified via the flat structure of ξ. Let $H^*(K,\xi)$ be the spaces of cohomology classes obtained from the cochain complex

$$C^0(K,\xi) \xrightarrow{d} C^1(K,\xi) \xrightarrow{d} C^2(K,\xi) \xrightarrow{d} \cdots$$

8.1 **Theorem:** $H^p(K,\xi) \approx H^p_\Delta(X,\xi)$ for all p.

In order to prove 8.1 the following well known result will be used

8.2 **Lemma:** ("the five lemma" [8]) Given a commutative diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc} G_5 & \to & G_4 & \to & G_3 & \to & G_2 & \to & G_1 \\ \downarrow\gamma_5 & & \downarrow\gamma_4 & & \downarrow\gamma_3 & & \downarrow\gamma_2 & & \downarrow\gamma_1 \\ H_5 & \to & H_4 & \to & H_3 & \to & H_2 & \to & H_1 \end{array}$$

in which each row is exact and γ_1, γ_2, γ_4 and γ_5 are isomorphisms, then γ_3 is an isomorphism.

Proof of 8.1: Define a map $\tau:C^p(X,\xi) \to C^p(K,\xi)$ as follows: Given any simplex $[v_0, v_1,\cdots, v_p] \in K$ there exists a unique $\sigma \in C_p(X,Z)$ that maps the i^{th} vertex of Δ^p to v_i. So define $\tau':C_p(K,Z) \to C_p(X,Z)$ to be the map which sends $[v_0, v_1, \cdots, v_p]$ to this unique singular simplex. Then for $\gamma \in C^p(X,\xi)$ set

$$(\tau\gamma)[v_0, v_1, \cdots, v_p] = \gamma(\tau'[v_0, v_1, \cdots, v_p]) = \gamma(\sigma).$$

Extend τ to a cochain map. It will be shown that this cochain map induces an isomorphism of cohomology.

Claim: Given any subcomplexes K_1 and K_2 of K, the rows of the following diagram are Mayer-Vietoris sequences.

$$\to H^p_\Delta(|K_1| \cup |K_2|, \xi) \to H^p_\Delta(|K_1|, \xi) \oplus H^p_\Delta(|K_2|, \xi) \to H^p_\Delta(|K_1| \cap |K_2|, \xi) \to$$

$$\to H^p(K_1 \cup K_2, \xi) \to H^p(K_1, \xi) \oplus H^p(K_2, \xi) \to H^p(K_1 \cap K_2, \xi) \to$$

Proof of claim: The first exact sequence exists because $|K_1|$ and $|K_2|$ are an excisive pair and sheaf cohomology theory guarantees the existence of a Mayer-Vietoris sequence for an excisive pair. The second exact sequence exists because $\{C^pK,\xi), d\}$ is a chain complex so a short exact sequence on the chain level gives rise to a long exact sequence on the cohomology level. Now $C^p(K_1 \cap K_2, \xi) = C^p(K_1, \xi) \cap C^p(K_2, \xi)$ and $C^p(K_1, \xi) + C^p(K_2, \xi) = C^p(K_1 \cup K_2, \xi)$. It follows from this that there exists the short exact sequence

$$0 \to C^p(K_1 \cup K_2, \xi) \to C^p(K_1, \xi) \oplus C^p(K_2, \xi) \to C^p(K_1 \cap K_2, \xi) \to 0$$

and the associated long exact sequence is the second Mayer-Vietoris sequence of the diagram.

The proof of 8.1 will now proceed by induction on the number of simplexes in K. Suppose K consists of only one simplex s. Then s must be a zero simplex. But a bundle over a one point space is trivial so $H^p_\Delta(|s|, \xi)$ and $H^p(s, \xi)$ are as in the constant coefficient case where it is known that $H^p(s, G) \approx H^p_\Delta(|s|, G)$ (G is a group). Now suppose the result is known for any complex containing less than n simplexes. Let s be a simplex of K having maximum dimension and let L be the subcomplex of K consisting of all simplexes except s. Then $K = L \cup s$ and $L \cap s = \dot{s}$ ($\dot{s}$ is the subcomplex consisting of proper faces of s). So $H^p_\Delta(|L|, \xi) \approx H^p(L, \xi)$ and $H^p_\Delta(|L \cap s|, \xi) \approx H^p(L \cap s, \xi)$ by the induction hypothesis. It is known that τ induces an isomorphism between singular cohomology groups and simplicial cohomology groups in the constant coefficient case, so τ induces an isomorphism between $H^p_\Delta(|s|, \xi)$ and $H^p(s, \xi)$ (this is because $|s|$ is connected and simply connected so ξ restricted to s is a trivial bundle, which reduces this to the constant coefficient case).

Thus by the Mayer-Vietoris sequences and 8.2 the desired result is obtained.

§9 Proof of Theorem 3.3

Suppose X, ξ, f, and ϕ are as before.

9.1 Lemma: Suppose $g:X \to X$ is continuous and g is homotopic to f, then g induces a map

$$G^*:H^*_\Delta(X,\xi) \to H^*_\Delta(X,\xi)$$

such that $G^* = F^*$.

Proof: Define maps $\tilde{g},\tilde{f}:X \to X \times I$ (I is the unit interval) by $\tilde{f}(x) = (x,0)$ and $\tilde{g}(x) = (x,1)$ for all $x \in X$. Suppose $h:X \times I \to X$ is the map which defines the homotopy of f and g. That is, $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Then $f = h\circ\tilde{f}$ and $g = h\circ\tilde{g}$. Since $h:X \times I \to X$, the "pull back" bundle $h^*\xi$ over $X \times I$ exists. Define maps $\tilde{F}:C^p(X{\times}I,h^*\xi) \to C^p(X,\xi)$, $\tilde{G}:C^p(X{\times}I,h^*\xi) \to C^p(X,\xi)$ and $H:C^p(X,\xi) \to C^p(X{\times}I,h^*\xi)$ as follows: First note that for $\gamma \in C^p(X{\times}I,h^*\xi)$ and σ a singular simplex in X, $(\tilde{F}\gamma)(\sigma)$ should be contained in $\xi|_{b_\sigma}$ and $\gamma(\tilde{f}\circ\sigma) \in h^*\xi|_{b_{\tilde{f}\circ\sigma}} = \tilde{f}^*h^*\xi|_{b_\sigma} = f^*\xi|_{b_\sigma}$. So define

$$(\tilde{F}\gamma)(\sigma) = \phi(\gamma(\tilde{f}\circ\sigma)) \in \xi|_{b_\sigma}.$$

Now $g \simeq f$ implies that $g^*\xi$ and $f^*\xi$ are naturally isomorphic vector bundles. Also for any fixed $t_0 \in I$, set

9.2 $h(x,t_0) = h_{t_0}(x)$, then $h_{t_0}: X \to X$ and $h_{t_0}{}^*\xi$ is naturally isomorphic to $f^*\xi$. So consider these bundles identified via the natural isomorphisms.

In light of this it makes sense to define

$$\tilde{G}: C^p(X \times I,\ h^*\xi) \to C^p(X,\xi) \text{ by}$$

$$(\tilde{G}\gamma)(\sigma) = \phi(\gamma(\tilde{g}\circ\sigma)).$$

As to H, notice $(H\gamma)(\sigma)$ should be contained in $h^*\xi|_{b_\sigma}$ (here σ is a singular simplex in $X \times I$ and $\gamma \in C^p(X,\xi)$), so set

$$(H\gamma)(\sigma) = \gamma(h\circ\sigma) \in \xi|_{b_{h\circ\sigma}} = h^*\xi|_{b_\sigma}.$$

Then $\tilde{F}H: C^p(X,\xi) \to C^p(X,\xi)$ and

$$[(\tilde{F}H)\gamma](\sigma) = [\tilde{F}(H\gamma)](\sigma)$$

$$= \phi(H\gamma(\tilde{f}\circ\sigma)) = \phi(\gamma(h\circ\tilde{f}\circ\sigma)) = \phi(\gamma(f\circ\sigma)) = (F\gamma)(\sigma).$$

Thus $\tilde{F}H = F$. Similarly $\tilde{G}H = H$. Hence $\tilde{F}^*H^* = F^*$ and $\tilde{G}^*H^* = G^*$, where

$$\tilde{G}^*, \tilde{F}^*: H^*_\Delta(X \times I, h^*\xi) \to H^*_\Delta(X,\xi)$$

and

$$H^*: H^*_\Delta(x,\xi) \to H^*_\Delta(X \times I, h^*\xi)$$

are the maps induced by, respectively, $\tilde{G}$,$\tilde{F}$ and H. So to prove $F^* = G^*$, it needs only to be shown that $\tilde{F}^* = \tilde{G}^*$.

For G any abelian group, there exists a map

$$p: C_p(X,G) \to C_{p+1}(X\times I,G)$$

called the prism operator [12]. It satisfies the following: If

$$\tilde{f}': C_p(X,G) \to C_p(X\times I,G)$$

$$\tilde{g}': C_p(X,G) \to C_p(X\times I,G)$$

are the maps induced on singular p-chains by $\tilde{f}$ and $\tilde{g}$ respectively, then

$$\tilde{g}' - \tilde{f}' = dp + pd.$$

Define $P: C^{p+1}(X\times I, h^*\boldsymbol{\xi}) \to C^p(X,\boldsymbol{\xi})$ by setting

$$(P\boldsymbol{\gamma})(\sigma) = \phi(\boldsymbol{\gamma}(p(\sigma))).$$

This definition makes sense in view of 9.2.

Claim: $\tilde{G} - \tilde{F} = dP + Pd$

Proof of claim: $[(\tilde{G} - \tilde{F})(\boldsymbol{\gamma})](\sigma) = \phi(\boldsymbol{\gamma}(\tilde{g} - \tilde{f})\circ\sigma)$. On the other hand,

$$[(dP + Pd)(\boldsymbol{\gamma})](\sigma) = [(dP)(\boldsymbol{\gamma})](\sigma) + [(Pd)(\boldsymbol{\gamma})](\sigma)$$

$$= \phi(\boldsymbol{\gamma}(p(d\sigma))) + \phi(\boldsymbol{\gamma}(dp(\sigma))) = \phi[\boldsymbol{\gamma}(pd(\sigma)) + \boldsymbol{\gamma}dp(\sigma))]$$

$$= \phi(\boldsymbol{\gamma}(pd + dp)(\sigma)),$$

since $\tilde{g} - \tilde{f} = pd + dp$ it follows that $\tilde{G} - \tilde{F} = Pd + dP$. Then, it follows from the claim that $\tilde{G}^* = \tilde{F}^*$.

9.3 **Definition:**[8] Let K and K′ be simplicial complexes and suppose $f:|K| \to |K'|$ is a continuous map. A simplicial map $h:K \to K'$ is called a simplicial approximation to f if for $x \in |K|$ the set $\{f(x), |h|(x)\}$ is contained in the closure of some $|s'|$ where s′ is a simplex of K′. ($|h|:|K| \to |K'|$ is the continuous map induced by h.)

9.4 **Theorem:**[8] Let $f:|K| \to |K'|$ be continuous. There exists a subdivision of K, call it L, which admits a simplicial approximation $h:|L| \to |K'|$ of f. And any subdivision of L also admits a simplicial approximation to f. Furthermore h is homotopic to f.

9.5 **Theorem:** If K′ is a subdivision of K, there exists a subdivision chain map $\gamma:C^p(K',\xi) \to C^p(K,\xi)$. If $\psi:K' \to K$ is a simplicial approximation to the identity then $\gamma^* = \psi^{*-1}:H^p(K',\xi) \to H^p(K,\xi)$.

9.6 **Theorem:** For a simplicial map $h:K \to K'$, the diagram

$$\begin{array}{ccc} H^*(K',\xi) & \xleftarrow{H^*} & H^*(K,\xi) \\ \wr\| & & \wr\| \\ H^*_\Delta(|K'|,\xi) & \xleftarrow{|H|^*} & H^*_\Delta(|K|,\xi) \end{array}$$

is commutative. (H^* and $|H|^*$ are the maps induced by h.)

The proofs of the last two theorem are as in the constant coefficient case [8].

Proof of 3.3: Recall $f:X \to X$ has a finite number of fixed points $\{x_i\}_{i=1}^{l}$ and K is a triangulation of X such that each x_i lies in the interior of a simplex of maximum dimension. Let K' be a subdivision of K fine enough so that there exists a simplicial map $h:K' \to K$ which is an approximation to f, and choose K' so that the fixed points still lie in the interior of simplexes of K' of maximum dimension. Observe that the simplex of K' containing x_i must be mapped by h onto the simplex of K containing x_i for all i.

Denote by X' the space obtained from X by deleting the interior of the simplexes of K' containing the fixed points. Then $f:X' \to X$ has no fixed points and because X is compact, there exists $\varepsilon > 0$ such that

$d(x,f(x)) > \varepsilon$ for all $x \in X'$ ($d(x,y)$ denotes the distance from x to y). Denote by L' a subdivision of K' restricted to X' of mesh less than $\varepsilon/2$, and let L be the triangulation of X which consists of the simplexes of L' together with the simplexes of K' containing the fixed points. Then there exists a simplicial approximation of f still denoted by $h:L \to K$ with the property that for any simplex $s \in L'$, $h(s) \cap s = \emptyset$.

9.6 Observe that this also means that for any simplex $s \in L$ of less than maximum dimension, $h(s) \cap s = \emptyset$.

Now let $\gamma':C^p(L) \to C^p(K)$ be the dual of the subdivision chain map. This induces a map still denoted $\gamma':C^p(L,\xi) \to C^p(K,\xi)$. Let $H:C^p(K,\xi) \to C^p(L,\xi)$ be the map induced by h. Then

$$(H \circ \gamma')^p:C^p(L,\xi) \to C^p(L,\xi).$$

Let $\{s_i\}_{i=1}^{l}$ be the simplexes of L of maximum dimension n and suppose that the fixed points $x_i \in s_i$ for $1 \le i \le l$. Denote by c_{ij} the basis for $C^n(L,\xi)$ where $c_{ij} = c_i \otimes e_j^i$, $c_i \in C^n(L)$ is such that $c_i(s_k) = \delta_{ik}$ and $\{e_j^i\}_{j=1}^{K}$ is a basis of locally constant sections for $\xi|_{s_i}$.

Because $h(s_i) \cap s_i = \emptyset$ for all i such that

$1 < i \leq m$, the trace of $(H \circ \gamma')^n : C^n(L,\xi) \to C^n(L,\xi)$ depends only on the action of $(H \circ \gamma')^n$ on $\{c_{ij}\}_{1\leq i\leq l}$ and for any $p \neq n$

9.7 $(H \circ \gamma')^p : C^p(L,\xi) \to C^p(L,\xi)$ has trace zero by 9.6.

Since $h(s_i) \supseteq s_i$ for $1 \leq i \leq l$ and $\xi|_{h(si)}$ is trivial (with fibers naturally identified via the flat structure of ξ), it follows that

$$(H\circ\gamma')^n(c_{ij}) = (H\circ\gamma')^n(c_i \otimes e_j^i) = [(h'\circ\gamma')^n(c_i)]\otimes\phi|_{X_i} e_j^i$$

for $1\leq i\leq l$. Here $h' : C^p(K) \to C^p(L)$ is induced by h and the map ϕ which is the "same" on all fibers of $\xi|_{h(s)}$, (this is because ϕ preserves the flat structure of ξ), is denoted by $\phi|_{X_i}$. So suppose

9.8 $$(h'\circ\gamma')^n c_i \otimes \phi|_{X_i} e_j^i = a_i c_i \otimes b_j^i e_j^i + \Sigma(\text{other terms})$$
$$= a_i b_j^i c_i \otimes e_j^i + \Sigma(\text{other terms}),$$

then the trace of $(H\circ\gamma')^n : C^n(L,\xi) \to C^n(L,\xi)$ is

$$\Sigma_{i=1}^{l} \Sigma_{j=1}^{k} a_i b_j^i = \Sigma_{i=1}^{l} a_i \operatorname{tr}\phi|_{X_i}$$

and from 9.7 it follows that

$$\Sigma_{r=0}^{n} (-1)^r \operatorname{tr}(H\circ\gamma')^r = \Sigma_{i=1}^{l} (-1)^n a_i \operatorname{tr}\phi|_{X_i}.$$

Also from 9.8 and 9.7 it follows

$$\Sigma_{r=0}^{n} (-1)^r \operatorname{tr}(h'\circ\gamma')^r = \Sigma_{i=1}^{l} (-1)^n a_i.$$

It remains to be shown that

$$\Sigma_r(-1)^r \mathrm{tr}(H\circ\gamma')^r = \Sigma_r(-1)^r \mathrm{tr}(F^r)^*.$$

The following diagram is commutative:

$$\begin{array}{ccccc} H^*(L,\xi) & \xrightarrow{\gamma^*} & H^*(K,\xi) & \xrightarrow{h^*} & H^*(L,\xi) \\ \wr\| & & \wr\| & & \wr\| \\ H^*_\Delta(X,\xi) & \xrightarrow{F^*} & H^*_\Delta(X,\xi) & \xrightarrow{Id.} & H^*_\Delta(X,\xi) \end{array}$$

That the diagram is commutative follows from theorems 9.5, 9.6 and lemma 9.1. This shows that

$$\Sigma(-1)^r \mathrm{tr}(H\circ\gamma')^{r*} = \Sigma(-1)^r \mathrm{tr}(F^r)^* = L(f).$$

However, it can be shown by purely algebraic means that $\Sigma(-1)^r \mathrm{tr}(H\circ\gamma')^{r*} = \Sigma(-1)^r \mathrm{tr}(H\circ\gamma')^r$ [4].

Similarly $\Sigma(-1)^r \mathrm{tr}(h\circ\gamma')^r = \Sigma(1-)^r \mathrm{tr}(f^r)^*$ so

$$\Sigma(-1)^r \mathrm{tr}(f^r)^* = \Sigma_i(-1)^n a_i.$$

Chapter 2.

§1 An Abstract Setting

Let X be a compact topological manifold and H_0 and H_1 Hilbert spaces. Denote by $L(H_i)$, the bounded operators from H_i into itself for i = 0,1. Suppose there exist *-homomorphisms σ_0 and σ_1 from the continuous functions on X into $L(H_0)$ and $L(H_1)$ respectively. Suppose also, that $F:H_0 \to H_1$ is a bounded Fredholm operator.

Let $f:X \to X$ be a continuous map with no fixed points, and suppose there exist trace class operators $T_i:H_i \to H_i$ for i =0,1 satisfying

1. $T_1F = FT_0$ and
2. $T_i\sigma_i(g) = \sigma_i(g \circ f)T_i$ for i =0,1 and g any continuous function on X.

It follows from condition 1. that T_0:KerF → KerF and T_1:ImF →ImF. Because T_1 maps ImF into itself, T_1 induces a map

$\tilde{T}_1$:CokerF $\to$ CokerF. (KerF = kernel of F, ImF = image of F, CokerF = cokernel of F.) Denote the restriction of T_0 to the kernel of F by $\tilde{T}_0$.

Since F is Fredholm KerF and CokerF are finite dimensional Hilbert spaces.

§2 The Theorem

Lemma: $TrT_1 - TrT_0 = Tr\tilde{T}_1 - Tr\tilde{T}_0$ (TrT denotes the trace of T).

Proof: Since F is Fredholm H_0 and H_1 have the following direct sum decompositions:

3. $$H_0 = KerF \oplus (KerF)^{\perp}$$
$$H_1 = ImF \oplus (ImF)^{\perp} .$$

Denote T_1 restricted to ImF by A_1, T_0 restricted to $(KerF)^{\perp}$ by B_0 and T_1 restricted to $(ImF)^{\perp}$ by B_1. Now F:$(KerF)^{\perp} \to$ ImF is bijective so from condition 1. it follows that $FB_0 = A_1F$ or

4. $$A_1 = FB_0F^{-1} .$$

Then from 3. and 4. it can be seen that T_0 and T_1 have the matrix representations

$$T_0 = \begin{pmatrix} \tilde{T}_0 & C_0 \\ 0 & B_0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} FB_0F^{-1} & C_1 \\ 0 & B_1 \end{pmatrix}$$

(The zeros in the bottom left hand corner come from the fact that KerF is invariant under T_0 and ImF is invariant under T_1.)

Let $\{e_n\}$ be any orthonormal basis for $(\mathrm{KerF})^{\perp}$ and $\{f_n\}$ an orthonormal basis for KerF. Then $\{e_n, f_n\}$ is an orthonormal basis for H_0 and

$$\mathrm{Tr}T_0 = \Sigma\langle e_n, B_0 e_n\rangle + \Sigma\langle f_n, \tilde{T} f_n\rangle$$

or

$$\Sigma\langle e_n, B_0 e_n\rangle = \mathrm{Tr}T_0 - \mathrm{Tr}\tilde{T}.$$

This shows that $\mathrm{Tr}T_0 = \mathrm{Tr}B_0 + \mathrm{Tr}\tilde{T}$, similarly $\mathrm{Tr}T_1 = \mathrm{Tr}A_1 + \mathrm{Tr}B_1$. Thus $\mathrm{Tr}T_1 - \mathrm{Tr}T_0 = \mathrm{Tr}A_1 + \mathrm{Tr}B_1 - \mathrm{Tr}\tilde{T} - \mathrm{Tr}B_0 = \mathrm{Tr}B_1 - \mathrm{Tr}\tilde{T}$.

(The last equality follows from 4.)

Now the diagram

$$\begin{array}{ccc} (\mathrm{ImF})^{\perp} & \overset{B_1}{\to} & (\mathrm{ImF})^{\perp} \\ \wr\Vert & & \wr\Vert \\ \mathrm{CokerF} & \overset{\tilde{T}_1}{\to} & \mathrm{CokerF} \end{array}$$

is commutative so $\mathrm{Tr}B_1 = \mathrm{Tr}\tilde{T}$ and so the lemma has been proved.

Theorem: $\mathrm{Tr}\tilde{T}_1 = \mathrm{Tr}\tilde{T}_0$

Proof: It will be shown that $\mathrm{Tr}T_1 = \mathrm{Tr}T_0 = 0$ and then the theorem follows from the lemma.

Denote by $\Gamma(f)$ the graph of f in $X \times X$ and let Δ denote the diagonal in $X \times X$. Because X is compact and f has no fixed points, there exists $\varepsilon > 0$ such that the distance from $\Gamma(f)$ to Δ is greater than ε. Choose a finite open cover $\{U_j\}$ of X such that the diameter of each U_j is less than ε, and let $\{\phi_j\}$ be a partition of unity subordinate to this cover. Then the ϕ_j have the property that $\mathrm{supp}\phi_j \cap f^{-1}(\mathrm{supp}\phi_j) = \emptyset$.

It is known that for A and B trace class and $C \in L(H)$ (i) $\mathrm{Tr}(A + B) = \mathrm{Tr}A = \mathrm{Tr}B$ and (ii) CA and AC are trace class and TrAC =TrCA. Thus

$$\mathrm{Tr}T_i = \mathrm{Tr}(T_i\Sigma\sigma_i(\phi_j))$$
$$= \Sigma\mathrm{Tr}T_i\sigma_i(\phi_j) = \Sigma\mathrm{Tr}T_i\sigma_i((\sqrt{\phi_j})^2)$$
$$= \Sigma\mathrm{Tr}\sigma_i(\sqrt{\phi_j})T_i\sigma_i(\sqrt{\phi_j}) = \Sigma\mathrm{Tr}\sigma_i(\sqrt{\phi_j})\sigma_i(\sqrt{\phi_j}\circ f)T_i$$
$$= \Sigma\mathrm{Tr}\sigma_i(\sqrt{\phi_j}(\sqrt{\phi_j}\circ f))T_i$$

but the $\mathrm{supp}(\sqrt{\phi_j}\circ f) = f^{-1}(\mathrm{supp}\phi_j)$ hence

$$\Sigma\mathrm{Tr}\sigma_i(\sqrt{\phi_j}(\sqrt{\phi_j}\circ f))T_i = 0.$$

Thus $\mathrm{Tr}T_i = 0$ for $i = 0,1$.

Chapter 3.

§1 The Smooth Case

Let X be a smooth compact Riemannian manifold and E and F real or complex vector bundles over X. Denote by $\Gamma(E)$ and $\Gamma(F)$ the smooth sections of E and F, respectively.

Let $f: X \to X$ be a smooth isometry with only finitely many fixed points $\{x_i\}_{i=1}^{l}$. Because f has only finitely many fixed points, these points are isolated. This in turn implies that there exists an $\varepsilon > 0$ such that, given any tubular neighborhood of the graph of f with radius less than 2ε, each piece of the diagonal in $X \times X$ intersecting this tubular neighborhood is contained in a neighborhood diffeomorphic to $\mathbb{R}^n$.

Let $D: \Gamma(E) \to \Gamma(F)$ be a pseudodifferential elliptic operator of degree zero. Suppose that D is ε - local, that is, the Schwartz kernel associated to D is supported on a tubular neighborhood of the diagonal in $X \times X$ of diameter less than ε. Since D is of order zero it extends to a bounded operator from $L^2(E)$ into $L^2(F)$ (the square integrable sections of E and F respectively). It follows from the ellipticity of D that the kernel of D and the cokernel of D are finite dimensional spaces. Call $Q: L^2(F) \to L^2(E)$ a **parametrix** for D if the operators $(1 - DQ): L^2(F) \to L^2(F)$ and $(1 - QD): L^2(E) \to L^2(E)$ are trace class operators.

The map f induces maps $f^*: \Gamma(E) \to \Gamma(f^*E)$ and $f^*: \Gamma(F) \to \Gamma(f^*F)$. Suppose there exist bundle homomorphisms $\phi_0: f^*E \to E$ and $\phi_1: f^*F \to F$ which cover the identity. Set $f_0 = \phi_0 f^*: \Gamma(E) \to \Gamma(E)$ and $f_1 = \phi_1 f^*: \Gamma(F) \to \Gamma(F)$. Suppose further that f_0 and f_1 extend to bounded operators on $L^2(E)$ and $L^2(F)$ respectively, and that the diagram

1. $$\begin{array}{ccc} L^2(E) & \xrightarrow{D} & L^2(F) \\ f_0 \downarrow & & f_1 \downarrow \\ L^2(E) & \xrightarrow{D} & L^2(F) \end{array}$$

commutes.

Let Q be a parametrix for D. Denote the trace of a trace class operator A by trA.

Lemma 1: [7] If A is trace class and B is a bounded operator on a Hilbert space, then both AB and BA are trace class and trAB is equal to trBA.

Definition: Set $\mathrm{tr}(f_0, f_1 | \Gamma(E), \Gamma(F)) =$ $\mathrm{tr}f_1(1 - DQ) - \mathrm{tr}f_0(1 - QD)$. Denote this trace by T. The following lemma is stonger than Lemma 1.

Lemma 2: [7] If A and B are bounded operators on a Hilbert space and if both AB and BA are trace class then trAB = trBA.

Proposition 1: T is independent of the choice of Q.

Proof: Let Q_1 and Q_2 be two parametrixs for D. Then $1 - Q_1D = R_0$, $1 - DQ_1 = R_1$, $1 - Q_2D = S_0$, $1 - DQ_2 = S_1$, where R_0, R_1, S_0 and $S_1 \in I_1$. (I_1 denotes the space of trace class operators.) It follows that

$$(Q_1 - Q_2)D = S_0 - R_0 \in I_1$$
$$D(Q_1 - Q_2) = S_1 - R_1 \in I_1.$$

Then

$$\begin{aligned}
&[\mathrm{tr}f_1(1 - DQ_1) - \mathrm{tr}f_0(1 - Q_1D)] \\
&- [\mathrm{tr}f_1(1 - DQ_2) - \mathrm{tr}f_0(1 - Q_2D)] \\
&= \mathrm{tr}[f_1(1 - DQ_1) - f_1(1 - DQ_2)] \\
&- \mathrm{tr}[f_0(1 - Q_1D) - f_0(1 - Q_2D)] \\
&= \mathrm{tr}f_1D(Q_2 - Q_1) - \mathrm{tr}f_0(Q_2 - Q_1)D \\
&= \mathrm{tr}f_1D(Q_2 - Q_1) - \mathrm{tr}(Q_2 - Q_1)Df_0
\end{aligned}$$

(This last step follows from lemma 1.)

$$= \mathrm{tr}Df_0(Q_2 - Q_1) - \mathrm{tr}(Q_2 - Q_1)Df_0$$

(This step follows from the commutativity of diagram 1.)

$$= \mathrm{tr}(Q_2 - Q_1)Df_0 - \mathrm{tr}(Q_2 - Q_1)Df_0$$
$$= 0$$

(The second to last equality follows from lemma 2.)

It follows from diagram 1. that f_0 induces a map denoted F_0:kerD $\to$ kerD and f_1 induces a map F_1: cokerD $\to$ cokerD.

Definition: The number $\mathrm{tr}F_1 - \mathrm{tr}F_0$ is called the Lefschetz number of f and is denoted L(f).

Theorem 1: $L(f) = \Sigma_{i=1}^{l} a_i$, where each a_i is a number which depends only on the action of f in a small neighborhood of x_i. In particular, if f has no fixed points than
L(f) = 0.

Proof: Let A denote the orthogonal complement of kerD in $L^2(E)$ and π the orthogonal projection π: $L^2(F) \to$ imD. D: A $\to$ imD is an isomorphism. Denote by G the Green's operator $D^{-1}\circ\pi$: $L^2(F) \to A$. Set

$R_0 = 1 - GD$ and $R_1 = 1 - DG$. Notice that R_0 is the projection onto the kernel of D and R_1 is the projection onto the orthogonal complement of imD. Now, kerD is a finite dimensional space and the orthogonal complement of imD is isomorphic to cokerD which is also finite dimensional, so R_0 and R_1 are trace class operators. Hence G is a parametrix for D.

Calculating T using G yields

$$T = \mathrm{tr}(f_1R_1) - \mathrm{tr}(f_0R_0).$$

However, $\mathrm{tr}(f_1R_1) = \mathrm{tr}F_1$ and $\mathrm{tr}(f_0R_0) = \mathrm{tr}F_0$. Thus

$$T = L(f).$$

Next choose a bounded operator $Q: L^2(F) \to L^2(E)$ such that $1 - QD$ and $1 - DQ$ are smoothing operators. Since smoothing operators are trace class, Q is a parametrix for D. Set $1 - QD = S_0'$ and $1 - DQ = S_1'$.

Let $\{U_\alpha\}$ be a finite open cover of X such that the diameter of each U_α is less than ε. Suppose $\{\phi_\alpha^2\}$ is a partition of unity subordinate to this cover. The operator $\Sigma_\alpha \phi Q \phi$ is still a parametrix for D and gives rise to smoothing operators S_0 and S_1. It follows from the assumption that D is ε - local, that S_0 and S_1 are 2ε - local.

Evaluating T using $\Sigma_\alpha \phi Q \phi$ yields

$$T = tr(f_1S_1) - tr(f_0S_0).$$

The following is well known:

If S is a smoothing operator with associated Schwartz kernel K(x,y) on X×X, then S is trace class and $trS = \int_X trK(x,x)$.

Now if S_i has associated Schwartz kernel $K_i(x,y)$, then the kernel associated to S_if_i is $K_i(f(x),y)$ for i = 1,2. Hence $trS_if_i = \int_X trK_i(f(x),x)$. Because S_i is 2ε - local, $K_i(x,y)$ is supported on a tubular neighborhood of the diagonal. This implies that $K_i(f(x),y)$ is supported on a tubular neighborhood (of diameter less than 2ε) of the graph of f. And because ε was chosen sufficiently small $K_i(f(x),x)$ is zero everywhere except on a small neighborhood of each fixed point of f; in fact, a neighborhood which can be taken to be diffeomorphic to $\mathbb{R}^n$. The theorem then follows from proposition 1.

§2 An Example

Suppose X is a smooth compact Riemannian manifold. Let E and F be smooth vector bundles over X. Denote by

π the projection $T^*X \to X$. Let $S^m(E,F)$ denote the standard symbol classes of order m, that is,

$$S^m(E,F) = \{p \in \Gamma(\pi^*\mathrm{Hom}(E,F)) \mid |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|} \text{ for all } \alpha,\beta \leq 0, \text{ where } C_{\alpha\beta} > 0\}.$$

Here (x,ξ) are the coordinates on T^*X.

Given any element p of $S^m(E,F)$, it is possible to construct, in an almost canonical way, a pseudo-differential operator from $\Gamma(E)$ into $\Gamma(F)$ which has p as its symbol. This construction given in [6] goes as follows:

Suppose $\lambda:\mathbf{R} \to \mathbf{R}$ is a bump function suported on $[-1,-1]$ and for any $\eta>0$ set $\lambda_\eta(x) = \lambda(x/\eta)$. Denote by $\rho:X\times X \to \mathbf{R}$ the geodesic distance. Then $\rho_\eta = \lambda_\eta \circ \rho$ is a function supported on a tubular neigborhood of the diagonal in $X\times X$. Let $\exp:TX \to X\times X$ denote the exponential map. Now given any $s(x) \in \Gamma(E)$, set $s'(v) = [\exp_x^* \rho_\eta(x,y)s(y)](v)$. So, s' is a section of the pull back bundle $\exp_x^*(E)$ over T_xX. Then for $p \in S^m(E,F)$ define the operator $\Theta p:\Gamma(E) \to \Gamma(F)$ by

$$(\Theta p)s(x) = (2\pi)^{-n} \int\int e^{-i\langle v,\xi\rangle} p(x,\xi) s'(v) dv d\xi.$$

Here the double integral is over $T_xX \times T_x^*X$. Notice that the distributional kernel of Θp is supported on a tubular neighborhood of the diagonal in $X \times X$ because ρ_η is supported there. The diameter of this tube can made arbitrarily small by taking η sufficiently small.

Given any $p \in S^1(E,F)$ it is possible to construct an operator of order zero by rescaling p. That is, if $p'(x,\xi)$ is given by $p'(x,\xi) = p(x,\xi/|\xi|)$, then $\Theta p'$ is a psuedodifferential operator of order zero.

Suppose $f:X \to X$ is an isometry with at most finitely many fixed points. Set $E = F = \Lambda^*T^*(X)$ the bundle of the exterior algebra of the cotangent bundle of X. Then f induces a bounded map $f^*:L^2(E) \to L^2(E)$. For any $p \in S^1(E,E)$, invariant under the isometry f, (an example of such a p is the symbol associated to the exterior derivative d), set $D = \Theta p'$. Then the following diagram is commutative:

$$\begin{array}{ccc} L^2(E) & \xrightarrow{D} & L^2(E) \\ f^*\downarrow & & f^*\downarrow \\ L^2(E) & \xrightarrow{D} & L^2(E) \end{array}$$

The diagram commutes because f is an isometry and the exponential map depends only on the Riemannian metric. So this f and D satisfy the hypotheses of theorem 1.

§3 The Lipschitz Case

It is possible to weaken the hypotheses of the theorem in the previous section. The manifold X can be taken to be Lipschitz instead of smooth. The spaces $L^2(E)$ and $L^2(F)$ can be replaced by more general Hilbert spaces H_0 and H_1 which are modules over the continuous functions on X. The map $f:X \to X$ is required to be a Lipschitz isometry. The operator $D:H_0 \to H_1$ is assumed to be bounded, Fredholm and ε-local. Again the existence of a commutative diagram:

$$\begin{array}{ccc} H_0 & \xrightarrow{D} & H_1 \\ f_0\downarrow & & f_1\downarrow \\ H_0 & \xrightarrow{D} & H_1 \end{array}$$

is postulated.

The example of the previous section of course still satisfies these more general hypotheses. However, other examples which do not require the assumption of smoothness have already been constructed by S. Donaldson-D. Sullivan [10] and are presently studied by D. Sullivan and N. Teleman in on going research. One of these include a "signature" operator associated to an even dimensional Lipschitz manifold. In this example the signature operator constructed in [11] is rescaled and localized.

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