The Euler Equations Of An Incompressible Ideal Fluid In A High-Dimensional Bounded Region

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Abstract of the Dissertation

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The author proves that the maximum norm of the vorticity controls the breakdown of smooth solutions of the Euler Equations of an incompressible ideal fluid in $\Omega = T^n \times [0,1]$, $n \geq 2$, a high-dimensional bounded domain. In other words, If a solution is initially smooth and loses its regularity at some later time, then the maximum norm of the vorticity must necessary grow without bound as critical time approaches.

To the memory of my father Hetong Yu.

To my mather Meiru Song,

my wife Sharon and my son Kevin.

Contents

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Introduction

Let Ω be an open set in $R^n (n \geq 2)$ and $\partial\Omega$ be its boundary. The set Ω can be either the whole $R^n (n \geq 2)$, or a bounded set with a sufficiently smooth boundary $\partial\Omega$. The motion of an inviscid incompressible fluid filling Ω is governed by the initial-boundary value problem of the Euler equation:

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u + \nabla p = 0, \qquad in \;\; \Omega \times [0, T], \tag{0.1}$$

$$\nabla \cdot u = 0, \qquad in \;\; \Omega \times [0, T], \tag{0.2}$$

$$< u, \nu >= 0, \qquad on \;\; \partial\Omega \times [0, T], \tag{0.3}$$

$$u(x, 0) = u_0(x), \qquad on \;\; \Omega, \tag{0.4}$$

where ν is the outward unit normal vector of $\partial\Omega$, u_0 is a given vector-valued function, while the velocity $u(x, t)$ and the pressure $p(x, t)$ are to be determined. Inside of the equation, ∇ indicates the spatial derivative $(\partial_1, ..., \partial_n)$, ∇p is the gradient of p, and $\nabla \cdot u$ is the divergence of u.

The existence and uniqueness of the solutions of the Euler equations has been considered by several authors including M. Wolibner [W], V. Judovich [J], T. Kato [K1], [K2], D. Ebin and J. Marsden [E-M], J. Bourguignon and H. Brezis [B-B], R. Teman [T1], and D. Ebin [E]. The regularity of the solutions was studied by R. Teman [T2], and C. Bardos and U. Frisch [B-F]. In [K1] Kato proved the existence and uniqueness of the global solutions in the two dimensional case, and a simplified proof by Ebin [E] followed. In the higher dimensional case ($n \geq 3$), all known theorems are local results which can be stated as follows: *Suppose an initial velocity field* u_0 *is specified in* $H^s(\Omega)$, $s > n/2 + 1$, *with*

$$\|u_0\|_{H^s(\Omega)} \leq N,$$

then there exists $T_0 > 0$, *depending on* N, *such that (0.1)-(0.4) have an unique solution in the class*

$$SC = C([0,T]; H^s(\Omega)) \cap C^1([0,T]; H^{s-1}(\Omega)), \tag{0.5}$$

at least for $T = T_0$. But such a result does not give an indication as to whether the solutions actually keep their regularity for all time as in the two dimensional case. For the three dimensional case, several numerical investigations by A. Chorin [C1] and [C2], U. Frisch, P. Sulem and M. Nelkin [F-S-N] and R. Morf, S. Orszag and U. Frisch [M-O-F] predict that the solutions of the fluid equations, which at first represent smooth flows, may develop singularities. Theoretically, the proof of the breakdown of smooth solutions for the high-dimensional Euler equations is still open. J. Beal, T. Kato and A. Majda [B-K-M] studied $\Omega = R^3$ case and established a mathematically rigorous link between the formation of singularities and the accumulation of the vorticity $w = \nabla \times u$, the curl of u. They proved that if the L^∞-norm of the vorticity w is integrable as a function of t, then the solution u will not lose its regularity; i.e., $\|u_0\|_{H^s(\Omega)} < \infty$ will give us $\|u\|_{H^s(\Omega)} < \infty$ for $t \in [0,T]$ if

$$\int_0^T \|w\|_{L^\infty(\Omega)} dt < \infty,$$

where $s \geq 3$. In other words, if the solution is initially smooth and loses its regularity at some later time, then the maximum norm of the vorticity, w, must necessarily grow without bound as the critical time approaches. It was pointed out in their paper that their method doesn't work for the fluid flow in a bounded region, and that a more involved proof using additional ideas is necessary.

In this paper, we announce that the result of [B-K-M] holds for $\Omega = T^n \times [0,1]$, where T^n is the n-dimensional torus. In what follows we only prove the result for the three dimensional case, i.e., $\Omega = T^2 \times [0,1]$. For $n \geq 3$, the proof will be similar. n here is different from n on p.1.

We denote the spatial differentiation vector by $D = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. Thus Du indicates the first derivative of u in spatial variables, and D^2u, D^3u indicate the second and the third spatial derivatives of u. The whole proof is divided into several steps as follows:

In Section 1, for any bounded region Ω with a smooth boundary $\partial\Omega$, we prove that the solution u of (0.1)-(0.4) satisfies the inequality

$$\frac{d}{dt}\|u\|_{H^s(\Omega)} \leq K_1(\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}, \tag{0.6}$$

where K_1 is a constant depending only on $\partial\Omega$, the diameter of Ω, and s, where $s \geq 3$.

According to the Sobolev embedding theorem, in R^3,

$$\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)} \leq C(\|u\|_{H^2(\Omega)} + \|Du\|_{H^2(\Omega)}) \leq C\|u\|_{H^3(\Omega)},$$

inequality (0.6) becomes

$$\frac{d}{dt}\|u\|_{H^s(\Omega)} \leq C\|u\|_{H^3(\Omega)}\|u\|_{H^s(\Omega)},$$

which leads us to two well-known results for the regularity of the solutions of the Euler equations in $\Omega \subseteq R^3$:

1. *If (0.1)-(0.4) has an H^3 solution for $t \in [0,T]$, then (0.6) guarantees that the H^3 solution is as smooth as the initial data u_0 for all time $t \in [0,T]$, i.e., $u_0 \in H^s(\Omega), s \geq 3$, gives $u(t) \in H^s(\Omega)$.*

2. *If u_0 is $C^\infty(\Omega)$, by using the local existence and uniqueness theorem for $H^3(\Omega)$, (0.1)-(0.4) has a C^∞ solution during the time interval $[0,T]$, where the H^3 solution exists.*

The above regularity results were proved by R. Teman using Sobolev estimates [T2] and by C. Bardos and U. Frisch using Schauder estimates [B-F].

In Section 2, we give the estimates for $\|u\|_{L^\infty(\Omega)}$ and $\|Du\|_{L^\infty(\Omega)}$, the terms on the right side of (0.6).

For a function u defined in a bounded region Ω with the cone property, assume that u satisfies

$$\|Du\|_{L^p(\Omega)} \leq K_2 p(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)}), \tag{0.7}$$

for $p \geq 2$, where K_2 only depends on Ω, and is independent of p. One can prove

$$\|Du\|_{L^\infty(\Omega)} \leq K_3(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})[1 + \ln(1 + \|u\|_{H^3(\Omega)})], \tag{0.8}$$

where K_3 is just a constant depending on Ω.

While (0.7) holds, we also have an estimate for u, the solution of (0.1)-(0.4):

$$\|u\|_{L^\infty(\Omega)} \leq K_4(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)}), \tag{0.9}$$

where K_4 is a constant depending only on Ω.

Remark: If the boundary $\partial\Omega$ is smooth, say at least C^1, the cone property is guaranteed.

In Section 3, we prove our main result: for $s \geq 3$,

$$\|u(t)\|_{H^s(\Omega)} \leq e^{-1}[e(\|u_0\|_{H^s(\Omega)} + 1)]^{F(t)} - 1, \tag{0.10}$$

where

$$F(t) = exp[K \int_0^t (\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)} + 1)d\tau],$$

with K is a constant depending on Ω and s; i.e., if a fluid flow with a smooth initial velocity field u_0, say u_0 belongs to $H^s(\Omega)$, then the fluid flow keeps that smoothness for all time t, provided that the maximum norm of the vorticity of the fluid flow remains bounded accumulatively for all time t.

In Section 4, we borrow some definitions and lemmas from [S] by E. Stein, and then we show that A_p, the constant in Lemma 10, can be refined. The result with the refined constant is very important in the proof of (0.7) in Section 5.

In Section 5, for a special region $\Omega = T^2 \times [0,1]$, we derive that the relation between u, the velocity, and it's vorticity, w, actually are Dirichlet problems or Neumann problems of the elliptic systems. We study the half-space case first and then use the partition of unity for the compact set to prove (0.7) for this special region . We believe that a more careful estimate for elliptic systems will lead to that inequality (0.7) holds for general bounded region Ω in R^n. If it is so, the main result can be extended to any bounded region with a smooth boundary in the high-dimensional case.

Section One

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq s$, and let

$$v = D^\alpha u = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}} u.$$

Applying D^α to (0.1), we get

$$\frac{\partial}{\partial t} v + (u \cdot \nabla) v + D^\alpha \nabla p = F_\alpha, \tag{0.11}$$

where

$$F_\alpha = D^\alpha((u \cdot \nabla)u) - u \cdot \nabla D^\alpha u. \tag{0.12}$$

Multiplying (0.11) by v and integrating it over Ω, we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} |v|^2_{L^2(\Omega)} = -\int_\Omega < (u \cdot \nabla) v, v > dx$$

$$-\int_\Omega < D^\alpha \nabla p, v > dx + \int_\Omega < F_\alpha, v > dx$$

$$= I_1 + I_2 + I_3. \tag{0.13}$$

A). Estimate I_1 in (0.13):

For any smooth function $h(x)$

$$\int_\Omega [(u \cdot \nabla) h] h dx$$

$$= \int_\Omega (u_1 \frac{\partial}{\partial x_1} h + u_2 \frac{\partial}{\partial x_2} h + u_3 \frac{\partial}{\partial x_3} h) h dx$$

$$= \int_{\partial\Omega} h^2 < u, \nu > dS - \int_\Omega h \nabla \cdot (hu) dx$$

$$= \int_{\partial\Omega} h^2 < u, \nu > dS - \int_\Omega h(h\nabla \cdot u + (u \cdot \nabla)h)dx.$$

By using equations (0.2) and (0.3):

$$\int_\Omega [(u \cdot \nabla)h]hdx$$

$$= \frac{1}{2}\int_{\partial\Omega} h^2 < u, \nu > dS - \frac{1}{2}\int_\Omega h^2 \nabla \cdot udx = 0. \quad (0.14)$$

So $I_1 = 0$.

B). Estimate I_3 in (0.13):

We need some lemmas.

Lemma 1 *Assume $f(x), g(x) \in H^s(\Omega)$, then for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $|\alpha| \leq s$, we have*

$$\|D^\alpha(fg)\|_{L^2(\Omega)}$$

$$\leq C_s(\|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)} + \|g\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)}), \quad (0.15)$$

and

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)}$$

$$\leq C_s(\|Df\|_{L^\infty(\Omega)}\|g\|_{H^{s-1}(\Omega)} + \|g\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)}), \quad (0.16)$$

where C_s only depends on s.

Proof: The proof of (0.15) and (0.16) for $\Omega = R^n$ can be found in [M] of Moser and in [K-M] of Klainerman and Majda by using the key inequality known as the Gagliardo-Nirenberg calculus inequality:

$$(G-N.1)\quad \|D^i f\|_{L^{2r/i}(\Omega)} \le C_N \|f\|^{1-i/r}_{L^\infty(\Omega)} \|D^r f\|^{i/r}_{L^2(\Omega)},$$

where $r \ge i \ge 0$. For a general bounded region Ω, this key inequality becomes

$$(G-N.2)\quad \|D^i f\|_{L^{2r/i}(\Omega)} \le C_N \|f\|^{1-i/r}_{L^\infty(\Omega)} \|D^r f\|^{i/r}_{L^2(\Omega)} + C_\Omega \|f\|_{L^2(\Omega)},$$

(see [N] Page 125). Applying first a Holder inequality and then (G-N.2) we can write

$$\|D^\alpha(fg)\|_{L^2(\Omega)} \le C \sum_{\beta+\gamma=\alpha} \|D^\beta f D^\gamma g\|_{L^2(\Omega)}$$

$$\le C \sum_{\beta+\gamma=\alpha} \|D^\beta f\|_{L^{2|\alpha|/|\beta|}(\Omega)} \|D^\gamma g\|_{L^{2|\alpha|/|\gamma|}(\Omega)}$$

$$\le C \sum_{\beta+\gamma=\alpha} (\|f\|^{1-|\beta|/|\alpha|}_{L^\infty(\Omega)} \|D^\alpha f\|^{|\beta|/|\alpha|}_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})(\|g\|^{1-|\gamma|/|\alpha|}_{L^\infty(\Omega)} \|D^\alpha g\|^{|\gamma|/|\alpha|}_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}).$$

Since Ω is bounded, we have

$$\|f\|_{L^2(\Omega)} \le C \|f\|_{L^\infty(\Omega)},$$

where C depends only on Ω. Writing

$$\|f\|_{L^2(\Omega)} \le C \|f\|^{1-|\beta|/|\alpha|}_{L^\infty(\Omega)} \|f\|^{|\beta|/|\alpha|}_{L^2(\Omega)},$$

we then obtain

$$\|f\|^{1-|\beta|/|\alpha|}_{L^\infty(\Omega)} \|D^\alpha f\|^{|\beta|/|\alpha|}_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}$$

$$\le C \|f\|^{1-|\beta|/|\alpha|}_{L^\infty(\Omega)} \|f\|^{|\beta|/|\alpha|}_{H^s(\Omega)}.$$

For the same reason

$$\|g\|_{L^\infty(\Omega)}^{1-|\gamma|/|\alpha|}\|D^\alpha g\|_{L^2(\Omega)}^{|\gamma|/|\alpha|} + \|g\|_{L^2(\Omega)}$$

$$\leq C\|g\|_{L^\infty(\Omega)}^{1-|\gamma|/|\alpha|}\|g\|_{H^s(\Omega)}^{|\gamma|/|\alpha|}.$$

Notice

$$|\beta| + |\gamma| = |\alpha|,$$

so we get:

$$\|D^\alpha(fg)\|_{L^2(\Omega)} \leq C \sum_{\beta+\gamma=\alpha} (\|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)})^{|\gamma|/|\alpha|}(\|g\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)})^{|\beta|/|\alpha|}$$

$$\leq C(\|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)} + \|g\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)}),$$

which is precisely (0.15).

To prove (0.16) we make the follow modifications:

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)} \leq C \sum_{\beta+\gamma=\alpha,\beta\neq 0} \|D^\beta f D^\gamma g\|_{L^2(\Omega)}$$

$$= C \sum_{|\delta|+|\gamma|=|\alpha|-1} \|D^\delta(Df)D^\gamma g\|_{L^2(\Omega)}.$$

We then proceed as before, replacing f by Df and $|\alpha|$ by $|\alpha| - 1$.

Q.E.D.

Lemma 2 *F_α as defined in (0.12) can be estimated by*

$$\|F_\alpha\|_{L^2(\Omega)} \leq C_1\|Du\|_{L^\infty(\Omega)}\|u\|_{H^s(\Omega)}, \tag{0.17}$$

where C_1 is a constant depending only on Ω and s.

Proof: We take $f = u$, $g = Du$. If both $f, g \in H^s \cap C^1(\Omega)$, by using Lemma 1, we have

$$\|F_\alpha\|_{L^2(\Omega)}$$
$$\leq C\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)}$$
$$\leq C\|Du\|_{L^\infty(\Omega)}\|u\|_{H^s(\Omega)}.$$

Since we only assume $u \in H^s \cap C^1(\Omega)$, we need (0.16) remain true with $g \in H^{s-1} \cap C(\Omega)$, even though $D^\alpha(fg)$ and $fD^\alpha g$ may not be in $L^2(\Omega)$. Notice that

$$D^\alpha(fg) - fD^\alpha g = \sum_{\beta+\gamma=\alpha, \beta\neq 0} C_\gamma D^\beta f D^\gamma g,$$

is in $L^2(\Omega)$.

Since H^s is dense in H^{s-1}, we can choose a sequence $g_n \in H^s \cap C(\Omega)$, such that $g_n \to g$ in $H^{s-1} \cap C(\Omega)$, as $n \to \infty$. Clearly, for any β, γ, with $|\gamma| \leq |\alpha| - 1 \leq s - 1, |\beta| \leq s$,

$$D^\beta f D^\gamma g_n \to D^\beta f D^\gamma g \quad (n \to \infty),$$

in $L_2(\Omega)$. Thus

$$D^\alpha(fg_n) - fD^\alpha g_n \to D^\alpha(fg) - fD^\alpha g \quad (n \to \infty)$$

in $L_2(\Omega)$. Using the sequence g_n, we have

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)}$$
$$= \|\sum_{\beta+\gamma=\alpha, \beta\neq 0} D^\beta f D^\gamma g\|_{L^2(\Omega)}$$
$$= \lim_{n\to\infty} \|\sum_{\beta+\gamma=\alpha, \beta\neq 0} D^\beta f D^\gamma g_n\|_{L^2(\Omega)}$$

$$= \lim_{n\to\infty} \|D^\alpha(fg_n) - fD^\alpha g_n\|_{L^2(\Omega)}$$

$$\leq \lim_{n\to\infty} C_s(\|Df\|_{L^\infty(\Omega)}\|g_n\|_{H^{s-1}(\Omega)} + \|g_n\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)})$$

$$\leq C_s(\|Df\|_{L^\infty(\Omega)}\|g\|_{H^{s-1}(\Omega)} + \|g\|_{L^\infty(\Omega)}\|f\|_{H^s(\Omega)}).$$

We have thus proved Lemma 2.

Q.E.D.

Consequently,

$$|I_3| \leq \int_\Omega | < F_\alpha, v > |dx$$

$$\leq \|F_\alpha\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

$$\leq C_1\|Du\|_{L^\infty(\Omega)}\|u\|_{H^s(\Omega)}\|v\|_{L^2(\Omega)}.$$

C). Estimate I_2 in (0.13):

Lemma 3 *If u and p satisfy (0.1)-(0.3), then the function p satisfies*

$$\Delta p = -\sum_{i,j} \frac{\partial}{\partial x_i} u_j \frac{\partial}{\partial x_j} u_i, \qquad in \quad \Omega \times [0,T],$$

$$\frac{\partial}{\partial \nu} p = -\sum_{i,j} u_i u_j \phi_{i,j}, \qquad on \quad \partial\Omega \times [0,T],$$

where the functions $\phi_{i,j}$ depend only on $\partial\Omega$.

Proof: See [T1], p.34-35.

Lemma 4 *p, defined as in Lemma 3, has an estimate*

$$\|\nabla p\|_{H^s(\Omega)} \leq C_2(\|Du\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)},$$

where C_2 is a constant depending on Ω and s, with $s \geq 3$.

Proof: By Lemma 3 and the well-known Agmon-Douglis-Nirenberg theory in [A-D-N], p satisfies the inequality

$$\|\nabla p\|_{H^s(\Omega)} \leq C(\|DuDu\|_{H^{s-1}(\Omega)} + \|uu\|_{H^{s-\frac{1}{2}}(\partial\Omega)}).$$

Applying (0.15), we get

$$\begin{aligned} \|DuDu\|_{H^{s-1}(\Omega)} &\leq C\|Du\|_{L^\infty(\Omega)}\|D^{s-1}(Du)\|_{L^2(\Omega)} \\ &\leq C\|Du\|_{L^\infty(\Omega)}\|u\|_{H^s(\Omega)}. \end{aligned} \tag{0.18}$$

By using the trace theorem and (0.15), we get

$$\begin{aligned} &\|uu\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\leq C\|uu\|_{H^s(\Omega)} \\ &\leq C\|u\|_{L^\infty(\Omega)}\|u\|_{H^s(\Omega)}. \end{aligned} \tag{0.19}$$

(0.18) and (0.19) complete the proof of Lemma 4.

Q.E.D.

Now we have estimate for I_2:

$$\begin{aligned} |I_2| &\leq \int_\Omega |< D^\alpha \nabla p, v >| dx \\ &\leq C_2\|\nabla p\|_{H^s(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq C_3(\|Du\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}\|v\|_{L^2(\Omega)}. \end{aligned}$$

We sum here over α with $0 \leq |\alpha| \leq s$. Based on argument A), B), C), we have our important estimate for $\|u\|_{H^s(\Omega)}$ in terms of $\|Du\|_{L^\infty(\Omega)}$ and $\|u\|_{L^\infty(\Omega)}$:

$$\frac{d}{dt}\|u\|^2_{H^s(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)})\|u\|^2_{H^s(\Omega)}.$$

i.e.,

$$\frac{d}{dt}\|u\|_{H^s(\Omega)} \leq K_2(\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}. \tag{0.6}$$

Section 2

In this section, we give the estimates for $\|u\|_{L^\infty(\Omega)}$ and $\|Du\|_{L^\infty(\Omega)}$.

Lemma 5 *Suppose $\Omega \subset R^n$ is a bounded region with the cone property. Assume a function $u(x)$ satisfies the inequality*

$$\|Du\|_{L^p(\Omega)} \le C_3 p(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)}), \qquad (0.7)$$

for $p \ge 2$, where C_3 is a constant independent of p. Then

$$\|Du\|_{L^\infty(\Omega)} \le K_3(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})[1 + \ln(1 + \|u\|_{H^3(\Omega)})], \qquad (0.8)$$

where K_3 depends only on Ω.

Proof: Here we prove (0.8) for $n = 3$. Actually for any $n \ge 2$, the proof is similar.

We use a result in [So], p. 487 by Sobolev: any C^1 function $v(x)$ on Ω, can be written in integral form over a small cone V_0 contained in Ω with x as its vertex:

$$v(x) = \int_{V_0} \xi_0 v(y)dy + \sum_{i=1}^{3} \int_{V_0} \frac{\xi_i}{|x-y|^2} \frac{\partial}{\partial y_i} v(y)dy, \qquad (0.20)$$

where $\xi_i, i = 0, 1, 2, 3$ are known smooth functions depending on Ω and

$$|\xi_i| \le C_4, \qquad i = 0, 1, 2, 3,$$

for some constant C_4 independent of $v(x)$. Thus for any k, the function $|v(x)|^k$ satisfies

$$|v(x)|^k = \int_{V_0} \xi_0 |v(y)|^k dy$$

$$+k \sum_{i=1}^{3} \int_{V_0} \frac{\xi_i}{|x-y|^2} |v(x)|^{k-2} v(y) \frac{\partial}{\partial y_i} v(y) dy$$

$$= I_0 + \sum_{i=1}^{3} I_i. \tag{0.21}$$

The first term I_0 in the above equation satisfies

$$|I_0| \leq C_4 \int_{\Omega} |v(x)|^k dx = C_4 \|v\|^k_{L^k(\Omega)}. \tag{0.22}$$

For the terms in the summation, we obtain, using the Holder inequality

$$|I_i| \leq C_5 k \int_{V_0} \frac{1}{|x-y|^2} |v(y)|^{k-1} |Dv(y)| dy$$

$$\leq C_5 k \|Dv\|_{L^4(\Omega)} [\int_{V_0} (\frac{1}{|x-y|^2} |v(y)|^{k-1})^q dy]^{\frac{1}{q}},$$

$$for \quad i = 1, 2, 3,$$

where

$$\frac{1}{4} + \frac{1}{q} = 1, \quad i.e., \quad q = \frac{4}{3}.$$

We replace q with $4/3$ in the above inequality. Notice

$$[\int_{V_0} (\frac{1}{|x-y|^{\frac{8}{3}}} |v(y)|^{\frac{4}{3}(k-1)}) dy]^{\frac{3}{4}}$$

$$\leq [(\int_{V_0} \frac{1}{|x-y|^{\frac{8}{3}\alpha}} dy)^{\frac{1}{\alpha}} \|v^{\frac{4}{3}(k-1)}\|_{L^\beta(\Omega)}]^{\frac{3}{4}},$$

where $\alpha > 1, \beta > 1$, and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

To determine α and β, we utilize the inside of the above integral. Let

$$E = \left(\int_{V_0} \frac{1}{|x-y|^{\frac{8}{3}\alpha}} dy\right)^{\frac{1}{\alpha}}.$$

E exists, if

$$-\frac{8}{3}\alpha + 3 \geq 0, \quad i.e., \quad if \quad \alpha \leq \frac{9}{8}.$$

Here we take $\alpha = \frac{17}{16}$, so that $\beta = 17$. Since Ω is compact, E is bounded by a constant independent of x. Thus for $i = 1, 2, 3$,

$$|I_i| \leq C_6 k \|Dv\|_{L^4(\Omega)} \|v\|^{k-1}_{L^{\frac{68}{3}(k-1)}(\Omega)},$$

$$for \quad i = 1, 2, 3. \tag{0.23}$$

Equation (0.21) and inequalities (0.22) and (0.23) give us

$$|v(x)|^k \leq C_7 \|v\|^k_{L^k(\Omega)} + C_7 k \|Dv\|_{L^4(\Omega)} \|v\|^{k-1}_{L^{\frac{68}{3}(k-1)}(\Omega)}. \tag{0.24}$$

Recall the well-known Taylor series

$$e^{\mu|v|} = 1 + \mu|v| + \frac{1}{2}\mu^2|v|^2 + \sum_{k=3}^{\infty} \frac{1}{k!}\mu^k |v|^k.$$

Since

$$\mu|v| \leq \frac{1}{2} + \frac{1}{2}\mu^2|v|^2,$$

we obtain:

$$e^{\mu|v|} \leq \frac{3}{2} + \mu^2|v|^2 + \sum_{k=3}^{\infty} \frac{1}{k!}\mu^k|v|^k, \tag{0.25}$$

where μ is a positive constant to be chosen.

Right now we let $v(x) = Du(x)$. Using assumption (0.7) and inequalities (0.24), (0.25), we obtain

$$e^{\mu|Du|} \leq \frac{3}{2} + \mu^2 C_8(\|Du\|^2_{L^2(\Omega)} + \|D^2u\|_{L^4(\Omega)}\|v\|_{L^{\frac{68}{3}}(\Omega)})$$

$$+C_8\sum_{k=3}^{\infty}\frac{1}{k!}\mu^k\|Du\|^k_{L^k(\Omega)} + C_8\|D^2u\|_{L^4(\Omega)}\sum_{k=3}^{\infty}\frac{1}{(k-1)!}\mu^k\|Du\|^{k-1}_{L^{\frac{68}{3}(k-1)}(\Omega)}$$

$$\leq \frac{3}{2} + C_9\sum_{k=2}^{\infty}\frac{1}{k!}\mu^k[K_2k(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})]^k$$

$$+ C_9\|D^2u\|_{L^4(\Omega)}\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\mu^k[K_2\frac{68}{3}(k-1)(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})]^{k-1}. \tag{0.26}$$

Using Sterling's formula:

$$k! \geq \sqrt{2k\pi}(\frac{k}{e})^k,$$

the inequality (0.26) can be written

$$e^{\mu|Du|} \leq C_{10}\sum_{k=0}^{\infty}[C_{11}(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})\mu]^k$$

$$+C_{10}\|D^2u\|_{L^4(\Omega)}\mu\sum_{k=0}^{\infty}[C_{11}(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})\mu]^k$$

$$\leq C_{10}[\frac{1}{1-\mu C_{11}(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})}(1+\mu\|D^2u\|_{L^4(\Omega)})]. \tag{0.27}$$

Let

$$\mu = \min(1, \frac{1}{2C_{11}(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})}),$$

then

$$\|Du\|_{L^\infty(\Omega)} \leq \frac{1}{\mu}\ln[2C_{10}(1+\mu\|D^2u\|_{L^4(\Omega)})]$$

$$\leq [1 + 2C_{11}(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})][\ln(1+\mu\|D^2u\|_{L^4(\Omega)}) + \ln(2C_{10})],$$

$$\leq K_3(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})[1 + \ln(1 + \|u\|_{H^3(\Omega)})]. \tag{0.8}$$

In the last inequality we have used

$$\|D^2u\|_{L^4(\Omega)} \le \|u\|_{H^3(\Omega)},$$

in [L] by O. A. Ladyzhenskaya. This proves Lemma 5.

Q.E.D.

Lemma 6 *Assume u, the solution of (0.1)-(0.4), satisfies condition (0.7); then u must satisfy*

$$\|u\|_{L^\infty(\Omega)} \le K_4(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)}), \qquad (0.9)$$

where K_4 is a constant depending only on Ω.

Proof: Recall the well-known inequality in [N], p.125 by L. Nirenberg:

$$\|D^ju\|_{L^p(\Omega)} \le constant(A)\|D^mu\|^a_{L^r(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} + constant(B)\|u\|_{L^2(\Omega)},$$

where

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \le a \le 1.$$

Here j, p, m, r, q are real numbers and n is the dimension of the space. The $constant(A)$ depends only on n, m, j, q, r, a, and the $constant(B)$ depends only on the bounded domain Ω. Since $n = 3$, we specifically choose $p = \infty, j = 0, m = 1, r = 12, q = 2, a = \frac{2}{3}$, which gives us

$$\|u\|_{L^\infty(\Omega)} \le constant(A)\|Du\|_{L^{12}(\Omega)}^{\frac{2}{3}}\|u\|_{L^2(\Omega)}^{\frac{1}{3}} + constant(B)\|u\|_{L^2(\Omega)}$$

$$\le C(\|Du\|_{L^{12}(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{0.28}$$

Notice that u is the solution of equations (0.1)-(0.4). Multiply (0.1) by u, and integrate over Ω. Using integration by parts, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\Omega)} = -\int_\Omega u(u\cdot\nabla)udx$$

$$+\int_\Omega p\nabla\cdot udx - \int_{\partial\Omega} < u,\nu > pdx.$$

The estimate I_1 in Section 1 shows that the first term on the right side is zero. Equations (0.2) and (0.3) show that the other terms are also zero. $\|u\|_{L^2(\Omega)}$ is thus conserved, i.e.,

$$\|u\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}. \tag{0.29}$$

The proof is completed by combining the equation (0.29) and inequalities (0.7) and (0.28).

Q.E.D.

Section 3

We can now prove quite easily our main result, using the estimates in the previous sections. The main result of this paper can be stated as follows:

Theorem 1 *Let $\Omega = T^2 \times [0,1]$, and let u be the solution of (0.1)-(0.4). If u exists for $t \in [0, T_0]$, the following inequality holds:*

$$\|u(t)\|_{H^s(\Omega)} \leq e^{-1}[e(\|u_0\|_{H^s(\Omega)} + 1)]^{F(t)} - 1, \qquad (0.10)$$

where

$$F(t) = exp[K \int_0^t (\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)} + 1)dt]$$

for $s \geq 3$, and K is just a constant depending on Ω and s.

Remark: The proof of this theorem depends on the inequality (0.7) which we prove only in the case $\Omega = T^2 \times [0,1]$. Hence we state the theorem only in that case. However, we believe that (0.7) is true for all bounded domains $\Omega \subseteq R^n$ which have smooth boundaries. If it is, then the above theorem is true for all such domains also.

Proof: Using inequalities (0.6), (0.8), (0.9), we get

$$\frac{d}{dt}\|u\|_{H^s(\Omega)} \leq K_2(\|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}$$

$$\leq K(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})[1 + \ln(1 + \|u\|_{H^3(\Omega)})]\|u\|_{H^s(\Omega)}. \qquad (0.30)$$

Let

$$y(t) = 1 + \|u\|_{H^s(\Omega)}.$$

From (.30), we then obtain

$$\frac{d}{dt}y(t) \leq K(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})(1 + \ln y(t))y(t).$$

Denote

$$z(t) = 1 + \ln y(t),$$

where $z(t)$ satisfies

$$\frac{d}{dt}z(t) \leq K(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})z(t).$$

Therefor, if Z(t) is the solution of the ordinary differential equation:

$$\frac{d}{dt}Z(t) = K(1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})Z(t),$$

$$Z(0) = z(0) = z_0,$$

then

$$z(t) \leq Z(t).$$

This gives us:

$$z(t) \leq z_0 exp\{K \int_0^t (1 + \|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)})dt\}.$$

Changing back to the form of $\|u\|_{H^s(\Omega)}$, we see that $\|u\|_{H^s(\Omega)}$ is bounded by some form of $\|w\|_{L^\infty(\Omega)}$ and $\|u_0\|_{L^2(\Omega)}$:

$$\ln(1 + \|u(t)\|_{H^s(\Omega)})$$

$$\leq [1 + \ln(1 + \|u_0\|_{H^s(\Omega)})]exp[K \int_0^t (\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)} + 1)dt] - 1.$$

This produces the inequality (.10).

Q.E.D.

The theorem has the following immediate consequences:

Corollary 1 *Let u be as in the above theorem, and suppose there is a time T_* such that the solution u can not be continued in the class*

$$SC = C([0,T]; H^s(\Omega)) \cap C^1([0,T]; H^{s-1}(\Omega)) \tag{0.5}$$

to $T = T_*$. Assume T_* is the first time that discontinuity occurs. Then

$$\int_0^{T_*} \|w\|_{L^\infty(\Omega)} dt = \infty,$$

and in particular

$$\lim_{t \to T_*} \sup \|w(t)\|_{L^\infty(\Omega)} = \infty.$$

Corollary 2 *Let u be a solution of (0.1)-(0.4), and suppose there are constants M_0 and T_*, such that on any interval $[0,T]$, with $T \leq T_*$, we have*

$$\int_0^{T} \|w\|_{L^\infty(\Omega)} dt \leq M_0.$$

Assume the initial data, u_0 is in $H^s(\Omega)$. The solution u then can be continued for $t \in [0, T_*]$ in the class SC.

Section 4

This section is the preparation of Section 5. We introduce some definitions and well-known lemmas borrowed from [S] by E. Stein, and then refine a constant in Lemma 10.

Definition 1 *Let Φ be a linear mapping from $L^p(R^n)$ to $L^q(R^n)$, where $1 \leq p, q \leq \infty$. Then*

1. *Φ is of type (p,q), if for any function $f \in L^p(R^n)$,*

$$\|\Phi(f)\|_{L^q(R^n)} \leq A\|f\|_{L^p(R^n)},$$

   where the constant A does not depends on f.

2. *Φ is of weak-type (p,q), if for any function $f \in L^p(R^n)$,*

$$measure\{x : |(\Phi(f))(x)| > \alpha\} \leq (\frac{A\|f\|_{L^p(R^n)}}{\alpha})^q.$$

Obviously, we have

Lemma 7 *If Φ is of type (p,q), then Φ is of weak-type (p,q).*

Proof: See [S], p. 20.

Definition 2 *$L^p(R^n) + L^q(R^n)$ is the space of all functions f such that $f = f_1 + f_2$, with $f_1 \in L^p(R^n)$ and $f_2 \in L^q(R^n)$.*

Lemma 8 *For all p such that* $p_1 \le p \le p_2$,

$$L^p(R^n) \subset L^{p_1}(R^n) + L^{p_2}(R^n).$$

Proof: See [S], p. 20.

Lemma 9 *Suppose* $1 < r \le \infty$, *and let* Φ *be a mapping from* $L^1(R^n) + L^r(R^n)$ *to the space of measurable functions on* R^n. *Suppose that for all f and* $g \in L^1(R^n) + L^r(R^n)$, *we have:*

1.

$$|\Phi(f+g)| \le |\Phi(f)| + |\Phi(g)|, \quad (sub-additive),$$

2.

$$measure\{x : |(\Phi(f))(x)| > \alpha\} \le \frac{A_1}{\alpha}\|f\|_{L^1(R^n)}, \quad (weak-type(1,1)),$$

3.

$$measure\{x : |(\Phi(f))(x)| > \alpha\} \le (\frac{A_r\|f\|_{L^r(R^n)}}{\alpha})^r, \quad (weak–type(r,r)).$$

Then

$$\|\Phi(f)\|_{L^p(R^n)} \le A_p\|f\|_{L^p(R^n)} \tag{0.31}$$

for all $f \in L^p(R^n)$ *with* $1 < p < r$, *where*

$$A_p = (\frac{2A_1}{p-1} + \frac{(2A_r)^r}{r-p})^{\frac{1}{p}}.$$

Moreover, as $p \to 1$,

$$A_p(p-1) \to C,$$

where C is a fixed constant depending only on A_1, A_r, *and r.*

Proof: See [S], p. 21 - 22.

Lemma 10 *Suppose*

$$(\Phi(f))(x) = \int_{R^n} K(x-y)f(y)dy, \tag{0.32}$$

where

$$K(x) = \frac{\Psi(x)}{|x|^n},$$

with $\Psi(x)$ *homogeneous of degree* 0*. Suppose* $\Psi(x)$ *also satisfies:*

1.

$$\int_{S^{n-1}} \Psi(x)d\sigma = 0,$$

where S^{n-1} *is the unit sphere in* R^n*.*

2. *if*

$$\sup_{|x-y|\leq\delta, |x|=|y|=1} |\Psi(x) - \Psi(y)| = \omega(\delta),$$

then

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta \leq \infty.$$

Remark: *This condition requires a certain continuity of* $\Psi(x)$*, which is known as 'Dini-type'. Of course, any* $\Psi(x)$ *which is of* C^1*, or even merely Lipschitz continuous, gives* $\omega(\delta) \leq C\delta^\alpha$*, for some* $\alpha > 0$*, and thus satisfies this condition.*

Then,

1. Φ *is weak-type (1,1).*

2. For any $f \in L^p(R^n)$, $1 < p < \infty$,

$$\|\Phi(f)\|_{L^p(R^n)} \leq A_p \|f\|_{L^p(R^n)}, \tag{0.33}$$

where A_p is a constant depending only on p.

Proof: See [S], p. 29 - 42.

We claim here:

Lemma 11 *the constant A_p in the above lemma can be dominated by Ap for some fixed constant A independent of p; i.e. (0.33) can be rewritten as*

$$\|\Phi(f)\|_{L^p(R^n)} \leq Ap\|f\|_{L^p(R^n)}. \tag{0.34}$$

Proof: We prove this lemma in two steps.

1). For $1 < p \leq 2$.

In fact, according to Lemma 10, Φ is weak-type (1,1) and weak-type (3,3). Also, Φ is sub-additive because of its form (0.32). Applying Lemma 9, we have

$$\|\Phi(f)\|_{L^p(R^n)} \leq B_p \|f\|_{L^p(R^n)}, \tag{0.35}$$

for any function $f \in L^p(R^n)$, $1 < p \leq 2$, with

$$B_p = \left(\frac{2A_1}{p-1} + \frac{(2A_3)^2}{3-p}\right)^{\frac{1}{p}}. \tag{0.36}$$

2). For $2 \leq p < \infty$.

We use the equivalent L^p norm

$$\|\psi\|_{L^p(R^n)} = \sup_{\|\phi\|_{L^q(R^n)}\leq 1} |\int_{R^n} \psi(x)\phi(x)dx|,$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For any $f \in L^p(R^n)$, $2 \leq p < \infty$, and $g \in L^q(R^n)$, with $\|g\|_{L^q(R^n)} \leq 1$, we have that

$$\begin{aligned}
&\int_{R^n} (\Phi(f))(x)g(x)dx \\
&= \int_{R^n} [\int_{R^n} K(x-y)f(y)dy]g(x)dx \\
&= \int_{R^n} [\int_{R^n} K(x-y)g(x)dx]f(y)dy.
\end{aligned}$$

Applying 1). to g, we get

$$\begin{aligned}
&|\int_{R^n} (\Phi(f))(x)g(x)dx| \\
&\leq \|\Phi(g(-x))\|_{L^q(R^n)} \|f\|_{L^p(R^n)} \\
&\leq B_q \|g\|_{L^q(R^n)} \|f\|_{L^p(R^n)} \\
&\leq B_q \|f\|_{L^p(R^n)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\Phi(f)\|_{L^p(R^n)} &= \sup_{\|g\|_{L^q(R^n)}\leq 1} |\int_{R^n} (\Phi(f))(x)g(x)dx| \\
&\leq B_q \|f\|_{L^p(R^n)}, \qquad (0.37)
\end{aligned}$$

where

$$B_q = (\frac{2A_1}{q-1} + \frac{(2A_3)^2}{3-q})^{\frac{1}{q}}.$$

Since

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we then obtain

$$B_q = [2A_1(p-1) + (2A_3)^2 \frac{2p-3}{p-1}]^{\frac{p-1}{p}} \leq Ap, \tag{0.38}$$

for some fixed constant A independent of p. Therefor (0.35), (0.37) and (0.38) give (0.34).

Q.E.D.

Section 5

In this section, we prove the inequality (0.7) for the region $\Omega = T^2 \times [0,1]$. A similar inequality

$$\|Du\|_{L^p(\Omega)} \leq C_p(\|w\|_{L^p(\Omega)} + \|u_0\|_{L^2(\Omega)})$$

may have been proven somewhere before for some constant C_p changing with p, although the author hasn't seen it. Here, we need C_p to be bounded by $K_2 p$ for some constant K_2 independent of p.

Now we turn our attention to the relation between the velocity u, the solution of (0.1)-(0.4), and its vorticity $w = \nabla \times u$. Since we assume $\Omega = T^2 \times [0,1]$, $u(x_1, x_2, x_3)$ is periodic in (x_1, x_2). Assume the period is (1,1), where the region has its boundary in the x_3 direction. Applying $\nabla\times$ to w, and using (0.2) and (0.3), we get three elliptic equations, which are Dirichlet problems or Neumann problems:

For u_1:

$$\Delta u_1 = \frac{\partial}{\partial x_2} w_3 - \frac{\partial}{\partial x_3} w_2,$$
$$\frac{\partial}{\partial x_3} u_1|_{x_3=0,1} = w_2|_{x_3=0,1}.$$

For u_2:

$$\Delta u_2 = \frac{\partial}{\partial x_3} w_1 - \frac{\partial}{\partial x_1} w_3,$$
$$\frac{\partial}{\partial x_3} u_2|_{x_3=0,1} = -w_1|_{x_3=0,1}.$$

For u_3:

$$\Delta u_3 = \frac{\partial}{\partial x_1} w_2 - \frac{\partial}{\partial x_2} w_1,$$

$$u_3|_{x_3=0,1} = 0.$$

We study the above problems by starting with the half-plane $x_3 \geq 0$. Since the region Ω is compact, we can then prove the inequality by using the partition of unity method.

Now we consider the Neumann problem in the half -plane $x_3 \geq 0$ for u_1 with the property that $u_1(p)$ vanishes as the point p goes to infinity. We have

$$\Delta u_1 = \frac{\partial}{\partial x_2} w_3 - \frac{\partial}{\partial x_3} w_2, \tag{0.39}$$

$$\frac{\partial}{\partial x_3} u_1|_{x_3=0} = w_2|_{x_3=0}, \tag{0.40}$$

and here we assume that w is supported in the half ball B^+.

We give some notation before getting the explicit representation of the above solution. Let $x = (x_1, x_2, x_3)$, $\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3)$, and $\overline{x}^* = (\overline{x}_1, \overline{x}_2, -\overline{x}_3)$, the reflection point of $\overline{x}$. Also, let

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Choose the second Green's function

$$G(x, \overline{x}) = \frac{1}{2\pi}\left(\frac{1}{|x - \overline{x}|} + \frac{1}{|x - \overline{x}^*|}\right).$$

Clearly,

$$\frac{\partial}{\partial \overline{x}_3} G(x, \overline{x})|_{x_3=0} = 0.$$

Applying the Green's second formula, we obtain that

$$u_1(x) = \int_{R^3_+} G(x, \overline{x})\left[\frac{\partial}{\partial x_2} w_3(\overline{x}) - \frac{\partial}{\partial x_3} w_2(\overline{x})\right] d\overline{x}$$

$$+\frac{1}{2\pi}\int_{R^2}\frac{2w_2(\overline{x}_1,\overline{x}_2,0)}{\sqrt{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2}}d\overline{x}_1 d\overline{x}_2$$

$$=-\int_{R^3_+}[\frac{\partial}{\partial\overline{x}_2}G(x,\overline{x})w_3(\overline{x})-\frac{\partial}{\partial\overline{x}_3}G(x,\overline{x})w_2(\overline{x})]d\overline{x}$$

$$+\frac{2}{\pi}\int_{R^2}\frac{w_2(\overline{x}_1,\overline{x}_2,0)}{\sqrt{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2}}d\overline{x}_1 d\overline{x}_2. \tag{0.41}$$

Say the first integral is the interior term and the second one is the boundary term. The interior term is a linear combination of I_2, I_2^*, I_3, and I_3^*, where

$$I_i=\int_{R^3_+}\frac{\partial}{\partial\overline{x}_i}(\frac{1}{|x-\overline{x}|})w(\overline{x})d\overline{x}=\int_{R^3_+}\frac{x_i-\overline{x}_i}{|x-\overline{x}|^3}w(\overline{x})d\overline{x},$$

$$I_i^*=\int_{R^3_+}\frac{\partial}{\partial\overline{x}_i}(\frac{1}{|x-\overline{x}^*|})w(\overline{x})d\overline{x},$$

$$for \quad i=1,2,3.$$

Here, we ignore the indices of w. Apply derivatives to I_2 to obtain

$$\frac{\partial}{\partial x_1}I_2=\int_{R^3_+}\frac{3(x_1-\overline{x}_1)(x_2-\overline{x}_2)}{|x-\overline{x}|^5}w(\overline{x})d\overline{x}$$

$$=\int_{R^3_+}\frac{\Psi_1(x-\overline{x})}{|x-\overline{x}|^3}w(\overline{x})d\overline{x},$$

where

$$\Psi_1(x_1,x_2,x_3)=\frac{3x_1x_2}{x_1^2+x_2^2+x_3^2},$$

Also,

$$\frac{\partial}{\partial x_2}I_2=\int_{R^3_+}\frac{\Psi_2(x-\overline{x})}{|x-\overline{x}|^3}w(\overline{x})d\overline{x},$$

where

$$\Psi_2(x_1, x_2, x_3) = \frac{2x_2^2 - x_1^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2},$$

and

$$\frac{\partial}{\partial x_3} I_2 = \int_{R_+^3} \frac{\Psi_3(x - \overline{x})}{|x - \overline{x}|^3} w(\overline{x}) d\overline{x},$$

where

$$\Psi_3(x_1, x_2, x_3) = \frac{3x_2 x_3}{x_1^2 + x_2^2 + x_3^2}.$$

We extend $w(\overline{x}) = 0$ to the lower half-space $x_3 \leq 0$. The above integrals of DI_2 can be written as integrals over R^n. It is easy to check that all the integral kernels $\Psi_i, i = 1, 2, 3$, satisfy conditions (1) and (2) in Lemma 10. Thus,

$$\|DI_2\|_{L^p(\Omega)} \leq Ap\|w\|_{L^p(\Omega)}.$$

Now, observe I_i and I_i^*, for $i = 1, 2, 3$, and note that the integral forms of their derivatives will have the same properties as DI_2. So

$$\max\{\|DI_i\|_{L^p(\Omega)}, \|DI_i^*\|_{L^p(\Omega)}\} \leq Ap\|w\|_{L^p(\Omega)},$$

$$for \quad i = 1, 2, 3.$$

According to the Minkowski inequality,

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)},$$

we arrive at the estimate for the derivatives of the interior term of (0.41)

$$\|D\int_{R^3_+} D_{\overline{x}}G(x,\overline{x})w(\overline{x})d\overline{x}\|_{L^p(\Omega)} \leq Ap\|w\|_{L^p(\Omega)}, \tag{0.42}$$

for some constant A independent of p.

Now we should estimate the boundary term in equation (0.41). Let

$$J = \int_{R^2} \frac{w(\overline{x}_1,\overline{x}_2,0)}{\sqrt{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2}} d\overline{x}_1 d\overline{x}_2.$$

Let

$$M_0 = \sqrt{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2},$$

$$M = \sqrt{x_1^2+x_2^2+x_3^2}.$$

Taking the partial derivative of J with the respect to x_1, we obtain

$$\frac{\partial}{\partial x_1}J = \int_{R^2} \frac{-(x_1-\overline{x}_1)w(\overline{x}_1,\overline{x}_2,0)}{((x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2)^{\frac{3}{2}}} d\overline{x}_1 d\overline{x}_2$$

$$= \int_{R^2} \frac{\Gamma_1(\frac{x_1-\overline{x}_1}{M_0},\frac{x_2-\overline{x}_2}{M_0},\frac{x_3}{M_0})}{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2} w(\overline{x}_1,\overline{x}_2,0)d\overline{x}_1 d\overline{x}_2,$$

where

$$\Gamma_1(\frac{x_1}{M},\frac{x_2}{M},\frac{x_3}{M}) = -\frac{x_1}{M}.$$

Similarly, the partial derivatives of J with the respect to x_2 and x_3 are as follows:

$$\frac{\partial}{\partial x_2}J = \int_{R^2} \frac{\Gamma_2(\frac{x_1-\overline{x}_1}{M_0},\frac{x_2-\overline{x}_2}{M_0},\frac{x_3}{M_0})}{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2} w(\overline{x}_1,\overline{x}_2,0)d\overline{x}_1 d\overline{x}_2,$$

where

$$\Gamma_2(\frac{x_1}{M},\frac{x_2}{M},\frac{x_3}{M}) = -\frac{x_2}{M},$$

and

$$\frac{\partial}{\partial x_3}J = \int_{R^2}\frac{\Gamma_3(\frac{x_1-\overline{x}_1}{M_0},\frac{x_2-\overline{x}_2}{M_0},\frac{x_3}{M_0})}{(x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+x_3^2}d\overline{x}_1 d\overline{x}_2,$$

where

$$\Gamma_3(\frac{x_1}{M},\frac{x_2}{M},\frac{x_3}{M}) = -\frac{x_3}{M}.$$

We will derive the following in general in order to estimate $\|DJ\|_{L^p(\Omega)}$. Let $y = (y_1,\ldots,y_n)$, $z \geq 0$ and real, $|y| = \sqrt{y_1^2+\ldots+y_n^2}, P = (y,z)$, and $|P| = (|y|^2+z^2)^{\frac{1}{2}}$. $K(y,z)$ is a kernel defined in the upper-half space R_+^{n+1} with homogeneous degree $-n$, and is of the form

$$K(y,z) = \frac{\Gamma(\frac{y}{|P|},\frac{z}{|P|})}{(|y|^2+z^2)^{\frac{n}{2}}},$$

where $\Gamma(P)$ is homogeneous of degree 0 and satisfies:

$$(1) \quad |\Gamma(Q_1)-\Gamma(Q_2)| \leq \kappa\overline{Q_1Q_2}^{\alpha},$$

for any unit vectors $Q_1 = (y,0), Q_2 = (\overline{y},z)$ and some constant $\kappa, \alpha, (\alpha \leq 1)$, where $\overline{Q_1Q_2}$ represents the geodesic distance on the unit sphere from Q_1 to Q_2; and

$$(2) \quad \int_{|y|=1}\Gamma(y,0)d\omega_y = 0.$$

Write

$$K(y,z) = K_1(y,z) + K_2(y,z),$$

where

$$K_1(y,z) = \frac{\Gamma(\frac{y}{|(y,z)|}, \frac{z}{|(y,z)|}) - \Gamma(\frac{y}{|y|}, 0)}{(|y|^2 + z^2)^{\frac{n}{2}}},$$

and

$$K_2(y,z) = \frac{\Gamma(\frac{y}{|y|}, 0)}{(|y|^2 + z^2)^{\frac{n}{2}}}. \tag{0.43}$$

For $K_1(y,z)$**:**

Let θ be the angle between vectors Q_1 and Q_2. Thus

$$\sin(\theta) = \frac{z}{\sqrt{|y|^2 + z^2}}$$

By Condition (1) and the inequality

$$|\theta|^\alpha \leq C^\alpha \sin^\alpha(\theta)$$

for $0 \leq \theta \leq \frac{\pi}{2}$, where constant C^α is fixed, we have

$$\begin{aligned}
|K_1(y,z)| &\leq \kappa \overline{PQ}^\alpha \frac{1}{(|y|^2 + z^2)^{\frac{n}{2}}} \\
&\leq \kappa C^\alpha \frac{\sin^\alpha(\theta)}{(|y|^2 + z^2)^{\frac{n}{2}}} \\
&= \kappa C^\alpha \left(\frac{z}{\sqrt{|y|^2 + z^2}}\right)^\alpha \frac{1}{(|y|^2 + z^2)^{\frac{n}{2}}}.
\end{aligned}$$

For any fixed z, set $y = z\xi, dy = z^n d\xi$.

$$\int_{R_n} |K_1(y,z)|dy \leq \kappa C^{\alpha} \int_{R_n} \frac{d\xi}{(|\xi|^2+1)^{\frac{n+\alpha}{2}}} \leq C. \tag{0.44}$$

Here C is independent of z. Recall the well-known result about convolution:

Lemma 12 *For any p such that $1 \leq p \leq \infty$, if $f \in L^1$, then*

$$\|f * g\|_{L^p(\Omega)} \leq \|f\|_{L^1(\Omega)} \|g\|_{L^p(\Omega)},$$

for any $g \in L^p$.

Proof: See [Tr], p. 278 -280.

(0.44) shows that K_1 is L^1. Using the above lemma, we have

$$\begin{aligned} &\| \int_{R_n} K_1(y-\eta, z) w(\eta, 0) d\eta \|_{L^p(R^n), z} \\ &\leq \|K_1\|_{L^1(R^n)} \|w(y,0)\|_{L^p(R^n)} \\ &\leq C\|w(y,0)\|_{L^p(R^n)}, \end{aligned}$$

where C depends neither on z nor on p.

For $K_2(y,z)$:

We should use Agmon-Douglis-Nirenberg's

Lemma 13 *For any p such that $1 < p < \infty$, and any z, if $w(y,0) \in L^p(R^n)$, and K_2 is of the form as in (.43), then*

$$\| \int_{R_n} K_2(y-\eta, z) w(\eta, 0) d\eta \|_{L^p(R^n),z} \leq C_p \| w(y,0) \|_{L^p(R^n)}, \qquad (0.45)$$

where C_p depends on p and not on z.

Proof: For a detailed proof, see [A-D-N], p. 709-711.

The above inequality is not sharp enough to satisfy us. We claim:

Lemma 14 *The constant C_p in the inequality (0.45) can be replaced by Cp, for some constant C independent of z.*

Proof: Using the same method as Stein used in [S], p.29-31 and p.39-45, and noticing that

$$\frac{1}{|y|^2 + z^2} \leq \frac{1}{|y|^2},$$

we can prove that for any fixed z, $K_2(y, z)$ is of weak-type (1,1) with constants A_1 independent of z. Now combine Lemma 10, Lemma 11 and Lemma 13 to prove Lemma 14.

Q.E.D.

The estimates for K_1 and K_2 give us

$$\| \int_{R_n} K(y-\eta, z) w(\eta, 0) d\eta \|_{L^p(R^n),z} \leq Cp \| w(y,0) \|_{L^p(R^n)}, \qquad (0.46)$$

where C is a constant independent of z.

Notice that DJ are the integral forms with kernels satisfying the conditions mentioned above. By (0.46) we immediately get

$$\|DJ\|_{L^p(R^2),z} \leq Cp\|w(y,0)\|_{L^p(R^2)}.$$

If u is supported in B^+, so is Du. Let Ω_1 be an open set such that

$$supp(u) \subset \Omega_1 \subset B^+.$$

Let ϕ be a smooth function defined in R^3 with support in B^+, and suppose that $\phi \equiv 1$ in Ω_1, $|\phi| \leq 1$, and $|D\phi| \leq C$. Ignoring the indices, we notice that Du is a linear combination of DI and DI^*, the derivatives of the interior terms, and DJ, the derivatives of the boundary terms. That is

$$Du = const.DI + const.DI^* + const.DJ.$$

So

$$Du = \phi Du = const.\phi DI + const.\phi DI^* + const.\phi DJ.$$

Since

$$\begin{aligned}
\|\phi DJ\|^p_{L^p(B+)} &\leq \int_0^1 (\int_{R^2} |\phi DJ|^p dy)dz \\
&\leq \int_0^1 (\int_{R^2} |DJ|^p dy)dz \\
&\leq \int_0^1 (Cp\|w(y,0)\|_{L^p(R^2)})^p dz \\
&\leq (Cp\|w(y,0)\|_{L^p(R^2)})^p,
\end{aligned}$$

we have

$$\|Du_1\|_{L^p(B^+)} = \|\phi Du_1\|_{L^p(B^+)}$$

$$\leq C(\|DI\|_{L^p(R^3)} + \|DI^*\|_{L^p(R^3)} + \|\phi DJ\|_{L^p(B^+)})$$

$$\leq Cp(\|w\|_{L^p(R^3)} + \|w(y,0)\|_{L^p(R^2)})$$

$$\leq Cp(\|w\|_{L^p(B^+)} + \|w(y,0)\|_{L^p(B^+ \cap R^2)})$$

$$\leq Cp\|w\|_{L^\infty(B^+)}.$$

The last second step is due to the assumption that w has support in B^+.

The norm estimates above let us arrive at the following:

If u is the solution of (0.39), (0.40), and u and w are supported in the half ball B^+, then

$$\|Du\|_{L^p(B^+)} \leq Cp\|w\|_{L^\infty(B^+)}. \tag{0.47}$$

Notice that u_1 and u_2 satisfy the same type of problem (.39)-(.40). So the above result also holds for u_2.

To derive the estimate for u_3, we consider the Dirichlet Problem in R^3_+:

$$\Delta u_3 = \frac{\partial}{\partial x_1} w_2 - \frac{\partial}{\partial x_2} w_1, \tag{0.48}$$

$$u_3|_{x_3=0} = 0, \tag{0.49}$$

where $u(p)$ goes to zero as the point p goes to infinity. Choose the first Green's function

$$G(x,\overline{x}) = \frac{1}{2\pi}(\frac{1}{|x-\overline{x}|} - \frac{1}{|x-\overline{x}^*|}).$$

We still assume w is supported in B_+ here. Now, u_3 can be represented by the integral:

$$u_3(x) = \int_{R^3_+} G(x,\overline{x})(\frac{\partial}{\partial \overline{x}_1} w_2(\overline{x}) - \frac{\partial}{\partial \overline{x}_2} w_1(\overline{x}))d\overline{x}$$

$$= -\int_{R^3_+} \frac{\partial}{\partial \overline{x}_1} G(x,\overline{x}) w_2(\overline{x}) - \frac{\partial}{\partial \overline{x}_2} G(x,\overline{x}) w_1(\overline{x}) d\overline{x}. \qquad (0.50)$$

Inside the representation, there is no boundary term J, since the zero boundary condition (0.48) and the derivatives are only in the $\overline{x}_1$ and $\overline{x}_2$ directions. Du_3, the first derivatives of u_3, are of the form

$$const.DI + const.DI^*.$$

Here, we ignore the indices as before. So Du_3 has norm estimate (0.47) as well.

In addition, we have to solve

$$\Delta u = Dw \qquad (0.51)$$

in R^3, where the result will be used in the partition of unity. We assume w is supported in the unit ball B, and we want to find the solution u vanishing at infinity. Here we choose Green's function

$$G(x,\overline{x}) = \frac{1}{2\pi}\frac{1}{|x-\overline{x}|}.$$

Again the solution u can be written as a convolution integral. Taking the first derivative of u should give us

$$Du = const.DI + const.DI^*.$$

Then the above argument will produce

$$\|Du\|_{L^p(R^3)} \leq Cp\|w\|_{L^\infty(B^+)}. \tag{0.52}$$

Now, we can prove our main inequality.

Lemma 15 *If u is the solution of (.1)-(.4) in $\Omega = T^2 \times [0,1]$, and w is its vorticity, then*

$$\|Du\|_{L^p(\Omega)} \leq K_2 p(\|w\|_{L^\infty(\Omega)} + \|u_0\|_{L^2(\Omega)}), \qquad (.7)$$

for $p > 1$, where K_2 is a fixed constant.

Proof: Since Ω is compact, there exists a partition of unity (U_j, ϕ_j) for $j = 1, \ldots, r+s$, where the U_j are small neighborhoods and the ϕ_j are smooth functions supported in U_j with

$$\sum_{j=1}^{r+s} \phi_j \equiv 1, \qquad and \qquad \bigcup_{j=1}^{r+s} U_j \supset \Omega.$$

Also

$$|\phi_j| \leq 1, \quad |D\phi_j| \leq C, \quad |D^2\phi_j| \leq C, \qquad for \quad j = 1, \ldots, r+s.$$

Suppose that for $1 \leq j \leq r$, the U_j are mapped into the unit ball B by a shift, and for $r+1 \leq j \leq r+s$, the U_j are mapped into the half ball B_+ in such a way that the boundaries $U_j \cap \partial\Omega$ are mapped into the plane $x_3 = 0$. Let

$$V_i^j = \phi_j u_i, \quad for \quad i = 1, 2, 3; \;\; j = 1, \ldots, r + s.$$

Ignoring the indices, we see that, for $i = 1, 2; j = r + 1, \ldots, r + s$, the V_i^j satisfy

$$\Delta V = D(\phi w) + 2D((D\phi)u) - (D\phi)w - (D^2\phi)u, \tag{0.53}$$

with the boundary condition

$$\frac{\partial}{\partial x_3} V|_{x_3=0} = \phi w + (D\phi)u, \tag{0.54}$$

in the half-space $x_3 \geq 0$. Using the second Green's function, V will be written as an integral of the interior terms

$$(DG)[\phi w + (D\phi)u] + G[(D\phi)w + (D^2\phi)u]$$

plus the boundary term as J where w replaced by $\phi w + (D\phi)u$. Applying the derivative D to V, we get

$$DV = const.DI + const.DI^* + const.DJ + \int (DG)[\phi w + (D\phi)u] d\overline{x}.$$

Since the last integral is not a singular integral, also ϕw and $(D\phi)u$ are supported in B^+, then

$$\| \int (DG)[\phi w + (D\phi)u] d\overline{x} \|_{L^p(R^n)} \leq C(\|w\|_{L^\infty(B^+)} + \|u\|_{L^\infty(B^+)}).$$

Using the previous results for DI, $DI*$ and DJ, we get that for $i = 1, 2; j = r+1, \ldots, r+s$, V_i^j satisfy

$$\|DV_i^j\|_{L^p(B+)} \leq Cp(\|w\|_{L^\infty(B+)} + \|u\|_{L^\infty(B+)}). \tag{0.55}$$

For $i = 3; j = r+1, \ldots, r+s$, the V_i^j satisfy

$$\Delta V = \frac{\partial}{\partial x_1}(\phi w) - \frac{\partial}{\partial x_2}(\phi w) + 2D((D\phi)u) - (D\phi)w - (D^2\phi)u, \tag{0.56}$$

with the boundary condition

$$V|_{x_3=0} = 0. \tag{0.57}$$

Notice that the derivatives of ϕw are only in the x_1 and x_2 directions, and that V has the zero boundary condition. So, DV will have the form as u_3 in (0.50), i.e., no boundary terms. Also $V, \phi w, (D\phi)w$, and $(D^2\phi)u$ are supported in B^+. By the result for u_3 and the above argument, the V_i^j should satisfy the inequality (0.55) for $i = 3; j = r+1, \ldots, r+s$.

For $i = 1, 2, 3; j = 1, \ldots, r$, the V_i^j satisfy the elliptic system in R^3 as type (0.51):

$$\Delta V = D(\phi w) + 2D((D\phi)u) - (D\phi)w - (D^2\phi)u. \tag{0.58}$$

Applying (0.52), for $i = 1, 2, 3; j = 1, \ldots, r$, we have

$$\begin{aligned}\|DV_i^j\|_{L^p(B^+)} &\leq \|DV_i^j\|_{L^p(R^3)} \\ &\leq Cp(\|w\|_{L^\infty(B)} + \|u\|_{L^\infty(B)}).\end{aligned}$$

Thus, we obtain the important estimate

$$\|Du\|_{L^p(\Omega)} \le \sum_{i,j} \|DV_i^j\|_{L^p(\Omega)}$$

$$\le Cp(\|w\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}). \qquad (0.59)$$

Recall the well-known inequality

$$\|u\|_{L^\infty} \le constant(A)\|Du\|_{L^{12}}^{\frac{2}{3}}\|u\|_{L^2}^{\frac{1}{3}} + constant(B)\|u\|_{L^2}.$$

We obtain

$$\|u\|_{L^\infty} \le \epsilon\|Du\|_{L^{12}} + C_\epsilon\|u\|_{L^2}$$

$$\le \epsilon\|Du\|_{L^{12}} + C_\epsilon\|u_0\|_{L^2}, \qquad (0.60)$$

where ϵ is a positive constant to be determined. Inequalities (0.59) and (0.60) give us

$$\|Du\|_{L^p} \le Cp(\epsilon\|Du\|_{L^{12}} + \|w\|_{L^\infty} + C_\epsilon\|u_0\|_{L^2}), \qquad (0.61)$$

for any $p, 1 < p < \infty$. Choosing $p = 12$, we have

$$\|Du\|_{L^{12}} \le 12C(\epsilon\|Du\|_{L^{12}} + \|w\|_{L^\infty} + C_\epsilon\|u_0\|_{L^2}).$$

If ϵ is sufficiently small so that

$$12C\epsilon < \frac{1}{2},$$

we have

$$\|Du\|_{L^{12}} \le C(\|w\|_{L^\infty} + \|u_0\|_{L^2}), \qquad (0.62)$$

for some fixed constant C. Replacing $\|Du\|_{L^{12}}$ in (0.61) with the right side of (0.62), we finally prove

$$\|Du\|_{L^p} \leq K_2 p(\|w\|_{L^\infty} + \|u_0\|_{L^2}), \quad (0.7)$$

for general $p, 1 < p < \infty$.

Q.E.D.

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