Mixing Elements into Kleinian Groups

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Abstract of the Dissertation

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We consider the question of creating Kleinian groups by adjoining elements to groups which are known to be Kleinian; alternatively, we consider the question of how subgroups of a Kleinian group with intersecting limit sets can interact. Specifically, we develop necessary conditions for the group $\langle G, M \rangle$ to be Kleinian, where G is an analytically finite Kleinian group and M is any Möbius transformation of infinite order which has a fixed point in $\Lambda(G)$, the limit set of G. Our main result has two parts, depending on the type of M. If M is loxodromic, then some (positive) power of M must lie in G; if M is parabolic, then either some (positive)

power of M lies in G or there is a doubly cusped parabolic element of G which has the same fixed point as M.

We prove these results first in the case that $\Lambda(G)$ is connected; we make use of a classical result on the Poincaré metric on $\Omega(G)$ to estimate the area of the quotient $\Omega(G)/G$. We then make use of standard techniques in Kleinian group theory to extend the results to the more general case.

To my parents, Wyatt and Margaret, and my wife, Christa.

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Acknowledgements

Graduate school is a wonderful experience, but it is not a solitary one. There are many people who helped me through it, and it is fitting that they be recognized.

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Chapter 1

Definitions and History

We begin by stating the definitions needed in this work and by discussing results of other authors which have a bearing on the results proven herein.

A Möbius transformation of the Riemann sphere $\overline{\mathbb{C}}$ is a map $T : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ of the form $T(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \overline{\mathbb{C}}$ and $ad - bc = 1$; the group M of Möbius transformations is exactly the group of orientation preserving conformal homeomorphisms of $\overline{\mathbb{C}}$. M acts triply transitively on $\overline{\mathbb{C}}$; that is, given two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of distinct points of $\overline{\mathbb{C}}$, there is a unique element $T \in \mathsf{M}$ such that $T(z_j) = w_j$ for $1 \leq j \leq 3$. For each $T \in \mathsf{M}$, let $\mathbf{fix(T)}$ denote the set of fixed points of T in M.

Every nontrivial element T of M has either 1 or 2 fixed points. If T has a single fixed point, then T is conjugate (in M) to the transformation $P(z) = z + 1$; such a T is **parabolic**. If T has 2 fixed points, then T is conjugate (in M) to the transformation $S(z) = \lambda z$ for some nonzero complex number $\lambda \neq 1$. If $|\lambda| = 1$, then T is **elliptic**; otherwise, T is **loxodromic**. λ is the **multiplier** of the transformation T.

Let $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation which does not fix ∞; that is, $c \neq 0$. The circle $I_T = \{z \in \mathbb{C} : |z + \frac{d}{c}| = \frac{1}{|c|}\}$ is the **isometric circle** of T. The name comes from the fact that $diam_E(I_T) = diam_E(T(I_T))$, where $diam_E$ denotes Euclidean diameter. This fact follows immediately from the observation that I_T is the set of points at which $|T'(z)| = 1$. Two properties of isometric circles we shall need are that $T(I_T) = I_{T^{-1}}$ and that T maps the exterior of I_T onto the interior of $I_{T^{-1}}$ [4].

Let G be a subgroup of $\mathbf{M}$. Say that G acts **discontinuously** at a point $z \in \overline{\mathbb{C}}$ if there exists a neighborhood U of z so that $g(U) \cap U \neq \emptyset$ for only finitely many $g \in G$. Define the **ordinary set** $\Omega(G)$ of G to be the set of points of $\overline{\mathbb{C}}$ at which G acts discontinuously.

Throughout this paper, the term **Kleinian group** is reserved for those subgroups G of $\mathbf{M}$ which have a nonempty ordinary set. It should be noted that every Kleinian group is a discrete subgroup of $\mathbf{M}$.

Let $\Lambda(G)$ be the complement of $\Omega(G)$ in $\overline{\mathbb{C}}$. $\Lambda(G)$, the **limit set of** G, is a closed set which contains $0, 1, 2$, or uncountably many points. If $\Lambda(G)$ is finite, G is **elementary**; if $\Lambda(G)$ is infinite, G is **nonelementary**. $\Lambda(G)$ contains the fixed points of all elements of infinite order in G.

A **point of approximation** of G is, loosely, a limit point of G which behaves like a fixed point of a loxodromic element of G. More precisely, x is a point of approximation of G if there is a sequence $\{g_m\}$ of distinct elements of G so that, for every compact set $K \subset \overline{\mathbb{C}} - \{x\}$, there is a constant $\delta_K > 0$ so that the spherical distance $d_S(g_m(x), g_m(z)) \geq \delta_K$ for all $z \in K$.

Let $X \subseteq \overline{\mathbb{C}}$ be any set. The stabilizer subgroup of X in the Kleinian group G is defined as $\mathbf{stab_G(X)} = \{g \in G : g(X) = X\}$. If Δ is a (connected) component of $\Omega(G)$, define $G_\Delta = stab_G(\Delta)$.

For a subgroup H of G, X is said to be **precisely invariant** under H in G if $h(X) = X$ for all $h \in H$ and if $h(X) \cap X$ is empty for all $h \in G - H$.

An element g of the Kleinian group G is **primitive** if no root of g lies in G; that is, there does not exist an element $h \in G$ with $h^m = g$ for some $m > 1$.

Let $x \in \Lambda(G)$ be any point. Say that x is a **doubly cusped parabolic fixed point** of G if there exists a primitive parabolic $P_x \in G$ fixing x and if there exist two open circular discs (or half planes) B_1 and B_2 in $\Omega(G)$ so that $B_1 \cup B_2$ is precisely invariant under $stab_G(x)$ in G. If x is a doubly cusped parabolic fixed point, then either $stab_G(x) = \langle P_x \rangle$ or $stab_G(x) = \langle P_x, e \rangle$, where e is an elliptic element of order 2 fixing x. We say that a parabolic element P of G is a **doubly cusped parabolic** if $fix(P)$ is a doubly cusped parabolic fixed point of G.

We say that a point $x \in \Lambda(G)$ is a **rank 2 parabolic fixed point** if $stab_G(x)$ contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of finite index. This $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is necessarily purely parabolic, with every element fixing x.

One of the major tools used in this work is the natural metric on $\Omega(G)$, where G is a nonelementary Kleinian group; this is the **Poincaré metric.** The Poincaré metric is the unique, complete metric of constant curvature -1 on $\Omega(G)$ which is invariant under the action of G. For completeness' sake, we give here a brief description of the Poincaré metric on $\Omega(G)$ and of some of its properties. For ease of notation, assume that $\infty \in \Lambda(G)$; then, every component of $\Omega(G)$ is a plane domain. Let z denote the standard complex coordinate on $\mathbb{C}$; then, $|dz|$ is the line element of the standard Euclidean metric on $\mathbb{C}$. We will define the Poincaré metric in terms of $|dz|$.

Let Δ be a component of $\Omega(G)$; G is nonelementary, so $\partial\Delta$ contains more than 2 points and Δ then has the upper half plane $\mathbb{H}^2$ of $\mathbb{C}$ as its universal covering. Moreover, G_Δ acts on Δ by conformal homeomorphisms, and this action lifts to an action of a subgroup $H \subset PSL_2(\mathbb{R})$ on $\mathbb{H}^2$. There is a complete metric on $\mathbb{H}^2$ which has constant curvature -1 and is invariant under the action of $PSL_2(\mathbb{R})$; the line element of this metric is $ds = \frac{|dz|}{Im(z)}$. Projecting to Δ, we obtain a metric on Δ which is invariant under G_Δ.

Since any two coverings of Δ by $\mathbb{H}^2$ differ by composition by an element of $PSL_2(\mathbb{R})$, the Poincaré metric on Δ is independent of any particular choice of covering.

The Poincaré metric is conformally equivalent to the standard metric on $\Omega(G)$; this follows from the fact that measuring angles at a point using the Poincaré metric and the standard Euclidean metric give the same result. Therefore, we can write the line element of the Poincaré metric as $\rho(z)|dz|$; $\rho(z)$ is a real valued function on $\Omega(G)$, called the **Poincaré distortion.**

For a set $D \subset \Omega(G)$, let $\mathbf{area}_\rho(\mathbf{D})$ denote the area of D calculated in the Poincaré metric on $\Omega(G)$; $area_\rho(D) = \int_D \rho^2(z)dA$, where dA is the Euclidean area element. We also define $\mathbf{area_E(D)}$ to be the Euclidean area of D; $area_E(D) = \int_D dA$.

Let $\pi : \Omega(G) \to \Omega(G)/G$ be the standard projection. The Poincaré metric in invariant under the action of G and so descends to a metric on the quotient $\Omega(G)/G$. For a subset D^0 of $\Omega(G)/G$, we also let $area_\rho(D^0)$ denote the Poincaré area of D^0 on the quotient $\Omega(G)/G$. It will be clear from the context whether we are calculating the area of a subset of $\Omega(G)$ or whether we are calculating the area of a subset of the quotient $\Omega(G)/G$.

If $area_\rho(\Omega(G)/G)$ is finite, G is said to be **analytically finite**. By convention, elementary groups are also considered to be analytically finite.

We will need the following observation. Let $D \subset \Omega(G)$ be an open set which is precisely invariant under the identity in G. Then,

$$area_\rho(D) \leq area_\rho(\Omega(G)/G).$$

Analytic finiteness is the weakest condition one can place on a Kleinian group and still have some uniformity of behavior. Two other conditions which Kleinian groups can be asked to satisfy are that G be finitely generated and that G be geometrically finite. Every finitely generated Kleinian group is analytically finite; this is Ahlfors' finiteness theorem [7]. Stronger still than finitely generated is that a Kleinian group be geometrically finite; this is the condition that the group have a finite sided fundamental polyhedron for its action on $\mathbb{H}^3$.

At this point, we add the assumption that $\Lambda(G)$ is connected; coupled with our earlier assumption that $\infty \in \Lambda(G)$, this is equivalent to saying that every component of $\Omega(G)$ is a simply connected plane domain. On $\Omega(G)$, let $\delta(z)$ denote the Euclidean distance to $\Lambda(G)$, and let $\rho(z)$ be the Poincaré distortion. It is a well known consequence of the Koebe $\frac{1}{4}$-theorem that

$$\frac{1}{2\delta(z)} \leq \rho(z)$$

for all $z \in \Omega(G)$ (see, for example, [3]).

The results in this work belong to a family of results which contain necessary conditions for subgroups of $\mathbf{M}$ to be discrete or Kleinian. Possibly the first result of this type is the observation that, in any discrete subgroup of $\mathbf{M}$, parabolic and loxodromic elements cannot share fixed points [9]. Beardon and Maskit generalize this observation with the following theorem.

Theorem (Beardon and Maskit [2]) *Let G be a discrete subgroup of $\mathbf{M}$. No point of $\Lambda(G)$ can be both a parabolic fixed point of G and a point of approximation of G.*

The following theorem of Susskind is a corollary of a stronger result which plays a crucial role in his characterization of intersections of pairs of geometrically finite subgroups of a discrete subgroup of $\mathbf{M}$.

Theorem (Susskind [13]) *Let K be a discrete subgroup of $\mathbf{M}$, let G be any subgroup of K, and let L be a loxodromic element of K fixing a point of approximation of G. Then $L^n \in G$ for some $n > 0$.*

Necessary conditions in a different vein are the inequalities of Jørgensen [6], Tan [14], and Gehring and Martin [5]. These results quantify the intuitive notion that there are universal lower bounds on how close elements of a discrete subgroup of $\mathbf{M}$ can be to one another.

None of the work yet cited has made use of any assumption stronger than discreteness. As we shall see later, there is a major difference between groups which are merely discrete and groups which are Kleinian. One result which requires the assumption the groups be Kleinian is the following theorem of Maskit.

Theorem (Maskit [10]) *Let K be a Kleinian group and let $\Delta \subset \Omega(K)$ be any component whose quotient Δ/K_Δ is analytically finite. Let k be any element of K which has a fixed point in $\partial\Delta = \Lambda(K_\Delta)$. If k is loxodromic, then $k^n \in K_\Delta$ for some $n > 0$. If k is parabolic, then there is a parabolic $h \in K_\Delta$ with $fix(k) = fix(h)$.*

When discussing necessary conditions, it is natural to also consider the question of sufficient conditions. This is a much more difficult question. To date, the only general sufficient conditions for a subgroup of $\mathbf{M}$ to be Kleinian are Poincaré's polyhedron theorem [11] and the combination theorems of Klein and Maskit [9].

Chapter 2

Statement of Results

The work in this paper deals with the problem of analyzing how pairs of subgroups of a Kleinian group can interact. We analyze the case in which one of the subgroups is analytically finite, the other subgroup is infinite cyclic, and the subgroups have intersecting limit sets.

Another way of viewing the work in this paper is to say that we wish to develop necessary conditions for the group $\langle G, M \rangle$ to be Kleinian, where G is an analytically finite Kleinian group and M is an infinite order Möbius transformation fixing a point of $\Lambda(G)$.

To this end, we prove the following two theorems.

Main Theorem (Loxodromic Case) *Let K be a Kleinian group and let G be an analytically finite subgroup of K. Suppose there exists a loxodromic element L of K which has a fixed point in $\Lambda(G)$. Then, $L^n \in G$ for some positive integer n.*

Main Theorem (Parabolic Case) *Let K be a Kleinian group and let*

G be an analytically finite subgroup of K. Suppose there exists a parabolic element P of K whose fixed point lies in $\Lambda(G)$. *Then, either* $P^n \in G$ *for some positive integer n or* $fix(P)$ *is a doubly cusped parabolic fixed point of G.*

As corollaries of these results, we have the following.

Corollary *Let K be a Kleinian group and let G be an analytically finite subgroup of K. Suppose that* $x \in \Lambda(K)$ *is a rank 2 parabolic fixed point. If* $x \in \Lambda(G)$, *then x is either a rank 2 parabolic fixed point of G or a doubly cusped parabolic fixed point of G.*

Corollary *Let K be a Kleinian group and let G be a nonelementary analytically finite subgroup. Suppose that there is an element* $k \in K$ *of infinite order with* $k(\Lambda(G)) \subset \Lambda(G)$. *Then,* $k(\Lambda(G)) = \Lambda(G)$ *and* $k^n \in G$ *for some* $n > 0$.

Chapter 3

Adding Loxodromic Elements

In this chapter, we prove the loxodromic case of the main theorem under the additional assumption that the group G has connected limit set. We prove this result by contradiction. That is, we assume that no (nonzero) power of L lies in G and then construct an open subset D of $\Omega(G)$ which is precisely invariant under the identity in G and which has infinite Poincaré area. This contradicts the analytic finiteness of G.

We will need the following lemma in the construction of such a set.

Lemma (The Invariance Lemma) *Let K be a Kleinian group and let G be a subgroup of K. Assume there exists an element $M \in K$ of infinite order so that no (nonzero) power of M lies in G. Let $D \subset \Omega(K)$ be any set which is precisely invariant under the identity in K. Then, $A = \bigcup_{n\in\mathbf{Z}} M^n(D)$ is precisely invariant under the identity in G.*

Proof Assume that there exists an element $g \in G$ so that $g(A) \cap A$ is nonempty. Then, $g(M^n(D)) \cap M^l(D)$ is nonempty for some integers n and l;

the precise invariance of D under the identity in K then forces $g(M^n(D)) = M^l(D)$. Therefore, $M^{-l} \cdot g \cdot M^n(D) = D$, which forces $M^{-l} \cdot g \cdot M^n$ to be the identity. Consequently, $g = M^{l-n}$. Since no nonzero power of M lies in G, it must be that $l = n$ and g is the identity. □

We are now ready to complete the proof of the loxodromic case of the main theorem in the special case that G has a connected limit set.

Lemma (Loxodromic Special Case) *Let K be a Kleinian group and let G be an analytically finite subgroup of K with connected limit set. Suppose there exists a loxodromic element L of K which has a fixed point in $\Lambda(G)$. Then, $L^n \in G$ for some positive integer n.*

Proof First, we show that G cannot be elementary. G cannot be finite, since $\Lambda(G)$ is assumed to be nonempty. Were G to contain a purely parabolic subgroup of finite index, then $\langle G, L\rangle$ would be a discrete subgroup of M in which a parabolic and a loxodromic transformation share a fixed point; this cannot occur. G cannot contain a loxodromic cyclic subgroup of finite index, as $\Lambda(G)$ is assumed to be connected. By the classification of elementary Kleinian groups, these are the only possibilities. Therefore, G is nonelementary.

Assume that no (nonzero) power of L lies in G. Let x_0 be a fixed point of L contained in $\Lambda(G)$. After normalization (that is, after we conjugate K by an appropriate element of M), we can assume that $L(z) = \lambda z$, $x_0 = \infty$, and $1 \in \Lambda(G)$. We assume that $1 \in \Lambda(G)$ because we have no information on whether 0, the other fixed point of L, is a point of $\Omega(G)$ or of $\Lambda(G)$; we fix $1 \in \Lambda(G)$ solely to make the following calculation concrete.

If necessary, replace L by L^{-1} so that $|\lambda| > 1$. Since $\infty \in \Lambda(G)$ and $\Lambda(G)$ is connected, every component of $\Omega(G)$ is a simply connected plane domain.

Choose $z_0 \in \Omega(K)$ and $\epsilon > 0$ so that $B = B_\epsilon(z_0)$, the Euclidean disc of radius ϵ about z_0, is precisely invariant under the identity in K; define $B_m = L^m(B)$. Let ρ be the Poincaré distortion on $\Omega(G)$ and let δ be the Euclidean distance to $\Lambda(G)$; since $1 \in \Lambda(G)$, $\delta(z) \leq |z-1|$.

By the invariance lemma, $\bigcup_{m\in\mathbb{Z}} B_m$ is precisely invariant under the identity in G; the B_m are disjoint, so

$$area_\rho(\Omega(G)/G) \geq \sum_{m=0}^{\infty} area_\rho(B_m) = \sum_{m=0}^{\infty} \int_{B_m} \rho^2(z)dA \geq \frac{1}{4}\sum_{m=0}^{\infty} \int_{B_m} \frac{1}{\delta^2(z)}dA.$$

Choose an integer m_0 so that B_m lies outside the unit circle for $m \geq m_0$. Then, $\delta(z) \leq |z-1| \leq |z|+1 < 2|z|$ for any $z \in B_m$ with $m \geq m_0$, and so

$$\frac{1}{4}\int_{B_m} \frac{1}{\delta^2(z)}dA \geq \frac{1}{16}\int_{B_m} \frac{1}{|z|^2}dA.$$

At this point, we claim that $\int_{B_m} \frac{1}{|z|^2}dA$ is a positive constant independent of m. To see this, note that

$$\int_{B_m} \frac{1}{|z|^2}dA = \int_B \frac{1}{|L^m(z)|^2}((L^m)^*dA),$$

where $(L^m)^*$ is the standard pull-back of L^m.

Since $dA = \frac{1}{2i}dz \wedge d\bar{z}$, we have that

$$(L^m)^*(\frac{1}{2i}dz \wedge d\bar{z}) = \frac{1}{2i}d(L^m(z)) \wedge d\overline{(L^m(z))} = \frac{1}{2i}(\lambda^m dz) \wedge (\bar{\lambda}^m d\bar{z}) = |\lambda|^{2m}dA.$$

Thus,

$$\int_{B_m} \frac{1}{|z|^2} dA = \int_B \frac{1}{|L^m(z)|^2} (L^m)^* dA = \int_B \frac{1}{|\lambda|^{2m}|z|^2} |\lambda|^{2m} dA = \int_B \frac{1}{|z|^2} dA.$$

Therefore, $area_\rho(\Omega(G)/G)$ is infinite, a contradiction. □

Chapter 4

Adding Parabolic Elements

In this chapter, we prove the parabolic case of the main theorem under the additional assumption that the group G has connected limit set. Again we use proof by contradiction. We assume that no (nonzero) power of P lies in G and we show that either $fix(P)$ is a doubly cusped parabolic fixed point of G or there is an open set $D \subset \Omega(G)$ which is precisely invariant under the identity in G and which has infinite Poincaré area. The analytic finiteness of G then forces the former. For technical reasons, however, we are required to make a much more delicate choice of D than in the loxodromic case. We also need to make much less crude use of the estimate of $\rho(z)$ on $\Omega(G)$.

The following two lemmas are area calculations which we will make repeated use of in the argument to follow.

Lemma (The Disc Lemma) *Let G be a nonelementary Kleinian group with connected limit set. If there exists an open circular disc $D \subset \Omega(G)$ such that some point $x \in \Lambda(G)$ lies on the boundary of D and D is precisely* *invariant under the identity in G, then* $area_\rho(\Omega(G)/G)$ *is infinite.*

Proof Normalize G so that $x = 0$ and so that $\infty \in \Lambda(G)$; since $0 \in \Lambda(G)$, $\delta(z) \leq |z|$. We wish to estimate the Poincaré area of D. Let $\rho(z)$ be the Poincaré distortion on $\Omega(G)$ and let $\delta(z)$ be the Euclidean distance to $\Lambda(G)$. Since $\Lambda(G)$ is connected and contains ∞, we have the inequality

$$\frac{1}{2\delta(z)} \leq \rho(z).$$

Note that for some choice of angles $0 < \theta_1 < \theta_2 < 2\pi$ and $0 < r$, the wedge shaped region $W = \{z \in \mathbb{C} : \theta_1 < arg(z) < \theta_2,\ |z| < r\}$ is contained in D (see figure 1).

The precise invariance of W under the identity in G yields that

$$area_\rho(\Omega(G)/G) \geq area_\rho(W) = \int_W \rho^2(z)dA \geq \frac{1}{4}\int_W \frac{1}{\delta^2(z)}dA \geq \frac{1}{4}\int_W \frac{1}{|z|^2}dA.$$

Changing to polar coordinates, one see that this last integral is infinite. □

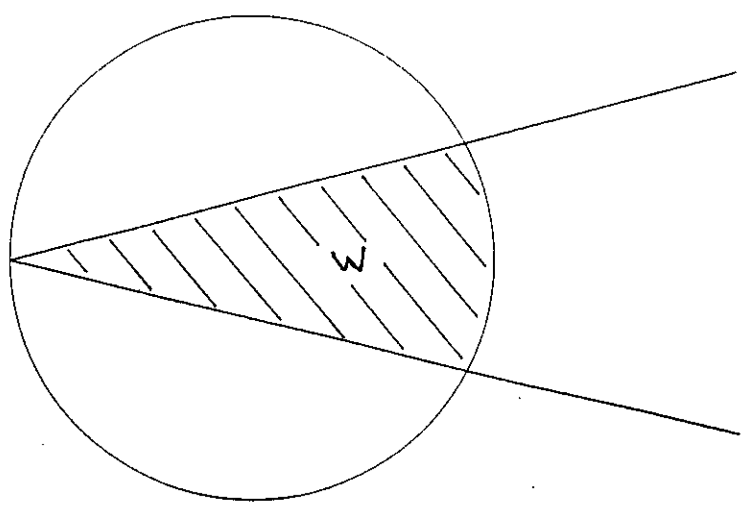


Figure 1

Lemma (The Area Lemma) *Let G be a nonelementary Kleinian group with connected limit set and $\infty \in \Lambda(G)$. Let $\{D_j\}_{j\in J}$ be a countable collection of pairwise disjoint open sets in $\Omega(G)$ such that $D = \bigcup_{j\in J} D_j$ is precisely invariant under the identity in G. If there are positive constants c and e such that $\delta(z) \leq c$ for all $z \in D$ and $area_E(D_j) \geq e$ for each $j \in J$, then $area_\rho(\Omega(G)/G)$ is infinite.*

Proof The proof is an area calculation. Since D is precisely invariant under the identity in G,

$$area_\rho(\Omega(G)/G) \geq area_\rho(D) = \sum_{j\in J} area_\rho(D_j).$$

Using the estimates on $\delta(z)$ and $area_E(D_j)$, we have

$$area_\rho(D_j) = \int_{D_j} \rho^2(z)dA \geq \int_{D_j} \frac{dA}{4\delta^2(z)} \geq \frac{area_E(D_j)}{4c^2} \geq \frac{e}{4c^2}$$

for each j. Therefore, $area_\rho(\Omega(G)/G)$ is infinite. □

Before going on to the next lemma, it is necessary to develop a bit of notation. Given $\tau \in \mathbb{C}$ with $Im(\tau) > 0$, let C_0^0 be the closed region in $\mathbb{C}$ bounded by the parallelogram with vertices 0, 1, τ, and $1+\tau$. For integers n and m, let $C_n^m = C_0^0 + n + m\tau$. The bricks C_n^m decompose the complex plane into an infinite checkerboard. We call this the **checkerboard corresponding to τ.**

For a nonempty closed subset A of $\mathbb{C}$ and a brick C of this infinite checkerboard, define $d_A(C)$ to be the minimum number of moves a (chess) king must make to get from C to a brick containing a point of A; call d_A the **king distance** for the set A. Let $\delta_A(z)$ denote the Euclidean distance from z to A.

For a brick C in our checkerboard and a point $z \in C$, we have that $\delta_A(z) \leq (d_A(C)+1)l_\tau$, where l_τ is the length of the longer diagonal of C_0^0. Also, the value of d_A can jump at most 1 each time we move from a brick to any of the eight bricks around it; in other words, if B and C are contiguous bricks in this checkerboard, then $|d_A(B) - d_A(C)| \leq 1$.

We are now ready to complete the proof of the parabolic case of the main theorem in the special case that G has a connected limit set.

Lemma (Parabolic Special Case) *Let K be a Kleinian group and let G be an analytically finite subgroup of K with connected limit set. Suppose there exists a parabolic element P of K whose fixed point lies in $\Lambda(G)$. If no (nonzero) power of P lies in G, there exists a doubly cusped parabolic $Q \in G$ with $fix(Q) = fix(P)$.*

Proof First, we take care of the case that G is elementary. G cannot be a finite group, since $\Lambda(G)$ is assumed to be nonempty. G cannot contain a loxodromic cyclic subgroup of finite index, since $\Lambda(G)$ is assumed to be connected.

If G contains a purely parabolic subgroup of finite index, then $\Lambda(G)$ consists of a single point x; since $fix(P) \in \Lambda(G)$, $\langle G, P\rangle$ is a Kleinian group whose limit set consists solely of the point x. Coupling the assumption that no (nonzero) power of P lies in G with the fact that Kleinian groups cannot contain $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ subgroups, we find that the purely parabolic subgroup of G has rank 1. The generator Q of the purely parabolic subgroup of G is then a doubly cusped parabolic element which has the same fixed point as P. This completes the elementary case. Hereafter, we can assume that G is nonelementary.

Normalize K so that $P(z) = z + 1$; choose $z_0 \in \Omega(K)$ and $\epsilon > 0$ so that the Euclidean disc $B = B_\epsilon(z_0)$ is precisely invariant under the identity in K. For ease of calculation, conjugate K by a translation so that $z_0 = \frac{1}{2}(1 + i)$. Let ρ be the Poincaré distortion on $\Omega(G)$ and let δ be the Euclidean distance to $\Lambda(G)$. Decompose the plane into the checkerboard corresponding to i and let $d = d_{\Lambda(G)}$. Let $B_n = P^n(B)$, and notice that $B_n \subset C_n^0$ (see figure 2).

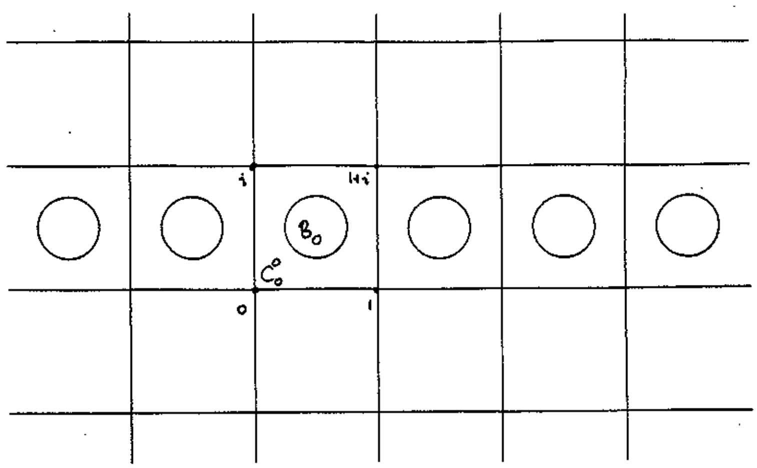


Figure 2

The first step is to show that there must exist a parabolic element Q of G with $fix(P) = fix(Q)$; assume not. The first case to consider is that no element of $G - \{1\}$ fixes ∞. Every element $g \in G - \{1\}$ then has a well defined isometric circle, whose center $g^{-1}(\infty)$ lies in $\Lambda(G)$. For $g \in G - \{1\}$, let r_g

be the radius of the isometric circle of g. K is a Kleinian group containing $P(z) = z + 1$, so $r_g \leq 1$ for all $g \in G - \{1\}$; this is a consequence of the Shimizu-Leutbecher lemma [9]. Every element of $G - \{1\}$ maps the exterior of its isometric circle onto the interior of the isometric circle of its inverse; therefore, the common exterior of the isometric circles of all elements of $G - \{1\}$ is precisely invariant under the identity in G. Therefore, the set $\{\delta(z) > 1\}$ is precisely invariant under the identity in G.

Let $V(n) = \bigcup_{m \in \mathbb{Z}} C_n^m$; $V(n)$ is the closed strip between the vertical lines $\{Re(z) = n\}$ and $\{Re(z) = n + 1\}$. Consider the $V(n)$ with $n > 0$; either finitely many of these strips contain points of $\Lambda(G)$ or infinitely many contain points of $\Lambda(G)$.

If only finitely many of the $V(n)$ contain points of $\Lambda(G)$, let $V(n_0)$ be the rightmost such. $\Lambda(G)$ is then contained in the half plane $\{Re(z) \leq n_0 + 1\}$. Thus, the half plane $\{Re(z) > n_0 + 3\}$ is contained in the set $\{\delta(z) > 1\}$ and so is precisely invariant under the identity in G. The disc lemma then implies that $area_\rho(\Omega(G)/G)$ is infinite.

In the case that infinitely many of the $V(n)$ contain points of $\Lambda(G)$, we would like to be able to exert some control. What we are going to do is to find an open set in each of these $V(n)$ so that the union of these open sets is precisely invariant under the identity in G and the Poincaré area of these open sets is bounded away from 0. This will imply that the Poincaré area of the union of these open sets is infinite.

However, it will suffice to find such open sets in an infinite subset of these $V(n)$; if we take any infinite subset of these open sets mentioned above, the union of the subsets will still be precisely invariant under the identity in G and will still have infinite Poincaré area.

Therefore, we need only work with a subsequence of these $V(n)$. By passing to an appropriate subsequence of these $V(n)$, we can assume that $d(C_n^0)$ behaves in one of two ways.

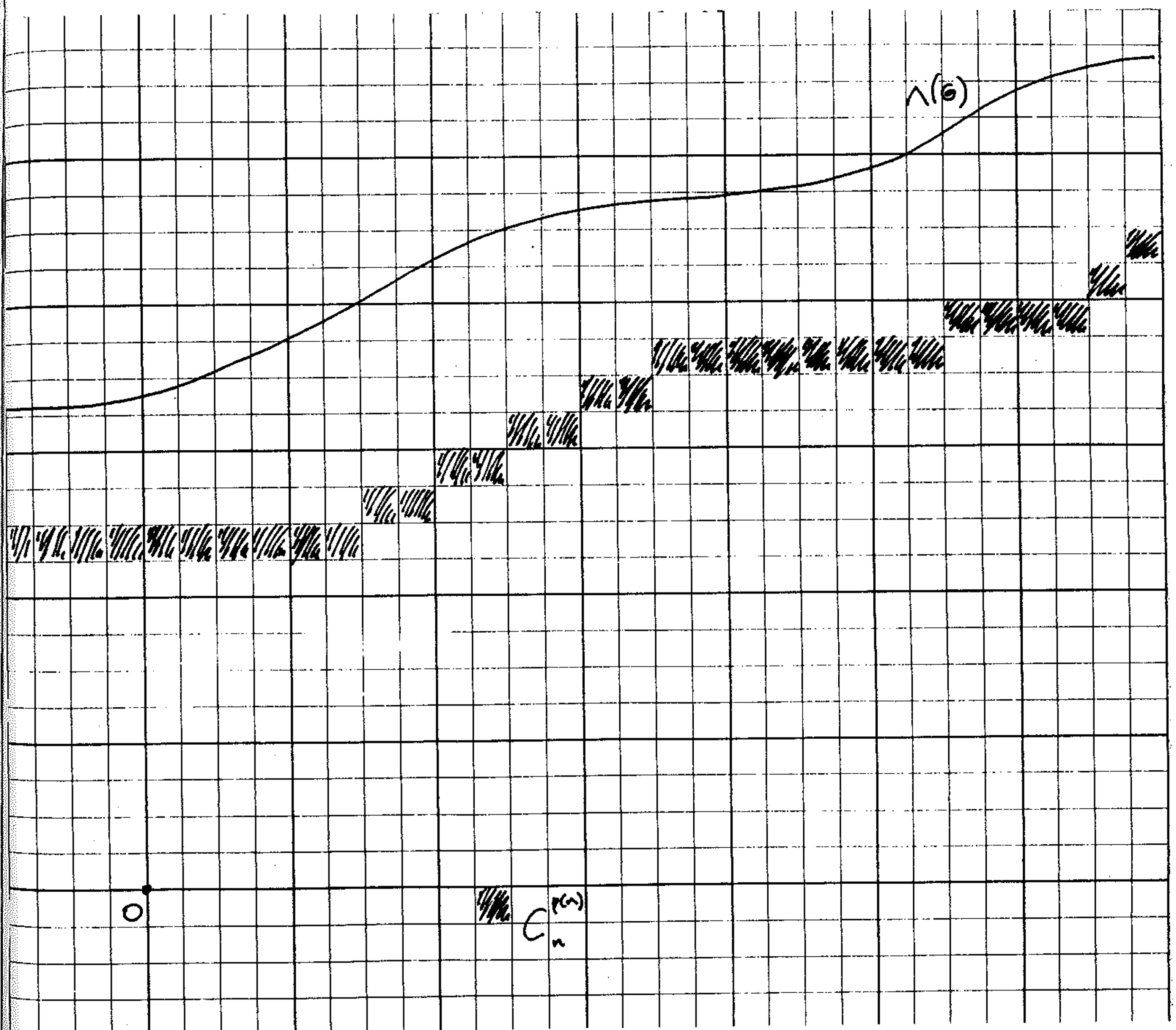
If we can find a subsequence on which $d(C_n^0)$ remains bounded, we can refine it to a subsequence on which $d(C_n^0)$ is constant. Otherwise, $d(C_n^0)$ eventually wanders off to infinity as $n \to \infty$; in this case, we can refine the subsequence so that $d(C_n^0) \to \infty$ monotonically as $n \to \infty$.

Having found such a subsequence, renumber the strips $V(n)$ in the following way. Ignore all the strips $V(n)$ which are not part of the subsequence we have just constructed, and number the strips in this subsequence consecutively; that is, as $V(1), V(2), \ldots, V(n), \ldots$. We renumber in this way solely to make the following piece of the argument easier to follow. Without this renumbering, we could proceed only with an unnecessary proliferation of subscripts and subsubscripts, which we would like to avoid.

First, consider the case that $d(C_n^0)$ remains constant as $n \to \infty$. That is, $d(C_n^0) = c$ for $n > 0$. Using our earlier estimate, we then have that $\delta(z) \leq (c+1)\sqrt{2}$ for all $z \in \bigcup_{n>0} P^n(B)$. Since $area_E(P^n(B))$ is constant for $n > 0$, the area lemma implies that $area_\rho(\Omega(G)/G)$ is infinite.

Next, consider the case that $d(C_n^0) \to \infty$ monotonically as $n \to \infty$. For each n, let $C_n^{m(n)}$ be the brick in the strip $V(n)$ so that $C_n^{m(n)}$ contains a point of $\Lambda(G)$ and is closest to the brick C_n^0 (see figure 3). Note that every brick in $V(n)$ strictly between $C_n^{m(n)}$ and C_n^0 is contained wholly in $\Omega(G)$.

Figure 3



Since $C_n^{m(n)}$ contains a point of $\Lambda(G)$, $d(C_n^{m(n)}) = 0$ for all n. $d(C_n^0) \to \infty$ monotonically as $n \to \infty$, so $d(C_n^0) \geq 5$ for n sufficiently large. Stepping vertically from $C_n^{m(n)}$ to $C_n^{m(n)-1}$ on down to C_n^0, d decreases at most 1 during each step. Therefore, for each n sufficiently large, we can find a brick $C_n^{p(n)}$ in $V(n)$, strictly between $C_n^{m(n)}$ and C_n^0, so that $d(C_n^{p(n)}) = 4$. $\delta(z)$ is then uniformly bounded as z ranges over all $C_n^{p(n)}$ with n sufficiently large. $area_E(int(C_n^{p(n)}))$ is constant, so the area lemma implies that $area_\rho(\Omega(G)/G)$ is infinite.

To recap, under the assumption that no element of $G - \{1\}$ fixes ∞, $area_\rho(\Omega(G)/G)$ is infinite. The analytic finiteness of G then forces there to exist an nontrivial element Q of G with $fix(Q) = fix(P) = \infty$. Q cannot be loxodromic, since loxodromic and parabolic elements of a Kleinian group cannot share a fixed point.

Assume that Q is elliptic; without loss of generality, $Q(0) = 0$ as well. Q is then a Euclidean isometry and so preserves both $\{\delta(z) > 1\}$ and $\Lambda(G)$. As before, we can assume that either $d(C_n^0)$ is constant or $d(C_n^0) \to \infty$ monotonically as $n \to \infty$.

In the case that $d(C_n^0)$ is constant, $area_\rho(\Omega(G)/G)$ is infinite. To see this, note that, since $d(C_n^0)$ is constant, we have a uniform bound on $\delta(z)$ as z ranges over $D = \bigcup_{n=0}^{\infty} P^n(B)$. The invariance lemma gives that D is precisely invariant under the identity in G. The Euclidean area of $P^n(B)$ is constant, so the area lemma yields that $area_\rho(C)$ is infinite.

It is only in the case that $d(C_n^0) \to \infty$ as $n \to \infty$ that the presence of an elliptic fixing ∞ causes us to be more cautious.

Proceed with the choosing of bricks $C_n^{p(n)}$ exactly as in the case that no element of $G - \{1\}$ fixes ∞. Let $C = \bigcup_{n=0}^{\infty} C_n^{p(n)}$, and note that the only elements g of G which can satisfy $g(C) \cap C \neq \emptyset$ are powers of Q. To see this, recall that the only elements of G which can fix ∞ are powers of Q. Therefore, if $g \in G$ is not a power of Q, then C lies outside the isometric circles of both g and g^{-1} (by construction) and so $g(C) \cap C = \emptyset$.

If we can find an infinite subsequence of the $C_n^{p(n)}$ whose union is precisely invariant under the identity in $\langle Q \rangle$, then this union will also be precisely invariant under the identity in all of G.

By the normalization on Q, Q preserves every disc centered at the origin. Moreover, each such disc can intersect only finitely many of the $C_n^{p(n)}$. Start off by choosing an open disc D_0 centered at the origin which contains $C_0^{p(0)}$. Since D_0 can intersect only finitely many of the $C_n^{p(n)}$, we can choose $C_{n_1}^{p(n_1)}$ outside D_0.

Q preserves D_0, so $C_0^{p(0)} \cup C_{n_1}^{p(n_1)}$ is precisely invariant under the identity in $\langle Q \rangle$. This implies that $C_0^{p(0)} \cup C_{n_1}^{p(n_1)}$ is precisely invariant under the identity in G.

Now, choose an open disc D_1 centered at the origin which contains $C_0^{p(0)} \cup C_{n_1}^{p(n_1)}$ and choose $C_{n_2}^{p(n_2)}$ outside D_1. Proceeding in this fashion, we construct a subsequence $C_{n_j}^{p(n_j)}$ of the $C_n^{p(n)}$ so that $C' = \bigcup_{j=0}^{\infty} C_{n_j}^{p(n_j)}$ is precisely invariant under the identity in G. The area lemma gives that $area_\rho(C')$, and hence $area_\rho(\Omega(G)/G)$, is infinite.

Therefore, the element $Q \in G$ fixing ∞ must be parabolic; without loss of generality, assume that Q is primitive. All that remains to prove is that Q is doubly cusped; assume not. Since we're given that no nonzero power of

P lies in G, the maximal purely parabolic subgroup of $stab_G(\infty)$ has rank 1. Therefore, either $stab_G(\infty) = \langle Q \rangle$ or $stab_G(\infty) = \langle Q, e \rangle$, where e is an elliptic of order 2 fixing ∞.

$Q(z) = z + \tau$ for some τ with nonzero imaginary part; if necessary, replace Q by Q^{-1} so that $im(\tau) > 0$. Choose $z_0 \in \Omega(K)$ and $\epsilon > 0$ so that the Euclidean disc $B = B_\epsilon(z_0)$ is precisely invariant under the identity in K; without loss of generality, we can take $z_0 = \frac{1}{2}(1 + \tau)$.

Decompose the complex plane into the checkerboard corresponding to τ and consider the slices $V(n) = \bigcup_{m \in \mathbb{Z}} C_n^m$. Let $B_n = P^n(B)$, and notice that $B_n \subset C_n^0$. Either finitely many of the $V(n)$ contain points of $\Lambda(G)$ or infinitely many of the $V(n)$ contain points of $\Lambda(G)$.

In the latter case, we can assume that infinitely many of the $V(n)$ with $n > 0$ contain a point of $\Lambda(G)$. If $V(n)$ is a strip containing a point of $\Lambda(G)$, then the brick C_n^0 contains a point of $\Lambda(G)$; all we need do is find some point of $\Lambda(G)$ in $V(n)$ and translate it along $V(n)$ to C_n^0 using an appropriate power of Q. Let $D = \bigcup_{\mathbb{Z}} P^n(B)$. $\delta(z)$ is uniformly bounded as z ranges over D and the Euclidean area of $P^n(B)$ is constant. Therefore, the area lemma implies that $area_\rho(D)$ is infinite.

In the former case, we show that Q is a doubly cusped parabolic. Let $V(l_0)$ be the leftmost strip containing a point of $\Lambda(G)$, $V(r_0)$ the rightmost strip containing a point of $\Lambda(G)$, and define D to be the union of the open regions to the left of $V(l_0 - 1)$ and to the right of $V(r_0 + 1)$ (see figure 4). If $stab_G(\infty) = \langle Q, e \rangle$, then replace D by $D \cap e(D)$. Regardless of whether $stab_G(\infty) = \langle Q \rangle$ or $stab_G(\infty) = \langle Q, e \rangle$, D is the union of two circular discs.

D is invariant under $stab_G(\infty)$ by construction. To see that D is precisely invariant under $stab_G(\infty)$ in G, note that D lies in the region $\{\delta(z) > 1\}$ and so D lies outside the isometric circle of any element of $G - stab_G(\infty)$.

Having chased through all possibilities, the only allowable possibility under the assumptions that no (nonzero) power of P lies in G and that G is analytically finite is that G contains a doubly cusped parabolic element Q with $fix(Q) = fix(P)$. □

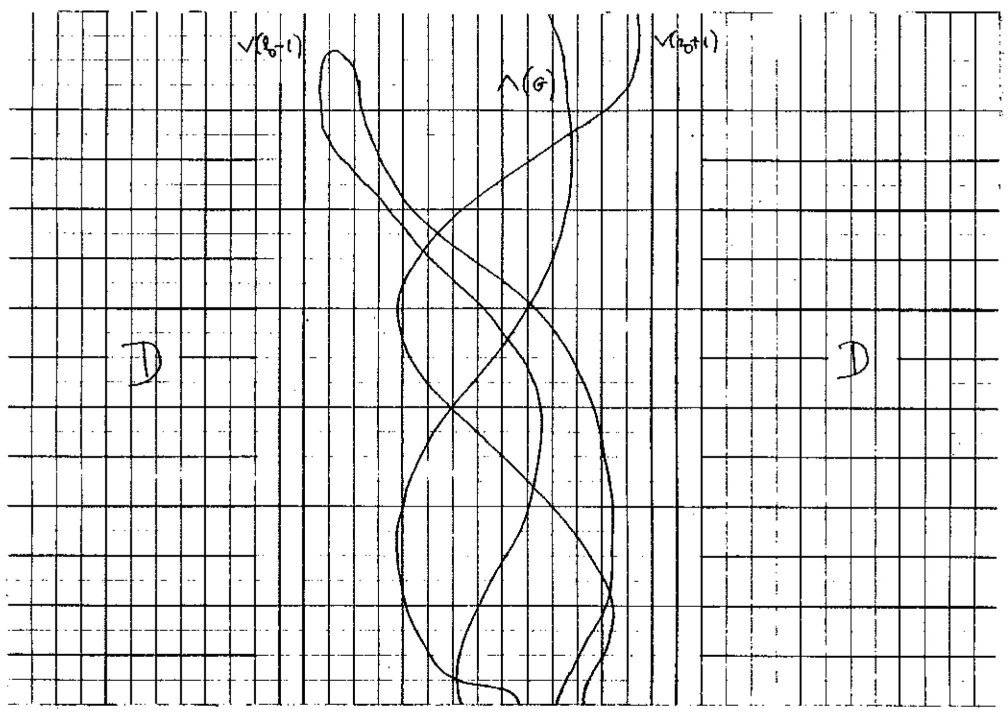


Figure 4

Chapter 5

The Reduction Step

The purpose of this chapter is to complete the proofs of the two cases of the main theorem. In order to accomplish this, we need some way of reducing the general situation to the special situation in which the group G has a connected limit set. To this end, we prove the following lemma.

Lemma (The Decomposition Lemma) *Let G be an analytically finite Kleinian group. There exist finitely many subgroups $G_1, \ldots, G_s$ of G so that each G_j is analytically finite with connected limit set, each point of $\Lambda(G)$ is either the translate of a point of some $\Lambda(G_j)$ or is a point of approximation of G, each parabolic $P \in G$ is conjugate to a parabolic element of some G_j, and P is doubly cusped in G if and only if its conjugate is doubly cusped in G_j.*

In order to prove this lemma, we will need to invoke the machinary of the planarity theorem and the combination theorems. The planarity theorem gives us topological information about the action of a Kleinian group on its ordinary set, and the combination theorems allow us to decompose the group into simpler groups using this topological information. We begin with the planarity theorem.

Theorem (Planarity Theorem) [9] *Let $p : \Delta \to S$ be a regular covering of the topologically finite Riemann surface S, where Δ is planar. Then there is a finite set $\{w'_m\}$ of disjoint loops on S, where each w'_m is the power of a simple geodesic loop, so that $p : \Delta \to S$ is the highest regular covering of S for which the loops $\{w'_m\}$ all lift to loops.*

In order to exploit the existence of such a set of loops, we need to use some sort of combination theorem. Before we state the theorem, we need to define our terms. The definitions are adapted from those in [12].

We say that a Jordan curve γ in $\overline{\mathbb{C}}$ is a **swirl for** G if γ lies in $\Omega(G)$ and if γ is precisely invariant under a finite cyclic subgroup $J(\gamma)$ of G. One important fact about a swirl σ is that the spherical diameters of any sequence of distinct translates of σ go to 0 [12].

Let Σ be a collection of swirls. A **panel for** Σ is a equivalence class of points of $\overline{\mathbb{C}} - \Sigma$, where two points are equivalent if they are separated by no swirl in Σ.

We say that Σ is a **system of swirls for** G if Σ is a G invariant collection of swirls so that Σ/G consists of finitely many curves, if no two swirls in Σ cross (they are allowed to touch), and if for every swirl σ on the boundary of two inequivalent panels P_1 and P_2, there is an element $g_i \in stab_G(P_i)$ with $g_i(\sigma) \neq \sigma$, $i = 1, 2$.

Let Σ be any system of swirls for the Kleinian group G. We will need information about the points of $\overline{\mathbb{C}}$ which belong to no swirl and to no panel. To this end, we prove the following proposition.

Proposition 1 *Let Σ be a system of swirls for the Kleinian group G. Then, a point of $\overline{\mathbb{C}}$ either belongs to a swirl in Σ or a panel for Σ, or is a point of approximation of G.*

Proof Let $x \in \overline{\mathbb{C}}$ be a point which belongs to neither a swirl nor a panel. There must then exist a sequence $\{\sigma_m\} \subset \Sigma$ such that the σ_m nest about x; that is, the spherical diameter $diam_S(\sigma_m)$ of σ_m goes to 0 as m goes to ∞ and σ_{m+1} separates σ_m from x (see figure 5). Since Σ/G contains only finitely many curves, we can assume that $\sigma_m = g_m(\sigma_0)$ for elements $g_m \in G$ and a fixed swirl $\sigma_0 \in \Sigma$.

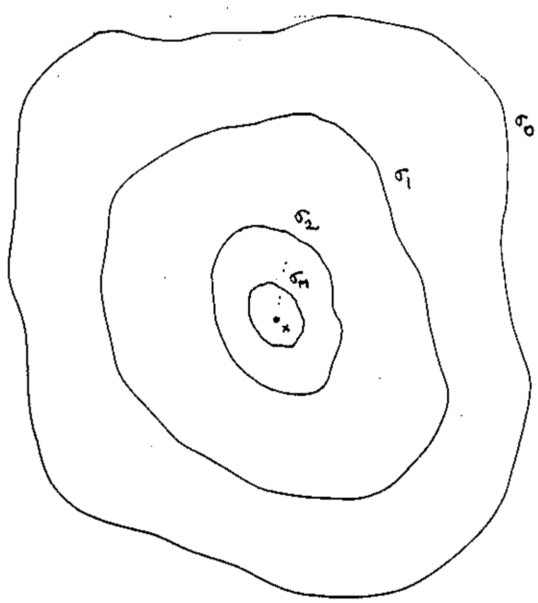


figure 5

We wish to show that x is a point of approximation of G. Recall that a point of approximation of G is a point $x \in \overline{\mathbb{C}}$ so that there is a sequence $\{g_m\} \subset G$ of distinct elements of G so that $d_S(g_m(x), g_m(z)) \geq \delta_K > 0$ for each compact set $K \subset \overline{\mathbb{C}} - \{x\}$ and each $z \in K$.

Let K be any compact subset of $\overline{\mathbb{C}} - \{x\}$. Since $diam_S(\sigma_m)$ goes to 0 as m goes to ∞, σ_m separates K from x for $m \geq m_K$.

Each point $g_m^{-1}(x)$ lies in $\Lambda(G)$ and $\Lambda(G)$ is closed, so every accumulation point of the $g_m^{-1}(x)$ also lies in $\Lambda(G)$. Since $\sigma_0 \subset \Omega(G)$, we have that

$$\epsilon = inf\{dist_S(g_m^{-1}(x), \sigma_0) : m \geq 0\} > 0.$$

Since σ_m separates K from x for $m \geq m_K$, σ_0 separates $g_m^{-1}(K)$ from $g_m^{-1}(x)$ for $m \geq m_K$; hence, $dist_S(g_m^{-1}(x), g_m^{-1}(K)) \geq \epsilon$ for $m \geq m_K$.

Let $\alpha = min\{dist_S(g_m^{-1}(x), g_m^{-1}(K)) : m < m_K\}$. Since $x \notin K$, $\alpha > 0$.

Setting $\delta_K = min(\alpha, \epsilon)$, we see that x is a point of approximation of G.

□

We will also need to know how the limit points of G are related to the limit points of the panel stabilizers. To this end, we prove the following proposition.

Proposition 2 *Let Σ be a system of swirls for the Kleinian group G. Let X be a panel for Σ. If $x \in \Lambda(G) \cap X$, then $x \in \Lambda(stab_G(X))$.*

Proof We first consider the possiblity that x is an interior point of X. Since $\Lambda(G)$ is perfect (i. e., every point of $\Lambda(G)$ is an accumulation point) and loxodromic fixed points are dense in $\Lambda(G)$, we can find loxodromic elements L_m of G which fix points x_m (respectively) so that the x_m converge to x.

Since x is an interior point of X, it must be that x_m is an interior point of X for m sufficiently large. Then, $L_m(X) = X$ for all m sufficiently large. x_m then belongs to $\Lambda(stab_G(X))$ for m sufficiently large, and so $x \in \Lambda(stab_G(X))$.

If x is not an interior point of X, there is a sequence σ_m of swirls which lies on the boundary of X and which converge to x (see figure 6). Since Σ/G is finite, we can assume that $\sigma_m = g_m(\sigma_0)$ for some fixed swirl σ_0 on the boundary of X and elements $g_m \in G$.

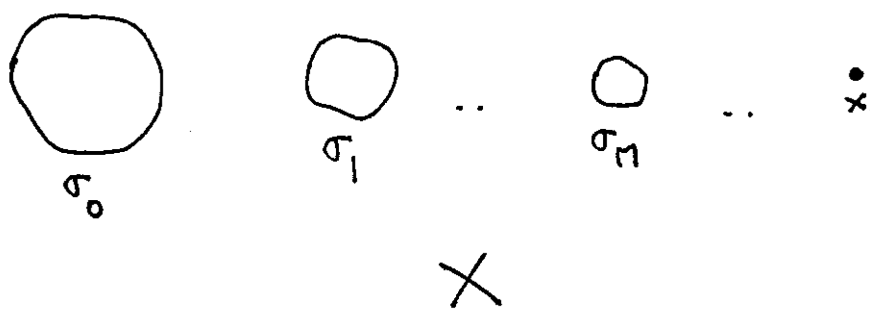


Figure 6

If $g_m(X) = X$ for infinitely many of the g_m, then $x \in \Lambda(stab_G(X))$.

If not, then $g_m(X) \neq X$ for all but finitely many m. By removing finitely many terms from the sequence, we can assume that $g_m(X) \neq X$ for all m. Let Y be the panel which lies across σ_0 from X. We then have that $g_m(Y) = X$ for all m.

In this case, $g_1^{-1} g_m(Y) = Y$ for all m. Since the $g_m(\sigma_0)$ converge to x as

$m \to \infty$, the $g_1^{-1}g_m(\sigma_0)$ converge to $g_1^{-1}(x)$. Therefore, $g_1^{-1}(x) \in \Lambda(stab_G(Y)) = \Lambda(stab_G(g_1^{-1}(X)))$. Thus, $x \in g_1(\Lambda(stab_G(Y))) = \Lambda(stab_G(X))$. □

We now state the combination theorem. The version we use here can be found in [12].

Theorem (Combination Theorem) [12] *Suppose there is a nonempty system of swirls for the Kleinian group G. Then, G is analytically finite if and only if each panel stabilizer is analytically finite, and every parabolic element of G belongs to some panel stabilizer.*

We are now ready to prove the decomposition lemma.

Lemma (The Decomposition Lemma) *Let G be an analytically finite Kleinian group. There exist finitely many subgroups $G_1, \ldots, G_s$ of G so that each G_j is analytically finite with connected limit set, each point of $\Lambda(G)$ is either the translate of a point of some $\Lambda(G_j)$ or is a point of approximation of G, and each parabolic $P \in G$ is conjugate to a parabolic element of some G_j, and P is doubly cusped in G if and only if its conjugate is doubly cusped in G_j.*

Proof We call the subgroups G_j the **factors** of G.

The quotient $\Omega(G)/G$ is the union of finitely many surfaces $S_1, \ldots, S_q$, where each S_j has finite genus and finitely many punctures. Let Δ_j be a component of $\Omega(G)$ which covers S_j.

By the planarity theorem, there are finitely many disjoint simple geodesic loops $\{w_m^j\}_{m=1}^{n(j)}$ on S_j and integers α_m^j so that $\pi : \Delta_j \to S_j$ is the highest regular covering for which all the $(w_m^j)' = (w_m^j)^{\alpha_m^j}$ lift to loops. Therefore, the set

$$S = \{(w_m^j)' : m = 1, \ldots, n(j); j = 1, \ldots, q\}$$

is a maximal set of loops on $\Omega(G)/G$, each of which lifts to a loop on $\Omega(G)$. Furthermore, the set S is empty only in the case that G has connected limit set.

For each j and each m, choose a lift γ_m^j of $(w_m^j)'$ in Δ_j, and define

$$\Gamma = \{\gamma_m^j : m = 1, \ldots, n(j); j = 1, \ldots, q\}.$$

The planarity theorem imposes a set of conditions on the $(w_m^j)'$ which translate into the following conditions on the γ_m^j.

Since the γ_m^j project to loops on $\Omega(G)/G$ which are powers of simple loops, each γ_m^j is precisely invariant under a finite cyclic group J_m^j of order α_m^j in G. Therefore, each γ_m^j is a swirl for G.

The γ_m^j project to disjoint loops, so the orbits of γ_m^j and γ_n^j, $m \neq n$, under G are disjoint. Obviously, the orbits of γ_m^j and γ_n^k, $j \neq k$, under G are disjoint.

In order to show that the set $G(\Gamma)$ is a system of swirls for G, we only need now show that the panel condition is met. Each panel covers a component of $S_j - \bigcup_{m=1}^{n(j)}(w_m^j)'$ for some j, so the panel stabilizers are infinite. Let $\sigma \in G(\Gamma)$ be a swirl on the boundaries of panels X_1 and X_2. The stabilizer $J(\sigma)$ of σ is finite, so there must exist elements $g_k \in stab_G(P_k)$ so that $g_k(\sigma) \neq \sigma$. Therefore, $G(\Gamma)$ is a system of swirls for G.

Let $X_1, \ldots, X_p$ be a complete set of inequivalent panels for the system $G(\Gamma)$, and let $G_j = stab_G(X_j)$.

We need to show that each of the G_j has connected limit set. This is a consequence of the planarity theorem. The set Γ is a maximal set of loops in $\Omega(G)$ in which each loop separates $\Lambda(G)$ and the elements of Γ project to homologically distinct loops on $\Omega(G)/G$. If some G_j has disconnected limit set, we would be able to use the planarity theorem to enlarge the set Γ; this cannot be done.

We are now ready to track the points of $\Lambda(G)$. Let $x \in \Lambda(G)$ be any point. One of two things can happen. It might be that there is a panel X so that $x \in X$. Invoking proposition 2, we see that $x \in \Lambda(stab_G(X))$, and so x is the translate of a point of some $\Lambda(G_j)$.

Suppose that x belongs to no panel. Then, there is a sequence of swirls in $G(\Gamma)$ nesting about x. Invoking the proposition 1, we see that x is a point of approximation of G.

The last thing we need to do is to track the parabolic elements. The combination theorem tells us that every parabolic in G stabilizes some panel; this immediately implies that every parabolic $P \in G$ is conjugate to a parabolic element of some G_j.

We only need to show that, if a parabolic element of some panel stabilizer is doubly cusped, then that parabolic is doubly cusped in G. This follows immediately from the fact the each swirl in $G(\Gamma)$ is contained in $\Omega(G)$. We can shrink the cusped regions of the parabolic until they miss all the swirls in $G(\Gamma)$. The panel stabilizers only interact at the swirl stabilizers, since the panels are precisely invariant under the panel stabilizers; so, the parabolic remains doubly cusped in the entire group G. □

We are now ready to complete the proofs of the two cases of the main theorem.

Main Theorem (Loxodromic Case) *Let K be a Kleinian group and let G be an analytically finite subgroup of K. Suppose there exists a loxodromic element L of K which has a fixed point in $\Lambda(G)$. Then, $L^n \in G$ for some positive integer n.*

Proof Let x_0 be a fixed point of L which is contained in $\Lambda(G)$. If x_0 is a point of approximation of G, then $L^n \in G$ for some positive integer n by Susskind's theorem [13]. Otherwise, there exists a factor H of G and an element $g \in G$ so that $x_0 \in g(\Lambda(H)) = \Lambda(gHg^{-1})$. By the lemma for the loxodromic special case, there then exists a positive integer n so that $L^n \in gHg^{-1} \subset G$. □

Main Theorem (Parabolic Case) *Let K be a Kleinian group and let G be an analytically finite subgroup of K. Suppose there exists a parabolic element P of K which has a fixed point in $\Lambda(G)$. Then, either $P^n \in G$ for some positive integer n or $fix(P)$ is a doubly cusped parabolic fixed point of G.*

Proof Let x_0 be the fixed point of P. By the Beardon-Maskit theorem

[2], x_0 cannot be a point of approximation of G.

By the decomposition lemma, there is a factor H of G and an element $g \in G$ so that $x_0 \in g(\Lambda(H)) = \Lambda(gHg^{-1})$. The lemma for the parabolic special case immediately implies that either $P^n \in gHg^{-1}$ for some $n > 0$ or $fix(P)$ is a doubly cusped parabolic fixed point of gHg^{-1}. In the latter case, we use the decomposition lemma again to see that $fix(P)$ is a doubly cusped parabolic fixed point of G. □

Chapter 6

Corollaries

In this chapter, we state and prove two corollaries to the two cases of the main theorem.

As an immediate corollary to the parabolic case of the main theorem, we have the following.

Corollary *Let K be a Kleinian group and let G be an analytically finite subgroup of K. Suppose that $x \in \Lambda(K)$ is a rank 2 parabolic fixed point. If $x \in \Lambda(G)$, then x is either a rank 2 parabolic fixed point or a doubly cusped parabolic fixed point of G.*

Proof Since x is a rank 2 parabolic fixed point of K, $stab_K(x)$ contains a rank 2 parabolic subgroup $\langle P, Q\rangle$. By the parabolic case of the main theorem, either both some power of P lies in G and some power of Q lies in G, or $stab_G(x)$ is a doubly cusped parabolic subgroup of G. □

As a second corollary to the main theorem, we have the following result about limits sets of Kleinian groups.

Corollary *Let K be a Kleinian group and let G be a nonelementary analytically finite subgroup. Suppose that there is an element $k \in K$ of infinite order with $k(\Lambda(G)) \subset \Lambda(G)$. Then, $k(\Lambda(G)) = \Lambda(G)$ and $k^n \in G$ for some $n > 0$.*

Proof Since $k(\Lambda(G)) \subset \Lambda(G)$, we have that $k^n(\Lambda(G)) \subset \Lambda(G)$ for all $n > 0$.

$fix(k) \subset \Lambda(K)$; if not, then some sufficiently high power k^m of k would satisfy $k^m(\Lambda(G)) \cap \Lambda(G) = \emptyset$.

If k is loxodromic, then the loxodromic case of the main theorem immediately implies that some positive power of k belongs to G.

If k is parabolic, then either $k^n \in G$ for some $n > 0$ or there is a doubly cusped parabolic element $Q \in G$ with $fix(Q) = fix(k)$. The latter case cannot occur; if there were a doubly cusped parabolic $Q \in G$ with $fix(Q) = fix(k)$, then $k^m(\Lambda(G)) \cap \Lambda(G) = \{fix(k)\}$ for $|m|$ sufficiently large. This is easist to see if we normalize so that $Q(z) = z + 1$; then, $k(z) = z + \tau$, where $Im(\tau) \neq 0$, and $\Lambda(G) \subset \{|Im(z)| < c\}$ for some constant $c > 0$. □

Chapter 7

Discussion of Results

This chapter begins with a discussion of the conclusions of both cases of the main theorem. We wish to show that these conclusions are the best possible. We do this by presenting examples of Kleinian groups and Möbius transformations which satisfy the hypotheses of the main theorem and which illustrate each of the possible conclusions.

We will also discuss the hypotheses of both cases of the main theorem. Here, what we wish to show is that a weakening of any of the hypotheses removes any possibility of having uniform control over how the element being added behaves.

7.1 Conclusions

In this section, we consider the conclusions of both cases of the main theorem and show that none of them are superfluous and that none of them can be weakened. We do this by constructing examples.

We start with the loxodromic case. The loxodromic case of the main theorem states that some positive power of L lies in G. In order to show that this statement cannot be weakened, we need to construct an analytically finite Kleinian group G and a loxodromic Möbius transformation L so that L has a fixed point in $\Lambda(G)$, the group $K = \langle G, L\rangle$ is Kleinian, and L is itself not an element of G.

Let K be a purely loxodromic, free, finitely generated Kleinian group. By a theorem of Maskit [8], any such a group is a Schottky group. Let $\{k_1, \ldots, k_p\}$ $(p > 1)$ be a minimal set of generators of K, and define $G = \langle k_1^2, k_2, \ldots, k_p\rangle$. G is finitely generated, hence analytically finite. Since G is free, k_1 is not an element of G, even though $\langle G, k_1\rangle$ is Kleinian.

The example in the parabolic case has a similar flavor. In this case, however, we need two examples. First, we need to construct an analytically finite Kleinian group G and a parabolic Möbius transformation P so that P has its fixed point in $\Lambda(G)$, the group $K = \langle G, P\rangle$ is Kleinian, and P is itself not an element of G.

Let $P(z) = z + 1$ and $Q(z) = \frac{z}{2z+1}$. The group $F = \langle P, Q\rangle$ is Kleinian, as it is a subgroup of $PSL_2(\mathbb{Z})$. Moreover, F is a free group; this follows immediately from the observation that every discrete subgroup of $PSL_2(\mathbb{R})$ is the fundamental group of a surface S, where S is either open (and so has free fundamental group) or is closed of genus at least 2 (in which case, its fundamental group has at least 4 generators).

Define $G = \langle P^2, Q\rangle$. G is finitely generated, hence analytically finite. Moreover, G is free, so P is not an element of G,. Finally, $F = \langle G, P\rangle$ is

Kleinian.

To illustrate the second conclusion, we need to construct an analytically finite Kleinian group G and a parabolic Möbius transformation P so that P has its fixed point in $\Lambda(G)$, the group $K = \langle G, P \rangle$ is Kleinian, and $fix(P)$ is a doubly cusped parabolic fixed point of G.

For this, let F be as given in the previous paragraph, and note that ∞ is a doubly cusped parabolic fixed point of F. Let $R(z) = z + 10i$. Using either the combination theorems or Poincaré's polyhedron theorem, it is easy to see that the group $K = \langle F, R \rangle$ is Kleinian.

Therefore, we have that, in some sense, the conclusions of both cases of the main theorem are as strong as can be.

7.2 Hypotheses

In this section, we consider the hypotheses of the main theorem. There are two hypotheses that we will examine. The first is that the group G be analytically finite. The second is that the group K be Kleinian.

What we will show is this section is that, if we weaken either of these hypotheses, we are no longer able to make any non-trivial uniform statement about how the element M behaves with respect to the group G.

The first hypothesis we will consider is that G is analytically finite. Here, we need to construct an analytically infinite Kleinian group G and an infinite order Möbius transformation M which has a fixed point in $\Lambda(G)$ so that the group $K = \langle G, M \rangle$ is Kleinian, but no (nonzero) power of M lies in G. We will construct an example in which M can be chosen to be either loxodromic or parabolic.

Let K be a nonelementary, analytically finite, free Kleinian group which contains parabolic elements, and let M be any element of K which has infinite order. Let $g \in K$ be any element of infinite order which is not conjugate to any power of M, and let N_m be the normal closure of $\langle M^m, g \rangle$ in K.

Define $G = \bigcap_{m=0}^{\infty} N_m$. G is a normal subgroup of K containing g, which implies that $\Lambda(G) = \Lambda(K)$. Moreover, G contains no (nonzero) power of M. This implies that G is an infinite index subgroup of K. The calculation of areas of fundamental domains shows that an infinite index, normal subgoup of an analytically finite Kleinian group is analytically infinite, so G is analytically infinite.

$\Lambda(G) = \Lambda(K)$, so $\Lambda(G)$ contains the fixed point set of M. Since $\langle G, M \rangle$ is a subgroup of K, it is Kleinian, even though no (nonzero) power of M lies in G. Note that this is independent of whether M is parabolic or loxodromic.

Therefore, the hypothesis that G be analytically finite is necessary to force any uniformity of behavior on M.

The next hypothesis is that the group K must be Kleinian as opposed to merely discrete. Here, we will present an example due to Peter Waterman (oral communication).

The example has the following form. We construct an analytically finite Kleinian group G and a loxodromic Möbius transformation L so that no (nonzero) power of L lies in G and the group $K = \langle G, L \rangle$ is discrete but not

Kleinian.

Let $L(z) = 2z$. Let $S = \{z \in \mathbb{C} : |z - \frac{3}{2}| = \frac{1}{8}\}$. Choose a finitely generated Fuchsian group of the first kind F fixing S so that the complement of some fundamental domain D for F lies in the annulus $A = \{1 < |z| < 2\}$ (see figure 7).

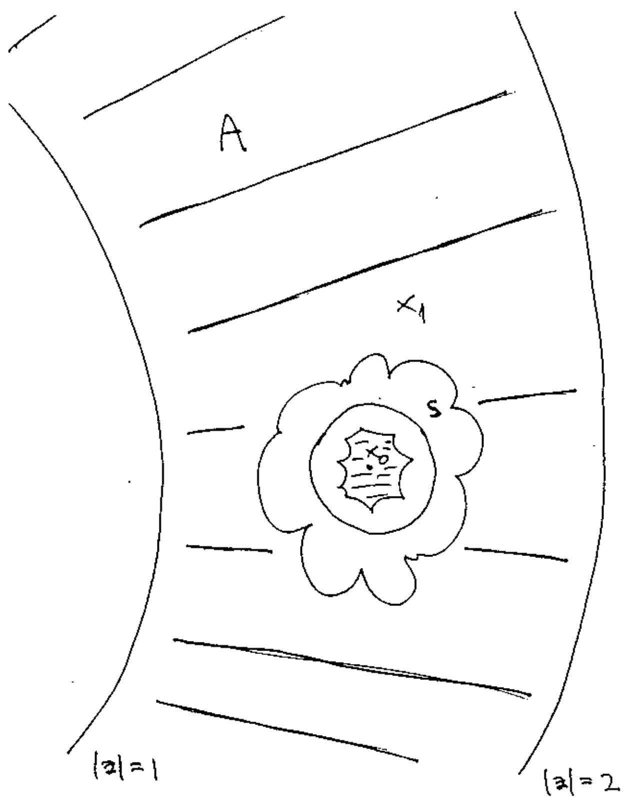


Figure 7

Let $X = A \cap D$. X has two connected components. X_0 lies inside S and X_1 lies outside S.

Let H_m be an infinitely generated Schottky group which fills out X_m for $m = 1, 2$; such a group is constructed in chapter 8 of [9].

Define

$$G = \langle (\cup_{n\in\mathbb{Z}} L^n F L^{-n}) \cup (\cup_{n\in\mathbb{Z}} L^n H_1 L^{-n}) \cup (\cup_{n\in\mathbb{Z}-\{0\}} L^n H_0 L^{-n}) \rangle.$$

G is a Kleinian group, by Poincaré's polyhedron theorem [11]. Moreover, $\Omega(G)/G = X_0/F$, so G is analytically finite.

Poincaré's theorem also yields that $K = \langle G, L \rangle$ is discrete but not Kleinian.

It is easy to modify the above example so that the element L is parabolic. In this case, we take $L(z) = z + 1$, and we replace the annulus A with the vertical strip $V = \{0 < Re(z) < 1\}$. The rest of the construction is unchanged.

Waterman's example also answers the following question. Let G be an analytically finite Kleinian group, let M be an infinite order Möbius transformation fixing a point of $\Lambda(G)$, where $K = \langle G, M \rangle$ is discrete. Need K be Kleinian? As we have just seen, the answer to this question is no.

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