K theoretic indices of Dirac type operators on complete manifolds and the Roe algebra

A Dissertation Presented

by

Guoliang Yu

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1991

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation

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Dirac type operators on complete open manifolds are generalized Fredholm operators in the sense that they are invertible modulo the algebra of locally traceable operators with bounded propagation, which is the Roe algebra. Hence Dirac type operators have indices in the K theory of the Roe algebra. In this thesis we study the computation and the geometric significance of such a K theoretic index, based on the study of the K theory for the Roe algebra.

To my family.

Contents

Acknowledgements

First and foremost, I wish to express my deep gratitude to my thesis advisor, Professor Ronald G. Douglas, for sharing with me his clear vision of mathematics and penetrating insights. He has been a constant source of inspiration and encouragement for my years at Stony Brook.

I am indebted to the Mathematics Department of SUNY at Stony Brook for providing me a wonderful opportunity of studying mathematics. In particular I want to thank Professors Nicholas Teleman, Dusa McDuff, Chih-Han Sah, Joel Pincus, Ralph Spatzier, Anthony Knapp and Michael Anderson for their generous help and wonderful lectures. I also have benefitted greatly from conversations with Professor Jerry Kaminker of IUPUI.

It is a pleasure to thank my fellow students G. Gong, D. Zheng, X. Jiang, S. Zhou, D. Gong, E. Park, P. Miegom, X. Dai and J. Anderson for enjoyable discussions. Particular thanks are due to G. Gong, D. Zheng and X. Jiang for extremely helpful discussions on topics related to this thesis.

Last but not the least, I wish to thank my master degree advisor, Professor Shunhua Sun, and my high school mathematics teacher Wenhu Liu for their constant spiritual encouragement.

Chapter 1

Introduction

Index theory is the study of topological invariants for elliptic differential operators on manifolds.

An elliptic operator D on a compact manifold is a Fredholm operator. The only topological invariant for a Fredholm operator is the Fredholm index [16], which is defined to be $dim(kerD) - dim(cokerD)$. The Atiyah-Singer index theorem calculates the Fredhom index in terms of the topological data. This theorem establishes a bridge between analysis, geometry and topology [2] [3]. The Fredholm index is often related to the geometry of the manifold. An example of this is that the nonnegativity of the scalar curvature implies the vanishing of the Fredholm index for the Dirac operator. Therefore a nonzero $\hat{A}$ genus is an obstruction to the existence of a metric with nonegative scalar curvature.

Index theorems have been generalized to noncompact manifolds of various sorts. Elliptic operators on noncompact manifolds are no longer Fredholm in the classical sense, but are Fredholm in a generalized sense with respect to certain operator algebras. The topological invariant for an elliptic operator is now the generalized Fredholm index, which lives in the K theory of an operator algebra. An early example of this was the index theorem for almost periodic Toeplitz operators, which computes partially a generalized Fredholm index (Coburn, Douglas, Schaeffer and Singer [11]). (Toeplitz operators are the odd analogue of elliptic operators.) Some other examples are the index theorem for coverings (Atiyah [1], Miscenko and Fomenko [27], Connes and Moscovici [14]), for foliations (Connes and Skandalis [15]), for homogeneous spaces of Lie groups (Connes and Moscovici [13]), and for complete manifolds of bounded geometry with regular exhaustions (Roe [31]) .

In the case of a complete manifold M, Dirac type operators on M are generalized Fredholm operators in the sense that they are invertible modulo the algebra of locally traceable operators with bounded propagation, which is the Roe algebra. Hence Dirac type operators have indices in the K theory of the Roe algebra [31] [33]. If M has bounded geometry with regular exhaustions, then there is a trace on the Roe algebra. Roe's index theorem calculates the pairing of this trace with the K theoretic index [31]. In some other special cases Roe constructs "higher traces" (cyclic cocycles) over the Roe algebra to compute their pairing with the K theoretic index [32, 33, 34].

The purpose of this thesis is to understand the K theoretic indices of Dirac operators on complete open manifolds and its geometric significance, based on the study of the K theory for the Roe algebra.

In this thesis we prove that a non-zero K theoretic index of such a Dirac operator on a "geometrically open" manifold is an obstruction to the existence of a metric in the strict quasi-isometry class with uniformly positive scalar curvature near infinity. In order to detect the K theoretic index we construct a class of K homology elements over the Roe algebra and compute their pairing with the K theoretic index, which can be used in detecting the nonvanishing of the K theoretic index on manifolds with a cone like end. We obtain a formula of the K theoretic index on the connected sum of manifolds. Also we answer a question of Roe on the nonexistence of metrics with nonnegative Ricci curvature near infinity.

For a finitely generated discrete group, a Roe algebra can be defined in an analogous manner by using the word length function defined by a finite generating set. When $\tilde{M}$ is the universal cover of a compact Riemannian manifold M, then the Roe algebra for $\tilde{M}$ can be shown to be Morita equivalent to the Roe algebra for the fundamental group of M. One of the purposes of this thesis is to understand the K theory and cyclic cohomology of the Roe algebra for reasonably "nice" discrete groups. We compute the K theory of the Roe algebra for various groups. As a consequence we prove that the K theory of the Roe algebra for Euclidean space and for the Poincare disc is huge while there exists a compact spin manifold M with nonzero $\hat{A}$ genus such that the K theory of the Roe algebra for the universal cover of M is trivial.

For groups of polynomial growth we obtain certain vanishing results. In particular we prove a conjecture of S. Hurder which states that the exotic cohomology $HX^q(\Gamma)=0$ for $q > d$, if Γ is a discrete group with polynomial growth of degree d. The exotic cohomology was introduced by Roe to construct cyclic cocycles over the Roe algebra [32] , [33]. Moreover, we shall prove that

every cyclic cocycle over the Roe algebra for such a group has dimension at most d. We also show that group cocycles can be used to construct cyclic cocycles over the Roe algebra of amenable groups. By using this cyclic cocycle method we show that the K theoretic index of the Dirac operator for the spin manifold $E\Gamma$ with Γ invariant metric is nonzero when Γ has polynomial growth and $B\Gamma$ is compact and oriented ($E\Gamma$ is topologically the universal cover of the classifying space $B\Gamma$ for Γ).

Chapter 2

K theoretic index and positive scalar curvature near infinity

2.1 The K theoretic indices of Dirac type operators

In this section we construct the K theoretic indices of Dirac type operators on complete manifolds introduced by Roe [31] [33] in the framework of Kasparov's KK theory. Such a KK theoretic interpretation of the index will be useful in computation.

Let M be a complete Riemannian manifold, and let $Cliff^c(M)$ be the complexified Clifford algebra bundle over M. There exists a natural connection ∇ over $Cliff^c(M)$ which is compatible with the Riemannian connection over TM, the tangent bundle of M.

A vector bundle S over M is called a Clifford bundle if S is a left module over $Cliff^c(M)$ and S is endowed with a Hermitian metric and a compatible connection such that:

(1) For each unit vector v in T_xM, its module action on S is an isometry.

(2) For all smooth sections c of $Cliff^c(M)$ and s of S, the connection ∇ is compatible with the module action, i.e, $\nabla(cs) = c\nabla s + (\nabla c)s$.

The Dirac operator D acting on S is defined by:

$$(Ds)_x = \Sigma e_k(\nabla_{e_k} s)_x,$$

where $\{e_1, ..., e_k\}$ is an orthonormal basis for T_xM and s is a smooth section of S.

We will assume that S is a graded Cifford bundle, i.e, there is a grading operator η such that $\eta^2 = 1$ and η anticommutes with the module action of tangent vectors. Hence S can be decomposed into the direct sum of the positive and negative eigenbundles of η:

$$S = S_+ \oplus S_-.$$

The Dirac operator D can be decomposed correspondingly:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

When M is a spin manifold, there is a natural Clifford bundle called the spin bundle. The Dirac operator acting on the spin bundle is just the usual Dirac operator. For more information see [18].

A positive operator b acting on $L^2(S)$ is said to locally traceble if, for all compactly supported continuous functions f on M, the operator fbf is of trace class. A general operator is locally traceble if it is a finite linear combination of positive locally traceble operators. The Roe algebra B_S consists of all locally traceble operators with bounded propagation. Recall that an operator b is said to have bounded propagation if there exists $r > 0$ such that for any $s \in L^2(S)$,

$$Supp(bs) \cup Supp(b^*s) \subseteq \{x | x \in M, dist(x, Supp(s)) \leq r\}.$$

Denote by $\overline{B}_S$ the operator norm closure of B_S.

The Roe algebra plays an important role in the index theory on a complete manifold since its K theory is the receptacle for the index.

The Dirac operator D on the Clifford bundle S is essentially selfadjoint under our assumption that M is complete [10]. If f is a continuous real-valued odd function on the real line such that $\lim_{n \to +\infty} f(x) = 1$ and $\lim_{n \to -\infty} f(x) = -1$, then $f(D)$ is a well defined bounded operator and we have the following lemma, which is implicit in Roe [33].

Lemma 1 *$f(D)^2 - 1$ is in $\overline{B}_S$.*

We have the following decomposition of $f(D)$ since f is odd.

$$f(D) = \begin{pmatrix} 0 & f(D)_- \\ f(D)_+ & 0 \end{pmatrix}.$$

By lemma 1 we know that $f(D)_-$ is a parametrix of $f(D)_+$. More precisely, we have

$$f(D)_+ f(D)_- - 1 = C_-$$

$$f(D)_- f(D)_+ - 1 = C_+$$

where

$$C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix}$$

is in $\overline{B}_S$. Let

$$L = \begin{pmatrix} C_+ & f(D)_- - C_+ f(D)_- \\ f(D)_+ & -C_- \end{pmatrix}.$$

It is not difficult to check that L has an inverse:

$$L^{-1} = \begin{pmatrix} C_+ & f(D)_-(1 - C_-) \\ f(D)_+ & -C_- \end{pmatrix}.$$

Now the index of D is defined to be:

$$IndD = L \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} C_+^2 & f(D)_- C_-(1 - C_-) \\ f(D)_+ C_+ & -C_-^2 \end{pmatrix},$$

which lives in $K_0(\overline{B}_S)$. It is not difficult to check that $IndD$ does not depends on the choice of f.

Notice that $IndD$ is 0 if D_+ is invertible. Hence $IndD$ can be viewed as a topological obstruction to the invertibility of D_+.

The motivation for such a definition is the following. Let A_S be the C^* algebra of bounded operators acting on $L^2(S)$ with bounded propagation. It can be shown that $\overline{B}_S$ is a $*$-ideal of A_S. Hence we have the following short exact sequence:

$$0 \longrightarrow \overline{B}_S \longrightarrow A_S \longrightarrow A_S/\overline{B}_S \longrightarrow 0$$

which induces the following exact sequence of K groups:

$$\begin{array}{ccccc} K_1(\overline{B}_S) & \longrightarrow & K_1(A_S) & \longrightarrow & K_1(A_S/\overline{B}_S) \\ \uparrow & & & & \delta\downarrow \\ K_0(A_S/\overline{B}_S) & \longleftarrow & K_0(A_S) & \longleftarrow & K_0(\overline{B}_S) \end{array}$$

where δ is the connecting map. By lemma 1 $[f(D)]$ is invertible in $A_S/\overline{B}_S$ and therefore represents an element in $K_1(A_S/\overline{B}_S)$. Clearly $[f(D)] = [L]$ in $A_S/\overline{B}_S$. Now we can see that $IndD$ is the same as $\delta(f(D))$ [6].

Let A and B be graded C^* algebras. Recall that a Kasparov module for (A, B) is a triple (E, ϕ, F), where E is a countably generated graded Hilbert module over B, ϕ is a graded $*$-homomorphism from A to $B(E)$, and F is an operator in $B(E)$ of degree 1, such that $[F, \phi(a)]$, $(F^2-1)\phi(a)$, and $(F-F^*)\phi(a)$ are all in $K(E)$ for all $a \in A$, where $[\ ]$ is the graded commutator and $K(E)$ is the algebra of compact operators on E with respect to B [6].

The set of all Kasparov modules for (A, B) can be made into a group $KK_0(A, B)$, the Kasparov bivariant K group. When $A = \mathbb{C}$, $KK_0(\mathbb{C}, B)$ can be naturally identified with $K_0(B)$, the K theory for B. When $B = \mathbb{C}$, $KK_0(A, \mathbb{C})$ can be naturally identified with $K^0(A)$, the K homology for A [6].

There is a natural $\mathbb{Z}_2$ grading on $\overline{B}_S$ which is induced by the grading on S. $\overline{B}_S$ is a Hilbert module over itself by right multiplication. $f(D)$ is a module map by the multiplication on $\overline{B}_S$. Let ϕ be the graded $*$-homomorphism from $\mathbb{C}$ to $B(\overline{B}_S)$ such that $\phi(1)$ is the identity operator on $\overline{B}_S$. By lemma 1 it is easy to see $(\overline{B}_S,\ f(D),\ \phi)$ is a Kasparov module for $(\mathbb{C}, \overline{B}_S)$. We have the following natural result.

Proposition 1 *IndD is equivalent to the Kasparov module $(\overline{B}_S,\ f(D),\ \phi)$ for $(\mathbb{C}, \overline{B}_S)$ by the natural identification of $K_0(\overline{B}_S)$ with $KK(\mathbb{C}, \overline{B}_S)$.*

Proof: Cf [6].

Recall that the metrics g_1 and g_2 are said to be quasi-isometric on the manifold M if $c_1 g_1 \leq g_2 \leq c_2 g_1$ for some $c_1, c_2 > 0$. g_1 and g_2 are strictly quasi-isometric if, in addition, the difference of their Riemmannian connections is bounded. Notice that the Roe algebra for a spin bundle depends only on the quasi-isometry class of the metric.

The following result is implicit in Roe [32].

Proposition 2 *Let M be a spin manifold, and g_1 and g_2 be two complete strictly quasi-isometric metrics. Assume D_1 and D_2 are the Dirac operators on M induced by g_1 and g_2. Then $IndD_1 = IndD_2$ in $K_0(\overline{B}_S)$, where S is the spin bundle of M.*

2.2 Obstruction to uniform positive scalar curvature outside a compact set

Throughout this section M is an even dimensional complete open Riemmannian spin manifold with positive injectivity radius and S is the spin bundle on M. Recall that the injectivity radius for M is defined to be the supremum of the r for which the exponential map exp_m is an embedding on the open ball of radius r, in T_mM for all $m \in M$.

The following result on an obstruction to the existence of a metric with uniform positive scalar curvature near infinity is the main result of this section.

Theorem 2.2.1 *Let M have sectional curvature bounded from above. If there exists a metric in the strict quasi-isometry class on M whose scalar curvature is uniformly positive outside a compact set, then the K theoretic index of the Dirac operator is zero in $K_0(\overline{B}_S)$.*

The above theorem presents a phenomenon peculiar to noncompact manifolds, which is not possible when the manifold is compact.

Recall that a regular exhaustion on M is an increasing sequence (M_i) of compact subsets such that the union of M_i is M and for each $r > 0$

$$\lim_{i\to\infty} Vol\{x \in M|dist(x, M_i) \leq r\}/Vol(M\backslash\{x \in M|dist(x, M\backslash M_i) \leq r\}) = 1.$$

The existence of an exhaustion amounts to a certain kind of amenability of M. In the case that M is a universal cover of a compact Riemannian manifold X the existence of an exhaustion is equivalent to the amenability for the fundamental group of X. When M has a regular exhaustion (M_i) a trace τ can be defined on B_S as follows:

$$\tau(k) = \lim_i \int_{M_i} trk(x,x)/volM_i,$$

where the limit is taken as linear functionals over B_S in the weak* topology [31]. If M has infinite volume it is easy to see $\tau(k)$ depends only on the value of $k(x, x)$ outside a compact set. This fact is used by Roe to prove that $\tau(IndD)$ is an obstruction to to the existence of a metric with positive scalar curvature on a "large set" [31]. This result is the motivation for theorem 2.2.1.

There are several different concepts of an end for a complete manifold. In this section an end for the manifold M is simply a connected component of $M\backslash C$ for some compact set $C \in M$.

Using corollary 2.3.1 in section 2.3 we can specialize theorem 1 to obtain:

Corollary 2.2.1 *If M has an end isometric to the cone $\mathbb{R}^+ \times N$ with metric $dr^2 + f(r)^2 g^N$ where g^N is the metric on the compact oriented manifold N and $\lim_{r\to+\infty} f(r) = +\infty$, then there exists no metric in the strict quasi-isometry class on M whose scalar curvature is uniformly positive outside a compact set.*

Proof: An examination of the proof of lemma 2 shows that theorem 2.2.1 is still true if we replace the conditions of theorem 2.2.1 on injectivity radius and sectional curvature by the following condtion: there exists an infinite sequence of balls $B(x_i, r)$ for some $r > 0$ such that $Volume(B(x_i, r)) > c$ for certain constant $c > 0$ and $B(x_i, r) \cap B(x_j, r) = \emptyset$ for $i \neq j$, where $B(x_i, r)$ is the ball with center x_i and radius r. This is satisfied by the manifold under consideration. Now our result follows from corollary 2.3.1. QED

The following lemma plays a key role in the proof of Theorem 2.2.1.

Lemma 2 *Let M be as in theorem 1. If K is the algebra of all compact operators acting on* $L^2(S)$*, then the natural inclusion* i_**:* $K_0(K) \longrightarrow K_0(\overline{B}_S)$ *is zero.*

Proof: Since M is a complete open manifold, there exists a ray in M, i.e. a curve γ: $[0, +\infty) \rightarrow M$ such that γ is a geodesic and γ minimizes the distance between any pair of points on itself [8]. Let $r > 0$ be the injectivity radius and $B(x, R)$ be the ball with center x and radius R. By our assumption on sectional curvature and the volume comparison theorem [8] we have

$$Volume(B(\gamma(nr), r/2)) > c$$

for some $c > 0$. Therefore there exists a sequence of uniformly bounded cross sections $\{f_n\}_{n=1}^{\infty} \in L^2(S)$ satisfying $\| f_n \|_{L^2(S)} = 1$ and $Suppf_n \subset B(\gamma(nr), r/2)$. Let $f_{n+1} \otimes f_n$ be the operator: $(f_{n+1} \otimes f_n)h = < h, f_n > f_{n+1}$, for any $h \in L^2(S)$. Define U by $U = \sum_{i=1}^{\infty} f_{n+1} \otimes f_n$ in the strong operator topology. It is not difficult to check that U is in $\overline{B}_S$ and U is a partial isometry since the f_n are uniformly bounded and $Suppf_m \cap Suppf_n = \emptyset$ for $m \neq n$.

Now U^*U is the projection onto the subspace of $L^2(S)$ spanned by $\{f_n\}_{n=1}^{\infty}$ and UU^* is the projection onto the subspace of $L^2(S)$ spanned by $\{f_n\}_{n=2}^{\infty}$. This implies that $f_1 \otimes f_1$ is equivalent to 0 in $K_0(\overline{B}_S)$. The lemma follows from the fact that $f_1 \otimes f_1$ is the generator of $K_0(K)$. QED.

Now we are ready to prove theorem 2.2.1.

Proof of theorem 2.2.1:

Take

$$f(x) = \begin{cases} \sqrt{1 - e^{-x^2/2}} & \text{for } x > 0 \\ \\ -\sqrt{1 - e^{-x^2/2}} & \text{for } x \leq 0 \end{cases}$$

in the definition of the index of D. We have

$$IndD = \begin{pmatrix} e^{-D_+^* D_+} & e^{-\frac{1}{2}D_+^* D_+}(I + e^{-D_+^* D_+})f(D)_+^* \\ \\ -e^{-\frac{1}{2}D_+ D_+^*} f(D)_+ & -e^{-D_+ D_+^*} \end{pmatrix},$$

which is an idempotent in $M_2(\mathbb{C}) \otimes \overline{B}_S$. This idempotent is homotopic to the following idempotent :

$$(IndD)_t = \begin{pmatrix} e^{-tD_+^* D_+} & e^{-\frac{t}{2}D_+^* D_+}(I + e^{-tD_+^* D_+})f_t(D)_+^* \\ \\ -e^{-\frac{t}{2}D_+ D_+^*} f_t(D)_+ & -e^{-tD_+ D_+^*} \end{pmatrix}$$

for $t > 0$, where

$$f_t(x) = \begin{cases} \sqrt{1 - e^{-tx^2/2}} & \text{for } x > 0 \\ -\sqrt{1 - e^{-tx^2/2}} & \text{for } x \leq 0 \end{cases}.$$

We have

$$(IndD)_t = IndD$$

in $K_0(\overline{B}_S)$. Hence by lemma 2 it is enough to prove that $(IndD)_t$ converges to a compact idempotent in operator norm as t goes to ∞. Now since

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

we have

$$e^{-\frac{t}{2}D^2}(I - e^{-tD^2})f_t(D) = \begin{pmatrix} A_t & e^{-\frac{t}{2}D_+D_+^*}(I + e^{-tD_+D_+^*})f_t(D)_+ \\ e^{-\frac{t}{2}D_+^*D_+}(I + e^{-tD_+^*D_+})f_t(D)_+^* & B_t \end{pmatrix},$$

$$e^{-\frac{t}{2}D^2}f_t(D) = \begin{pmatrix} A_t' & e^{-\frac{t}{2}D_+D_+^*}f_t(D)_+ \\ e^{-\frac{t}{2}D_+^*D_+}f_t(D)_+^* & B_t' \end{pmatrix},$$

where A_t, B_t, A_t', and B_t' are bounded operators. Therefore it is enough to prove that $e^{-\frac{t}{2}D^2}(I - e^{-tD^2})f_t(D)$ and $e^{-\frac{t}{2}D^2}f_t(D)$ converge to compact operators in operator norm as t goes to ∞. We have

$$(e^{-\frac{t}{2}D^2}(I - e^{-tD^2})f_t(D))^2 = e^{-tD^2}(1 + e^{-tD^2})(1 - e^{-t/2D^2})$$

$$(e^{-\frac{t}{2}D^2} f_t(D))^2 = e^{-tD^2}(1 - e^{-t/2D^2}).$$

Combining the above equalities and the fact that $e^{-\frac{t}{2}D^2}(I - e^{-tD^2})f_t(D)$ and $e^{-\frac{t}{2}D^2} f_t(D)$ are selfadjoint operators we see that it is enough to prove that e^{-tD^2} converges to a compact operator in operator norm as t goes to ∞. We have the well known Weitzenbock formula [24] [25]

$$D^2 = \nabla^*\nabla + k$$

where $\nabla^*\nabla$ is the connection Laplacian and k is the scalar curvature. Assume $k \geq c > 0$ outside some compact set K. Let χ be a smooth nonnegative function on M such that

$$\chi = \begin{cases} 0 & \text{if } x \in M \text{ and } dist(x, K) \leq 1 \\ \\ 1 & \text{if } x \in M \text{ and } dist(x, K) \geq 2 \end{cases}$$

and $\sqrt{\chi}$ is also smooth. Let $s \in L^2(S)$, and $s_t = e^{-tD^2}s$. Then by the Weitzenbock formula we have

$$\|D^2 s_t\|\|s_t\| \geq \int_M (D^2 s_t, s_t)\chi$$

$$\geq c\int_M (s_t, s_t)\chi + \int_M (\nabla^*\nabla s_t, s_t)\chi$$

$$= c\int_M (s_t, s_t)\chi + \int_M (\nabla s_t, (d\chi)s_t) + \int_M (\nabla s_t, \chi\nabla s_t)$$

$$\geq c\int_M (s_t, s_t)\chi + \int_M (\nabla s_t, (d\chi)s_t)$$

But

$$|\int_M (\nabla s_t, (d\chi)s_t)| \le c_1 \sqrt{(\int_{M\setminus K} \|\nabla s_t\|^2} \|s_t\| \le c_1 \|Ds_t\| \|s_t\|,$$

here again we are using the Weitzenbock formula and $c_1 = \|d\chi\|_\infty$. Combining the two inequalities we have

$$\int_M (s_t, s_t)\chi \le c_1/c \|Ds_t\| \|s_t\| + 1/c \|D^2 s_t\|$$

which goes to 0 as t goes to ∞ since

$$Ds_t = De^{-tD^2} s = 1/\sqrt{t}(\sqrt{t}De^{-tD^2} s),$$

$$D^2 s_t = D^2 e^{-tD^2} s = 1/t(tD^2 e^{-tD^2} s),$$

and $\sqrt{t}De^{-tD^2}, tDe^{-tD^2}$ are uniformly bounded with respect to t for $t \ge 0$ by the spectral theorem. Therefore $\sqrt{\chi}e^{-tD^2}\sqrt{\chi}$ goes to 0 in operator norm as t goes to ∞. We have

$$\|\sqrt{\chi}e^{-tD^2}\|^2 = \|(\sqrt{\chi}e^{-tD^2})(\sqrt{\chi}e^{-tD^2})^*)\| = \|\sqrt{\chi}e^{-2tD^2}\sqrt{\chi}\|.$$

It follows that $\sqrt{\chi}e^{-tD^2}$ goes to 0 in operator norm as t goes to ∞. Hence

$$(1 - \sqrt{\chi})e^{-tD^2}\sqrt{\chi} = (1 - \sqrt{\chi})(\sqrt{\chi}e^{-tD^2})^*$$

goes to 0 in operator norm as t goes to ∞. By lemma 1.2 in [30] we know that the kernel of e^{-tD^2} converges to 0 on compact subsets of M as t goes to ∞. Therefore $(1 - \sqrt{\chi})e^{-tD^2}(1 - \sqrt{\chi})$ converges to a compact operator as t goes to ∞. Finally we conclude that

$$e^{-tD^2} = (1-\sqrt{\chi})e^{-tD^2}(1-\sqrt{\chi}) + (1-\sqrt{\chi})e^{-tD^2}\sqrt{\chi} + \sqrt{\chi}e^{-tD^2}\sqrt{\chi} + \sqrt{\chi}e^{-tD^2}(1-\sqrt{\chi})$$

converges to a compact operator in operator norm as t goes to ∞. QED.

2.3 Detection of the K theoretic index

In this section we construct a class of K homology elements on the Roe algebra $\overline{B_S}$ and obtain a formula for their pairing with the index of the Dirac operator. This can be used to detect nonzero index.

Recall that any compactly supported vector bundle on M can be represented by a pair: (α, E), where $E = E_0 \oplus E_1$ is a graded bundle on M which is trivial outside a compact set and α is a continuous E bundle endomorphism of grading 1 such that $\alpha^2 = 1$ outside a compact set [2]. Now a special compactly supported vector bundle on M is a compactly supported vector bundle represented by a pair (α, E) such that: (1) E is trivial as a vector bundle. (2) E is endowed with a Hermitian structure such that α is selfadjoint. (3) $\lim_{x\to\infty} \sup_{y\in B(x,r)} \|\alpha(y) - \alpha(x)\| = 0$ for any $r > 0$.

Example 1 *Let $M = \mathbb{R}^2 = \mathbb{C}$ and $E = \mathbb{C}\times M \oplus \mathbb{C}\times M$ with the usual Euclidean metric and*

$$\alpha = \begin{pmatrix} 0 & \frac{z}{|z|}\phi(z) \\ \frac{\bar{z}}{|z|}\phi(z) & 0 \end{pmatrix}$$

for $z \in \mathbb{C}$, where ϕ is a smooth function on M such that $\phi(z) = 1$ outside a compact set and $\phi(0) = 0$. Then the pair (α, E) is a special compactly supported vector bundle on M.

More generally, if M is a simply connected nonpositively curved manifold, then the Miscenko element is a special compactly supported vector bundle on

M. We omit the definition of Miscenko element since it is not used in this thesis. (Cf [26] or [22] for the definition of Miscenko element.)

Let $C_0(M)$ be the algebra of all continuous functions vanishing at infinity. $C_0(M)$ is equiped with the trivial grading. $L^2(S)$ and $L^2(E)$ can be considered as graded modules over $C_0(M)$. The graded tensor product $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ can be identified with $L^2(S)\hat{\otimes}V$ for some graded finite dimensional vector space V by choosing a finite basis for the $C_0(M)$ module of all continuous compactly supported cross sections of E (Notice that E is a trivial bundle). (Cf [6] for the concept of graded tensor product).

A special compactly supported vector bundle (α, E) on M induces a K homology element τ over $\overline{B}_S$: $(L^2(S)\hat{\otimes}_{C_0(M)}L^2(E), \phi, 1\hat{\otimes}\alpha)$, where $\overline{B}_S$ acts on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E) \cong L^2(S)\hat{\otimes}V$ by $\phi(b)(s\hat{\otimes}v) = bs\hat{\otimes}v$ for $s \in L^2(S), v \in V, b \in \overline{B}_S$, and $\mathbb{C}$ acts on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ from the right in a trivial way.

Lemma 3 .

$$\tau = (L^2(S)\hat{\otimes}_{C_0(M)}L^2(E), \phi, 1\hat{\otimes}\alpha)$$

is a Kasparov module for $(\overline{B}_S, \mathbb{C})$.

Proof: All we have to check is that $\phi(b)((1\hat{\otimes}\alpha)^2-1)$ and $\phi(b)(1\hat{\otimes}\alpha)-(1\hat{\otimes}\alpha)\phi(b)$ are compact for $b \in B_S$. The compactness of $b((1\hat{\otimes}\alpha)^2 - 1)$ follows from the condition that α^2-1 is compactly supported. The compactness of $\phi(b)(1\hat{\otimes}\alpha)-(1\hat{\otimes}\alpha)\phi(b)$ is a consequence of proposition 5.18 in [32]. QED.

The pairing of a K homology element with the K theoretic index gives rise to a numerical invariant, which can be used to detect the nonvanishing of the K theoretic index. The pairing can be computed by using Kasparov's KK product (see [6] for more information on the KK product).

Our main result of this section is the following:

Theorem 2.3.1 . *If τ is as above, then $(IndD)\hat{\otimes}_{\overline{B}_S}\tau = [(\alpha, E)]\hat{\otimes}_{C_0(M)}[D]$ where the left hand side is the pairing between the K homology element τ and the K theory element IndD and the righthand side is the pairing between the K theory element in $K_0(C_0(M))$ induced by (α, E) and the K homology element in $K^0(C_0(M))$ induced by D.*

Remark. The right hand side can be computed explicitly by the Atiyah-Singer index formula [3].

Proof: Recall that $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ is identified with $L^2(S)\otimes V$ for some graded vector space V in order to define the $\overline{B}_S$ action on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$. We have a well defined operator $f(D)\hat{\otimes}1$ acting on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ by using such an identification. $f(D)\hat{\otimes}1$ is obviously a $f(D)$ connection. We can also choose α such that $\|\alpha\| \leq 1$. By proposition 18.10.1 in [6] we have

$$[(\alpha, E)]\hat{\otimes}_{C_0(M)}[D] = indexF_+,$$

where

$$F = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix} \doteq 1\hat{\otimes}\alpha + \sqrt{1 - 1\hat{\otimes}\alpha^2}f(D)\hat{\otimes}1$$

acts on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$. The Fredholmness of F_+ follows from the following identity which can be easily verified:

$$F^2 = 1 \ mod \ K,$$

where K is the algebra of all compact operators on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$.

$(F, \psi, L^2(S\hat{\otimes}_{C_0(M)}L^2(E))$ is obviously a Kasparov module for $(\mathbb{C}, \mathbb{C})$, where ψ is the trivial $*$ homomorphism from $\mathbb{C}$ to $B(L^2(S)\hat{\otimes}_{C_0(M)}L^2(E))$, the algebra of all bounded operators acting on $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$. We shall prove that $(F, \psi, L^2(S\hat{\otimes}_{C_0(M)}L^2(E))$ can be identified as the Kasparov product $(IndD)\hat{\otimes}_{\overline{B}_S}\tau$. Notice that $\overline{B}_S\hat{\otimes}_{\overline{B}_S}L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ is equivalent to $L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$ as a $(\mathbb{C}, \mathbb{C})$ bimodule. Hence $f(D)\hat{\otimes}1\hat{\otimes}1$ is equivalent to $f(D)\hat{\otimes}1$ and

$$[f(D)\hat{\otimes}1, F] = 2\sqrt{1 - 1\hat{\otimes}\alpha^2}f(D)^2\hat{\otimes}1) \geq 0$$

since $\|\alpha\| \leq 1$, where the graded commutator [ ] contributes to the cancellation. Now the only thing we have to check is that F is a $1\hat{\otimes}\alpha$ connection. If we set $T_x y = x\hat{\otimes}_{\overline{B}_S}y$ for $x \in \overline{B}_S$ and $y \in L^2(S)\hat{\otimes}_{C_0(M)}L^2(E)$, then we have

$$T_x(1\hat{\otimes}\alpha) - (-1)^{degx}FT_x = -\sqrt{1 - 1\hat{\otimes}\alpha^2}f(D)x\hat{\otimes}1,$$

which is compact since $1 - 1\hat{\otimes}\alpha^2$ is compactly supported. We are done by definition 18.4.1 in [6]. QED.

Let $C_h(M)$ be the algebra of all bounded continuous function α on M such that $\lim_{x\to\infty}\sup_{y\in B(x,r)}\|\alpha(y) - \alpha(x)\| = 0$ for any $r > 0$ [21] [32]. By Gelfand's theorem this algebra can be identified with the algebra of all continuous functions on a compact Hausdorff space $\overline{M}$ in which M is naturally embeded. ($\overline{M}\backslash M$ is called the Higson corona of M in [33].) We have the following short exact sequence of algebras

$$0 \longrightarrow C_0(M) \longrightarrow C_h(M) \longrightarrow C(\overline{M}\backslash M) \longrightarrow 0$$

which induces the following exact sequence of K groups:

$$\begin{array}{ccccc} K_1(C_0(M)) & \longrightarrow & K_1(C_h(M)) & \longrightarrow & K_1(C(\overline{M}\backslash M)) \\ \uparrow & & & & \downarrow \delta \\ K_0(C(\overline{M}\backslash M)) & \longleftarrow & K_0(C_h(M)) & \longleftarrow & K_0(C_0(M)) \end{array}$$

The special K group of M is defined to be the subgroup of $K_0(C_0(M))$ generated by all special compactly supported vector bundles on M.

The following result says that all special K elements come from the Higson corona.

Proposition 3 . *The special K group of M is the image of the connecting map δ in the above diagram.*

Proof: The connecting map δ can be realized as follows: Given any $[u] \in K_1(C(\overline{M}\backslash M))$ (u is a unitary in $M_n(C(\overline{M}\backslash M))$, there exists a trivial vector bundle V with a Hermitian structure on $\overline{M}\backslash M$ such that u is a continuous unitary endomorphism on V. V can be extended to a trivial vector bundle $\overline{V}$ with a Hermitian structure on $\overline{M}$. It is not difficult to see that u can be extended to a continuous endomorphism $\overline{u}$ of $\overline{V}$ in such way that $\overline{u}^*\overline{u} - 1$ and $\overline{u}\overline{u}^* - 1$ are compactly supported on M when $\overline{u}$ is restricted to M. (We can extend u to a continuous endomorphism u_1 of $\overline{V}$. Notice that $u_1^* u_1$ is uniformly positive in a neighborhood O of $\overline{M} \backslash M$ in $\overline{M}$. Hence $log(u_1^* u_1)$ is well defined on O. Now $log(u_1^* u_1)$ can be extended to a continuous endomorphism h of $\overline{V}$. Take $\overline{u} = u_1 e^{-h/2}$.) Now $\delta(u)$ can be defined as the pair (α, E) where $E = E_0 \oplus E_1$, $E_0 = E_1 = \overline{V}|_M$ and

$$\alpha = \begin{pmatrix} 0 & \overline{u}|_M \\ \overline{u}^*|_M & 0 \end{pmatrix}.$$

Now the pair (α, E) satisfies the condition (3) in the definition of special compactly supported vector bundle since α can be extended to $\overline{M}$. It is easy to see that the pair (α, E) satifies the other two conditions in the definition of special compactly supported vector bundle and (α, E) (as an element in the special K group) does not depend on the choice of the extension of u. It follows that (α, E) is a special compactly supported vector bundle on M. Conversely , if the pair (α, E) is a special compactly supported vector bundle on M, then by condition (3) α can be extended to a continuous endormorphism of $\overline{E}$, where E is extended to a trivial bundle $\overline{E}$ over $\overline{M}$. Now by the above explicit construction of the connecting map δ it is not difficult to see that the pair (α, E) is in the image of δ. QED.

Proposition 3 shows that theorem 2.3.1 is closely related to the results of Roe [33]. But the exact relation is unclear since the Fredholm modules we construct may not be finitely summable. This is closely related to an open question of Connes (see 2.4(3) for further remarks).

The formalism of this section be used to obtain the following nonvanishing result for the K theoretic index of the Dirac operator on manifold with a cone like end.

Corollary 2.3.1 *If M is even dimensional and has an end isometric to the end of the cone $\mathbb{R}^+ \times N$ with metric $dr^2 + f(r)^2 g^N$ where g^N is the metric on the compact oriented manifold N and $\lim_{r\to+\infty} f(r) = +\infty$, then the K theoretic index of the Dirac operator is nonzero.*

This result can also be obtained by applying results in [33].

Proof: For any $m \in \overline{M}\backslash M$, there exists an induced multiplicative linear functional m^* on $C_h(M)/C_0(M)$ by $m^*(g) = g(m)$ for any $g \in C_h(M) \cong C(\overline{M}\backslash M)$. Given a continuous function $h(x)$ on N, we can construct a function $g(x) \in C_h(M)$ such that (1) $g(x)$ vanishs outside the end. (2) $g(x) = h(n)$ for $x = (r, n)$ when r is large enough, where (r, n) is the product coordinates for $\mathbb{R}^+ \times N$. The fact that $g(x)$ is in $C_h(M)$ follows from the condition $\lim_{r\to+\infty} f(r) = +\infty$. We can define a multiplicative linear functional $p(m)_*$ on $C(N)$ by $p(m)_*(h) = m^*(g)$. It is not difficult to check that $p(m)_*(h)$ does not depends on the choice of g. Hence we have a continous map p from $\overline{M}\backslash M$ to N.

Now we are ready to construct a special compactly supported vector bundle to detect the K theoretic index. By the dimension assumption on M we know that N is odd dimensional. Hence there is an element $u \in K^1(N)$ such that its image under the Chern map chu is the fundamental element in the cohomology of N. The pullback $p^*(u)$ lies in $K^1(\overline{M}\backslash M)$. Its image $\delta p^*(u)$ under the connecting map in proposition 3 is a special K element by proposition 3. Let τ be the K homology element on $\overline{B_S}$ induced by the special K element $\delta p^*(u)$. Notice that the connecting map δ is explicitly constructed in the proof of proposition 3. Hence we can apply theorem 2.3.1 and the Atiyah-Singer index formula [3] to obtain

$$(IndD)\hat{\otimes}_{\overline{B_S}}\tau = 1.$$

Hence $IndD$ is nonzero. QED.

2.4 On the behavior of the K theoretic index under the connected sum of manifolds

In this section we show that the K theoretic index behaves extremely well with respect to the connected sum operation of manifolds.

Let M and N be open complete spin manifolds with positive injectivity radius. Let $M\#N$ be their connected sum. $M\#N$ is equiped with a Riemannian structure which is compatible with the Riemannian structures of M and N outside a compact set.

Lemma 4 *If M and N have sectional curvature bounded from above, then we have*

$$K_0(\overline{B}_{S_{M\#N}}) \cong K_0(\overline{B}_{S_M}) \oplus K_0(\overline{B}_{S_M}),$$

where $S_{M\#N}, S_M$ and S_N are correspondingly the spin bundles on $M\#N$, M and N.

Proof: Let $K_{M\#N}$,K_M and K_N be the algebras of compact operators acting correspondingly on $L^2(S_{M\#N}), L^2(S_M)$ and $L^2(S_N)$. We have the following short exact sequence of algebras

$$0 \longrightarrow K_{M\#N} \longrightarrow \overline{B}_{S_{M\#N}} \longrightarrow \overline{B}_{S_{M\#N}}/K_{M\#N} \longrightarrow 0$$

which induces the following exact sequence of K groups:

$$\begin{array}{ccccc} K_0(K_{M\#N}) & \xrightarrow{i_*} & K_0(\overline{B}_{S_{M\#N}}) & \longrightarrow & K_0(\overline{B}_{S_{M\#N}}/K_{M\#N}) \\ \uparrow & & & & \downarrow \\ K_1(\overline{B}_{S_{M\#N}}/K_{M\#N}) & \longleftarrow & K_1(\overline{B}_{S_{M\#N}}) & \longleftarrow & K_1(K_{M\#N}) \end{array}$$

By lemma 2 we know that $i_* = 0$. We also have the well known equality $K_1(K_{M\#N}) = 0$. Combining the above equality with the exact sequence we know that the natural map: $K_0(\overline{B}_{S_{M\#N}}) \to K_0(\overline{B}_{S_{M\#N}}/K_{M\#N})$, is an isomorphism. Similarly we can prove that $K_0(\overline{B}_{S_M})$ and $K_0(\overline{B}_{S_N})$ are naturally isomorphic to $K_0(\overline{B}_{S_M}/K_M)$ and $K_0(\overline{B}_{S_N}/K_N)$. It is not difficult to see that $\overline{B}_{S_{M\#N}}/K_{M\#N}$ is naturally isomorphic to $\overline{B}_{S_M}/K_M \oplus \overline{B}_{S_N}/K_N$. Now our lemma follows. QED.

Lemma 5 *If f is a smooth real-valued odd function such that* $\lim_{n\to+\infty} f(x) = 1$ *and* $\lim_{n\to-\infty} f(x) = -1$, *then f can be uniformly approximated by smooth functions of the form g such that (1)* $\lim_{n\to+\infty} g(x) = 1$ *and* $\lim_{n\to-\infty} g(x) = -1$, *(2) g^2-1 and $g(g^2-1)$ are rapidly decreasing and their Fourier transforms are compactly supported.*

Proof: Let's first notice that f can be approximated by smooth odd functions of the form f_1 such that (1) $\lim_{n\to+\infty} f_1(x) = 1$ and $\lim_{n\to-\infty} f_1(x) = -1$, (2)$f_1'$ can be uniformly approximated by rapidly decreasing smooth real-valued functions of the form h such that $\hat{h}$ is compactly supported (since $\lim_{n\to+\infty} f_1'(x) = 0$ and $\lim_{n\to-\infty} f_1(x) = 0$), where the 'hat' denotes the Fourier transform. Moreover h can be chosen such that $\int_{-\infty}^{\infty} h = 2$ since $\int_{-\infty}^{\infty} f_1' = 2$. Now take

$$g(x) = -1 + \int_{-\infty}^{x} h(t)dt.$$

By the properties of h we see that (1) $\lim_{n\to+\infty} g(x) = 1$, $\lim_{n\to-\infty} g(x) = -1$, and (2) $\hat{g'}$ is compactly supported. Notice that f_1 can be approximated by functions of the form g since $f_1(x) - g(x) = \int_{-\infty}^{x}(f_1'(t) - h(t))dt$, and f_1', h are rapidly decreasing functions. Now $ix\hat{g}(x) = \hat{h}$. Hence $\widehat{g^2 - 1} = \hat{g} * \hat{g} - \delta$ and $g(\widehat{g^2 - 1}) = \hat{g} * \hat{g} * \hat{g} - \hat{g}$ are compactly supported as distributions. Notice that $g^2 - 1$ and $g(g^2 - 1)$ are rapidly decreasing. It follows that $\widehat{g^2 - 1}$ and $g(\widehat{g^2 - 1})$ are smooth functions. QED.

The main result of this section is the following.

Theorem 2.4.1 *Let $D_{M\#N}, D_M$ and D_N be correspondingly the Dirac operators on the manifolds $M\#N, M$ and N. If M and N have sectional curvature bounded from above, then we have the following index formula:*

$$IndD_{M\#N} = IndD_M \oplus IndD_N,$$

where $IndD_M \oplus IndD_N$ is identified as an element in $K_0(\overline{B}_{S_{M\#N}})$ by lemma 4.

The idea behind the above theorem is that the K theoretic index is only related to the asymptotic behavior of the manifold at infinity.

Proof: Let $f(x)$ be a smooth real-valued odd function such that $\lim_{n\to+\infty} f(x) = 1$ and $\lim_{n\to-\infty} f(x) = -1$. Take $f(x)$ to be the function in the definition of $IndD_{M\#N}$, $IndD_M$ and $IndD_N$. We shall prove that

$$IndD_{M\#N} = \begin{pmatrix} (C_{M\#N})_+^2 & f(D_{M\#N})_-(C_{M\#N})_-(1-(C_{M\#N})_-) \\ f(D_{M\#N})_+(C_{M\#N})_+ & -(C_{M\#N})_-^2 \end{pmatrix}$$

can be decomposed correspondingly according to the decomposition given in lemma 4. By lemma 5 f can be uniformly approximated by smooth functions of the form g such that (1) $\lim_{n\to+\infty} g(x) = 1$ and $\lim_{n\to-\infty} g(x) = -1$, (2) g^2-1 and $g(g^2-1)$ are rapidly decreasing and their Fourier transform are compactly supported. Hence $C_{M\#N} = f(D_{M\#N})^2 - 1, C_M = f(D_M)^2 - 1$ and $C_N = f(D_N)^2 - 1$ can be correspondingly approximated by $g(D_{M\#N})^2-1, g(D_M)^2-1$ and $g(D_N)^2-1$. We have

$$g(D_{M\#N})^2 - 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\widehat{g^2-1})(t)e^{itD_{M\#N}}dt,$$
$$g(D_M)^2 - 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\widehat{g^2-1})(t)e^{itD_M}dt,$$
$$g(D_N)^2 - 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\widehat{g^2-1})(t)e^{itD_N}dt,$$

where the 'hat' denotes the Fourier transform. Since $\widehat{g^2-1}$ is compactly supported and $e^{itD_{M\#N}}$, e^{itD_M} and e^{itD_N} have finite propagation speed [10] [9], there exists a compact subset $K \subset M\#N$ such that

$$L^2(S_{M\#N}) \cong L^2(S_M|_{M\setminus K}) \oplus L^2(S_N|_{N\setminus K}) \oplus L^2(S_{M\#N}|_K)$$

and

$$g(D_{M\#N})^2 - 1 = \begin{pmatrix} (g(D_M)^2 - 1|_{M\setminus K}) & A_1 & A_2 \\ B_1 & (g(D_N)^2 - 1|_{N\setminus K}) & B_2 \\ C_1 & C_2 & C_3 \end{pmatrix},$$

where $A_1, A_2, B_1, B_2, C_1, C_2$, and C_3 are compact operators. The above formula together with the approximation process shows that

$$C_{M\#N} \cong C_M \oplus C_N$$

in $\overline{B}_{S_{M\#N}}/K_{M\#N} \cong \overline{B}_{S_M}/K_M \oplus \overline{B}_{S_N}/K_N$. Similarly we can prove that

$$\begin{pmatrix} 0 & f(D_{M\#N})_-(C_{M\#N})_-(1-(C_{M\#N})_-) \\ f(D_{M\#N})_+(C_{M\#N})_+ & 0 \end{pmatrix}$$

$$\cong \begin{pmatrix} 0 & f(D_M)_-(C_M)_-(1-(C_M)_-) \\ f(D_M)_+(C_M)_+ & 0 \end{pmatrix}$$

$$\oplus \begin{pmatrix} 0 & f(D_N)_+(C_N)_-(1-(C_N)_-) \\ f(D_N)_+(C_N)_+(1-(C_N)_+) & 0 \end{pmatrix}$$

in $\overline{B}_{S_{M\#N}}/K_{M\#N} \cong \overline{B}_{S_M}/K_M \oplus \overline{B}_{S_N}/K_N$ by using the fact that the Fourier transform of $g(g^2-1)$ is compactly supported. Now our theorem follows from the proof of lemma 4. QED.

2.5 On a problem of Roe

In section 2 we proved that the K theoretic index of the Dirac operator is an obstruction to the existence of metrics with uniform positive scalar curvature outside a compact set. The same result is not true in general if we replace "uniform positivity" by "positivity". Roe introduces a trace on the Roe algebra when there exists an "exhaustion" on the manfold. The pairing of the trace with the K theoretic index is an obstruction to the existence of metrics with positive scalar curvature on a "large" set due to the continuity of the trace [31]. The pairing of the trace with the K theoretic index can be obtained from Roe's index formula.

The index of the de Rham operator is analogously an obstruction to the existence of metrics with positive Ricci curvature outside a compact set in special cases [31]. The following result is an example of this.

Theorem 2.5.1 *Let X be a compact oriented 2-manifold or 4-manifold with Euler number $\chi(X) < 0$, and let M be an infinite covering of X. Then there is no metric in the strict quasi-isometry class on M whose Ricci curvature is nonnegative outside a compact set.*

The above result solves a problem of Roe. The same result was proved by Roe [31] in the case of an amenable covering.

Proof. Assume $Ric \geq 0$ outside the compact set $K \subset M$. There exists a closed ball $B(p,r)$ with center p and radius r such that $K \subseteq B(p,r)$. Choose R such that $dist(\partial B(p,R), B(p,r)) > 10r$, where $\partial B(p,R)$ is the boundary of $B(p,R)$. Now there are finitely many balls $\{B(x_i,r)\}_{i=1}^n$ such that $x_i \in \partial B(p,R)$ and $\partial B(p,R) \subseteq \cup_{i=1}^n B(x_i,r)$. Let $M_i = \{x | x \in M$ and x can be connected to x_i by a minimal geodesic without intersecting $B(p,r)\}$. We claim

$$M - B(p,R) \subseteq \cup_{i=1}^n M_i.$$

Proof of the claim: For any $x \in M - B(p,R)$ there is a minimal geodesic γ connecting x and p. Then γ intersects $B(x_i,r)$ for some i. Consider the minimal geodesic β connecting x with x_i. β does not intersect $B(p,r)$ since

$$length\beta \leq (length\gamma - 10r) + r = length\gamma - 9r.$$

Hence x is in M_i. So we have proved our claim. Now M_i are star shaped sets and $Ric|_{M_i} \geq 0$. By the volume comparison theorem [5] we have

$$Volume(B(x_i,r_1) \cap M_i)/Volume(B(x_i,r_2) \cap M_i) \leq V^0(r_1)/V^0(r_2),$$

where $V^0(r)$ is the volume of the Euclidean ball of radius r. Notice that the volume for the Euclidean space has polynomial growth, i.e. $V^0(d) \leq kd^n$ for some fixed $k > 0, n > 0$ and arbitrary $d > 0$. Combining this fact and the above volume inequality with the claim we see that the volume of M has polynomial growth, i.e. there exists $c > 0, n > 0$ such that for a fixed $x \in M$ $Volume(B(x,d)) \leq cd^n$ for any d. Hence there exists an exhaustion on M by proposition 6.2 in [31]. Now Roe's index formula [31] can be applied to the deRham operator to obtain the desired result as in the proof of proposition 2.9 in [31]. QED

Chapter 3

The algebraic topology of Roe algebras

In order to compute the K theoretic index explicitly we are faced with the problem of computing the K theory of the Roe algebra. In this chapter we shall compute the K theory of Roe algebras for various spaces to see the scope of complexity. In particular, we prove that the K theory of the Roe algebra for Euclidean space and the Poincare hyperbolic disc is the direct product of countably many copies of $\mathbb{Z}$. We also construct an even dimensional compact spin manifold M with nonzero $\hat{A}$ genus such that the K theory of the Roe algebra for the universal covering space of M is trivial. Therefore even if the $C_r^*(\pi_1(M))$ index of the Dirac operator is nontrivial [27], the K theoretic index of the Dirac operator living in the K theory of the Roe algebra may be trivial.

While a complete computation of the K theoretic index is not always possible, we can compute a partial index by constructing K homology elements or cyclic cocycles over the Roe algebra and then computing their pairing with the index. One other purpose of this chapter is to understand the cyclic cohomology of the Roe algebra for reasonably nice spaces.

For a finitely generated discrete group Γ, we define a Roe algebra in an analogous manner by using the word length function defined by a finite generating set. When $\tilde{M}$ is the universal cover of a compact Riemannian manifold M, then we show that the Roe algebra for $\tilde{M}$ is Morita equivalent to the Roe algebra for the fundamental group of M. When Γ has polynomial growth, we propose a inductive way of computing the continuous cyclic cohomology of a certain natural smooth subalgebra of the Roe algebra. In particular, we prove that if Γ has polynomial growth of degree d, then every cyclic cocycle over the smooth subalgebra has dimension at most d. The same method can be used to prove a conjecture of S. Hurder which states that the exotic cohomology $HX^q(\Gamma)$=0 for $q > d$ if Γ is a discrete group with polynomial growth of degree d. The exotic cohomology was introduced by Roe to construct cyclic cocycles over the Roe algebra [32] [33]. We also show that group cocycles can be used to construct cyclic cocycles over the Roe algebra of amenable groups. By using this cyclic cocycle method we show that the K theoretic index of Dirac operator is nonzero for $E\Gamma$ spin manifold with Γ invariant metric when Γ has polynomial growth and $B\Gamma$ is compact and oriented ($E\Gamma$ is topologically the universal cover of the classifying space $B\Gamma$ for Γ).

3.1 K theory of Roe algebras

Let Γ be a finitely generated discrete group. A length function l can be naturally defined on Γ by choosing a finite generating set on Γ and letting $l(\gamma)$ equal the minimal length of a word in these generators which represents

γ. Then l induces the metric d on Γ by $d(\gamma_1,\gamma_2) = l(\gamma_1\gamma_2^{-1})$ for $\gamma_1,\gamma_2 \in \Gamma$. Now the Roe algebra B_Γ for Γ consists of the operators acting on $l^2(\Gamma)$ in the following way:

$$(kf)(\gamma) = \sum_{\gamma_1 \in \Gamma} k(\gamma,\gamma_1)f(\gamma_1),$$

where f is in $l_2(\Gamma)$ and $k(\gamma_1,\gamma_2)$ is a bounded function on $\Gamma \times \Gamma$ which vanishes when the distance between γ_1 and γ_2 is greater than some constant depending on the operator k. It is not difficult to verify that B_Γ does not depend on the choice of the generating set in defining the length function.

Denote by $\overline{B}_\Gamma$ the operator norm closure of B_Γ. The motivation for considering $\overline{B}_\Gamma$ is the following:

Lemma 6 *Let M be a compact spin manifold and $\tilde{M}$ be its universal cover with spin bundle S endowed with a $\pi_1(M)$ invariant metric. Then the Roe algebra $\overline{B}_S$ is Morita equivalent to $\overline{B}_{\pi_1(M)}$.*

One consequence of the above result is that $\overline{B}_S$ does not depends on the choices of the invariant metric and will therefore be denoted by $\overline{B}_{\tilde{M}}$.

Proof: Let M be endowed with the metric whose pullback is the given metric on $\tilde{M}$. Let S_1 be the spin bundle on M. Denote $\pi_1(M)$ by Γ. We can use a fundamental domain U to make the identification

$$L^2(S) \cong l^2(\Gamma) \otimes L^2(S|_U) \cong l^2(\Gamma) \otimes L^2(S_1).$$

Hence each bounded operator on $L^2(S)$ can be decomposed correspondingly. Now it is not difficult to see

$$\overline{B}_S \cong \overline{B}_\Gamma \otimes K,$$

where K is the algebra of all compact operators on $L^2(S_1)$. QED.

There is a natural Γ action on $l^\infty(\Gamma)$:

$$(g^*f)(\gamma) = f(g^{-1}\gamma)$$

for any $g \in \Gamma, f \in l^\infty(\Gamma)$. We have the following structure result for $\overline{B}_\Gamma$:

Proposition 4 *$\overline{B}_\Gamma$ is isomorphic to $l^\infty(\Gamma) \rtimes \Gamma$, the reduced crossed product algebra.*

Proof: Let's first prove

$$B_\Gamma \cong l^\infty(\Gamma) \rtimes_a \Gamma$$

the algebraic crossed product. Any $a \in B_\Gamma$ has a kernel $a(\gamma_1, \gamma_2)$ on $\Gamma \times \Gamma$. For any $g \in \Gamma$ let f_g be the function on Γ defined by $f_g(\gamma) = a(\gamma, g^{-1}\gamma)$. Then we have

$$a = \sum_{\gamma \in \Gamma} M_{f_g} g$$

as operators on $l^2(\Gamma)$ where M_{f_g} is the multiplication operator on $l^2(\Gamma)$ by f_g and g acts on $l^2(\Gamma)$ as follows: $(gf)(\gamma) = f(g^{-1}\gamma)$ for any $g \in \Gamma$ and $f \in l^2(\Gamma)$. $\sum_{\gamma \in \Gamma} M_{f_g} g$ can be viewed as an element in $l^\infty(\Gamma) \rtimes_a \Gamma$. It is not difficult to see such a correspondence is one to one. Next let's check that the result holds at the level of C^* algebras. The norm on the crossed product is defined by the regular representation on $l^2(\Gamma) \otimes l^2(\Gamma) = l^2(\Gamma \times \Gamma)$:

$$((ag)f)(s,t) = a(ts)f(s, g^{-1}t)$$

for any $a \in l^\infty(\Gamma)$ and $g \in \Gamma$. $l^2(\Gamma \times \Gamma)$ is isomorphic to $\oplus_{\gamma \in \Gamma} l^2(\Gamma)$ by the map:

$$f(s,t) \to \oplus_{\gamma \in \Gamma} f(\gamma, t).$$

ag can be decomposed correspondingly:

$$(ag)(\oplus_{\gamma\in\Gamma}\xi_\gamma)(t) = \oplus_{\gamma\in\Gamma}a(t\gamma)\xi_\gamma(g^{-1}t).$$

Now it is easy to see that the operator norm of $\sum_i a_i g_i$ (for $a_i \in l^\infty(\Gamma), g_i \in \Gamma$) on each subspace of the decomposition is equivalent to the operator norm of $\sum_i a_i g_i$ as an element in B_Γ. QED.

Now we are ready to compute the K theory of the Roe algebra for Euclidean space $\mathbb{R}^n$ with the usual metric.

Proposition 5 *$K_0(\overline{B}_{\mathbb{R}^n})$ is isomorphic to the direct product of countably many copies of $\mathbb{Z}$.*

Proof: Combining the above lemma and proposition 4 we have

$$K_0(\overline{B}_{\mathbb{R}^n}) = K_0(l^\infty(\mathbb{Z}^n) \rtimes \mathbb{Z}^n) = K_0(((l^\infty(\mathbb{Z}^n) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}) \rtimes \cdots),$$

the iterated crossed product. Now our result follows by repeatedly applying the Pimsner-Voiculescu exact sequence [30]. QED.

By a similar method we have $K_1(\overline{B}_{\mathbb{R}}) = \mathbb{Z}$. The generator of $K_1(\overline{B}_{\mathbb{R}})$ can be detected by the cyclic cocycle constructed from partitioning $\mathbb{R}$ [34]. Hence the pairing of this cyclic cocycle with the K theoretic index $IndD$ detects the K thoeretic index completely. This implies that there is only one index theorem for the Dirac operator on $\mathbb{R}$ with the usual metric, which is the one given by Roe [34].

The K theory for the Roe algebra of Euclidean space is huge. The following result provides another extreme example.

Proposition 6 $K_0(\overline{B}_{\mathbb{F}_n}) = 0$, *where* $\mathbb{F}_n$ *is the free group with* n *generators and* $n > 1$.

In particular, the above proposition implies that the identity element is 0 in $K_0(\overline{B}_{\mathbb{F}_n})$. If Γ is an amenable group, then the identity element is nonzero in $K_0(\overline{B}_\Gamma)$ since a Γ invariant mean on $l^\infty(\Gamma)$ can be used to construct a trace tr on $\overline{B}_\Gamma$ such that $tr(1) = 1$. Hence it is tempting to conjecture that the identity element is 0 in $K_0(\overline{B}_\Gamma)$ for nonamenable groups.

Proof of proposition 6: For simplicity take $n = 2$. From proposition 4 we have the following exact sequence of Pimsner and Voiculescu [30]:

$$K_0(l^\infty(\mathbb{F}_2)) \oplus K_0(l^\infty(\mathbb{F}_2)) \xrightarrow{\sigma} K_0(l^\infty(\mathbb{F}_2)) \xrightarrow{i_*} K_0(\overline{B}_{\mathbb{F}_2}) \rightarrow$$

$$\rightarrow K_1(l^\infty(\mathbb{F}_2)) \oplus K_1(l^\infty(\mathbb{F}_2)) = 0,$$

where i_* is induced by the inclusion i: $l^\infty(\mathbb{F}_2) \rightarrow \overline{B_{\mathbb{F}_2}} = l^\infty(\mathbb{F}_2) \rtimes \mathbb{F}_2$, $\sigma(\phi_1 \oplus \phi_2) = (1 - \alpha_1^*)(\phi_1) + (1 - \alpha_2^*)(\phi_2)$ for $\phi_1, \phi_2 \in l^\infty(\mathbb{F}_2)$ and α_1^*, α_2^* are induced by the action of two generators a and b for $\mathbb{F}_2$ on $l^\infty(\mathbb{F}_2)$.

By the exactness of the above sequence, we see that i_* is onto. Notice that $K_0(l^\infty(\mathbb{F}_2))$ is generated by projections in $l^\infty(\mathbb{F}_2)$. Hence it is enough to prove that $i_*(P) = 0$ in $K_0(\overline{B}_{\mathbb{F}_2})$ for any projection P in $l^\infty(\mathbb{F}_2)$. Let e_1 and e_2 be the projections in $l^\infty(\mathbb{F}_2)$ such that

$$e_1(\gamma) = \begin{cases} 1 & \text{if } \gamma = a^m b^n \cdots \text{ for } m \geq 0,\, n \neq 0 \text{ and } P(a^{-m}\gamma) = 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

and

$$e_2(\gamma) = \begin{cases} 1 & \text{if } \gamma = b^m a^n \cdots \text{ for } m \geq 0,\, n \neq 0 \text{ and } P(b^{-m}\gamma) = 1 \\ 0 & \text{otherwise} \end{cases},$$

where γ is written in the reduced form. Then it is easy to see:

$$(1 - \alpha_1^*)(e_1 \oplus 0)(\gamma) = \begin{cases} 1 & \text{if } \gamma = b^n \cdots \text{ for } n \neq 0 \text{ and } P(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(1 - \alpha_2^*)(0 \oplus e_2)(\gamma) = \begin{cases} 1 & \text{if } \gamma = a^n \cdots \text{ for } n \neq 0 \text{ and } P(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Hence we have

$$\sigma((e_1 \oplus 0) + (0 \oplus e_2)) = P - e_0,$$

where

$$e_0(\gamma) = \begin{cases} P(\gamma) & \text{if } \gamma = 1 \\ 0 & \text{otherwise} \end{cases},$$

where γ is written in the reduced form. It follows that $i_*(P) + i_*(e_0) = 0$ in $K_0(\overline{B}_{\mathbb{F}_2})$. We can prove $i_*(e_0) = 0$ in $K_0(\overline{B}_{\mathbb{F}_2})$ by using an argument similar to that used in lemma 2. Hence we have

$$i_*(P) = 0$$

in $K_0(\overline{B}_{\mathbb{F}_2})$. QED.

Corollary 3.1.1 *There exists a compact Riemannian manifold M such that*

$$K_0(\overline{B}_{\tilde{M}}) = 0,$$

where $\tilde{M}$ is the universal covering of M.

Proof of the corollary: Take $M = (S^1 \times S^2)\sharp(S^1 \times S^2)$, where S^n is the n dimensional sphere and $\sharp$ is the connected sum. The fundamental group of M is $\mathbb{F}_2$. Hence our result follows from lemma 6 and the above proposition. QED.

Let $\tilde{M}$ be the universal covering of a compact spin manifold M and D be the Dirac operator on $\tilde{M}$. Miscenko and Fomenko defined the $C_r^*(\pi_1(M))$ index of D: $indD \in K_0(C_r^*(\pi_1(M)))$ [27]. By lemma 6 we have a natural inclusion i_*: $K_0(C_r^*(\pi_1(M))) \to K_0(\overline{B}_{\tilde{M}})$. The image of $indD$ under i_* is clearly $IndD \in K_0(\overline{B}_{\tilde{M}})$. Nonzeroness of $IndD$ has a stronger implication than the nonzeroness of $indD$ (e.g. see theorem 1.) The reason is that $IndD$ is an invariant of the strict quasi-isometry class while $indD$ is only an invariant of the class of $\pi_1(M)$ invariant metrics (actually $indD$ is only defined for $\pi_1(M)$ invariant metrics.) But $indD$ is sometimes easier to compute. Hence it is natural to ask if the nonzeroness of ind D implies the nonzeroness of Ind D (such a possibility is suggested in [31]). The following result indicates that it is not true in general.

Proposition 7 *There exists a compact spin manifold M with nonzero $\hat{A}$ genus such that $K_0(\overline{B}_{\tilde{M}}) = 0$, where S is the spin bundle over the universal cover $\tilde{M}$ of M and is endowed with a $\pi_1(M)$ invariant metric.*

Notice that the nonvanishing of $\hat{A}$ implies the nonvanishing of $indD$ by Atiyah's L^2 index theorem for the covering space [1].

Proof: Consider $M_1 = (S^1 \times S^3)\sharp(S^1 \times S^3)$, where S^n is the n dimensional sphere and $\sharp$ is the connected sum. It is easy to see that M_1 is a spin manifold with fundamental group $\pi_1(M_1) = \mathbb{F}_2$. It follows from the behavior of the $\hat{A}$ under the connected sum [17] that the $\hat{A}$ genus for M_1 is 0. There exists a 4 dimensional spin simply connected manifold V with nonzero $\hat{A}$ genus (see p 91 [24]). Take $M = M_1\sharp V$. Then M is a spin manifold with nonzero $\hat{A}$ genus and $\pi_1(M) = \mathbb{F}_2$. By lemma 6 and proposition 6 M is the desired manifold. QED.

The last example we want to compute is the K theory of the Roe algebra for the Poincare hyperbolic disc D with the metric:

$ds^2 = 4|dz|^2/(1-|z|^2)^2$.

Proposition 8 *$K_0(\overline{B}_D)$ is naturally isomorphic to the direct product of countably many copies of $\mathbb{Z}$.*

Proof: D is the universal cover of a Riemann surface M with genus $g \geq 2$ and a metric of constant curvature -1. By lemma 6 we know $K_0(\overline{B}_D) = K_0(\overline{B}_\Gamma)$ for $\Gamma = \pi_1(M)$. Γ can be realized as the amalgamated free product in the following way:

Let $a_1, b_1, \cdots, a_{g-1}, b_{g-1}$ be free generators for the free group H and a, b be free generators for the free group G. Let $L = \mathbb{Z}$ be the abelian group with the generator [u]. Then L can be identified with subgroups of H and K by:

$[u] \to [a_1, b_1] \cdots [a_{g-1}, b_{g-1}] \in H$

$[u] \to [a, b] \in K$, where $[\ ,\ ]$ is the group commutator.

Now Γ is isomorphic to $H *_L K$, the free amalgamation of H and K over L. Hence we have the Pimsner exact sequence [28]:

$$\begin{array}{ccccc} K_0(l^\infty(\Gamma) \rtimes L) & \longrightarrow & K_0(l^\infty(\Gamma) \rtimes H) \oplus K_0(l^\infty(\Gamma) \rtimes K) & \longrightarrow & K_0(\overline{B}_\Gamma) \\ \uparrow & & & & \downarrow \delta \\ K_1(\overline{B}_\Gamma) & \longleftarrow & K_1(l^\infty(\Gamma) \rtimes H) \oplus K_1(l^\infty(\Gamma) \rtimes K) & \overset{i_*}{\longleftarrow} & K_1(l^\infty(\Gamma) \rtimes L) \end{array}$$

where i_* is induced by the difference of inclusions.

Notice that $l^\infty(\Gamma) \rtimes H$ is isomorphic to the direct product of countably many copies of $l^\infty(H) \rtimes H$. It follows from propositions 4 and 6 that $K_0(l^\infty(\Gamma) \rtimes H) = 0$. Similarly we have $K_0(l^\infty(\Gamma) \rtimes K) = 0$. It also can be seen that $l^\infty(\Gamma) \rtimes L$ is the direct product of countably many copies of $l^\infty(L) \rtimes L$. By the exact sequence of Pimsner and Voiculescu we have $K_1(l^\infty(L) \rtimes L) = \mathbb{Z}$. Hence $K_1(l^\infty(\Gamma) \rtimes L)$ is isomorphic to the direct product of countably many copies of $\mathbb{Z}$. By the above explicit computation of $K_1(l^\infty(\Gamma) \rtimes L)$ and the way that L is embeded in H and K we know that $i_* = 0$. Hence our result follows from Pimsner's exact sequence. QED.

3.2 Cyclic cohomology of Roe algebras

The pairing of a cyclic cocycle over a smooth subalgebra of the Roe algebra with the K theoretic index gives rise to a numerical index invariant.

(A smooth subalgebra is a dense $*$-subalgebra which is stable under the holomorphic functional calculus.) In order to carry out this process we need first to construct a natural smooth subalgebra of the Roe algebra on which there should exist nontrivial cyclic cocycles, second to construct cyclic cocycles, and finaly, to compute their pairing with the K theoretic index. In this section we shall carry out this program for the Roe algera of the universal cover $\tilde{M}$ of a compact manifold M when $\Gamma = \pi_1(M)$ has polynomial growth. We construct a smooth subalgebra B_Γ^∞ of the Roe algebra $\overline{B}_\Gamma$ when $\Gamma = \pi_1(M)$ has polynomial growth. We propose an inductive method of computing the cyclic cohomology of B_Γ^∞. A method of constructing cyclic cocycles over B_Γ^∞ is also given by using group cocyles over Γ. Pairing of such cyclic cocycles with the index of Dirac operators can be computed by using Connes and Moscovici's index theorem [14].

Let Γ be a finitely generatedly discrete group and l be the length function on Γ defined by a finite generating set.

Definition 1 *B_Γ^∞ consists of all elements of the form:*
$\sum_{\gamma\in\Gamma} \phi(\gamma)\gamma$, where $\phi(\gamma) \in l^\infty(\Gamma)$ and $\sum_{\gamma\in\Gamma} \|\phi(\gamma))\|_{l^\infty(\Gamma)}(1 + l(\gamma))^{2s}$ converges for any $s > 0$.

Proposition 9 *If Γ has polynomial growth, then B_Γ^∞ is a smooth subalgebra of the Roe algebra $\overline{B}_\Gamma$, i.e. B_Γ^∞ is a dense $*$-subalgebra which is stable under the holomorphic functional calculus in the Roe algebra.*

Proof: The proof is completely analogous to the proof of the corresponding result for $C_r^*(\Gamma)$ in [23] and is therefore omitted. QED.

Corollary 3.2.1 $K_0(B_\Gamma^\infty) = K_0(\overline{B_\Gamma})$.

Proof: Cf [6].

Definition 2 *The dimension of a cyclic cocycle over an algebra A is defined to be the minimal n for which its class in the periodic cyclic cohomology $HC^*(A)$ comes from $H_\lambda^n(A)$.*

When Γ has polynomial growth we obtain the following result as a consequence of our inductive method of computing the continuous cyclic cohomology for B_Γ^∞.

Theorem 3.2.1 *If Γ is a finitely generated discrete group with polynomial growth of degree d, then every continuous cyclic cocycle over B_Γ^∞ has dimension at most d.*

The proof of the theorem will require considerable preparatory lemmas and a deep theorem of Gromov.

If Γ is amenable we have the following method for constructing cyclic cocycles over B_Γ by using group cocycles over Γ.

Recall that a normalized cocycle c on Γ is a map from Γ^n to $\mathbb{C}$ satisfying [14]:

(1). $c(g_2, \cdots, g_{n+1}) + \sum_{j=1}^{n} c(g_1, \cdots, g_j g_{j+1}, \cdots, g_{n+1})$

$+ (-1)^{n+1} c(g_1, \cdots, g_n) = 0$

for all $g_1, \cdots, g_{n+1} \in \Gamma$.

(2). $c(g_1, \cdots, g_n) = 0$

when $g_1 \cdots g_n = 1$ or $g_j = 1$ for some j.

Let c be a normalized group cocycle over Γ and m be an invariant mean on $l^\infty(\Gamma)$. Then we can define the induced cyclic cocycle as:

$$\tau_c(f_0,\cdots,f_n) = \sum_{g_0\cdots g_n=1} m(f_0(g_0)g_0^*f_1(g_1)\cdots(g_0\cdots g_n)^*f_n(g_n))c(g_0,\cdots,g_n),$$

where $f_i \in B_\Gamma = l^\infty(\Gamma) \rtimes_a \Gamma$, the algebraic crossed product of $l^\infty(\Gamma)$ with $\Gamma, f_i(g_i) \in l^\infty(\Gamma)$, g^* is the translation action on $l^\infty(\Gamma)$ induced by g, and $g_i \in \Gamma$.

Lemma 7 *If Γ has polynomial growth, then τ_c can be extended to a cyclic cocycle over B_Γ^∞.*

Proof: The lemma follows from the fact that every normalized group cocycle over a discrete group with polynomial growth has polynomial growth. QED.

Recall that $E\Gamma$ is the universal cover for the classifying space $B\Gamma$ for Γ.

Corollary 3.2.2 *If Γ has polynomial growth and $B\Gamma$ is an even dimensional compact oriented manifold, then the K theoretic index of the Dirac operator on an $E\Gamma$ spin manifold with Γ invariant metric is nonzero when Γ has polynomial growth.*

Proof: Choose the group cocycle corresponding to the fundamental cohomology element on $B\Gamma$. By Connes-Moscovici's index theorem [13] we know $\tau_c(indD) = 1$. Now our corollary follows from above lemma. QED.

Let G be a finitely generated discrete group and l be the length function on G defined by a finite generating set. The space $C^\infty(G)$ of rapidly decreasing functions on G consists of all elements of the form: $\sum_{\gamma\in\Gamma} c_\gamma\gamma$, where $c_\gamma \in \mathbb{C}$

and $\sum_{\gamma\in\Gamma}|c_\gamma|^2 l(1+(\gamma))^{2s}$ converges for all $s>0$ [23]. When Γ has polynomial growth $C^\infty(G)$ can be made into a complete locally convex topological algebra by the seminorms: $p_s(\sum_{\gamma\in\Gamma} c_\gamma\gamma) = (\sum_{\gamma\in\Gamma}|c_\gamma|^2(1+l(\gamma))^{2s})^{1/2}$ for any $s>0$ [23].

Let G be a finitely generated discrete group with polynomial growth. If A is a complete locally convex topological algebra with a G action by automorphisms, then we define the topological crossed product $A \rtimes_\pi G$ as the completion of the algebraic crossed product $A \rtimes_a G$ in $A\hat{\otimes}_\pi C^\infty(G)$, where $\hat{\otimes}_\pi$ means the projective tensor product. $A \rtimes_\pi G$ is a complete locally convex topological algebra with the algebra structure coming from $A \rtimes_a G$.

If Γ is a finitely generated nilpotent discrete group, then there exists a filtration $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{k+1} = \{1\}$ such that Γ_i is a normal subgroup of Γ_{i-1} and Γ_i/Γ_{i-1} has a single generator. Notice that a finitely generated nilpotent discrete group has polynomial growth. Let $G_i = \Gamma_i/\Gamma_{i-1}$, we have :

Lemma 8 *If Γ is a finitely generated nilpotent discrete group, then $B_\Gamma^\infty = (((l^\infty(\Gamma) \rtimes_\pi G_k) \rtimes_\pi G_{k-1})\cdots) \rtimes_\pi G_1$, the iterated topological crossed product, where $l^\infty(\Gamma)$ is endowed with the localy convex topology defined in this section and G_i is either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$.*

Proof: Let $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{k+1} = \{1\}$ be a filtration of Γ such that Γ_i is a normal subgroup of Γ_{i-1} and Γ_i/Γ_{i-1} with a single generator. Let $G_i = \Gamma_i/\Gamma_{i-1}$, then we have $\Gamma_{i-1} = \Gamma_i \rtimes G_i$, the semidirect product of Γ_i with G_i, where G_i acts on Γ_i in the following way: $g^n\gamma = \tilde{g}^{-n}\gamma\tilde{g}^n$ for a fixed generator g of G_i, a fixed preimage $\tilde{g}$ of g in Γ_{i-1} and any $\gamma \in \Gamma_i$. This action is well defined since Γ_i is normal in Γ_{i-1}. Hence we can write $\Gamma = G_k \rtimes \cdots \rtimes G_1$, the iterated semidirect product of groups. Now by the proof of proposition 4 we see that B_Γ is isomorphic to $l^\infty(\Gamma) \rtimes_a \Gamma$. Our lemma follows from taking completion with respect the appropriate topologies and the fact that Γ is a finitely generated nilpotent discrete group. QED.

We need the following deep theorem of Gromov [19] on the the structure of finitely generated groups of polynomial growth.

Theorem 3.2.2 *(Gromov). If Γ is a finitely generated group of polynomial growth, then Γ has a nilpotent subgruop Γ' of finite index.*

We need the following lemma in our proof.

Lemma 9 *If Γ and Γ' are groups as in the above theorem, then Γ is again finitely generated and has polynomial growth of the same degree as Γ.*

Proof: Consider the finite set $F = \{\gamma_1, \cdots, \gamma_n\}$ in Γ such that $\{[\gamma_1], \cdots, [\gamma_n]\} = \Gamma/\Gamma'$. Define the subset $S \subseteq \Gamma'$ by:

$$S = \{\gamma | \gamma \in \Gamma', dist(\gamma_i\gamma, F) \leq c, \ for \ \ some \ \gamma_i \in F\},$$

where the distance is induced by the length function on Γ and c is a positive constant. S is obviously finite.

Claim: $\exists c$ such that S generates Γ'.

Proof of the claim: Let $[S]$ be the group generated by S in Γ. Let $\{g_1, \cdots, g_m\}$ be the generating set of Γ used in defining the length fuction. Without loss of generality we may assume that if $g_i \in \{g_1, \cdots, g_m\}$, then $g_i^{-1} \in \{g_1, \cdots, g_m\}$.

Let $\gamma_i' \in \Gamma'$ be such that

$$g_i = \gamma_{n_i}\gamma_i'$$

for some $\gamma_{n_i} \in F$. Let $\gamma_{k,l}', \gamma_{k,l}'' \in \Gamma'$ be such that we have the following commutation relations

$$\gamma_l'\gamma_k = \gamma_{k,l,1}\gamma_{k,l}',$$

$$\gamma_l\gamma_k = \gamma_{k,l,2}\gamma_{k,l}'',$$

for some $\gamma_{k,l,1}, \gamma_{k,l,2} \in F$. Take

$$c = max_{k,l}\{l(\gamma_{k,l,1}), l(\gamma_{k,l,2}), l(\gamma_i')\}.$$

Then $\gamma_{k,l}', \gamma_{k,l}'' \in S, \gamma_i' \in S$. Now we can prove $F[S] = \Gamma$ as follows, where $F[S] = \{gh | g \in F, h \in [S]\}$. For any $g = g_{m_1}^{n_1} \cdots g_{m_m}^{n_m} \in \Gamma$ notice that n_i can be chosen to be positive by the assumption on the generating set. Hence we can use the commutation relations to obtain

$$g = (\gamma_{k_1}\gamma_{m_1}')^{n_1} \cdots (\gamma_{k_m}\gamma_{m_m}')^{n_m}$$

$$= (\gamma_{k_1}\gamma_{l_1}') \cdots (\gamma_{k_t}\gamma_{l_t}')$$

$$= \gamma_{k_1}\gamma_{l_1}' \cdots (\gamma_{l_{t-1}}'\gamma_{k_t})\gamma_{l_t}'$$

$$= \gamma_{k_1}\gamma_{l_1}' \cdots \gamma_{k_t,l_{t-1},1}\gamma_{k_t,l_{t-1}}'\gamma_{l_t}'$$

(where $\gamma_{k_t,l_{t-1}}' \in [S]$.)

$$\vdots$$

$$= \gamma\gamma'$$

where $\gamma \in F, \gamma' \in [S]$. With the identity $F[S] = \Gamma$ it is easy to conclude that S generates Γ'. By choosing $F \cup S$ as the generating set in defining the length function we see that Γ' has polynomial growth of the same degree as Γ. We are done since the degree of polynomial growth does not depends on the choice of generating set. QED.

Lemma 10 *If Γ is a finitely generated discrete group and Γ' is a subgroup in Γ with finite index, then $B_\Gamma \cong B_{\Gamma'} \otimes M_n(\mathbb{C})$ for $n = |\Gamma/\Gamma'|$.*

Proof: By lemma 9 Γ' is finitely generated. Let $\{\gamma_1', \cdots, \gamma_m'\}$ be a generating set for Γ'. Let $F = \{\gamma_1, \cdots, \gamma_n\}$ in Γ such that $\{[\gamma_1], \cdots, [\gamma_n]\} = \Gamma/\Gamma'$. Let S be as in the proof of the above lemma. Choose $T = F \cup S$ as the generating set to define the length function on Γ. Now Γ is isomorphic to $\Gamma/\Gamma' \times \Gamma'$ as sets by the map: $\gamma \longrightarrow [\gamma] \times \gamma'$, where $\gamma' = \gamma_i^{-1}\gamma$ for the $\gamma_i \in F$ such that $[\gamma] = [\gamma_i]$. Let l and l' be the length functions on Γ and Γ' defined by the generating sets T and S, respectively. Define the distance d_1 on $\Gamma/\Gamma' \times \Gamma'$ by:

$$d_1((\gamma_1, \gamma_1'), (\gamma_2, \gamma_2')) = \begin{cases} 2 + l'(\gamma_1' \gamma_2'^{-1}) & \text{if } \gamma_1 \neq \gamma_2 \\ \\ l'(\gamma_1' \gamma_2'^{-1}) & \text{if } \gamma_1 = \gamma_2 \end{cases}$$

Denote by d the distance on Γ defined by the length function l. Then it is easy to see

$$d(x, y) \leq d_1(x, y)$$

for any $x, y \in \Gamma = \Gamma/\Gamma' \times \Gamma'$. Assume $S = \{s_1, \cdots, s_k\}$. Without loss of generality we can assume further that $s^{-1} \in S$ whenever $s \in S$. Then we have the following commutation relations:

$$s_i \gamma_j = \gamma_{1,i,j} \gamma_{1,i,j}'$$

for some $\gamma_{1,i,j} \in F, \gamma'_{1,i,j} \in \Gamma'$

$$s_i \gamma_j^{-1} = \gamma_{2,i,j}\gamma'_{2,i,j}$$

for some $\gamma_{2,i,j} \in F, \gamma'_{2,i,j} \in \Gamma'$

$$\gamma_i \gamma_j = \gamma_{3,i,j}\gamma'_{3,i,j}$$

for some $\gamma_{3,i,j} \in F, \gamma'_{3,i,j} \in \Gamma'$

$$\gamma_i \gamma_j^{-1} = \gamma_{4,i,j}\gamma'_{4,i,j}$$

for some $\gamma_{4,i,j} \in F, \gamma'_{4,i,j} \in \Gamma'$

$$\gamma_i^{-1} \gamma_j^{-1} = \gamma_{5,i,j}\gamma'_{5,i,j}$$

for some $\gamma_{5,i,j} \in F, \gamma'_{5,i,j} \in \Gamma.'$ Take $M = max_{i,j,k}\{l'(\gamma_{k,i,j})\}$. It is not difficult to see that the Roe algebra B_Γ with respect to the distance d_1 is isomorphic to $B_{\Gamma'} \otimes M_n(\mathbb{C})$. Hence it is enough to prove that for any $c > 0$ there exists $c_1 > 0$ such that:

$$\{(x,y)|x,y \in \Gamma, d(x,y) \leq c\} \subseteq \{(x,y)|x,y \in \Gamma, d_1(x,y) \leq c_1\}.$$

If $(x,y) \in \Gamma \times \Gamma$ is such that $l(xy^{-1}) \leq c$, then xy^{-1} can be written as

$$s_{i_1}^{m_1}\gamma_{j_1}^{n_1} \cdots s_{i_t}^{m_t}\gamma_{j_t}^{n_t}$$

for $s_i \in S, \gamma_j \in F$ such that $\sum_{i=1}^t |m_i| + \sum_{i=1}^t |n_i| \leq c$.

By using the commutation relations we can shuffle all γ_j in the first place by beginning with γ_{j_t}. We finally get:

$$xy^{-1} = \gamma g$$

for some $\gamma \in F, g \in \Gamma'$ such that

$$l'(g) \leq M^{c^2+c} l(xy^{-1}).$$

Suppose

$$x = g_1 h_1, y = g_2 h_2$$

for some $g_i \in F, h_i \in \Gamma'$. Then we have

$$h_1 h_2^{-1} = g_1 x y^{-1} g_2 = g_1 \gamma g g_2^{-1}.$$

By using the commutation relations and the shuffling process we have

$$l'(h_1 h_2^{-1}) \leq M^{c+2} l'(g) \leq M^{c^2+2c+2} l(xy^{-1}) \leq c_1 = M^{c^2+2c+2} c.$$

QED.

The following topological analogue of lemma 10 follows from the above proof.

Lemma 11 *If Γ is a finitely generated discrete group and Γ' is a subgroup in Γ with finite index, then $B_\Gamma^\infty \cong B_{\Gamma'}^\infty \otimes M_n(\mathbb{C})$ for $n = |\Gamma/\Gamma'|$.*

The above lemma together with lemma 9 makes it possible for us to compute inductively the continuous cyclic cohomology of B_Γ^∞ (Cf the following proof).

Now we are ready to prove theorem 3.2.1.

Proof of the theorem 3.2.1: Step 1 : When Γ is nilpotent, we shall prove by induction that every cyclic cocycle over $B_\Gamma^n = (((l^\infty(\Gamma) \rtimes_\pi G_1) \rtimes_\pi G_2) \cdots) \rtimes_\pi G_n$

has dimension less than or equal to $\sum_{i=1}^{n} rankG_i$, where

$$rankG_i = \begin{cases} 1 & \text{if } G_i = \mathbb{Z} \\ 0 & \text{if } G_i = \mathbb{Z}_n \end{cases}$$

Notice $B_\Gamma^\infty = B_\Gamma^k$. Assume by induction that the result holds for $B_\Gamma^{(n)}$. This is true when $n = 0$. Now consider $B_\Gamma^{(n+1)} = B_\Gamma^{(n)} \rtimes_\pi G_{n+1}$.
First case: $G_{n+1} = \mathbb{Z}_p$ for some positive integer p. We have

$$B_\Gamma^{(n+1)} = l^\infty(\Gamma) \rtimes_\pi \Gamma_{n+1},$$

where $\Gamma_{n+1} = G_1 \rtimes G_2 \cdots \rtimes G_{n+1}$. Notice that $l^\infty(\Gamma) \rtimes_\pi \Gamma_{n+1}$ is isomorphic to the direct product of $card|\Gamma/\Gamma_{n+1}|$ copies of $l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_{n+1} = B_{\Gamma_{n+1}}^\infty$. By lemma 11 $B_{\Gamma_{n+1}}^\infty$ is isomorphic to $B_{\Gamma_n}^\infty \otimes M_p(\mathbb{C})$, which is Morita equivalent to $B_{\Gamma_n}^\infty$. It follows from the induction hypothesis that every cyclic cocycle over $B_{\Gamma_{n+1}}^\infty$ has dimension at most $rank\Gamma_n = rank\Gamma_{n+1}$.
Second case: $G_{n+1} = \mathbb{Z}$. As in the previous case $B_\Gamma^{(n+1)}$ is isomorphic to the direct product of $card|\Gamma/\Gamma_{n+1}|$ copies of $l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_{n+1} = B_{\Gamma_{n+1}}^\infty$. We know

$$l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_{n+1} \cong (l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z}.$$

It is not diffcult to see that our topological crossed product is equivalent to Nest's smooth crossed product [28]. By Nest's result [28] we have the following long exact sequence:

$$\cdots \to H_\lambda^{m-1}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \to H_\lambda^{m-1}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \to H_\lambda^m((l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z})_{hom}$$

$$\to H_\lambda^m(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \to H_\lambda^m(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \to \cdots$$

where $H_\lambda^m((l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z})_{hom}$ is the "homogeneous" cyclic cohomology for $(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z}$. Recall that the "homogeneous" cyclic cohomology for $(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z}$ is the cohomology of homogeneous cyclic cochains, i.e., such that given $a_0, \cdots, a_m \in l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n$,

$$\phi(u^{n_0}a_0, \cdots, u^{n_m}a_m) = \phi(u^{n_0}a_0, \cdots, u^{n_m}a_m)\delta_{n_0+\cdots+n_m,0},$$

where ϕ is a cyclic cochain over $(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z}$, u is the generator for $\mathbb{Z}$ and

$$\delta_{n,0} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

A result of Nest [28] implies that the periodic cyclic cohomology of $(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z}$ can be computed by the "homogeneous" cyclic cohomology. We have the following commutative diagram:

$$\begin{array}{ccc}
H_\lambda^{m-1}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) & \longrightarrow & H_\lambda^{m-1}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \\
\downarrow \delta_1 & & \downarrow \delta_2 \\
HC^{(m-1)mod2}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) & \longrightarrow & HC^{(m-1)mod2}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n)
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & H_\lambda^m((l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z})_{hom} & \longrightarrow & H_\lambda^m(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \\
& \downarrow \delta_3 & & \downarrow \delta_4 \\
\longrightarrow & HC^{m\ mod2}((l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \rtimes_\pi \mathbb{Z})_{hom} & \longrightarrow & HC^{m\ mod2}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n)
\end{array}$$

$$\begin{array}{cc}
\longrightarrow & H_\lambda^m(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n) \\
& \downarrow \delta_5 \\
\longrightarrow & HC^{m\ mod2}(l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n)
\end{array} .$$

Every cyclic cocycle over $l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n$ has dimension at most $rank\Gamma_n$ since $l^\infty(\Gamma_{n+1}) \rtimes_\pi \Gamma_n$ is isomorphic to the direct product of $card|\Gamma/\Gamma_{n+1}|$ copies of $l^\infty(\Gamma_n) \rtimes_\pi \Gamma_n$. Hence δ_1, δ_2, δ_4, and δ_3 are onto if we take $m = rank\Gamma_{n+1}$. Now it follows from a standard diagram chasing argument in the above commutative diagram that δ_5 is onto.

Second step: Reduction of the general case to the case of nilpotent group. By lemma 9 and 10 B_Γ^∞ is Morita equivalent to $B_{\Gamma'}^\infty$ for a finitely generated nilpotent group Γ' with the same degree of polynomial growth. QED.

3.3 On a conjecture of S. Hurder

In their approach to the Novikov conjecture Connes and Moscovici use Alexander-Spanier cohomology to construct "local" cyclic cocycles. Motivated by this idea Roe introduces exotic cohomology to construct cyclic cocycles over the Roe algebra. Roughly speaking the exotic cohomology of a metric space describes how bounded sets patch together at infinity. So it is natural to ask how the growth of the metric space at infinity is related to the exotic cohomology of the metric space. In this aspect we prove the following conjecture of S. Hurder [33]:

Theorem 3.3.1 *If Γ is a finitely generated discrete group with polynomial growth of degree d, then*

$$HX^q(\Gamma) = 0 \ for \ q > d.$$

Let's first recall the definition of Roe's exotic cohomolgy $HX^*(X)$ for a metric space X [32] [33]. Let M^{q+1} be the Cartesian product of $q+1$ copies of M. A function ϕ: $M^{q+1} \to \mathbb{R}$ is said to be ω-bounded if it is bounded on every bounded subset. The exotic complex $CX^*(M)$ is defined as follows: $CX^q(M)$ is the space of ω-bounded Borel functions ϕ: $M^{q+1} \to \mathbb{R}$ which satisfy the following support conditon: for each $r > o$, the set

$$Supp(\phi) \cap \{x | x \in M^{q+1}, dist(x, \triangle)\}$$

is relatively compact in M^{q+1}, where $\triangle$ is the diagonal in M^{q+1}. The complex $CX^*(M)$ is equipped with the coboundary map

$$\partial\phi(x_0, \cdots, x_{q+1}) = \sum_{i=0}^{q+1}(-1)^i\phi(x_0, \cdots, \hat{x}_i, \cdots, x_{q+1}),$$

where the 'hat' denotes the omission of the specified term. Now the exotic cohomology $HX^*(M)$ is defined to be the cohomology of this complex.

The importance of exotic cohomology is that there is a natural map λ from the exotic cohomology $HX^*(M)$ to the cyclic cohomology of the Roe algebra on M. In the case that M is a finitely generated discrete group with the metric induced by the word length function, the map λ from the exotic cohomology $HX^q(\Gamma)$ to $H^q_\lambda(B_\Gamma)$ is defined as follows: for ϕ a cocycle in the exotic complex $CX^*(\Gamma)$ define

$$\lambda(\phi)(a_0, a_1, \cdots, a_n)$$

$$= \sum_{\gamma_0, \cdots, \gamma_n \in \Gamma} a_0(\gamma_0, \gamma_1) \cdots, a_n(\gamma_n, \gamma_0)\phi(\gamma_0, \cdots, \gamma_n)$$

for any $a_0, \cdots, a_n \in B_\Gamma$, where $a_i(x, y)$ is the kernel for a_i. It is not difficult to check that $\lambda(\phi)$ is indeed a cyclic cocycle over B_Γ.

The strategy of the proof of theorem 3.3.1 is as follows: we endow B_Γ with a locally convex topology t such that the map from $HX^*(\Gamma)$ to $H^*_\lambda(\overline{B}^t_\Gamma)$ is injective and preserves the grading, where $\overline{B}^t_\Gamma$ is the completion of B_Γ under the topology t. By the same argument as in the previous section we can see that $\overline{B}^t_\Gamma$ has homological dimension less than or equal to d. Hence our desired result follows.

Endow $B_\Gamma = l^\infty(\Gamma) \rtimes_a \Gamma$ with the locally convex topology t defined by the seminorms:

$$p_{c,\alpha}(x) = \sum_{\gamma\in\Gamma} c_\gamma p'_\alpha(x_\gamma)$$

for any $x = \sum_{\gamma\in\Gamma} x_\gamma \gamma \in l^\infty(\Gamma) \rtimes_a \Gamma$, where $p'_\alpha(a) = \sum_{\gamma\in\Gamma} \alpha_\gamma |a_\gamma|$ for a nonnegative function α on Γ with nonzero values at most a finite number of points and any $a \in l^\infty(\Gamma)$, and c is a nonnegative function on Γ.

Denote by $\overline{B}^t_\Gamma$ the completion of B_Γ with respect to the t topology. The reason for choosing such a topology is the following sequence of lemmas:

Lemma 12 $\overline{B}^t_\Gamma = l(\Gamma) \rtimes_a \Gamma$, *where* $l(\Gamma)$ *is the algebra of all functions on* Γ.

Proof: The proof is completely analogous to that of proposition 4 and is therefore omitted. QED.

By the above result it is not difficult to see that the cyclic cocycles over B_Γ constructed from exotic cohomology classes can be extended to continuous cyclic cocycles over $\overline{B}^t_\Gamma$.

Lemma 13 *The natural map from* $HX^n(\Gamma)$ *to* $H^n_\lambda(\overline{B}^t_\Gamma)$*, the continuous cyclic cohomology of* $\overline{B}^t_\Gamma$*, is injective.*

Proof: Assume $[\phi] \in HX^n(\Gamma)$ and that its image in $H^n_\lambda(\overline{B}^t_\Gamma)$ is 0, i.e. there exists a continuous multilinear functional τ over $\overline{B}^t_\Gamma$ such that

$$\lambda(\phi)(a_0, a_1, \cdots, a_n)$$

$$= \sum_{\gamma_0,\cdots,\gamma_n \in \Gamma} a_0(\gamma_0, \gamma_1) \cdots a_n(\gamma_n, \gamma_0)\phi(\gamma_0, \cdots, \gamma_n)$$

$$= (b\tau)(a_0, a_1, \cdots, a_n),$$

where $a_i(x, y)$ is the kernel of a_i. Let ψ be the function on $\Gamma^n = \Gamma \times \cdots \times \Gamma$ (n times) defined by:

$$\psi(y_0, \cdots, y_{n-1}) = \tau(\delta_{y_0,y_1}, \delta_{y_1,y_2}, \cdots, \delta_{y_{n-1},y_0})$$

where $\delta_{x,y}$ is in B_Γ with kernel defined by:

$$\delta_{x,y}(g, h) = \begin{cases} 1 & \text{if } (g, h) = (x, y) \\ 0 & \text{if } (g, h) \neq (x, y) \end{cases}$$

By the continuity of τ we know that τ is defined on $\overline{B}^t_\Gamma \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \overline{B}^t_\Gamma$ (n times) $\cong \overline{B}^t_{\Gamma^n}$. For any $c \in l(\Gamma^n)$, S a finite subset of Γ^n, x and $y \in \Gamma^n$, we define $a_S = \sum_{g\in S} c(g)\delta_{xg,yg}$. The net a_S converges to $cy^{-1}x \in \overline{B}^t_{\Gamma^n} \cong l(\Gamma^n) \rtimes_a \Gamma^n$ in the t topology, where $x = (x_1, \cdots, x_n), g = (g_1, \cdots, g_n)$, and $xg = (x_1g_1, \cdots, x_ng_n)$. Hence $\tau(\sum_{g\in\Gamma^n} c(g)\delta_{xg,yg})$ is well defined. By the continuity of τ we have

$$\tau(cy^{-1}x) = \lim_S \sum_{g\in S} c(g)\tau(\delta_{xg,yg})$$

for any $c \in l(\Gamma^n)$. It follows that $\lim_S \sum_{g\in S} c(g)\tau(\delta_{xg,yg})$ is well defined for any $c \in l(\Gamma^n)$. This fact implies that $\tau(\delta_{xg,yg})$ is nonzero for at most a finite number of $g \in \Gamma^n$ for fixed x and $y \in \Gamma^n$. Therefore

$$Supp\psi \cap \{z | z \in \Gamma^n, \ and\ dist(z, \Delta) \leq r\}$$

is finite for any $r > 0$, where Δ is the diagonal of Γ^n. Now it is routine to check $\tau_\phi = b\tau_\psi$. It follows that $\phi = \partial\psi$, where ∂ is the coboundary map in exotic cohomology. QED.

Lemma 14 *The map λ from $HX^*(\Gamma)$ to $H^*_\lambda(\overline{B}^t_\Gamma)$ preserves the filtration, i.e. if $\lambda[\phi] = S\tau$ for $[\phi] \in HX^n(\Gamma)$ and some $\tau \in H^n_\lambda(\overline{B}^t_\Gamma)$, then $[\phi] = 0$, where S is the suspension map in cyclic cohomology.*

Proof: Suppose $\lambda[\phi] = \tau_\phi = S\tau$. Define the function ψ on Γ^{n-1} by:

$$\psi(y_0, \cdots, y_{n-2}) = \tau(\delta_{y_0,y_1}, \cdots, \delta_{y_{n-3},y_{n-2}}, \delta_{y_{n-2},y_0}),$$

where $\delta_{x,y}$ is the same as in the proof of the previous lemma. By the same argument as in the proof of the previous lemma we know that $Supp\psi \cap \{z | z \in \Gamma^{n-1}, \ and\ dist(z, \Delta) \leq r\}$ is finite for each $r > 0$. Now routine calculation shows:

$$\tau_\phi = S\tau_\psi$$

and it follows that

$$\phi(x_0, \cdots, x_n) = \sum_{i,j} (-1)^{i-j} \psi(x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n)$$

$$= \sum_i (-1)^i f(x_0, \cdots, \hat{x}_i, \cdots, x_n)$$

where

$$f(x_0, \cdots, x_{n-1}) = \sum_j (-1)^j \psi(x_0, \cdots, \hat{x}_j, \cdots, x_{n-1})$$

and the 'hat' denotes the ommission of the specified term. Hence ϕ is a coboundary in exotic cohomology. QED.

Now we are ready to prove the main result of this section.

Proof of theorem 3.3.1: For any complete locally convex topological algebra A with a $\mathbb{Z}$ action by automorphisms Nest's method can be used to prove the following exact sequence:

$$\cdots \to H_\lambda^{m-1}(A) \to H_\lambda^{m-1}(A) \to H_\lambda^m(A \rtimes_a \mathbb{Z})_{hom} \to H_\lambda^m(A) \to H_\lambda^m(A) \to \cdots$$

and the fact that the periodic cyclic cohomology of $A \rtimes_a \mathbb{Z}$ can be computed by the "homogeneous" cyclic cohomology (the proof in this case is even simpler than the case in [28] since we have the algebraic tensor product here). Now the same argument as in proving theorem 3.2.1 can be used to prove that every continuous cyclic cocycle over $\overline{B}_\Gamma^t$ has dimension at most d. Combining this fact with the previous two lemmas we see that S. Hurder's conjecture is proved. QED.

3.4 Open Questions

Results in sections 2,3,4 of chapter 2 indicate that the K theory of the Roe algebra and the K theoretic index of the Dirac operator are related only to the asymptotic behavior of the manifold at infinity. It hints that the K theory of the Roe algebra is intimately related to a certain "geometric boundary" of the manifold.

(1) Can one establish an explict relation between the K theory of the Roe algebra and the topology of a certain (computable) "geometric boundary" of the manifold? If the above question has a positive answer, it is natural to ask: can one express $IndD$ in terms of the topological data of this "geometric boundary"?

If Γ is a finitely generated group, then one can prove that

$$k_0(\overline{B}_\Gamma) \cong K_0(C(\beta\Gamma \setminus \Gamma) \rtimes \Gamma),$$

where $\beta\Gamma$ is the Čech compactification of Γ (Γ is endowed with the discrete topology) and the Γ action on $C(\beta\Gamma\setminus\Gamma)$ is induced by the left translation action on Γ. $C(\beta\Gamma \setminus \Gamma) \rtimes \Gamma$ can be considered as the "noncommutative boundary" of Γ. Unfortunately the computation of $K_0(C(\beta\Gamma \setminus \Gamma) \rtimes \Gamma)$ is not always easy.

(2) We know that the cyclic cohomology of the group algebra $\mathbb{C}\Gamma$ can be expressed in terms of the topology of Γ [7]. Can one similarly compute the cyclic cohomology of $B_\Gamma \cong \mathbb{C}\Gamma \rtimes \Gamma$? An answer to this question might shed some light on the first question.

(3) Does there exist a nontrivial p-summable Fredholm module over the Roe algebra which is induced by a special compactly supported vector bundle. A positive answer to this question would lead to the solution of the following open problem proposed by A. Connes: Does there exist a nontrivial p-summable Fredholm module over $C_c^\infty(G)$, the convolution algebra for a real semisimple Lie group G [12].

(4) Is the converse of theorem 2.2.1 true? i.e. if M is a complete spin manifold

and the K theoretic index of the Dirac operator on M is zero, does there exist a metric with uniformly positive scalar curvature near infinity in the strict quasi-isometry on M? I suspect that this is not true in general.

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