Geometric and topological properties of manifolds with completely integrable geodesic flows

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To my parents, Selva and Alejandro

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Introduction

The study of completely integrable geodesic flows (and Hamiltonian systems in general) has regained momentum in recent years, as new techniques have been discovered to construct examples. Let us recall that a geodesic flow is said to be completely integrable if it admits a maximal number of independent conservation laws (i.e. first integrals) that Poisson-commute. Classical examples are given by n-dimensional ellipsoids with different principal axes (Jacobi, 1838), left invariant metrics on $SO(3)$ (Euler, 1765), surfaces of revolution ("Clairaut's first integral"), and flat tori.

In part due to Poincaré's realization that complete integrability was a rare phenomenon, the subject went through a period in which very little development occured. In the past decades the study of Hamiltonian actions and the geometry of the moment map provided the necessary framework for a solid theory of symmetries. As a consequence, new examples appeared. In 1978, Mishchenko and Fomenko [28] constructed left invariant metrics on semisimple Lie groups with completely integrable geodesic flows. Then Thimm [33] devised a new method for constructing first integrals in involution on homogeneous spaces. In particular he was able to show that the geodesic flow on real or complex Grassmannians is completely integrable. Guillemin and Sternberg [18] strengthened this method and obtained further examples. Very recently Spatzier and the author [31] constructed the first non-homogeneous examples using riemannian submersions. We were able to show that spaces like Eschenburg's strongly inhomogeneous 7-manifold [7], $\mathbf{CP^n}\#\mathbf{CP^n}$ for n odd and the exotic 7-sphere constructed by Gromoll and Meyer [12], have metrics with completely integrable geodesic flows.

A natural question arises: What are the geometric and topological properties of a compact riemannian manifold whose geodesic flow is completely integrable? Some topological features are shared by all the previous examples and we would like to draw attention to them. Following Grove and Halperin [15] we will say that a compact manifold M^n with finite fundamental group is *rationally elliptic* if the total rational homotopy of M, $\pi_*(M)\otimes\mathbf{Q}$ is finite dimensional. Homogeneous spaces are known to have this property, although is rather restrictive (cf. Section 2.1).

Rational ellipticity is shared by all the known examples of manifolds with completely integrable geodesic flows. Also in all of them $\pi_1(M)$ has polynomial growth. We should mention that recently, Taymanov [10] proved that in the real analytic case, $\pi_1(M)$ is almost abelian.

Before we state our results let us set some terminology.

Let G be a compact connected Lie group acting by Hamiltonian transformations on a symplectic manifold X with moment map $\phi : X \to g^*$ (cf. Section 1.1 for definitions and properties of the moment map). We will say that the action has *multiplicity* k if for generic $x \in X$, the symplectic reduction of $Ker\ d\phi_x$ (i.e. the quotient of $Ker\ d\phi_x$ by its null subspace) has dimension k. Since the symplectic reduction of a subspace is naturally symplectic, k can take only even values. If $k=0$, then $Ker\ d\phi_x$ is isotropic for generic $x \in X$ and we obtain the notion of *multiplicity free action* introduced and studied by

Guillemin and Sternberg in [18, 19].

Let H be a G-invariant Hamiltonian, ξ_H its Hamiltonian vector field and $H^{-1}(a) = N$ a compact regular level surface. Let $h_{top}(H)$ denote the topological entropy (cf. Section 2.2) of the flow of ξ_H restricted to N.

Theorem 4.3.2 *If the action of G has multiplicity zero or two, then $h_{top}(H) = 0$.*

Let us now describe some of the interesting consequences that Theorem 4.3.2 has in the case of geodesic flows. Let M be a compact riemannian manifold. If the topological entropy of the geodesic flow is zero then $\pi_1(M)$ has sub-exponential growth [6]. Moreover, we will see in Section 2.3 that if $\pi_1(M)$ is finite, M is rationally elliptic (Corollary 2.3.2).

Thus from Theorem 4.3.2 we obtain:

Theorem 4.4.1 *Let M be a compact manifold whose cotangent bundle admits a compact Hamiltonian G-action with multiplicity $k \leq 2$. Assume the set of G-invariant functions on T^*M contains the Hamiltonian associated with some riemannian metric. Then $\pi_1(M)$ has sub-exponential growth and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Observe that Theorem 4.4.1 and thus Theorem 4.3.2 are false for $k \geq 4$. For example $M = S^2 \times S^2 \# S^2 \times S^2$ is not a rationally elliptic manifold, and admits a 2-torus action (cf. Section 2.1) . The lift of this action to the cotangent bundle of M has mutiplicity $k = 4$. Any riemannian metric invariant under the torus action gives rise to a geodesic flow with positive topological entropy.

The idea behind Theorem 4.4.1 is very simple. If the geodesic flow admits a sufficiently large group of symmetries ($k = 0, 2$), then M has severe topological restrictions (rational ellipticity).

Let us now describe briefly why actions with multiplicity ≤ 2 are relevant to complete integrability. A function of the form $f \circ \phi$, for $f : g^* \to \mathbf{R}$ is called *collective*. We will prove (cf. Proposition 4.1.5) that if there exist $f_1, ..., f_s$ in $C^\infty(g^*)$ such that $f_1 \circ \phi, ..., f_s \circ \phi$ are s-independent functions that Poisson-commute on X^{2n}, then the multiplicity of the action is $\leq 2(n - s)$. Observe that if $s = n$, that is, if we can find a full set of commutative collective Hamiltonians, then the action is multiplicity free. This was proved in [18]. Note also that a G-invariant Hamiltonian H is also completely integrable if it admits $n - 1$ independent commuting collective integrals besides H. In this case the action has multiplicity ≤ 2.

Most of the known examples of completely integrable geodesic flows arise by considering collective integrals as above. The Thimm method (cf. [18, 33]) fits into this framework.

Let (M^n, g) be a compact riemannian manifold whose geodesic flow is completely integrable with first integrals $F_1 = \| \cdot \|_g, F_2, ..., F_n$. We will say that the geodesic flow is completely integrable with *collective integrals* if the functions F_i, $2 \leq i \leq n$ are collective with respect to the action of some compact Lie group G that leaves the Hamiltonian associated with the riemannian metric invariant.

Theorem 4.4.4 *Let M^n be a compact riemannian manifold whose geodesic flow is completely integrable with collective integrals. Then $\pi_1(M)$ has sub-exponential growth and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Suppose X is the cotangent bundle of a manifold M and G acts by derivatives. If the action is multiplicity free then M is a homogeneous space G/K [19, pag 43]. In this case one calls (G, K) a *Gelfand pair*. When G is simple, all possible Gelfand pairs have been classified by Kramer in [23]. For $SU(n)$ for example, the possible K are $SO(n)$, $Sp([n/2])$, $U(1).Sp([n/2])$ and $S(U(k) \times U(l))$ where $k + l = n$. If the action of G does not arise from an action on M, then M does not need to be homogeneous. In Section 4.2 we give examples of this situation. In fact we will prove that on the cotangent bundle of $\mathbf{CP^n} \# - \mathbf{CP^n}$ there is a multiplicity free action of the group $SU(n) \times T^2$. Certain sphere bundles over the Grassmannian $G_{n-1,2}(\mathbf{R})$ provide examples as well.

In the case of a 4-dimensional symplectic manifold the vanishing of the topological entropy can be obtained under different hypothesis on the integrals.

Theorem 3.3.1 *Let H be a Hamiltonian system on a 4-dimensional symplectic manifold. Suppose H is completely integrable, and on some compact regular level surface N the integral f satifies either one of the following conditions:*

(a) f is real analytic.

(b) The connected components of the set of critical points of f form submanifolds.

Then the system restricted to N has topological entropy $h_{top} = 0$.

Combining this theorem with results of Dinaburg [6] we obtain:

Corollary 3.3.2 *Let M^2 be a compact connected surface. Assume M^2 supports a geodesic flow that is completely integrable by means of an integral as in Theorem 3.3.1. Then $\chi(M^2) \geq 0$.*

Corollary 3.3.2 was proved by Kozlov [22] in the case of a real analytic function by completely different methods. If we assume condition (b), the integral could even be of class C^1.

Now take two points p and q in M. Denote by $n(p,q,\lambda)$ the number of geodesics connecting p and q with length $< \lambda$. Define

$$N(\lambda) = \int_{M\times M} n(p,q,\lambda)$$

(we will prove in Section 2.4 that if p and q are not conjugate $n(p,q,\lambda)$ is finite and that the set of points for which this does not happen has measure zero). The growth of this function can be viewed as a measure of the complexity of the geometry of geodesics. Basic properties of this function as well as its connection with the topological entropy are studied in Chapter 2.

We also obtain:

Theorem 4.4.5 *Let G/K be a homogeneous space such that the action of G on $T^*(G/K)$ has multiplicity zero or two. Then for any left invariant metric on G/K, $N(\lambda)$ grows sub-exponentially.*

Examples of homogeneous spaces such that the action of G on $T^*(G/K)$

has multiplicity two are the Stiefel manifold $SO(n+1)/SO(n-1)$ and the Wallach manifold $SU(3)/T^2$ (cf. Example 4.1.4).

In Chapter 5, using quite different techniques, we will prove that $N(\lambda)$ grows at most like λ^2 for any compact convex surface of revolution. In general, we will prove that in the real analytic case we have:

Theorem 5.3.1 *If M^n admits a codimension-one torus action by isometries and $Ric_M > 0$, then $N(\lambda)$ grows at most like λ^n.*

As an added consequence of some of the ideas used in the proof of the last theorem we get a result on closed geodesics that we now describe:

Consider a left invariant metric on $SO(3)$ defined by

$$< X, X >= \frac{X_1^2}{I_1} + \frac{X_2^2}{I_2} + \frac{X_3^2}{I_3}.$$

Let $SO(2)$ be any one-parameter subgroup. Then $SO(2)$ acts on $SO(3)$ from the left by isometries. The quotient, M_{I_1,I_2,I_3} is a 2-sphere and we endow it with the submersion metric. This corresponds to the classical "Poisson reduction" and M is called *the Poisson sphere* [2]. It follows from a theorem of Lusternik and Schnirelmann [21] and estimates of Klingenberg and Toponogov that any convex metric on S^2 whose Gaussian curvature satisfies $1/\Delta < K < \Delta$, has at least three geometrically different closed geodesics with length in $(2\pi/\sqrt{\Delta}, 2\pi\sqrt{\Delta})$. That this is optimal is shown by a result of Morse:

Given any constant $N > 2\pi$ there exists an $\epsilon > 0$ such that any prime closed geodesic on an ellipsoid

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 1, \;\; a_1 < a_1 < a_3$$

and $\mid 1 - a_i \mid < \epsilon$, is either a principal ellipse or is larger than N.

We will prove a similar result for the Poisson sphere:

Theorem 5.5.1 *Given $N > 2\pi$ there exists an $\epsilon > 0$ such that any prime closed geodesic on the Poisson sphere M_{I_1,I_2,I_3} with $\mid 1 - I_i \mid < \epsilon$ has length $> N$ except for three closed geodesics with length close to 2π.*

Complete integrability guarantees via Liouville's Theorem the existence of an open dense subset U of the unit tangent bundle of M that is foliated by tori, and the geodesic flow on these tori is quasiperiodic. We can also prove:

Theorem 5.6.1 *Let M^n be a compact riemannian manifold whose geodesic flow is completely integrable. Suppose for some p in M the unit sphere at p is contained in U. Then $\pi_1(M)$ has polynomial growth of degree $\leq n$ and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Chapter 1

The moment map

1.1 Hamiltonian actions and moment map

All the results in this chapter are taken from [17].

Let X be a symplectic manifold with symplectic form ω. Given a C^∞-function f on X let ξ_f be the associated Hamiltonian vector field, i.e. $\omega(\xi_f, *) = df(*)$. The map

$$\tau : C^\infty \to Symplectic\ vector\ fields$$

sending $f \to \xi_f$ is a morphism of the Poisson algebra $C^\infty(X)$ into the Lie algebra of symplectic vector fields. Suppose now that a connected Lie group G, acts symplectically on X. Let g be its Lie algebra. Associated with the action of G is a Lie algebra morphism

$$\beta : g \to Symplectic\ vector\ fields.$$

The action is said to be *Hamiltonian* if there exists a Lie algebra morphism

$$\alpha : g \to C^\infty(X)$$

such that $\tau \circ \alpha = \beta$. To each point, $x \in X$ we can associate an element l_x of the dual space g^*, by the identity

$$l_x(\zeta) = \phi^\zeta(x)\ ,\ \forall \zeta \in g$$

where $\phi^\zeta = \alpha(\zeta) \in C^\infty(X)$. If we let x vary, we get a map

$$\phi : X \to g^*, \quad x \to l_x.$$

This map is called the *moment map*. It is easily seen to be G-equivariant if we consider on g^* the coadjoint action. Thus if G_c denotes the stabilizer of a point $c \in g^*$, then G_c leaves $\phi^{-1}(c)$ invariant. Another way of describing the moment map is as follows. Let $\zeta \in g$ and let $e_\zeta : g^* \to \mathbf{R}$ be the linear form on g^*: $f \in g^* \to f(\zeta)$. Let $\phi^\zeta = e_\zeta \circ \phi$ and let $\zeta^\sharp$ be the vector field on X associated with ζ by $\zeta^\sharp = \beta(\zeta)$. Then

$$\omega(\zeta^\sharp, *) = d\phi^\zeta(*).$$

This equation determines ϕ^ζ up to an additive constant.

Let us now describe some of the most relevant properties of the moment map.

Proposition 1.1.1 *The symplectic perpendicular to the tangent space to the orbit at x is $Ker\ d\phi_x$. The image of $d\phi_x$ is the annihilator of the Lie algebra of the stabilizer group of x. In particular, $d\phi_x$ is surjective if and only if the stabilizer of x is discrete.*

Recall the following definition. Let $f : X \to Y$ be a smooth map between two differentiable manifolds, and suppose W is an embedded submanifold of Y. We say that the map f intersects W *cleanly* if

(i) $f^{-1}(W)$ is a submanifold of X,

(ii) at each $x \in f^{-1}(W)$, $T_x f^{-1}(W) = df_x^{-1}(T_{f(x)}W)$.

Now we have the following proposition:

Proposition 1.1.2 *If $\phi : X \to g^*$ intersects an orbit $\mathcal{O}$ cleanly, then $\phi^{-1}(\mathcal{O})$ is coisotropic and the null foliation through a point $x \in \phi^{-1}(\mathcal{O})$ is the orbit of x under $G^o_{\phi(x)}$, the connected component of the isotropy group of $\phi(x)$.*

Assume now that the group G is compact. Then G_c is always connected. Let X^* denote the set of principal orbits of G. As is well known, X^* is an open connected dense set of X. Let B_x denote the vector subspace of T_xX spanned by $\xi_f(x)$ where f is a G-invariant function. Then we have:

Proposition 1.1.3 *The restriction of ϕ to X^* intersects every orbit in $\phi(X^*)$ cleanly. Moreover, for each $x \in X^*$, $Ker\ d\phi_x = B_x$. In addition, if dim $\mathcal{O}_{\phi(x)}$ is maximal among all G-orbits in $\phi(X^*)$, then $G_{\phi(x)}/G^o_x$ is abelian.*

Next, let $H \subset G$ be a closed subgroup and let $X_H = \{x \in X : G_x = H\}$.

Proposition 1.1.4 *Let the compact Lie group G act as a group of symplectic transformations of the symplectic manifold X and let H be a closed subgroup of G. Then X_H is a symplectic submanifold. If the action of G is Hamiltonian with moment map ϕ, then ϕ maps each connected component of X_H into an affine subspace of g^* of the form $p + h^o$, where h^o denotes the annihilator of h in g^*. If H is a normal subgroup, then X_H is G-invariant, and the restriction of the moment map to each component is a submersion onto an open subset of the affine subspace $p + h^o$.*

1.2 Collective motion

Let G be a connected Lie group acting on a symplectic manifold X by Hamiltonian transformations. Let $\phi : X \to g^*$ denote the associated moment map. A Hamiltonian H on X is said to be *collective* if it is a pullback by the moment map of a smooth function f on g^*, i.e. $H = f \circ \phi$.

Let us explain now, what ingredients go into the solution of a collective Hamiltonian. Recall that a function $f \in C^\infty(g^*)$ defines a map $L_f : g^* \to g$ by the formula: $L_f(c)(\alpha) = df_c(\alpha)$. The map L_f is sometimes known as the *Legendre transformation* associated with f. The following important relation holds:

$$\xi_H(x) = L_f(\phi(x))^\sharp(x) \quad for\ all\ x \in X. \tag{1.1}$$

It follows from equation (1.1) that if $x(t)$ denotes the trajectory of the Hamiltonian system ξ_H with $x(0) = x$, then $x(t)$ lies entirely on the orbit of G through x and hence $\phi(x(t))$ lies entirely on the orbit $\mathcal{O}$ through $\phi(x)$. Moreover $\gamma(t) = \phi(x(t))$ is a solution of the Hamiltonian system corresponding to $f_{\mathcal{O}} = f/\mathcal{O}$. Set $\zeta(t) = L_f(\gamma(t))$, then equation (1.1) says that $x'(t) = \zeta(t)^\sharp(x(t))$. So we can find the solution curve by applying the following three steps:

(1) Find the orbit $\mathcal{O}$ through $\phi(x)$.

(2) Find the solution to the Hamiltonian system on $\mathcal{O}$ corresponding to $f_{\mathcal{O}}$ passing through $\phi(x)$ at $t = 0$. Call this curve $\gamma(t)$.

(3) Compute the curve $\zeta(t) = L_f(\gamma(t))$. This is a curve in g. Solve the differential equation (i.e. find the curve in G satisfying)

$$a'(t) = \zeta(t)a(t), \quad a(0) = e.$$

Then $a(t)x$ is the desired solution curve.

Remark 1.2.1 Suppose for instance that f is invariant. Then, on each orbit $\mathcal{O}$, $\gamma(t)$ is constant, but the map L_f need not be trivial. Thus $\zeta(t)$ will be a constant element of g, and $a(t)$ will be a one-parameter subgroup. Therefore the motion corresponding to $f \circ \phi$ when f is invariant is given by the action of a one-parameter subgroup, the one-parameter subgroup depending on x.

Chapter 2

Entropy and rational homotopy

2.1 Rationally elliptic manifolds

In what follows, manifolds will always be assumed to be closed connected and simply connected.

As it was proved by Felix and Halperin [8], the class of n-manifolds M^n (or more generally the simply connected topological spaces of the rational homotopy type of a CW-complex) is divided into two subclasses: either

(a) $\pi_p(M)$ is finite for all $p > 2n-1$, or

(b) the integers $\rho_p = \sum_{q \leq p} dim\ \pi_q(M) \otimes \mathbf{Q}$ grow exponentially in p (i.e. $\exists C > 1$, $\exists k \in \mathbf{N}$: $p > q \Rightarrow \rho_p \geq C^p$).

Following Grove and Halperin [15] manifolds in class (a) are called *rationally elliptic*; the rest (those in class (b)) are called *rationally hyperbolic.*

The "generic" manifold is rationally hyperbolic; rational ellipticity is a severely restrictive condition, albeit satisfied by all simply connected homogeneous spaces and manifolds that admit a codimension-one compact action

[16].

For instance if M is rationally elliptic then:

(1) $dim\ \pi_*(M) \otimes \mathbf{Q} \leq n$,

(2) $dim\ H_*(M, \mathbf{Q}) \leq 2^n$,

(3) the Euler-Poincaré characteristic $\chi_M \geq 0$,

(4) $\chi_M > 0$ if and only if $H_p(M, \mathbf{Q}) = 0$ for all p odd.

Let $b_i(\Omega M, \mathbf{Q})$ denote the Betti numbers $dim\ H_i(\Omega M, \mathbf{Q})$. It is pointed out in [15] that rational ellipticity is equivalent to the property that the integers $\mu_p = \sum_{q \leq p} b_q(\Omega M, \mathbf{Q})$ grow only sub-exponentially in p. We outline the argument for this equivalence. By a theorem of Milnor and Moore, $\pi_*(\Omega M) \otimes \mathbf{Q}$ is isomorphic to the Lie algebra of primitive elements of the Hopf algebra $H_*(\Omega M, \mathbf{Q})$. Then it is enough to prove that $H_*(\Omega M, \mathbf{Q})$ is finitely generated. Using a structure theorem for Hopf algerbras we can write $H_*(\Omega M, \mathbf{Q})$ as a tensor product of a polynomial algebra with only even dimensional generators and an exterior algebra with only odd dimensional generators. Now it is not hard to see that if these generators are not finite, the sum of the Betti numbers of the loop space grows exponentially.

Interesting examples of rationally hyperbolic manifolds are $S^k \times S^l \# S^k \times S^l$, $k \geq l \geq 2$ [16]. They admit a codimension two-action of the group $SO(k) \times SO(l)$.

2.2 Topological entropy

The topological entropy $h_{top}(g)$ of a continuous flow g_t on a compact metric space (X, d) may be defined as $h_{top}(g_1)$ using the entropy of the time one-map or it may be defined in either of the following ways. All three definitions give the same number h_{top} which is independent of the choice of metric [27].

A set $Y \subset X$ is called a (T, δ)-separated set if given different points $y, y' \in Y$ there exists $t \in [0, T]$, such that $d(g_t y, g_t y') \geq \delta$. Let $N(T, \delta)$ denote the maximal cardinality of a (T, δ)-separated set. Then

$$h_{top}(g) = sup_{\delta > 0} limsup_{T \to \infty} \frac{1}{T} log N(T, \delta).$$

A set $Z \subset X$ is called a (T, δ)-spanning set if for all $x \in X$ there exists $z \in Z$ such that $d(g_t x, g_t z) \leq \delta$ for $t \in [0, T]$. Let $M(T, \delta)$ denote the minimal cardinality of a (T, δ)-spanning set. Then

$$h_{top}(g) = sup_{\delta > 0} limsup_{T \to \infty} \frac{1}{T} log M(T, \delta).$$

Remark 2.2.1 Given a compact subset $K \in X$ (not necessarily invariant) we can define the topological entropy of the flow respect to the set K, $h_{top}(g, K)$, simply by considering separated (spanning) sets of K [4].

The topological entropy of a flow g_t is a number which roughly measures the orbit structure complexity of g_t. Flows with positive entropy exhibit complicated dynamics, and the larger the entropy is, the more complicated the dynamics is.

Let $\omega(\gamma)$ denote the ω-limit set of the orbit γ. If $\gamma \subset \omega(\gamma)$, then γ is said to be *recurrent*. If γ is not a critical orbit (i.e. a fixed point or a closed orbit) then it is said to be *non-trivially recurrent.*

The following proposition gives an idea of the dynamical significance of the topological entropy.

Proposition 2.2.2 *The topological entropy verifies the following properties:*
(i) For any two closed subsets Y_1, Y_2 in X,

$$h_{top}(g, Y_1 \cup Y_2) = max_{i=1,2} h_{top}(g, Y_i).$$

(ii) If $Y_1 \subset Y_2$ then $h_{top}(g, Y_1) \leq h_{top}(g, Y_2)$.
(iii) Let $g_t^i : X_i \to X_i$ for $i = 1, 2$ be two flows and let $\pi : X_1 \to X_2$ be a continuous map commuting with g^i i.e. $g_t^2 \circ \pi = \pi \circ g_t^1$. If π is onto, then $h_{top}(g^1) \geq h_{top}(g^2)$ and if π is finite-to-one, then $h_{top}(g^1) \leq h_{top}(g^2)$.
(iv) Suppose that X is separable. If g_t only has trivial recurrence, then $h_{top}(g) =$ 0.

Proof: The first three properties are fairly standard, so we omit the corresponding proofs. To prove (iv) assume $h_{top}(g) > 0$. The variational property of the entropy [25] says

$$h_{top}(g) = sup_{\mu \in \mathbf{M_{erg}}} h_\mu(g),$$

where $h_\mu(g)$ denotes the measure-theoretical entropy with respect to μ and $\mathbf{M_{erg}}$ the set of all g_t-invariant ergodic probability measures. Therefore we can assume that for some $\mu \in \mathbf{M_{erg}}$, $h_\mu(g) > 0$.

Since X is separable, the Poincaré recurrence Theorem [25] implies that the set of recurrent orbits has full μ-measure. But we only have trivial recurrence and μ is ergodic, then μ has to be supported on a single orbit, which in turn implies that $h_\mu(g) = 0$.

$\diamond$

Next we will state some results of Bowen that we will need later.

Theorem 2.2.3 [4, Theorem 17] *Let (X, d) and (Y, e) be compact metric spaces and $g_t : X \to X$, $f_t : Y \to Y$ continuous flows. Let $\pi : X \to Y$ be a surjective continuous map so that $\pi \circ g_t = f_t \circ \pi$. Then*

$$h_{top}(g) \leq h_{top}(f) + sup_{y \in Y} h_{top}(g, \pi^{-1}(y)).$$

Corollary 2.2.4 [4, Corollary 18] *Let (X, d) and (Y, e) be compact metric spaces and $g_t : X \to X$ a flow. Suppose $\pi : X \to Y$ is a continuous surjective map such that $\pi \circ g_t = \pi$. Then*

$$h_{top}(g) = sup_{y \in Y} h_{top}(g, \pi^{-1}(y)).$$

Let (X, d) be a compact metric space and G a compact Lie group that acts on X by isometries. Then the quotient space $Y = X/G$ is a compact metric space with the induced topology. Let $\pi : X \to Y$ denote the projection map.

Theorem 2.2.5 [4, Theorem 19] *Let $g_t : X \to X$ and $f_t : Y \to Y$ be continuous flows such that $\pi \circ g_t = f_t \circ \pi$. Suppose g_t commutes with the action of*

G. Then

$$h_{top}(g) = h_{top}(f).$$

2.3 Geodesic entropy and topological entropy

Let M be a n-dimensional compact riemannian manifold. Fix $\lambda > 0$ and $p \in M$. For each $q \in M$ define $n_{p,\lambda}(q)$ as the number of geodesics connecting p and q with length $< \lambda$. Set

$$I_p(\lambda) = \int_M n_{p,\lambda}.$$

In [3] it was proved that this integral is well defined and

$$I_p(\lambda) = \int_0^\lambda dt \int_{S_p} \mid det A_v(t) \mid d\nu, \tag{2.1}$$

where $A_v(t)$ is the unique family of linear maps along the geodesic defined by v verifing the Jacobi equation with initial conditions $A_v(0) = 0$ and $A_v'(0) = Id$. The unit sphere at p is denoted by S_p and ν stands for its canonical measure.

We define the *geodesic entropy at* p by

$$\sigma_p = limsup_{\lambda \to +\infty} \frac{1}{\lambda} log I_p(\lambda).$$

Let $h_{top}(g)$ denote the topological entropy of the geodesic flow on the unit tangent bundle. The following theorem is an improvement of Manning's inequality [27]. It is basically contained in [13].

Theorem 2.3.1 *For every $p \in M$ we have*

$$\sigma_p \leq h_{top}(g).$$

Proof: We will make use of Yomdin's Theorem. It gives a lower bound for $h_{top}(g)$ in terms of growth rates of volumes of iterates submanifolds under g_t. Fix $p \in M$ and set $Y = S_pM$. According to Yomdin's Theorem (see [34]) we have:

$$h_{top}(g) \geq limsup_{t \to +\infty} \frac{1}{t} log Vol(g_t Y) \stackrel{\text{def}}{=} \Delta$$

where Vol stands for the $n-1$-dimensional Riemannian volume on the unit tangent bundle. Hence we only need to prove that $\Delta \geq \sigma_p$. Set $V(t) = Vol(g_t Y)$ and observe using equation (2.1) that

$$I_p(t) = \int_0^t dr \int_{S_pM} \mid det\ A_v(r) \mid d\nu$$

$$\leq \int_0^t dr \int_{S_pM} \mid det\ (dg_r \mid_{T_v S_p M}) \mid d\nu$$

$$= \int_0^t V(r) dr.$$

Given $\epsilon > 0$ there exists $T(\epsilon)$ such that if $t \geq T(\epsilon)$ then $V(t) \leq e^{(\Delta+\epsilon)t}$. Thus

$$I_p(t) \leq \int_0^{T(\epsilon)} V(r) dr + \int_{T(\epsilon)}^t V(r) dr$$

$$\leq \int_0^{T(\epsilon)} V(r) dr + \frac{e^{(\Delta+\epsilon)t}}{\Delta + \epsilon}.$$

Hence we get

$$limsup_{t \to +\infty} \frac{1}{t} log I_p(t) \leq \Delta + \epsilon$$

for all $\epsilon > 0$ i.e.

$$\sigma_p \leq \Delta.$$

$\diamond$

Let $B_{\hat{p}}(\lambda)$ be a geodesic ball in $\hat{M}$ the universal covering of M. Let $pr : \hat{M} \to M$ denote the covering projection. From equation (2.1) for $I_p(\lambda)$ it is clear that $Vol(B_{\hat{p}}(\lambda)) \leq I_{pr(\hat{p})}(\lambda)$. Therefore if μ denotes the growth rate of volume of balls in the universal covering, we clearly have $\mu \leq \sigma_p$. Hence Theorem 2.3.1 sharpens Manning's inequality. As we will see below the geodesic entropy is particulary relevant in the simply connected case, while $\mu = 0$ automatically. We also observe that if M does not have conjugate points then by the results in [11] the three quantities are the same: $\mu = \sigma_p = h_{top}(g)$.

Corollary 2.3.2 *Let M be a compact riemannian manifold with finite fundamental group. If $h_{top}(g) = 0$ then M is rationally elliptic.*

Proof: In [14] Gromov proved that if M is compact and its fundamental group is finite there exists a constant c depending only on the geometry of M such that whenever p and q are not conjugate

$$n_{p,\lambda}(q) \geq \sum_{i=1}^{c(\lambda-1)} b_i(\Omega M, \mathbf{Q}),$$

where $b_i(\Omega M, \mathbf{Q})$ are the rational Betti numbers of the loop space of M. Hence we deduce that ($\lambda \geq 1$):

$$V(M) \sum_{i=1}^{c(\lambda-1)} b_i(\Omega M, \mathbf{Q}) \leq I_p(\lambda)$$

and this implies

$$limsup_{m \to +\infty} \frac{1}{m} log(\sum_{i=1}^{m} b_i(\Omega M, \mathbf{Q})) \leq \frac{\sigma_p}{c}.$$

Since $h_{top}(g) = 0$, we get $\sigma_p = 0$ by Theorem 2.3.1. This implies that the sum of the Betti numbers of the loop space grows sub-exponentially. As we know (cf. Section 2.1) this property is equivalent to rational ellipticity.

$\diamond$

2.4 Average growth of geodesics

Fix $\lambda > 0$. Take two points p and q in a compact riemannian manifold M and define $n_\lambda(p,q)$ to be the number of geodesics connecting p and q with length $< \lambda$. Consider the set $A = \{(p,q) \in M \times M : p \textit{ and } q \textit{ are conjugate}\}$. We claim that A has measure zero in $M \times M$. Call μ the standard measure on M and $\nu = \mu \times \mu$. Set $A_p = \{q \in M : (p,q) \in A\}$. Sard's theorem implies that $\mu(A_p) = 0$ because $q \in A_p$ if and only if exp_p has q as a singular value. Define $g(p) = \mu(A_p)$. Then if $\mu(M) < +\infty$ a standard result from measure theory says that g is measurable and $\int g d\mu = \nu(A)$. Hence $g \equiv 0$ and $\nu(A) = 0$. Standard Morse Theory now guarantees that if (p,q) is not in A, then $n_\lambda(p,q)$ is finite. Hence n_λ is a well defined measurable function on $M \times M$ for each $\lambda > 0$. Hence if we set

$$N(\lambda) \stackrel{\text{def}}{=} \int_{M\times M} n_\lambda,$$

Fubini's Theorem implies

$$N(\lambda) = \int_M I_p(\lambda) d\mu(p)$$

and using Fubini's Theorem again and equation (2.1) we get

$$N(\lambda) = \int_0^\lambda dt \int_{SM} \mid detA_v(t) \mid d\sigma, \tag{2.2}$$

where σ denotes the standard measure on SM. Hence we deduce that N is a C^1-function and its derivative is given by

$$\int_{SM} \mid detA_v(\lambda) \mid d\sigma.$$

Remark 2.4.1 Take $v \in T_pM$ with unit norm. Take the geodesic $\gamma(t) = exp_p tv$ and an orthonormal basis $\{v, e_2, ..., e_n\}$ in T_pM. Take Jacobi fields J_i such that $J_i(0) = 0$ and $J_i'(0) = e_i$. Recall that $d(exp_p)_{tv}v = \gamma'(t)$ and $d(exp_p)_{tv}te_i = J_i(t)$. Then $\mid detA_v(t) \mid = \sqrt{\mid det < J_i(t), J_j(t) > \mid}$. Thus

$$\mid detA_v(t) \mid = \sqrt{\mid det < J_i(t), J_j(t) > \mid} \leq \parallel J_2(t) \parallel ... \parallel J_n(t) \parallel .$$

The growth of $N(\lambda)$ can be viewed as a measure of the complexity of the geometry of geodesics. It also has the property of being a significant global geometric invariant. In general, the growth is very hard to predict, although as we will see below for "almost all" manifolds, i.e. for rationally hyperbolic manifolds the growth is exponential. Recall from the previous section the definition of σ_p, the geodesic entropy at p. Let us now define the *geodesic entropy*, $e(M)$ as:

$$e(M) = limsup_{\lambda \to +\infty} \frac{1}{\lambda} logN(\lambda).$$

Note that it follows from Jensen's inequality that $e(M)$ and σ_p are related by: $e(M) \geq \frac{1}{\mu(M)} \int_M \sigma_p d\mu(p)$.

Next we will prove:

Proposition 2.4.2 *The following properties hold:*

(a) For a compact symmetric space M^n, $N(\lambda)$ does not grow faster than λ^n. If the space has rank one, $N(\lambda)$ grows linearly.

(b) If M^n is flat, then $N(\lambda) = \frac{\lambda^n}{n} vol(SM)$.

(c) If M is rationally hyperbolic or $\pi_1(M)$ has exponential growth, then $e(M) >$ 0.

(d) If M has no conjugate points, $e(M) = h_{top}(g) = \sigma_p$ for all $p \in M$.

(e) If M is a homogeneous space, $e(M) = \sigma_p$ for all $p \in M$.

Proof: (a) On a compact symmetric space the Jacobi equation can be solved easily. The solutions have at most linear growth. Hence $\mid detA_v(t) \mid$ can be bounded uniformly in v by a polynomial of degree $n-1$. Then it follows from equation (2.2) and Remark 2.4.1 that $N(\lambda)$ grows at most like λ^n. If the symmetric space has rank one then all the orthogonal Jacobi fields are bounded, thus $N(\lambda)$ grows linearly.

(b) For flat manifolds the Jacobi equation reads $J'' = 0$. Then $\mid detA_v(t) \mid = t^{n-1}$ and the claim in (b) follows from equation (2.2).

(c) The same proof as in Corollary 2.3.2 shows:

$$limsup_{m\to+\infty} \frac{1}{m} log(\sum_{i=1}^{m} b_i(\Omega M, \mathbf{Q})) \leq \frac{e(M)}{c}.$$

Hence if M is rationally hyperbolic the sum of the Betti numbers of the loop space grows exponentially, thus $e(M) > 0$.

Next observe that since $\sigma_p \geq \mu$ we deduce that $e(M) \geq \mu$. But if $\pi_1(M)$ has exponential growth then $\mu > 0$, thus $e(M) > 0$.

(d) Suppose M has no conjugate points. Let K denote a fundamental domain for the action of $\pi_1(M)$ on the universal covering of M. Then it follows that

$$N(\lambda) = \int_K vol(B_{\hat{p}}(\lambda)).$$

But since the volume growth is independent of $\hat{p} \in K$ [27], from the last equation we deduce that $e(M) \leq \mu$. But we already know that $e(M) \geq \mu$. Thus $e(M) = \mu$. The results in [11] now imply

$$e(M) = \mu = h_{top}(g) = \sigma_p \quad for\ all\ p \in M.$$

Assertion (e) is trivial.

$\diamond$

Remark 2.4.3 As we just saw, if M does not have conjugate points then σ_p does not depend on p. In general the behaviour of the map $p \to \sigma_p$ is not very clear to us.

Next, we will determine the growth of $N(\lambda)$ for a certain class of homogeneous spaces, the so called *naturally reductive*. Let $M^n = G/K$ be a homogeneous space equipped with a left-invariant metric. Let $\pi : G \to G/K$ denote the canonical projection. Let g be the Lie algebra of G and k the Lie algebra of K. We can assume that G/K is *reductive*, i.e. there exists a complement p of k in g: $g = k \oplus p$ so that $Ad(K)$ leaves p invariant. The metric on $T_{[K]}G/K$ induces a metric on p denoted by $< , >$. M is called *naturally reductive* (with respect to the complement p) if $[X, .]_p : p \to p$ is skew-symmetric for all $X \in p$. Naturally reductive spaces include normal homogeneous spaces and hence symmetric spaces. An important property of naturally reductive spaces is that

$$Exp_{\pi(e)} = \pi \circ exp \mid_p \tag{2.3}$$

where Exp denotes the riemannian exponential map on M. Thus, geodesics in M are images of one-parameter subgroups of G [36].

Proposition 2.4.4 *Let $M^n = G/K$ be a naturally reductive homogeneous space with G compact. Then $N(\lambda)$ does not grow faster than λ^n.*

Proof: Since M^n is homogeneous it is enough to prove that $I_e(\lambda)$ does not grow faster than λ^n. We know that the Jacobi field along the geodesic defined by the unit vector $v \in p$ with initial conditions $J(0) = 0$ and $J'(0) = W \in p$ is given by

$$J(t) = d(Exp_{\pi(e)})_{tv}(tW).$$

Using equation (2.3) we get

$$J(t) = d\pi_{exptv} \circ dexp_{tv}(tW).$$

Endow G with a bi-invariant metric and denote its norm on tangent spaces by $\mid * \mid_G$. Since G is compact the last equation implies

$$\parallel J(t) \parallel \leq L \mid dexp_{tv}(tW) \mid_G$$

for some constant L. But since G endowed with a bi-invariant metric is a symmetric space we know that $\mid dexp_{tv}(tW) \mid_G$ is bounded uniformly by $(at + b) \mid W \mid_G$ for some constants a and b. Since norms in euclidean space are equivalent there exists c such that $| W |_G \leq c$ for any vector W such that $< W, W >= 1$. Hence from (2.1) and Remark 2.4.1 we get

$$I_e(\lambda) \leq (Lc)^{n-1} \int_0^\lambda dt \int_{S_e} (at+b)^{n-1} d\nu.$$

$\diamond$

Next, we will show that certain metrics over non-homogeneous spaces have $N(\lambda)$ also with polynomial growth. To be precise we will prove:

Proposition 2.4.5 *Let $\pi : M^n \to B^k$ be a riemannian submersion where $M = G/K$ is a naturally reductive homogeneous space with G compact. Then $N_B(\lambda)$ does not grow faster than λ^k.*

Proof: Take a point $x \in B$ and a unit vector $v \in T_xB$. Consider the geodesic γ_o defined by v. Let $J_o(t)$ be a Jacobi field along γ_o with $J_o(0) = 0$, $< J_o(0), \gamma_o'(0) >= 0$ and $< J_o'(0), J_o'(0) >= 1$. Since π is a riemmanian submersion we can lift γ_o to a horizontal geodesic γ in M. Moreover we can lift J_o to a Jacobi field (not necessarily horizontal) J along γ with $J(0) = 0$, $< \gamma'(0), J'(0) >= 0$ and $< J'(0), J'(0) >= 1$. Hence

$$\| J_o(t) \| \leq \| J(t) \| .$$

Now recall that in the proof of the previous proposition we proved that $\| J(t) \|$ is bounded uniformly by a linear polynomial $p(t)$. Thus

$$N_B(\lambda) \leq \int_0^\lambda dt \int_{SB} p(t)^{k-1}.$$

$\diamond$

Example 2.4.6 Let G be a compact Lie group endowed with a bi-invariant metric. Recall that $G \times G$ acts on G as follows. Given g_1, g_2 in G we have $(g_1, g_2).x = g_1 x g_2^{-1}$. Hence if $H \subset G \times G$ is a closed subgroup that acts freely on G we can consider the riemmanian submersion $\pi : G \to B = G/H$. Now the last proposition implies that for all the spaces thus constructed, $N_B(\lambda)$ does not grow faster than $\lambda^{dim\ B}$.

Concrete examples of this situation are the exotic 7-sphere constructed by Gromoll and Meyer [12] $\Sigma^7 = Sp(2)/\Delta$, and examples of Eschenburg [7]. These examples are *strongly inhomogeneous,* i.e. they do not have the homotopy type of any homogeneous space.

A natural question arises: What can be said about the growth of $N(\lambda)$ for an invariant metric on a compact homogeneous space? For example, $SO(3)$ with a generic left invariant metric is not naturally reductive because there exist Jacobi fields with exponential growth [35]. Hence direct estimates for $N(\lambda)$ as in the proof of the previous propositions become impossible. Here is where the topological entropy becomes very useful. As we pointed out vanishing of the toplogical entropy implies –in the homogeneous case– $e(M) = 0$, proving at least sub-exponential growth for $N(\lambda)$. We will address these questions in the next chapters.

Chapter 3

Some entropy computations

3.1 Entropy formula for collective Hamiltonians

Let G be a compact Lie group . Let X be a symplectic manifold on which G acts by Hamiltonian transformations with moment map $\phi : X \to g^*$. Take a collective Hamiltonian $H = f \circ \phi$ and let g_t denote the flow of ξ_H. In what follows, for a subset $A \subset X$, we will denote $h_{top}(g, A)$ also by $h_{top}(H, A)$. Recall that $\mathcal{O}_c$ denotes the orbit through c under the coadjoint action, and $f_{\mathcal{O}_c}$ stands for the restriction of f to $\mathcal{O}_c$.

Proposition 3.1.1 *If $A \subset X$ is any compact g_t-invariant subset we have*

$$h_{top}(H, A) = sup_{c \in \phi(A)} h_{top}(f_{\mathcal{O}_c}, \mathcal{O}_c \cap \phi(A)).$$

Proof: As we mention in Section 1.2, g_t leaves the orbits of G invariant. Hence it follows from Corollary 2.2.4 that

$$h_{top}(H, A) = sup_{x \in A} h_{top}(H, A \cap \mathcal{O}_x), \tag{3.1}$$

where $\mathcal{O}_x$ denotes the orbit of G through x. Let us compute now $h_{top}(H, A \cap \mathcal{O}_x)$.

Consider the map $\pi = \phi / \mathcal{O}_x : \mathcal{O}_x \to \mathcal{O}_{\phi(x)}$. By Theorem 2.2.3 we deduce that

$$h_{top}(H, \mathcal{O}_x) \leq h_{top}(f_{\mathcal{O}_{\phi(x)}}, \mathcal{O}_{\phi(x)}) + sup_{c \in \mathcal{O}_{\phi(x)}} h_{top}(H, \pi^{-1}(c)).$$

But now, according to the description of collective motion that we gave in Section 1.2, the curve $a(t)$ is the same for every $x \in \phi^{-1}(c)$. Hence for any $x \in \phi^{-1}(c)$, $g_t x = a(t)x$ and since G is compact this clearly implies $h_{top}(H, \pi^{-1}(c)) = 0$. Thus

$$h_{top}(H, \mathcal{O}_x) \leq h_{top}(f_{\mathcal{O}_{\phi(x)}}, \mathcal{O}_{\phi(x)}).$$

But according to part (iii) in Proposition 2.2.2 the reverse inequality holds. Thus

$$h_{top}(H, \mathcal{O}_x) = h_{top}(f_{\mathcal{O}_{\phi(x)}}, \mathcal{O}_{\phi(x)}),$$

and also

$$h_{top}(H, \mathcal{O}_x \cap A) = h_{top}(f_{\mathcal{O}_{\phi(x)}}, \mathcal{O}_{\phi(x)} \cap \phi(A)).$$

This equation together with equation (3.1) implies

$$h_{top}(H, A) = sup_{x \in A} h_{top}(f_{\mathcal{O}_{\phi(x)}}, \mathcal{O}_{\phi(x)} \cap \phi(A)) = sup_{c \in \phi(A)} h_{top}(f_{\mathcal{O}_c}, \mathcal{O}_c \cap \phi(A)).$$

$\diamond$

Corollary 3.1.2 *Suppose the energy surface $H^{-1}(a) = \phi^{-1}(f^{-1}(a))$ is compact. Then*

$$h_{top}(H, H^{-1}(a)) = sup_{c \in f^{-1}(a)} h_{top}(f_{\mathcal{O}_c}, f_{\mathcal{O}_c}^{-1}(a)).$$

$\diamond$

Let us discuss some applications.

Example 3.1.3 Let X be a compact Hamiltonian $SO(3)$-space. In this case the coadjoint orbits are two-spheres. Hence for any smooth function on $so(3)^*$ we have $h_{top}(f_{\mathcal{O}_c}) = 0$. Thus we deduce that for any collective Hamiltonian H, $h_{top}(H) = 0$.

Example 3.1.4 Let G be a compact Lie group endowed with a left invariant metric. Then it is known that its associated Hamiltonian is collective for the right action [17, pag 219]. Let f denote the quadratic form on g^* that defines the left invariant metric. Then, Corollary 3.1.2 implies that the topological entropy of the geodesic flow defined by the left invariant metric is given by

$$h_{top}(g) = sup_{c \in f^{-1}(1)} h_{top}(f_{\mathcal{O}_c}, f_{\mathcal{O}_c}^{-1}(1)).$$

We deduce for example, that for $G = SO(3)$, $h_{top}(g) = 0$.

Example 3.1.5 Let X be a compact Hamiltonian G-space with moment map $\phi : X \to g^*$. Let $K \subset G$ be a closed subgroup. The inclusion $k \to g$ induces a projection $\pi : g^* \to k^*$. This projection, restricted to a coadjoint orbit $\mathcal{O}$, can be viewed as the moment map corresponding to Hamiltonian action of K on $\mathcal{O}$. Now let $f : k^* \to \mathbf{R}$ be a function invariant under the coadjoint action of K on k^*. Set $H = f \circ \pi \circ \phi$. Apply now Proposition 3.1.1 twice; once to deduce that $h_{top}(f \circ \pi_{\mathcal{O}_c}, \mathcal{O}_c) = 0$ and again to obtain $h_{top}(H) = 0$.

Collective functions like H, i.e., Hamiltonians defined by means of a subalgebra and the corresponding projection, where introduced by Thimm [33] to prove the complete integrability of certain geodesic flows on homogeneous spaces.

3.2 Submersions and collective metrics

Let M be a riemannian manifold on which the group K acts freely, discontinuously and by isometries. Consider the quotient $B = M/K$ and let $\pi : M \to B$ denote the canonical projection. Endow B with the submersion metric. The metrics on M and B induce canonical maps $TM \xrightarrow{\chi_1} T^*M$ and $TB \xrightarrow{\chi_2} T^*B$. Suppose now that G is a group acting on M and its action commutes with the action of K. Then there is a naturally induced action on B. In this way, by lifting to the corresponding cotangent bundles, we have two moment maps: $\phi_G^1 : T^*M \to g^*$ and $\phi_G^2 : T^*B \to g^*$.

Proposition 3.2.1 *The equality $\phi_G^2 \circ \chi_2 \circ d\pi = \phi_G^1 \circ \chi_1$ holds in the set of horizontal vectors on TM.*

Proof: Recall that the moment map ϕ_G^1 is given by $\phi_G^1(v)(\zeta) = v(\zeta^\sharp(p_1 v))$, where $p_1 : T^*M \to M$ is the canonical projection. Similarly $\phi_G^2(u)(\zeta) = u(\zeta^\sharp(p_2 u))$ for $p_2 : T^*B \to B$. Hence we need to prove that if $v \in TM$ is horizontal, then

$$< d\pi(v), \zeta^\sharp(p_2 d\pi v) > = < v, \zeta^\sharp(p_1 v) > .$$

But observe that $p_2 \circ d\pi = \pi \circ p_1$ and that $\zeta^\sharp(\pi \circ p_1 v) = d\pi\zeta^\sharp(p_1 v)$. Hence

$$< d\pi(v), \zeta^\sharp(p_2 d\pi(v)) > = < d\pi(v), d\pi\zeta^\sharp(p_1 v) > .$$

But since v is horizontal by definition of the submersion metric

$$< d\pi(v), d\pi\zeta^\sharp(p_1 v) > = < v, \zeta^\sharp(p_1 v) > .$$

$\diamond$

We apply the proposition to the following situation. Let G be a Lie group with a right-invariant metric. Then $TG \xrightarrow{\chi_1} T^*G$ gives rise to a right-invariant Hamiltonian on T^*G. Then it is known that the latter is collective for the left action of G on T^*G [17, pag 219]. In other words our Hamiltonian can be written as $f \circ \phi_G^1$ where f is some positive definite quadratic form on g^*. Now let K be a subgroup of G acting from the right. Then we can endow G/K with the submersion metric. Clearly there is also an induced action of G on $T^*(G/K)$ with moment map ϕ_G^2. Then from the proposition we deduce that $f \circ \phi_G^2$ is the Hamiltonian associated with the submersion metric on G/K. We have proved:

Corollary 3.2.2 *The Hamiltonian associated with the submersion metric on G/K is collective for the canonical action of G on $T^*(G/K)$, and its defining function is the same one that defines the right-invariant metric on G.*

3.3 Zero entropy and completely integrable systems on 4-manifolds

Let X^4 be a four-dimensional symplectic manifold and H a Hamiltonian on X. Let N be a non-singular, compact level surface of H. Suppose the system is completely integrable; i.e., suppose there exists an additional function on X which is independent of H (almost everywhere) and is in involution with H (such a function is called an integral). Restricting this integral to N gives a smooth function f. The main result of this section is:

Theorem 3.3.1 *Let X^4 be a smooth symplectic manifold. Suppose that the Hamiltonian H is completely integrable and on some non-singular compact level surface N the integral f satisfies either one of the following conditions:*

(a) f is real analytic.

(b) The connected components of the set of critical points of f form submanifolds.

Then $h_{top}(H, N) = 0$.

Corollary 3.3.2 *Let M^2 be a compact connected surface. Assume M^2 supports a geodesic flow that is completely integrable by means of an integral as in Theorem 3.3.1. Then $\chi(M^2) \geq 0$.*

Remark 3.3.3 Corollary 3.3.2 was proved by Kozlov [22] in the case of an analytic integral by completely different methods. If we assume condition (b) the integral could even be of class C^1.

Observe that the class of functions considered in (b) includes the Bott integrals studied by Fomenko in [9].

Let us start with the proof of the theorem. We first state a result of Katok [20, Corollary 4.3] that we will use:

Theorem 3.3.4 *If g is a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact two-dimensional manifold and $h_{top}(g) > 0$ then g has a hyperbolic periodic point with a transversal homoclinic point and consequently there exists a g-invariant hyperbolic set Λ such that the restriction of g to Λ is topologically conjugate to a subshift of finite type.*

We note that Theorem 3.3.4 extends to flows without singularities on 3-manifolds. Theorem 3.3.1 follows from Theorem 3.3.4 and the following lemma:

Lemma 3.3.5 *Under the hypothesis of Theorem 3.3.1 there are no transversal homoclinic orbits.*

Proof: If the function is real analytic, the lemma was proved by Moser in [29]. Therefore assume that condition (b) is verified. Denote by $Crit(f)$ the set of critical points of f. Since f is an integral, the flow of ξ_H leaves $Crit(f)$ invariant. Condition (b) says that $Crit(f)$ is a disjoint union of circles and compact connected surfaces. These surfaces are tori and Klein bottles because ξ_H is never zero.

Suppose now that there is a transversal homoclinic orbit. Then we have the analogue for flows of the hyperbolic set Λ in Theorem 3.3.4. We will also call it Λ (for the properties of shifts and suspended horseshoes, we refer to [32]). We claim that there exists a surface X^2 in $Crit(f)$ such that $\Lambda \subset X^2$. To prove this observe first that since f is an integral it follows that if γ is a hyperbolic closed orbit then $\gamma \subset Crit(f)$. Otherwise the symplectic gradient of f would generate a non-zero eigenvector with eigenvalue one for the Poincaré map of γ. This idea can be traced back to Poincaré (see [22] for details). But the hyperbolic closed orbits in Λ are dense and $Crit(f)$ is a closed set therefore $\Lambda \subset Crit(f)$. Moreover since the flow on Λ is transitive (i.e there is a dense orbit) we deduce the claim.

We now argue considering the flow of ξ_H restricted to X^2. Since ξ_H is never zero, for any closed orbit γ, $X^2 - \gamma$ is a cylinder or a Mobius band. By a Poincaré-Bendixson argument (see [30]) we deduce that ξ_H has no non-trivial recurrent orbits, i.e. if $\omega(\gamma)$ denotes the limit set of the orbit γ and $\gamma \subset \omega(\gamma)$ then $\omega(\gamma)$ is a closed orbit. But this is absurd because dense orbits in Λ have non-trivial recurrence. The lemma is proved.

$\diamond$

Remark 3.3.6 Note that the proof of the lemma still works if we allow the surfaces to have boundary. We also note that Moser in [29] proves that $\Lambda \subset Crit(f)$ with a different argument which only requires f to be of class C^1.

Proof of Corollary 3.3.2: In [6] Dinaburg proved that if $\pi_1(M^2)$ has exponential growth, then $h_{top}(g) > 0$. Hence from Theorem 3.3.1 we get that

$\pi_1(M^2)$ cannot have exponential growth and therefore $\chi(M^2) \geq 0$.

$\diamond$

Chapter 4

Hamiltonian actions with low multiplicity

4.1 Multiplicity k actions

Let G be a compact connected Lie group acting by Hamiltonian transformations on a symplectic manifold X with moment map $\phi : X \to g^*$. As we mention in the introduction, we will say that the action has *multiplicity* k if for generic $x \in X$, the symplectic reduction of $Ker\ d\phi_x$ (i.e. the quotient of $Ker\ d\phi_x$ by its null subspace) has dimension k. Since the symplectic reduction of a subspace is naturally symplectic, k can only take even values. If $k = 0$, then $Ker\ d\phi_x$ is isotropic for generic $x \in X$ and we obtain the notion of *multiplicity free action* introduced and studied by Guillemin and Sternberg in [18, 19].

Let B_x denote the vector subspace of T_xX spanned by $\{\xi_f(x) : f\ is\ G-invariant\}$. Let us denote by $E(x)$ the tangent space at x to the orbit of G through x. In what follows $\perp$ stands for the orthogonal complement with respect to ω. On a generic set we have the relations (cf. Propositions 1.1.1

and 1.1.3):

$$B_x = E(x)^{\perp} = Ker\ d\phi_x. \tag{4.1}$$

Recall that if K is a subspace of a symplectic vector space, the symplectic reduction of K is defined as: $K^{red} = K/K \cap K^{\perp}$. Hence from (4.1) we deduce that the action has multiplicity k if and only if generically $dim\ B_x^{red} = k$. We also note that this is equivalent to saying that for generic c in the image of ϕ, the Marsden-Weinstein reduced space $\phi^{-1}(c)/G_c$ has dimension k. We will need the following lemma:

Lemma 4.1.1 *Let $K \subset E$ be subspaces of a symplectic vector space. Then*

$$dim\ K^{red} \leq dim\ E^{red}.$$

Proof: Write $K = (K \cap K^{\perp}) \oplus V$ where V is some complement. Assume $dim\ V > dim\ E - dim\ E^{\perp} \cap E$. Then $V \cap (E^{\perp} \cap E) \neq \{0\}$. Take $0 \neq \eta \in V \cap (E^{\perp} \cap E)$. Since $K^{\perp} \supset E^{\perp}$, we get that $\eta \in K^{\perp} \cap K$. This is impossible since $V \cap (K^{\perp} \cap K) = \{0\}$.

◇

Now let $H \subset G$ be a closed subgroup and let $X_H = \{x \in X : G_x = H\}$. We know from Proposition 1.1.4 that X_H is a symplectic sub manifold of X. Moreover ϕ maps each connected component of X_H into an affine subspace of g^* of the form $p + h^o$, where h^o denotes the annihilator of h in g^*. Let N_H denote the normalizer of H in G.

Proposition 4.1.2 *Suppose the action of G on X has multiplicity k. Then the action of N_H on X_H has multiplicity $\leq k$.*

Proof: First observe that since G is compact by averaging we can always extend any N_H-invariant function on X_H to a G-invariant function on X.

Take $x_o \in X_H$ and let B_{H,x_o} denote the subspace spanned by the Hamiltonians at x_o of N_H-invariant functions on X_H. We choose x_o so that $dim\ B_{H,x_o}$ is maximal. We need to show that $dim\ B^{red}_{H,x_o} \leq k$. Let $\xi_{f_1}(x_o), ..., \xi_{f_r}(x_o)$ be a basis of B_{H,x_o}. As we mentioned before, N_H-invariant functions are restrictions of G-invariant functions, thus we can think each ξ_{f_i} as defined on all of X. Now set $B_{H,x} = span\{\xi_{f_1}(x), ..., \xi_{f_r}(x)\}$ for $x \in X$.

Since all the ξ_{f_i}'s are independent at x_o, there exists an open set U of X containing x_o so that, $dim\ B_{H,x}$ is constant on U. Now let $V \subset U$ be an open set on which $dim\ (B_{H,x} + B^{\perp}_{H,x})$ is maximal. Since $dim\ B^{red}_x = k$ on a generic set, we can find $y \in V$, so that $dim\ B^{red}_y = k$. But clearly $B_{H,y} \subset B_y$. Hence using Lemma 4.1.1 we get $dim\ B^{red}_{H,y} \leq k$. But $dim\ B_{H,y} = dim\ B_{H,x_o}$ and since $dim\ (B_{H,y} + B^{\perp}_{H,y}) \geq dim\ (B_{H,x_o} + B^{\perp}_{H,x_o})$ we deduce that $dim\ (B_{H,y} \cap B^{\perp}_{H,y}) \leq dim\ (B_{H,x_o} \cap B^{\perp}_{H,x_o})$. Thus $dim\ B^{red}_{H,x_o} \leq dim\ B^{red}_{H,y} \leq k$. $\diamond$

We will denote by $ind\ G$ the index of G defined as $ind\ G = min_{c \in g^*} dim\ G_c$. If G is semi-simple the index is the same as the rank.

Lemma 4.1.3 *Assume for some $x \in X$, the isotropy group G_x is discrete. Then $k = dim\ X - dim\ G - ind\ G$.*

Proof: From Proposition 1.1.1 we know that if $dim G_x = 0$, then the moment map is a submersion at x. Therefore for generic c in the image of ϕ we get:

$$k = dim\ \phi^{-1}(c) - dim\ G_c = dim\ X - dim\ G - ind\ G$$

$\diamond$

Example 4.1.4 Let us check that the action of $SU(3)$ on $T^*(SU(3)/T^2)$ has multiplicity two. First let us give a detailed description of the orbit structure.

The Lie algebra of $SU(3)$ consists of all the skew hermitian matrices with trace zero. Denote by t the Lie algebra of T^2. It consists of all the matrices Y of the form:

$$Y = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

where α, β and γ are purely imaginary and their sum is zero.

Consider the Killing metric on $SU(3)$ i.e. $(X, Y) = -\frac{1}{2} Re\ tr(XY)$. With respect to this product $t^{\perp}$ is the subset of $su(3)$ given by the matrices with zero entries on the diagonal. By homogeneity it suffices to study the action of T^2 on the cotangent plane at the coset of the identity $[T^2]$. We may identify $T^*_{[T^2]}M$ with $t^{\perp}$, and under this identification the action is given by $Ad_x : t^{\perp} \to t^{\perp}$, $x \in T^2$. Therefore the dimension of the orbit of T^2 through $X \in t^{\perp}$ is given by the dimension of the image of the map

$$Y \to ad_X(Y) = [X, Y]\,, \qquad Y \in t.$$

Let $X \in t^{\perp}$ be given by

$$X = \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix}.$$

Take $Y \in t$ as before and compute $[X, Y]$. We get

$$[X, Y] = \begin{pmatrix} 0 & a(\beta - \alpha) & b(\gamma - \alpha) \\ \bar{a}(\beta - \alpha) & 0 & c(\gamma - \beta) \\ \bar{b}(\gamma - \alpha) & \bar{c}(\gamma - \beta) & 0 \end{pmatrix}.$$

To compute the kernel of ad_X we set $a(\beta - \alpha) = b(\gamma - \alpha) = c(\gamma - \beta) = 0$. Therefore whenever two components of X are different from zero, the kernel of ad_X is trivial and the orbit through X is 2-dimensional. Otherwise , the orbit is 1-dimensional if X is not in the zero section.

Thus since $dim\ SU(3) = 8$ and $ind\ SU(3) = 2$ we conclude that $k = 2$.

Similarly we can also check easily that the lift of the 2-torus action to the cotangent bundle of $S^2 \times S^2 \# S^2 \times S^2$ has multiplicity $k = 4$.

Now, we will study the relation between multiplicity k actions and collective functions.

Proposition 4.1.5 *Suppose that $f_1, ..., f_s$ are functions on g^* such that $f_1 \circ \phi, ..., f_s \circ \phi$ are s-independent functions that Poisson-commute on X^{2n}. Then if k denotes the multiplicity of the G-action, we have that $k \leq 2(n-s)$.*

Proof: Let $F : X \to \mathbf{R}^s$ be the function $F = (f_1 \circ \phi, ..., f_s \circ \phi)$. Generically $F^{-1}(a)$ is a submanifold and the vector fields $\xi_{f_1 \circ \phi}, ..., \xi_{f_s \circ \phi}$ are tangent to it. Moreover, since $f_1 \circ \phi, ..., f_s \circ \phi$ Poisson-commute, the associated Hamiltonian fields span a null subspace. Thus generically $dim\ Ker\ dF_x^{red} \leq 2n - 2s$. But the level surface $\phi^{-1}(c)$ is contained in $F^{-1}(f_1(c), ..., f_s(c))$, hence $Ker\ d\phi_x \subset Ker\ dF_x$. Apply now Lemma 4.1.1.

◇

Remark 4.1.6 Observe that if $s = n$, i.e., if we can find a full set of commutative collective Hamiltonians, then the action is multiplicity free. This was proved in [18].

Next we will study the following special case. Suppose $X = T^*M$ and the action of G on T^*M is the lift of an action of G on M.

Proposition 4.1.7 *Let G/K be a principal orbit for the action of G on M and let l be its codimension. If k denotes the multiplicity of the action of G on T^*M, then the multiplicity of the action of G on $T^*(G/K)$ is $k - 2l$.*

Proof: Let $M_{(K)}$ denote the set of principal orbits. Then there exists an invariant neighborhood N of G/K diffeomorphic to $G/K \times U$ where U is an open set of $M_{(K)}/G$. Moreover G acts on U trivially and on G/K by left translations. Hence T^*N is symplectomorphic to $T^*(G/K \times U)$ and the symplectomorphism is equivariant. Next observe that T^*N is an open subset of T^*M, thus the action of G on T^*N has multipicity k. But since G acts trivially on U it is clear that the action of G on $T^*(G/K) \times T^*U$ has multiplicity k if and only if the action of G on $T^*(G/K)$ has multiplicity $k - 2l$ where $l = dim\ U$.

$\diamond$

Corollary 4.1.8 *Suppose the action of G on M has codimension one. Then $k = 2$ if and only if (G, K) is a Gelfand pair.*

Note that it also follows from the proposition that $k \geq 2l$. Hence for $k = 0$, the action is transitive (this was proved in [19]) and for $k = 2$ it has codimension ≤ 1.

Let us see some examples:

Example 4.1.9 Take $M = S^n$ and the standard codimension one action of $G = SO(n)$. The principal orbit $SO(n)/SO(n-1)$ is a symmetric pair, thus a Gelfand pair. Therefore the action of $SO(n)$ on T^*S^n has multiplicity two.

Example 4.1.10 Imbed $SU(2n)$ into $SU(2n+1)$ as the upper left $2n \times 2n$-corner, to obtain the canonical embedding $Sp(n) \subset SU(2n) \subset SU(2n+1)$. Consider the 1-parameter subgroup of $SU(2n+1)$ given by

$$diag(e^{-i\theta/2n}, ..., e^{-i\theta/2n}, e^{i\theta}).$$

Since the latter centralizes $SU(2n)$, it induces a free action of $U(1)$ on $SU(2n+1)/Sp(n)$. Let $U(1)$ act on S^2 by rotations, and consider the manifold $M = SU(2n+1)/Sp(n) \times_{U(1)} S^2$ obtained by the quotient of $SU(2n+1)/Sp(n) \times S^2$ by the diagonal action. Clearly M admits a codimension one action of the group $SU(2n+1)$ with principal orbit $SU(2n+1)/Sp(n)$. Since $(SU(2n+1), Sp(n))$ is a Gelfand pair [23, Tabelle 1] we deduce that the action of $SU(2n+1)$ on the cotangent bundle of M has multiplicity two.

Example 4.1.11 Let S^1 act on the right on $SU(2)$ and on S^2 by rotations. Consider the space $M = SU(2) \times_{S^1} S^2$ obtained by taking the quotient of $SU(2) \times S^2$ by the diagonal action of S^1. The space thus constructed is $\mathbf{CP}^2\# - \mathbf{CP}^2$ which is not diffeomorphic to any homogeneous space [5]. Now observe that the natural action of the group $SU(2) \times S^1$ on $SU(2)$ descends to an action on M. The principal orbits, $SU(2) \times S^1/S^1$ are Gelfand pairs. Hence the action of $SU(2) \times S^1$ on the cotangent bunlde of $\mathbf{CP}^2\# - \mathbf{CP}^2$ has multiplicity two.

4.2 More examples

Let X be a Hamiltonian G-space with G compact. Suppose the action is multiplicity free. If X is the cotangent bundle of a manifold M and G acts by derivatives then we saw that M is a homogeneous space G/K. If the action of G does not arise from an action on M then M does not need to be homogeneous. We will now give examples of this situation.

First let us describe a multiplicity free action on $X = T^*S^2$ which is not the lift of an action on S^2. Since T^*S^2 can be written as $SO(3) \times_{S^1} \mathbf{R}^2$ we note that X admits a Hamiltonian action of the group $SO(3) \times S^1$. Let T^2 be a maximal torus of this group. Then the action of T^2 clearly has multiplicity zero, and obviously does not arise from an action on S^2.

Let Q be a manifold such that T^*Q admits a multiplicity free G-action. Suppose S^1 acts on Q without fixed points. Let S^1 act on the 2-sphere S^2 by rotations. Consider now the diagonal action of S^1 on $X = Q \times S^2$. Since the latter is free we can consider the quotient manifold $M = Q \times_{S^1} S^2$. Let $\pi : X \to M$ denote the projection map.

Proposition 4.2.1 *Suppose the action of G on T^*Q and the induced action of S^1 on T^*Q commute. Then T^*M admits a multiplicity free $G \times S^1$-action.*

Proof: Let S^1_o be the Hamiltonian circle action coming from the S^1-factor of the group $SO(3) \times S^1$ that acts on T^*S^2. This circle action commutes with the lift of the rotation. Observe now that by obvious extensions we get an action of $\hat{G} = G \times S^1_o$ on T^*X. Let ϕ_{S^1} denote the moment map associated with the S^1-action on T^*X. By assumption the latter action commutes with the $\hat{G}$-action. Thus $\hat{G}$ leaves $\phi_{S^1}^{-1}(0)$ invariant and descends to a Hamiltonian action on $\phi_{S^1}^{-1}(0)/S^1$ making $d\pi$ equivariant. Since T^*M and $\phi_{S^1}^{-1}(0)/S^1$ are symplectomorphic [2, pag. 95] we only need to prove that the action of $\hat{G}$ on $\phi_{S^1}^{-1}(0)/S^1$ is multiplicity free. By the equivalences stated in [18] it is enough to prove that for generic points $\hat{v} \in \phi_{S^1}^{-1}(0)/S^1$ the orbit $O(\hat{v})$ of $\hat{G}$ through $\hat{v}$ is coisotropic. Let ω denote the symplectic form on T^*X and let $\hat{\omega}$ denote the symplectic form on $\phi_{S^1}^{-1}(0)/S^1$. Assume that $\hat{\omega}(\hat{\zeta}, \hat{\eta}) = 0$ for all $\hat{\eta} \in T_{\hat{v}}O(\hat{v})$. Then we need to prove that $\hat{\zeta} \in T_{\hat{v}}O(\hat{v})$. Set $p = d\pi$. Take a point $v \in \phi_{S^1}^{-1}(0)$ such that $p(v) = \hat{v}$. Since the action of $\hat{G}$ is equivariant for each $\hat{\eta} \in T_{\hat{v}}O(\hat{v})$ we can take $\eta \in T_vO(v)$ such that $dp(\eta) = \hat{\eta}$. Also write $dp(\zeta) = \hat{\zeta}$ for some $\zeta \in T_v\phi_{S^1}^{-1}(0)$. We have

$$\hat{\omega}(\hat{\zeta}, \hat{\eta}) = \hat{\omega}(dp(\zeta), dp(\eta)) = \omega(\zeta, \eta) = 0$$

Since $\phi_{S^1}^{-1}(0) \subset T^*Q \times T^*S^2$ write $\zeta = (\zeta_1, \zeta_2)$, $\eta = (\eta_1, \eta_2)$ and $v = (v_1, v_2)$. Hence $\omega(\zeta, \eta) = 0$ implies $\omega_1(\zeta_1, \eta_1) + \omega_2(\zeta_2, \eta_2) = 0$ where ω_1 and ω_2 are the symplectic forms on T^*Q and T^*S^2 respectively. Also $T_{(v_1,v_2)}O(v) = T_{v_1}Gv_1 \oplus T_{v_2}S^1_o v_2$. Take $\eta_2 = 0$ hence $\omega_1(\zeta_1, \eta_1) = 0$ for all $\eta_1 \in T_{v_1}Gv_1$. Since the action of G on T^*Q is multiplicity free and the projection of $\phi_{S^1}^{-1}(0)$ onto T^*Q is dense, we deduce that for generic v_1, with v_1 the first component of a pair in $\phi_{S^1}^{-1}(0)$, $\zeta_1 \in T_{v_1}Gv_1$.

Let W be the tangent vector field to the orbits of S^1 on $\phi_{S^1}^{-1}(0)$. Then W can be written as $(-W_1, W_2)$ where W_1 denotes the tangent field to the S^1-orbits on T^*Q and W_2 the corresponding tangent field to the S^1-orbits on T^*S^2. Note that since $\omega(\zeta, W(v)) = 0$ and $\omega_1(\zeta_1, W_1(v_1)) = 0$ we deduce $\omega_2(\zeta_2, W_2(v_2)) = 0$. Also note that $\omega_2(\zeta_2, \eta_2) = 0$ for all $\eta_2 \in T_{v_2}S^1_o v_2$. Suppose now, we choose v_2 so that $W_2(v_2) \notin T_{v_2}S^1_o v_2$. This a generic condition. Then if S denotes the linear subspace spanned by η_2 and $W_2(v_2)$ we have $dim\ S = 2$, and therefore S is a Lagrangian subspace. This implies that $\zeta_2 \in S$. Write $\zeta_2 = a\eta_2 + bW_2(v_2)$. Observe that $\omega_1(W_1(v_1), T_{v_1}Gv_1) = 0$ and $T_{v_1}Gv_1$ is coisotropic, thus $W_1(v_1) \in T_{v_1}Gv_1$. Now write

$$\zeta = (\zeta_1, \zeta_2) = (\zeta_1 + bW_1(v_1), 0) + (0, a\eta_2) + bW(v)$$

But since $dp(W(v)) = 0$, $(\zeta_1 + bW_1(v_1), 0) \in T_{v_1}Gv_1$ and $(0, a\eta_2) \in T_{v_2}S^1_o v_2$ we deduce that $dp(\zeta) \in T_{\hat{v}}O(\hat{v})$.

$\diamond$

Let us mention some concrete examples:

Example 4.2.2 Consider the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{CP^n}$. The canonical action of $SU(n+1)$ on S^{2n+1} commutes with the S^1-action. It is known that the action of $G = SU(n+1) \times S^1$ on T^*S^{2n+1} is multiplicity free [33]. Then clearly all the hypotheses of the proposition are satisfied. Thus the cotangent bundle of $M = S^{2n+1} \times_{S^1} S^2$ has a multiplicity free $SU(n+1) \times T^2$-action. Note that M is diffeomorphic to $\mathbf{CP^{n+1}} \# - \mathbf{CP^{n+1}}$. For $n = 1$ we get $\mathbf{CP^2} \# - \mathbf{CP^2}$ which is not diffeomorphic to any homogeneous space [5].

We will now describe a class of riemannian metrics whose associated quadratic forms are invariant under $SU(n+1) \times T^2$. Denote by g_t the metric on S^{2n+1} which is obtained from the standard metric by multiplying with t^2 in the directions tangent to the S^1-orbits. Endow $X = S^{2n+1} \times S^2$ with the product metric and M with the submersion metric $\hat{g}_t$. Since the action of $SU(n+1) \times S^1$ on (S^{2n+1}, g_t) is by isometries it is easy to verify that $SU(n+1) \times T^2$ leaves the Hamiltonians associated with the metrics $\hat{g}_t$ on M invariant for all real t.

Incidentally, it follows from the Thimm method that the geodesic flow on $(M, \hat{g}_t)$ is completely integrable. See [31] for details.

Example 4.2.3 Let $G_{n-1,2}(\mathbf{R}) = SO(n+1)/SO(n-1) \times SO(2)$ denote the Grassmannian of 2-planes in $n+1$-space. Consider the fibration $S^1 \to SO(n+1)/SO(n-1) \to G_{n-1,2}(\mathbf{R})$, where S^1 acts on $SO(n+1)/SO(n-1)$ by right translations. The action of $SO(n+1) \times S^1$ on the cotangent bundle of $SO(n+1)/SO(n-1)$ is multiplicity free [33]. Thus using the Proposition we deduce that the cotangent bundle of $M = SO(n+1)/SO(n-1) \times_{S^1} S^2$ has a multiplicity free $SO(n+1) \times T^2$-action. As well as in the previous example, a family of metrics $\hat{g}_t$ on M can be constructed so that the corresponding Hamiltonians are invariant under the $SO(n+1) \times T^2$-action.

4.3 The Main Theorem

Let H be a G-invariant Hamiltonian, ξ_H its Hamiltonian vector field and $H^{-1}(a) = N$ a compact regular level surface. If g_t denotes the flow of ξ_H, then G and g_t leave N invariant. Set $\varphi = \phi/N$.

We say that $x \in X$ defines a *stationary motion* if there exists a 1-parameter subgroup ψ_t of G such that $\psi_t x = g_t x$. We denote by $St(G)$ the set of all $x \in X$ that define stationary motions.

Lemma 4.3.1 *If g_x^o denotes the annihilator of g_x in $g*$, then*

$$Im\ d\varphi_x = g_x^o$$

if x is not in $St(G)$.

Proof: It follows from Proposition 1.1.1 and the fact that if $x \notin St(G)$ then $Im\ d\varphi_x = Im\ d\phi_x$. $\diamond$

Theorem 4.3.2 *If the action of G has multiplicity zero or two, then $h_{top}(H, N) = 0$.*

Proof: Suppose first that the action is multiplicity free. We claim that $St(G) = X$.

Since the action of G on X is multiplicity free we know that for generic $x \in X$ the orbit of G through x is coisotropic [18]. Denote by $E(x)$ the tangent space at x of the orbit of G through x. Now observe that since g_t commutes with G we have $\omega(\xi_H(x), \zeta) = 0$ for all $\zeta \in E(x)$. So, if $E(x)$ is coisotropic we deduce that $\xi_H(x) \in E(x)$. This in turn implies that generically, $x \in St(G)$. Since $St(G)$ is a closed set we deduce $St(G) = X$.

Now apply Theorem 2.2.5 with $Y = N/G$. Since $St(G) = X$, the induced flow fixes every point in Y. Thus $h_{top}(H, N) = 0$.

Next, let us prove the Theorem in the multiplicity two case.

Set again $Y = N/G$, call π the canonical projection and let $\hat{g}_t$ be the induced flow on Y. According to Theorem 2.2.5 we only need to show that $h_{top}(\hat{g}) = 0$. We will prove more: $\hat{g}_t$ has only trivial recurrence (cf. Proposition 2.2.2).

Let $\hat{\gamma}$ denote an orbit of $\hat{g}_t$, i.e. $\hat{\gamma}(t) = \hat{g}_t\hat{x}$ for some $\hat{x} \in Y$. Take $x \in \pi^{-1}(\hat{x})$ and consider the orbit of g_t through x. Thus $\pi \circ \gamma(t) = \hat{\gamma}(t)$. Let $H = G_x$. Then since g_t commutes with the G-action, we deduce that $\gamma \subset X_H$. Let $\phi_{N_H} : X_H \to n_H^*$ denote the moment map corresponding to the action of N_H on X_H. Recall that in fact ϕ_{N_H} takes values on a subspace of n_H^* of the form $p + h^o$ where h^o is the annihilator of h in n_H^*. Set $c = \phi_{N_H}(\gamma)$ and

$\varphi = \phi_{N_H}/X_H \cap N$.

Observe now that Lemma 4.3.1 says that c is a regular value of φ if $\varphi^{-1}(c) \cap St(N_H)$ is empty. Set $Q_c = \varphi^{-1}(c) - (\varphi^{-1}(c) \cap St(N_H))$. We have now two possible cases:

(a) $x \in St(N_H)$. If this happens, then clearly $\hat{\gamma}$ is a fixed point and hence trivially recurrent.

(b) $x \notin St(N_H)$. In this case Q_c is a non-empty submanifold of $X_H \cap N$ and $\gamma \subset Q_c$. From now on we will work with the connected component of Q_c containing γ. Let K_c denote the identity component of the stabilizer at c of the coadjoint action of N_H on n_H^*. Since the action of G on X has multiplicity two by Proposition 4.1.2, the action of N_H on X_H has multiplicity at most two. But it cannot be zero if $x \notin St(N_H)$ (recall the claim we proved at the very begining: if an action is multiplicity free, the set of stationary motions of any invariant Hamiltonian is the whole symplectic space). Thus $dim\, Q_c/K_c = 1$. Now we also have two possible cases:

(b1) Q_c/K_c is a circle. In this case it follows inmediately that $\hat{\gamma}$ is a closed orbit and hence trivially recurrent.

(b2) Q_c/K_c is an open interval I. Then Q_c is diffeomorphic to $\mathcal{O} \times I$, where $\mathcal{O}$ denotes a principal orbit for the action of K_c on Q_c. Also γ intersects every orbit of K_c once and only once. Thus if we assume that $\hat{\gamma}$ is not a closed orbit it follows that every G-orbit in X that intersects Q_c, does it in a single K_c-orbit. Hence we can find a G-invariant neighborhood W of x in X so that there exists $T > 0$ with the property that $\gamma(t) \notin W$ for $t \geq T$. But this implies that $\hat{\gamma}(t) \notin \pi(W)$ for $t \geq T$ and thus $\hat{x} \notin \omega(\hat{\gamma})$, proving that $\hat{\gamma}$ is not recurrent.

$\diamond$

Remark 4.3.3 We observe that from the proof of Theorem 4.3.2 is possible to obtain an acurate picture of the dynamics of the induced flow on $Y = N/G$. For example, we can carry out this analysis in full detail in the case of the geodesic flow corresponding to a left invarint metric on $SU(3)/T^2$ using the information obtained in Example 4.1.4

4.4 Consequences of the Main Theorem

Let us now describe some of the interesting consequences that Theorem 4.3.2 has in the case of geodesic flows. Let M be a compact riemannian manifold. Recall that if the topological entropy of the geodesic flow is zero then $\pi_1(M)$ has sub-exponential growth [6]. Moreover if $\pi_1(M)$ is finite, M is rationally elliptic (Corollary 2.3.2).

Thus from Theorem 4.3.2 we get:

Theorem 4.4.1 *Let M be a compact manifold whose cotangent bundle admits a compact Hamiltonian G-action with multiplicity $k \leq 2$. Assume the set of G-invariant functions on T^*M contains the Hamiltonian associated with some riemannian metric. Then $\pi_1(M)$ has sub-exponential growth and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Remark 4.4.2 Observe that Theorem 4.4.1 and thus Theorem 4.3.2 are false for $k \geq 4$. For example $M = S^2 \times S^2 \# S^2 \times S^2$ is non-rationally elliptic manifold that admits a 2-torus action (cf. Section 2.1) . From Example 4.1.4 we know that the lift of this action to the cotangent bundle of M has mutiplicity $k = 4$. Any riemannian metric invariant under the torus action, gives rise to a geodesic flow with positive topological entropy.

We consider another consequence.

Let (M^n, g) be a compact riemannian manifold whose geodesic flow is completely integrable with first integrals $f_1 = \| \cdot \|_g, f_2, ..., f_n$. We will say that the geodesic flow is completely integrable with *periodic integrals* if the Hamiltonian vector field associated with f_i generates a circle action for $2 \leq i \leq n$.

Corollary 4.4.3 *Let M^n be a compact riemannian manifold whose geodesic flow is completely integrable with periodic integrals. Then $\pi_1(M)$ has sub-exponential growth and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Proof: If the geodesic flow is completely integrable with periodic integrals, then there exists a torus action with $k = 2$ that leaves the Hamiltonian associated with the metric invariant. The Corollary follows from Theorem 4.4.1.

◇

Let (M^n, g) be a compact riemannian manifold whose geodesic flow is completely integrable with first integrals $f_1 = \| \cdot \|_g, f_2, ..., f_n$. We will say that the geodesic flow is completely integrable with *collective integrals* if the functions f_i, $2 \leq i \leq n$ are collective with respect to the action of some compact Lie group G that leaves the Hamiltonian associated with the riemannian metric invariant.

Theorem 4.4.4 *Let M^n be a compact riemannian manifold whose geodesic flow is completely integrable with collective integrals. Then $\pi_1(M)$ has sub-exponential growth and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Proof: Since T^*M admits $n-1$ independent collective integrals, we deduce from Proposition 4.1.5 that the action of G has multiplicity ≤ 2. Now apply Theorem 4.4.1.

$\diamond$

As we mentioned in Proposition 2.4.2 the vanishing of the topological entropy implies –in the homogeneous case– that $e(M) = 0$. Hence we also obtain the following geometrical result:

Theorem 4.4.5 *Let G/K be a homogeneous space such that the action of G on $T^*(G/K)$ has multiplicity zero or two. Then for any left invariant metric on G/K, $N(\lambda)$ grows sub-exponentially.*

Examples of homogeneous spaces such that the action of G on $T^*(G/K)$ has multiplicity two are the Stiefel manifold $SO(n+1)/SO(n-1)$ and the Wallach manifold $SU(3)/T^2$ (cf. Example 4.1.4).

Chapter 5

Geometry of geodesics

5.1 The first return time

Let X be a Hamiltonian G-space with G compact and associated moment map $\phi : X \to g^*$. Let H be a G-invariant Hamiltonian and $N = H^{-1}(a)$ a compact regular level surface. If g_t denotes the flow of ξ_H, then clearly G and g_t leave N invariant. Set $\varphi = \phi/N$. Let X^* denote the set of principal orbits of G. Moving a a little if necessary we can assume that $N^* = N \cap X^*$ is non-empty. Then N^* is an open connected dense set of N. Now define $Y = \varphi(N) - \varphi((N - N^*) \cup (St(G) \cap N))$ and $U = \varphi^{-1}(Y)$. Assume U is non-empty, i.e. there exists $x \in N$ such that $x \notin St(G)$.

Observe that U is an open G-invariant and g_t-invariant set. Also the set Y has the following property. If $c \in Y$, then it follows from Proposition 1.1.3 and Lemma 4.3.1 that φ intersects $\mathcal{O}_c$ cleanly.

Now we will prove:

Lemma 5.1.1 *Suppose the action of G has multiplicity two. Then for each*

$x \in U$, there exists a continuous return time $\tau(x)$ of the flow g_t to the orbit of x under G. Moreover, there exists $\psi \in G_{\varphi(x)}$ such that $\psi x = g_{\tau(x)}x$.

Proof: Take $x \in U$ and let $c = \varphi(x)$. Since φ intersects $\mathcal{O}_c$ cleanly, $\varphi^{-1}(c)$ is a compact submanifold of N. Moreover, since the action has multiplicity two, $dim(\varphi^{-1}(c)/G_c) = 1$. But g_t leaves $\varphi^{-1}(c)$ invariant, hence the orbit of x under g_t descends to a closed orbit with period, let us say $\tau(x)$. This is the same as saying that for some $\psi \in G_c$, $\psi x = g_{\tau(x)}x$. Now considering that the only elements of G that leave $\varphi^{-1}(c)$ invariant are the elements in G_c, we deduce that $\tau(x)$ is the first return time to the orbit of x under G. The continuity of τ follows from the continuity of the flow g_t and the action.

$\diamond$

5.2 The first return time and $N(\lambda)$

Let M^n be a compact riemannian manifold and let SM be its unit tangent bundle endowed with its standard riemannian structure $\ll\ ,\ \gg$. Let $\pi : SM \to M$ denote the canonical projection. If $v \in SM$, denote by $N(v)$ the space of vectors $\omega \in T_{\pi(v)}M$ such that $< v, \omega >= 0$. Define $S(v) = d\pi^{-1}(N(v))$. $S(v)$ is orthogonal to $E_g(v)$ where $E_g(v)$ stands for the one dimensional subbundle generated by the geodesic flow. Note that $d\pi E_g(v)$ is the one dimensional subspace generated by v. Moreover $S(v)$ is invariant under dg_t, i.e. $dg_t(S(v)) = S(g_t v)$ for all t and v. The vertical subspace is defined as $V(v) = d\pi_v^{-1}(\{0\})$. The horizontal subspace $H(v)$ is the orthogonal complement of $V(v)$ in $S(v)$. The map $d\pi$ restricted to $H(v)$ is an isometry between $H(v)$ and $N(v)$. Moreover there exists an isometry $J_v : S(v) \to S(v)$ such that $J_v^2 = -Id$, $J_v V(v) = H(v)$ and $J_v H(v) = V(v)$. The symplectic form $\Omega_v : S(v) \times S(v) \to R$ defined by $\Omega_v(u_1, u_2) = \ll u_1, J_v u_2 \gg$ is invariant under dg_t and is nothing but the restriction of the symplectic form ω of TM to $S(v)$, obtained by pulling back the standard symplectic form on T^*M using the riemannian metric.

It is clear that $dim\ H(v) = dim\ V(v) = dim\ N(v) = n - 1$. Let $orth\ E$ denote the orthogonal complement of a subspace $E \subset S(v)$ with respect to Ω_v.

Suppose now, the compact group G acts on TM by Hamiltonian transformations, leaving the norm of vectors invariant. Let $\beta_v : g \to T_v SM$ be the map $\beta_v(\zeta) = \zeta^\sharp(v)$. Denote by $P_v : T_v SM \to S(v)$ the orthogonal projection onto $S(v)$. We define two natural subbundles. For $c \in g^*$ let g_c denote the Lie algebra of G_c. Set

$$E(v) = P_v \beta_v(g),$$

$$F(v) = P_v \beta_v(g_{\varphi(v)}).$$

Lemma 5.2.1 *Both subbundles are invariant under the geodesic flow and $F(v) \subset orth\ E(v)$. Moreover, if the action has multiplicity two and $v \in U$ then $F(v) = orth\ E(v)$.*

Proof: The invariance of the subbundles follows from the formulas $\beta_{g_t v}(\zeta) = dg_t(\beta_v(\zeta))$ and $dg_t \circ P_v = P_{g_t v} \circ dg_t$, together with the fact that φ is constant along the orbits of g_t.

Observe now that if ω denotes the symplectic form of TM then

$$\Omega_v(P_v u_1, P_v u_2) = \omega_v(u_1, u_2),$$

where Ω is the symplectic form described above. Therefore we get

$$\Omega_v(P_v\beta_v(\zeta_1), P_v\beta_v(\zeta_2)) = \omega_v(\beta_v(\zeta_1), \beta_v(\zeta_2)) =$$

$$= \omega_v(\xi_{\alpha(\zeta_1)}(v), \xi_{\alpha(\zeta_2)}(v)) = \alpha(v)([\zeta_1, \zeta_2]),$$

where α is the morphism that gives rise to the moment map (cf. Section 1.1).

Now observe that if $\zeta_1 \in g_{\varphi(v)}$, then $\alpha(v)([\zeta_1, \zeta_2]) = 0$ for all $\zeta_2 \in g$. Hence $F(v) \subset orth\ E(v)$.

Next, note that if $v \in U$, $dim\ E(v) = dim\ \beta_v(g)$ and $dim\ F(v) = dim\ \beta_v(g_{\varphi(v)})$. But from Proposition 1.1.1 we know that $\beta_v(g)^{\perp} = Ker\ d\phi_v$. If the action has multiplicity two, $dim\ Ker\ d\phi_v - dim\ \beta_v(g_{\varphi(v)}) = 2$. Thus $dim\ orth\ E(v) = dim\ F(v)$. $\diamond$

Remark 5.2.2 Note that on U, E is a continuous subbundle because its dimension remains constant.

Next, we will prove the main result of this section.

Proposition 5.2.3 *Suppose the action of G has multiplicity two. Then if $0 < a < \tau(v) < b < \infty$ on U and U is dense, $N(\lambda)$ does not grow faster than λ^n.*

First we need to prove some lemmas. Let (W,Ω) denote a symplectic space with a compatible positive inner product $< \, , \, >$, i.e. there exists an isometry $J : W \to W$ such that $J^2 = -Id$ and $\Omega(v,\omega) =< v, J\omega >$ for all $v, \omega \in W$.

Lemma 5.2.4 *Suppose W is a symplectic linear space and $\theta : W \to W$ a symplectic linear map. Assume there exists a coisotropic subspace E such that E and orth E are invariant under θ. Set $k = dim$ orth E and $l = dim\ E$. Take $e_1, ..., e_k$ an orthonormal basis of orth E. Extend it to an orthonormal basis $e_1, ..., e_k, e_{k+1}, ..., e_l$ of E. Then in the basis $\mathcal{B} = \{e_1, ..., e_k, e_{k+1}, ..., e_l, Je_1, ..., Je_k\}$ the matrix of θ has the form*

$$\begin{pmatrix} A & B \\ 0 & (A_1^{-1})^t \end{pmatrix}, \quad \textit{where} \quad A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}.$$

Proof: Because E and *orth* E are invariant, the matrix of θ in the basis $\mathcal{B}$ has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \text{ where } A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}.$$

We only have to prove that $C = (A_1^{-1})^t$.

For this observe that in the basis $\mathcal{B}$ the matrix of J has the form

$$J = \begin{pmatrix} M & J_2 \\ & \\ J_1 & 0 \end{pmatrix}, \text{ where}$$

$$J_1 = \begin{pmatrix} Id_{k\times k} & 0_{k\times(l-k)} \end{pmatrix}, \quad J_2 = \begin{pmatrix} -Id_{k\times k} \\ \\ 0_{(l-k)\times k} \end{pmatrix},$$

and

$$M_{l\times l} = \begin{pmatrix} 0_{k\times k} & M_1 \\ & \\ 0 & M_2 \end{pmatrix}.$$

The map θ is symplectic, therefore since $\mathcal{B}$ is an orthonormal basis, we have in terms of matrices

$$\theta^t J \theta = J.$$

Doing the corresponding operations we get

$$B^t M A + C^t J_1 A = J_1.$$

Write $B^t = (B_1 \mid B_2)$. Then $B^t M A = (0 \mid *)$ and

$$C^t J_1 A = (C^t A_1 \mid C^t A_2).$$

Hence $C^t A_1 = Id$ as we wanted. $\diamond$

Fix $v \in U$ and consider ψ as in Lemma 5.1.1.

Define a linear map $\theta_v : S(v) \to S(v)$ by

$$\theta_v = P_v \circ d\psi_v^{-1} \circ dg_{\tau(v)}(v)$$

θ_v is well defined because $\psi v = g_{\tau(v)} v$

Lemma 5.2.5 *The following properties hold:*

(a) $dg_t(v) = Q_t(v) \circ exp(tD_v)$, *where* $Q_t(v) : S(v) \to S(g_t v)$ *verifies*

$$Q_{t+\tau(v)}(v) = P_{g_{t+\tau(v)} v} \circ d\psi_{g_t v} \circ Q_t(v),$$

and

$$D_v = \frac{1}{\tau(v)} log\theta_v.$$

(b) θ_v *is a symplectic map and leaves* $E(v)$ *invariant. Moreover,*

$$\theta_v / F(v) = Id.$$

(c) All the eigenvalues of θ_v *have absolute value one and the Jordan blocks of its canonical form, cannot have size bigger than two.*

Proof: To prove (a) define $Q_t(v)$ by

$$Q_t(v) = dg_t \circ exp(-tD_v).$$

If we compute $Q_{t+\tau(v)}(v)$ the statement follows in a straightfoward fashion from the definitions and the fact that ψ commutes with the geodesic flow.

It is clear that θ_v is symplectic because $dg_{\tau(v)}$ and $P_v \circ d\psi_v^{-1}$ are. Now we observe the formulas

$$d\psi_v(\beta_v(\zeta)) = \beta_{\psi v}(Ad_\psi(\zeta)),$$

and

$$\theta_v(P_v\beta_v(\zeta)) = P_v\beta_v(Ad_{\psi^{-1}}(\zeta)). \tag{5.1}$$

Clearly this implies that θ_v preserves $E(v)$. Since $\psi \in G_{\varphi(v)}$ and $G_{\varphi(v)}/G_v^o$ is abelian (cf. Proposition 1.1.3), we deduce that $\beta_v(Ad_{\psi^{-1}}(\zeta)) = \beta_v(\zeta)$ (recall that $Ker\,\beta_v = g_v$). Hence $\theta_v/F(v) = Id$. This proves (b).

To prove (c) we will use the compactness of G. We observe first that θ_v verifies the hypotheses of the previous lemma. Moreover with the notation of the lemma we have that $A_1 = Id$. Then on some basis the matrix of θ_v has the form

$$\begin{pmatrix} A & B \\ 0 & Id \end{pmatrix}.$$

From (5.1) we obtain

$$\theta_v^n(P_v\beta_v(\zeta)) = P_v\beta_v(Ad_{\psi^{-1}})^n(\zeta). \tag{5.2}$$

But $(Ad_{\psi^{-1}})^n = Ad_{\psi^{-n}}$. The map $\psi \to \parallel Ad_\psi \parallel$ is continuous and G is a compact group. This implies that for all n, $\parallel Ad_{\psi^{-n}} \parallel$ stays bounded. Thus from equation (5.2) we deduce that $\parallel (\theta_v/E)^n \parallel = \parallel A^n \parallel$ is bounded. Then all the eigenvalues of θ_v have absolute value one. Now we look at the matrix of θ_v^n. We have

$$\begin{pmatrix} A^n & (A^{n-1} + A^{n-2} + ... + Id)B \\ 0 & Id \end{pmatrix}.$$

Since the norm of A^n is bounded we get

$$\| \theta_v^n \| \leq c_1(n-1) + c_2,$$

where c_1 and c_2 are constants independent of n. This in turn implies that the Jordan blocks of θ_v cannot have size bigger than two. $\diamond$

Lemma 5.2.6 *Under the hypothesis of Proposition 5.2.3 we have*

$$sup_{v \in U} \| dg_t(v) \| \leq p(t),$$

where $p(t)$ is a linear polynomial.

Proof: We divide the proof in three steps. First set $b = sup_{v \in U} \tau(v) < +\infty$.

Step (i): We claim that $sup_{v \in U} \| D_v \| = A < +\infty$. From the definition of θ_v we get using the fact that P_v is a projection,

$$\| \theta_v \| \leq \| d\psi_v^{-1} \| \| dg_{\tau(v)} \| .$$

Since G is compact the map $(\psi, v) \rightarrow \| d\psi_v \|$ is bounded. Observe that $dg_t(v)$ is defined and continuous for every t and and every $v \in SM$. Since $\tau(v)$ is bounded on U we get $sup_{v \in U} \| \theta_v \| < +\infty$. But log is continuous and $\tau > a > 0$, so the claim follows.

Step (ii): We assert that $sup_{v \in U, t \geq 0} \| Q_t(v) \| < +\infty$. Set $g(t) = max_{v \in SM} \| dg_t(v) \|$. g is continuous. Using the definition of $Q_t(v)$ and Step (i) we get

$$\| Q_t(v) \| \leq g(t) e^{tA} \quad for \ all \ v \in U \ and \ t \geq 0.$$

Take any $t \geq 0$ and write it as $t = n\tau(v) + q$, where $q \in [0, \tau(v)]$ and n is an integer. Then using Lemma 5.2.5 part (a) we have

$$Q_t(v) = P_{g_{n\tau(v)+q}v} \circ d\psi_{g_{n\tau(v)+q}v} \circ ... \circ P_{g_{\tau(v)+q}v} \circ d\psi_{g_q v} \circ Q_q(v).$$

Since

$$P_{g_{i\tau(v)+q}v} \circ d\psi_{g_{i\tau(v)+q}v} \circ P_{g_{(i-1)\tau(v)+q}v} = P_{g_{i\tau(v)+q}v} \circ d\psi_{g_{i\tau(v)+q}v},$$

we deduce that

$$Q_t(v) = P_{g_{n\tau(v)+q}v} \circ d\psi_{g_{n\tau(v)+q}v} \circ d\psi_{g_{(n-1)\tau(v)+q}v} \circ ... \circ d\psi_{g_q v} \circ Q_q(v) =$$

$$= P_{g_{n\tau(v)+q}v} \circ d(\psi^n)_{g_q v} \circ Q_q(v).$$

But the projections are norm decreasing and G is compact, hence we can find a constant c such that

$$\| Q_t(v) \| \leq c \| Q_q(v) \| .$$

Therefore we get

$$sup_{v\in U, t\geq 0} \| Q_t(v) \| \leq sup_{t\in[0,b]} cg(t)e^{tA} < +\infty$$

by continuity of g.

Step (iii): According to Step (ii) it is enough to prove that $sup_{v\in U} \| exp(tD_v) \|$ is bounded by a linear polynomial. But this follows from Step(i) and the fact that all the eigenvalues of D_v are purely imaginary and all its Jordan blocks have size less than two. $\diamond$

Remark 5.2.7 If the group is commutative then $F(v) = E(v)$, and D_v is nilpotent of order two.

Proof of Proposition 5.2.3: Take $v \in S_p$. Consider the geodesic $\gamma(t) = exp_p(tv)$ and an orthonormal basis $\{v, e_2, ..., e_n\}$ in T_pM. Take Jacobi fields J_i such that $J_i(0) = 0$ and $J_i'(0) = e_i$. Then as we observed in Remark 2.4.1,

$$| \, det A_v(t) \, | = \sqrt{| \, det < J_i(t), J_j(t) > |}.$$

Hence we have

$$| \, det A_v(t) \, | \leq \| \, J_2(t) \, \| \ldots \| \, J_n(t) \, \| \, .$$

But we also have that $\| \, J_i(t) \, \| \leq \| \, dg_t \, \|$. Therefore from equation (2.2) and the last lemma we obtain that

$$N(\lambda) \leq vol(SM) \int_0^\lambda p(t)^{n-1} dt.$$

◇

5.3 The commutative case

We will now state the main result of this section. Let Ric_M denote the Ricci curvature of the compact riemannian manifold M. Recall the definition of the set U at the begining of Section 5.1.

Theorem 5.3.1 *Let M^n be a compact riemannian manifold with $Ric_M > 0$. Suppose the isometry group of M^n contains a torus acting with codimension* *one. Then if U is dense and has finite connectivity, $N(\lambda)$ does not grow faster than λ^n.*

Remark 5.3.2 Note that in the real analytic category, U has full measure and finite connectivity.

Before proving Theorem 5.3.1 we will need some lemmas.

Suppose G is a compact subgroup of the isometry group of M, whose action on the tangent bundle has multiplicity two.

Let us make a definition: Take $v \in SM$ and $F \subset S(v)$ an isotropic subspace. We will say that t_o is a singular point of F if $dg_{t_o}F \cap V(g_{t_o}v) \neq \{0\}$.

Remark 5.3.3 It is known that if F is a lagrangian subspace, the set of singular points is discrete [26]. Since any isotropic subspace is contained in a lagrangian subspace we deduce that the set of singular points of an isotropic subspace is also discrete.

Lemma 5.3.4 *The function $n : U \to \mathbf{Z}$ given by*

$$n(v) = \textit{number of singular points of } F(v) \textit{ in } [0, \tau(v)),$$

is continuous. Moreover, if $n(v) > 0$ then $\tau(v) \geq inj(M)$ where $inj(M)$ denotes the injectivity radius of M.

Proof: Observe that in this case $\psi = d\varphi$ where φ is some isometry. Then $\varphi \circ \pi = \pi \circ \psi$ which implies that

$$d\psi(V(v)) = V(\psi v) = V(g_{\tau(v)}v).$$

Since $d\psi$ preserves F we deduce that t_o is a singular point if and only if $t_o+\tau(v)$ is a singular point.

Denote by $t_1(v), ..., t_{n(v)}(v)$ the singular points of $F(v)$ in $[0, \tau(v))$. Fix $v_o \in U$. We argue by contradiction. Suppose $n(v) < n(v_o)$. This implies, using the continuity of τ and the previous observation that there exists $1 \leq i \leq n(v_o)$ such that $t_i(v_o)$ is not a singular point for $F(v)$. F and V are continuous subbundles, therefore

$$F(g_{t_i(v_o)}v) \cap V(g_{t_i(v_o)}v) = \{0\}$$

implies that $F \cap V = \{0\}$ in a neighborhood of $g_{t_i(v_o)}v$. Now the continuity of g_t says that $g_{t_i(v_o)}v$, and $g_{t_i(v_o)}v_o$ are close. But

$$F(g_{t_i(v_o)}v_o) \cap V(g_{t_i(v_o)}v_o) \neq \{0\}.$$

This is not possible.

Suppose now that $n(v) > 0$. Then there exists $t_o \in [0, \tau(v)]$ such that t_o is a singular point for $F(v)$. Take $\zeta \in F(v)$ such that $dg_{t_o}\zeta \in V(g_{t_o}v)$. From part (b) of Lemma 5.2.5 we get $\theta_{g_{t_o}v}(dg_{t_o}\zeta) = dg_{t_o}\zeta$. Then from the definition of θ_v and the fact that ψ is an isometry of SM we obtain

$$dg_{t_o+\tau(v)}\zeta = d\psi_{g_{t_o}v} \circ dg_{t_o}\zeta.$$

But since $d\psi$ preserves V and F we have that $dg_{t_o+\tau(v)}\zeta \in V(g_{t_o+\tau(v)}v)$. Hence $J(t) = d\pi \circ dg_t\zeta$ is a Jacobi field that vanishes at t_o and $t_o + \tau(v)$. Therefore $\tau(v) \geq inj(M)$.

$\diamond$

Corollary 5.3.5 *Denote by $d(v)$ the supremum of the distance between two consecutive singular points of $F(v)$ on $[0, \tau(v)]$. If $n(v) > 0$ then*

$$\tau(v) \leq n(v)d(v).$$

Furthermore, on the connected component of U containing v, τ is bounded if d is bounded.

$\diamond$

Lemma 5.3.6 *Let $F \subset S(v)$ a lagrangian subspace. If $Ric_M \geq \delta > 0$, then the distance between two consecutive singular points of F is $\leq \frac{\pi}{\sqrt{\delta}}$.*

Proof: In [26] it is proved that if $dg_t F \cap V(g_t v) = \{0\}$ for $t \in [0, a]$ then the geodesic arc $\pi g_t v$, $t \in [0, a]$ does not contain conjugate points. But if $Ric_M \geq \delta > 0$, the distance between two consecutive conjugate points is $\leq \frac{\pi}{\sqrt{\delta}}$. The result follows.

$\diamond$

Corollary 5.3.7 *Suppose G is commutative and U has a finite number of connected components. If $Ric_M \geq \delta > 0$ then*

$$inj(M) \leq \tau(v) \leq m\frac{\pi}{\sqrt{\delta}},$$

where $m = sup_{v \in U} n(v) < +\infty$.

◇

Proof of Theorem 5.3.1: It follows directly from Proposition 5.2.3 and Corollary 5.3.7

◇

Finally let us discuss some applications of Theorem 5.3.1.

Example 5.3.8 Consider a 2-sphere with a convex metric with an S^1 action by isometries. In this case all the orbits of the S^1-action on the unit tangent bundle of S^2 are principal. Observe also that $St(S^1)$ consists only of two closed orbits. These two orbits project on the 2-sphere as a parallel with maximum curvature. Obviously this implies that U is connected and has full measure. In this case $n(v) = 2$ for every $v \in U$, and $\tau \leq \frac{2\pi}{\sqrt{\delta}}$ if the curvature is $\geq \delta > 0$.

Thus we have proved:

For a convex surface of revolution, $N(\lambda)$ does not grow faster than λ^2.

Note that if the surface of revolution is not convex, there can be hyperbolic closed geodesics, and hence τ is not bounded above (cf. Lemma 5.2.6). We do not know the precise growth of $N(\lambda)$ in this case.

Example 5.3.9 Consider the standard action of $SU(2) \times SU(2)$ on $M = SU(2) = S^3$ (cf. Example 2.4.6). Then any maximal torus T^2 in $SU(2)\times SU(2)$ acts on M with codimension one principal orbits. Consider a real analytic metric invariant under T^2 with $Ric_M > 0$. Then from Theorem 5.3.1 we deduce that for a metric as above, $N(\lambda)$ does not grow faster than λ^3.

5.4 The return time for $SO(3)$

For the case of $SO(3)$ with a left invariant metric the return time can be explicitly computed. Consider the canonical identification of $so(3)$ with $\mathbf{R}^3$ and let

$$< X, X >= \frac{X_1^2}{I_1} + \frac{X_2^2}{I_2} + \frac{X_3^2}{I_3}$$

be a positive inner product. To be definite, we will assume in what follows that $I_3 > I_2 > I_1 > 0$.

The above inner product gives rise to a left invariant metric on $SO(3)$. Let E be the ellipsoid on $\mathbf{R}^3$ given by $< X, X >= 1$. Using left translations we can identify the unit bundle of $SO(3)$ with $SO(3) \times E$. Hence the geodesic flow can be reduced to a flow on E. This flow is given by the Euler equations [1]. In this case the Euler equations can be solved in terms of elliptic functions. These computations are very classical and they can be found for example in [24]. From them one gets right away a formula for the return time. If we set $x = X_1^2 + X_2^2 + X_3^2$ then we have

$$\tau(x) = 4\sqrt{I_1 I_2 I_3} \int_0^{\pi/2} \frac{du}{\sqrt{(I_3 - I_2)(x - I_1) - (I_2 - I_1)(I_3 - x)sin^2 u}}, \quad x \in (I_2, I_3). \tag{5.3}$$

For $x \in (I_1, I_2)$ the formula for τ is obtained from the above by permuting the indices 1 and 3.

The derivative of τ is easy to compute. One finds that for $x \in (I_1, I_2)$, $\frac{d\tau}{dx}(x) > 0$ and for $x \in (I_2, I_3)$, $\frac{d\tau}{dx}(x) < 0$. We also get

$$\tau(I_1^+) = lim_{x \to I_1^+} \tau(x) = 2\pi \sqrt{\frac{I_1 I_2 I_3}{(I_1 - I_2)(I_1 - I_3)}}, \tag{5.4}$$

$$\tau(I_3^-) = lim_{x \to I_3^-} \tau(x) = 2\pi \sqrt{\frac{I_1 I_2 I_3}{(I_3 - I_2)(I_3 - I_1)}}, \tag{5.5}$$

$$lim_{x \to I_2} \tau(x) = \infty. \tag{5.6}$$

It is trivial from the above information to get the graph of τ.

One can check that $\tau(I_1^+) \geq \tau(I_3^-)$ if and only if $I_2 \leq \frac{I_1 + I_3}{2}$.

Remark 5.4.1 It follows from equation (5.6) that Corollary 5.3.7 is false in the non-commutative case. Moreover, Proposition 5.2.3 cannot be applied. Although we know that $N(\lambda)$ grows sub-exponentially because the topological entropy is zero, we do not have a better estimate for the growth as yet.

5.5 The Poisson sphere

There is an interesting application of some of the calculations in the last section.

Consider a left invariant metric on $SO(3)$ defined by

$$< X, X >= \frac{X_1^2}{I_1} + \frac{X_2^2}{I_2} + \frac{X_3^2}{I_3}$$

Let $SO(2)$ be any one-parameter subgroup. Then $SO(2)$ acts on $SO(3)$ from the left by isometries. The quotient, M_{I_1,I_2,I_3} is a 2-sphere, and we endow it with the submersion metric. This corresponds to the classical "Poisson reduction" and M is called *the Poisson sphere* [2]. It follows from a theorem of Lusternik and Schnirelmann [21] and estimates of Klingenberg and Toponogov that any convex metric on S^2 whose Gaussian curvature satisfies $1/\Delta < K <$

Δ, has at least three geometrically different closed geodesics with length in $(2\pi/\sqrt{\Delta}, 2\pi\sqrt{\Delta})$. That this is optimal is shown by a result of Morse:

Given any constant $N > 2\pi$ there exists an $\epsilon > 0$ such that any prime closed geodesic on an ellipsoid

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 1, \;\; a_1 < a_1 < a_3$$

and $\mid 1 - a_i \mid < \epsilon$, is either a principal ellipse or is larger than N.

We will prove a similar result for the Poisson sphere:

Theorem 5.5.1 *Given $N > 2\pi$ there exists an $\epsilon > 0$ such that any prime closed geodesic on the Poisson sphere M_{I_1,I_2,I_3} with $\mid 1 - I_i \mid < \epsilon$ has length $> N$, except for three closed geodesics with length close to 2π.*

Proof: According to Corollary 3.2.2 the Hamiltonian associated with the metric on M_{I_1,I_2,I_3} is collective for the canonical action of $SO(3)$ on $T^*(M)$ and its defining function is $f = \frac{X_1^2}{I_1} + \frac{X_2^2}{I_2} + \frac{X_3^2}{I_3}$. Consider the sphere bundle S in $T^*(M)$. Then the moment map ϕ of the $SO(3)$-action on $T^*(M)$ is a submersion from S to E where E is the ellipsoid $\frac{X_1^2}{I_1} + \frac{X_2^2}{I_2} + \frac{X_3^2}{I_3} = 1$. Let us apply the description of collective motion from Section 1.2. It is known that the Hamiltonian flow of f restricted to E has six critical points, 4 heteroclinic connections and closed orbits with period $\tau(x)$ where $x = X_1^2 + X_2^2 + X_3^2$ [1].

The six critical points, give rise to geodesics which are orbits of one-parameter subgroups, namely the one-parameter subgroups generated by $(\pm\sqrt{I_1}, 0, 0)$, $(0, \pm\sqrt{I_2}, 0)$ and $(0, 0, \pm\sqrt{I_3})$. Geometrically we only get three different closed geodesics whose length is clearly close to 2π if $\mid 1 - I_i \mid < \epsilon$.

Note that since $\phi : S \to E$ is a submersion, those are the only geodesics which are orbits of one-parameter subgroups.

Now, suppose $x(t)$ is a closed geodesic with length L, different from the ones described above. Then $\phi(x(t))$ is a closed curve in E. Thus $L \geq \tau(x)$ for all $x \in (I_1, I_2)$ and all $x \in (I_2, I_3)$. In other words

$$L \geq min\{\tau(I_3^-), \tau(I_1^+)\}$$

But from the equations (5.4) and (5.5) we see that given N there exists $\epsilon > 0$ so that if $| 1 - I_i | < \epsilon$ then $min\{\tau(I_3^-), \tau(I_1^+)\} > N$.

$\diamond$

5.6 More about complete integrability

Complete integrability guarantees via Liouville's Theorem the existence of an open dense subset U of the unit tangent bundle of M that is foliated by tori, and the geodesic flow on these tori is quasiperiodic. We will prove:

Theorem 5.6.1 *Let M^n be a compact riemannian manifold whose geodesic flow is completely integrable. Suppose for some p in M the unit sphere at p is contained in U. Then $\pi_1(M)$ has polynomial growth of degree $\leq n$ and if $\pi_1(M)$ is finite, M is rationally elliptic.*

Proof: For completely integrable geodesic flows we have the following property [2]: given $v \in U$ there exists an open invariant set W containing v

and a diffeomorphism $\psi : W \to V \times T^n$ where V is an open set in $\mathbf{R}^{\mathbf{n-1}}$ and T^n is an n-dimensional torus. Moreover,

$$\psi \circ g_t \circ \psi^{-1}(I, \varphi) = (I, \varphi + \omega(I)t),$$

where $\omega(I)$ is differentiable function and $(I, \varphi \ mod \ 2\pi)$ are coordinates in $V \times T^n$.

We can deduce that $d\psi \circ dg_t \circ d\psi^{-1}$ can be represented by the matrix

$$\begin{pmatrix} Id & \frac{\partial \omega}{\partial I}(I)t \\ \\ 0 & Id \end{pmatrix}.$$

Therefore we conclude that for $v \in U$, $\| \ dg_t(v) \ \|$ grows at most linearly, i.e. $\| \ dg_t(v) \ \| \leq at + b$. But the constants a and b depend on the torus and we do not know how they behave when we approach the boundary of U. But if $S_p \subset U$ for some p, we can cover S_p by a finite number of sets like W and in all of them $\| \ dg_t \ \|$ is bounded uniformly by a linear polynomial. Thus $I_p(\lambda)$ does not grow faster than λ^n (cf. proof of Proposition 5.2.3). As we know this implies that $\pi_1(M)$ has polynomial growth degree $\leq n$ and if $\pi_1(M)$ is finite, M is rationally elliptic.

$\diamond$

Bibliography

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag 1984.

[2] V. I. Arnold (Ed.), *Dynamical Systems III*, Encyclopaedia of Mathematical Sciences, Springer Verlag: Berlin 1988.

[3] M. Berger, R. Bott, *Sur les variétés à courbure strictement positive*, Topology **1** (1962), 302-311.

[4] R. Bowen, *Entropy for Group Endomorphisms and Homogeneous spaces*, Trans. of Am. Math. Soc. **153** (1971), 401-414.

[5] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Diff. Geom. **8** (1972), 623–628.

[6] E. I. Dinaburg, *On the relations among various entropy characteristics of dynamical systems*, Math USSR Izv. **5** (1971) 337-378.

[7] J.-H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. **66** (1982), 469–480.

[8] Y. Felix, S. Halperin, *Rational Lusternik-Schnirelmann category and its applications*, Trans. Amer. Soc. **273** (1982), 1-37.

[9] A.T. Fomenko, *The topology of surfaces of constant energy in integrable Hamiltonian systems, and obstructions to integrability*, Math. USSR Izv., **79**(1987) 629-650.

[10] A. T. Fomenko, *Integrability and nonintegrability in geometry and dynamics*, Kluwer Academic Publishers, Dordrecht, 1988.

[11] A. Freire, R. Mañé, *On the entropy of the geodesic flow in manifolds without conjugate points*, Invent. Math. **69** (1982), 375-392.

[12] D. Gromoll, W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974), 401–406.

[13] M. Gromov, *Entropy, homology and semialgebraic Geometry*, Séminarie Bourbaki 38éme année, 1985-86 n^o 663, 225-240.

[14] M. Gromov, *Homotopical effects of dilatation*, J. Diff. Geom. **13** (1978), 303-310.

[15] K. Grove, S. Halperin, *Contributions of Rational Homotopy Theory to global problems in Geometry*, Publ. Math. I.H.E.S. **56** (1982), 379-385.

[16] K. Grove, S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J. Diff. Geom. , **26** (1987) 429-459.

[17] V. Guillemin, S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge 1984.

[18] V. Guillemin, S. Sternberg, *On collective complete integrability according to the method of Thimm*, Ergod. Th. and Dyn. Syst. **3** (1983), 219–230.

[19] V. Guillemin, S. Sternberg, *Multiplicity-free spaces*, J. Diff. Geom. **19** (1984), 31–56.

[20] A. Katok, *Liapunov exponents, entropy and periodic orbits of diffeomorphisms*, Publ. Math. IHES, **51** (1980) 137-174.

[21] W. Klingenberg, *Riemannian Geometry*, De Gruyter, Berlin-New York, 1982.

[22] V. V. Kozlov, *Integrability and non-integrability in Hamiltonian Mechanics*, Russian Math. Surveys **38** (1983) 1-76.

[23] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Mathematica **38** (1979), 129–153.

[24] L.D. Landau, E.M. Lifshitz, *Mechanics*, Oxford, New York, Pergamon Press 1969.

[25] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, A Series of Modern Surveys in Math., Springer-Verlag: Berlin-Heidelberg 1987.

[26] R. Mañé, *On a Theorem of Klingenberg*, Pitman Research Notes in Math. Series **160**, 319-345.

[27] A. Manning, *Topological entropy for geodesic flows*, Ann. Math. **110** (1979), 567-573.

[28] A. S. Mishchenko, A. T. Fomenko, *Euler equations on finite-dimensional Lie groups*, Izv. Akad. Nauk SSSR, Ser. Mat. **42** (1978), 396–415.

[29] J. Moser, *Stable and Random Motions in Dynamical Systems*, Ann. Math. Studies n^0 77, Princeton, NJ: Princeton Univ. Press 1973

[30] J. Palis jr. and W. de Melo, *Geometric Theory of Dynamical Systems*, Springer-New York, 1982.

[31] G. P. Paternain, R. J. Spatzier, *New examples of manifolds with completely integrable geodesic flows*, Preprint series of the Stony Brook Inst. for the Math. Sciences (1990).

[32] S. Smale, *Differentiable Dynamical Systems*, Bull. Amer. Math. Soc., **73** (1967) 747-817.

[33] A. Thimm, *Integrable geodesic flows on homogeneous spaces*, Ergod. Th. and Dyn. Syst. **1** (1981), 495–517.

[34] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 287-300.

[35] W. Ziller, *Closed geodesics on homogeneous spaces*, Math. Z. **152** (1976), 67-88.

[36] W. Ziller, *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comm. Math. Helv. **52** (1977), 573-590.