Finite Topological Type and Vanishing Theorems for Riemannian Manifolds

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Abstract of Dissertation

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In this thesis we establish some vanishing and finitness theorems for the topological type of complete open riemannian manifolds under certain positivity conditions for curvature. Key tools are comparison techniques and Morse Theory of Busemann and distance functions. We also obtain some related results in the case of closed riemannian manifolds.

To My Parents In China

To My Wife Tianping

Contents

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Introduction

One of the most important aspects of Riemannian Geometry deals with the relationship between the curvature properties of a riemannian manifold and its topological structure.

A typical theorem is the Sphere Theorem (cf. [CE] for references) which says that if the sectional curvature of a closed simply connected n-manifold satisfies $\frac{1}{4} < K_M \leq 1$, then M is homeomorphic to the n-sphere.

In this thesis, we prove a vanishing theorem for homotopy groups which generalizes the Sphere Theorem. Examples show that Theorem A is sharp.

Theorem A. *Let M be a closed simply connected n-manifold whose kth Ricci curvature, for some $1 \leq k \leq n-2$, and sectional curvature satisfy*

$$Ric_{(k)} > \frac{1}{4} \quad and \quad K_M \leq 1.$$

Then $\pi_i(M) = 0$ for $1 \leq i \leq n-k$.

For a riemannian n-manifold M, we say the kth Ricci curvature of M, for some $1 \leq k \leq n-1$, satisfies $Ric_{(k)} \geq$ (resp. $>$) H for some constant H, if for every point $x \in M$ and every $(k+1)$-dimensional subspace $V \subset T_xM$,

$$\sum_{i=1}^{k+1} \langle R(e_i, v)v, e_i \rangle \geq \text{ (resp. } >) \; kH, \quad v \in V,$$

where $\{e_1, \cdots, e_{k+1}\}$ is any orthonormal basis for V.

So far, most of results are for closed manifolds. In this thesis, we mainly study complete open riemannian manifolds. We say a complete open n-

manifold M is proper if for some $p \in M$, and some sequence of closed subsets $\mathcal{C} = \{C_n\}_{n=0}^{\infty}$ with $r_n := d(p, C_n) \to +\infty$, the sequence of functions $b_n(x) = r_n - d(x, C_n)$, $x \in M$, converges to a proper Lipschitz continuous function $b_{\mathcal{C}}$ on M, as $r_n \to +\infty$. One notices that any complete open n-manifold M of nonnegative sectional curvature outside a compact subset must be proper. Moreover in this case M has finite topological type [CG1][GW2]. H. Wu proved that any complete open n-manifold M of positive kth Ricci curvature for some $1 \leq k \leq n-1$ and nonnegative sectional curvature outside a compact subset has the homotopy type of a CW complex with finite many cells each of dimension $\leq k-1$. By using the techniques developed in [SH][W2], this theorem can be sharpened in the following verson.

Theorem B. *Let M be a complete open proper n-manifold whose kth Ricci curvature satisfies $Ric_{(k)} > 0$ for some $k, 1 \leq k \leq n-1$. Then M has the homotopy type of a CW complex with cells each of dimension $\leq k-1$. In particular, $H_i(M; \mathbb{Z}) = 0$, for $i \geq k$.*

Theorem B should be viewed as a generalized version of the Gromoll-Meyer theorem [GM1] which states that every complete open riemannian manifold of positive sectional curvature is diffeomorphic with $\mathbb{R}^n$. In the case of $k = n-1$, Theorem B tells us that any complete open proper n-manifold of positive Ricci curvature has the homotopy type of a CW complex with cells each of dimension $\leq n-2$. Hence $H_{n-1}(M; \mathbb{Z}) = 0$. This is an analogue of a vanshing theorem for closed manifolds which says that any closed n-manifold M of positive Ricci curvature satisfies $H_1(M;\mathbb{R}) = H_{n-1}(M;\mathbb{R}) = 0$(cf. e.g. [BY]). Recently, Sha-Yang [SY1,2] constructed n-dimensional open manifolds of infinite topological type for each $n \geq 4$, on which the metrics can be chosen to be complete proper, and of positive Ricci curvature. Topologically, these examples are obtained by removing infinitely many disjoint balls D_i^{p+1}, $i = 0, 1, \cdots, +\infty$, in $\mathbb{R}^{p+1}$ and then gluing $S^{n-p-1} \times (\mathbb{R}^{p+1} \setminus \coprod_{i=0}^{\infty} D_i^{p+1})$ with $D^{n-p} \times \coprod_{i=0}^{\infty} S_i^p$ together by the indentity maps along the corresponding boundaries, where $2 \leq p \leq n-2$. Let $M_{n,p}$ denote the resulting manifolds. Clearly, the singular homology groups $H_{n-2}(M_{n,n-2};\mathbb{Z})$ are infinitely generated. In this sense, Theorem B is sharp.

It seems to be difficult to determine whether a complete open riemannian manifold M is proper or not, even if M has nonnegative Ricci curvature. However, if (with respect to a point) M has small diameter growth of ends, then M is proper.

There are several definitions for the (essential) diameter of ends (cf. [AG] [S1]). Let us give the easiest one here. Let M be a complete open riemannian manifold with finitely many ends. We define the diameter of ends $w_M(p,r)$ at distance r from a point $p \in M$ in the following way. Suppose M has N ends. Let $R > 0$ be a number such that $M \setminus \overline{B(p,R)}$ has exactly N unbounded connected components, say, $U_1, ..., U_N$. Then for $r > R$, $w_M(p,r)$ is defined as

$$w_M(p,r) = \sup_{1 \leq i \leq N} diam_M(U_i \cap \partial B(p,r)).$$

In particular, if M has only one end, then $w_M(p,r)$ is defined for all $r > 0$, and

$$w_M(p,r) = diam_M(\partial B(p,r)), \quad r > 0,$$

where $diam_M(\partial B(p,r)) = \sup_{x,y\in\partial B(p,r)} d(x,y)$ denotes the diameter of the geodesic sphere of radius r around p. By the splitting theorem of Cheeger-Gromoll [CG2], one concludes that any complete open manifold M of positive Ricci curvature has only one end, and M of nonnegative Ricci curvature has at most two ends. The diameter growth can control the behavior of the Busemann functions. In particular one has the following

Proposition. *Let M be a complete open riemannian manifold with one end. Suppose for some point $p \in M$,*

$$\limsup_{r\to+\infty} diam_M(\partial B(p,r))/r = \zeta < 1.$$

Then M is proper.

Therefore one has

Theorem C. *Let M be a complete open n-manifold with positive kth Ricci curvature for some $1 \leq k \leq n-1$. Suppose that for some $p \in M$,*

$$\limsup_{r\to+\infty} diam_M(\partial B(p,r))/r = \zeta < 1.$$

Then M has the homotopy type of a CW complex with cells each of dimension $\leq k-1$. In particular, $H_i(M;\mathbb{Z}) = 0$ for $i \geq k$.

In the case of $k = n-1$, Theorem C tells us that if a complete open proper n-manifold M of positive Ricci curvature satisfies

$$\limsup_{r\to+\infty} diam_M(\partial B(p,r))/r = \zeta < 1,$$

then M has the homotopy type of a CW complex with cells each of dimension $\leq n-2$.

It was proved by M. Gromov [G1] that there is a constant $C(n)$ depending on only n such that for any closed n-manifold M of nonnegative sectional curvature, the total Betti number of M with respect to any field $\mathbb{F}$ satisfies

$$\sum_{k=0}^{n} b_i(M;\mathbb{F}) \leq C(n).$$

By the Soul Theorem of Cheeger-Gromoll [CG1] , this theorem is also valid for complete open n-manifolds of nonnegative sectional curvature. Examples in [SY1, 2] and [AKL], however, show that this theorem does not hold for complete n-mainfold of nonnegative Ricci curvature. The Soul Theorem of Cheeger-Gromoll says that for any complete open n-manifold M, there is a closed totally geodesic submanifold S, to be called a soul, such that M is diffeomorphic with the normal bundle $\nu(S)$ of S in M (the diffeomorphism does not come from the exponential map of S, in general). In particular, M has finite topological type. Recently, Abrech-Gromoll [AG] proved that a complete open n-manifold M of nonnegative Ricci curvature has finite topological type if M has diameter growth of order $o(r^{\frac{1}{n}})$, provided that the curvature is bounded from below. We remark that their condition for diameter growth is weaker than that $w_M(p,r) = o(r^{\frac{1}{n}})$, as $r \to +\infty$. A modification of their argument gives the following

Theorem D. *Let M be a complete open n-manifold of nonnegative kth Ricci curvature for some $2 \le k \le n-1$. Suppose that the sectional curvature $K_M \ge -K$ for some constant $K > 0$, and for some point $p \in M$,*

$$\limsup_{r \to +\infty} \frac{w_M(p,r)}{r^{\frac{1}{k+1}}} < C(k) K^{-\frac{k}{2(k+1)}},$$

where $C(k) = [\frac{2(k+1)}{k}(\frac{(k-1)\ln 2}{2k})^k]^{\frac{1}{k+1}}$. Then M is homeomorphic to the interior of a compact manifold with boundary.

On the manifolds M of Sha-Yang's examples [SY1,2] of infinite topological type, the metrics can be chosen to be of positive Ricci curvature and bounded curvature. But the diameter growth condition is violated.

Without the restriction of diameter growth, one can still obtain some topological obstruction to complete open manifolds with nonnegative Ricci curvature and bounded curvature. Let M be a complete open riemannian n-manifold and let $p \in M$. For any $r > 0$, let $b_i(p,r)$ denote the rank of $i_* : H_i(B(p,r); \mathbb{F}) \hookrightarrow H_i(M; \mathbb{F})$, where $\mathbb{F}$ is an arbitary field. We will prove

Theorem E. *Let M be a complete open n-manifold with Ricci curvature $Ric(M) \ge 0$ and sectional curvature $K_M \ge -1$. Then there is a constant $C(n)$ depending only on n such that*

$$\sum_{i=0}^{n} b_i(p,r) \le C(n)(1+r)^n, \quad r > 0.$$

M. Gromov [G4] proved that for a complete manifold M of sectional curvature $-1 \le K_M < 0$, if M has finite volume, then M is diffeomorphic with the interior of a compact manifold with boundary. We will prove the following related result.

Theorem F. *Let M be a complete open manifold with sectional curvature $K_M \geq -K$ for some constant $K > 0$. Suppose that M has finite many ends and for some $p \in M$*

$$\limsup_{r \to +\infty} w_M(p, r) < \frac{\ln 2}{\sqrt{K}}.$$

then M is homeomorphic to the interior of a compact manifold with boundary.

Most of the results in this thesis were announced in [S1].

Chapter 1

Basic Riemannian Geometry

1.1 Riemannian Manifolds

We begin with some notations and basic facts. Let (M, g) be an n-dimensional riemannian manifold. Let ∇ denote the Levi-Civita connection of g. The curvature tensor R is defined as

$$R(x,y)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad x, y, z \in T_pM$$

where $X, Y, Z \in C^\infty(TM)$ with $X_p = x, Y_p = y$ and $Z_p = z$, respectively.

Let $\varphi : N \to M$ be a smooth map. Let $\varphi^*(TM)$ denote the induced bundle, $\varphi^*(TM) = \cup_{x \in N} T_{\varphi(x)}M$. In the local coordinates (x^α) and (x^i) at $x \in N$ and $\varphi(x) \in M$, respectively, a smooth section W along φ can be expressed locally as

$$W = f^i(x) \frac{\partial}{\partial x^i} \,|_{\varphi(x)}$$

for some smooth functions f^i in (x^α). If $V \in C^\infty(TN)$, we can define $\nabla_{\varphi_*(V)} W \in C^\infty(\varphi^*(TM))$, the covariant derivative of W in the direction

V. In the above local coordinates, if $V = a^\alpha(x)\frac{\partial}{\partial x^\alpha}$

$$\nabla_{\varphi_*(V)}W = a^\alpha(x)\frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial x^i}\mid_{\varphi(x)} + a^\alpha(x)f^i(x)\frac{\partial \varphi^j}{\partial x^\alpha}(x)\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}\mid_{\varphi(x)}.$$

Suppose $\varphi : N \to M$ is an immersion. Given a unit vector $\xi \perp \varphi_*(T_pN)$, set

$$h_\xi(X,Y) = -\langle \nabla_{\varphi_*(X)}\varphi_*(Y), \xi\rangle,$$

where $X, Y \in C^\infty(TN)$. It is easy to check that h_ξ depends only on X_p and Y_p. h_ξ is called the second fundamental form of φ at $p \in N$ in the direction of ξ.

Given a smooth curve $c : [a,b] \to M$ and a vector field V along c. Let $V'(t) = \nabla_{\dot{c}}V(t)$. c is called a geodesilc if $\nabla_{\dot{c}}\dot{c} = 0$. Clearly a geodesic c must be parametrized proportionally to arclength by $[a,b]$. Given a geodesic $\gamma : [a,b] \to M$, a vector field J along γ is called a Jacobi field if

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J + R(J,\dot\gamma)\dot\gamma = 0$$

Jacobi fields come from variations. Let $\alpha : [a,b] \times [c,d] \to M$ be any map with the property that for each $s \in [c,d]$, $\alpha_s := \alpha(\cdot, s)$ is a geodesic. Then $J = \frac{\partial \alpha}{\partial s}\mid_{s=0}$ is a Jacobi field along $\alpha_o := \alpha(\cdot, 0)$.

Lemma 1 (cf. e.g. [K; p99-91]). *Let (M,g) be a complete n-manifold with injectivity radius $inj(M) \geq i_o$. Fix $0 < r < i_o$. For any $q \in \partial B(p,r)$, let $\gamma : [0,r] \to M$ be the normal minimal geodesic issuing from p to q with $\dot\gamma(0) = w$ and $\dot\gamma(r) = \xi$. Then the second fundamental form of $\partial B(p,r)$ at q in the direction of ξ satisfies*

$$h_\xi(v,v) = \int_0^r \{|J'(t)|^2 - \langle R(J(t),\dot\gamma(t))\dot\gamma(t), J(t)\rangle\}dt, \quad v \in T_q\partial B(p,r),$$

where J is the Jacobi field along γ with $J(0) = 0$ and $J(r) = v$.

Proof:

Given $v \neq 0 \in T_q\partial B(p,r)$. Define a variation $\alpha : [0,r] \times [-\varepsilon, \varepsilon] \to M$ as

$$\alpha(t,s) = \exp_p t\eta(s),$$

where $\eta : [-\varepsilon,\varepsilon] \to S^{n-1}(r) \subset T_pM$ with the following proerties: (i) $\eta(0) = w$; (ii) $|\eta(s)| = r,\ s \in [-\varepsilon,\varepsilon]$; (iii) $(\exp_p)_*|_w\dot{\eta}(0) = v \in T_q\partial B(p,r)$. Let $T = \frac{\partial\alpha}{\partial t}$ and $J = \frac{\partial\alpha}{\partial s}$. Then $\dot{\gamma}(t) = T(t,0)$, and $J(t) := J(t,0)$ is the Jacobi field along γ with $J(0) = 0$ and $J(r) = v$. Let $L(s)$ be the arclength of curves $\alpha_s := \alpha(\cdot, s)$. Clearly, $L(s)$ is a constant. By the second variation formula [CE], one has

$$\begin{aligned} 0 &= L''(0) \\ &= \langle \nabla_J J, T\rangle|_0^r + \int_0^r \{\langle \nabla_T J, \nabla_T J\rangle - \langle R(J,T)T, J\rangle - (T\langle J,T\rangle)^2\}dt. \end{aligned}$$

By Gauss lemma, one has that $\langle J,T\rangle = 0$. Thus

$$\begin{aligned} h_\xi(v,v) &= -\langle \nabla_J J, \xi\rangle_q \\ &= \int_0^r \{|J'(t)|^2 - \langle R(J(t),\dot{\gamma}(t))\dot{\gamma}(t), J(t)\rangle\}dt. \end{aligned}$$

Q.E.D.

For $1 \leq k \leq n-1$, the kth-Ricci curvature Ric_p^V at $p \in M$ in a $(k+1)$-dimensional subspace $V \subset T_pM$ is defined as

$$Ric_p^V(x,y) = \sum_{i=1}^{k+1} \langle R(x,e_i)e_i, y\rangle, \quad x,y \in V$$

where $\{e_i\}_{i=1}^{k+1}$ is an orthonormal basis for V. We say the kth Ricci curvature $Ric_{(k)} \geq H$ (resp. $> H$) in a subset U of M for some constant H if at any point $p \in U$, in any $(k+1)$-dimensional subspace $V \in T_pM$,

$$Ric_p^V(x,x) \geq (resp. >) kH\langle x,x\rangle, \quad x \in V.$$

Recall that for a plane $P = span\{x,y\} \subset T_pM$, the sectional curvature $K_p(P)$ in the direction of P is defined as

$$K_p(P) = \frac{\langle R(x,y)y,x\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2},$$

and the Ricci curvature Ric_p at p is defined as

$$Ric_p(x,y) = \sum_{k=1}^{n}\langle R(x,e_i)e_i,y\rangle, \quad x,y \in T_pM.$$

We write $Ric_M \geq$ (resp.$>$) H if $Ric_p(x,x) \geq$ (resp. $>$) $(n-1)H\langle x,x\rangle$, $p \in M, x \in T_pM$. Thus $Ric_{(1)} \geq H$ (resp. $> H$) if and only if $K_M \geq H$ (resp. $> H$), and $Ric_{(n-1)} \geq H$ (resp.$> H$) if and only if $Ric_M \geq H$ (resp. $> H$). One notices that if $Ric_{(k)} \geq H$ for some $1 \leq k \leq n-1$, then $Ric_{(i)} \geq H$ for all $k \leq i \leq n-1$.

Lemma 2 *Let M be a riemannian $n-$manifold. Suppose at some point $p \in M$ the sectional curvature and the kth-Ricci curvature, for some $1 \leq k \leq n-1$, satisfy $|K_p| \leq K$ and $Ric_{(k)} \geq H$ for some constants K and H, respectively. Then for any orthonormal set $\{e_1, ..., e_k\}$ in T_pM and any unit vector v in T_pM,*

$$\sum_{i=1}^{k}\langle R(v,e_i)e_i,v\rangle \;\geq\; -(k-1)K(\alpha^2 + 4\alpha\beta) + kH\beta^2,$$

where $\alpha = \sqrt{\sum_{i=1}^{k}\langle v,e_i\rangle^2}$ *and* $\beta = \sqrt{1 - \sum_{i=1}^{k}\langle v,e_i\rangle^2}$.

Proof:

Let $V = span\{e_1, ..., e_k\}$ and $v = v_1 + v_2$ such that $v_1 \in V$ and $v_2 \perp V$. Clearly, $|v_1|^2 = \sum_{i=1}^{k} \langle v, e_i \rangle^2$ and $|v_2|^2 = 1 - \sum_{i=1}^{k} \langle v, e_i \rangle^2$. Let $\{f_1, ..., f_k\}$ be another orthonormal basis for V such that $f_1 = |v_1| f_1$. Let f_{k+1} be the unit vector such that $f_{k+1} \perp V$ and $v_2 = |v_2| f_{k+1}$. Consider the following identity for $i = 2, ..., k$

$$\begin{aligned} 2\langle R(f_i, f_1) f_{k+1}, f_i \rangle &= \langle R(f_i,\ (f_1 + f_{k+1}))(f_1 + f_{k+1}), f_i \rangle \\ &\quad - \langle R(f_i, f_1) f_1, f_i \rangle - \langle R(f_i, f_{k+1}) f_{k+1}, f_i \rangle. \end{aligned}$$

It turns out that

$$2 |\langle R(f_i, f_1) f_{k+1}, f_i \rangle| \leq 4K.$$

Thus

$$\begin{aligned} \sum_{i=1}^{k} \langle R(v, e_i) e_i, v \rangle &= \sum_{i=1}^{k} \langle R(f_i, v) v, f_i \rangle \\ &= |v_1|^2 \sum_{i=2}^{k} \langle R(f_i, f_1) f_1, f_i \rangle \\ &\quad + 2|v_1||v_2| \sum_{i=2}^{k} \langle R(f_i, f_1) f_{k+1}, f_i \rangle \\ &\quad + |v_2|^2 \sum_{i=1}^{k} \langle R(f_i, f_{k+1}) f_{k+1}, f_i \rangle \\ &\geq -(k-1) K (|v_1|^2 + 4|v_1||v_2|) + kH|v_2|^2. \end{aligned}$$

Q.E.D.

1.2 The Class $C(k)$

In [SH][W2], J. Sha and H. Wu independently studied a class of functions $C(k)$, $1 \leq k \leq n$, on a riemannian n-manifold. Let (M, g) be a riemannian $n-$manifold (not necessary to be complete) and let $p \in M$. Let f be a continuous function defined in a neighborhood of $p \in M$. Let $\gamma : (-a, a) \to M$ be a normal geodesic with $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = v \in T_pM$. Define the following extended real number:

$$Cf(p; v) = \liminf_{r \to 0} \frac{1}{r^2}\{f \circ \gamma(r) + f \circ \gamma(-r) - 2f \circ \gamma(0)\}.$$

We say f belongs to $C(k)$ at $p \in M$ for some k, $1 \leq k \leq n$, if f is Lipschitz continuous in a neighborhood W of p, and there are positive constants ε and η such that if $x \in W$ and $\{v_1, ..., v_k\}$ is set in T_xM with $|\langle v_i, v_j\rangle - \delta_{ij}| < \varepsilon$, then

$$\sum_{i=1}^{k} Cf(x; v_i) \geq \eta.$$

We say f belongs to $C(k)$ on a subset A of M if f is defined on a neighborhood U of A, such that $f \in C(k)$ at every point $p \in U$. Similarly, a function f is said to be C^∞ on a subset A, if f is defined on a neighborhood U of A such that $f \in C^\infty(U)$.

Clearly, a C^2 function $f : M \to \mathbb{R}$ belongs to $C(k)$ on M if and only if

$$\sum_{i=1}^{k} \nabla^2 f(V_i, V_i) > 0,$$

for any set of orthonormal vector fields $\{V_1, \cdots, V_k\}$ locally defined in M. Thus if a smooth Morse function $f : M \to \mathbb{R}$ belongs to $C(k)$ on M, then the index of f at each critical point satisfies that ind $(f) \leq k - 1$. In [W2] H. Wu proved the following properties for $C(k)$.

A) **The maximum-closure property:**

For any two continuous functions $f_1,\ f_2 : M \to \mathbb{R}$, if $f_1,\ f_2$ belong to $C(k)$ at $p \in M$, then $\max(f_1, f_2)$ belongs to $C(k)$ at p.

B) **The C^∞ stability property:**

For any compact subset K of M, any $f \in C(k)$ on K, there exists an $\varepsilon > 0$ such that for any $\varphi \in C^\infty(K)$ which is ε-close to zero on K in the C^2 topology, $f + \varphi$ belongs to $C(k)$ on K

C) **The semilocal approximation property:**

For any positive constant ε and any $f \in C(k)$ on a compact subset K of M, which is C^∞ in a (possibly empty) subset K_1 of K, there is a C^∞ function $F \in C(k)$ on K, such that $|F - f| < \varepsilon$ on K, and F is ε-close to f on K_1 in the C^∞ topology.

By Theorem 1.1 in [GW1], Wu concludes the following

Theorem 1 ([W2]) *Let (M, g) be a riemannian n-manifold . Let $f : M \to \mathbb{R}$ belong to $C(k)$ on M, and $\xi : M \to \mathbb{R}$ be a positive continuous function. Then there exists a C^∞ function $F : M \to \mathbb{R}$ which belongs to $C(k)$ such that*

$$|F - f| < \xi.$$

For applications in §2.2 below, one needs a refinement of Theorem 1 for proper functions $f : M \to \mathbb{R}$. A function $f : M \to \mathbb{R}$ is said to be proper if all the sets $\{x \in M;\ \ f(x) \leq c\}$ are compact for $c \in \mathbb{R}$.

Proposition 1 *Let (M, g) be a riemannian n-manifold . Let $f : M \to \mathbb{R}$ be proper and belong to $C(k)$ on M, and $\xi : M \to \mathbb{R}$ be a positive continuous* *function. Then there exists a proper Morse function $F : M \to \mathbb{R}$ which belongs to $C(k)$ on M such that*

$$|F - f| < \xi.$$

The proof of Proposition 1 strongly relies on the argument in [M1]. We begin with the following

Lemma 3 *Let W be a compact domain in M with boundary $\partial W = N_0 \cup N_1$ (resp. $\partial W = N$), where N_0, N_1 are disjoint. Suppose a smooth function $f : W \to [a, b]$ has the following properties*

(1) $f^{-1}(a) = N_0$ and $f^{-1}(b) = N_1$ (resp. $\min f = a$ and $f^{-1}(b) = N$);

(2) f has no critical points in a neighborhood of ∂W;

(3) f belongs to $C(k)$ on W.

Then for every $\varepsilon > 0$, there is a Morse function $F : W \to [a, b]$ (resp. $[a - \varepsilon, b]$), such that

(a) F coincides with f in a neighborhood of ∂W;

(b) $\|F - f\|_{C^2(W)} < \varepsilon$;

(c) F belongs to $C(k)$ on W;

where $\|\cdot\|_{C^2(W)}$ denotes the C^2-norm with respect to a fixed coordinate system for W.

Proof:

We will only handle the case of $\partial W = N_0 \cup N_1$. The proof is based on the argument in [M1]. Let U, V be open neighborhoods of ∂W such that $\bar{V} \subset U$. Let $\{U_\alpha\}$ be a finite cover of W by coordinate neighborhoods such that each set U_α lies in U or $W \setminus \bar{V}$. Take a compact refinement $\{C_\alpha\}$ of $\{U_\alpha\}$ and let C_0 be the union of all those C_α for which U_α lie in U. Let $U_1, \cdots, U_k$ be the coordinate neighborhoods in $W \setminus \bar{V}$. For each $h : W \to [a, b]$ and each $\delta > 0$, let

$$N_h(\delta) = \{\varphi : W \to [a, b];\ \|h - \varphi\|_{C^2(W)} \le \delta \ \text{ and } \ \varphi|_V = h|_V\}.$$

By the C^∞ stability property of $C(k)$ above and Lemma B in §2 in [M1] there is δ_0, $0 < \delta_0 \le \varepsilon$ such that for any φ with $\|\varphi - f\|_{C^2(W)} \le \delta_0$, (i) φ belongs to $C(k)$ on W; (ii) φ has no degenerate critical points in C_0. For each i, $1 \le i \le k$, let $\eta_i : U_i \to \mathbb{R}^n$ be the corresponding coordinate, and let $\lambda_i : M \to [0, 1]$ be smooth functions such that $\lambda_i = 1$ in a neighborhood of C_i and $\lambda_i = 0$ in a neighborhood of $M \setminus U_i$. For almost all choices of linear maps $L : \mathbb{R}^n \to \mathbb{R}$ the function $f_1(x) = f(x) + \lambda_1(x) L \circ \eta_1(x)$, $x \in M$ has no degenerate critical points in C_1. Notice that $f_1|_V = f|_V$. By choosing a smaller δ_0 if necessary, one can find a linear map L such that $f_1 \in N_f(\delta_0)$ has no degenerate critical points in C_1 (hence in $C_0 \cup C_1$). By induction, we can find a smooth function $f_i \in N_{f_{i-1}}(\delta_{i-1}) \subset \cdots \subset N_f(\delta_0)$ such that f_i has no degenerate critical points in $C_0 \cup \cdots \cup C_i$. Finally, $F := f_k \in N_{f_{k-1}}(\delta_{k-1}) \subset \cdots \subset N_f(\delta_0)$ has no degenerate critical points in $C_0 \cup \cdots \cup C_k = W$. Since $F \in N_f(\delta_0)$, F satisfies (a)(b) (c) above. We complete the proof. Q.E.D.

Proof of Proposition 1: By Theorem 1, there is a smooth function $\tilde{f} : M \to \mathbb{R}$ which belongs to $C(k)$ on M such that

$$|\tilde{f} - f| < \frac{1}{2} \min(\xi, 1).$$

Since f is proper, so is $\tilde{f}$. By Sard's Theorem, there is a sequence of numbers $\min \tilde{f} = a_0 < a_1 < a_2 < \cdots \to +\infty$, such that $a_i's$, $i \geq 1$, are the regular values of $\tilde{f}$. Thus the sets $V_i := \tilde{f}^{-1}(a_i)$, $i \geq 1$ are smoothly embedded and closed hypersurfaces. Set

$$\begin{aligned} W_0 &= \{x \in M;\ \tilde{f} \leq a_1\}, \\ W_i &= \{x \in M;\ a_i \leq \tilde{f} \leq a_{i+1}\}, \quad i = 1, 2, \cdots. \end{aligned}$$

Let $\varepsilon_i = \frac{1}{2}\min_{x\in W_i} \xi > 0$, $i = 0, 1, 2, \cdots$. By Lemma 3, there are smooth Morse functions $F_i : W_i \to [a_i, a_{i+1}]$ (resp. $F_0 : W_0 \to [a_0 - \varepsilon_0, a_1]$), $i = 1, 2, \cdots$, such that for all $i = 0, 1, 2, \cdots$,

(1) $\|F_i - \tilde{f}\|_{C^2(W)} \leq \varepsilon_i \leq \frac{1}{2}\xi$;

(2) F_i coincides with $\tilde{f}$ in a neighborhood of ∂W_i in W_i;

(3) F_i belongs to $C(k)$ on W_i.

We glue up F_i to construct a smooth Morse function F on M, i.e.,

$$F|_{W_i} = F_i|_{W_i}, \qquad i = 0, 1, 2, \cdots.$$

Clearly, F is the function as desired. Q.E.D.

The following algebraic lemma is elementary. It is useful to verify that a locally Lipschitz function f belong to $C(k)$ at a point $p \in M$.

Lemma 4 *Let V be an inner product space of dimension n. Let S be a symmetric bilinear form on V. Suppose that for some k, $1 \leq k \leq n$, and some positive numbers η and A, S satisfies*

(i) $\sum_{i=1}^{k} S(e_i, e_i) \geq \eta$ *for any orthonormal set $\{e_1, ..., e_k\}$ in V,*

(ii) $|S(v,v)| \leq A|v|^2$ *for* $v \in V$.

Then there is $\varepsilon > 0$ *depending only on* k, η *and* A *such that for any set* $\{v_1, ..., v_k\}$ *in* V *with* $|\langle v_i, v_j\rangle - \delta_{ij}| < \varepsilon$,

$$\sum_{i=1}^{k} S(v_i, v_i) \geq \frac{\eta}{2}.$$

1.3 Diameter of Ends

There are several definitions for the diameter of ends (cf. [AG] [S1]). Let us give the easiest one here, and the others will be discussed in Appendix. Let M be any complete open riemannian manifold with *finitely* many ends. For any subset $A \subset M$, let $diam_M(A)$ denote the diameter of A in M, i.e. $diam_M(A) = \sup_{x,y\in A} d(x,y)$. Suppose M has N ends with a fixed point $p \in M$. Let $R > 0$ be a number such that $M \setminus \overline{B(p,R)}$ has exactly N unbounded connected components $\{U_i\}_{i=1}^{N}$. Then the diameter of ends at distance r around p, $w_M(p,r)$, is defined as

$$w_M(p,r) = \max_{1\leq i\leq N} diam_M(\partial B(p,r) \cap U_i), \quad r > R.$$

In case M has only one end, then $w_M(p,r)$ is defined for all $r > 0$, and

$$w_M(p,r) = diam_M(\partial B(p,r)), \ \ r > 0.$$

Clearly, by definition, $w_M(p,r) \leq 2r$ for all $r > R$. A simple argument shows that for any $0 < \alpha \leq 1$, the following number, to be denoted by $diam_\alpha(M)$, is independent of choices of p and R.

$$diam_\alpha(M) = \limsup_{r\to+\infty} \frac{w_M(p,r)}{r^\alpha}.$$

We say M has diameter growth of order $o(r^{\alpha})$ (resp. $O(r^{\alpha})$) for some $0 < \alpha \leq 1$, if $diam_{\alpha}(M) = 0$ (resp. $diam_{\alpha}(M) < +\infty$).

In § 1.4 below, we will show that the growth of $w_M(p, r)$ can control the behavior of Busemann functions near infinity.

By Cheeger-Gromoll's splitting theorem [CG2], one can conclude that any complete open manifold M of nonnegative Ricci curvature has no more than two ends. In addition, if M has positive Ricci curvature at some point, then M has only one end. Thus in this case $w_M(p, r)$ is well defined.

Remark 1 In Appendix, we will give two other definitions of diameter of ends for complete open manifolds (possibly with infinitely many ends), both were given in [AG] and [S1], respectively. Since it is easier to estimate $w_M(p, r)$ than the others , we mainly consider $w_M(p, r)$ throughout this thesis.

1.4 Busemann Functions

Let M be a complete open riemannian n-manifold and let $p \in M$. Recall that the Busemann function B_{γ} associated with a ray γ issuing from p is defined as $B_{\gamma}(x) = \lim_{t \to +\infty} t - d(x, \gamma(t))$, $x \in M$. For arbitrary $t \geq 0$, let $R_t(p) = \{\gamma(t);\ \gamma \text{ is a ray issuing from } p\}$, which is a closed subset of the geodesic sphere $\partial B(p, t)$. Set $B_p^t(x) = t - d(x, R_t(p))$, $x \in M$. It is clear that $B_p^t(x)$ is increasing in t and $|B_p^t(x)| \leq d(p, x)$, $x \in M$. The Busemann function B_p is defined as $B_p(x) = \lim_{t \to +\infty} B_p^t(x)$, which is a Lipschitz function with Lipschtz constant 1. In fact B_p is just $\sup_{\gamma} B_{\gamma}$, where the supremum is taken over all rays γ issuing from p.

Set $L_p(x) = d(x, R_t(p))$, where $t = d(p,x)$. Since $B_p^t(x)$ is increasing in t , it is easy to see that

$$d(p,x) - L_p(x) \leq B_p(x) \leq d(p,x), \quad x \in M. \tag{1.1}$$

Set $E_p(x) = d(p,x) - B_p(x)$, $x \in M$. Then E_p is called the *excess function* associated with $p \in M$.

Now we are going to define the *generalized Busemann function* b_p associated with $p \in M$. The family of functions $b_p^t : M \mapsto R$ defined as $b_p^t(x) = t - d(x, \partial B(p,t))$, $t \in [0, +\infty)$ are Lipschitz continuous (with Lipschitz constant 1) and also satisfy $|b_p^t(x)| \leq d(p,x)$ (by the triangle inequality). Thus it is an equi-continuous family uniformly bounded on compact subsets. By Ascoli's theorem, there exists a subsequence of b_p^t, to be denoted by $b_p^{t_n}$, converging to a continuous function b_p on M, with its convergence being uniform on compact subsets of M. The function b_p is called the generalized Busemann function associated with the point p. Set $e_p(x) = d(p,x) - b_p(x)$, $x \in M$. Then e_p is called the *generalized excess function* associated with $p \in M$. By (1.1), one obtains

$$\begin{aligned} d(p,x) - L_p(x) &\leq B_p(x) \leq b_p(x) \leq d(p,x), \quad x \in M & (1.2) \\ e_p(x) \leq E_p(x) &\leq L_p(x), \qquad\qquad x \in M. & (1.3) \end{aligned}$$

The question is when the generalized Busemann function b_p is proper. By (1.2) we notice that if B_p is proper, then so is b_p. If for some point $p \in M$, b_p^t subconverges to a proper function b_p, then for every point $q \in M$, b_q^t subconverges to a proper function b_q (cf. Proposition 2 below).

A complete open riemannian manifold M is said to be *proper*, if there is a sequence of pointed closed subsets $\mathcal{C} = \{C_n, p\}_{n=0}^{\infty}$ with $r_n := d(p, C_n) \to$

$+\infty$, such that the sequence of functions $b_n := r_n - d(\cdot, C_n)$ converges to a proper function $b_{\mathcal{C}}$ at every point. It is not difficult to prove the following

Proposition 2 *Let M a complete proper open riemannian manifold. Then for any point $q \in M$, the family of functions $b_q^t = t - d(\cdot, \partial B(q,t))$ sub-converges to a proper Lipschitz continuous function b_q, as $t \to +\infty$, with the convergence being uniform on compact subsets.*

Proof:

Suppose that for some sequence of pointed closed subsets $\mathcal{C} = \{C_n, p\}_{n=0}^{\infty}$ with $r_n := d(p, C_n) \to +\infty$, $b_n(\cdot) := r_n - d(\cdot, C_n)$ converges to a proper function $b_{\mathcal{C}}$. Let $q \in M$ be any point. Let $t_n = d(q, C_n)$. By the same argument as above, one may assume that a subsequence of $b_q^{t_n}$, to be denoted by the same $b_q^{t_n}$, converges to a Lipschitz function b_q, with the convergence being uniform on compact subsets. Clearly, for any point $x \in M$ with $d(q,x) < t_n$, one has $x \in B(q, t_n)$ and $d(x, \partial B(q, t_n)) \leq d(x, C_n)$. Thus

$$\begin{aligned} b_q^{t_n}(x) &= t_n - d(x, \partial B(q, t_n)) \\ &\geq d(q, C_n) - d(x, C_n) \\ &= -(r_n - d(q, C_n)) + (r_n - d(x, C_n)). \end{aligned}$$

Letting $r_n \to +\infty$, one obtains

$$b_q(x) \geq -b_{\mathcal{C}}(q) + b_{\mathcal{C}}(x), \quad x \in M.$$

Since $b_{\mathcal{C}}$ is proper, one concludes that b_q is proper too. Q.E.D.

Remark 2 It was proved in [CG1][GW2] that any complete open manifolds with nonnegative sectional curvature outside a compact subset is proper. In particular [LT], for any point $p \in M$,

$$\lim_{d(p,x)\to+\infty} \frac{b_p(x)}{d(p,x)} = \lim_{d(p,x)\to+\infty} \frac{B_p(x)}{d(p,x)} = 1.$$

Conjecture 1 *Let M be a complete open riemannian $n-$manifold. If M has nonnegative Ricci curvature outside a compact subset, or asymptotically nonnegative Ricci curvature in the sence of* [AG], *then M is proper.*

In the rest part of this section, we will prove the following important result which tells us how the diameter growth control the behavior of Busemann functions near infinity. In particular, we prove that if a complete open manifold M has diameter growth of order $o(r)$, then M is proper. We begin with the following

Lemma 5 *Suppose M is a complete riemannian n-manifold with finitely many ends. Then there is R_M such that for any $x \in M \setminus \overline{B(p, R_M)}$,*

$$L_p(x) \;\; \leq \;\; w_M(p, d(p,x)). \tag{1.4}$$

Proof:

Suppose M has N ends. Let $R > 0$ be a number such that $M \setminus \overline{B(p,R)}$ has exactly N unbounded connected components $\{U_i\}_{i=1}^N$ (see § 1.3). Clearly, there are only finitely many *bounded* connected components , to be denoted by $V_1, \cdots, V_K$, such that $R_i := \sup_{x\in V_i} d(p,x) \geq 2R$. Thus for $R_M := \max_{1\leq i\leq K} R_i \geq 2R$,

$$M \setminus \overline{B(p,R_M)} = \cup_{1\leq i\leq N} U_i \setminus \overline{B(p,R_M)}. \tag{1.5}$$

By (1.5), for any $x \in M \setminus \overline{B(p, R_M)}$, there is an unbounded connected component U_{i_o} of $M \setminus \overline{B(p,R)}$ such that $x \in U_{i_o}$. Take a ray γ issuing from p such that $\gamma|_{(R,+\infty)} \subset U_{i_o}$. Let $r_o = d(p,x)$.

$$\begin{aligned} L_p(x) &= d(x, R_{r_o}(p)) \leq d(x, \gamma(r_o)) \\ &\leq diam_M(U_{i_o} \cap \partial B(p, r_o)) \leq w_M(p, r_o). \end{aligned}$$

Q.E.D.

Theorem 2 *Let M be a complete open manifold with finitely many ends. Suppose that for some $p \in M$,*

$$\limsup_{r \to \infty} \frac{w_M(p,r)}{r} = \zeta < 1.$$

Then

$$1 \geq \liminf_{d(p,x) \to \infty} \frac{b_p(x)}{d(p,x)} \geq \liminf_{d(p,x) \to \infty} \frac{B_p(x)}{d(p,x)} \geq 1 - \zeta.$$

In this case, Busemann functions B_p and b_p are proper. Thus M is proper.

Proof:

Theorem 2 follows from (1.4) and (1.2). Q.E.D.

The following lemma is important for further study.

Lemma 6 *Let M be a complete open riemannian n-manifold and let $p \in M$. Then for any point $q \in M$, there is a ray $\sigma_q : [0, +\infty) \to M$ issuing from q such that for all $t \geq 0$, $b_p^{q,t}(x) := b_p(q) + t - d(x, \sigma_q(t))$, $\quad x \in M$, supports $b_p(x)$ at q, i.e., $b_p^{q,t}(x) \leq b_p(x)$ for all $x \in M$ and $b_p^{q,t}(q) = b_p(q)$. Moreover,*

$$b_p(\sigma_q(t)) = b_p(q) + t, \quad t \geq 0. \tag{1.6}$$

Proof:

For each $t_n > 0$, there is a point $x_n \in \partial B(p, t_n)$ such that $d(q, x_n) = d(q, \partial B(p, t_n))$. Take a normal minimal geodesic σ_n issuing from q to x_n. By passing to a subsequence if necessary, one can assume that $\dot{\sigma}_n(0)$ converges to a unit vector $v \in T_qM$. Set $\sigma_q(s) = \exp_q sv$. It is clear that σ_q is a ray. Notice that for sufficiently large t_n

$$d(q, \partial B(p, t_n)) = t + d(\sigma_n(t), \partial B(p, t_n)).$$

Thus one obtains

$$\begin{aligned}
b_p(x) - b_p^{q,t}(x) &= b_p(x) - b_p(q) - t + d(x, \sigma_q(t)) \\
&= \lim_{t_n \to +\infty} d(q, \partial B(p, t_n)) - d(x, \partial B(p, t_n)) - t + d(x, \sigma_q(t)) \\
&= \lim_{t_n \to +\infty} d(\sigma_n(t), \partial B(p, t_n)) - d(x, \partial B(p, t_n)) + d(x, \sigma_q(t)) \\
&\geq \lim_{t_n \to +\infty} -d(\sigma_n(t), x) + d(x, \sigma_q(t)) \\
&\geq \lim_{t_n \to +\infty} -d(\sigma_n(t), \sigma_q(t)) = 0.
\end{aligned}$$

It is obvious that $b_p^{q,t}(q) = b_p(q)$. The equality (1.6) was proved by Wu in [W1]. Q.E.D.

Chapter 2

Vanishing Theorems

2.1 Manifolds with $Ric_{(k)} > \frac{1}{4}$ and $K_M \leq 1$

In the 1940's, S. Bochner devised an analytic technique to obtain vanishing theorems for some topological invariants (e.g. Betti numbers) on a closed riemannian manifold, under some curvature assumption. Roughly speaking, if let $\Lambda^k T^*M$ be the bundle of k-forms on a closed n-manifold M, Δ_H the Hodge-de Rham Laplacian, then one has the following Weitzenböck formula:

$$\Delta_H \alpha = D^* D\alpha + \mathcal{R}_k \alpha, \qquad \alpha \in C^\infty(\Lambda^k T^*M),$$

where $\mathcal{R}_k$, which is expressed in terms of the curvature operator, is a symmetric endomorphism of $\Lambda^k T^*M$, and D^* is the formal-adjoint of the differential operator $D : C^\infty(\Lambda^k T^*M) \to C^\infty(T^*M \otimes \Lambda^k T^*M)$. Since Δ_H is a elliptic and M is closed, from Hodge-de Rham Theorem it follows that the dimension of the kernel of Δ_H is equal to the kth Betti number $b_k(M)$ of

M. Thus if $\mathcal{R}_k$ is positive everywhere, then $b_k(M) = 0$ (cf. [BY]). S. Gallot and D. Meyer [GM] proved that if the manifold M has positive curvature operator, all the Betti numbers $b_k(M) = 0$, $1 \leq k \leq n-1$.

In this section we will establish a vanishing theorem for homotopy groups which generalizes the Sphere Theorem.

Theorem 3 *Let M be a complete and simply connected manifold of dimension $n \geq 3$ whose sectional curvature and kth Ricci curvature, for some $1 \leq k \leq n-2$, satisfy*

$$Ric_{(k)} > \frac{1}{4} \quad and \quad K_M \leq 1.$$

Then $\pi_i(M) = 0$ for $1 \leq i \leq n-k$.

Corollary 1 *Let M be a complete and simply connected n-dimensional manifold whose sectional curvature and Ricci curvature satisfy*

$$Ric_M > \delta(n) \quad and \quad K_M \leq 1,$$

where $\delta(n) = \frac{5}{8} - \frac{3}{8(n-1)}$, for even n, and $\delta(n) = \frac{5}{8} - \frac{3}{4(n-1)}$, for odd n. Then M is homeomorphic to the n-sphere S^n.

The Sphere Theorem says that, if M is a complete, simply connected n-dimensional manifold with $\frac{1}{4} < K_M \leq 1$, where K_M is its sectional curvature, then M is homeomorphic to the n-sphere S^n (cf. e.g. [CE] for references). If one assumes that a complete riemannian manifold M has Ricci curvature $Ric(M) \geq 1$ and sectional curvature $K_M \geq -K$ for some $K \geq 0$, then, whenever its volume is close to the volume of the unit sphere,

M is topologically similar to the sphere. See [S] and [I] for further information. We note that it is necessary to assume a lower bound for the volume in both [S] and [I].

We begin with Hartman's theorem [H].

Theorem 4 (Hartman). *Let M be a complete and simply connected manifold of dimension $n \geq 3$. If $K_M \leq 1$ and $Ric_{(n-2)} \geq \frac{1}{4}$, then the injectivity radius of M satisfies $inj(M) \geq \pi$.*

Let N be any compact n-dimensional riemannian manifold with boundary ∂N. For any $1 \leq k \leq n-1$, let $\Lambda_k : \partial N \to \mathbb{R}$ be the function on ∂N, $\Lambda_k(q) =$ the minimum of all sums of any k eigenvalues of the second fundamental form h_ξ of ∂N at q with respect to the outward-pointing unit narmal ξ. ∂N is called k-convex if $\Lambda_k > 0$ on ∂N. H.Wu [W2] proved the following

Theorem 5 *If an n-dimensional compact riemannian manifold N with $Ric_{(k)} > 0$ has k-convex boundary for some $1 \leq k \leq n-1$, then N has the homotopy type of a CW complex obtained from ∂N by attaching a finite number of cells each of dimension $\geq n-k+1$.*

J. Sha in his dissertation [SH] independently proved a weaker statement than Theorem 5. He assumed $\Lambda_k > 0$ on ∂N and nonnegative sectional curvature in N. But he also proved the very interesting result that the converse of this weaker statement in fact holds.

The following lemma tells us when does $N = M \setminus B_r$ have k-convex boundary.

Lemma 7 *Let M be a complete, simply connected riemannian manifold of dimension $n \geq 3$ with sectional curvature $K_M \leq 1$ and $Ric_{(k)} \geq \delta > \frac{1}{4}$ for some $1 \leq k \leq n-1$. Suppose that $inj(M) \geq \pi$. Then, $M \setminus B(p,r)$, for $\frac{\pi}{2\sqrt{\delta}} < r < \pi$, is a compact manifold with k-convex boundary.*

Proof:

Notice that $M \setminus B(p,r)$ has smooth boundary for all r, $\frac{\pi}{2\sqrt{\delta}} < r < \pi$. Let h_ξ be the second fundamental form for the boundary of $M \setminus B(p,r)$ with respect to the outward-pointing unit normal ξ, i.e., $h_\xi(x,y) = \langle \nabla_x \xi, y \rangle$, for $x, y \in T\partial B(p,r)$, where ∇ is the Levi-Civita connection of $M \setminus B(p,r)$. Let q be any point on the boundary of $M \setminus B(p,r)$ and $\gamma(t)$ the normal geodesic from p to q with $\dot\gamma(r) = -\xi$. Since $h_{-\xi} = -h_\xi$ is the second fundamental form for the boundary of $B(p,r)$ with respect to the outward-pointing unit normal to $\partial B(p,r)$, by Lemma 1 in §1.1 one has

$$h_\xi(J(r), J(r)) = -\int_0^r \{|J'(t)|^2 - \langle R(J(t), \dot\gamma(t))\dot\gamma(t), J(t)\rangle\} dt \qquad (2.1)$$

for any Jacobi field $J(t)$ along $\gamma(t)$ with $J(0) = 0$ and $J(t) \perp \dot\gamma(t)$. For any k orthonormal vectors $x_1, \cdots, x_k$ in $T_q\partial B(p,r)$, we can take k Jacobi fields $J_1, \cdots, J_k$ along $\gamma(t)$ with $J_i(0) = 0, J_i(r) = x_i$ and $J_i(t) \perp \dot\gamma(t), 1 \leq i \leq k$. Let $W_i(t) = f(t)X_i(t)$, where $X_i(t)$ is the parallel vector field along $\gamma(t)$ with $X_i(r) = x_i$ and $f(t) = \frac{\sin\sqrt{\delta}t}{\sin\sqrt{\delta}r}$. It follows from (2.1) and the basic index lemma [CE] that for $\frac{\pi}{2\sqrt{\delta}} < r < \pi$,

$$\sum_{i=1}^k h_\xi(x_i, x_i) \;=\; -\sum_{i=1}^k \int_{i=1}^r \{|J_i'(t)|^2 - \langle R(J_i(t), \dot\gamma(t))\dot\gamma(t), J_i(t)\rangle\} dt$$

$$
\begin{aligned}
&\geq -\sum_{i=1}^{k}\int_0^r \{|W_i'(t)|^2 - \langle R(W_i(t),\dot\gamma(t))\dot\gamma(t), W_i(t)\rangle\}dt \\
&= -k\int_0^r \{|f'(t)|^2 - \frac{1}{k}\sum_{i=1}^{k}\langle R(X_i(t),\dot\gamma(t))\dot\gamma(t), X_i(t)\rangle f(t)^2\}dt \\
&\geq -k\int_0^r \{|f'(t)|^2 - \delta f(t)^2\}dt \\
&= -k\sqrt{\delta}\frac{\cos\sqrt{\delta}r}{\sin\sqrt{\delta}r} \\
&> 0.
\end{aligned}
$$

Therefore $M\backslash B(p,r)$ has k-convex boundary for $\frac{\pi}{2\sqrt{\delta}} < r < inj(M)$. Q.E.D.

Proof of Theorem 3: Since M has $K_M \leq 1$ and $Ric_{(k)} \geq \delta > \frac{1}{4}$ for some constant δ, it follows from Theorem 4 and Lemma 7 above that $inj(M) \geq \pi$, and $M \setminus B_r$ is a compact manifold with k-convex boundary for $\frac{\pi}{2\sqrt{\delta}} < r < \pi \leq inj(M)$, where B_r is an open r-ball on M. From Theorem 5 we conclude that $M \setminus B_r$ has the homotopy type of a CW complex obtained from $\partial B_r \simeq S^{n-1}$, the $(n-1)$-sphere, by attaching a finite number of cells each of dimension $\geq n-k+1$. Thus M has the homotopy type of a relative CW complex obtained from $B_r \simeq D^n$, the $n-$disk, by attaching a finite number of cells each of dimension $\geq n-k+1$. By Theorem 6.1 in [Sw], one has $\pi_i(M,B_r) = 0$ for $1 \leq i \leq n-k$. Consider the following homotopy exact sequence (cf. e.g. Theorem 3.9 in [Sw]):

$$
\begin{aligned}
\cdots \to \pi_{n-k}(B_r) &\to \pi_{n-k}(M) \to \pi_{n-k}(M,B_r) \to \\
\cdots \to \pi_1(B_r) &\to \pi_1(M) \to \pi_1(M,B_r) \to \pi_0(B_r) \to \pi_0(M)
\end{aligned}
$$

Since $\pi_i(B_r) = \pi_i(M,B_r) = 0$ for $i \leq n-k \leq n-1$, one obtains

$$\pi_i(M) = 0. \qquad \text{for } \ i \leq n-k.$$

Q.E.D.

Proof of Corollary 1: For $n = 2$ Corollary 1 is obvious. For $n = 3$ Corollary 1 follows from Hamilton's work [HA]. Thus, from now on we assume that $n \geq 4$. Note that $Ric_M > \delta(n)$ with $K_M \leq 1$ implies that $Ric_{(k)} > \frac{1}{4}$ for $k = n - [\frac{n}{2}] \leq n - 2$. It follows from Theorem 3 that $\pi_i(M) = 0$ for $1 \leq i \leq [\frac{n}{2}]$. By the basic argument in algebraic topology, one can conclude that M is homeomorphic to the n-sphere S^n.

Q.E.D.

Following examples show that Theorem 3 is sharp.

Recall that any n-dimensional (normalized) symmetric space of rank one, M, has the following property [Ch] that there is a number $0 \leq \lambda \leq n-1$ such that for any unit vector $x \in T_pM$, the linear mapping $R_x := R(\cdot, x)x : x^\perp \to x^\perp$ has λ eigenvalues 1 and $n - 1 - \lambda$ eigenvalues $\frac{1}{4}$. λ's do not depend on the choices of $x \in T_pM$ and $p \in M$. In this sence we call M is of type λ. The classification theorem says that the only possibilities for λ are $\lambda = 0,\ 1,\ 3,\ 7,\ n-1$. The corresponding riemannian symmetric spaces are real projective spaces $\mathbb{R}P^n$, complex projective spaces $\mathbb{C}P^n$, quaternionic projective spaces $\mathbb{H}P^n$, Caley projective plane CaP^2, and n-sphere S^n. The non-trivial examples for Theorem 3 are quaternionic projective spaces and Caley projective plane. Notice that $\mathbb{H}P^n$ and CaP^2 are simply connected.

One can check that $M = \mathbb{H}P^n$ with a normalized metric has

$$\begin{aligned} K_M &\leq 1, \\ Ric_{(k)} &\geq \frac{1}{4} + \frac{3(k-n+4)}{4k}, \quad k \geq n-4. \end{aligned}$$

By Theorem 3, one obtains that $\pi_i(M) = 0$, for $i \leq 3$. The CW-structure on $M = \mathbb{H}P^n$ shows that $\pi_4(M) \neq 0$.

One can check that $M = CaP^2$ with a normalized metric has

$$\begin{aligned} K_M &\leq 1, \\ Ric_{(k)} &\geq \frac{1}{4} + \frac{3(k-n+8)}{4k}, \qquad k \geq n-8. \end{aligned}$$

Notice that $n = \dim M = 16$. By Theorem 3, one obtains that $\pi_i(M) = 0$, for $i \leq 7$. Clearly, $\pi_8(M) \neq 0$. Otherwise M is homeomorphic with the n-sphere by the generalized Poincaré conjecture. Thus Theorem 3 is sharp.

Corollary 1 is also sharp in dimensions $n = 4$, 8 and 16. With standard metrics, $\mathbb{C}P^2$ has $\max K_M = 1$ and $\min Ric_M = \frac{1}{2} = \delta(4), \mathbb{H}P^2$ has $\max K_M = 1$ and $\min Ric_M = \frac{4}{7} = \delta(8)$ and Cayley projective plane CaP^2 has $\max K_M = 1$ and $\min Ric_M = \frac{3}{5} = \delta(16)$.

2.2 Open Manifolds with $Ric_{(k)} > 0$

It was proved by H. Wu in [W1] that if a complete open n-manifold has nonnegative sectional curvature everywhere and positive sectional curvature outside a compact subset, then it is diffeomorphic to $\mathbb{R}^n$. This is a slight generalization of the Gromoll-Meyer theorem [GM1] which states that every complete open n-manifold of positive sectional curvature is diffeomorphic to $\mathbb{R}^n$. The Gromoll-Meyer theorem does not hold for complete open manifolds of positive Ricci curvature. However R. Schoen and S. T. Yau proved that any complete open 3-dimensional manifold of positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$ (see [SHY]). Since Busemann functions are not proper in general, it is difficult to obtain some topological obstructions for complete open manifolds with positive or nonnegative kth-Ricci curvature.

Now let M be any complete open n-manifold with nonnegative kth Ricci curvature everywhere and positive kth Ricci curvature outside a compact subset. It was proved by Wu [W2] that for any ray γ issuing from a point $p \in M$, the associated Busemann function B_γ belongs to $C(k+1)$ on M. Thus $B_p = \sup B_\gamma$ also belongs to $C(k+1)$. If B_p is proper, by Proposition 1 in § 1.2 and Morse theory one obtains that M has the homotopy type of a CW complex with cells each of dimension $\leq k$. However, as one can see, this conclusion is not sharp. Wu [W2] proved that if in stead of B_p is proper, M has nonnegative sectional curvature outside a compact subset, then M has the homotopy type of a CW complex with *finitely many* cells each of dimension $\leq k-1$. In this case M must have finite topological type. Thus most of examples of positive Ricci curvature do not satisfy this condition for sectional curvature. Using the techniques in [SH][W2], in fact one can establish a sharp vanishing theorem for complete proper manifolds of positive kth Ricci curvature (Theorem 6). Then a more adaptable condition (diameter growth) for complete n-manifold of positive kth Ricci curvature implies the same vanishing theorem for CW structure (Theorem 7).

Theorem 6 *Let M be a complete proper open n-manifold. Suppose that for some $1 \leq k \leq n-1$, the kth-Ricci curvature is nonnegative everywhere and positive curvature outside a compact subset. Then M has the homotopy* *type of a CW complex with cells each of dimension $\leq k-1$. In particular,*

$$H_i(M;\mathbb{Z}) = 0, \quad \textit{for } i \geq k.$$

Recall that a complete open n-manifold M is proper if and only if for some point $p \in M$, the family of functions $b_p^t(x) = t - d(x, \partial B(p,t))$, $t > 0$, sub-converges to a proper function b_p on M (see § 1.4).

Corollary 2 *Let M be a complete proper open n-manifold of positive Ricci curvature. Then M has the homotopy type of a CW complex with cells each of dimension $\leq n-2$. In particular, $H_{n-1}(M;\mathbb{Z}) = 0$.*

By Theorem 2 in §1.4 and Theorem 6 above, one has the following

Theorem 7 *Let M be a complete open n-manifold. Suppose that for some $1 \leq k \leq n-1$, the kth-Ricci curvature is nonnegative everywhere and positive curvature outside a compact subset of M. Assume that for some $p \in M$,*

$$\limsup_{r \to +\infty} \frac{diam_M(\partial B(p,r))}{r} = \zeta < 1.$$

Then M has the homotopy type of a CW complex with cells each of dimension $\leq k-1$. In particular,

$$H_i(M;\mathbb{Z}) = 0, \quad \textit{for } i \geq k.$$

Corollary 3 *Let M be a complete open n-manifold of positive Ricci curvature . Suppose that for some $p \in M$,*

$$\limsup_{r + \infty} \frac{diam_M(\partial B(p,r))}{r} = \zeta < 1.$$

Then M has the homotopy type of a CW complex with cells each of dimension $\leq n-2$. In particular, $H_{n-1}(M;\mathbb{Z}) = 0$.

Remark 3 That $H_n(M;\mathbb{Z}) = 0$ is valid for every open n-manifold. More precisely, on any open n-manifold M, one always can construct a proper Morse function f such that its index, ind (f), is not greater than $n-1$ at every critical point of f. Then by Morse Theory one obtains that M has the homotopy type of a CW complex with cells each of dimension $\leq n-1$. In particular, $H_n(M;\mathbb{Z}) = 0$ (cf. e.g. [Ph]).

Theorem 6 should be viewed as a generalized version of the Gromoll-Meyer's theorem [GM1]. Corollary 2 is an analogue of a vanishing theorem for closed manifolds which says that any closed n-manifold M of positive Ricci curvature satisfies $H_1(M;\mathbb{R}) = H_{n-1}(M;\mathbb{R}) = 0$(cf. e.g. [BY]). Recently, for all $n \geq 4$, Sha-Yang [SY1,2] constructed n-dimensional open manifolds of infinite topological type, on which the metrics can be chosen to be complete proper, and of positive Ricci curvature. Topologically, these examples are obtained by removing infinitely many disjoint balls D_i^{m+1}, $i = 0, 1, \cdots, +\infty$, in $\mathbb{R}^{m+1}$ and then gluing $S^{n-m-1} \times (\mathbb{R}^{m+1} \setminus \coprod_{i=0}^{\infty} D_i^{m+1})$ with $D^{n-m} \times \coprod_{i=0}^{\infty} S_i^m$ together by the indentity maps along the corresponding boundaries, where $2 \leq m \leq n-2$. Let $M_{n,m}$ denote the resulting manifolds. Clearly, the singular homology groups $H_{n-2}(M_{n,n-2};\mathbb{Z})$ are infinitely generated. In this sense, Theorem 6 is sharp.

In [An] using different methods, M. Anderson proved that if M is a complete open $n-$manifold of nonnegative Ricci curvature, then the first Betti number $b_1(M) = \dim H_1(M;\mathbb{Q}) \leq n-1$. For further information see [An].

We start with the following

Proposition 3 *Let M be a complete open n-manifold. Suppose that for some $1 \leq k \leq n-1$ the kth-Ricci curvature nonnegative everywhere and positive outside a compact subset. Assume that for some point $p \in M$, the generalized Busemann function b_p is proper. Then there exists a function $\chi \in C^2(\mathbb{R})$ such that $\chi \circ b_p$ is a proper function and $\chi \circ b_p$ belongs to $C(k)$ on M.*

Proof:

Let $a = \min_{x \in M} b_p(x)$. Let $R \geq a$ be a number such that such that $Ric_{(k)} > 0$ on $\{x \in M;\ b_p(x) \geq R\}$. Clearly, there is a positive continuous function $H(r)$ on $[a, +\infty)$ such that

$$Ric_{(k)} \geq H(r) \quad \text{ on } \{x \in M;\ r - a + R \leq b_p(x) \leq r - a + 2R\}.$$

Take $t(r) = \max\{\frac{16}{H(r)R};\ 2(2R - a)\}$. Let $K(r)$ be a positive continuous function on $[a, +\infty)$ suth that

$$|K_M| \leq K(r), \quad \text{ on } \{x \in M;\ b_p(x) \leq r + t(r)\}.$$

Take a positive continuous function $C(r)$ on $[a, +\infty)$ such that the following quadratic form is nonnegative

$$(C(r) - \frac{1}{8}kH(r)R)\alpha^2 - \frac{1}{3}(k-1)K(r)t(r)(\alpha^2 + 4\alpha\beta) + \frac{1}{8}kH(r)R\beta^2 \geq 0. \quad (2.2)$$

For example, one can choose $C(r) = \frac{1}{2kH(r)R}(\frac{8}{3}(k-1)K(r)t(r) + \frac{1}{2}kH(r)R)^2$. Set

$$\chi(t) = \int_a^t \exp(\int_a^s C(\tau)d\tau)ds + a$$

It is easy to check that χ is of class C^2 and has the following properties:

(i) $\chi'(r) \geq 1$ for all $r \in [a, +\infty)$

(ii) $\chi''(r) = C(r)\chi'(r)$ for all $r \in [a, +\infty)$

Clearly, (i) above implies $\chi \circ b_p$ is also proper. We claim that $\chi \circ b_p$ belongs to $C(k)$ on M.

From Lemma 6 in § 1.4 it follows that for each point $q \in M$, there exists a ray $\sigma_q(t)$ issuing from q such that

$$b_p^{q,t}(x) = b_p(q) + t - d(x, \sigma_q(t)), \quad t \geq 0,$$

supports $b_p(x)$ at q, and

$$b_p(\sigma_q(t)) = b_p(q) + t, \quad t \geq 0.$$

Fix any point $q \in M$ with $b_p(q) = r$. For each $v \in T_qM$, let $v(s)$ be the parallel vector field along $\sigma_q(s)$ such that $v(0) = v$. Then define θ : $T_qM \times [0, t(r)] \to M$ as

$$\theta(v, s) = \exp_{\sigma_q(s)}(1 - \frac{s}{t(r)})v(s).$$

Set

$$f_r^q(v) = r + t(r) - \int_0^{t(r)} |\frac{\partial \theta}{\partial s}(v, s)| ds.$$

Clearly, $f_r^q \circ \exp_q^{-1}$ supports b_p at q. For any orthonormal set $\{e_1, ..., e_k\}$ in T_qM, let $e_i(s)$ be the parallel vector fields along $\sigma_q(s)$ such that $e_i(0) = e_i$, $i = 1, ..., k$. Let α and β be nonnegative numbers such that $\alpha^2 + \beta^2 = 1$ and $\alpha^2 = \sum_{j=1}^k \langle e_j, \dot\sigma_q(0)\rangle^2$. By Lemma 2 in § 1.1, one has

$$\sum_{i=1}^k \langle R(e_j(s), \dot\sigma_q(s))\dot\sigma_q(s), e_j(s)\rangle \geq$$
$$-(k-1)\max|K_M||_{\sigma_q(s)}(\alpha^2 + 4\alpha\beta) + k \min Ric_{(k)}|_{\sigma_q(s)}\beta^2.$$

Notice that $1-\frac{s}{t(r)}\geq\frac{1}{2}$, for $s\in[R-a,2R-a]$. Thus

$$\begin{aligned}
C(r)\sum_{j=1}^{k}|e_jf_r^q|^2+\sum_{j=1}^{k}\nabla^2f_r^q(e_j,e_j) = &\\
& C(r)\sum_{j=1}^{k}\langle e_j,\dot\sigma_q(0)\rangle^2-\frac{1}{t(r)}\sum_{j=1}^{k}(1-\langle e_j,\dot\sigma_q(0)\rangle^2)\\
& +\sum_{j=1}^{k}\int_0^{t(r)}(1-\frac{s}{t(r)})^2\langle R(e_j(s),\dot\sigma_q(s))\dot\sigma_q(s),e_j(s)\rangle ds\\
\geq\ & C(r)\alpha^2-\frac{k}{t(r)}\\
& -(k-1)\int_0^{t(r)}(1-\frac{s}{t(r)})^2\max|K_M|(\alpha^2+4\alpha\beta)ds\\
& +k\int_0^{t(r)}(1-\frac{s}{t(r)})^2\min Ric_{(k)}\beta^2ds\\
\geq\ & C(r)\alpha^2-\frac{k}{t(r)}\\
& -(k-1)K(r)(\alpha^2+4\alpha\beta)\int_0^{t(r)}(1-\frac{s}{t(r)})^2ds\\
& +kH(r)\beta^2\int_{R-a}^{2R-a}(1-\frac{s}{t(r)})^2ds\\
\geq\ & C(r)\alpha^2-\frac{1}{16}kH(r)R-\frac{1}{3}(k-1)K(r)t(r)(\alpha^2+4\alpha\beta)+\frac{1}{4}kH(r)R\beta^2\\
\geq\ & \frac{1}{16}kH(r)R>0.
\end{aligned}$$

The last inequality above follows from (2.2). Clearly,

$$\begin{aligned}
\sum_{j=1}^{k}\nabla^2(\chi\circ f_r^q)(e_j,e_j) &= [C(r)\sum_{j=1}^{k}|e_jf_r^q|^2+\sum_{j=1}^{k}\nabla^2f_r^q(e_j,e_j)](\chi'\circ f_r^q)(q)\\
&\geq \frac{1}{16}kH(r)R(\chi'\circ b_p)(q)\\
&= \frac{1}{16}kH(r)R\chi'(r)\geq\frac{1}{16}kH(r)R.
\end{aligned}$$

Similarly one can check that there is a positive continuous function $A(r)$ on

$[a, +\infty)$, for example, $A(r) = \max(C(r), \frac{1}{t(r)} + \frac{1}{3}K(r)t(r))\chi'(r)$, such that

$$|\nabla^2(\chi \circ f_r^q)(v, v)| \leq A(r)|v|^2. \quad v \in T_qM.$$

It follows from Lemma 4 in §1.2 that there is a positive continuous function $\varepsilon(r)$ on $[a, +\infty)$ depending only on $kH(r)R$ and $A(r)$, such that for all vector set $\{v_1, \cdots, v_k\}$ in T_qM with $|\langle v_i, v_j\rangle - \delta_{ij}| < \varepsilon(r)$,

$$\sum_{j=1}^{k} \nabla^2(\chi \circ f_r^q)(v_j, v_j) \geq \frac{1}{32}kH(r)R.$$

Since $\chi \circ f_r^q$ supports $\chi \circ b_p$ at $q \in M$, one obtains

$$\sum_{j=1}^{k} C(\chi \circ b_p)(q; v_j) \geq \frac{1}{32}kH(r)R.$$

By definition, $\chi \circ b_p$ belongs to $C(k)$ on M. Q.E.D.

Proof of Theorem 6: Since M is proper, by Proposition 2 in § 1.4 , there is a point $p \in M$, such that the family of functions $b_p^t = t - d(\cdot, \partial B(p, t))$ sub-converges to a proper (generalized) Busemann function b_p. Then it follows from Proposition 3 above that there is a function $\chi \in C^2(\mathbb{R})$ such that $\chi \circ b_p$ is proper and belongs to $C(k)$ on M. By Proposition 1 in §1.2, there is a smooth proper Morse function F which belongs to $C(k)$ on M. Clearly, the index of F at each critical point satisfies that ind $(F) \leq k - 1$. Then Theorem 6 follows from the standard Morse theory [M2]. Q.E.D.

It is a long time conjecture that any complete open $n-$manifold of nonnegative Ricci curvature admits a sequence of compact domains $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $M = \cup_{i=1}^{k}\Omega_i$ and each Ω_i has smooth boundary with positive mean curvature. This conjecture is affirmative in case that M is a complete proper open manifold of positive Ricci curvature. In fact, we will prove the following

Proposition 4 *Let M be a complete proper n-manifold of positive kth Ricci curvature for some $1 \leq k \leq n-1$. Then M admits a sequence of compact domains with smooth boundary $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $M = \cup_{i=1}^{\infty} \Omega_i$ and each Ω_i has k-convex boundary.*

Proof:

Since M is proper, it follows from Propositions 2 and 3 that there is a proper smooth function $f : M \to \mathbb{R}$ such that f belongs to $C(k)$ on M. Choose a sequence of numbers $a_1 < a_2 < \cdots \to +\infty$ such that each a_i is a regular value of f. Set

$$\Omega_i = \{x \in M;\ \ f(x) \leq a_i\}.$$

Then each Ω_i has a smooth boundary $\partial\Omega_i$. Let $\{e_1, \cdots, e_k\}$ be any orthonormal set in $T_q\partial\Omega_i$, $q \in \partial\Omega_i$, and $\{\tilde{e}_1, \cdots, \tilde{e}_k\}$ be any extension of $\{e_1, \cdots, e_k\}$ to a set of tangent vector fields to $\partial\Omega_i$ near q. Since f belongs to $C(k)$, one has $\sum_{i=1}^k \nabla^2 f(e_i, e_i) > 0$. Notice that $\sum_{i=1}^k \nabla^2 f(e_i, e_i) = \sum_{i=1}^k \{\tilde{e}_i(\tilde{e}_i f)$ $-(\nabla_{\tilde{e}_i}\tilde{e}_i)f\} = -\sum_{i=1}^k (\nabla_{\tilde{e}_i}\tilde{e}_i)f$. Thus

$$-\sum_{i=1}^k (\nabla_{\tilde{e}_i}\tilde{e}_i)f > 0.$$

Let ξ be the outward-pointing unit normal at $q \in \partial\Omega_i$. Then the second fundamental form h_ξ at q satisfies

$$\begin{aligned}
\sum_{i=1}^{k} h_\xi(e_i, e_i) &= -\sum_{i=1}^{k} \langle \nabla_{\tilde{e}_i} \tilde{e}_i, \xi \rangle \\
&= -\frac{1}{\|grad\ f\|} \sum_{i=1}^{k} \langle \nabla_{\tilde{e}_i} \tilde{e}_i, grad\ f \rangle \\
&= -\frac{1}{\|grad\ f\|} \sum_{i=1}^{k} (\nabla_{\tilde{e}_i} \tilde{e}_i) f > 0.
\end{aligned}$$

Thus Ω_i has k-convex boundary. Q.E.D.

Corollary 4 *Let M be complete open $n-$manifold of positive kth Ricci curvature for some $1 \leq k \leq n-1$. Suppose that for some point $p \in M$,*

$$\limsup_{r \to +\infty} \frac{diam_M(\partial B(p,r))}{r} = \zeta < 1.$$

Then M admits a sequence of compact domains $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $M = \cup_{i=1}^{\infty} \Omega_i$ and each Ω_i has has k-convex smooth boundary.

Chapter 3

Finite Topological Type Theorems

3.1 Upper Bounds of the Betti Numbers

Let M be an n-dimensional connected closed n-manifold with sectional curvature $K_M \geq -K$ for some constant $K \geq 0$, and diameter $diam(M) \leq d$. Then M. Gromov [G1] proved that the total Betti numbers of M (with respect to any field $\mathbb{F}$) satisfies

$$\sum_{k=0}^{n} b_k(M) \leq C(n)^{1+\sqrt{K}d},$$

where $C(n) > 1$ is a constant depending only on $n = \dim M$. In particular, if M has nonnegative sectional curvature, then the total Betti numbers of M satisfies

$$\sum_{k=0}^{n} b_k(M) \leq C(n).$$

with the same constant $C(n)$ as above. Examples of Sha and Yang [SY1,2] show that this estimate does not hold if one assumes that M satisfies $Ric(M) \geq -H$ instead of $K_M \geq -K$. In this case, however, one can prove that the first Betti number of M (with respect to the field $\mathbb{R}$) satisfies $b_1(M) \leq C(n)^{1+\sqrt{H}d}$(see [G2]). For further information see [G3] [Li] [B].

The purpose of this section is to give an upper estimate of the Betti numbers for certain closed riemannian manifolds with curvature bounded from below, while the diameter can be arbitrarily large. The case of complete open manifolds with nonnegative Ricci curvature is also discussed. Before we state the main results, we would like to introduce a new concept of *width* for closed riemannian manifolds.

Let M be a complete riemannian manifold and let $d = diam(M)$ denote the diameter of M. For points $p, q \in M$ of maximal distance, set

$$w_{p,q} = \max\{\sup_{0\leq r\leq d} diam_M(\partial B(p,r)), \quad \sup_{0\leq r\leq d} diam_M(\partial B(q,r))\}.$$

The width of M, $w(M)$, is defined as

$$w(M) = \inf w_{p,q},$$

where the infimum is taken over all points $p, q \in M$ of maximal distance.

Theorem 8 *Given n, there is a constant $C(n) > 1$ depending only on n such that if a closed n-manifold M with sectional curvature $K_M \geq -K$, for some constant $K \geq 0$, satisfies $\sqrt{K}w(M) < \frac{1}{2}$, then the total Betti numbers of M satisfies*

$$\sum_{k=0}^{n} b_k(M) \leq C(n).$$

Let $H_k(X;\mathbb{F})$ denote the kth singular homology group of a subset X in a riemannian manifold M, where $\mathbb{F}$ is any fixed field. For any two subsets $i : X \subset Y \subset M$, let $b_k(X,Y)$ denote the rank of $i_* : H_k(X;\mathbb{F}) \hookrightarrow H_k(Y;\mathbb{F})$, and $b_k(X) = b_k(X,X) = \dim H_k(X;\mathbb{F})$. Notice that for subsets $X \subset \tilde{X} \subset \tilde{Y} \subset Y$ in M, $b_k(X,Y) \leq b_k(\tilde{X},\tilde{Y})$. In [G1], Gromov proved the following remarkable theorem. One can also refer to [A] for the details.

Theorem 9 (Gromov). *Let M be an n-dimensional complete riemannian manifold with sectional curvature $K_M \geq -1$. Then there is a constant $C(n) > 1$ depending only on n such that for any $0 < \varepsilon \leq 1$ and any bounded subset $X \subset M$,*

$$\sum_{k=0}^{n} b_k(X, U_\varepsilon X) \leq (1 + diam_M(X)/\varepsilon)^n C(n)^{1+diam_M(X)},$$

$U_\epsilon X$ denotes the ϵ-neigborhood of X in M.

In [GS], Grove-Shiohama introduced the concept of *critical points of distance functions*, which was used by Gromov [G1] to prove Theorem 9.

To prove Theorem 8, we consider points $p, q \in M$ of maximal distance $d = diam(M)$, and the functions $\zeta_p(x) : = \frac{1}{2}\{d(p,x) - d(q,x) + d\}$ and $\zeta_q(x) : = \frac{1}{2}\{d(q,x) - d(p,x) + d\}$. A point $x \in M$ is called a reqular point for ζ_p, ζ_q (or simply, p, q) if there exists $v \in T_xM$ such that

$$\langle v, \dot{\sigma}(0) - \dot{\tau}(0) \rangle > 0,$$

for all minimal geodesics σ and τ from x to p and q, respectively. A non-regular point is calle a critical point. It is easy to check that at any critical point x for p, q, there are minimal geodesics σ and τ from x to p and

q, repectively, such that $\angle(\dot{\sigma}(0), \dot{\tau}(0)) \leq \frac{\pi}{2}$ (cf. e.g. [E]). Notice that $0 \leq \zeta_p(x), \zeta_q(x) \leq d$, for $x \in M$. For $0 \leq r \leq d$, by $D(p,r)$ (resp. $D(q,r)$) we denote the subset $\{x \in M;\ \zeta_p(x) < r\}$ (resp. $\{x \in M;\ \zeta_q(x) < r\}$.

Lemma 8 (Isotopy Lemma). *If $0 < r_1 < r_2 < d$ and if $\overline{D(p,r_2)} \setminus D(p,r_1)$ contains no critical point for p, q. Then $\overline{D(p,r_2)} \setminus D(p,r_1)$ is homeomorphic to $\partial D(p,r) \times [r_1, r_2]$, for any r, $r_1 \leq r \leq r_2$.*

The idea of the proof can be found in [GS][G1].

Lemma 9 *Let M be a closed $n-$manifold with sectional curvature $K_M \geq -1$ and $d = diam(M) \geq 20$. Let $p, q \in M$ be any points of maximal distance d. If $w_{p,q} < \frac{1}{2}$, then $\overline{D(p,d-5)} \setminus D(p,5) = \overline{D(q,d-5)} \setminus D(q,5)$ contains no critical points for p, q.*

Proof:

Our proof is based on the idea of [AG] by applying Toponogov's Theorem to a thin triangle. Assume the contrary that $\overline{D(p,d-5)} \setminus D(p,5)$ contains a critical point x for p, q. Then there are minimal geodesics σ and τ from x to p and q, respectively, such that $\angle(\dot{\sigma}(0), \dot{\tau}(0)) \leq \frac{\pi}{2}$. Take a minimal geodesic γ from p to q, and let

$$e_{p,q} = \max_{x \in M}\{d(p,x) + d(q,x) - d\}.$$

It is easy to see that for $x_o \in M$ with $e_{p,q} = d(p,x_o) + d(q,x_o) - d$,

$$e_{p,q} \;=\; d(p,x_o) - d(p, \gamma(d - d(q,x_o))$$

$$\begin{aligned} &\le d(x_o, \gamma(d - d(q, x_o)) \\ &\le diam_M(\partial B(q, d(q, x_o))) \\ &\le w_{p,q}. \end{aligned} \tag{3.1}$$

Applying Toponogov's Theorem (cf. e.g. [CE]) to the triangle formed by σ, τ and γ, one obtains

$$\cosh d \le \cosh d(p, x) \cosh d(q, x).$$

Thus

$$\begin{aligned} 2 &\le 4e^{-d} \cosh d \\ &\le 4 \cosh d(p, x) \cosh d(q, x) \\ &\le e^{e_{p,q}} + e^{2(\zeta_p(x) - d)} + e^{2(\zeta_q(x) - d)} + e^{-2d} \\ &\le e^{\frac{1}{2}} + 3e^{-10} < 2. \end{aligned}$$

It is a contradiction. Q.E.D.

Proof of Theorem 8. For the sake of simplicity, we assume that $K_M \ge -1$. Let $C(n)$ be the same constant as in Theorem 9. Let $p, q \in M$ be any points of maximal distance $d = diam(M)$ such that $w_{p,q} < \frac{1}{2}$.

In case of $d < 20$, it is done by Theorem 9, i.e.,

$$\begin{aligned} \sum_{k=0}^{n} b_k(M) &= \sum_{k=0}^{n} b_k(B(p, 20), U_1 B(p, 20) \\ &\le 41^n C(n)^{41}. \end{aligned}$$

From now on we always assume $d \ge 20$. It follows from Lemma 9 that $\overline{D(p, d-5)} \setminus D(p, 5) = \overline{D(q, d-5)} \setminus D(q, 5)$ contains no critical points for

p, q. Since $w_{p,q} < \frac{1}{2}$, one has

$$\begin{aligned} \overline{D(p,5)} &\subset U_1\overline{D(p,5)} \subset \overline{D(p,7)}, \\ \overline{D(q,5)} &\subset U_1\overline{D(q,5)} \subset \overline{D(q,7)}, \end{aligned}$$

and

$$\partial D(p,7) \subset U_1\partial D(p,7) \subset \overline{D(p,9)} \setminus D(p,5).$$

Notice that $diam_M(\overline{D(p,5)})$, $diam_M(\overline{D(q,5)})$, $diam_M(\partial D(p,7)) \leq 20$. From Lemma 8, one concludes that

$$\begin{aligned} b_k(\overline{D(p,d-5)}) &= b_k(\overline{D(p,5)}) \\ &= b_k(\overline{D(p,5)}, \overline{D(p,7)}) \\ &\leq b_k(\overline{D(p,5)}, U_1\overline{D(p,5)}) \\ &\leq 21^n C(n)^{21}. \end{aligned}$$

Similarly,

$$b_k(\overline{D(q,d-5)}) = b_k(\overline{D(q,5)}) \leq 21^n C(n)^{21},$$

and

$$\begin{aligned} b_k(\overline{D(q,d-5)} \setminus D(p,5)) &= b_k(\partial D(p,7)) \\ &= b_k(\partial D(p,7), \overline{D(p,9)} \setminus D(p,5)) \\ &\leq b_k(\partial D(p,7), U_1\partial D(p,7)) \\ &\leq 21^n C(n)^{21}. \end{aligned}$$

Let $A_1 = \overline{D(p,d-5)}$ and $A_2 = \overline{D(q,d-5)}$. Notice that $A_1 \cup A_2 = M$ and $A_1 \cap A_2 = \overline{D(p,d-5)} \setminus D(p,5)$. The Mayer-Vietoris sequence,

$$\to H_k(A_1) \oplus H_k(A_2) \to H_k(A_1 \cup A_2) \to H_{k-1}(A_1 \cap A_2) \to$$

leads immediately to the estimate

$$\begin{aligned} b_k(M) &\le b_k(\overline{D(p,d-5)}) + b_k(\overline{D(q,d-5)}) \\ &\quad + b_{k-1}(\overline{D(p,d-5)} \setminus D(p,5)) \\ &\le 3 \cdot 21^n C(n)^{21}. \end{aligned}$$

This completes the proof. Q.E.D.

Remark 4 Let M be a closed riemannian manifold. The excess of M, $e(M)$, is defined as

$$e(M) = \inf e_{p,q},$$

where the infimum is taken over all points p, q of maximal distance (cf. [GP]). Clearly, by (3.1), one obtains

$$e(M) \le w(M).$$

The same argument as above shows that there is a constant $C(n) > 1$, if a closed $n-$manifold M with sectional curvature $K_M \ge -K$ for some constant $K > 0$, satisfies

$$\sqrt{K} e(M) < \frac{1}{2},$$

then

$$\sum_{k=0}^{n} b_k(M) \le C(n).$$

The author would like to thank Professor Peter Petersen V. for pointing out this to him. The above argument for Theorem 8 is also due to him and greatly simplies the original proof of the author.

Corollary 5 *Given $n, K \geq \lambda > 0$, there are constants $C(n)$ and $\varepsilon(n)(= 20^{-n})$ depending only on n such that if a closed $n-$manifold M satisfies the bounds : $Ric_M \geq \lambda$, $K_M \geq -K$, and*

$$(\frac{K}{\lambda})^{\frac{n}{2}}(\pi - \sqrt{\lambda} diam(M)) < \varepsilon(n),$$

then

$$\sum_{k=0}^{n} b_k(M) \leq C(n).$$

Proof:

The argument in [E] shows that there is a small number $\mu_n(= 4^{-n})$ such that if

$$\pi - \sqrt{\lambda} diam(M) < \mu_n,$$

then

$$\sqrt{K} e(M) \leq 10\sqrt{\frac{K}{\lambda}}(\pi - \sqrt{\lambda} diam(M))^{\frac{1}{n}}.$$

Then Corollary 5 follows from Remark 4 above. Q.E.D.

Remark 5 $\sqrt{K} diam(M)$ can be arbitrarily large, thus Theorem 9 above does not give a universal bound (depending only on dimension) for the total Betti numbers of M.

In the rest part of this section, we will study the "topological growth" of the geodesic balls in complete open manifolds of nonnegative Ricci curvature. We will prove the following

Proposition 5 *Let M be a complete open n-manifold and let $p \in M$. Suppose that M has Ricci curvature $Ric(M) \geq 0$ and sectional curvature $K_M \geq -1$. Let $b_i(p,r)$ denote the rank of $i_* : H_i(B(p,r);\mathbb{F}) \to H_i(M;\mathbb{F})$. Then $b_i(p,r)$, $0 \leq i \leq n$, satisfy*

$$\sum_{i=0}^{n} b_i(p,r) \leq C(n)(1+r)^n, \quad r > 0,$$

where $C(n)$ is a consant depending only on n.

Proof:

Let B be any ball in M with radius r and let $\rho > 1$. Denote by ρB the concentric ball of B with radius ρr. By Theorem 9, there is a constant $C_1(n)$ depending only on n such that for all balls B with radius $r \leq 1$ in M,

$$\sum_{i=0}^{n} b_i(B, 5B) \leq C_1(n). \tag{3.2}$$

The rest part of proof will rely on the following Topological Lemma which was proved by Gromov [G1].

Lemma 10 ([G1][A]). *Let M be a complete riemannian n-manifold and let $p \in M$. For any fixed numbers $r > 0$ and $r_o \leq 7^{-n-1}$, let $B_j^0 = B(p_j, r_o)$, $j = 1, \cdots, N$, be a ball covering of $B(p,r)$ with $p_j \in B(p,r)$. Let $B_j^k = 7^k B_j^0$, $k = 0, \cdots, n+1$. Then*

$$\sum_{i=0}^{n} b_i(B(p,r), B(p,r+1)) \leq$$

$$(e-1)Nt^n \sup\{\sum_{i=0}^{n} b_i(B_j^k, 5B_j^k);\ \ 0 \leq k \leq n,\ 1 \leq j \leq N\},$$

where t is the smallest number such that each ball B_j^n intersects at most t other balls $B_{j'}^n$.

Take $r_o = 7^{-n-1}$, and let $B(p_j, \frac{1}{2}r_o)$, $j = 1, \cdots, N$, be a maximal set of disjoint balls with $p_j \in B(p,r)$. Then $B_j^0 := B(p_j, r_o)$, $j = 1, \cdots, N$, is a covering of $B(p,r)$. By Bishop-Gromov's volume comparison theorem (cf. e.g. [G3]), one obtains

$$N \leq (1 + 4\frac{r}{r_o})^n \leq 4^n 7^{n^2+n}(1+r)^n.$$

Let $B_j^k = 7^k B_j^0$, $k = 0, \cdots, n+1$. Assume that B_j^n intersects t other balls $B_{j'}^n$. By the same volume comparison argument, one obtains

$$t \leq 5^n.$$

Since each ball B_j^k has radius ≤ 1, it follows from (3.2) and Lemma 10 above that

$$\sum_{i=0}^{n} b_i(B(p,r), M) \leq \sum_{i=0}^{n} b_i(B(p,r), B(p, r+1)) \leq C(n)(1+r)^n.$$

Q.E.D.

This proposition gives a topological obstruction to complete open manifolds M with nonnegative Ricci curvature and sectional curvature bounded from below.

3.2 Open Manifolds with $K_M \geq -K$

M. Gromov [G4] proved that for a complete manifold M of sectional curvature $-1 \leq K_M < 0$, if M has finite volume, then M is diffeomorphic with the interior of a compact manifold with boundary. We will prove the following related result.

Theorem 10 *Let M be a complete open riemannian manifold with sectional curvature $K_M \geq -K$ for some constant $K > 0$. Suppose that M has finitely many ends and for some $p \in M$,*

$$\limsup_{r\to\infty} w_M(p,r) < \frac{\ln 2}{\sqrt{K}};$$

Then M is homeomorphic to the interior of a compact manifold with boundary.

Remark 6 Theorem 10 above holds if instead of $w_M(p,r)$, M has the same growth as above for $\widetilde{diam}(p,r)$ which is defined in [AG].

Remark 7 In Theorem 10, the upper bound $\ln 2/\sqrt{K}$ must depend on K. Otherwise the connected sum of infinitely many copies of $S^2 \times S^2$ provides an easy counterexample.

Given a point p in a complete riemannian manifold M, a point $q \neq p$ is called a critical point of d_p if for any unit vector $v \in T_qM$ there exists a minimal geodesic γ issuing from q to p such that $\dot{\gamma}(0)$ and v make an angle at most $\frac{\pi}{2}$. We have the following version of Gromov's isotopy lemma [G1].

Lemma 11 (Isotopy Lemma) *Let M be a complete open riemannian n-manifold. Suppose that for some point $p \in M$, there is no critical point of d_p outside a compact subset of M. Then M is homeomorphic to the interior of a compact manifold with boundary.*

This lemma was known to Gromov [G1].

Lemma 12 *Let M be a complete open riemannian manifold with sectional curvature $K_M \geq -K$ for some constant $K > 0$, and let $p \in M$ be fixed. Suppose that $q \neq p$ is a critical point of d_p. Then*

$$e_p(q) \ \geq \ \frac{1}{\sqrt{K}} \ln \frac{\exp \sqrt{K} d(p,q)}{\cosh \sqrt{K} d(p,q)}. \tag{3.3}$$

Proof:

Suppose $\{b_p^{t_n}\}$ is a subsequence of $b_p^t = t - d(\cdot, \partial B(p,t))$, for which $b_p^{t_n}(x) = t_n - d(x, \partial B(p,t_n))$ converges to $b_p(x)$ on M. Let $x_n \in \partial B(p,t_n)$ be a point for which $d(q,x_n) = d(q, \partial B(p,t_n))$. Take a minimal geodesic γ issuing from p to x_n, and a minimal geodesic σ issuing from q to x_n. Since q is an $\alpha-$critical point of d_p, there exists a minimal geodesic τ issuing from q to p such that $\dot\sigma(0)$ and $\dot\tau(0)$ make an angle at most $\frac{\pi}{2}$. Applying Toponogov's Theorem [CE] to the triangle formed by γ, σ and τ, we have

$$\cosh \sqrt{K} t_n \ \leq \ \cosh \sqrt{K} d(q,x_n) \cosh \sqrt{K} d(p,q). \tag{3.4}$$

Multiplying (3.4) by $2 \exp \sqrt{K}(d(p,q) - t_n)$, and letting $t_n \to +\infty$, we obtain

$$\exp \sqrt{K} d(p,q) \ \leq \ \exp \sqrt{K} e_p(q) \cosh \sqrt{K} d(p,q). \tag{3.5}$$

Then Lemma 12 follows from (3.5). Q.E.D.

Proof of Theorem 10: By hypothesis, there is $R > 0$, such that

$$w_M(p,r) < \frac{1}{\sqrt{K}} \ln \frac{\exp \sqrt{K} r}{\cosh \sqrt{K} r}, \quad r \geq R.$$

It follows from Lemma 5 in §1.4 and (1.3) that there is an $R_o \geq R$, such that for $x \in M \setminus \overline{B(p, R_o)}$,

$$e_p(x) \leq L_p(x) < \frac{1}{\sqrt{K}} \ln \frac{\exp \sqrt{K} d(p,x)}{\cosh \sqrt{K} d(p,x)}.$$

Thus by Lemma 12 above there is no critical point of d_p in $M \setminus \overline{B(p, R_o)}$. Then Theorem 10 follows from Lemma 11 above. Q.E.D.

3.3 Open Manifolds with $Ric_{(k)} \geq 0$

It is well known result in Riemannian Geometry that any complete open riemannian manifold with nonnegative sectional curvature has finite topological type. This is a weak version of a theorem of Cheeger-Gromoll (cf. [CG1]). This kind of finiteness result does not hold for complete riemannian manifolds with nonnegative Ricci curvature (cf. [SY1,2]). The additional assumptions are therefore required. Recently U. Abresch and D. Gromoll [AG] proved that for any complete open Riemannian manifold M with nonnegative Ricci curvature and diameter growth of order $o(r^{\frac{1}{n}})$, M has finite topological type, where $n = \dim M$. Here the diameter growth means the growth of the essential diameter of ends, $\widetilde{diam}(p, r)$, which was defined in [AG].

The purpose of this section is to give a generalized version of Abresch-Gromoll's theorem [AG].

Theorem 11 *Let M be a complete open riemannian manifold of dimension n. Assume that the sectional curvature $K_M \geq -K$ for some constant $K >$*

0, and the kth Ricci curvature $Ric_{(k)} \geq 0$ for some $2 \leq k \leq n-1$. Suppose that for some point $p \in M$,

$$\limsup_{r \to +\infty} \frac{w_M(p,r)}{r^{\frac{1}{(k+1)}}} < C(k) K^{-\frac{k}{2(k+1)}};$$

where $C(k) = [\frac{2(k+1)}{k}(\frac{(k-1)\ln 2}{2k})^k]^{\frac{1}{k+1}}$. Then M is homeomorphic to the interior of a compact manifold with boundary.

Remark 8 Theorem 11 above holds if instead of $w_M(p,r)$, M has the same growth as above for $\widetilde{diam}(p,r)$.

Remark 9 In the case of $k = n-1$, as noted in [AG] the condition for diameter growth in Theorem 11 is violated in the Sha-Yang examples of positive Ricci curvature [SY1]. On these seven dimensional manifolds of infinite topological type, metrics can be chosen to have diameter growth of order $O(r^{\frac{2}{3}})$. However, the condition for diameter growth in Theorem 11 holds for Gromoll-Meyer examples [GM2]. All these examples have sectional curvature bounded from below.

Proof of Theorem 11. By the same argument as in § 3.2, Theorem 11 follows from Lemma 12 in § 3.2 and Lemma 14 below. Q.E.D.

We start with the following

Lemma 13 *Let M be a complete n-manifold whose kth-Ricci curvature $Ric_{(k)} \geq 0$ for some k, $1 \leq k \leq n-1$. Let $\rho = d(p,\cdot)$ be the distance function from some point $p \in M$. Let C_p be the cutlocus of p. Then ρ*

is smooth at any point $q \in \Omega_p := M \setminus C_p \cup \{p\}$ *and for any orthonormal set* $\{e_1, \cdots, e_{k+1}\}$ *in* T_qM *with grad* $\rho|_q \in span\{e_1, \cdots, e_{k+1}\}$,

$$\sum_{j=1}^{k+1} \nabla^2 \rho(e_j, e_j) \leq \frac{k}{\rho}.$$

Proof:

The proof is quite standard. Let γ be the minimal normal geodesic issuing from p with $\gamma(r) = q,\ r = d(p,q)$. For each $u \in T_qM$, let $u(t)$ be the parallel vector field along γ with $u(r) = u$. Then define $\alpha : T_qM \times [0,r] \to M$ as

$$\alpha(u,t) = \exp_{\gamma(t)} \frac{t}{r} u(t).$$

Set

$$f(u) = \int_0^r |\frac{\partial \alpha}{\partial t}(u,t)| dt.$$

Clearly, $\rho \circ \exp_q^{-1}$ supports f at $u = 0$, i.e., $\rho \circ \exp_q^{-1} u \leq f(u)$, for all u close to 0, and the equality holds at $u = 0$. Hence by the second variation formula [CE] one obtains

$$\begin{aligned} \nabla^2 \rho(u,u) &\leq \frac{d^2}{ds^2} f(su)|_{s=0} \\ &= \frac{1}{r}(1 - \langle u, grad\ \rho\rangle^2) - \int_0^r (\frac{s}{r})^2 \langle R(u(t), \dot\gamma(t))\dot\gamma(t), u(t)\rangle dt \end{aligned}$$

Thus

$$\sum_{j=1}^{k+1} \nabla^2 \rho(e_j, e_j) \leq \frac{1}{r}(k + 1 - \sum_{i=1}^{k+1} \langle e_j, grad\ \rho\rangle^2) = \frac{k}{r}.$$

Q.E.D.

Using a modification of the argument given in [AG], we prove the following

Lemma 14 *Suppose that M has nonnegative kth-Ricci curvature for some $2 \leq k \leq n-1$. Then for all $q \in M$ with $L_p(q) < d(p,q)$*

$$e_p(q) \leq \frac{2k}{k-1}\left[\frac{k}{2(k+1)} \times \frac{L_p(q)^{k+1}}{d(p,q)-L_p(q)}\right]^{\frac{1}{k}}.$$

Proof:

For $2 \leq k \leq n-1$ and $r > 0$, set

$$\varphi_r(t) = \frac{1}{(k-1)(k+1)}(t^{1-k} - r^{1-k})r^{k+1} + \frac{1}{2(k+1)}(t^2 - r^2)$$

It is easy to check that

a) $\varphi_r''(t) + \frac{k}{t}\varphi'(t) = 1$

b) $\varphi_r'(t) < 0 \quad \text{for } 0 < t < r$

c) $\varphi_r(r) = 0$

Now for any point $q \in M$ with $L_p(q) < d(p,q)$, fix r and C such that $\ell := L_p(q) < r < d(p,q)$ and $C > \frac{k}{d(p,q)-r} > C_o := \frac{k}{d(p,q)-\ell}$. Define a $f : \overline{B(q,r)} \to \mathbb{R}$ as

$$f(y) = C\varphi_r(d(q,y)) - e_p(y), \quad y \in \overline{B(q,r)},$$

where $e_p(y) = d(p,y) - b_p(y)$ is the excess function associated with p (see § 1.4). We claim that f has no locally maximal point in $B(q,r) \setminus \{q\}$. We prove it by contradiction. Suppose f has a locally maximal value at some point $x \in B(q,r) \setminus \{q\}$. Take a normal minimal geodesic γ issuing from p to x and a normal minimal geodesic τ issuing from q to x. By triangle inequality one can prove that $-\varepsilon - d(\cdot, \gamma(\varepsilon))$ supports $-d(\cdot, p)$ at x and $-\varepsilon - d(\cdot, \tau(\varepsilon))$ supports $-d(\cdot, q)$ at x, respectively. By Lemma 6, there is a ray σ_x issuing from x such that $b_p^{x,t}(y) := b_p(x) + t - d(y, \sigma_x(t))$ supports $b_p(y)$ at x. Therefore for small $\varepsilon > 0$

$$f_x^\varepsilon(y) := C\varphi_r(\varepsilon + d(y, \tau(\varepsilon))) + b_p(x) + \frac{1}{\varepsilon} - d(y, \sigma_x(\frac{1}{\varepsilon})) - \varepsilon - d(y, \gamma(\varepsilon))$$

is smooth near x and supports $f(y)$ at x. Thus f_x^ε is locally maximal at x and

$$\nabla^2 f_x^\varepsilon(v, v) \leq 0, \qquad v \in T_x M.$$

Let $\{e_1, ..., e_{k+1}\}$ be arbitraty orthornormal set in $T_x M$ such that $\dot{\gamma}(d(p, x))$, $\dot{\tau}(d(q, x))$ and $\dot{\sigma}_x(0)$ is in span $\{e_1, ..., e_{k+1}\}$. By Lemma 13, one has

$$\begin{aligned} 0 \geq \sum_{i=1}^{k+1} \nabla^2 f_x^\varepsilon(e_i, e_i) \;\; &\geq \;\; C[1 + \varphi_r'(d(q, x))(\frac{k}{d(x, \tau(\varepsilon))} - \frac{k}{d(q, x)})] \\ &\quad -k\varepsilon - \frac{k}{d(x, \gamma(\varepsilon))}. \end{aligned} \tag{3.6}$$

Since $C > \frac{k}{d(p,q)-r} > \frac{k}{d(p,x)}$, for $x \in B(q, r) \setminus \{q\}$, the right side of (3.6) is positive for sufficicently small $\varepsilon > 0$. It is a contradiction. Therefore one concludes that f has no locally maximal point in $B(q, r)$. Clearly, there is $z \in R_{r_o}(p)$ such that $d(q, z) = L_p(q) < r$, where $r_o = d(p, q)$. Thus, one has

$$\begin{aligned} f(z) \;\; &= \;\; C\varphi_r(d(q, z)) > 0 \\ f|_{\partial B(q,r)} \;\; &= \;\; -e_p|_{\partial B(q,r)} \leq 0. \end{aligned}$$

Therefore for any ρ, $0 < \rho < \ell = L_p(q)$

$$0 < f(z) \leq \max_{B(q,r) \setminus B(q,\rho)} f = \max_{\partial B(q,\rho)} f = C\varphi_r(\rho) - \min_{\partial B(q,\rho)} e_p,$$

which implies

$$\begin{aligned} e_p(q) \;\; &\leq \;\; \min_{\partial B(q,\rho)} e_p + 2\rho \\ &\leq \;\; 2\rho + C\varphi_r(\rho). \end{aligned}$$

Letting $r \to \ell = L_p(q)$ and $C \to C_o = \frac{k}{d(p,q)-\ell}$, one obtains

$$e_p(q) \;\leq\; \min_{0<\rho<\ell}(2\rho + C_o\varphi_\ell(\rho)), \quad 0 < \rho < \ell.$$

Notice that $h(\rho) := 2\rho + C_o\varphi_\ell(\rho)$ satisfies that $\lim_{\rho\to 0^+} h(\rho) = +\infty$ and $\lim_{\rho\to+\infty} h(\rho) = +\infty$. Then $h(\rho)$ has a minimal point $\rho_o \in (0, +\infty)$.

$$h'(\rho_o) = 2 + \frac{C_o}{k+1}(\rho_o - \rho_o^{-k}\ell^{k+1}) = 0. \tag{3.7}$$

It follows from (3.7) that

$$\rho_o < (\frac{C_o}{2(k+1)}\ell^{k+1})^{\frac{1}{k}}, \tag{3.8}$$

and

$$\rho_o < \ell. \tag{3.9}$$

By (3.7), (3.8) and (3.9), one obtains

$$\begin{aligned} e_p(q) &\leq \min_{0<\rho<\ell} h(\rho) \\ &= h(\rho_o) \\ &= \frac{2k}{k-1}\rho_o + \frac{C_o}{2(k-1)}(\rho_o^2 - \ell^2) \\ &\leq \frac{2k}{k-1}\rho_o \\ &\leq \frac{2k}{k-1}(\frac{C_o}{2(k+1)}\ell^{k+1})^{\frac{1}{k}}. \end{aligned}$$

Q.E.D.

Finally, we give the following applications of Theorem 6 in § 2.2 and Theorem 11 without proof.

Theorem 12 *Let M be a complete open riemannian manifold of dimension n. Assume that the sectional curvature satisfies $K_M \geq -K$ for some constant $K > 0$ and the kth Ricci curvature satisfies $Ric_{(k)} > 0$ for some $2 \leq k \leq n-1$. Suppose for some $p \in M$,*

$$\limsup_{r \to +\infty} \frac{w_M(p,r)}{r^{\frac{1}{(k+1)}}} < C(k) K^{-\frac{k}{2(k+1)}};$$

where $C(k) = [\frac{2(k+1)}{k}(\frac{(k-1)\ln 2}{2k})^k]^{\frac{1}{k+1}}$. Then M has the homotopy type of a CW complex with finitely many cells each of dimension $\leq k-1$.

Corollary 6 *Let M be a complete open riemannian manifold of dimension n. Assume that the sectional curvature satisfies $K_M \geq -K$ for some constant $K > 0$ and the Ricci curvature satisfies $Ric(M) > 0$. Suppose that for some $p \in M$,*

$$\limsup_{r \to +\infty} \frac{w_M(p,r)}{r^{\frac{1}{n}}} < C(n) K^{-\frac{n-1}{2n}};$$

where $C(n) = [\frac{2n}{n-1}(\frac{(n-2)\ln 2}{2(n-1)})^{n-1}]^{\frac{1}{n}}$. Then M has the homotopy type of a CW complex with finitely many cells each of dimension $\leq n-2$.

Appendix

Essential Diameter of Ends

In this appendix, we will give the other definitions of (essential) diameter of ends for complete open manifolds. First we will give the one defined in [S1].

Let M be a complete open riemannian n-manifold and let $p \in M$. For any $r > 0$, let $B(p,r)$ denote the geodesic ball of radius r around p. Let $C(p,r)$ denote the union of all unbounded connected components of $M \setminus \overline{B(p,r)}$. For $r_2 > r_1 > 0$, set $C(p;r_1,r_2) = C(p,r_1) \cap B(p,r_2)$. Let $1 > \alpha > \beta > 0$ be fixed numbers. For any two points $x, y \in C(p,\beta r)$, consider the distance $d_r(x,y) = \inf \mathrm{Length}(\phi)$ between x and y in $C(p,\beta r)$, where the infimum is taken over all smooth curves $\phi \subset C(p,\beta r)$ from x to y. Set $diam(\sum \cap \partial B(p,r), C(p,\beta r)) = \sup d_r(x,y)$, where $x, y \in \Sigma \cap \partial B(p,r)$, for any connected component $\sum$ of $C(p;\alpha r, \frac{1}{\alpha} r)$. Then the essential diameter of ends at distance r from p is defined as

$$diam(p,r) = \sup diam(\Sigma \cap \partial B(p,r), C(p,\beta r)),$$

where the supremum is taken over all connected components $\sum$ of $C(p;\alpha r, \frac{1}{\alpha} r)$.

Now we are going to define the essential diameter of ends, $\widetilde{diam}(p,r)$, which is given in [AG]. With the same notation as above, set $\widetilde{diam}(\tilde{\sum}, C(p,\beta r))$

$= \sup_{x,y \in \tilde{\sum}} d_r(x,y)$, for any connected component $\tilde{\sum}$ of $\partial C(p,r)$. Then $\widetilde{diam}(p,r)$ is defined as

$$\widetilde{diam}(p,r) = \sup \widetilde{diam}(\tilde{\Sigma}, C(p, \beta r)),$$

where the supremum is taken over all connected components $\tilde{\sum}$ of $\partial C(p,r)$. It is easy to see that

$$\widetilde{diam}(p,r) \le diam(p,r), \quad r > 0.$$

We will prove that the growth of $diam(p,r)$ can control the the behavior of Busemann functions near infinity. But in general, the growth condition for $\widetilde{diam}(p,r)$ does not give any information about Busemann functions. In fact we will prove that if M satisfies that $diam(p,r) = o(r)$ as $r \to +\infty$, i.e.,

$$\lim_{r \to +\infty} \frac{diam(p,r)}{r} = 0,$$

then M has finitely many ends and $diam(p,r) = w_M(p,r)$ for sufficiently large r. Conversely, if M has finitely many ends and satisfies that $w_M(p,r) = o(r)$ as $r \to +\infty$, then $diam(p,r) = w_M(p,r)$ for sufficiently large r . One notices that it is much easier to estimate $w_M(p,r)$ than $diam(p,r)$ or $\widetilde{diam}(p,r)$.

Proposition 6 (Abresch-Gromoll). *Let M be a complete open n-manifold. Suppose that M has nonnegative Ricci curvature . Then for any point $p \in M$,*

$$diam(p,r) \le C(n, \alpha, \beta) r,$$

where $C(p, \alpha, \beta)$ is a constant depending only on n, α, and β.

Proof:

For any connected component $\sum$ of $C(p; \alpha r, \frac{1}{\alpha} r)$, and any two points $x, y \in \sum \cap \partial B(p, r)$, there is a continuous curve $\phi : [0,\ 1] \mapsto \sum$ with $\phi(0) = x$ and $\phi(1) = y$. We choose a maximal set of disjoint geodesic balls $\{B(\phi(t_i), \varepsilon r)\}_{i=0}^{N}$ centered at ϕ with $0 = t_0 < \cdots < t_N = 1$, where $\varepsilon = \frac{\alpha - \beta}{2}$. Note that for any k, $0 \leq k \leq N$,

$$\bigcup_{i=0}^{N} B(\phi(t_i), \varepsilon r) \subset B(p, (\frac{1}{\alpha} + \varepsilon) r) \subset B(\phi(t_k), (\frac{2}{\alpha} + \varepsilon) r).$$

By the Bishop-Gromov Comparison Theorem one obtains that for some k, $0 \leq k \leq N$,

$$N + 1 \leq \frac{vol(B(\phi(t_k), (\frac{2}{\alpha} + \varepsilon) r))}{vol(B(\phi(t_k), \varepsilon r))} \leq C_1(n, \alpha, \beta).$$

Since the curve ϕ can be covered by $\{B(\phi(t_i), 2\varepsilon r)\}_{i=0}^{N} \subset C(p, \beta r)$, one can find a piecewise continuous geodesic contained in $C(p, \beta r)$ joining x and y with length at most $4\varepsilon N r \leq C(n, \alpha, \beta) r$. This implies that $d_r(x, y) \leq C(n, \alpha, \beta) r$, and therefore

$$diam(p, r) \leq C(n, \alpha, \beta) r.$$

Q.E.D.

Lemma 15 *Let M be a complete open manifold with a fixed point p. Suppose M ha diameter growth of order $o(r)$, i.e.*

$$\limsup_{r \to +\infty} \frac{diam(p, r)}{r} = 0.$$

Then there is $R > 0$ such that if U is an unbounded connected component of $M \setminus \overline{B(p, \alpha R)}$, then U has the following prperties:

(i) *for any* $r \geq R$, $U \cap \partial B(p,r) = \sum \cap \partial B(p,r)$ *for some connected component* $\sum$ *of* $C(p; \alpha r, \frac{1}{\alpha} r)$, *and* $d_r(x,y) = d(x,y)$ *for* $x, y \in \sum \cap \partial B(p,r)$;

(ii) *for any* $r \geq R$, $U \setminus \overline{B(p, \alpha r)}$ *has only one unbounded connected component, i.e.* U *has only one end.*

Proof:

Let $\varepsilon < \frac{1}{2}\min\{1-\alpha, \ \frac{1}{\alpha}-1\}$ be any small positive number. By hypothesis, there is $R > 0$ such that

$$diam(p,r) \leq \varepsilon r, \quad \text{for } r \geq \alpha R.$$

To prove (i) we choose a ray γ issuing from p such that $\gamma|(\alpha R, +\infty) \subset U$. For any $r \geq R$, let $\sum_r$ denote the connected component containing $\gamma(r)$ in $C(p; \alpha r, \frac{1}{\alpha} r)$. Fixing any $r_o \geq R$, we claim that $U \cap \partial B(p, r_o)$ is connected in $C(p, \alpha r, \frac{1}{\alpha})$. For any $x \in U \cap \partial B(p, r_o)$, since U is connected, there is a continuous curve ϕ :[0, 1]$\mapsto U$ with $\phi(0) = \gamma(r_o)$ and $\phi(1) = x$. For the simplicity's sake, let r(t) denote d(p,$\phi(t)$). Note that $r(t) > \alpha R$ for $t \in$[0, 1], and $r(0) = r(1) = r_o$. Let A denote $\{t \in [0,1]; \phi(t) \in \sum_{r(t)} \cap \partial B(p, r(t))\}$. Note that $\phi(0) = \gamma(r_o) \in \sum_{r_o} \cap \partial B(p, r_o) = \sum_{r(0)} \cap \partial B(p, r(0))$, which implies that $0 \in A$. An elementary argument shows that A is closed and open in [0, 1]. Therefore $1 \in A$, i.e. $x = \phi(1) \in \sum_{r(1)} \cap \partial B(p, r(1)) = \sum_{r_o} \cap \partial B(p, r_o)$. Since x is an arbitrary point in $U \cap \partial B(p, r_o)$, thus we conclude that $U \cap \partial B(p, r_o) = \sum_{r_o} \cap \partial B(p, r_o)$. Clearly, for any $x, y \in \sum_{r_o} \cap \partial B(p, r_o)$, $d(x,y) \leq d_{r_o}(x,y) \leq \varepsilon r_o$. Take a minimal geodesic segment σ joining x and y. Since $\varepsilon < \frac{1}{2}\min\{1-\alpha, \ \frac{1}{\alpha}-1\}$, it is easy to see that

σ is contained in the connected component $\sum_{r_o}$ of $C(p; \alpha r_o, \frac{1}{\alpha} r_o)$. Thus $d(x, y) = d_{r_o}(x, y)$.

To prove (ii) we suppose that for some $r \geq R$, $U \setminus \overline{B(p, \alpha r)}$ has at least two unbounded connected components, say, U_1 and U_2. It follows from (i) that $(U_1 \cup U_2) \cap \partial B(p, r) \subset U \cap \partial B(p, r)$ is contained in some connected component of $C(p; \alpha r, \frac{1}{\alpha} r)$. Thus $U_1 = U_2$. This is a contradiction. Q.E.D.

Proposition 7 *Let M be a complete open n-manifold. Suppose that for some $p \in M$,*

$$\limsup_{r \to +\infty} \frac{diam(p, r)}{r} = 0.$$

Then M has finitely many ends. Furthermore

$$diam(p, r) = w_M(p, r) \tag{3.10}$$

for sufficiently large r. Conversely, if M has finitely many ends and for some $p \in M$,

$$\limsup_{r \to +\infty} \frac{w_M(p, r)}{r} = 0,$$

then (3.10) holds for sufficiently large r.

Proof:

Let R be as in Lemma 15, and let $\{U_i\}_{i=1}^N$ be the set of all unbounded connected components of $M \setminus \overline{B(p, \alpha R)}$. It follows from Lemma 15 (ii) that M has N ends. Thus $w_M(p, r)$ is defined for $r > \alpha R$. From Lemma 15 (i) it follows that for $r \geq R$, $1 \leq i \leq N$, there is a connected component $\sum_i$ of

$C(p, \alpha r, \frac{1}{\alpha}r)$, such that $U_i \cap \partial B(p,r) = \sum_i \cap \partial B(p,r)$, and $d_r(x,y) = d(x,y)$ for $x, y \in \sum_i \cap \partial B(p,r)$. Thus

$$diam(\sum_i \cap \partial B(p,r), C(p, \beta r)) = diam_M(U_i \cap \partial B(p,r)).$$

Therefore by definition,

$$diam(p,r) = w_M(p,r), \quad \text{for} \quad r \geq R.$$

We omit the proof of the last statement. Q.E.D.

Lemma 16 *Let M be a complete open manifold with a fixed point p. Suppose that*

$$\limsup_{r \to \infty} \frac{diam(p,r)}{r} = 0.$$

Then there is an $\tilde{R} > 0$, such that for any $x \in M \setminus \overline{B(p, \tilde{R})}$,

$$L_p(x) \;\; \leq \;\; diam(p, d(p,x)), \tag{3.11}$$

Proof:

Let R be as in Lemma 15. it is easy to see that there are finitely many *bounded* connected components $\{V_i\}_{i=1}^K$ of $M \setminus \overline{B(p, \alpha R)}$, such that $R_i := \sup_{x \in V_i} d(p,x) \geq R$. Let $\tilde{R} = \max_{1 \leq i \leq K} R_i \geq R$. Then for any $x \in M \setminus \overline{B(p, \tilde{R})}$, there is an unbounded connected component U of $M \setminus \overline{B(p, \alpha R)}$, such that $x \in U$. Let γ be a ray emanating from p such that $\gamma|(\alpha R, \infty) \subset U$. Let $r_o = d(p,x) > R$. Lemma 15 (i) shows that

$$U \cap \partial B(p, r_o) = \sum \cap \partial B(p, r_o),$$

for the connected component $\sum$ of $C(p, \alpha r_o, \frac{1}{\alpha} r_o)$, which contains $\gamma(r_o)$. Thus

$$\begin{aligned} L_p(x) &= d(x, R_{r_o}(p)) \\ &\leq d(x, \gamma(r_o)) \\ &\leq diam(p, r_o) = diam(p, d(p, x)). \end{aligned}$$

Q.E.D.

Theorem 13 *Let M be a complete open manifold. Suppose that for some $p \in M$,*

$$\limsup_{r \to \infty} \frac{diam(p, r)}{r} = 0.$$

Then

$$\lim_{d(p,x) \to \infty} \frac{b_p(x)}{d(p, x)} = \lim_{d(p,x) \to \infty} \frac{B_p(x)}{d(p, x)} = 1.$$

In this case, Busemann functions B_p and b_p are proper. Thus M is proper.

Proof:

Theorem 13 follows from (3.11) and (1.2). Q.E.D.

Bibliography

[A] U. Abresch, *Lower curvature bounds, Toponogov's theorem, and bounded topology,* II, Ann. Scient. Éc. Norm. Sup., 20 (1987), 475-502.

[An] M. Anderson, *On the topology of complete manifolds of nonnegative Ricci curvature,* preprint.

[AKL] M. Anderson, P. Kronheimer and C. LeBrun, *Complete Ricci-flat Kähler manifolds of infinite topological type,* preprint.

[AG] U. Abresch and D. Gromoll, *On complete manifolds with nonnegative Ricci curvature*, to appear in J. Amer. Math. Soc. (1990).

[B] P. Berard, *From vanishing theorems to estimating theorems: the Bochner technique revisited,* Bull. Amer. Math. Soc. 19(1988), 371-406.

[BY] S. Bochner and K. Yano, *Curvature and Betti numbers,* Ann. of Math. Studies, no. 32, Princeton Univ. Press, 1953.

[CE] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry,* American Elsevier, New York, 1975.

[CG1] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature,* Ann. of Math., 96(1974), 413-443.

[CG2] J. Cheeger and D. Gromoll, *The splitting theorems for manifolds of nonnegative Ricci curvature,* J. Diff. Geom., 6(1971), 119-128.

[Ch] I. Chavel, *Riemannian symmetric spaces of rank one*, New York, M. Dekker, 1972.

[E] J. -H. Eschenburg, *Diameter, volume and topology for positive Ricci curvature,* preprint.

[G1] M. Gromov, *Curvature,diameter and Betti numbers,* Comment. Math. Helvetici, 56 (1981), 179-195.

[G2] M. Gromov, J. Lafontaine and P. Pansu, *Structure Métrique pour les Variétés Riemanniennes,* Cedic/Fernant Nathan(1981).

[G3] M. Gromov, *P. Levy's isoperimetric inequality,* preprint, Inst. Hautes Etudes Sci., 1980.

[G4] M. Gromov, *Manifolds of negative curvature,* J. Diff. Geom. 13(1978), 223-230.

[GM1] D. Gromoll and W. T. Meyer, *Complete open manifolds of positive curvature,* Ann. of Math., 96 (1969), 75-90.

[GM2] D. Gromoll and W. T. Meyer, *Examples of complete manifolds with positive Ricci curvature,* J. Diff. Geom., 21(1985), 195-211.

[GM] S. Gallot and D. Meyer, *Opórateur de courbure et laplacien des formes différentielles d'une variété riemannienne,* J. Math. Pures Appl. 54(1975), 259-284.

[GP] K. Grove and P. Petersen V.,*On the excess of metric spaces and manifolds,* preprint.

[GS] K. Grove and K. Shiohama, *A generalized sphere theorem,* Ann. of Math. 106(1977), 201-211.

[GW1] R. E. Greene and H. Wu, C^∞ *approximation of convex, subharmoinc and plurisubharmoinc functions,* Ann. Sci. École Norm. Sup. 12(1979), 47-84.

[GW2] R. E. Greene and H. Wu, *Integrals of subharmonic function on manifolds of nonnegative curvature,* Invent. Math., 27(1974), 265-298.

[H] P. Hartman, *Oscillation criteria for self-adjoint second-order differential systems and "principal sectional curvature",* J. Diff. Eq., 34(1979), 326-338.

[HA] R. Hamilton, *Three manifolds with positive Ricci curvature,* J. Diff. Geom., 17(1982), 255-306.

[I] Y. Itokawa, *On certain Riemannian manifolds with positive Ricci curvature,* Ph.D. Dissertation, State Univ. of New York at Stony Brook, 1982.

[K] W. Klingenberg, *Riemannian Geometry*, Berlin; New York: de Gruyter, 1982.

[Li] P. Li, *On the Sobolev constant and p-spectrum of a compact riemannian manifold,* Ann. Sci. École Norm. Sup. 13(1980), 451-469.

[LT] P. Li and L. F. Tam, *Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set,* Ann. Math., 125(1987), 171-207.

[M1] J. Milnor, *Lectures on the h-Cobordism Theorem,* Princeton Univ. Press, Pronceton(1965).

[M2] J. Milnor, *Morse theory,* Princeton Univ. Press, Princeton (1975).

[Ph] A. Phillips, *Submersions of open manifolds,* Topology 6(1966), 171-206.

[S1] Z. Shen, *Finiteness and vanishing theorems for complete open riemannian manifolds,* Bull. Amer. Math. Soc. (N.S.) 21(1989), 241-244.

[S2] Z. Shen, *A sphere theorem for manifolds of positive Ricci curvature,* Indiana Univ. Math. J., 38(1989), 299-233.

[S] K. Shiohama, *A sphere theorem for manifolds of positive Ricci curvature,* Trans. Amer. Math. Soc., 275(1983), 811-819.

[SHY] R. Schoen and S. T. Yau, *Complete three dimensional manifolds with positive Ricci curvature and scalar curvature, Seminar on Differential Geometry,* Ann. Math. Studies , ed. S. T. Yau, Princeton 102(1982), 209-228

[SY1] J. Sha and D. Yang, *Examples of manifolds of positive Ricci curvature,* To appear in J. Diff. Gem., (1989).

[SY2] J.Sha and D. Yang, *Positive Ricci curvature on the connected sums of* $S^n \times S^m$, preprint.

[SH] J. Sha, *p-convexity of manifolds with boundary*, Ph.D. Dissertation, State Univ. of New York at Stony Brook, 1986.

[Sw] R. Switzer, *Algebraic Topology – Homotopy and Homology* Springer-Verlag, Berlin Heidelberg New York (1975).

[W1] H. Wu, *An elementary method in the study of nonnegative cuvature,* Acta Math., 142(1979), 57-78.

[W2] H. Wu, *Manifolds of partially positive curvature,* Indiana Univ. Math. J., 36(1987), 525-548.