Rationality of Limiting η-Invariants of Collapsed 3-Manifolds

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Abstract of Dissertation

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We study the topological aspects of a sufficiently collapsed 3-manifold with bounded covering geometry. Based on the recent work of Cheeger and Gromov on collapsing Riemannian manifolds and F-structure theory, we find the existence of injective F-structures on such manifolds.

As an application, we prove a Cheeger and Gromov conjecture on rationality of limiting η-invariants associated to a volume collapse with bounded covering geometry in the 3-dimensional case. Another application is that we are able to give an explicite residue formula for computing the limiting η-invariants in terms of the information derived from an injective F-structure. We also discuss the uniqueness of injective F-structures and obtain a finite-

ness result on limiting η-invariants.

To my parents and Bin

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Introduction

In this thesis, we study the rationality of limiting η- invariants for collapsed Riemannian manifolds. These were defined and previously studied in [CG2]. We prove a conjecture of Cheeger and Gromov (see below) in the 3-dimensional case.

Let (M, g) be an n-dimensional complete Riemannian manifold with bounded sectional curvature, $|K| \leq 1$. Let $\alpha(g)$ denote one of the following geometric quantities associated to g: the upper bound on the injectivity radii; the volume of M; or the diameter of M. If α is sufficiently small relative to the sectional curvature, we will say that $\alpha(g)$ is sufficiently collapsed. If M admits a family of metrics $\{g_\delta\}$ so that the family $\{\alpha(g_\delta)\}$ converges to zero as $\delta \to 0$, one says that M admits an α-collapse.

Understanding the interplay between the collapsing geometry and the topology of M has been very fruitful in recent years. This can be viewed as studying the questions complimentary to those of controlling the topology by putting bounds on certain geometric quantities (for instance, Cheeger's finiteness theorem [Ch]). The basic questions are:

1) What kind of structures and invariants can be attached to a sufficiently α-collapsed manifold?

2) Does a sufficiently α-collapsed manifold imply an α-collapse?

Starting with [Gr1], there has been considerable progress in this field; for instance, Gromov's almost flat manifolds ([Gr1]) for sufficiently small diameters, the F-structure theory for uniformly small injectivity radii ([CG3], [CG4]) and the bundle structure theorems for manifolds which collapse to a manifold of lower dimension ([Fu1], [Fu2]) and more recently, the Nil-structure theorem ([CFG]).

Important progress has been made, concerning the implication of a volume collapse, in the work of Cheeger and Gromov in [CG1], [CG2] and [CG5]. There, they gave generalizations of Gauss-Bonnet theory to open manifolds of bounded sectional curvature and finite volume (note that such manifolds have their volume collapsed at infinity).

A volume collapse $\{g_\delta\}$ with bounded curvature is said to have bounded covering geometry (BCG) if the pull-back metrics of the family $\{g_\delta\}$ on the universal covering of M have a uniform lower bound on their injectivity radii.

Let N be a compact oriented $(4k-1)$-manifold. Consider the *η-invariant* of N in the sense of Atiyah, Patodi and Singer ([APS]). Cheeger and Gromov [CG2] proved that the limit of η-invariants with respect to a volume collapse with BCG exists and is a topological invariant (i.e., it does not depend on the specific collapsing sequence of metrics). We call this limit the *limiting η-invariant* and denote it by $\eta_{(2)}(N)$. Cheeger and Gromov made the following conjecture.

Conjecture I: The limiting η-invariant $\eta_{(2)}(N)$ is a rational number.

The difficulty in proving Conjecture I stems from the fact that the metrics in a given volume collapse with BCG may not be obviously related to one another. In addition, it is very difficult to compute the η-invariant explicitly except in circumstances which are special in one way or another. In this thesis, we prove Conjecture I for 3-manifolds.

Theorem A. *Let N be a compact oriented 3-manifold. Suppose N admits a volume collapse with BCG, then the limiting η-invariant $\eta_{(2)}(N)$ is a rational number.*

Our approach to Theorem A is to show that sufficiently collapsed 3-manifolds with BCG admit a certain topological structure, the injective F-structure. Roughly an F-structure can be thought of as a family of local torus actions satisfying a certain consistency condition. It was introduced in [CG3]. An F-structure partitions the underlying manifold into orbits. An injective F-structure is a F-structure satisfying the condition that the fundamental group of each orbit injects into the fundamental group of the total space. We will prove

Theorem B. *There exists a constant $\epsilon > 0$ such that if a compact 3-manifold N admits a Riemannian metric g satisfying:*

(i) $|K_g(N)| \leq 1$

(ii) $Injrad(N, g) < \epsilon$

(iii) $\widetilde{geo}(N, g) \geq 1$ *(BCG)*

then either N admits an injective F-structure or N (possibly some doubling cover of it) is diffeomorphic to a lens space. In particular, a compact 3-manifold N admits a volume collapse with BCG if and only if N admits an *injective F-stucture.*

An F-structure is said to be polarized, if the local orbits have the same dimension as the local group which acts. We observe that when specialized to the case of an injective F-structure the volume collapse constructed using a polarized F-structure as in [CG3] has BCG. This together with Theorem B implies that an injective F-structure on a 3-manifold fully reflects the volume collapse with BCG. A consequence of this is that one can compute $\eta_{(2)}(N)$ with respect to a volume collapse compatible with an injective F-structure (in general, the limiting η-invariant associated to a volume collapse compatible with a polarized F-structure $\mathcal{F}$ is called the limiting η-invariant associated to $\mathcal{F}$ and is denoted by $\eta(N, \mathcal{F})$. $\eta(N, \mathcal{F})$ is independent of invariant volume collapse with or without BCG). By filling in a compact 4-manifold M and extending the injective F-structure to M by using the so-called equivariant plumbing technique, the residue theory for the secondary characteristic classes as developed in [Ya] implies the rationality of $\eta(N, \mathcal{F})$ and thus the rationality of $\eta_{(2)}(N)$.

In proving Theorem B our starting point is the result obtained in [CG4]. There Cheeger and Gromov associate an F-structure $\mathcal{F}_g$ (topological structure) to a sufficiently injectivity radius collapsed metric g. $\mathcal{F}_g$ is constructed on the scale of small injectivity radius of the manifold, priori, it reflects neither small volume nor BCG. Using some theorems from 3-dimensional topology and some property of $\mathcal{F}_g$ which depends on BCG, we find the existence of an injective F-structure satisfying the BCG condition. Note that the existence of an injective F-structure may not be deduced from the geometrical construction in [CG4] (also see example 7).

We also disuss the uniqueness of injective F-structures on a 3-manifold. Two injective F-structures on a 3-manifold are said to be weakly isomorphic, if they are different only on a saturated area which is homeomorphic to $T^2 \times I$ (see Definition 9). Based on the classification theorem for graph manifolds obtained in [Wa1] and [Wa2], we obtain the following.

Theorem C. *Let N be a closed orintable 3-manifold which is not homeomorphic to either $S^2 \times S^1$ or a nilmanifold. Suppose N admits an injective T-structure $\mathcal{F}$. Then $\mathcal{F}$ is unique up to weak isomorphism.*

From our approach to Theorem A, not only does one get the rationality of the limiting η-invariant, but one can compute explicitly the value $\eta_{(2)}(N)$ and see its topological meaning in the 3-dimensional case.

In the second part of this thesis, we give the explicit residue formula for $\eta_{(2)}(N)$ in terms of topological data derived from the injective F-structure (see Theorem 11). Theorem C implies that except for a few cases such a residue formula is intrinsic in a certain sense. For instance, we express $\eta_{(2)}(N)$ for N an injective Seifert manifold in terms of Seifert invariants (we refer to [Or] for Seifert manifolds).

Theorem D. *Let $N = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$ be the Seifert invariant of an injective Seifert manifold N. Then*

$$\eta_{(2)}(N) = -\sum_{i=1}^{r} s_i + \epsilon(b, r) + \frac{1}{3}(b + r) + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{ij} + \frac{1}{3}\sum_{i=1}^{r} \frac{\alpha_{i,s_i-1}}{\alpha_i} \quad (.1)$$

where $\frac{\alpha_i}{\alpha_i-\beta_i} = [b_{i,1}, ..., b_{i,s_i}]$ *is the continued fraction* ($b_{i,j} \geq 2$), $\alpha_{i,j} = b_{i,j}\alpha_{i,j-1} - \alpha_{i,j-2}$; $\alpha_{i,0} = 1$, $\alpha_{i,1} = b_{i,1}$ *and*

$$\epsilon(b,r) = \begin{cases} -1 & if\ b+r>0 \\ +1 & if\ b+r \leq 0. \end{cases}$$

A consequence of Theorem D is that the invariants $\eta_{(2)}(N)$ can take any rational number in 3-dimension. However, using Gromov's pre-compactness theprem, the result in [Fu2] and Theorem D, we prove a finiteness result for limiting η-invarants as follows:

Theorem E. *For a real number* $D > 0$*, let* $\mathcal{M}^3(D)$ *be the collection of closed orientable 3-manifolds which admit volume collapses with BCG for which the diameters stay bounded by* D*. Then* $\{\eta_{(2)}(N) \ \ mod\, Z|\ N \in \mathcal{M}^3(D)\}$*is a finite set.*

Note that $\mathcal{M}^3(D)$ contains infinitely many diffeomorphic types for any $D > 0$. For instance, 3-dimensional nilmanifolds.

The result in the last part of this thesis is about a bounded version of the main result in [CG5]. We will use this result to study rationality of geometric signatures for open manifolds (see below). First, let us recall the following.

Theorem 12([CG5]) *Let* M^n *be a complete manifold with bounded sectional curvature,* $|K| \leq 1$*. Given* $X \subset M^n, 0 < r \leq 1$*, there is a sub-*

manifold U^n with smooth boundary ∂U^n such that for some constant $c(n)$ depending on n,

$$X \subset U^n \subset T_r(X)$$
$$Vol(\partial U) \leq c(n) Vol(T_r(X)) r^{-1}$$
$$||II_{\partial U}|| \leq c(n) r^{-1}$$

Moreover U^n can be choosen to be invariant under $I(r, X)$, where $I(r, X)$ denotes the group of isometries of $T_r(X)$ which fix X.

We show the following fact:

Theorem F. *Under the same assumptions as in Theorem 12. Let $D = diam(X) = sup\{dist(x, y) | \ x, y \in X\}$. Then there exists a constant, $c(n, r, D)$, depending on n, r and D, such that*

$$diam(\partial U) \leq c(n, r, D).$$

An immediate consequence of Theorem F is what can be viewed a bounded version of Cheeger and Gromov's good chopping theorem (see Theorem 14).

In fact, our results in this thesis can be further exploited to prove a Cheeger and Gromov conjecture on rationality of geometric signatures. We now explain this.

By the work of Cheeger and Gromov, the *Gauss-Bonnet-Chern theorem* has been extended to complete manifolds of bounded sectional curvature and finite volume ([CG1] and [CG2]). They considered the *geometric Euler* *numbers* and the *geometric signatures* of such manifolds. These are defined as the integrals of the *Euler form* and the *signature form* on these manifolds. (Note that examples show that the *geometric signatures* of a $4k$- manifold can be an arbitrary real number in general.) They proved that, under the assumption that the metrics have bounded covering geometry over a neighborhood of infinity (denoted by $\widetilde{geo}_\infty(M) \geq 1$), the *geometric Euler number* is a homotopy invariant and the *geometric signature* is a proper homotopy invariant. More precisely, they are equal to the L_2*-Euler number* and L_2*-signature*, denoted by $\chi_{(2)}(M)$ and $\sigma_{(2)}(M)$ respectively. Using the so-called good chopping technique ([CG5]), Cheeger and Gromov proved that the geometric *Euler numbers* are always integers (without assuming BCG). Furthermore they made the following conjecture.

Conjecture II : *$\sigma_{(2)}(M)$ is a rational number.*

It turns out that using Theorem A, Theorem E, Theorem 14 and the results in [CFG], we can prove Conjecture II for 4-manifolds under an additional assumption that the manifolds have finite diameters at infinity. A complete manifold M is said to has finite diameter at infinity, if there is point $p \in M$ so that $\sup_{r>0}\{\sum_\alpha diam((T_1(\partial B_r(p)))_\alpha)\} < \infty$, where the sume runs over all components of the 1-tubular neighborhood of the matric sphere $\partial B_r(p)$. We will only give the outline of the proof in §3.2. The detailed argument will appear elsewhere.

The thesis is organized as follows.

In §1.1, we briefly review the results in [CG3], [CG4], [Fu1] and [Fu2] which we shall use in the sequel. We also introduce injective F-structures and show that a volume collapse compatible with a polarized F-structure $\mathcal{F}$ has BCG if and only if $\mathcal{F}$ is injective (Theorem 5).

In §1.2, we give a quick proof of Theorem A by assuming Theorem B (proved in §1.3) and a result from §1.6. Some readers may wish to read §1.3 and §1.6 before reading §1.4 and §1.5.

In §1.3 we first identify the collapsed 3-manifolds as graph manifolds in general, and the collapsed 3-manifolds with bounded diameters as Seifert manifolds and solvable manifolds up to a double covering (Propositions 4 and 5). Then we give the characterization of injective F-structures on 3-manifolds (Theorem 8). Furthermore, we give a sufficient and necessary condition which guarentees that a polarized F-structure which can be modified to be injective (Theorem 9). Based on this we prove Theorem B.

In §1.4, we prove Theorem C.

In §1.5, we describe the idea of an alternative proof of that a limiting η-invariant is a topological invariant in dimension 3. This uses mainly Theorem A, B and C (note that this is totally different from the one given in [CG2]).

Given a pair $(N, \mathcal{F})$, we fill in N with a compact 4-manifold M and extend the polarized F-structure of N to M by using the invariant plumbing technique (Theorem 10). This is accomplished in §1.6.

Section §2.1 is devoted to explicit residue formulas for $\eta_{(2)}(N)$. We carry out the general residue theory of [Ya] in our special circumstance. Theorem D is proved and the general residue formula is given (Theorem 11).

Theorem E is proved in §2.2.

In §3.1 we prove Theorem F.

Finally in §3.2 we describe the outline of our approach to Conjecture II in 4-dimensional case.

Chapter 1

Rationality of Limiting η-invariants of 3-manifolds

1.1 Collapsing and F-structures

In this section we will review the fundamental results on collapsed Riemannian manifolds and F-structures obtained in [CG3], [CG4], [Fu1] and [Fu2]. Then we will discuss some basic properties of injective F-structures.

Let M be a manifold. A sequence of complete Riemannian metrics $\{g_i\}$ on M is said to determine an injectivity radius (or volume) collapse, if $\{g_i\}$

satisfies the conditions

(i) $|K_g(M)| \leq 1$,

(ii) $\lim_{i\to\infty} sup\{InjRad(x, g_i)| \ \ x \in M\} = 0$ $(or \ \ \lim_{i\to\infty} Vol(M, g_i) = 0)$.

A volume collapse $\{g_i\}$ is said to have bounded diameter, if $diam(M, g_i) \leq D$ for some constant $D > 0$.

A volume collapse $\{g_i\}$ is said to have BCG, if the pull back metrics of $\{g_i\}$ on the universal covering of M have a lower bound (say at least 1) on the injectivity radius.

The topological aspect of collapsing phenomena on a manifold M is to a large extent captured by the so-called F-structures which were first introduced in [CG3].

Definition 1 *Let M be C^∞ manifold. An F-structure $\mathcal{F}$ on M stands for $\{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ (called an atlas for $\mathcal{F}$) such that,*

(i) $\{U_i\}$ is a locally finite open cover of M,

(ii) $\pi_i : \tilde{U}_i \to U_i$ is a finite Galois covering with Galois (deck transformation) group$=G_i$.

(iii) T^{k_i} is a k_i-dimensional torus and $\phi_i : T^{k_i} \to Diff(\tilde{U}_i)$ is an effective and smooth action,

(iv)$\psi_i : G_i \to Aut(T^{k_i})$ is a homomorphism satisfying

$$g_i(\phi_i(\gamma_i)(x)) = \phi_i((\psi_i(g_i)(\gamma_i))(g_i x))$$

for each $g_i \in G_i, \gamma_i \in T^{k_i}$ and $x \in \tilde{U}_i$,

(v) if $U_i \bigcap U_j \neq \emptyset$, then there is a common covering V_{ij} of $\pi_i^{-1}(U_i \bigcap U_j)$ and $\pi_j^{-1}(U_j \bigcap U_j)$ so that the lifting actions of T^{k_i} and T^{k_j} on V_{ij} commute.

Remark 1 *In view of (iv) in Definition 1, the orbits of the local action ϕ_i on U_i are well defined. By (v) in Definition 1, the orbits of $\mathcal{F}$ at the point $x \in M$ are defined as the union of all the orbits of ϕ_i through x. The rank of $\mathcal{F}$ at $x \in M$ is defined as the dimension of the orbit through x. $\mathcal{F}$ has positive rank, if all orbits of $\mathcal{F}$ have positive dimensions.*

Definition 2 *A T-structure on M is an F-structure such that we can choose $\tilde{U}_i = U_i$ for all i as in Definition 1.*

Definition 3 *An F-structure $\mathcal{F}$ is called pure, if for all $i, j, k_i = k_j$. $\mathcal{F}$ is called polarized, if the local action ϕ_i has finite isotropy group at all points of $\tilde{U}_i$.*

Definition 4 *A polarization of $\mathcal{F}$ is a collection of connected subgroups $H_i \subset T^{k_i}$ such that the dimension of each H_i-orbit is equal to $dim H_i$. If H_i is a compact subgroup of T^{k_i} for all i, we call the polarization a polarized substructure of $\mathcal{F}$.*

Definition 5 *A T-structure $\mathcal{F} = \{(U_i, T^{k_i}, \phi_i)\}$ is isomorphic to the T-structure $\mathcal{F}' = \{(U_i', T^{k_i}, \phi_i')\}$, if there is a diffeomorphism ϕ of N such that (U_i, T^{k_i}, ϕ_i) is isomorphic to (U_i', T^{k_i}, ϕ_i') under the restriction of ϕ for all i.*

The relations between positive rank F-structures and the injectivity radius collapse on a manifold M is given in the following theorems.

Theorem 1 *([CG3]) Let (M, g) be a complete n-manifold with $|K| \leq 1$. There exists a constant $c(n)$ depending only on n, so that if the injectivity* *radii are smaller than* $c(n)$ *everywhere on* M*, then* M *admits a positive rank F-structure* $\mathcal{F}_g$ *which is almost compatible with the metric* g*. Conversely, suppose* M *admits a positive rank F-structure. Then one is able to construct an invariant injective radius collapse.*

If a manifold M admits a polarized F-structure, then one can obtain a volume collapse.

Theorem 2 *([CG4]) (1) Let* M *be a manifold and* $\mathcal{F}$ *be a polarized F-structure of* M*. Then* M *admits an invariant volume collapse.*

(2) Suppose M *admits a polarized F-structure* $\mathcal{F}$ *outside a compact subset* C *of* M*, then* M *admits a complete metric* g *with bounded sectional curvature, finite volume and compatible with* $\mathcal{F}$*.*

Remark 2 *The converse of Theorem 2 may not be true in general. In fact, an affirmative answer would solve an open question on vanishing minimal volume (see [Gr2]).*

The next two theorems are not needed in proving Theorem A but only in proving Proposition 5 in §1.3 and Theorem E in §2.2. Some readers may omit them on the first reading.

Theorem 3 *([Fu2]) Let* M_i *be a sequence of closed* n*-manifolds satisfying the following conditions:*

(i) $|K(M_i)| \leq 1$

(ii) $diam(M_i) \leq D$.

Assume there exists a metric space Y *such that* $\lim_{i\to+\infty} d_H(M_i, Y) = 0$. *Then we have the following:*

There exists a smooth manifold N with a $C^{1,\alpha}$-metric g_N, on which there is a smooth and isometric action of the orthogonal group $O(n)$ such that

(1) Y is isometric to $N/O(n)$,

(2) For each $p \in N$, the isotropy group is an extension of a torus T^k by a finite group.

Theorem 4 *([F2]) Under the same assumption as in Proposition 3, for sufficiently large i, there exists maps $f_i : M_i \to Y$ and $\tilde{f}_i : FM_i \to N$ satisfying:*

(1) $\tilde{f}_i$ is a fibration with fiber infranilmanifold $\approx N_i/\Lambda$.

(2) The structure group is contained in

$$\frac{CentN_i}{CentN_i \bigcap \Lambda} \propto Aut\Lambda$$

(3) $\tilde{f}_i$ is an almost Riemannian submersion. In other words, for each $V \in TM_i$ perpendicular to the fiber of $\tilde{f}_i$, we have

$$e^{-o(i)} < \frac{|\tilde{f}_{i*}(V)|}{|V|} < e^{o(i)}$$

where $o(i)$ satisfies $\lim_{i\to\infty} o(i) = 0$.

(4) The following diagram commutes:

$$\begin{array}{ccc} FM_i & \xrightarrow{\tilde{f}_i} & N \\ \pi_i \downarrow & & \pi_i \downarrow \\ M_i & \xrightarrow{f_i} & Y = N/O(n) \end{array}$$

Note that (2) of Theorem 4 implies that, for sufficiently large i, N_i admits a pure positive rank F-structure which is given by the projection of structure group in (2). Note that in general such pure F-structure is not polarized.

For examples of F-structures we refer to [CG3], [CG4] and [Fu3].

To study a volume collapse with BCG, we introduce the injectivity of an F-structure as follows.

Definition 6 *Let $\mathcal{F} = \{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ be a positive rank F-structure of N. For any $x \in N$ and $(\tilde{U}_{i_0}, U_{i_0}, T^{k_{i_0}}, \phi_{i_0}, \psi_{i_0}) \supset \mathcal{O}_x$ ($dim(\mathcal{O}_x) = k_{i_0}$), consider the following diagram:*

$$\begin{array}{ccc} & \phi_{i_0} & \\ (T^{k_{i_0}}, e) & \longrightarrow & (\tilde{U}_{i_0}, \tilde{x}) \\ \tilde{\phi}_{i_0} = \pi \circ \phi_{i_0} & \searrow & \downarrow \pi \\ & & (U_{i_0}, x) \subset (N, x). \end{array}$$

$\mathcal{F}$ is called injective if the induced map $(\tilde{\phi}_{i_0})_* : \pi_1(T^{k_{i_0}}, e) \to \pi_1(N, x)$ is injective.

Remark 3 *The injective F-structure is to some degree a generalization of an injective torus action studied in [CR1], [CR2], and [LR].*

Injective F-structures are polarized. The basic topological feature of an injective F-structure is the local splitting result for its universal covering space.

Lemma 1 *Let $\mathcal{F}$ be an injective F-structure of M, let $\pi : \tilde{M} \to M$ be the universal covering. For any $x \in M$ there exists an invariant tubular neighborhood U of $\mathcal{O}_x$ such that $\pi^{-1}(U)$ is homeomorphic to $D^{n-k} \times R^k (k = dim(\mathcal{O}_x))$.*

Remark 4 . *To be consistent with usage, Lemma 1 can be viewed as the local version of the so-called splitting result for the injective torus actions in [CR1] and [CR2].*

Proof of Lemma 1. First we assume that $\mathcal{F}$ is a T-structure. For any $x \in M$, let (U, T^k, ϕ) be a stratum of $\mathcal{F}$ with $U \supset \mathcal{O}_x$. Let Γ be the finite isotropy subgroup of $\mathcal{O}_x$ and let $S \approx D^{n-k}$ be the slice of $\mathcal{O}_x$ at x. Then $\mathcal{O}_x$ determines the invariant tubular neighborhood U which is diffeomorphic to

$$U \approx D^{n-k} \times_\Gamma T^k.$$

The universal covering of U is $D^{n-k} \times R^k$ and the covering group R^k of T^k acts on $\pi^{-1}(U)$ by the addition of the R^k-factor with the following commutative diagram

$$\begin{array}{ccc} & \tilde{\phi} & \\ R^k \times (D^{n-k} \times R^k) & \to & D^{n-k} \times R^k \\ \pi_t \times \pi \downarrow & & \pi \downarrow \\ & \phi & \\ T^k \times (D^{n-k} \times_\Gamma T^k) & \to & D^{n-k} \times_\Gamma T^k. \end{array}$$

We claim that $\pi^{-1}(U)$ is simply connected (thus it is diffeomorphic to $D^{n-k} \times R^k$). This follows from $\pi_1(U) \approx \pi_1(\mathcal{O}_x)$ and the injectivity of the local T^k-action.

Now consider $\mathcal{F}$ to be a general F-structure. Take a stratum $(\tilde{U}, U, T^k, \phi, \psi)$ which contains $\mathcal{O}_x$ $(dim\mathcal{O}_x = k)$. Pick up $\tilde{x} \in \tilde{U}$ with $\pi(\tilde{x}) = x$. Clearly, we can first find a tubular neighborhood of $\mathcal{O}_{\tilde{x}}$ as before and project it down by π to obtain the desired result. Q.E.D.

The basic geometrical consequence of the existence of an injective F-structure is the existence of a compatible volume collapse with BCG.

Theorem 5 *(1) Suppose M admits an injective F-structure. Then M admits an invariant volume collapse with BCG.*

(2) Suppose M (open) admits an injective F-structure outside some compact subset, then M admits an invariant metric g satisfying

(i) $|K_g(M)| \leq 1$,

(ii) $Vol(M, g) < +\infty$,

(iii) $g\tilde{e}o_{\infty}(M, g) \geq 1$.

Proof. Given an injective F-structure $\mathcal{F}$, the construction for a volume collapse $\{g_\delta\}(0 \leq \delta \leq 1)$ compatible with $\mathcal{F}$ is given in [CG3]. What we shall do is observe the BCG, that is actually a consequence of the local splitting property of $\mathcal{F}$. Let $\pi : \tilde{M} \to M$ be the universal covering, let $\tilde{g_\delta} = (\pi)^*(g_\delta)$ be the pull-back metric. Fix a point $\tilde{x} \in \tilde{M}$ and $x = \pi(\tilde{x})$. Let $\mathcal{O}_x$ be the orbit through x and $T_x^{\perp}$ be the orthogonal complement of the subspace $T_x\mathcal{O}_x$ in T_xM with respect to the initial metric g_1. We can find $\rho_x > 0$ such that $S_x = exp_x B_{\rho_x}^{\perp} (\approx D^{n-k})(k = dim(\mathcal{O}_x))$ is a slice of $\mathcal{O}_x$ where $B_{\rho_x}^{\perp}$ is the ball of radius ρ_x in $T_x^{\perp}$. Let U be the invariant neighborhood determined by S_x. Applying Lemma 1, $\pi^{-1}(U) \approx S \times R^k$.

Let $\tilde{g}_\delta^\perp$ denote the restriction of $\tilde{g}_\delta$ to S_x. Following the proof of Theorem 2 in [CG4], one sees that the metrics $\{\tilde{g}_\delta\}$ converge to a C^∞ product metric $\tilde{g}_0 = \tilde{g}_0' + \tilde{g}_0''$, where $\tilde{g}_0'' = fg_e$ (f is a C^∞ function on the S_x, g_e is the *Euclidean metric* and $\tilde{g}_0'$ is the limit of $\{\tilde{g}_\delta^\perp\}$). If $\mathcal{F}$ is pure, then $\tilde{g}_0' = g_\delta^\perp = g_1^\perp$ $(0 \leq \delta \leq 1)$. It follows that $InjRad(\tilde{x}, \tilde{g}_\delta) \geq \rho_x$ as $\delta \to 0$. In the case of $\mathcal{F}$ not being pure, one still has $InjRad(\tilde{x}, \tilde{g}_\delta) \geq \rho_x$ since here the $\tilde{g}_\delta^\perp$ is obtained basically by spanning $g_1^\perp$.

Now it is clear that if the initial invariant metric g_1 satisfies the condition that the ρ_x has a uniform lower bound for all $x \in M$, then $\{g_\delta\}$ has BCG. The result in [CG3] implies one can always construct such an invariant metric g_1. We refer to [CG3] for further details. Q.E.D.

Examples of injective F-structures.

Example 1 The catagery of the manifolds which admits a pure injective F-structure concides with the categray of the so-called injective Seifert fiber space with fiber either a flat manifold or a almost flat manifold. Seifert fiber space were first introduced in [CR1] and [CR2]. Seifert fiber space include the following:

1) classical Seifert space with infinite fundamental groups,

2) flat manifolds and almost flat manifolds(=infranilmanifolds). An n-dimensional flat manifold M admits a natural injective T^n-structure. By Bieberbach's theorem, there exists a finite covering of M which is a flat torus T^n. It is clear that the multiplication of T^n determines the pure T^n-structure which is also injective.

Similarly, an almost flat manifolds M admits an injective F-structure given by the center of a nilpotent group acting on a finite covering of it (see[Gr1]).

The following example is very important.

Example 2 All the 3-manifolds (or a double covering) which admits an injective F-structure can be obtained as follows:

Take finite Seifert fiber spaces with torii boudary components, $N_1, ..., N_r$. Suppose none of N_i is homeomorphic to a solid torus and the Seifert fiber structure of each N_i is trivial near ∂N_i. Form a closed by identifying the boundary components of N_i's in pairs (we assume the total number of boundary components are even). Then these S^1- Seifert fibering on N_i generate a polarized T-structure which is actually injective (see Theorem 8).

Example 3 Let M^n be a closed oriented manifold and let $f : K(\pi, 1)$ be the classifying map, where $\pi \approx \pi_1(M^n)$. We call M^n essential, if the fundamental class, $[M^n] \in H_n(M, R)$ satisfies $f_*([M^n]) \neq 0$. Suppose $\mathcal{F}$ is a pure positive rank F-structure on an essential manifold M^n. Then $\mathcal{F}$ is injective. (The proof of this is given in the Appendix of [CG3]).

Example 4 Let M be a complete Riemannian manifold satisfying:

(i) $-b^2 \leq K \leq -a^2$, for some constant $a, b > 0$,

(ii) $Vol(M) < \infty$.

Then M has finite topological type, i.e., there exists a compact manifold with boundary M_0, such that M is diffeomorphic to the interior of M_0. Moreover each component of ∂M_0 is an infranilmanifold. Thus M admits an injective F-structure outside a compact subset of M. We refer to [BGS] for more details.

1.2 Proof of Rationality of Limiting η-invariants Modulo §1.3 and §1.6

In this section, by employing the results in §1.3 and §1.6, we will give a simple proof of Theorem A. We also explain our idea to approach to Theorem A. First, let us recall the following result in [CG2].

Theorem 6 *([CG2]) Let N be a compact orientable $(4k-1)$ manifold. Suppose N admits a volume collapse $\{g_\delta\}$ with BCG. Then the limiting η-invariant $\eta_{(2)}(N)$, defined by*

$$\eta_{(2)}(N) = lim_{\delta\to 0}\eta(N, g_\delta)$$

exists (and thus $\eta_{(2)}(N)$ is a topological invariant).

Our starting point in proving Theorem A is Theorem 1, Theorem 5, Theorem 6 and the result in [Ya].

In [Ya], D. Yang took a different approach from [CG2] to realize the global and topological aspects of the secondary geometric invariants (in particular, η-invariant). Yang 's approach depends on the result on collapsing geometry in [CG3]. Essentially, he generalized the residue formula for characteristic numbers of closed manifolds in [Bo] and [BC], which expresses the characteristic numbers in terms of the data from the zero point set of a global Killing vector field, to compact manifolds whose boundary admits a polarized F-structure. Specializing to the η-invariant, with a little work, his result can be stated as follows.

Theorem 7 *([Ya]) Let N be an orientable compact $(4k-1)$-manifold and let $\mathcal{F}$ be a polarized F-structure of N. If $\{g_\delta\}$ is a volume collapse compatible with $\mathcal{F}$, then the limiting η-invariant,*

$$\eta(N, \mathcal{F}) = \lim_{\delta \to 0} \eta(N, g_\delta) \tag{1.1}$$

exists and is independent of this specific invariant volume collapse chosen. Thus $\eta(N, \mathcal{F})$ is a topological invariant depending only on N and $\mathcal{F}$, which is called the limiting η-invariant associated to $\mathcal{F}$. Further, suppose N is the boundary of some compact manifold M and $\mathcal{F}$ extends over M, $\tilde{\mathcal{F}}$. Let $Z = \bigcup Z_i$ be the singularity of polarization. Then

$$\eta(N, \mathcal{F}) = \lim_{\delta \to 0} \eta(N, g_\delta) = \sigma(M) + \sum_i Res(\alpha, Z_i) \tag{1.2}$$

where $\sigma(M)$ is the signature of M and $Res(\alpha, Z_i)$ is the residue of the cochain α, which is determined by the data from $\mathcal{F}$, at the component Z_i of Z. In particular, if all orbits of $\tilde{\mathcal{F}}$ are closed submanifolds of M, then $\eta(N, \mathcal{F})$ is a rational number.

Note that $\eta(N, \mathcal{F})$ is computable for an invariant volume collapse as long as N is a boundary and $\mathcal{F}$ is extendable. In addition, it follows from the concrete construction for α (see [Ya]) that $\eta(N, \mathcal{F})$ is a rational number if all orbits of $\tilde{\mathcal{F}}$ are closed submanifolds.

Now consider N as in Theorem 6. To get the rationality of $\eta_{(2)}(N)$ one needs to somehow find a method for computing $\eta_{(2)}(N)$ explicitly. We need to find a "good" volume collapse with BCG by means of $\eta_{(2)}(N)$ computable. Motivated by Theorem 6, Theorem 5 and Theorem 7, we would like to show the following:

From a sufficiently volum-collapsed metric g with BCG, one finds an injective T-structure $\mathcal{F}$. Using $\mathcal{F}$, one constructs an invariant volume collapse with BCG. Consequently, $\eta_{(2)}(N) = \eta(N, \mathcal{F})$. By filling N with a compact $4k$-manifold M and extending $\mathcal{F}$ to M one gets the rationality of $\eta_{(2)}(N)$.

Two questions arise here:

1) does a volume collapse with BCG imply the existence of injective F-structures $\mathcal{F}$?

2) can one always fill in a $(4k-1)$-manifold N with a compact $4k$-manfold M and extend $\mathcal{F}$ over M ?

In our special circumstances (3-dimensional case) the affirmative answer to 2) is not hard to achieve. However, 1) turns out to be considerably more difficult to establish.

The difficulties in 1) have two different sources. One is whether or not a sufficiently volume collapse metric leads to the existence of a polarized F-structure (see Introduction). In fact, according to Theorem 2, the affirmative answer would solve an open question on existence of the critical vanishing minimal volume (see [Gr2]). However, it is not hard to see that a sufficiently collapsed 3-manifold always admits a polarized F-structure (Corollary 1). Another difficulty is illustrated by the following observation:

3) The construction of $\mathcal{F}_g$ from a sufficiently injective radii collapsed metric g depends only on the local geometry of g ([CG4]). Although $\mathcal{F}_g$ fully reflects the injectivity radii collapse, $\mathcal{F}_g$ does not take into account either the small volume or BCG. Consequently, one can not expect $\mathcal{F}_g$ to be injective (see Example 7 and the comment 3) in §1.5).

4) Except for few cases, N admits a unique injective F-structure (if it admits any at all). Note that N may admit infinitely many different polarized F-structures. Thus the existence of an injective F-structure is a global and topological constraint on N (see §1.4).

Intuitively, what we need to do is first find some local property for $\mathcal{F}_g$ which reflects the BCG. Then use this to get the existence of an injective F-structure (the global topological information).

Following the construction for $\mathcal{F}_g$ in [CG4], one sees that locally, $\mathcal{F}_g$ is injective. By taking into account BCG, we find that the S^1-orbits of $\mathcal{F}_g$ are not homotopically contractible globally. It turns out, in the 3-dimensional situation, that such a local property is enough to guarantee the existence of an injective F-structure (here we assume $\pi_1(N)$ is infinite). Our basic method is to modify $\mathcal{F}_g$ to be injective and our basic tools are some theorems from 3-dimensional topology.

Now by assuming Theorem B and Theorem 10, we give a simple proof of Theorem A as follows.

Proof of Theorem A. Let N be a closed oriented 3-manifold which admits a volume collapse $\{g_\delta\}$ with BCG ($\delta \to 0$).

First, according to Theorem B, we can find an injective T-structure $\mathcal{F}$ on N. Using $\mathcal{F}$ we construct an invariant volume collapse $\{\tilde{g}_\delta\}$ with BCG on N (Theorem 5). From Theorem 6 and Theorem 7, we obtain

$$\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta) = \lim_{\delta \to 0} \eta(N, \tilde{g}_\delta) = \eta(N, \mathcal{F}).$$

From Theorem 10, we fill in N with a compact 4-manifold M_N and extend $\mathcal{F}$ to M_N so that all orbits of the extension are closed submanifolds. Finally we get the rationality of $\eta_{(2)}(N)$ by applying Theorem 7 to $\eta(N, \mathcal{F})$. Q.E.D.

1.3 Collapsed 3-Manifolds and Injective T-structures

Based on the general results in §1.1, we study systematically the collapsing phenomena and F-structures on 3-manifolds. Our main goal is to establish Theorem B.

a. Polarized T-structures on 3-manifolds

The basic fact of an F-structure on a 3-manifold is the following:

Proposition 1 *Any positive rank F-structure $\mathcal{F}$ on a 3-manifold contains a polarized substructure.*

A consequence of Theorem 1 and Proposition 1 is the following.

Corollary 1 *Let (N, g) be a complete 3-manifold with $|K| \leq 1$. There exists a constant $\epsilon > 0$, so that if the injectivity radii is smaller than ϵ everywhere on N, then N admits a polarized F-structure $\mathcal{F}_g$ which is almost compatible with the metric g.*

Remark 5 *Corollary 1 fails in higher dimensions. See Example 1.9 in [CG3].*

Proof of Proposition 1. Let $\mathcal{F} = \{(\tilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)\}$. If there is a stratum $(\tilde{U}_\alpha, U_\alpha, T^{k_\alpha}, \phi_\alpha)$ with $k_\alpha = 3$, then $U_\alpha = N$ and N is a T^3-manifold up to finite covering. Since the T^3-action ϕ_α has no fixed points, it follows that there is an S^1-subgroup of T^3 acting on N (or a finite cover of N) with only finite isotropy groups; that is, $\mathcal{F}$ contains a polarized S^1-substructure.

Now we assume all $k_\alpha \leq 2$. Let Z be a component of the singular set of $\mathcal{F}$ and let $(\tilde{U}_1, U_1, T^2, \phi_1, \psi_1), ..., (\tilde{U}_r, U_r, T^2, \phi_r, \psi_r)$ be all the strata of $\mathcal{F}$ which contain Z. By taking a common cover U of $\tilde{U}_1, ..., \tilde{U}_r$ and lifting the T^2-action on $\tilde{U}_i$ over U, we may assume $\mathcal{F}$ is locally a T-structure (i.e., we assume $Z \subset \tilde{U}_i \subseteq U$). Since $dimN = 3$ and $dimZ \geq 1$, it is easy to see that Z consists of a single S^1-orbit; i.e., $Z \approx S^1$ is isolated. Endowed with an invariant metric, one can find an invariant tubular neighborhood $T_\rho(Z)$ with radius ρ satisfying the conditions: (i) $T_\rho(Z)$ contains no other components of the singular set of $\mathcal{F}$, (ii) $T_\rho(Z) \approx D^2 \times S^1$ (slice theorem), and (iii) $T_\rho(Z) \subset \bigcap_{i=1}^r \tilde{U}_i$. Let S^1 be a subgroup of T^2 without fixed points under ϕ_1. By replacing the strata $(\tilde{U}_1, U_1, T^2, \phi_1), ..., (\tilde{U}_r, U_r, T^2, \phi_r)$ in $\mathcal{F}$ by $(T_\rho(Z), S^1, \phi_1|_{S^1}, \psi_1), (\tilde{U}_1', U_1', T^2, \phi_1, \psi_1), ..., (\tilde{U}_r', U_r', T^2, \phi_r, \psi_r)$ with $U_i' = U_i - T_{\rho/2}(Z)$, one gets a substructure of $\mathcal{F}$ which is polarized near Z.

Q.E.D.

A T-structure is easier to handle than an F-structure. We will see that an F-structure on an orientable 3-manifold has the same orbit structure as some T-structure (Proposition 3). In this sense one can reduce an F-structure to a T-structure in dimension 3. In fact, let us observe the following.

Proposition 2 *Let $\pi : M' \to M$ be a finite covering and let $\mathcal{F}$ be an F-structures of M. Then $\mathcal{F}$ lifts to an F-structure $\mathcal{F}'$. Moreover, $\mathcal{F}$ has the the following propoties if and only if $\mathcal{F}'$ does:*

(i) $\mathcal{F}$ is polarized;

(ii) $\mathcal{F}$ is pure;

(iii) $\mathcal{F}$ is injective.

Proof. Take any atlas $\{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$ for $\mathcal{F}$. For each i, consider the local pull-back covering bundle:

$$\begin{array}{ccc} & \pi^* & \\ \pi^*(\tilde{U}) & \to & \tilde{U}_i \\ \pi_i \downarrow & & \pi_i \downarrow \\ & \pi & \\ \pi^{-1}(U_i) & \to & U_i \end{array}$$

Since π is a finite covering, the pull-back $\pi^*(\tilde{U})$ is a finite covering over $\tilde{U}$. Thus the T^{k_i}-action ϕ_i on $\tilde{U}$ lifts to a (unique) T^{k_i}-action ϕ_i' (as the covering group of T^{k_i}) on $\pi^*(\tilde{U})$ with the commutative diagram:

$$\begin{array}{ccc} & \phi_i' & \\ T^{k_i} \times \pi^*(\tilde{U}_i) & \rightarrow & \pi^*(\tilde{U}_i) \\ 1 \times \pi^* \downarrow & & \pi^* \downarrow \\ & \phi_i & \\ T^{k_i} \times (\tilde{U}_i) & \rightarrow & \tilde{U}_i \end{array}$$

Now we need to check that the collection $\{\pi^*(\tilde{U}_i), \pi^{-1}(U_i), T^{k_i}, \phi_i'\}$ satisfies Definition 1. This is straightforward from the above commutative diagrams and the definition of $\{(\tilde{U}_i, U_i, T^{k_i}, \phi_i, \psi_i)\}$. Q.E.D.

A consequence of Proposition 2 is that, as far as the properies (i), (ii) and (iii) are concerned in practice, one can always assume the manifolds are orientable (see Proposition 3).

Given a polarized F-structure $\mathcal{F}$ on an orientable 3-manifold, let N' and N'' be the union of 1-dimensional orbits and 2-dimensional orbits respectively. Let $\bar{N}' = \bigcup N_i$ and $\bar{N}'' = \bigcup X_j$ where N_i and X_j are connected components. Since N is orientable a 2-dimensional orbit of $\mathcal{F}$ is a torus and each X_j is homeomorphic to $T^2 \times I$ with $T^2 \times t$ corresponding to the 2-orbit (I is a closed interval). Thus we may view N as decomposed into pieces $N_1, ..., N_r$ which satisfy the conditions:

(i) Each N_i is a Seifert fiber space with tori boundary components such that the Seifert fiber structure is trivial near the boundary,

(ii) By identifying the corresponding boundary components of N_i's in pairs, these S^1-Seifert fibrations produce T^2-orbits near the identified boundaries.

We call the above natural decomposition of the pair $(N, \mathcal{F})$ and denote it by $\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_r\}$.

To find a T-structure whose natural decomposition coincides with the natural decomposition of $\mathcal{F}$, it is enough to realize that, locally, a Seifert fiber structure coincides with the orbits of a local S^1-action. This is true because an S^1-fibered solid torus (and also a solid Klein bottle) admits an S^1-action leaving the fibration invariant (see [Sc]). As a summary of the above discussion, we give the following proposition.

Proposition 3 *The orbit structure of a polarized F-structure on an orientable 3-manifold N coincides with the orbit structure of some T-structure on N.*

Note that $K^2 \times S^1$ (K^2 a Klein bottle) supports an F-structure of rank three but does not admit a T-structure of rank three. Thus the orientability is a nessessary condition in Proposition 3.

To characterize the 3-manifolds which admit polarized T-structures, we recall graph manifolds as follows.

Definition 7 *Let N be a compact 3-manifold. Suppose N decomposes into pieces $N_1, ..., N_r$ satisfying the following conditions*

(i) each N_i is a manifold with torus boundary components,

(ii) each N_i supports an S^1-bundle which is trivial near ∂N_i.

Then $N_G = \{N_1, ..., N_r\}$ is said to be a graph structure of N and N is called a graph manifold with the graph structure N_G.

The topological classification for graph manifolds was obtained in [Wa1] and [Wa2]. We shall use it to discuss the uniqueness of injective F-structures on 3-manifolds in §1.4.

Note that the natural decomposition $\mathcal{D}(N,\mathcal{F}) = \{N_1, ..., N_r\}$ of a pair $(N,\mathcal{F})$ may not be a graph structure since the Seifert fibration on N_i may not be an S^1-bundle structure (e.g. suppose N_i has exceptional orbits). But if one cuts out a tubular neighborhood around each exceptional orbit of N_i (i.e., further decomposes N_i), then one can actually obtain a graph structure from $\mathcal{D}(N,\mathcal{F})$.

On the other hand, for a given graph structure $N_G = \{N_1, ..., N_r\}$, by assigning an S^1-structure on each $N_i \in N_G$ (which is not necessarily the original S^1-bundle structure) one then constructs a polarized T-structure which is called a polarized T-structure associated to the graph structure N_G (note that this associated polarized T-structure may not have rank two near ∂N_i's).

As a summary of the above discussion one can characterize the 3-manifolds which admit polarized F-structures as follows.

Proposition 4 *Let N be an orientable 3-manifold and let $\mathcal{F}$ be a polarized T-structure. Then N is a graph manifold and $\mathcal{F}$ is associated to the graph structure of N.*

The next proposition is about the sufficiently collapsed 3-manifolds with bounded diameter; it is needed only to prove Theorem E.

Proposition 5 *For each real number $D > 0$, there exists a constant, $\epsilon_3(D) >$*

0, depending only on D, such that if a 3-manifold N satisfies the conditions:

$$|sectional\ \ curvature| \leq 1, diameter \leq D, and\ \ Volume < \epsilon_3(D),$$

then N admits a pure polarized F-structure. Moreover, N is diffeomorphic to either a Seifert manifold or a solve manifold up to a double covering.

To prove Proposition 5, we need a lemma as follows.

Lemma 2 *Let G be a compact Lie group acting on a manifold. If the principal orbits have codimension 2, then the orbit space is a manifold (with boundary).*

Proof. See 4.1 Lemma in [Br]. Q.E.D.

Proof of Proposition 5. We first assume that N admits a pure polarized T^k-structure $\mathcal{F}$. If $k = 1$, this amounts to saying that N admits an S^1-foliation; i.e., N is a Seifert fibre space ([Sc]). If $k = 2$, then the orbit space $N/\mathcal{F} \approx S^1$ since $\mathcal{F}$ is pure polarized. Thus the projection $\pi : N \to N/\mathcal{F}$ is actually a bundle map with fiber either T^2 or K^2 (Klein bottle), equivalently, N (or its double covering if N is not orientable) is solvable.

Now we show that N does admit a pure polarization. We prove by contradiction. Supposing the opposite, we then obtain a sequence of closed 3-manifolds $\{N_i\}$ satisfying the conditions: 1) $|K(N_i)| \leq 1$, 2) $diam(N_i) \leq D$, 3) $Vol(N_i) \leq \frac{1}{i}$, and 4) N_i does not admit any pure polarized F-structure. By Gromov's pre-compactness theorem one can chose a subsequence of $\{N_i\}$

which converges to a metric space Y under the Hausdoff distance. Note that 3) implies $dim(Y) < 3$. We may assume that $\lim_{i\to+\infty} d_H(N_i, Y) = 0$. Applying Theorem 3, $Y \approx N/O(3)$ with $dim(Y) \leq 2$. From Lemma 2, Y is a manifold (with boundary). If $dim(Y) = 2$, Theorem 4 shows that N_i admits an S^1-fibration and hence a Seifert manifold ([Sc]) (otherwise Y is not a manifold) for i sufficiently large. This contradicts our assumption 4) above. If $dim(Y) = 1$, then $Y \approx S^1$ or $[0,1]$. In the first case N_i is a fiber bundle with fiber a 2-infranilmanifold (Theorem 4); that is, either T^2 or K^2. So N_i admits a pure polarized T^2-structure and this contradicts 4) again. If $Y \approx [0,1]$, the N_i are actually T^2-manifolds without fixed points. It is easy to see that one can pick up an S^1-subgroup of T^2 which also has no fixed points; i.e., N_i admits an S^1 polarization. Again this contradicts 4). By now our proof is complete. Q.E.D.

b. Characterization of injective F-structures on 3-manifolds

We first observe the simple case where a 3-manifold N admits a pure injective F-structure. In such a case, N, or possibly its double covering, is either a Seifert manifold (for rank one) or a solve manifold (for rank two). Note that a solve manifold is the total space of a T^2-bundle over S^1 and the T^2-bundle structure is a pure injective T^2-structure. Moreover we have

Proposition 6 *Let N be a Seifert manifold and $\mathcal{F}$ the pure S^1-structure which coincides with the Seifert fiber structure. Then $\mathcal{F}$ is injective if and only if $\pi_1(N)$ is infinite.*

Proof. See [Or]. Q.E.D.

The next lemma is crucial for establishing the characterization of 3-manifolds which admit injective F-structures (Theorem 9).

Let Σ be a boundary component of a 3-manifold N. Σ is called incompressible if Σ is not S^2 or P^2 and the natural map $\pi_1(\Sigma) \to \pi_1(N)$ is injective.

Lemma 3 *The boundary of a Seifert fibre space N_s is incompressible unless N_s is homeomorphic to a solid torus or a solid Klein bottle.*

Proof. See Corollary 3.3 in [Sc]. Q.E.D.

The criterion for a polarized T-structure to be injective is given below.

Theorem 8 *Let $\mathcal{F}$ be a polarized T-structure on a 3-manifold N. Then $\mathcal{F}$ is injective if and only if the natural decomposition $\mathcal{D}(N, \mathcal{F})$ of $\mathcal{F}$ contains no piece N_i which is homeomorphic to a solid torus.*

Proof. One direction in the proof is obvious. Suppose $\mathcal{F}$ is injective. Let $\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_r\}$ be the natural decomposition of $\mathcal{F}$. From Lemma 3, the injectivity of $\mathcal{F}$ implies that no N_i is a solid torus.

To prove the opposite direction we need some results from the combinatorial group theory.

Let G be a group, and let A and B be subgroups of G with $\phi : A \longrightarrow B$ an isomorphism. The HNN extension of G relative to A, B and ϕ is the group

$$G^* =< G, t; t^{-1}at = \phi(a), a \in A > . \quad (1.3)$$

The group G is called the base of G^*, t is called the stable letter, and A and B are called the associated subgroups. The relation between G and G^* is the following:

Lemma 4 *The group G is embedded in G^* by the natural inclusion map.*

Proof. See Theorem 2.1 in [LV]. Q.E.D.

HNN extensions arise in the following topological context. Let X be a space, and U and V be subspaces. Suppose $\pi_1(U) \to \pi_1(X)$ and $\pi_1(V) \to \pi_1(X)$ are both injective. Let $h : U \to V$ be a homeomorphism. Construct a new space Z by identifying U and V via h. Then $\pi_1(Z)$ is an HNN-extension of $\pi_1(X)$ associated to $\pi_1(U)$ and $\pi_1(V)$. Note that here t represents a path in X from u to $h(u)$ ($u \in U$).

Corollary 2 *Let $i : X \to Z$ be the natural inclusion. Then the induced map $i_* : \pi_1(X) \to \pi_1(Z)$ is injective.*

A different situation from above is when V is a subspace of another space Y. We still suppose $\pi_1(V) \longrightarrow \pi_1(Y)$ is injective. Let Z be the space formed by identifying U and V via h. According to Van Kampen's theorem, the fundamental group $\pi_1(Z)$ is:

$$\pi_1(Z) =< \pi_1(X) * \pi_1(Y) : \quad \pi_1(U) = \pi_1(V), h_* > \quad (1.4)$$

Correspondingly, let B be a subgroup of another group H. The group G^* is called the free product of G and H with amalgamation ϕ, if

$$G^* =< G * H, \quad A = B, \phi > . \tag{1.5}$$

Lemma 5 *G and H are embedded in G^* by the natural inclusions.*

Proof. See Theorem 2.6 in [LV]. Q.E.D.

Corollary 3 *Let $i_1 : X \to Z$ and $i_2 : Y \to Z$ be the natural inclusions. Then the induced maps $i_{1*} : \pi_1(X) \to \pi_1(Z)$ and $i_{2*} : \pi_1(Y) \to \pi_1(Z)$ are injective.*

Now we continue our proof of Theorem 8.

Assume each N_i in the natural decomposition $\{N_1, ..., N_r\}$ of $(N, \mathcal{F})$ is not homeomorphic to a solid torus. We shall show that $\mathcal{F}$ is injective.

First we claim that the S^1-fiberation of N_i is injective. This can be seen from Lemma 3. Note that assuming the S^1-fiber is not injective implies that the ∂N_i must be compressible. Consequently, N_i is a solid torus (Lemma 3).

When glueing $\{N_i\}$ together, these S^1-fibers generate T^2-orbits around ∂N_i's. We need to show that these T^2-orbits are also injective. We start at N_1 and glue one component of ∂N_1 with its partner. There are two possibilities. One is that the partner is another component of ∂N_1 itself. In this case, since ∂N_i is incompressible, we use the HNN extension (Corollary 2) to conclude that the produced T^2-orbits are injective. For the other case we use (Corollary 3) to get the same concluson. Note that the boundary of the glueing result still remains incompressible in both cases. By repeating the process finitely many times we then complete the proof. Q.E.D.

c. Existence of injective F-structures on 3-manifolds

We begin to prove our main result, Theorem B.

First, we take the constant $\epsilon = c(3)$, where $c(3)$ is given in Theorem 1. By Theorem 1 one finds a positive rank F-structure $\mathcal{F}_g$ which is almost compatible with g. From Propositions 1, 2 and 3, we can assume N is orientable and $\mathcal{F}_g$ is a polarized T-structure. As we have already explained in §1.2, $\mathcal{F}_g$ may not be injective in general (also see Example 9 and 3) in §1.5). What we shall do is to modify $\mathcal{F}_g$ so as to obtain an F-structure which is injective.

Let $\mathcal{D}(N, \mathcal{F}_g) = \{N_1, ..., N_r\}$ be the natural decomposition of $(N, \mathcal{F}_g)$. If each N_i is not homeomorphic to a solid torus then $\mathcal{F}_g$ is already injective (Theorem 8). Thus the appearance of solid torii in $\mathcal{D}(N, \mathcal{F})$ violates the injectivity of $\mathcal{F}_g$. We shall get rid of these defect by changing the S^1-structure on N_i which is homeomorphic to a solid torus.

Lemma 6 *Let N_1 be a Seifert fiber space with r disjoint torus boundary components such that the fiber structure is trivial near the boundary. Form a manifold N with boundary by attaching solid tori $D_1 \times S^1, ..., D_{r'} \times S^1$ to N_1 along their boundaries ($r' < r$). If the S^1-fiber of N_1 is not homotopically contractible in each $D_i \times S^1$ and N is not a solid torus, then N admits an injective S^1-structure.*

Proof. Since the S^1-fiber of N_1 is not homotopically contractible in each $D_i \times S^1$, it follows that the S^1-fibration of N_1 extends over N. So N is a Seifert fibre space with boundary $(r' < r)$. We need to show that the S^1-fiber has infinite order in $\pi_1(N)$. Since N is not a solid torus, ∂N is incompressible (Lemma 3). Thus $\pi_1(\partial N) \to \pi_1(N)$ is injective. This implies that the S^1-fiber of N is injective. Q.E.D.

Before proceeding further, we give an interesting example.

Example 5 ([So], [CG1]). This example shows that when consided as an graph manifold, a solid torus may have a rather complicated standard decomposition. First, let Σ denote the surface formed by removing three disjoint disks from S^2. Take two copies of $D^2 \times S^1$ and attach them to $\Sigma \times S^1$ along their boundaries. We require that the attachment satisfies the conditions that the S^1-factor of $\Sigma \times S^1$ is not identified with ∂D^2, and ∂N is compressible. Note that N is a Seifert fiber space with boundary. Applying Lemma 3 one recognizes N as a solid torus.

Now taking r copies, $\Sigma_1 \times S^1, ..., \Sigma_r \times S^1$ of $\Sigma \times S^1$, and $r+1$ solid tori, $D_0 \times S^1, ..., D_r \times S^1$, we play the following game. The first step is to attach $D_0 \times S^1$ and $D_1 \times S^1$ to $\Sigma_1 \times S^1$ as above and to denote the result (a solid torus) by N_1. The second step is to attach N_1 and $D_2 \times S^1$ to $\Sigma_2 \times S^1$. Continuing in this manner, it is obvious that one actually obtains the natural decomposition of $(D \times S^1, \mathcal{F})$ for some polarized T-structure $\mathcal{F}$.

Motivated by Example 5, we introduce the following.

Definition 8 *Let $\mathcal{D}(N, \mathcal{F})$ be the natural decomposition. A maximal solid torus chain of $\mathcal{D}(N, \mathcal{F})$ is a subset $\mathcal{T}_{max}$ in $\mathcal{D}(N, \mathcal{F})$ such that the total space of $\mathcal{T}_{max}$ is homeomorphic to a solid tori and $\mathcal{T}_{max}$ is a maximal subset with this property.*

Note that $\mathcal{D}(N, \mathcal{F})$ may contain no solid torus chain at all.

From the proof of Theorem 8 one observes that each maximal solid torus contains at least one $D_{i_0} \times S^1$ for some i_0. To form a maximal solid torus chain, one starts with $D_{i_0} \times S^1$ and looks at its glueing partner. If the partner is also a solid torus, then the maximal solid torus chain consists of a single $D_{i_0} \times S^1$ (and N is homeomorphic to a lens space). Otherwise one attaches $D_{i_0} \times S^1$ to the partner and denotes the result by N_1. Next one attaches all the solid torus, which are the partners of N_1, to N_1 and denotes the result by N_2. If N_2 is not a solid tori then we claim that the maximal solid torus chain is $D_{i_0} \times S^1$. This can be seen easily from Theorem 8. Otherwise we then repeat the above program by starting at the solid torus N_2. After a finite number of steps one obtains the maximal solid torus chain containing $D_{i_0} \times S^1$. Clearly, if two distinct maximal solid torus chains in $\mathcal{D}(N, \mathcal{F})$ have nonempty intersection, then N is formed by glueing two solid tori along their boundaries. It is well-known that such a space is homeomorphic to a lens space (which includes S^3 and $S^2 \times S^1$).

We have actually proved the following lemma.

Lemma 7 *Let $\mathcal{T}_{max}$ and $\mathcal{T}'_{max}$ be two distinct maximal solid torus chains of $\mathcal{D}(N, \mathcal{F})$. If $\mathcal{T}_{max} \bigcap \mathcal{T}'_{max} \neq \emptyset$, then N is homeomorphic to a lens space.*

Theorem 9 *Let $\mathcal{D}(N, \mathcal{F})$ be the natural decomposition in which all distinct maximal solid tori chains are disjoint. Suppose all S^1-orbits of $\mathcal{F}$ are not homotopically contractible in N. Then N admits an injective T-structure.*

Proof. Let $\{\mathcal{T}_{max,k}\}$ be the collection of all maximal solid torus chains of $\mathcal{D}(N, \mathcal{F})$. We obtain a new decomposition $\mathcal{D}_1(N)$ for N by simply replacing $\{\mathcal{T}_{max,k}\}$ in $\mathcal{D}(N, \mathcal{F})$ by their total spaces $\{D_k \times S^1\}$. Attaching all $D_k \times S^1$ in $\mathcal{D}_1(N)$ to their partners, we get two possible results. One is that the result is N itself. In this case all $\{D_k \times S^1\}$ attach to a single N_{i_0}. It follows from the condition that the S^1-fibration extends to an injective S^1-structure of N (Lemma 6). Another possible result is that we obtain the decompositon $\mathcal{D}_2(N)$ of N satifying the condition of Theorem 8. This can be seen as follows. First, each piece in $\mathcal{D}_2(N)$ is not a solid torus (otherwise it would contradict the maximal solid tori chains we defined). Secondly, our assumption that all S^1-orbits of $\mathcal{F}$ are not homotopically contractible implies that each piece in $\mathcal{F}_2(N)$ admits an injective Seifert fiber structure (Lemma 6). We then complete the proof by applying Theorem 8. Q.E.D.

Now we can finish the proof of Theorem B.

From Theorem 3 we only need to check that the S^1-orbit of $\mathcal{F}_g$ is not homotopically contractible. Note that so far we have not used anything related to BCG. But now this property enters. Suppose there is an S^1-orbit $\mathcal{O}_x$ of $\mathcal{F}_g$ which is homotopically trivial. From the concrete construction for $\mathcal{F}_g$ given in [CG4], one sees that $\mathcal{O}_x$ is not homotopically trivial in a metric ball of certain size at x. Say, $B_r(x) \supset \mathcal{O}_x$ for some constant $0 < r < 1$. Let

$\pi : \tilde{N} \to N$ be the universal covering $\tilde{N}$ of N and $\tilde{g} = \pi^* g$ the pull-back metric. Pick up a point $\tilde{x}$ in $\pi^{-1}(x)$ and denote by $\tilde{\mathcal{O}}_{\tilde{x}}$ the lifting of $\mathcal{O}_x$ at $\tilde{x}$. Then $\tilde{\mathcal{O}}_{\tilde{x}} \subset B_r(\tilde{x})$ since $\mathcal{O}_x$ is trivial in N. Note that the injectivity radius of $\tilde{M}$ at $\tilde{x}$ is at least 1 (BCG). It follows that $\tilde{\mathcal{O}}_{\tilde{x}}$ is homotopically trivial in $B_r(\tilde{x})$. So is the projection $\mathcal{O}_x$ of $\tilde{\mathcal{O}}_{\tilde{x}}$ in $B_r(x)$. We then get an obvious contradiction. Q.E.D.

We conclude this section by giving several examples. The first one will show that Theorem B is sharp.

Example 6. Let S^3 be the standard 3-sphere. Take a sequence of prime integers p_i with $p_i \to +\infty$. For each p_i, define a free Z_{p_i}-action on S^3 as follows: first, parametrize $S^3 \subset C^1 \times C^1$ and let φ_{p_i} be the generator of Z_{p_i}, then the action is given by $\varphi_{p_i}(\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i\theta_2}) = (\rho_1 e^{\frac{2\pi i\theta_1}{p_i}}, \rho_2 e^{\frac{2\pi i\theta_2}{p_i}})$. It is well-known that the quotient S^3/Z_{p_i} is the lens space $L(p_i, 1)$. Since Z_{p_i} acts as isometries, the standard metric of S^3 projects to a metric on $L(p_i, 1)$ of sectional curvature $= 1$. Note that $Vol(L(p_i, 1)) = Vol(S^3)/p_i \to 0$ as $p_i \to +\infty$.

The next example shows that in general, one cannot expect to obtain injective F-structures by means of geometrical constructions as given in both [CG4] and [Fu3].

Example 7. Take the standard S^2 and S^1 and make $S^2 \times S^1$. Consider the S^1 subgroup H of $SO(3)$:

$$H = \{ \begin{pmatrix} cost & -sint & 0 \\ sint & cost & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \leq t < 2\pi \}.$$

$T^2 = H \times S^1$ acts on $S^2 \times S^1$ by H on the first factor and multiplication on the second factor. Let R^1_θ be a R^1-subgroup of T^2, determined by an irrational angle θ. We obtain a volume collapse $\{g_\delta\}$ (with bounded diameter) by simply shrinking the standard product metric on $S^2 \times S^1$ along the orbits of R^1_θ. Note that the universal covering space of $S^2 \times S^1$ is $S^2 \times R^1$. It follows that the pull-back metrics have a uniform lower bound on injectivity radii. Since R^1_θ is dense in T^2, the limiting space of this collapse is a closed interval. This implies that for any sufficiently collapsed metric g_δ, the F-structure $\mathcal{F}_{g_\delta}$, constructed from g_δ by using either the local short geodesics technique ([CG4]) or the frame-bundle technique ([F2]), has to contain 2-dimensional orbits. In particular, $\mathcal{F}_{g_\delta}$ is not injective.

1.4 Uniqueness of Injective F-structures on 3-manifolds

Using the classification results for Seifert manifolds and for graph manifolds obtained in [Se] and in [Wa1] and [Wa2] respectively, we discuss the uniqueness of injective T-structures and prove Theorem C.

The question on the uniqueness of injective F-structures on a closed orientable 3-manifold N is suggested by the following observation:

Suppose N admits two injective F-structures $\mathcal{F}_1$ and $\mathcal{F}_2$. From our previous work one gets

$$\eta(N, \mathcal{F}_1) = \eta_{(2)}(N) = \eta(N, \mathcal{F}_2). \tag{1.6}$$

The explicit residue formula for limiting η-invariant associated to a polarized T-structure shows that $\eta(N, \mathcal{F})$ is computed using the topological data derived from $(N, \mathcal{F})$.

To be precise, let $\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_r\}$ be the natural decomposition. An S^1-orbit of $\mathcal{F}$ is said to be an exceptional orbit if it is an exceptional orbit for some N_i (as a Seifert fiber space). The rational portion of $3\eta(N, \mathcal{F})$ is determined by the exceptional orbits of $\mathcal{F}$ and the integer portion of $3\eta(N, \mathcal{F})$ is contributed by the twisting around the boundaries of the N_i's (see §1.7 for further details).

Thus (1.6) suggests that in some sense $\mathcal{F}_1$ may not be very "different" from $\mathcal{F}_2$. It turns out that except for few cases, if a closed orientable 3-manifold admits an injective F-structure $\mathcal{F}$ then it is unique up to weak isomorphism (see Definition 9). Before giving the precise definition for weak isomorphism, let us first describe all the exceptional cases.

Example 8 Let N be a 3-nilmanifold. The action of the center of N on N gives a (unique) pure S^1-structure $\mathcal{F}_1$ which is injective. Note that N may also be viewed as a torus bundle over S^1 and thus N supports a pure T^2-structure $\mathcal{F}_2$ which is also injective. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ have different ranks $\mathcal{F}_1$ is not isomorphic to $\mathcal{F}_2$.

Example 9 Let N be a solve manifold. N is the total space of a T^2-bundle over S^1, $\pi : T^2 \to N \to S^1$. Take a finite open cover, $U_1, ..., U_r (r \geq 2)$ of the base S^1 and denote $\tilde{U}_i = \pi^{-1}(U_i)$. By choosing pairwise different S^1-actions ϕ_i on $\tilde{U}_i$, we then obtain a mixed injective T-structure $\mathcal{F} = \{(\tilde{U}_i, S^1, \phi_i)\}_{i=1}^r$. It is obvious that in such a way one can construct infinitely many non-isomorphic classes of injective T-structures on N.

Note that all the injective T-structures in above examples do not have exceptional orbits. The next example is not of the case.

Example 10 Any S^1-action without a fixed point on $S^2 \times S^1$ is injective. Thus $S^2 \times S^1$ supports infinitely many non-isomorphic pure injective S^1-structures. Note that the Seifert invariants corresponding to the S^1-action is $\{b, (o, 0), (\alpha, \beta), (\alpha, \alpha - \beta)\}$.

Two exceptional orbits of a T-structure $\mathcal{F}$ on an oriented manifold N are said to be conjugate if their corresponding Seifert invariants (α_1, β_1) and (α_2, β_2) satisfying $\alpha_1 = \alpha_2$ and $\beta_1 = \alpha_2 - \beta_2$. Our explicit residue formula in §1.6 will show that a pair of conjugate exceptional orbits in $\mathcal{F}$ contributes only integers to $3\eta(N, \mathcal{F})$.

Definition 9 *Let N be a 3-manifold and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two injective T-structures. Let $\mathcal{F}(N, \mathcal{F}_i) = \{N_{i,1}, ..., N_{i,r_i}\}$ be the natural decompositions of $\mathcal{F}_i$ $(i = 1, 2)$. N_i is called a trival piece if N_i is homeomorphic to $T^2 \times I$. $\mathcal{F}_1$ is said to be weakly isomorphic to $\mathcal{F}_2$ if there is an automorphism ϕ of N*

such that for each non-trivial piece $N_{1,j}$*, there is an* $N_{2,k(j)}$ *such that* $\phi_{|N_{1,j}} : N_{1,j} \to N_{2,k(j)}$ *is an embedding which prevents the Seifert fiber structure.*

Remark 6 *Note that 2) actually says that if* $\mathcal{F}_1$ *is weakly isomorphic to* $\mathcal{F}_2$*, then they are different only on the trivial pieces. Thus weakly isomorphic injective T-structures have the same exceptional orbits. The expilict residue formula for* $\eta(N, \mathcal{F})$ *in §1.7 will show that* $\eta(N, \mathcal{F})$ *depends only on the weak isomorphism class of* $\mathcal{F}$*.*

Note that the injective T-stuctures in Example 9 are in the same weak isomorphism class since their natural decompositions contain only trivial pieces. We shall see later that Examples 8 and 10 are the only cases where a 3-manifold N supports more than one non weak isomorphism classes of injective T-structures. First, let us check this for the simplest case where the injective T-structure is pure.

Proposition 7 *Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be two pure injective* T^k*-structure on a manifold* N $(k = 1, 2)$*. Suppose* N *is not diffeomorphic to* $S^2 \times S^1$*. Then* $\mathcal{F}_1$ *is isomorphic to* $\mathcal{F}_2$ *up to isomorphism.*

Proof. First we check the rank one case; that is, the case when N is an injective Seifert manifold. It is well-known that a large Seifert manifold supports a unique Seifert fibration up to isomorphism (see Theorem 6 of §5, [Or]). Since N is not homeomorphic to $S^2 \times S^1$ and $\pi_1(N)$ is not finite, one easily checks that N is actually a large Seifert manifold. Consequently, N admits a unique injective S^1-structure.

In the case of rank two, N is a solve manifold. By the classification result for solve manifolds ([RV]) one concludes with the uniqueness of injective T^2-structures.

Q.E.D.

Now we are ready to prove Theorem C.

Proof of Theorem C. Suppose N is homeomorphic neither to $S^2 \times S^1$ nor a nilmanifold. Let $\mathcal{F}_i$ be two injective T-structures on N and let $\mathcal{D}(N, \mathcal{F}_i) = \{N_{i,1}, ..., N_{i,r_i}\}$ be the natural decomposition of $\mathcal{F}_i$, $(i = 1, 2)$. If $r_1 = 1$, then either $N = N_1$, or N is homeomorphic to $T^2 \times I$, or N is not homeomorphic to $T^2 \times I$. The first two cases are actually reduced to Proposition 7. The last possibility will be treated in the case of $r_i \geq 2$.

Now assume either $r_1 = 1$ and N_1 is not homeomorphic to $T^2 \times I$, or $r_i \geq 2$ (*).

First we put an additional assumption that both $\mathcal{D}(N, \mathcal{F}_1)$ and $\mathcal{D}(N, \mathcal{F}_2)$ contain no trivial piece. Under (*) and the above additional assumption, it is easy to check that by cutting out a tubular neighborhood around each exceptional orbit of N_i one can actually obtain a so-called simple graph structure in the sense of [Wa1] and [Wa2] (also see §8 of [Or]). Consequently, there is an automorphism ϕ of N which is isotopic to the identity map such that $\phi(N_{1,j}) = N_{2,j}$ and $r_1 = r_2$ (see Theorem 5 of §8, [Or]).

Note that from Proposition 7 each $N_{i,j}$ admits a unique Seifert fiber structure up to isomorphism. Thus $\mathcal{F}_1$ is actually isomorphic to $\mathcal{F}_2$.

Now we allow $\mathcal{D}(N, \mathcal{F}_i)$ to contain trivial pieces. What we shall do is glue the trival pieces in $\mathcal{D}(N, \mathcal{F}_i)$ to produce a simple decomposition as in the previous case.

We first glue the trivial pieces in each $\mathcal{D}(N, \mathcal{F}_i)$ which are supposed to glue to each other. Clearly the results are disjoint trivial pieces. Then we attach each of these disjoint trival pieces to a non-trivial piece in $\mathcal{D}(N, \mathcal{F}_i)$ (there are two choices here). We denote the result by $\{\tilde{N}_{i,1}, ..., \tilde{N}_{i,\tilde{r}_i}\}$ ($i = 1, 2$). Since none of $\tilde{N}_{i,j}$ are trival, from our previous discussion we conclude that there is an automorphism ϕ of N such that $\phi(\tilde{N}_{1,i}) = \tilde{N}_{2,i}$ and $\tilde{r}_1 = \tilde{r}_2$. This implies that $\mathcal{F}_1$ is weakly isomorphic to $\mathcal{F}_2$. Q.E.D.

1.5 Further Discussion

Besides the uniqueness of injective T-structures, there is another question arising from our proof of Theorem A. To state it clearly, let us recall the following:

Let $\{g_\delta\}$ be a volume collapse with BCG on a compact orientable 3-manifold. Then $\eta_{(2)}(N) = \lim_{\delta \to 0} \eta(N, g_\delta)$. For a fixed sufficiently collapsed metric g_δ, one constructs a polarized F-structure $\mathcal{F}_\delta$ which is almost compatible with g_δ (Corollary 1). Then,

$$\eta_{(2)}(N) \approx \eta(N, g_\delta) \approx \eta(N, \mathcal{F}_\delta). \tag{1.7}$$

Note that the first approximation in (1.7) follows from the estimate below which was established in [CG2]:

$$|\eta_{(2)}(N) - \eta(N, g_\delta)| \leq C \cdot Vol(N, g_\delta)$$

The second approximation follows from the fact that g_δ is sufficiently collapsed and almost compatible with $\mathcal{F}_\delta$. Here "almost compatible" means that there exists an invariant metric closed to g_δ (see [CG4]).

On the other hand, by performing the modification process on $\mathcal{F}_\delta$ as in the proof of Theorem 9 , one obtains an injective F-structure $\mathcal{F}$. From $\eta(N,\mathcal{F}) = \eta_{(2)}(N)$ and (1.7) we deduce that

$$\eta(N,\mathcal{F}_\delta) \approx \eta(N,\mathcal{F}). \tag{1.8}$$

Note that (1.8) may suggest that $\mathcal{F}_\delta$ is weakly isomorphic to $\mathcal{F}$ when g_δ is sufficiently collapsed. (Here, we assume N is neither homeomorphic to $S^2 \times S^1$ nor a nilmanifold and thus the weak isomorphism class of $\mathcal{F}$ is independent of δ (Theorem C).) The following example supports this.

Example 11 Let N be an injective Seifert manifold. To simplify our discussion below we asume N has a single exceptional orbit $\mathcal{O}$. As a model, we may form N as follows:

Let Σ be a torus with a disc removed and let $N_1 = \Sigma \times S^1$. Glueing in a twisted manner a solid torus $D \times S^1$ to N_1 along their boundaries, one obtains a closed injective Seifert manifold N with a single exceptional orbit which is the central orbit of $D \times S^1$.

Now we start with an invariant metric g_1 and simply shrink g_1 along the Seifert fiber while keeping g_1 fixed on the orthogonal direction to the orbits. Thus we obtain an invariant volume collapse $\{g_\delta\}$ with BCG (Theorem 5).

Then we fix a sufficiently collapsed metic g_δ and construct $\mathcal{F}_\delta$ as in [CG4]. Note that by the invariance of g_δ the exceptional orbit $\mathcal{O}$ is the shortest closed geodesic in a neighborhood U of $\mathcal{O}$. From the concrete construction of $\mathcal{F}_\delta$ as in [CG4] one sees that $\mathcal{F}_\delta$ is actually isomorphic to $\mathcal{F}$

on U and on N_1. On $N-(U\bigcup N_1)$ which is homeomorphic to $T^2\times I$, $\mathcal{F}_\delta$ is actually a mixed T-structure just as in Example 9. Thus we see that in this situation $\mathcal{F}_\delta$ is actually weakly isomorphic to $\mathcal{F}$.

We would like to make the following comments.

1) Note that the above discussion for Example 11 is valid for any 3-manifold N as long as the volume collapse with BCG is compatible with an injective T-structure on N. Although a general volume collapse $\{g_\delta\}$ with BCG may not be constructed as in Example 11, it can be thought of as a perturbation of an invariant volume collapse with respect to an injective T-structure. Thus the associated polarized T-structures $\mathcal{F}_\delta$ is "almost " weakly isomorphic to $\mathcal{F}$ in the sense that $\mathcal{F}_\delta$ is weakly isomorphic to $\mathcal{F}$ as $\delta\to 0$. Consequently,

$$\lim_{\delta\to 0}\eta(N,\mathcal{F}_\delta)=\eta(N,\mathcal{F}) \tag{1.9}$$

2) Based on the observation in 1) and our previous work in §1.3, §1.4 and §1.6, we can give a topological proof of Theorem 6 for 3-manifolds that is totally different from [CG2].

We first assume N is not homeomorphic to either $S^2\times S^1$ or a nil-manifold. We start with a volume collapse $\{g_\delta\}$ with BCG on a closed oriented 3-manifold N. For each sufficiently small δ, we obtain a polarized T-structure $\mathcal{F}_\delta$ (Corollary 1). Since each metric g_δ has BCG, we can modify each $\mathcal{F}_\delta$ to an injective T-structure $\mathcal{F}$ as in the proof of Theorem B (note that by our assumption on N, $\mathcal{F}$ is unique up to weak isomorphism

(Theorem C)). From (1.7) and (1.9) we obtain

$$\lim_{\delta \to 0} \eta(N, g_\delta) = \lim_{\delta \to 0} \eta(N, \mathcal{F}_\delta) = \eta(N, \mathcal{F}.) \tag{1.10}$$

This implies that the limiting η-invariant with respect to a volume collapse with BCG is a topological invariant.

As for when N is either $S^2 \times S^1$ or a nilmanifold, although N admits non-weak isomorphic classes of injective T-structures, a simple computations show that $\eta(N, \mathcal{F})$ is independent of $\mathcal{F}$ as along as $\mathcal{F}$ is injective. Now the proof is complete.

3) Note that in Example 11, one actually gets a maximal solid torus chain of $(N, \mathcal{F}_\delta)$ on $N - (U \bigcup N_1)$. In fact, this maximal solid torus chain is a trival one, i.e., except for the initial U, all other pieces in this chain are homeomorphic to $T^2 \times I$. It is conceivable that more complicated maximal solid torus chains may be formed in the polarized T-structure constructed from geometry as in ([CG4]). Also, since $\mathcal{F}_\delta$ contains a maximal solid torus chain for all sufficiently small δ, one sees again that an injective T-structure cannot be produced from the geometrical construction given in [CG4].

1.6 Filling 3-manifolds via Equivariant Plumbings

In part **c** of this section we will prove the following fact

Theorem 10 *Let N ba a closed orientable 3-manifold and let $\mathcal{F}$ be a polarized T-structure of N. Then there is a 4-manifold M_N and a T-structure $\tilde{\mathcal{F}}$ satisfying:*

1) $\partial M_N \approx N$,

2) $\tilde{\mathcal{F}}_{|_N} \approx \mathcal{F}$,

3) all orbits of $\tilde{\mathcal{F}}$ are closed submanifolds of M_N.

Note that any closed orientable 3-manifold can be viewed as a boundary since the cobodism ring of such manifolds is trivial. To insure that we can extend $\mathcal{F}$ to M_N, we will construct M_N by the so-called equivariant plumbing technique.

a. Equivariant Plumbings

Let us recall the equivariant plumbing technique from [Or]. The principal S^1-bundles over a closed orientable 2-manifold Y are classified by $H^2(Y, Z) = Z$. Denote the associated D^2-bundles indexed by $b \in Z$ as $\xi = (M_b, \pi, Y), \pi : M_b \to Y$. The compact 4-manifold M_b has the homotopy type of Y. Let the zero section $\nu : M_b \to Y$ represent the positive generator $\alpha \in H_2(M_b, Z)$. Then the self intersection number $\alpha \cdot \alpha = b$ is the *Euler number*.

Given two such bundles $\xi_i = (M_{b_i}, \pi_i, Y_i)(i = 1, 2)$, we plumb them together as follows. Choose 2-disks $B_i \subset Y_i$ and the bundles over them $\pi^{-1}(B_i)$. Since they are trivial bundles there are natural identifications $\mu_i : D^2 \times D^2 \to \xi_i$. Let $s : D^2 \times D^2 \to D^2 \times D^2, s(x, y) = (y, x)$ be the reflection and define $f : \pi^{-1}(B_1) \to \pi^{-1}(B_2)$ by $f = \mu_2 \circ s \circ \mu^{-1}$. Pasting ξ_1

and ξ_2 together along $\pi^{-1}(B_1)$ and $\pi^{-1}(B_1)$ by the map f is called plumbing. It yields a topological 4-manifold with corners that may be smoothed. The resulting smooth structure is independent of the choices involved.

A graph Γ is a finite, 1-dimensional, connected simplicial complex. Let $A_0, ..., A_n$ denote its vertices. A *star* is a contractible graph where at most one vertex, say A_0, is connected with more than two other vertices. If there is such a vertex, call it the center. A *weighted graph* is a graph where a non-negative integer g_i (the genus) and an integer b_i (the weight) is associated with each vertex A_i.

Given a weighted graph Γ we define a compact 4-manifold M_Γ as follows: For each vertex (A_i, g_i, b_i) take a D^2-bundle $\xi_i = (Y_{b_i}, \pi_i, M_i)$ where M_i is a closed, orientable 2-manifold of genus g_i. If an edge connects A_i and A_j in Γ then perform plumbing on ξ_i and ξ_j. If A_i is connected with more than one other vertex, choose pairwise disjoint disks on M_i to perform the plumbing. Finally smooth the resulting manifold to obtain M_Γ.

We shall now define an S^1-action on the building blocks. For $g > 0$ let S^1 act trivially on the base and as a rotation on each fiber. For $g = 0$ we define S^1-actions on $\xi = (M_b, \pi, S^2)$ in general. A plumbing is equivariant if the trivializing maps μ_i and identifying map s are equivariant.

b. Seifert manifolds as the results of plumbings

In his classical paper, Seifert [Se] classified the class of closed 3-manifolds (Seifert manifolds) satisfying the conditions:

(S1) the manifold decomposes into a collection of simple closed curves called fibers so that each point lies on a unique fiber, (S2) each fiber has a tubular neighborhood U consisting of fibers so that U is a fibred solid torus. A fibred solid torus is finitely covered by a trivial fibred solid torus.

His main result states that a Seifert manifold N is determined up to a fiber-preserving homeomorphism by the following Seifert invariants:

$$N = \{b; (\epsilon, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\} \tag{1.11}$$

where g is the genus of the base space Y of N, the symbols $\epsilon = o, o_1, n_1$ and n_2 represent the situation when both N and Y are orientable, N is orientable and Y is not, N is not orientable and Y is, neither N nor Y are orientable respectively. (α_i, β_i) is the Seifert invariant of the exceptional orbit which is determined as follows. An S^1-orbit $\mathcal{O}$ is called exceptional, if its isotrpy group is nontrivial, say Z_p. In a tubular neighborhood $D \times S^1$ of $\mathcal{O}$ as in (S2), the S^1-fibration is equivalent to the orbits of the local S^1-action (after normalization):

$$\theta(r, \theta_1, \theta_2) = (r, \theta_1 + q\theta, \theta_2 + p\theta)$$

Then the oriented Seifert invariant of $\mathcal{O}$ is determined by

$$\alpha = p, \quad \beta q \equiv 1 \ mod\, \alpha \quad 0 < \beta < \alpha$$

If N is orientable then the fiber preserving homeomorphisms preserves orientation. Note that a change of orientation of N gives the Seifert invariants

$$-M = \{-b - r; (\epsilon, g); (\alpha_1, \alpha_1 - \beta_1), ..., (\alpha_r, \alpha_r - \beta_r)\} \tag{1.12}$$

Lemma 8 *Consider the star S below with each $b_{i,j} \geq 2$ and $g_{i,j} = 0$ except for the center:*

$$
\begin{array}{lll}
 & \diagup & (-b_{1,1}) \text{---} (-b_{1,2}) \text{---} \cdots\cdots \text{---} (-b_{1,s_1}) \\
[-b-r, g] & \text{---} & (-b_{2,1}) \text{---} (-b_{2,2}) \text{---} \cdots\cdots \text{---} (-b_{2,s_2}) \\
 & & \vdots \\
 & \diagdown & (-b_{r,1}) \text{---} (-b_{r,2}) \text{---} \cdots\cdots \text{---} (-b_{r,s_r})
\end{array}
\tag{1.13}
$$

The result, $N_S = \partial M_S$, of the equivariant boundary plumbing has Seifert invariants

$$N_S = \{b; (o, g); (\alpha_1, \beta_1), ..., (\alpha_s, \beta_s)\} \tag{1.14}$$

where

$$\frac{\alpha_j}{\alpha_j - \beta_j} = [b_{j,1}, ..., b_{j,s_j}], \qquad j = 1, ..., r.$$

are the continued fractions. Conversely, every Seifert manifold as in (1.14) is the result of an equivariant plumbing according to a star S as in (1.13).

Proof. See Lemma 3 and Corollary 5, §2 of [Or]. Q.E.D.

Remark 7 *Lemma 8 is valid for the Seifert manifolds whose base spaces are non-orientable. Let $\xi = (M_{-b-r}, \pi, Y)$ be the D^2-bundle corresponding to the center $[-b-r, g]$ in (1.13) and let $\tilde{\xi} = (\tilde{M}_{-b-r}, \tilde{\pi}, \tilde{Y})$ have the same meaning as $\xi = (M_{-b-r}, \pi, Y)$ except that $\tilde{Y}$ is non-orientable. It is not hard to see that if in the plumbing process of Lemma 1.13, we replace M_{-b-r} by $\tilde{M}_{-b-r}$ and leave all others unchanged, then the plumbing result is the Seifert manifold whose Seifert invariant is $N = \{b; (n_2, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$ (i.e., the base space of N is non-orientable).*

Remark 8 *Let* $N = \{b; (o,g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$. *Then* $-N = \{-b - r; (o,g); (\alpha_1, \alpha_1 - \beta_1), ..., (\alpha_r, \alpha_r - \beta_r)\}$. *If we denote by* M_N *and* M_{-N} *the fillings for* N *and* $-N$ *respectively as in Lemma 8, then* $M_N \neq M_{-N}$ *since* $|\sigma(M_N)| \neq |\sigma(M_{-N})|$ *(see Theorem D). This implies that* M_N *depends not only on the topology of* N *but also on the orientation of* N.

c. Proof of Theorem 10

Let $(N, \mathcal{F})$ be as in Theorem 10. From now on, we will assume the natural decomposition of $(N, \mathcal{F})$ has the following form

$$\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_k; \phi_1, ..., \phi_l\}$$

where

1) Each N_e is an orientable Seifert fiber space (with boundary) such that the Seifert fiber structure is trivial near ∂N_e and $\mathcal{F}$ is generated by these Seifert fibration of N_e's $(1 \leq e \leq k)$.

and

2) the total number of boundary components of $\{N_e\}_{e=1}^k$ is $2l$. For each normalized pair $((\partial N_e)_f, (\partial N_{e'})_f)$ $((\partial N_e)_f = S_e^1 \times S_f^1)$ which are glued via ϕ_f, we may assume the glueing map ϕ_f $(1 \leq f \leq l)$ is given by the matrix

$$\phi_f = \begin{pmatrix} u_f & v_f \\ p_f & q_f \end{pmatrix} : S_e^1 \times S_f^1 \to S_{e'}^1 \times S_f^1; \quad det(\phi_f) = -1 \tag{1.15}$$

where the second factors are fibers. We will form M_N in four steps.

Step 1. By 1) one can close up each component of ∂N_e by attaching a solid torus $D_e \times S_f^1$ to $(\partial N_e)_f$ in the obvious way. We denote by $\tilde{N}_e$ the closed Seifert manifold produced from N_e. According to the classification result in [Se] we may assume $\tilde{N}_e$ is given by

$$\tilde{N}_e = \{b_e; (\epsilon_e, g); (\alpha_{e,1}, \beta_{e,1}), ..., (\alpha_{e,s_e}, \beta_{e,s_e})\}$$

where $\epsilon_e = o$ if the base of $\tilde{N}_e$ is orientable. Otherwise $\epsilon_e = n_2$.

Step 2. Fill in each $\tilde{N}_e$ with a 4-manifold $M_{\tilde{N}_e}$ as in Lemma 8 or Remark 7 depending on $\epsilon_e = o$, or n_2.

Step 3. Form M_N by making plumbings among the $M_{\tilde{N}_e}$'s according to the ϕ_f $(1 \leq f \leq l)$. For each ϕ_f we glue $M_{\tilde{N}_e}$ with $M_{\tilde{N}_{e'}}$ by making equivariant plumbings from $D_e \times D_f (D_e \times \partial D_f = D_e \times S^1_f)$ to $D_{e'} \times D_f (D_{e'} \times \partial D_f = D_{e'} \times S^1_f)$ successively according to the linear graph

$$(-c_{f,1}) \rule{2em}{0.4pt} (-c_{f,2}) \rule{2em}{0.4pt} \cdots\cdots \rule{2em}{0.4pt} (-c_{f,s_f})$$

which is determined by the continious fraction $p_f/q_f = [c_{f,1}, ..., c_{f,s'_f}]$.

Note that the boundary of the total space of the sequence of plumbings from $D_e \times D_f$ to $D_{e'} \times D_f$ is the lens space $L(p_f, q_f)$.

Clearly, ∂M_N has a decomposition $(N_1, ..., N_k, \phi'_1, ..., \phi'_l)$ where

$$\phi'_f = \begin{pmatrix} u'_f & v'_f \\ p_f & q_f \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ c_{f,s'_f} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ c_{f,1} & 1 \end{pmatrix}.$$

To see $\partial M_N \approx N$, it is enough to show the following:

Lemma 9 *ϕ_f is isotopic to ϕ'_f $(1 \leq f \leq l)$.*

Proof. Since

$$det \begin{pmatrix} u_f & v_f \\ p_f & q_f \end{pmatrix} = det \begin{pmatrix} u'_f & v'_f \\ p_f & q_f \end{pmatrix} = -1$$

i.e.,

$$u_f q_f - p_f v_f = -1 \tag{1.16}$$

$$u'_f q_f - p_f v'_f = -1, \tag{1.17}$$

by subtracting (1.17) from (1.16), one gets

$$(u_f - u'_f)q_f = (v_f - v'_f)p_f. \tag{1.18}$$

Since p_f and q_f are coprime integers and $p_f|(u_f - u'_f)q_f$, then $p_f|(u_f - u'_f)$. Put $u_f - u'_f = p_f m_f$. It follows that $v_f - v'_f = m_f q_f$. Therefore we define an isotopy $\phi_{f,t}$ by

$$\phi_{f,t} = \begin{pmatrix} u'_f + tm_f p_f & v'_f + tm_f q_f \\ p_f & q_f \end{pmatrix} \qquad 0 \leq t \leq 1.$$

Q.E.D.

Step 4. Extending $\mathcal{F}$ over M_N. First, we observe the following:

(i) $M_N = (\bigcup_{e=1}^k M_{\tilde{N}_e}) \bigcup (\bigcup_{f=1}^l M_{L(p_f,q_f)})$,

(ii) $M_{\tilde{N}_e} \bigcap M_{\tilde{N}_{e'}} = \emptyset$, $(e \neq e')$

(iii) $M_{\tilde{N}_e} \bigcap M_{L(p_f,q_f)} = \emptyset$ or $D_e \times D_f$.

Let $\mathcal{F}_e$ be the restriction of $\mathcal{F}$ to N_e and let $\mathcal{F}'_f$ be the restriction of $\mathcal{F}$ to $L(p_f, q_f)$ (note that $\mathcal{F}'_f$ is actually the T-structure given in **c** of §2.1 with spesifying $a_1 = a_2 = 0$).

We then make the extension of $\mathcal{F}$ over M_N by simply extending $\mathcal{F}_e$ over $M_{\tilde{N}_e}$ and $\mathcal{F}'_f$ over $M_{L(p_j,q_j)}$. Note that the above extension are not unique in any sense. Clearly, all orbits of the extended T-structure $\tilde{\mathcal{F}}$ are closed submanifolds of M_N (note that $\tilde{\mathcal{F}}$ may not have positive rank).

Now our proof of Theorem 10 is complete. Q.E.D.

Chapter 2

Computation of Limiting η-invariants in Dimension 3

2.1 Residue Formula of Limiting η-invariants in 3-dimension

This section is devoted to explicit topological residue formulas for $\eta_{(2)}(N)$ where N is an oriented closed 3-manifold which admits a volume collapse with BCG. We shall express $\eta_{(2)}(N)$ in terms of the data from an injective T-structure $\mathcal{F}$ of N; more precisely, the data associated to the natural decomposition of the pair $(N, \mathcal{F})$. Since in most cases such decompositions are unique up to weak isomorphism (Theorem C), the corresponding residue formulas are intrinsic (e.g Theorem D).

We should point out that the methods used in this section are valid for computing $\eta(N, \mathcal{F})$ where $\mathcal{F}$ is a polarized T-structure (possibly without BCG).

As we have already explained in §1.2, we shall carry out the formula (1.2) of Theorem 7. We organize our computation in the following order:

a. residue formula for $N = \{b; (o, g); (\alpha, \beta)\}$,

b. residue formula for Seifert manifolds,

c. residue formula for $\eta(L(p, q), \mathcal{F})$,

d. residue formula for graph manifolds.

a. Residue formula for $N = \{b; (o, g); (\alpha, \beta)\}$

Before giving the residue formula for general injective Seifert manifolds, let us first consider the simpler case of Seifert manifolds which have only a single exceptional orbit.

Proposition 8 *Given an oriented injective Seifert manifold $N = \{b; (o, g), (\alpha, \beta)\}$, then*

$$\eta_{(2)}(N) = -s + \epsilon(b) + \frac{1}{3}\sum_{i=1}^{s} b_i + \frac{\alpha_{s-1}}{3\alpha} \tag{2.1}$$

where $\frac{\alpha}{\alpha-\beta} = [b_1, ..., b_s]$, $\alpha_i = b_i\alpha_{i-1} - \alpha_{i-2}$ with $\alpha_0 = 1$ and $\alpha_1 = b_1$, and $\epsilon(b) = -1$ or 1 if $b > -1$ or $b \leq -1$ respectively.

Remark 9 *Note that M_N is not homeomorphic to M_{-N} (see Remark 8). Thus $\eta_{(2)}(-N) = -\eta_{(2)}(N)$ cannot be seen from (2.1).*

Proof. By applying Lemma 8, we obtain M_N. The injective S^1-action on N extends to M_N. Note that M_N is formed by plumbing the bundles $\xi = (M_{-b-1}, \pi, Y)$, $\xi_i = (M_{-b_i}, \pi_i, S^2)$ successively. Decomposing each base $S_i^2 = B_{i,1} \times D_{i,1} \bigcup_{\phi_i} B_{i,2} \times D_{i,2}$, where

$$\phi_i = \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}$$

we can express M_N as

$$\begin{array}{llll}
M_N = M_{-b-1} \perp B_{1,1} \times D_{1,1} \displaystyle\bigcup_{\phi_1} B_{1,2} \times D_{1,2} & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad s \downarrow & & & \\
\qquad\qquad\qquad\qquad B_{2,1} \times D_{2,1} \displaystyle\bigcup_{\phi_2} B_{2,2} \times D_{2,2} & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \downarrow & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dots & & & \\
\qquad\qquad\qquad\qquad\qquad\qquad B_{s,1} \times D_{s,1} \displaystyle\bigcup_{\phi_s} B_{s,2} \times D_{s,2} & & &
\end{array}$$

and

$$\begin{pmatrix} \alpha_{s-1} & \beta_{s-1} \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ b_s & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}.$$

Let X be the velocity vector field of the S^1-action of M_N and let X_i be the restriction of X on $B_{i,1} \times D_{i,1}$ $(1 \leq i \leq s)$. Choosing multipolar coordinates on $B_{i,1} \times D_{i,1}$ we write

$$X_i = \alpha_i \frac{\partial}{\partial \theta_{i,1}} - \alpha_{i+1} \frac{\partial}{\partial \theta_{i,2}}, \qquad \alpha_0 = 1, \alpha_1 = b_1.$$

We claim that $\alpha_i > \alpha_{i-1} > 0$. We prove this by induction. Assume $\alpha_k > \alpha_{k-1}$ for $k = 1, ..., i-1$. Since $b_i \geq 2$ and $\frac{\alpha_{i-1}}{\alpha_i} < 1$, then $\alpha_i = b_i\alpha_{i-1} - \alpha_{i-2} =$

$\alpha_i(b_i - \frac{\alpha_{i-1}}{\alpha_i}) > \alpha_i$. From $\alpha_i > 0$ it follows that the fixed point set of X consists of $Z_0 \bigcup \{0_{1,1} \times 0_{1,1}, 0_{2,1} \times 0_{2,1}, ..., 0_{s,1} \times 0_{s,1}, 0_{s,2} \times 0_{s,2}\}$, where $Z_0 \approx Y$ and each $0_{i,j} \times 0_{i,j}$ is the center of $B_{i,j} \times D_{i,j}$. According to formula (1.2) of Theorem 7,

$$\begin{aligned} \eta_{(2)}(N) = \; & \sigma(M_N) + Res(X, Z_0) + \textstyle\sum_{i=1}^{s} Res(X, 0_{i,1} \times 0_{i,2}) \\ & + Res(X, 0_{s,2} \times 0_{s,2}). \end{aligned} \tag{2.2}$$

We shall compute each term in (2.2) in the following lemmas.

Lemma 10 $\sigma(M_N) = -s + \epsilon(b)$.

Proof. From §2 of [Or] one sees that $\sigma(M_N)$ is equal to the signature of the following matrix:

$$A = \begin{pmatrix} -b-1 & 1 & & & \\ 1 & -b_1 & 1 & & \\ & 1 & & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix},$$

and A is negative definite if and only if $-b-1 < 0$. So we only need to show that if $-b-1 \geq 0$ then $sig(A) = -s+1$. First assume $-b-1 > 0$; that is, $-b-1 \geq 1$. Given two $n \times n$ matrics A_1 and A_2. If there is a invertible matrx C such that $A_1 = CA_2C^T$, then we say that A_1 is congruent to A_2

and denote this by "$A_1 \sim A_2$". It is easy to see that

$$A \sim \begin{pmatrix} -b-1 & 0 & & & \\ 0 & -b_1 + \frac{1}{b+1} & 1 & & \\ & 1 & & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}.$$

Since $b_1 \geq 2$ and $-b-1 \geq 1$, then $-b_1 + \frac{1}{-b_1-1} < 0$. Consequently, the matrix

$$A_1 = \begin{pmatrix} -b_1 + \frac{1}{-b_1-1} & 1 & & & \\ 1 & -b_2 & 1 & & \\ & 1 & & & \\ & & & \ddots & 1 \\ & & & 1 & -b_s \end{pmatrix}$$

is negative definite; i.e., $sig(A) = -s+1$.

Now we consider $b = -1$, i.e., $-b-1 = 0$. In this case

$$A \sim \begin{pmatrix} -b_1 & 0 & & & & \\ 0 & \frac{1}{b_1} & 0 & & & \\ & 0 & -b_2 + \frac{1}{b_1} & 1 & & \\ & & 1 & & & \\ & & & & \ddots & 1 \\ & & & & 1 & -b_s \end{pmatrix}.$$

Note that $-b_2 + \frac{1}{b_1} < 0$ since $b_i \geq 2$. By the same reason as above we conclude $sig(A) = -s+1$.

Q.E.D.

The next lemma is concerned with the local residue in the following situations:

1) a pure rank one structure at an isolated singular point,

2) a pair of pure structures at the isolated singular S^1-orbit.

Take $D_2^2 \times D_2^2 \subset R^2 \times R^2$ (D_2^2 is a 2-disc with radius 2) and let $(\gamma_1, \theta_1, \gamma_2, \theta_2)$ be the multipolar coordinate of R^4. Put

$$\begin{aligned} X &= a\tfrac{\partial}{\partial\theta_1} + b\tfrac{\partial}{\partial\theta_2} & (ab \neq 0) \\ X_i &= a_i\tfrac{\partial}{\partial\theta_1} + b_i\tfrac{\partial}{\partial\theta_2} & (i = 1, 2). \end{aligned}$$

Suppose $a_1b_1 \neq 0$ and $b_2 \neq 0$. Put $U_1 = D_1^2 \times D_1^2$ and $U_2 = D^2 \times D^2 - U_1$. The local models of 1) and 2) are (U_1, X) and $\{(U_1, X_1), (U_2, X_2)\}$ respectively. Their sigularities are 0×0 and $(0 \times \partial D_1^2) \bigcup (\partial D_1^2 \times 0)$ respectively.

Lemma 11 *Suppose* (U, X) *and* $\{(U_1, X_1), (U_2, X_2)\}$ *are given as above. Under the standard orientation of* R^4,

$$\begin{aligned} Res(X, 0 \times 0) &= \tfrac{a}{b} + \tfrac{b}{a}, \\ Res(\{X_1, X_2\}, 0 \times \partial D_1^2) &= -\tfrac{a_1}{b_1} + \tfrac{a_2}{b_2}, \\ Res(\{X_1, X_2\}, \partial D_1^2 \times 0) &= \tfrac{a_1}{b_1} - \tfrac{a_2}{b_2}. \end{aligned} \tag{2.3}$$

Proof. See [Ya].

Q.E.D

Now we can finish the proof of Proposition 8. From Lemma 11 we get

$$\begin{aligned} Res(X, 0_{i,1} \times 0_{i,2}) &= \tfrac{\alpha_{i+1}}{\alpha_i} + \tfrac{\alpha_i}{\alpha_{i+1}} \\ &= b_i + \tfrac{\alpha_i}{\alpha_{i+1}} - \tfrac{\alpha_{i-1}}{\alpha_i}. \end{aligned} \tag{2.4}$$

Substituting Lemma 10, (2.4) and $Res(X, Z_0) = \frac{b+r}{3}$ into (2.2), a simplification of (2.2) gives (2.1).

Q.E.D

b. Residue formula for Seifert manifolds

First, let us restate Theorem D as follows.

Theorem D. *Let* $N = \{b; (o,g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$ *be an injective Seifert manifold, then*

$$\eta_{(2)}(N) = -\sum_{i=1}^{r} s_i + \epsilon(b,r) + \frac{1}{3}(b+r) + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{ij} + \frac{1}{3}\sum_{i=1}^{r}\frac{\alpha_{i,s_i-1}}{\alpha_i} \tag{2.5}$$

where $\frac{\alpha_i}{\alpha_i-\beta_i} = [b_{i,1}, ..., b_{i,s_i}]$, $\alpha_{i,j} = b_{i,j}\alpha_{i,j-1} - \alpha_{i,j-2}$; $\alpha_{i,0} = 1$, $\alpha_{i,1} = b_{i,1}$ *and*

$$\epsilon(b,r) = \begin{cases} -1 & if\ b+r>0 \\ +1 & if\ b+r \le 0. \end{cases}$$

Note that an injective Seifert manifold admits a volume collapse with BCG (Theorem 5). Thus (2.5) makes sense. Also by Theorem 6, for any volume collapse $\{g_\delta\}$ with BCG, the limit $\lim_{\delta\to 0}\eta(N, g_\delta)$ is given by (2.5).

Corollary 4 *Given* $N = \{b; (o,g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$ *and* $N' = \{b'; (o,g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$. *Then*

$$3\eta_{(2)}(N) = 3\eta_{(2)}(N') \quad mod\ Z. \tag{2.6}$$

Remark 10 *Corollary 4 is valid for* $N = \{b; (n_2, g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$. *By the same argument as in the proof of Theorem D one can get*

$$\eta_{(2)}(N) = \sigma(M_N) + \frac{1}{3}(b+r) + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{ij} + \frac{1}{3}\sum_{i=1}^{r}\frac{\alpha_{i,s_i-1}}{\alpha_i} \tag{2.7}$$

where $\frac{\alpha_i}{\alpha_i-\beta_i} = [b_{i,1}, ..., b_{i,s_i}]$, $\alpha_{i,j} = b_{i,j}\alpha_{i,j-1} - \alpha_{i,j-2}$; $\alpha_{i,0} = 1$, $\alpha_{i,1} = b_{i,1}$.

Remark 11 *Due to the same reason as was given in Remark 8, the anti-invariance of* $\eta_{(2)}(N)$ *under the change of orientation is not obvious from (2.5). However, we can show* $\eta_{(2)}(N) = -\eta_{(2)}(-N)$ *as follows by using the anti-invariance in Proposition 8.*

Given $N = \{b; (o,g); (\alpha_1, \beta_1), ..., (\alpha_r, \beta_r)\}$, *then* $-N = \{b; (o,g); (\alpha_1, \alpha_1 - \beta_1), ..., (\alpha_r, \alpha_r - \beta_r)\}$. *Assume* $b + r > 0$. *Putting* $N_1 = \{b; (o,g); (\alpha_1, \beta_1)\}$ *and* $N_i = \{0; (o,g); (\alpha_i, \beta_i)\}$ *we can write (compare Proposition 8 with Theorem D)*

$$\eta_{(2)}(N) = \sum_{i=0}^{r} \eta_{(2)}(N_i) + (r-1)$$

and

$$\eta_{(2)}(-N) = \sum_{i=0}^{r} \eta_{(2)}(-N_i) - (r-1).$$

Therefore $\eta_{(2)}(-N) = -\eta_{(2)}(N)$ *since* $\eta_{(2)}(-N_i) = -\eta_{(2)}(N_i)$ *for all* i.

Proof of Theorem D. From Lemma 8 we have a 4-manifold M_N such that $\partial M_N \approx N$. Note that the S^1-action on N which gives the Seifert fiber structure extends uniquely over M_N. Applying formula (1.2) of Theorem 7, and using essentially same argument as in the proof of Proposition 8, we deduce the following:

$$\eta_{(2)}(N) = \sigma(M_N) + \frac{1}{3}\sum_{i=1}^{r}\sum_{j=1}^{s_i} b_{i,j} + \frac{1}{3}(b+r) + \sum_{i=1}^{r} \frac{\alpha_{i,s_i-1}}{3\alpha_i}.$$

Again from [Or] one sees that $\sigma(M_N)$ is equal to the signature of the following symetric matrix ϕ

$$\begin{pmatrix} -b-r & 1 & 1 & & 1 \\ & \begin{bmatrix} -b_{1,1} & 1 & & \\ 1 & -b_{1,2} & & \\ & & \ddots & 1 \\ & & 1 & -b_{1,s_1} \end{bmatrix} & & & \\ & & \begin{bmatrix} -b_{2,1} & 1 & \\ 1 & \ddots & \\ & 1 & -b_{1,s_2} \end{bmatrix} & & \\ & & & \ddots & \\ & & & & \begin{bmatrix} -b_{r,1} & 1 & \\ 1 & \ddots & \\ & 1 & -b_{1,s_r} \end{bmatrix} \end{pmatrix}$$

where each unfilled entry equals zero. Since $b_{i,j} \geq 2$ for all i, j this matrix is easily seen to be negative definite if and only if $-b-r < 0$. What we shall do is to show that if $-b-r \geq 0$ then $\sigma(M_N) = -\sum_{i=1}^{r} s_i + 1$. We first prove an algebraic lemma.

Lemma 12 *Suppose we are given*

$$A = \begin{pmatrix} A_1 - \alpha_0 E_{11}^1 & -\alpha_0 E_{12}^1 & \cdots\cdot & -\alpha_0 E_{1r}^1 \\ -\alpha_0 E_{21}^1 & A_2 - \alpha_0 E_{22}^1 & \cdots\cdot & -\alpha_0 E_{2r}^1 \\ \vdots & \vdots & \cdots\cdot & \vdots \\ -\alpha_0 E_{r1}^1 & -\alpha_0 E_{r2}^1 & \cdots\cdot & A_r - \alpha_0 E_{rr}^1 \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} -b_{i,1} & 1 & & & \\ 1 & -b_{i,2} & 1 & & \\ & 1 & & & \\ & & & \ddots & 1 \\ & & & 1 & -b_{i,s_i} \end{pmatrix}_{m_i \times m_i},$$

and $E_{ij}^k = (a_{ij})$ is the $m_i \times m_j$ matrix with $a_{k1} = 1$ and all other entries equal to zero. Suppose $\alpha_0 > 0$ and $b_{i,j} \geq 2$. Then A is negative definite.

Proof. We proceed by induction on r. The case of $r = 1$ is just Lemma 10. Assume Lemma 12 holds for $r - 1$. In what follows we shall perform a congruence transformation on A to make A_1^1 diagnal and E_{1i}^1 and E_{i1}^1 vanish simultaneously ($1 \leq i \leq r$). In the first step we see the following

$$A \sim A^1 = \begin{pmatrix} A_1^1 & -a_1 E_{12}^2 & \cdots & -a_1 E_{1r}^2 \\ -a_1 E_{21}^2 & A_2 - (\alpha_0 - \alpha_1) E_{22}^1 & \cdots & -(\alpha_0 - \alpha_1) E_{2r}^1 \\ \vdots & \vdots & \cdots & \vdots \\ -a_1 E_{r1}^2 & -(\alpha_0 - \alpha_1) E_{r2}^1 & \cdots & A_r - (\alpha_0 - \alpha_1) E_{rr}^1 \end{pmatrix}$$

where

$$A_1^1 = \begin{pmatrix} -(b_{1,1} + \alpha_0) & & & & \\ & -(b_{1,2} - \tilde{a}_1) & 1 & & \\ & 1 & -b_{1,3} & & \\ & & & \ddots & 1 \\ & & & 1 & -b_{i,s_1} \end{pmatrix}_{m_1 \times m_1},$$

$\tilde{a}_1 = \frac{1}{b_{1,1}+\alpha_0}$, $\tilde{a}_1 = \frac{\alpha_0}{b_{1,1}+\alpha_0}$ and $\alpha_1 = \frac{\alpha_0^2}{b_{1,1}+\alpha_0}$. In the second step we get

$$A \sim A^1 \sim A^2 = \begin{pmatrix} A_1^2 & -a_2 E_{12}^3 & \dots & -a_2 E_{1r}^3 \\ -a_2 E_{21}^3 & A_2 - (\alpha_0 - \alpha_1 - \alpha_2) E_{22}^1 & \dots & (\alpha_0 - \alpha_1 - \alpha_2) E_{2r}^1 \\ \vdots & \vdots & \dots & \vdots \\ -a_2 E_{r1}^3 & -(\alpha_0 - \alpha_1 - \alpha_2) E_{r2}^1 & \dots & A_r - (\alpha_0 - \alpha_1 - \alpha_2) E_{rr}^1 \end{pmatrix}$$

where

$$A_1^2 = \begin{pmatrix} -(b_{1,1}+\alpha_0) & & & & & \\ & -(b_{1,2}-\tilde{a}_1) & & & & \\ & & -(b_{1,3}-\tilde{a}_2) & 1 & & \\ & & 1 & & & \\ & & & & \ddots & 1 \\ & & & & 1 & -b_{i,s_1} \end{pmatrix}_{m_1 \times m_1},$$

$\tilde{a}_2 = \frac{1}{b_{1,2}-\tilde{a}_1}$, $a_2 = \frac{\alpha_1^2}{b_{1,2}-\tilde{a}_1}$ and $\alpha_2 = \frac{\alpha_1^2}{b_{1,2}-\alpha_2}$. After the s_1-steps we find

$$A \sim A^{s_1} = \begin{pmatrix} A_1^{s_1} & 0 & \dots & 0 \\ 0 & A_2 - (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{22}^1 & \dots & -(\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{2r}^1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & -(\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{r2}^1 & \dots & A_r - (\alpha_0 - \sum_{i=1}^{s_1} \alpha_i) E_{rr}^1 \end{pmatrix}$$

with

$$A_1^{s_1} = \begin{pmatrix} -(b_{1,1}+\alpha_0) & & & & \\ & -(b_{1,2}-\tilde{a}_1) & & & \\ & & -(b_{1,3}-\tilde{a}_2) & & \\ & & & \ddots & \\ & & & & -(b_{i,s_1}-\tilde{a}_{s_1-1}) \end{pmatrix}_{m_1 \times m_1},$$

where

$$\begin{aligned} \alpha_1 &= \tfrac{\alpha_0^2}{b_{1,1}+\alpha_0}, \quad \alpha_{i+1} = \tfrac{a_i^2}{b_{1,i+1}-\tilde{a}_i} \\ a_1 &= \tfrac{\alpha_0}{b_{1,1}+\alpha_0}, \quad a_{i+1} = \tfrac{a_i}{b_{1,i}-\tilde{a}_i} \qquad\qquad (i = 1, ..., s_1 - 1)\ . \\ \tilde{a}_1 &= \tfrac{1}{b_{1,1}+\alpha_0}, \quad \tilde{a}_{i+1} = \tfrac{1}{b_{1,i+1}-\tilde{a}_i}. \end{aligned}$$

To apply the inductive assumption on A^{s_1} we only need to verify

$$\alpha_0 - \sum_{i=1}^{s_1} \alpha_i > 0. \tag{2.8}$$

We first estimate α_i by starting at α_2;

$$\alpha_2 = \frac{a_1^2}{b_{1,2} - \tilde{a}_1} = \frac{(\frac{\alpha_0}{b_{1,1}+\alpha_0})^2}{b_{1,2} - \frac{1}{b_{1,1}+\alpha_0}} = \frac{\alpha_1}{b_{1,2}(b_{1,1} + \alpha_0) - 1}.$$

Put

$$c(\alpha_0) = \frac{1}{b_{1,2}(b_{1,1} + \alpha_0) - 1}$$

and we have $\alpha_2 = \alpha_1 c(\alpha_0)$. We claim that $\alpha_k < \frac{\alpha_1 c(\alpha_0)}{2^{k-2}}$ for $k = 2, ..., s_1$. We prove this by induction. Assume $\alpha_k < \frac{\alpha_1 c(\alpha_0)}{2^{k-2}}$. Then

$$\begin{aligned} \alpha_{k+1} &= \tfrac{a_k^2}{b_{1,k+1}-\tilde{a}_k} = \tfrac{(\frac{a_{k-1}}{b_{1,k}-\tilde{a}_k})^2}{b_{1,k+1}-\frac{1}{b_{1,k}-\tilde{a}_{k-1}}} \\ &= \tfrac{a_{k-1}^2}{b_{1,k}-\tilde{a}_{k-1}} \cdot \tfrac{1}{b_{1,k+1}(b_{1,k}-\tilde{a}_{k-1})-1} = \tfrac{\alpha_k}{b_{1,k+1}(b_{1,k}-\tilde{a}_{k-1})-1} \\ &= \tfrac{\alpha_1 c(\alpha_0)}{2^{k-2}} \cdot \tfrac{1}{b_{1,k+1}(b_{1,k}-\tilde{a}_{k-1})-1} \leq \tfrac{\alpha_1 c(\alpha_0)}{2^{k-1}} \end{aligned}$$

since $b_{i,j} \geq 2$ and $\tilde{a}_i < 1/2$ and hence $b_{1,k+1}(b_{1,k} - \tilde{a}_{k-1}) - 1 \geq 2$. Then we estimate (2.7) as follows:

$$\begin{aligned} \alpha_0 - \textstyle\sum_{i=1}^{s_1} \alpha_i &> \alpha_0 - \alpha_1(1 + c(\alpha_0) \textstyle\sum_{i=2}^{s_1} \frac{1}{2^{i-2}}) \\ &> \alpha_0 - \alpha_1(1 + 2c(\alpha_0)) \\ &= \alpha_0 - \tfrac{\alpha_0^2}{b_{1,1}+\alpha_0} \cdot (1 + \tfrac{2}{b_{1,1}(b_{1,2}+\alpha_0)-1}) \\ &= \alpha_0(1 - \tfrac{\alpha_0}{b_{1,1}+\alpha_0} \cdot \tfrac{b_{1,1}(b_{1,2}+\alpha_0)+1}{b_{1,1}(b_{1,2}+\alpha_0)-1}) \geq 0 \end{aligned}$$

if

$$
\begin{aligned}
& \frac{\alpha_0}{b_{1,1}+\alpha_0} \cdot \frac{b_{1,1}(b_{1,2}+\alpha_0)+1}{b_{1,1}(b_{1,2}+\alpha_0)-1} \leq 1 \\
\Leftrightarrow \quad & \alpha_0(b_{1,1}b_{1,2} + b_{1,1}\alpha_0 + 1) \leq (b_{1,1} + \alpha_0)(b_{1,1}b_{1,2} + b_{1,1}\alpha_0 + 1) \\
\Leftrightarrow \quad & 2\alpha_0 \leq b_{1,1}^2\alpha_0 + b_{1,1}^2 b_{1,2} - b_{1,1}.
\end{aligned} \tag{2.9}
$$

The last inequality in (2.8) is obvious since $b_{i,j} \geq 2$. Since $\alpha_0 - \sum_{i=1}^{s_1} \alpha_i > 0$, our proof is complete by our inductive assumption. Q.E.D.

Now we return to the proof of Theorem D.

First assume $b + r < 0$, i.e., $b + r \leq -1$. It is easy to see that

$$
\phi \sim \begin{pmatrix}
-b-r & 0 & \cdots & 0 \\
0 & A_1 + \frac{1}{b+r}E_{11}^1 & \cdots & \frac{1}{b+r}E_{1r}^1 \\
0 & \frac{1}{b+r}E_{21}^1 & \cdots & \frac{1}{b+r}E_{2r}^1 \\
\vdots & \vdots & \cdots & \vdots \\
0 & \frac{1}{b+r}E_{r1}^1 & \cdots & A_r + \frac{1}{b+r}E_{rr}^1
\end{pmatrix}.
$$

Since $b + r \leq -1$, by applying Lemma 12 we obtain $sig(\phi) = -\sum_{i=1}^{r} s_i - 1$.

Now consider $b + r = 0$. By performing the same diagonalization procedure to ϕ as done in the proof of Lemma 12 we find

$$
\phi \sim \begin{pmatrix}
A_1^{s_1} & 0 & \cdots & 0 \\
0 & A_2 - \alpha_0 E_{22}^1 & \cdots & -\alpha_0 E_{2r}^1 \\
\vdots & \vdots & \cdots & \vdots \\
0 & -\alpha_0 E_{r2}^1 & \cdots & A_r - \alpha_0 E_{rr}^1
\end{pmatrix};
$$

where

$$A_1^{s_1} = \begin{pmatrix} -b_{1,1} & & & & \\ & \frac{1}{b_{1,1}} & & & \\ & & -(b_{1,3} - \tilde{a}_2) & & \\ & & & \ddots & \\ & & & & -(b_{i,s_1} - \tilde{a}_{s_1-1}) \end{pmatrix}_{m_1 \times m_1}$$

$\alpha_0 = b_{1,1} - \sum_{i=2}^{s_1} \alpha_i$; $\tilde{a}_2 = a_2 = \alpha_2 = \frac{1}{b_{1,1}}$; $\tilde{a}_k = \frac{1}{b_{1,k} - \tilde{a}_{k-1}}$; $a_k = \frac{a_{k-1}}{b_{1,k} - \tilde{a}_{k-1}}$ and $\alpha_k = \frac{a_{k-1}^2}{b_{1,k} - \tilde{a}_{k-1}}$ $(k = 3, ..., s_1)$. By using the essentially the same method as used in the proof of Lemma 12, one gets $\alpha_k < \frac{1}{2^{k-2}}$. Thus

$$\begin{aligned} \alpha_0 &= b_{1,1} - \textstyle\sum_{i=2}^{s_1} \alpha_i = b_{1,1} - \alpha_2(1 + \sum_{i=3}^{s_1} \frac{1}{2^{k-2}}) \\ &> b_{1,1} - \frac{1}{b_{1,1}}(1 + \frac{1}{2^{k-2}}) > b_{1,1} - \frac{2}{b_{1,1}} \geq 1. \qquad (b_{1,1} \geq 2) \end{aligned}$$

Consequently $sig(\phi) = -\sum_{i=1}^{r} s_i + 1$ (Lemma 12). The proof of Theorem D is now complete.

Q.E.D.

c. Residue formula for $\eta(L(p,q), \mathcal{F})$

To prepare for deriving the topological residue formula for limiting η-invariants in general, we need to compute the limiting η-invariant associated with a non-pure polarized T-structure on a lens space.

Taking two copies of a solid torus $D \times S^1$ and parameterizing them in multipolar coordinates $(r_i, \theta_{i,1}, \theta_{i,2})$ $(i = 1, 2)$, one forms a lens space $L(p,q)$ by glueing $D_1 \times S_1^1$ with $D_2 \times S_2^1$ along their boundaries by the matrix

$$\begin{pmatrix} u & v \\ p & q \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ b_r & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix};$$

where $\frac{p}{q} = [b_1, ..., b_s]$. Let

$$X_i = a_i \frac{\partial}{\partial \theta_{i,1}} + \tilde{a}_i \frac{\partial}{\partial \theta_{i,1}} \qquad (i = 1, 2)$$

be defined in a neighborhood U_i of $D_i \times S_i^1$. Assume $\tilde{a}_1 \tilde{a}_2 \neq 0$. Then $\mathcal{F}(X_1, X_2) = \{(U_1, X_1), (U_2, X_2)\}$ determines a non-pure polarized T-structure of $L(p, q)$.

We shall compute the precise value for the limiting η-invariant associated to $\mathcal{F}(X_1, X_2)$ in the fashion of [Ya] (we refer to [Ya] for more details).

First we fill in $L(p, q)$ with a 4-manifold M_N by using equivariant plumbing. Suppose we are given a linear graph $\Gamma[b_1, ..., b_s]$:

$$(-b_1) \text{———} (-b_2) \text{———} \cdots\cdots \text{———} (-b_s)$$

The following lemma is elementary.

Lemma 13 *The boundary of the equivariant linear plumbing M_Γ according to the graph $\Gamma[b_1, ..., b_s]$ above, is the lens space $L(p, q)$ where*

$$\frac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ldots, - \cfrac{1}{b_r}}}} = [b_1, ..., b_s], \qquad b_i \geq 2. \tag{2.10}$$

Proof. See Lemma 1 of §2, [Or]. Q.E.D.

To see $M_{L(p,q)}$ clearly, we express it as follows.

$$\begin{array}{llllll}
M_{L(p,q)} = B_{1,1} \times D_{1,1} \bigcup_{\phi_1} B_{1,2} \times D_{1,2} & & & \\
 & s \downarrow & & \\
 & B_{2,1} \times D_{2,1} \bigcup_{\phi_2} B_{2,2} \times D_{2,2} & & \\
 & & s \downarrow & \\
 & & \cdots & \\
 & & & B_{r,1} \times D_{r,1} \bigcup_{\phi_s} B_{r,2} \times D_{r,2}
\end{array}$$

and

$$\phi_i = \begin{pmatrix} -1 & 0 \\ b_i & 1 \end{pmatrix}.$$

We make the following observation:

1) $\sigma(M_{L(p,q)}) = -s - 1$. (Lemma 10).

2) An S^1-action on $B_{1,1} \times \partial D_{1,1}(B_{r,2} \times \partial D_{r,2}) \subset L(p,q)$ extends to a unique S^1-action over $M_{L(p,q)}$.

3) Consider the natural inclusion map $(i,j) : B_{i,j} \times D_{i,j} \hookrightarrow R^2 \times R^2 = R^4$ $(1 \leq i, j \leq r)$. Clearly, either (i,j) does preserve orientation for all (i,j) or does not for all (i,j). The orientation convention of $M_{L(p,q)}$ is taken so that $(i,1)$ is a orientation preserving under the standard orientation on R^4.

From 2) one may view X_1 (X_2) as defined on a neighborhood of $B_{1,1} \times \partial D_{1,1}$ $(B_{r,2} \times \partial D_{r,2})$. We may choose $\tilde{\mathcal{F}} = \{(U_0, X_0), (U_1, X_1), (U_2, X_2)\}$

where U_0 is the interior of $M_{L(p,q)}$ and X_0 is generated by

$$\frac{\partial}{\partial\theta_{1,1}} + \frac{\partial}{\partial\theta_{1,1}}$$

on $B_{1,1} \times D_{1,1}$.

Let $X_0^{(i)}$ denote the restriction of X_0 to $B_{i,1} \times D_{i,1}$ and put

$$X_0^{(i)} = \alpha_i \frac{\partial}{\partial\theta_{i,1}} - \alpha_{i+1}\frac{\partial}{\partial\theta_{i,2}}.$$

We then have $0 < \alpha_i = b_i\alpha_{i-1} - \alpha_{i-2}$; $\alpha_0 = 1$ and $\alpha_1 = 1$ $(i = 2, ..., s)$ (see page 54).

The stratified set of $M_{L(p,q)}$ subordinate to $\{U_1, U_2, U_3\}$ and compatible with $\tilde{\mathcal{F}}$, is $\{M_0, M_1, M_2, M_{(0,1)}, M_{(0,2)}\}$ where

$$\begin{aligned}
M_1 &= B_{1,1}(1/2) \times A_{1,1}; \quad A_{1,1} = D_{1,1} - D_{1,1}(1/2)\\
M_2 &= B_{r,2}(1/2) \times A_{r,2}; \quad A_{r,2} = D_{r,2} - D_{r,2}(1/2)\\
M_0 &= M(p,q) - M_1 \bigcup M_2\\
M_{(0,1)} &= \bar{M}_0 \bigcap \bar{M}_1 = B_{1,1}(1/2) \times \partial D(1/2)\\
M_{(0,2)} &= \bar{M}_0 \bigcap \bar{M}_2 = B_{r,2} \times \partial D_{r,2}(1/2)
\end{aligned}$$

and the singularity Z of $\tilde{\mathcal{F}}$ is $Z_0 \bigcup Z_{(0,1)} \bigcup Z_{(0,2)}$, where

$$\begin{aligned}
Z_0 &= \{0_{1,1} \times 0_{1,1}, 0_{2,1} \times 0_{2,1}, ..., 0_{r,1} \times 0_{r,1}, 0_{r,2} \times 0_{r,2}\} \subseteq M_0\\
Z_{(0,1)} &= 0_{1,2} \times \partial D_{1,2}(1/2) \subseteq M_{(0,1)}\\
Z_{(0,2)} &= 0_{r,2} \times \partial D_{r,2}(1/2) \subseteq M_{(0,2)},
\end{aligned}$$

$B_{1,1}(1/2)$ is the ball of radius $1/2$, and $0_{i,1} \times 0_{i,1}$ is the center of $B_{i,1} \times D_{i,1}$.

Under the same orientation convention as in 3), from Lemma 11 one gets

$$\begin{aligned} Res(X_0^{(i)}, 0_{i,1} \times 0_{i,1}) &= \tfrac{\alpha_i}{\alpha_{i+1}} + \tfrac{\alpha_{i+1}}{\alpha_i} = b_i + \tfrac{\alpha_i}{\alpha_{i+1}} - \tfrac{\alpha_{i-1}}{\alpha_i} \\ Res(\{X_0^{(1)}, X_1\}, 0_{1,2} \times \partial D_{1,2}(1/2)) &= -\tfrac{a_1}{\tilde{a}_1} + \tfrac{\alpha_0}{\alpha_1} \\ Res(\{X_0^{(s)}, X_2\}, 0_{r,2} \times \partial D_{s,2}(1/2)) &= -\tfrac{a_s}{\tilde{a}_s} + \tfrac{\alpha_{s-1}}{\alpha_s}. \end{aligned} \tag{2.11}$$

Substituting 1) and (2.10) into formula (1.2) of Theorem 7, one deduces:

$$\begin{aligned} \eta(L(p,q), \mathcal{F}) &= \sigma(M_{L(p,q)}) + \tfrac{1}{3} \textstyle\sum_{i=1}^{s} Res(X_0^{(i)}, 0_{i,1} \times 0_{i,1}) \\ &\quad + Res(\{X_0^{(1)}, X_1\}, 0_{1,2} \times \partial D_{1,2}(1/2)) \\ &\quad + Res(\{X_0^{(s)}, X_2\}, 0_{r,2} \times \partial D_{s,2}(1/2)) \\ &= -s - 1 + \tfrac{1}{3} \textstyle\sum_{i=1}^{s} (b_i + \tfrac{\alpha_i}{\alpha_{i+1}} - \tfrac{\alpha_{i-1}}{\alpha_i}) - \tfrac{a_1}{3\tilde{a}_1} + \tfrac{\alpha_0}{3\alpha_1} - \tfrac{a_s}{3\tilde{a}_s} + \tfrac{\alpha_{s-1}}{3\alpha_s} \\ &= -s - 1 + \tfrac{1}{3} \textstyle\sum_{i=1}^{s} b_i - \tfrac{1}{3}(\tfrac{a_1}{\tilde{a}_1} + \tfrac{a_2}{\tilde{a}_2}). \end{aligned} \tag{2.12}$$

We summarize the above discusion in the following lemma.

Lemma 14 *Given $L(p,q)$ and $\mathcal{F}(X_1, X_2)$ as above, then under the orientation convention,*

$$\eta(L(p,q), \mathcal{F}(X_1, X_2)) = -s - 1 + \frac{1}{3} \sum_{i=1}^{s} b_i - \frac{1}{3}(\frac{a_1}{\tilde{a}_1} + \frac{a_2}{\tilde{a}_2}). \tag{2.13}$$

d. Residue formula for graph manifolds

Let N be an oriented closed 3-manifold and let $\mathcal{F}$ be an injective T-structure of N. Taking the natural decomposition $\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_k, \phi_1, ..., \phi_l\}$ of $(N, \mathcal{F})$ as in part **c** of §1.6, we fill in N with the 4-manifold M_N and extend $\mathcal{F}$ to $\tilde{\mathcal{F}}$ over M_N. We recall the following:

1) There are submanifolds $M_{\tilde{N}_e}$ and $M_{L(p_f,q_f)}$ of M_N where $M_{\tilde{N}_e}$ is the filling of the Seifert manifold $\tilde{N}_e = \{b_e; (\epsilon, g_e); (\alpha_{e,1}, \beta_{e,1}), ..., (\alpha_{e,r_e}, \beta_{e,r_e})\}$ as in Lemma 8 and $M_{L(p_f,q_f)}$ is the filling of the lens space $L(p_f, q_f)$ as in Lemma 13. Note that $M_{\tilde{N}_e} \bigcap M_{L(p_f,q_f)} \approx D_e \times D_f$.

2) Denote by $\mathcal{F}_e$ and $\mathcal{F}'_f$ the restriction of $\mathcal{F}$ to N_e and $L(p_f, q_f)$ respectively. Then the restriction $\tilde{\mathcal{F}}_e$ of $\tilde{\mathcal{F}}$ to $M_{\tilde{N}_e}$, is as in Theorem D and the restriction $\tilde{\mathcal{F}}'_f$ of $\tilde{\mathcal{F}}$ on $M_{L(p_f,q_f)}$, is as in Lemma 13 with specifying $a_{e,1} = a_{e,2} = 0$. From formula (1.2) of Theorem 7 we can write

$$\eta_{(2)}(M_N) = \sigma(M_N) + \frac{1}{3}[\sum_{e=1}^{k} Res(M_{\tilde{N}_e}, \tilde{\mathcal{F}}_e) + \sum_{f=1}^{l} Res(M_{L(p_f,q_f)}, \tilde{\mathcal{F}}'_f)]. \quad (2.14)$$

From the proof of Theorem D and Lemma 13, we have

$$\begin{array}{c} Res(M_{\tilde{N}_e}, \tilde{\mathcal{F}}_e) = \frac{1}{3}(\sum_{i=1}^{r_e} \sum_{j=1}^{s_i} b_{e,i,j} + b_e + r_e + \sum_{i=1}^{r_e} \frac{\alpha_{e,i,s_i-1}}{\alpha_{e,i}}) \\ Res(M_{L(p_f,q_f)}, \tilde{\mathcal{F}}'_f) = \frac{1}{3} \sum_{j=1}^{r'_f} \epsilon_f c_{f,j} \end{array} \quad (2.15)$$

where $\epsilon_f = 1$, or -1 depending on whether or not the induced orientation on $L(p_f, q_f)$ agrees with the orientation convention of $L(p_f, q_f)$ (see 3), part c of §2.1). Substituting (2.15) into (2.14) and simplifying (2.14) we then obtain the following theorem.

Theorem 11 *Let N be an oriented compact 3-manifold and let $\mathcal{F}$ be an injective T-structure. For the natural decomposition $\mathcal{D}(N, \mathcal{F}) = \{N_1, ..., N_l, \phi_1, ..., \phi_l\}$ of $(N, \mathcal{F})$ as above, we have*

$$\begin{aligned} \eta_{(2)}(N) &= \eta(N, \mathcal{F}) \\ &= \sigma(M_N) + \tfrac{1}{3}[\textstyle\sum_{e=1}^{k}(\sum_{i=1}^{r_e} \sum_{j=1}^{s_i} b_{e,i,j} + (b_e + r_e) \\ &\quad + \textstyle\sum_{i=1}^{r_e} \frac{\alpha_{e,i,s_i-1}}{\alpha_{e,i}})] + \frac{1}{3} \sum_{f=1}^{l} \sum_{j=1}^{r'_f} \epsilon_f c_{f,j}. \end{aligned} \quad (2.16)$$

Remark 12 *If $\mathcal{F}$ has no exceptional orbits that is, each N_i has no exceptional orbit as a Seifert fiber space for all i then $3\eta_{(2)}(N)$ is an integer. Equivalently, the rational portion of $3\eta_{(2)}(N)$ is determined by the exceptional orbits of $\mathcal{F}$.*

Corollary 5 *Under the same assumption as in Theorem 11, if N is a solve manifold then $3\eta_{(2)}(N) = 0 \ mod \ Z$.*

Proof. To apply Theorem 4, we only need to find an injective T-structure of N without exceptional orbits. Note that an oriented 3-dimensional solve manifold N_ϕ is a T^2-bundle over S^1 with monodromy matrix:

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad det(\phi) = -1.$$

N is formed by identifying the boundary of $[0,1] \times T^2$ via φ. Writing T^2 as $S^1_1 \times S^1_2$, we choose an injective T-structure $\mathcal{F}_0$ of N_ϕ as follows. Let $\partial/\partial\theta_i (i = 1, 2)$ be the velocity vector of rotation on S^1_i and let U be an invariant neighborhood of $\{0\} \times T^2$ in N_φ. Since $\mathcal{F}_0$ is determined by $\{((\epsilon, 1-\epsilon) \times T^2, \partial/\partial\theta_2), (U, \ \partial/\partial\theta_1, \partial/\partial\theta_2)\}$, then $\mathcal{F}_0$ has no exceptional orbits. Note that the natural decomposition of $(N_\varphi, \mathcal{F}_0)$ is given by:

$$\mathcal{D}(N, \mathcal{F}_0) = \{[0,1] \times S^1_1 \times S^1_2, \phi\}. \tag{2.17}$$

Q.E.D.

2.2 A finiteness Result for Limiting η-Invariants

For a real number $D > 0$, let $\mathcal{M}^3(D)$ be the collection of closed orientable 3-manifolds which admit a volume collapse with BCG. According to Theorem

E, $\{\eta_{(2)}(N) \ mod\, mZ| \quad N \in \mathcal{M}^3(D)\}$ is a finite set. Before giving the proof of Theorem E, we make some remarks.

Remark 13 *1) The set $\mathcal{M}^3(D)$ contains infinitely many topological types for any $D > 0$. For instance, a 3-nilmanifold admits a diameter collapse (hence a volume collapse) with BCG and there are infinitely many non-differmorphisc classes of 3- nilmanifolds.*

2) The size of $\{\eta_{(2)}(N) \ mod\, Z| \quad N \in \mathcal{M}^3(D)\}$ depends on the number D. For example, if $D < exp(-exp(exp9))$, then $\{\eta_{(2)}(N) \ mod\, mZ| \quad N \in \mathcal{M}^3(D)\} = \{0, \frac{1}{3}, \frac{2}{3}\}$ (Corollary 5) since the elments in $\mathcal{M}^3(exp(-exp(exp9)))$ are nilmanifolds ([Gr1]).

Proof of Thoerem E. First, from Proposition 5 and Theorem B we identify the elements in $\mathcal{M}^3(N)$ as either injective Seifert manifolds or solve manifolds. Since $3\eta_{(2)}(N) \ mod\, Z = 0$ (Corollary 5), we only need to consider the elements in $\mathcal{M}^3(D)$ which are injective Seifert manifolds.

We proceed by contradiction. Assuming the opposite, we then have a sequence of injective Seifert manifolds $\{N_i\}$ in $\mathcal{M}^3(D)$ such that $\{\eta_{(2)}(N_i) \ mod\, Z\}$ is an infinite set. By Gromov's pre-compactness theorem ([GLP]), we may assume $\{N_i\}$ converges to a lower-dimensional metric space Y, since each N_i admits a sufficiently volume collapsed metric with diameter bounded by D.

We split the following proof according to $dim(Y)$.

Case I. $dim(Y) = 0$. This amounts to saying that $diam(N_i) \to 0$. It follows from a well-known result in [Gr1] that all but finitely many of $\{N_i\}$ are identified as infranilmanifols. Thus $\{\eta_{(2)}(N_i) \ \ modZ\} = \{0, 1/3, 2/3\}$

for i sufficiently large (Corollary 4). This contradicts our assumption on $\{N_i\}$.

Case II. $dim(Y) = 1$. Y is homeomorphic to either an interval $[0, 1]$ or S^1. Note that if $Y \approx S^1$, then N_i are solve manifolds and Thoerem E have been proved in this case (Corollary 5). If $Y \approx [0, 1]$, then N_i are trivial T^2-bundles with isolated S^1-orbits over 0 and 1 (Theorem 6). So the N_i are actually T^2-manifolds for large i. Since $\pi_1(N_j)$ are not finite we then identify N_i as either $S^2 \times S^1$ or T^3 ([Ne]). Thus we get the same contradiction as in case I.

Case III. $dim(Y) = 2$. In this case, the limit space Y can be viewed as the base space of the the injective Seifert manifold N_i for all large i (Proposition 5). Note that Y is a orbifold since $Y \approx N/O(3)$ as in Theorem 3. Denote by $y_1, ..., y_k$ all the singular points of Y. Thus each N_i has exactly k exceptional orbits and the same base space. Write

$$N_i = \{b_i; (\epsilon_i, g); (\alpha_{i,1}, \beta_{i,1}), ..., (\alpha_{i,k}, \beta_{i,k})\} \tag{2.18}$$

where g is the genus of Y. From Corollary 4 one sees that the invariant b_i contributes only integers to $3\eta_{(2)}(N_i)$. Therefore, to show $\{\eta_{(2)}(N_i) \quad modZ\}$ is finite it is enough to show that the Seifert invariants (2.18) is finite up to the invariant b_i for all i.

For convenience, we define the e-norm, "$|| \ ||_e$", for a Seifert manifold

$$N = \{b; (\epsilon, g); (\alpha_1, \beta_1), ..., (\alpha_k, \beta_k)\}.$$

We define

$$||N||_e = max\{\alpha_1, ..., \alpha_k\}. \tag{2.19}$$

We claim that $\{||N_i||_e\}$ is a finite set (thus, (2.18) is finite up to the invariants b_i). We prove this by contradiction.

Assuming the opposite, we then have a sequence $\{(\alpha_{i,j}, \beta_{i,j})\}((\alpha_{i,j}, \beta_{i,j}) \in N_i)$ such that $\alpha_{i,j} \to \infty(j \to \infty)$. We can also assume that the exceptional orbits in N_i corresponding to the $(\alpha_{i,j}, \beta_{i,j})$ are the preimages of some fixed y_j, say, y_1.

Choose a small metric balls $B_\delta(y_1)$ (with δ fixed) at y_1 such that $\pi_i^{-1}(B_\delta(y_1))$ in N_i is a solid torus for all i sufficiently large. Note that for sufficiently large i, $B_\delta(y_1)$ can be viewed as the base space of the S^1-action on $\pi_i^{-1}(B_\delta(y_1))$ determined by the exceptional orbit invariants $(\alpha_{i,1}, \beta_{i,1})$ with the isotropy group $Z_{\alpha_{i,1}}$. Since $diam(\pi_i^{-1}(B_\delta(y_1))) \leq D$ and $\alpha_{i,1} \to \infty$, the limiting of the orbit spaces $\pi^{-1}(B_\delta(y_1))/Z_{\alpha_i}$ must be of dimension 1. This contradicts $dim(B_\delta(y_1)) = dim(Y) = 2$.

By now the proof of Theorem E has been completed. Q.E.D.

Chapter 3

Bounded Good Choppings

3.1 Bounded Good Choppings

In this section, we wil prove Theorem F. As an application of Theorem F, we obtain a bounded version of Cheeger-Gromov's existence of good choppings theorem ([CG5]). First, let us recall the main result in [CG5] as follows.

Theorem 12 *([CG5]). Let M^n be a complete manifold with bounded sectional curvature, $|K| \leq 1$. Given $X \subset M^n, 0 < r \leq 1$, there is a submanifold U^n with smooth boundary ∂U^n such that for some constant $c(n)$*

depending on n,

$$X \subset U^n \subset T_r(X) \tag{3.1}$$

$$Vol(\partial U) \leq c(n) Vol(T_r(X)) r^{-1} \tag{3.2}$$

$$||II_{\partial U}|| \leq c(n) r^{-1} \tag{3.3}$$

Moreover U^n can be choosen to be invariant under $I(r, X)$, where $I(r, X)$ denote the group of isometries of $T_r(X)$ which fix X.

Remark 14 *If M^n admit a positive rank F-structure $\mathcal{F}$ and X is invariant, then U^n can be chosen invariant under $\mathcal{F}$.*

Theorem F. *Under the same assumptions as in Theorem 12. Let $D = diam(X) = sup\{dist(x, y)| \ x, y \in X\}$. Then there exists a constant, $c(n, r, D)$, depending on n, r and D, such that*

$$diam(\partial U) \leq c(n, r, D).$$

An important application of Theorem F is a bounded version of the following Cheeger-Gromov's good chopping theorem.

Theorem 13 *([CG5]) Let M^n be complete, $|K| \leq 1$, $Vol(M^n) < \infty$. Then M^n admits an exhaustion $M^n = \bigcup_{i=1}^{\infty} M_j^n$, by manifolds with smooth boundary, such that*

$$\lim_{j \to \infty} Vol(\partial M_j^n) = 0,$$
$$||II_{M_j^n}|| \leq c(n).$$

In order to state a bounded version of Theorem 12, we define the following.

Definition 10 *Let M be a complete manifold. M is said have finite diameter at infinity, if there is a point $p \in M$ so that $sup_{r>0}\{\sum_\alpha diam((T_1(\partial B_r(p)))_\alpha)\} < \infty$, where $B_r(p)$ is the metric ball of radius r with center at p and $(T_1(\partial B_r(p)))_\alpha$ is a component of the 1-tubular neighborhood of $\partial B_r(p)$.*

Let S be a connected submanifold of M. For $x, y \in S$, let $\gamma_{x,y}$ be a path in S from x to y. We define

$$dist_s(x,y) = inf_{\gamma_{x,y}}\{L(\gamma_{x,y})\}.$$

Theorem 14 *Under the same assumption as in Theorem 13. Suppose M has finite diameter at infinity. Then the good chopping choppings as in Theorem 12 can be choosen so that the set $\{diam_s(\partial M_j)\}$ is bounded. Where $diam_s(\partial M_i) = \sum_\alpha diam_s((M_i)_\alpha)$ and the sum runs over all components of ∂M_i.*

Proof. That M has finite diameter at infinity means that there is a constant $D > 0$ and a point $p \in M$ so that $sup_{r>0}\{\sum_\alpha diam((T_1(\partial B_r(p)))_\alpha)\} \leq D$. Since $T_1(\partial B_r(p))$ has at most $[D]$ (the integer part of D) components, applying Theorem F to $\partial B_r(p)$, we complete the proof. Q.E.D.

We need several lemmas to prove Theorem F. The first lemma is about the local isolate property of ∂U^n.

Lemma 15 *Under the same assumption as Theorem 12. Then there is a constant $\rho(n,r) > 0$ depending only on n and r so that for $x \in \partial U^n$, $B_{\rho(n,r)}(x) \bigcap \partial U^n$ is connected.*

Proof. The proof is based on the concrete construction for U^n given in [CG5]. Let us first recall the following from [CG5]:

Let g be the Riemannian metric of M. Then there exists a Riemannian metric $\tilde{g}$ such that

$$(\frac{5}{6})^2 g \leq \tilde{g} \leq (\frac{6}{5})^2 g$$

Denote by "$\| \ \|$" the norm determined by $\tilde{g}$. Clearly, it is enough to prove the lemma only for the metric $\tilde{g}$.

Roughly, ∂U^n is a level set of the smooth function $F : M \to R^1$ which satisfies the conditions:

(i) there are constants $0 < \delta(n) < 1$ and $\epsilon(n) > 0$ depending only on n such that

$$\begin{aligned} &\|grad\ F(x)\| \leq 2 \qquad x \in M, \\ &\|grad\ F(x)\| \geq \epsilon(n) \qquad x \in F^{-1}([0, \delta(n)r]), \end{aligned} \tag{3.4}$$

(ii) $\partial U^n = F^{-1}(y)$, $y \in [\frac{1}{3} \cdot \delta(n)r, \frac{2}{3} \cdot \delta(n)r]$.

(i) and (iii) implies

(iii) ∂U^n has a tubular neighborhood $F^{-1}([y - \frac{1}{4} \cdot \delta(n)r, y + \frac{1}{4} \cdot \delta(n)r])$. where the

Then we observe the following:

For $x \in \partial U^n$, let x' be in the $gradF$-flow line through x such that $x' \in F^{-1}([0, y - \frac{1}{4}\delta(n)r] \bigcup [y + \frac{1}{4}\delta(n)r, \delta(n)r])$. Let $\gamma : [0, l] \to M$ be the integral curve of $\frac{gradF}{\|gradF\|}$, $\gamma(0) = x$ and $\gamma(l) = x'$. We claim

$$\frac{\delta(n)r}{8} \leq l \tag{3.5}$$

Our claim is proved in the following:

$$\begin{aligned}\frac{\delta(n)r}{4} &\leq |F(x)-F(x')| = |F(\gamma(0))-F(\gamma(l))| \\ &= \int_0^l |\dot{F}(\gamma(t))|dt = \int_0^l |< gradF(\gamma(t)), \dot{\gamma}(t) > |dt \\ &\leq \int_0^l ||gradF(\gamma(t))||dt \leq 2\cdot l \qquad (by(3.4)).\end{aligned}$$

This implies that starting at ∂U^n, one can flow along the integral curves of $gradF$ at least $\frac{\delta(n)r}{8}$ units without hitting ∂U^n again.

Now we begin the proof. We shall first put an additional assumption that the injectivity radii is at least 1 everywhere on M. Then we will explain how we can reduce this case by lifting to the tangent space.

First we observe that Lemma 15 is trivial where M^n is *Euclidean* space and $F^{-1}(x)$ are hyperplanes for $x \in F^{-1}([0,\delta(n)r])$. Since in that situation the $(grad\,F)$-flow are straight lines perpendicular to $F^{-1}(x)$ and $l = dist(x,x')$. By (3.4) we can choose $\rho(n,r) = \frac{\delta(n)r}{8}$.

Roughly, the conditions that $geo(M) \geq 1$ (our additional assumption) and $||II_{\partial U^n}|| \leq c(n)r^{-1}$ ((3.3)) imply that by multiplying the metric by a large number one is able to reduce the general case to the above trivial situation.

Now we proceed by contradiction. Assuming the opposite, we then find a sequence of the quadrupla $(M_i, U_i, g_i, , B_{1/i}(x_i))$ which satisfies the conditions:

1) (M_i, U_i, g_i) satisfies the conditions of Lemma 15,

2) $B_{1/i}(x_i)\bigcap\partial U^n$ has at least two components $(x \in \partial U_i)$

Rescaling the metric g_i by i, we actually have the sequence $(M_i, U_i, i\cdot g_i, x_i)$ satisfying:

1') $|K(M_i)| \to 0$, $InjRad(M_i) \to \infty$ and $||II_{\partial U_i}|| \to 0$.

2') $B_1(x_i) \bigcap \partial U_i$ has at least two components ($x \in \partial U_i$)

Note that for sufficiently large i, (3.5) becomes

$$2 << l_i. \tag{3.6}$$

Consider the pointed Hausdorff limit (M_0, x_0) of (M_i, x_i) and $(\partial U_0, x_0)$ of $(\partial U_i, x_i)$ respectively. From 1') we see that M_0 is *Euclidean* space and ∂U_0 is a hyperplane in M_0. Then (3.6) implies that $B_1(x_0) \bigcap \partial U_0 = \emptyset$. This contradicts 2').

Now we consider M in general, i.e., we do not have $geo(M) \geq 1$. For any $x \in \partial U^n$, consider the tangent space at x, $T_x(M)$ which equipt with the pull-back metric $\tilde{g}'$ under the exponential map, $exp_x : T_x(M) \to M$. Let $B_{\frac{\pi}{2}}(0)$ be the metric ball at $0 \in T_x(M)$ of redius $\frac{\pi}{2}$. Then $(B_\pi(0), \tilde{g}')$ has bounded geometry, i.e., $\forall z \in B_{\frac{\pi}{2}}(0)$, $InjRad(T_x(M), z) \geq 1$. By lifting $B_{\frac{\pi}{4}}(x) \bigcap \partial U^n$ to $B_{\frac{\pi}{2}}(0)$, applying previous result we then obtain the desired constant $\rho(n, r)$.

Q.E.D

Lemma 16 *Under the same condition as in Theorem 12. Then there is a constant $\sigma(n,r) > 0$ depending only on n and r, such that each component of $B_\sigma(x) \bigcap \partial U^n$ has the diameter, $diam_s$, less than 1.*

Proof. The proof of Lemma 16 is essentially the same as the proof of Lemma 15. So we omit the details. We first observe that Lemma 16 is true for M a *Euclidean* space and ∂U^n a hyperplane. By rescalling the metric of M^n by a sufficiently large number we may assume M^n and ∂U^n are almost as in above situation. Therefore Lemma 16 is true in general (Otherwise we would get a contradiction). Q.E.D

Lemma 17 *Let M^n be a complete manifolds with $Ricc_{M^n} \geq 1$. For $D > 0$ and $r > 0$, there is natural number $N(n, r, D)$ depending on n, r and D, so that if U is a submanifold with boundary such that if $diam(U) \leq D$, then ∂U can be covered by $N(n, r, D)$ balls of radius r with centers in ∂U^n.*

Proof. See the Covering Lemma in [CG1]. Q.E.D

Proof of Theorem F. First we take $\epsilon(n, r) = min\{\rho(n, r), \sigma(n, r)\}$ where $\rho(n, r)$ and $\sigma(n, r)$ are as in Lemma 15 and 16 respectively. Thus we have

1) $B(x_i) \bigcap \partial U^n$ is connected (Lemma 15),

2) $diam_s(B(x_i) \bigcap \partial U^n) \leq 1$ (Lemma 16).

Since $diam(T_1(X)) \leq D$, we can find at most $N(n, r, D)$ metric balls of radius $\epsilon(n, r)$, $B(x_1), ..., B(x_{N(n,r,D)})$ $(x_i \in \partial U^n)$ which covers ∂U^n (Lemma 17). Bombining with 1) and 2), we deduce

$$diam_s(\partial U^n) \leq \sum_{i=1}^{N(n,r,D)} diam_s(B(x_i) \bigcap \partial U^n) = N(n, r, D).$$

Now the proof is complete. Q.E.D.

3.2 Some Progress on Conjecture II

Based on the results in this thesis we have made some progress in proving Conjecture II for 4-manifolds (see Introduction). In fact, using Theorem

A, Theorem E and Theorem 14 we can prove the following result:

Let (M,g) be an oriented complete 4-manifold, $|K_g| \leq 1$, $Vol(M,g) < \infty$ and $\widetilde{geo}_\infty(M) \geq 1$. Suppose M has finite diameter at infinity. Then the geometric signature, defined by

$$\sigma_{(2)}(M) = \int_M P_L(\Omega)$$

is a rational number.

Note that there are 4-manifolds of infinite topological type which satisfy the above conditions. Also if one can eliminite the assumption that M has finite diameter at infinity, then one would prove Conjecture II for 4-manifolds. Here we only give the outline of the proof. More detailed argument will appear elsewhere.

Sketch of Proof. Let us start with the general situation. Let M be a $4k$-complete manifold, $|K| \leq 1$, $Vol(M^n) < \infty$. Consider the geometric signature $\sigma(M,g)$ of (M,g), defined by

$$\sigma(M,g) = \int_M P_L(\Omega)$$

Suppose M has BCG in a neighborhood of infinity, $\widetilde{geo}_\infty(M,g) \geq 1$. As we have already explained in Introduction, the above integal, also denoted by $\sigma_{(2)}(M)$, is a proper homotopy invariant.

Taking a good choppings $\{M_j\}$ of M and then applying the Atiyah-Patodi-Singer index formula ([APS]) to $\{M_j\}$, one relates the $\sigma_{(2)}(M)$ to the asymptotic behavior of the sequence $\eta(\partial M_j, g_j)$ (g_j the induced metric on ∂M_j).

$$\begin{aligned} \sigma_{(2)}(M) &= \int_M P_L(\Omega) \\ &= \lim_{i\to\infty} \int_{M_i} P_L(\Omega) \\ &= \lim_{i\to\infty}(\sigma(M_i) - \eta(\partial M_i, g_i) - II_\sigma(\partial M_i, g_i)) \\ &= \lim_{i\to\infty}(\sigma(M_i) - \eta(\partial M_i, g_i)) \end{aligned} \tag{3.7}$$

where $II_\sigma(\partial M_j, g_i)$ are certain locally computable expression involving the second fundamental form of ∂M_j. Note that it follows from Theorem 12 that $\lim_{j\to\infty} II_\sigma(\partial M_i, g_j)) = 0$. Since we are interested in the rationality of $\sigma_{(2)}(M)$ (Conjecture II), we take for convience $mod\, Z$ on both sides of (3.7),

$$\sigma_{(2)}(M) \ mod\, Z = \lim_{i\to\infty}(\eta(\partial M_i, g_i) \ mod \ Z). \tag{3.8}$$

To prove Conjecture II one needs to study the asymptotic behavior of the sequence $\{\eta(\partial M_j, g_j) \ mod \ Z\}$. Here the difficulties are: (a) there are not obvious relations among ∂M_i in general; (b) computation for $\eta(\partial M_j, g_j)$.

Now consider the case when $k = 1$. In order to reduce the difficulty (a) and (b), we suppose the 4-manifold M has finite diameter at infinity, i.e., there is a constant $D > 0$ and $p \in M$ such that $sup_{r>0}\{\sum_\alpha diam((T_1(\partial B_r(p)))_\alpha)\} \leq D$. We claim that there exists a good choppings $\{M_i\}$ satisfying the following conditions:

1) there exists a positive integer $K(D)$ depending only on D, such that each ∂M_i has at most components. We number its components as $(\partial M_i)_1, ..., (\partial M_i)_K$ where we allow that some components are empty set.

2) the sequence of the jth-component $\{(\partial M_i)_j\}$ converges to a metric

space X_j under the Hausdorff distance, $\lim_{i\to\infty} dist_H((\partial M_i)_j, X_j) = 0$. Note that X_j is a topologicasl manifold of dimension less or equals to 2.

1) can be seen from the proof of Theorem F, and 2) can be proved by using Gromov's precompactness theorem and a simple induction argument. Since

$$\eta(\partial M_i, g_i) \ mod\ Z = \sum_{j=1}^{K(D)} \eta((\partial M_i)_j, g_{i,j}) \ mod\ Z,$$

it is enough to prove rationality of (3.8) for the case when $K(D) = 1$.

We split the rest proof according to $dim(X)$ (note that $X = X_1$).

Case I. $dim(X) = 0$ or 2 or $X \approx S^1$.

We can show that, in this case, $\pi_1(\partial M_i)$ is infinite. We also observe that $(\partial M_i, g_i)$ has BCG. Here we use the condition that $||II_{\partial M_i}||$ is bounded and the Guass Lemma. The consequence of these two facts is that each ∂M_i admits an injective F-structure for i sufficiently large (Theorem B), i.e., $\eta_{(2)}(\partial M_i)$ makes sense. Thus, as we has already seen in §1.5,

$$\lim_{i\to\infty} \eta(\partial M_i, g_i) = \lim_{i\to\infty} \eta_{(2)}(\partial M_i). \tag{3.9}$$

From (3.8) and (3.9), we have

$$\sigma_{(2)}(M) = \lim_{i\to\infty} \eta_{(2)}(\partial M_i) \ mod\ Z \tag{3.10}$$

From $diam(\partial M_i, g_i) \leq D$ we conclude that $\{\eta_{(2)}(\partial M_i) \ mod\ Z\}$ is a finite set (Theorem E). From (3.10) and Theorem A, we obtain the rationality of $\sigma_{(2)}(M)$.

Case II. $X \approx I$ where I is an interval.

We then identify that each ∂M_i is a lens space for i sufficiently large (see the proof of Proposition 5).

First, by applying Theorem 1 one construct, depending on g, a positive rank T-structure $\mathcal{F}_g$ outside some compact subset of M. We can assume that our previous good chopping is compactible with $\mathcal{F}_g$ and the metric g is invariant ([CFG]). Since ∂M_i converges to an interval, we observe the following:

3) $\mathcal{F}_g$ is pure T^2-structure in a neighborhood of ∂M_i. Let $\mathcal{O}_i$ be a singular S^1-orbit on ∂M_i. Then the singular set Z of $\mathcal{F}_g$ has non-empty intersection with ∂M_i as i sufficinetly large.

4) the diameters of the above T^2-orbits around $\mathcal{O}_i$ converges to zero as $i \to \infty$.

By studying the structure of the singularity Z and using the result in [CFG], we can show that in this case the T^2 orbits of $\mathcal{F}_g$ which are near to Z have a difinite size diameters (note that here we need the condition BCG). This contracts 4).

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