Spaces Of Real Algebraic Cycles And Homotopy Theory

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Abstract of the Dissertation

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In this thesis we study the spaces of real algebraic cycles on a real algebraic subset $X \subset \mathbb{P}^n_{\mathbb{C}}$ by using the construction of Lawson on Chow varieties. Our main result is an 'Algebraic Suspension Theorem' which in the fundamental case when $X = \mathbb{P}^n_{\mathbb{C}}$ provides a cycle-theoretic construction of the 'universal total Stiefel-Whitney class'. In analogy with the works of Lawson-Michelsohn, Friedlander-Mazur, we study the algebraic join operation on these cycle spaces which leads to interesting relations between these spaces and Whitney duality, and to the construction of a bigraded module associated to X.

To my parents

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1 Introduction and Statement of Results

The study of algebraic cycles on an algebraic variety is one of the central subject in algebraic geometry. Much attention has been devoted to the investigation of various 'adequate equivalence relations'([FW], [HR], [KS1,2], [SP2]) on the group of algebraic cycles and the invariants associated to them. Recently, a new theory based on the homotopy groups of the Chow varieties(i.e., the components of the algebraic space of effective algebraic cycles on a complex projective variety) was discovered through the foundational work of Lawson([LB1,2]). The new invariants obtained in this theory are called the Lawson homology groups by Friedlander, who has also successfully generalized Lawson's result to varieties over an algebraically closed field of arbitrary characteristic. The works of Friedlander([F1,2]), Friedlander and Mazur([FH]), and Lima-Filho([LFP]) have provided foundational properties and some interesting examples of this homology theory.

In ([LM]), Lawson and Michelsohn studied the complex join operation on algebraic cycles and showed that the spaces of algebraic cycles on $\mathbb{P}^n_{\mathbb{C}}$ have interesting relations with Bott periodicity and Chern classes. In particular, they obtained a 'Chern Characteristic Map'

$$c : BU_q \longrightarrow \prod_{k=1}^{q} K(\mathbb{Z}, 2k)$$

which represents the total Chern class of the universal complex q-plane bundle over BU_q. By taking limit as $q \longrightarrow \infty$, they obtained a total Chern class map

$$c : BU \longrightarrow K(\mathbb{Z}, even) \stackrel{def}{=} \prod_{k=1}^{\infty} K(\mathbb{Z}, 2k).$$

In fact, the cycle-theoretic construction in [LM] showed that $K(\mathbb{Z}, even)$ can be constructed via Chow varieties. In particular, the complex join operation on algebraic cycles induces on $K(\mathbb{Z}, even)$ the structure of an associative, homotopically commutative H-space which is compatible with the H-space structure on BU arising from the Whitney sum operation. In the recent work of Boyer, Lawson, Mann and Michelsohn([BLMM]), the map c has been shown to be an infinite loop map. This answered a question in the theory of infinite loop spaces raised by G. Segal.

For a complex projective variety X defined over $\mathbb{R}$, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts naturally on the set of algebraic cycles on X. In this thesis, we study the topological structure of the space of Galois invariant cycles, following the construction of [LB1,2]. In particular, we establish an 'Algebraic Suspension Theorem' which may provide the foundation for an analogous theory for real algebraic varieties. We also construct a 'Whitney Characteristic Map' and obtain results analogous to the work in [LM].

We begin by providing some basic terminologies.

1.1 Spaces of Real Algebraic Cycles

The free abelian group generated by the set of all p-dimensional irreducible subvarieties in the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is called the **group of algebraic p-cycles** on $\mathbb{P}^n_{\mathbb{C}}$. As a set, it consists of all formal finite sums $\sum n_\alpha V_\alpha$ where, for each α, n_α is an integer and V_α is an irreducible p-dimensional subvariety of $\mathbb{P}^n_{\mathbb{C}}$. The degree of an algebraic p-cycle $\sum n_\alpha V_\alpha$ on $\mathbb{P}^n_{\mathbb{C}}$ is the integer $\sum n_\alpha deg V_\alpha$. Reall that the **degree** of a p-dimensional irreducible subvariety $V \subset \mathbb{P}^n_{\mathbb{C}}$ is the number $degV$ of points in the intersection of V with a generic (n-p)-dimensional linear subspace in $\mathbb{P}^n_{\mathbb{C}}$.

An algebraic p-cycle $\sum n_\alpha V_\alpha$ is called **effective** if n_α is a positive integer for each α .

Let X be an algebraic subset of $\mathbb{P}^n_{\mathbb{C}}$. An effective algebraic p-cycle $\sum n_\alpha V_\alpha$ is said to be **supported on** X if $V_\alpha \subset X$ for each α. Its **support** is given by $\bigcup_\alpha V_\alpha$. For any effective algebraic cycle c, we write $c \subset X$ if the cycle c is supported on X. We also write $c \nvDash X$ if no irreducible component of c is contained in X, and $c \vDash X$ otherwise.

On $\mathbb{P}^n_{\mathbb{C}}$, there is a standard real structure given by the conjugation map $\tau : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$, defined in homogeneous coordinates by $\tau([z_o, \cdots, z_n]) = [\bar{z}_o, \cdots, \bar{z}_n]$. The **conjugate algebraic subset** $\tau(X)$ is defined by the conjugate of the homogeneous defining ideal of X.

Definition 1.1.1 *An algebraic subset X of $\mathbb{P}^n_{\mathbb{C}}$ is said to be* **real** *if $X = \tau(X)$. An effective algebraic p-cycle $\sum n_\alpha V_\alpha$ on X is said to be a* **real cycle** *if $\sum n_\alpha V_\alpha = \sum n_\alpha \tau(V_\alpha)$. The* **Galois sum** *of $\sum n_\alpha V_\alpha$ is defined as the cycle $\sum n_\alpha V_\alpha + \sum n_\alpha \tau(V_\alpha)$.*

Remark 1.1.1 Clearly an algebraic subset X is real if and only if its homogeneous defining ideal can be generated by homogeneous polynomials with real coefficients.

Denote respectively by $\mathbf{C}_p(X)$, $\mathbf{RC}_p(X)$ and $\mathbf{DC}_p(X)$ the sets of effective algebraic p-cycles, effective real algebraic p-cycles and Galois sums of effective algebraic p-cycles supported on a real algebraic subset $X \subset \mathbb{P}^n_{\mathbb{C}}$. Then we have

$$\begin{aligned}
\mathbf{C}_p(X) &= \coprod_{d\geq o} \mathbf{C}_{p,d}(X) \\
\mathbf{RC}_p(X) &= \coprod_{d\geq o} \mathbf{RC}_{p,d}(X) \\
\mathbf{DC}_p(X) &= \coprod_{d\geq o} \mathbf{DC}_{p,d}(X)
\end{aligned}$$

where $\mathbf{DC}_{p,d}(X) \subset \mathbf{RC}_{p,d}(X) \subset \mathbf{C}_{p,d}(X)$ are the subsets of degree d cycles in $\mathbf{DC}_p(X)$, $\mathbf{RC}_p(X)$ and $\mathbf{C}_p(X)$ respectively. Note that $\mathbf{DC}_{p,d}(X) = \emptyset$ if d is an odd integer.

It is well-known that $\mathbf{C}_{p,d}(X)$ admits the structure of an algebraic subset in some complex projective space (see [SP1], [Sh]). Moreover, the formal addition of cycles

$$\mathbf{C}_{p,d}(X) \times \mathbf{C}_{p,d'}(X) \xrightarrow{+} \mathbf{C}_{p,d+d'}(X)$$

is an algebraic map. If in addition X is a real algebraic subset, so is $\mathbf{C}_{p,d}(X)$, and $\mathbf{RC}_{p,d}(X)$ is then the real locus. With the analytic topology, the inclusions $\mathbf{DC}_p(X) \subset \mathbf{RC}_p(X) \subset \mathbf{C}_p(X)$ are continuous homomorphisms of abelian topological monoids. $\mathbf{C}_p(X)$ is called the **Chow monoid** in [F1,2]. Also note that these abelian topological monoids are compactly generated spaces (see [SN], [McM]) since $\mathbf{DC}_{p,d}(X)$, $\mathbf{RC}_{p,d}(X)$, $\mathbf{C}_{p,d}(X)$ are all triangulable compact spaces.

The topological structure of the Chow varieties are quite complicated in general. For example, see the discussion of $\mathbf{C}_{1,d}(\mathbb{P}^3_{\mathbb{C}})$ in [Sh]. However, in his foundational work [LB1,2], Lawson initiated a stabilizing procedure to study the topological structure of $\mathbf{C}_p(X)$ and obtained a **'Complex Suspension Theorem'**. The general setting of this stabilizing procedure was then pointed out by Friedlander who went on to generalize Lawson's results to varieties over an algebraically closed field of arbitrary characteristic, using the machinery of etale homotopy theory and Bousfield-Kan's work on completion and homotopy limit.([F1]) In fact, Friedlander noticed that the original construction of Lawson could be fitted into the general procedure of group completion studied extensively in homotopy theory.

The study of topological monoids, or more generally, H-spaces, can be considered as a generalization of the homotopy theory of Lie groups. The original work of Whitney on sphere bundles initiated the construction of classifying spaces for Lie groups (see [SJ], [SN1]). The construction of a **universal bundle**

$$G \longrightarrow EG \longrightarrow BG$$

for a general topological group G was first obtained by Milnor ([M]).

As a consequence of this construction, there is a homotopy equivalence $G \xrightarrow{\sim} \Omega BG$ if G is a connected CW complex and the group operation is cellular. From then on, classifying spaces for various H-spaces were constructed , notably the works of Dold and Lashof, Stasheff, Sugawara, Milgram, Boardman and Vogt, Segal, May, and many others. These constructions showed that any reasonable H-space, for example, a connected associative cellular H-space, has the homotopy type of a loop space.

Consider a Hausdorff, compactly generated topological monoid M which has the homotopy type of a CW-complex. The canonical map $M \longrightarrow \Omega BM$ is not necessary a homotopy equivalence in general. It is true if and only if translations by elements of M are homotopy equivalences, or equivalently if $\pi_o(M)$ is a group. What this map does in general has been extensively studied in algebraic topology. Roughly speaking, provided that M is **'sufficiently homotopically commutative'**, the map $M \longrightarrow \Omega BM$ has the effect of localizing the action of $\pi_o(M)$ on the Pontryagin ring $H_*(M)$. This is called the **Group Completion Theorem.** We will give some more details on group completion in Section 2.

Remark 1.1.2 ΩBM inherits the sturcture of a topological monoid from M, and is abelian if M is so. Moreover, the path-components of ΩBM are all homotopically equivalent to each other since $\pi_o(\Omega BM) = \pi_1(BM)$ is a group, i.e., $\Omega BM \sim \pi_o(\Omega BM) \times (\Omega BM)_o$ where $(\Omega BM)_o$ is the identity component of ΩBM.

We now define cycle spaces, following the setting of Friedlander and Lawson, by taking group completions.

Definition 1.1.2 *Let X be a real algebraic subset in $\mathbb{P}^n_{\mathbb{C}}$. Then*

$$\begin{aligned} \mathcal{C}_p(X) &\stackrel{def}{=} \Omega B\mathbf{C}_p(X) \\ \mathcal{RC}_p(X) &\stackrel{def}{=} \Omega B\mathbf{RC}_p(X) \\ \mathcal{DC}_p(X) &\stackrel{def}{=} \Omega B\mathbf{DC}_p(X). \end{aligned}$$

On the other hand, associated to any abelian topological monoid M, there is a **naive group completion** $\widetilde{M}$, namely the **Grothendieck group** of (equivalence classes of) formal differences of elements of M, endowed with the weak topology given by a family of closed subsets $\delta(M_\alpha \times M_\beta)$ where $\delta(a,b) = a - b$ is the 'difference map'.

Formally, $\widetilde{M}$ is the set of equivalence classes $[(a,b)]$ of pairs of elements in M such that

$$(a,b) \sim (a',b') \iff a + b' + c = a' + b + c$$
$$for\ some\ c \in M$$

Note that there is a canonical morphism of abelian topological monoids

$$\iota : M \longrightarrow \widetilde{M}$$

given by $\iota(a) = a - 0$. Furthermore, ι is an inclusion if M has cancellation law.

The naive group completion has the **universal property** that given any morphism $f : M \longrightarrow G$ into an abelian topological group, there is a morphism of topological groups $\tilde{f} : \widetilde{M} \longrightarrow G$ such that $f = \tilde{f} \circ \iota$.

In particular, we denote by $\widetilde{\mathcal{C}}_p(X)$, $\widetilde{\mathcal{RC}}_p(X)$ and $\widetilde{\mathcal{DC}}_p(X)$ the naive group completions of $\mathbf{C}_p(X)$, $\mathbf{RC}_p(X)$ and $\mathbf{DC}_p(X)$ respectively. Note that the topology of these spaces are generated by the families of closed subsets $\delta(\mathbf{C}_{p,d}(X) \times \mathbf{C}_{p,e}(X))$, $\delta(\mathbf{RC}_{p,d}(X) \times \mathbf{RC}_{p,e}(X))$ and $\delta(\mathbf{DC}_{p,d}(X) \times \mathbf{DC}_{p,e}(X))$ respectively. It is clear that there are induced degree maps on these spaces given by

$$\deg([(a,b)]) = \deg a - \deg b.$$

In [LB1,2], the homotopy type of $\mathcal{C}_p(\mathbb{P}^n_{\mathbb{C}})$ ($\widetilde{\mathcal{C}}_p(\mathbb{P}^n_{\mathbb{C}})$ resp.) was determined completely via the 'Complex Suspension Theorem'. The connection of $\mathcal{C}_p(\mathbb{P}^n_{\mathbb{C}})$ with Bott periodicity and Chern Classes was investigated in [LM].

The main direction of this thesis is to established analogous results for the spaces of real algebraic cycles defined as above.

1.2 Algebraic Suspension and The Algebraic Join Operation

To state our main results, we first recall the algebraic suspension map and algebraic join operation for algebraic subsets and algebraic cycles.

For the rest of this section, we will always assume that $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^m_{\mathbb{C}}$ are two disjoint real linear subspaces in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$ with the induced real structures, unless otherwise stated.

Definition 1.2.1 *Let $V \subset \mathbb{P}^n_{\mathbb{C}}, W \subset \mathbb{P}^m_{\mathbb{C}}$ be two algebraic subsets. The union of all projective lines in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$ joining points of V to points of W is an algebraic subset of $\mathbb{P}^{n+m+1}_{\mathbb{C}}$, denoted by $V\#_{\mathbb{C}}W$, and is called the* **algebraic join** *of V and W. In particular, if $W = \mathbb{P}^m_{\mathbb{C}}$, $\Sigma\!\!\!/^{m+1} V \stackrel{def}{=} V\#_{\mathbb{C}}\mathbb{P}^m_{\mathbb{C}}$ is called the $(m+1)$-fold* **algebraic suspension** *of V.*

It is clear that the defining equations of V also define $\Sigma\!\!\!/^{m+1} V$ in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$ and that $\Sigma\!\!\!/^{m+1} V = \Sigma\!\!\!/(\Sigma\!\!\!/(\cdots(\Sigma\!\!\!/ V)\cdots))$ $(m+1$ times). Also note that when $m = 0$, $\Sigma\!\!\!/ V$ is simply the Thom space of the hyperplane bundle $\mathcal{O}_V(1)$ on V.

If V and W are irreducible subvarieties, then so is $V\#_{\mathbb{C}}W$. Furthermore, $degV\#_{\mathbb{C}}W = degV \cdot degW$ and $dimV\#_{\mathbb{C}}W = dimV + dimW + 1$. Therefore, by extending the algebraic join operation biadditively to algebraic cycles, and noticing that the algebraic join operation preserves the spaces of real algebraic cycles, we have the following commutative diagram of continuous maps

$$\begin{array}{ccc}
\mathbf{C}_{p,d}(X) \times \mathbf{C}_{r,e}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{C}_{p+r+1,de}(X\#_{\mathbb{C}}Y) \\
\uparrow & & \uparrow \\
\mathbf{RC}_{p,d}(X) \times \mathbf{RC}_{r,e}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{RC}_{p+r+1,de}(X\#_{\mathbb{C}}Y) \\
\uparrow & & \uparrow \\
\mathbf{DC}_{p,d}(X) \times \mathbf{DC}_{r,e}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{DC}_{p+r+1,de}(X\#_{\mathbb{C}}Y)
\end{array}$$

for any two real algebraic subsets $X \subset \mathbb{P}^n_{\mathbb{C}}, Y \subset \mathbb{P}^m_{\mathbb{C}}$.

Passing to the corresponding monoids, we have the following commutative diagram of biadditive maps

$$\begin{array}{ccc}
\mathbf{C}_{p}(X) \times \mathbf{C}_{r}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{C}_{p+r+1}(X\#_{\mathbb{C}}Y) \\
\uparrow & & \uparrow \\
\mathbf{RC}_{p}(X) \times \mathbf{RC}_{r}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{RC}_{p+r+1}(X\#_{\mathbb{C}}Y) \\
\uparrow & & \uparrow \\
\mathbf{DC}_{p}(X) \times \mathbf{DC}_{r}(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathbf{DC}_{p+r+1}(X\#_{\mathbb{C}}Y)
\end{array}$$

which descends to maps of their smash products since $0\#_{\mathbb{C}}b = a\#_{\mathbb{C}}0 = 0$ is the 0 (empty) cycle for any cycle a on X and b on Y.

By the functorial properties of classifying spaces(see Section 2), we then have an induced commutative diagram of continuous maps between the cycle spaces :

$$\begin{array}{ccc}
\mathcal{C}_p(X) \times \mathcal{C}_r(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathcal{C}_{p+r+1}(X \#_{\mathbb{C}} Y) \\
\uparrow & & \uparrow \\
\mathcal{RC}_p(X) \times \mathcal{RC}_r(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathcal{RC}_{p+r+1}(X \#_{\mathbb{C}} Y) \\
\uparrow & & \uparrow \\
\mathcal{DC}_p(X) \times \mathcal{DC}_r(Y) & \xrightarrow{\#_{\mathbb{C}}} & \mathcal{DC}_{p+r+1}(X \#_{\mathbb{C}} Y)
\end{array}$$

In particular, if $Y = \mathbb{P}_{\mathbb{C}}^{m-1}$ and $r = m-1$, we have the **algebraic suspension maps**

$$\begin{array}{ccc}
\mathcal{C}_p(X) & \xrightarrow{\Sigma^m} & \mathcal{C}_{p+m}(\Sigma^m X) \\
\uparrow & & \uparrow \\
\mathcal{RC}_p(X) & \xrightarrow{\Sigma^m} & \mathcal{RC}_{p+m}(\Sigma^m X) \\
\uparrow & & \uparrow \\
\mathcal{DC}_p(X) & \xrightarrow{\Sigma^m} & \mathcal{DC}_{p+m}(\Sigma^m X)
\end{array}$$

Alternatively, by the universal properties of the universal topological groups $\widetilde{\mathcal{C}}_p(X)$, $\widetilde{\mathcal{RC}}_p(X)$ and $\widetilde{\mathcal{DC}}_p(X)$, there are corresponding algebraic suspension maps

$$\begin{array}{ccc}
\widetilde{\mathcal{C}}_p(X) & \xrightarrow{\widetilde{\Sigma^m}} & \widetilde{\mathcal{C}}_{p+m}(\Sigma^m X) \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\widetilde{\Sigma^m}} & \widetilde{\mathcal{RC}}_{p+m}(\Sigma^m X) \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{DC}}_p(X) & \xrightarrow{\widetilde{\Sigma^m}} & \widetilde{\mathcal{DC}}_{p+m}(\Sigma^m X)
\end{array}$$

The *'Complex Suspension Theorem'* in [LB2] states that the algebraic suspension maps

$$\mathcal{C}_p(X) \xrightarrow{\Sigma\!\!\!/^m} \mathcal{C}_{p+m}(\Sigma\!\!\!/^m X)$$
$$\tilde{\mathcal{C}}_p(X) \xrightarrow{\widetilde{\Sigma\!\!\!/^m}} \tilde{\mathcal{C}}_{p+m}(\Sigma\!\!\!/^m X)$$

are homotopy equivalences.

Our first main result is the following.

Theorem 1.2.1 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a real algebraic subset. Then the algebraic suspension maps*

$$\begin{array}{ccc} \mathcal{RC}_p(X) & \xrightarrow{\Sigma\!\!\!/^m} & \mathcal{RC}_{p+m}(\Sigma\!\!\!/^m X) \\ \mathcal{DC}_p(X) & \xrightarrow{\Sigma\!\!\!/^m} & \mathcal{DC}_{p+m}(\Sigma\!\!\!/^m X) \end{array}$$

are homotopy equivalences for every dimension p and every positive integer m. So also are the algebraic suspension maps

$$\begin{array}{ccc} \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\widetilde{\Sigma\!\!\!/^m}} & \widetilde{\mathcal{RC}}_{p+m}(\Sigma\!\!\!/^m X) \\ \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{\widetilde{\Sigma\!\!\!/^m}} & \widetilde{\mathcal{DC}}_{p+m}(\Sigma\!\!\!/^m X). \end{array}$$

1.3 Mod 2 Cycle Spaces

Given a monoid morphism $N \xrightarrow{f} M$, there is an induced right N-action on M given by

$$g \cdot x \stackrel{def}{=} x \cdot f(g)$$

for all $g \in N$ and $x \in M$.

Let $B(M,N,f) \stackrel{def}{=} B(M,N,*)$ where $B(\cdot,\cdot,\cdot)$ is the **'triple bar construction'** in [May2]. Consider the inclusions of abelian topological monoids

$$\begin{array}{ccc} \mathbf{RC}_p(X) & \stackrel{2}{\longrightarrow} & \mathbf{RC}_p(X) \\ \mathbf{DC}_p(X) & \stackrel{2}{\longrightarrow} & \mathbf{DC}_p(X) \\ \mathbf{DC}_p(X) & \stackrel{i}{\longrightarrow} & \mathbf{RC}_p(X), \end{array}$$

where $a \stackrel{2}{\longrightarrow} 2a$ and $a \stackrel{i}{\longrightarrow} a$, which induce morphisms of their group completions

$$\begin{array}{ccc} \mathcal{RC}_p(X) & \stackrel{2}{\longrightarrow} & \mathcal{RC}_p(X) \\ \mathcal{DC}_p(X) & \stackrel{2}{\longrightarrow} & \mathcal{DC}_p(X) \\ \mathcal{DC}_p(X) & \stackrel{i}{\longrightarrow} & \mathcal{RC}_p(X). \end{array}$$

Definition 1.3.1 *The* **mod 2 real cycle spaces** *are defined as follow :*

$$\begin{array}{rcl} \mathcal{RC}_p(X)\otimes\mathbb{Z}_2 & \stackrel{def}{=} & B(\mathcal{RC}_p(X),\mathcal{RC}_p(X),2) \\ \mathcal{DC}_p(X)\otimes\mathbb{Z}_2 & \stackrel{def}{=} & B(\mathcal{DC}_p(X),\mathcal{DC}_p(X),2) \\ \mathcal{E}_p(X) & \stackrel{def}{=} & B(\mathcal{RC}_p(X),\mathcal{DC}_p(X),i) \end{array}$$

The algebraic suspension maps on $\mathcal{RC}_p(X)$, $\mathcal{DC}_p(X)$ then induce algebraic suspension maps on these mod 2 real cycle spaces :

$$\begin{array}{rcl} \mathcal{RC}_p(X)\otimes\mathbb{Z}_2 & \stackrel{\Sigma^m}{\longrightarrow} & \mathcal{RC}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2 \\ \mathcal{DC}_p(X)\otimes\mathbb{Z}_2 & \stackrel{\Sigma^m}{\longrightarrow} & \mathcal{DC}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2 \\ \mathcal{E}_p(X) & \stackrel{\Sigma^m}{\longrightarrow} & \mathcal{E}_{p+m}(\Sigma^m X) \end{array}$$

Alternatively, we can consider the *quotient groups* of the naive group completions :

Definition 1.3.2 *The* **naive mod 2 real cycle spaces** *are definded as follow :*

$$\begin{aligned}\widetilde{\mathcal{RC}}_p(X)\otimes\mathbb{Z}_2 &\stackrel{def}{=} \widetilde{\mathcal{RC}}_p(X)/\!\!/2\widetilde{\mathcal{RC}}_p(X)\\ \widetilde{\mathcal{DC}}_p(X)\otimes\mathbb{Z}_2 &\stackrel{def}{=} \widetilde{\mathcal{DC}}_p(X)/\!\!/2\widetilde{\mathcal{DC}}_p(X)\\ \widetilde{\mathcal{E}}_p(X) &\stackrel{def}{=} \widetilde{\mathcal{RC}}_p(X)/\!\!/\widetilde{\mathcal{DC}}_p(X)\end{aligned}$$

where each of these quotients is endowed with the quotient topology.

Similarly, there are induced algebraic suspension maps

$$\begin{aligned}\widetilde{\mathcal{RC}}_p(X)\otimes\mathbb{Z}_2 &\xrightarrow{\widetilde{\Sigma^m}} \widetilde{\mathcal{RC}}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2\\ \widetilde{\mathcal{DC}}_p(X)\otimes\mathbb{Z}_2 &\xrightarrow{\widetilde{\Sigma^m}} \widetilde{\mathcal{DC}}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2\\ \widetilde{\mathcal{E}}_p(X) &\xrightarrow{\widetilde{\Sigma^m}} \widetilde{\mathcal{E}}_{p+m}(\Sigma^m X)\end{aligned}$$

on these naive mod 2 real cycle spaces. Our next result is :

Theorem 1.3.1 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a real algebraic subset. Then the algebraic suspension maps*

$$\begin{aligned}\mathcal{RC}_p(X)\otimes\mathbb{Z}_2 &\xrightarrow{\Sigma^m} \mathcal{RC}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2\\ \mathcal{DC}_p(X)\otimes\mathbb{Z}_2 &\xrightarrow{\Sigma^m} \mathcal{DC}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2\\ \mathcal{E}_p(X) &\xrightarrow{\Sigma^m} \mathcal{E}_{p+m}(\Sigma^m X)\end{aligned}$$

are homotopy equivalences for every dimension p and every positive integer *m. Similarly, the same is true for the algebraic suspension maps*

$$\begin{array}{rcl} \widetilde{\mathcal{RC}}_p(X)\otimes\mathbb{Z}_2 & \xrightarrow{\widetilde{\Sigma^m}} & \widetilde{\mathcal{RC}}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2 \\ \widetilde{\mathcal{DC}}_p(X)\otimes\mathbb{Z}_2 & \xrightarrow{\widetilde{\Sigma^m}} & \widetilde{\mathcal{DC}}_{p+m}(\Sigma^m X)\otimes\mathbb{Z}_2 \\ \tilde{\mathcal{E}}_p(X) & \xrightarrow{\widetilde{\Sigma^m}} & \tilde{\mathcal{E}}_{p+m}(\Sigma^m X). \end{array}$$

By the algebraic suspension theorems above and Dold-Thom's theorem [DT1,2], we have

Theorem 1.3.2 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a real algebraic subset with connected real locus $X_{\mathbb{R}}$. Then for any positive integer m , there are isomorphisms*

$$\begin{array}{rcl} \pi_*(\mathcal{RC}_m(\Sigma^m X)\otimes\mathbb{Z}_2) & \cong & H_*(X_{\mathbb{R}} \vee ((X/\mathbb{Z}_2)/X_{\mathbb{R}}), \mathbb{Z}_2) \\ \pi_*(\mathcal{DC}_m(\Sigma^m X)) & \cong & H_*(X/\mathbb{Z}_2, \mathbb{Z}) \\ \pi_*(\mathcal{DC}_m(\Sigma^m X)\otimes\mathbb{Z}_2) & \cong & H_*(X/\mathbb{Z}_2, \mathbb{Z}_2) \\ \pi_*(\mathcal{E}_m(\Sigma^m X)) & \cong & H_*(X_{\mathbb{R}}, \mathbb{Z}_2) \end{array}$$

where $X/\mathbb{Z}_2$ is the quotient space of X with respect to the conjugation map. Furthermore, they are homotopically equivalent to products of corresponding Eilenberg-MacLane spaces.

Of particular interest here is the case when $X = \mathbb{P}^n_{\mathbb{C}}$.

Corollary 1.3.1 *There are isomorphisms*

$$\begin{array}{rcl} \pi_k(\mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})\otimes\mathbb{Z}_2) & \cong & H_k(\mathbb{P}^{n-p}_{\mathbb{R}} \vee ((\mathbb{P}^n_{\mathbb{C}}/\mathbb{Z}_2)/\mathbb{P}^{n-p}_{\mathbb{R}}), \mathbb{Z}_2) \\ \pi_k(\mathcal{DC}_p(\mathbb{P}^n_{\mathbb{C}})) & \cong & H_k(\mathbb{P}^{n-p}_{\mathbb{C}}/\mathbb{Z}_2, \mathbb{Z}) \\ \pi_k(\mathcal{DC}_p(\mathbb{P}^n_{\mathbb{C}})\otimes\mathbb{Z}_2) & \cong & H_k(\mathbb{P}^{n-p}_{\mathbb{C}}/\mathbb{Z}_2, \mathbb{Z}_2) \\ \pi_k(\mathcal{E}_p(\mathbb{P}^n_{\mathbb{C}})) & \cong & H_k(\mathbb{P}^{n-p}_{\mathbb{R}}, \mathbb{Z}_2). \end{array}$$

Remark 1.3.1 Similarly, Theorem 1.3.2 and Corollary 1.3.1 still holds if we replace the mod 2 real cycle spaces by the naive mod 2 real cycle spaces defined in Definition 1.3.2. Also note that Theorem 1.3 shows that these cycle spaces have only finitely many nonvanishing homotopy groups.

In particular, we have

Corollary 1.3.2 *There are homotopy equivalences*

$$\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}}) \sim \widetilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}) \sim K(\mathbb{Z}_2\ ,0) \times K(\mathbb{Z}_2\ ,1) \times \cdots \times K(\mathbb{Z}_2\ ,q)$$

for each positive integer $q \leq n$, *where* $\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}}) \stackrel{def}{=} \mathcal{E}_{n-p}(\mathbb{P}^n_{\mathbb{C}}), \widetilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}) \stackrel{def}{=} \widetilde{\mathcal{E}}_{n-p}(\mathbb{P}^n_{\mathbb{C}})$ *and* $K(\mathbb{Z}_2\ ,k)$ *is the standard Eilenberg-MacLane space.*

Theorem 1.3.2 shows that the homology groups of the real locus $X_{\mathbb{R}}$ of a real algebraic have close relationship with the homotopy groups of the cycle spaces $\widetilde{\mathcal{E}}^q(\sum^k X)$. Following the construction of Friedlander and Mazur in [FM], we let $k \longrightarrow \infty$ and define $\mathfrak{E}^q(X) \stackrel{def}{=} \lim_{k\to\infty} \widetilde{\mathcal{E}}^q(\sum^k X)$. The algebraic join operation then induces a pairing

$$\mathfrak{E}^q(X) \times \mathfrak{E}^{q'}(Y) \xrightarrow{\#_{\mathbb{C}}} \mathfrak{E}^{q+q'}(X \#_{\mathbb{C}} Y)$$

which descends to the smash product

$$\mathfrak{E}^q(X) \wedge \mathfrak{E}^{q'}(Y) \xrightarrow{\#_{\mathbb{C}}} \mathfrak{E}^{q+q'}(X \#_{\mathbb{C}} Y).$$

Hence there is an induced pairing on the homotopy groups

$$\pi_r(\mathfrak{E}^q(X)) \otimes \pi_s(\mathfrak{E}^{q'}(Y)) \xrightarrow{\#_{\mathbb{C}}} \pi_{r+s}(\mathfrak{E}^{q+q'}(X \#_{\mathbb{C}} Y)).$$

In particular, when $X = \mathbb{P}^o_{\mathbb{C}}$, we have a bigraded ring $\pi_*(\mathfrak{C}^*(\mathbb{P}^o_{\mathbb{C}}))$. Then, for a general real algebraic subset X, $\pi_*(\mathfrak{C}^*(X))$ admits the structure of a bigraded module over the bigraded ring $\pi_*(\mathfrak{C}^*(\mathbb{P}^o_{\mathbb{C}}))$. In analogy with the work in [FM], this provides an intriguing filtration on the $\mathbb{Z}_2$-homology groups of algebraic varieties over $\mathbb{R}$. We will discuss some properties of this bigraded module in Section 5.

1.4 The Relative Theory

Let $Y \subset X \subset \mathbb{P}^n_{\mathbb{C}}$ be a pair of algebraic subsets in $\mathbb{P}^n_{\mathbb{C}}$. Then we have an inclusion of Chow monoids $\mathbf{C}_p(Y) \subset \mathbf{C}_p(X)$. A relative theory can then be defined as

$$\mathcal{C}_p(X,Y) \stackrel{def}{=} B(\mathcal{C}_p(X), \mathcal{C}_p(Y)),$$

or one can consider the quotient of the universal cycle groups :

$$\tilde{\mathcal{C}}_p(X,Y) \stackrel{def}{=} \tilde{\mathcal{C}}_p(X) /\!\!/ \tilde{\mathcal{C}}_p(Y).$$

The complex suspension theorem also holds for these spaces (see [LB2]).

The homotopy groups of $\mathcal{C}_p(X)$ and $\mathcal{C}_p(X,Y)$ form an interesting homology theory. They are called the Lawson homology groups. See [F1,2], [LFP] for more details.

If Y and X are both real algebraic subsets, we can defined relative real cycle spaces in an analogous way.

Definition 1.4.1 *For any real algebraic subsets $Y \subset X \subset \mathbb{P}^n_{\mathbb{C}}$,*

$$\begin{aligned}
\mathcal{RC}_p(X,Y) &\stackrel{def}{=} B(\mathcal{RC}_p(X), \mathcal{RC}_p(Y)) \\
\mathcal{DC}_p(X,Y) &\stackrel{def}{=} B(\mathcal{DC}_p(X), \mathcal{DC}_p(Y)) \\
\mathcal{E}_p(X,Y) &\stackrel{def}{=} B(\mathcal{E}_p(X), \mathcal{E}_p(Y)) \\
\widetilde{\mathcal{RC}}_p(X,Y) &\stackrel{def}{=} \widetilde{\mathcal{RC}}_p(X) /\!\!/ \widetilde{\mathcal{RC}}_p(Y) \\
\widetilde{\mathcal{DC}}_p(X,Y) &\stackrel{def}{=} \widetilde{\mathcal{DC}}_p(X) /\!\!/ \widetilde{\mathcal{DC}}_p(Y) \\
\tilde{\mathcal{E}}_p(X,Y) &\stackrel{def}{=} \tilde{\mathcal{E}}_p(X) /\!\!/ \tilde{\mathcal{E}}_p(Y).
\end{aligned}$$

Then, in a similar manner, we have

Theorem 1.4.1 *The algebraic suspension maps for these relative real cycle spaces are also homotopy equivalences.*

Remark 1.4.1 From [F1,2], a proper morphism $f : X \longrightarrow Y$ induces a continuous rational map

$$f_* : \mathbf{C}_p(X) \longrightarrow \mathbf{C}_p(Y).$$

Similarly, a flat morphism $f : X \longrightarrow Y$ of relative dimension r induces a continuous rational map

$$f^\# : \mathbf{C}_p(Y) \longrightarrow \mathbf{C}_{p+r}(X).$$

Clearly if $f : X \longrightarrow Y$ is real, i.e., f is a τ-equivariant morphism, then the spaces of real cycles we considered are preserved by proper push forward, or flat pullback. This provides fundamental functorial properties for the spaces defined in Definition 1.4.1.

Remark 1.4.2 In [LFP], an excision theorem was obtained which served as the main tool for the computation of Lawson homology groups for an interesting class of algebraic varieties. It seems likely that for pairs of real algebraic subsets (X, Y), (X', Y'), a Galois equivariant isomorphism $X - Y \cong X' - Y'$ will also induce homotopy equivalences between these relative real cycle spaces.

1.5 The Whitney Characteristic Map

Noticing that $\mathbf{RC}_{p,1}(\mathbb{P}_{\mathbb{C}}^{p+q})$ can be identified with the real Grassmannian of (p+1)-planes in $\mathbb{R}^{n+1}$, we consider the composition map $\mathbf{RC}_{p,1}(\mathbb{P}_{\mathbb{C}}^{p+q}) \longrightarrow \mathcal{RC}_p(\mathbb{P}_{\mathbb{C}}^{p+q}) \longrightarrow \mathcal{E}^q(\mathbb{P}_{\mathbb{C}}^{p+q})$. By letting $p \to \infty$, we obtain a map

$$BO_q \xrightarrow{w} K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, q)$$

via Corollary 1.3.2.

Recall that $K(\mathbb{Z}_2\,, k)$ is the classifying space for the cohomology functor $H^k(\cdot, \mathbb{Z}_2)$. In analogy with the results in [LM], we have

Theorem 1.5.1 *The map w represents the total Stiefel-Whitney class of the universal q-plane bundle on BO_q.*

In Section 5, we discuss the relation between the complex join operation on real cycle spaces and the Whitney duality.

2 Group Completion and Cycle Spaces

In this section, we provide some background materials on classifying spaces and group completions of topological monoids. For more details, see [A], [FM], [May 1,2], [MS] and [S1,2]. An alternative formulation of cycle spaces is then given. We also discuss some basic examples of cycle spaces.

2.1 Classifying Spaces and Quasifibrations

All topological spaces here are assumed to be compactly generated weak Hausdorff spaces.(cf. [McM], [SN2]) Products of spaces are endowed with the weak product topology. Let G be a topological monoid and let M be a right G-space. The **classifying space** $B(M,G)$ is defined as the geometric realization of the simplicial space with $M \times G^i$ as its i-simplices and with appropriate face and degeneracy operators.(see [May2] for details)

The construction $B(\cdot,\cdot)$ is functorial in the sense that for all pair of morphisms $(f,g) : (M,G) \longrightarrow (M',G')$ where f is g-equivariant, there is a naturally induced morphism $B(f,g) : B(M,G) \longrightarrow B(M',G')$. Furthermore, we have

1. If f and g are homoptopy equivalences, then so is $B(f,g)$.
2. For $(M,G),(M',G')$, the projections define a natural homeomorphism
$$B(M \times M', G \times G') \longrightarrow B(M,G) \times B(M',G').$$
3. If M and G have the homotopy type of CW-complexes, then so is $B(M,G)$.

4. $B(M,G)$ is n-connected if G is (n-1)-connected and M is n-connected.

In particular, if G acts trivially on $M = *$, then $BG \stackrel{def}{=} B(*,G)$. The monoid structure on G induces a monoid structure on BG. Moreover, if G is abelian, so is BG.

The monoid structure on G naturally induces a monoid structure on $\pi_o(G)$. G is said to be **grouplike** if $\pi_o(G)$ is a group. For example, any loop space is grouplike since $\pi_o(\Omega Y) = \pi_1(Y)$.

Proposition 2.1.1 ([May2]) *If G is grouplike and M is a right G-space, then there is a sequence of quasifibrations*

$$G \longrightarrow M \longrightarrow B(M,G) \longrightarrow BG.$$

Of particular interest here is the case when M is also a topological monoid and the right action of G on M is given by the right translation by elements of G through a monoid morphism $G \xrightarrow{f} M$.

Proposition 2.1.2 ([May2]) *Let G, M be grouplike topological monoids and let $G \xrightarrow{f} M$ be a monoid morphism. Then there is a sequence of quasifibrations*

$$G \longrightarrow M \longrightarrow B(M,G) \longrightarrow BG \xrightarrow{Bf} BM.$$

Furthermore, $B(M,G)$ is weakly homotopically equivalent to the homotopy fiber of $BG \xrightarrow{Bf} BM$.

Applying the proposition to the morphisms of abelian topological monoids

$$\begin{array}{rcl} \mathcal{RC}_p(X) & \xrightarrow{2} & \mathcal{RC}_p(X) \\ \mathcal{DC}_p(X) & \xrightarrow{2} & \mathcal{DC}_p(X) \\ \mathcal{DC}_p(X) & \xrightarrow{i} & \mathcal{RC}_p(X), \end{array}$$

we have the following corollary :

Corollary 2.1.1 *There are quasifibrations*

$$\begin{array}{ccccc} \mathcal{RC}_p(X) & \xrightarrow{2} & \mathcal{RC}_p(X) & \longrightarrow & \mathcal{RC}_p(X)\otimes\mathbb{Z}_2 \\ \mathcal{DC}_p(X) & \xrightarrow{2} & \mathcal{DC}_p(X) & \longrightarrow & \mathcal{DC}_p(X)\otimes\mathbb{Z}_2 \\ \mathcal{DC}_p(X) & \xrightarrow{i} & \mathcal{RC}_p(X) & \longrightarrow & \mathcal{E}_p(X). \end{array}$$

Moreover, $\mathcal{RC}_p(X)\otimes\mathbb{Z}_2$, $\mathcal{DC}_p(X)\otimes\mathbb{Z}_2$ and $\mathcal{E}_p(X)$ are homotopically equivalent to the homotopy fibers of the maps

$$\begin{array}{ccc} B\mathbf{RC}_p(X) & \xrightarrow{B2} & B\mathbf{RC}_p(X) \\ B\mathbf{DC}_p(X) & \xrightarrow{B2} & B\mathbf{DC}_p(X) \\ B\mathbf{DC}_p(X) & \xrightarrow{Bi} & B\mathbf{RC}_p(X) \end{array}$$

respectively.

Similarly, for the universal cycle groups, we have

Theorem 2.1.1 *The natural projection homomorphisms*

$$\begin{array}{ccccc} \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{2} & \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\rho} & \widetilde{\mathcal{RC}}_p(X)\otimes\mathbb{Z}_2 \\ \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{2} & \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{\rho} & \widetilde{\mathcal{DC}}_p(X)\otimes\mathbb{Z}_2 \\ \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{i} & \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\rho} & \widetilde{\mathcal{E}}_p(X) \end{array}$$

are principal fibrations.

Proof For the proof of each case, it is sufficient to construct a local section of the projection on a neighborhood of [0] in the quotient group. The proof follows closely the inductive construction of [DT2].

The quotient group $\widetilde{\mathcal{RC}}_p(X)\otimes\mathbb{Z}_2$ has a filtering by closed subsets $Q_d = \rho(\delta(F_d))$ where

$$F_d = \coprod_{k\leq d} \mathbf{RC}_{p,k}(X) \times \coprod_{k'\leq d} \mathbf{RC}_{p,k'}(X).$$

Fix a triangulation such that each F_{d-1} is a subcomplex of F_d. To inductively construct a local section on some neighborhood of $[0]$, we let $s_o([0]) = 0$ and assume that we already have a local section

$$s_{d-1} : U_{d-1} \longrightarrow \widetilde{\mathcal{RC}}_p(X),$$

where U_{d-1} is an open neighborhood of $[0]$ in Q_{d-1}. Defined a map

$$\sigma_{d-1} : \widetilde{U}_{d-1} \longrightarrow \widetilde{\mathcal{RC}}_p(X)$$

by

$$\sigma_{d-1}(x) = s_{d-1}(\rho(\delta(x))) - \delta(x)$$

where $\widetilde{U}_{d-1} = F_d \cap (\rho\circ\delta)^{-1}(U_{d-1})$ is open in $F_d \cap (\rho\circ\delta)^{-1}(Q_{d-1})$. Clearly, $\sigma_{d-1}(x) \in \rho^{-1}([0])$. Since $F_d \cap (\rho\circ\delta)^{-1}(Q_{d-1})$ is a subcomplex of F_d, there is an open set $\widetilde{U}_d$ in F_d and a retraction from $\widetilde{U}_d$ to $\widetilde{U}_{d-1}$. Therefore we can extend σ_{d-1} to

$$\tilde{\sigma}_d : \widetilde{U}_d \longrightarrow \rho^{-1}([0]) \subset \widetilde{\mathcal{RC}}_p(X).$$

Note that the map

$$\widetilde{U}_d - (\rho\circ\delta)^{-1}(Q_{d-1}) \longrightarrow Q_d - Q_{d-1}$$

is injective. Then the map $\tilde{s}_d = \tilde{\sigma}_d + id$ descends to a section s_d on a open subset U_d in Q_d. Hence we can proceed inductively to obtain a local section on a neighborhood of $[0]$.

The other cases can be proved in the same way. ∎

Remark 2.1.1 Similar construction shows that there are corresponding quasifibrations and principal fibrations for the relative real cycle spaces.

2.2 Group Completions

We now give a brief summary of some facts about group completions. Roughly speaking, an idealization procedure in algebraic topology usually means a kind of natural construction which provides us with a new space, which has some wanted 'nice' properties, associated to a given one which may not be 'nice'. Group completion of topological monoids is one of these procedures which idealizes the action of a topological monoid on itself by right (or left) translations. More precisely, let $\rho_z : M \longrightarrow M$ denote the map of right multiplication by $z \in M$. Then ρ_z is not a homotopy equivalence in general, unless z represents an invertible element in $\pi_o(M)$. Note that the induced homomorphism ρ_{z*} on $H_*(M)$ is independent of the choice of z within its path component. Therefore we have $\pi_o(M)$ acting on the Pontryagin ring $H_*(M)$.

Similarly, the induced homomorphism ρ_{z*} is not necessary an isomorphism. However, one can try to add 'formal inverses' of the elements in $\pi_o(M)$ to $H_*(M)$ such that these homomorphisms extend to isomorphisms. This procedure bears the name 'localization', which can be considered as a generalization of the usual notion of localization of modules over commutative rings. It is then a natural question whether there is a canonical construction which provides us with a space M^+ whose Pontryagin ring is exactly the localized Pontryagin ring of M. In this sense, the Group Completion Theorem can be viewed as a satisfactory answer to this question. Formally,

Definition 2.2.1 *A map of H-spaces* $M \xrightarrow{\rho} M^+$ *is said to be a* **(homological) group completion** *if the following properties are satisfied :*

1. $\rho_* : \pi_o(M) \longrightarrow \pi_o(M^+)$ *is the universal group of the monoid* $\pi_o(M)$,

2. $\rho_* : H_*(M) \longrightarrow H_*(M^+)$ *can be identified with the canonical homomorphism* $\rho_* : H_*(M) \longrightarrow H_*(M) \otimes_{\mathbb{Z}[\pi_o(M)]} \mathbb{Z}[\pi_o(M^+)]$.

Theorem 2.2.1 (Group Completion Theorem) *The canonical map of H-spaces* $M \longrightarrow \Omega BM$ *is a group completion of* M *if* $\pi_o(M)$ *lies in the center of the Pontryagin ring* $H_*(M)$ *(or more generally, if* $H_*(M)[\pi_o(M)^{-1}]$ *can be constructed by right fractions. See* [MS]*).*

In particular, if M is abelian, this property of $\pi_o(M)$ is satisfied. Furthermore, M^+ has the homotopy type of an infinite loop space ([FM]).

The Group Completion Theorem plays an important role in the study of the outputs of infinite loop space machineries. For applications, proofs and variants of this theorem, see [A], [BP], [J], [May2], [MS].

Actually, there is a standard way to construct group completions. For simplicity, let us assume that $\pi_o(M)$ is countable. Then the localization of the Pontryagin ring $H_*(M)$ with respect to the multiplicative system $\pi_o(M)$ can be realized as a direct limit of translations by elements of $\pi_o(M)$. It follows from the general fact that the localization of a right R-module A at a countable multiplicative system S lying in the center of the ring R can be obtained as the limit of a sequence of right translations

$$\cdots \longrightarrow A \xrightarrow{s_i} A \longrightarrow \cdots,$$

where s_i is an enumeration of S such that each element of S appears infinitely often in the sequence. It is then rather clear that one can construct M^+ as the colimit of a sequence of right translations of M by elements of M. More precisely,

Proposition 2.2.1 (cf. [F2], [FM]) *Let z_i be a sequence of elements in M such that each path component M_α of M appears infinitely often in the sequence M_{z_i} of path components containing z_i. Then any group completion $M \xrightarrow{\rho} M^+$ of M factors up to homotopy through a homology equivalence*

$$Tel\{M, \{z_i\}\} \xrightarrow{\tilde{\rho}} M^+,$$

where $Tel\{M, \{z_i\}\}$ is the **mapping telescope** *associated to the sequence of right translations by z_i :*

$$\cdots \longrightarrow M \xrightarrow{\rho_{z_i}} M \longrightarrow \cdots.$$

In particular, when M is abelian, $Tel\{M, \{z_i\}\} \xrightarrow{\tilde{\rho}} \Omega BM$ is a homotopy equivalence, and hence the identity component $Tel\{M, \{z_i\}\}_o$ is homotopically equivalent to $(\Omega BM)_o$.

2.3 The Telescope Description of Cycle Spaces

Proposition 2.2.1 suggests that we can re-defined cycle spaces by the telescope construction :

Definition 2.3.1 *For any real algebraic subset in $\mathbb{P}^n_{\mathbb{C}}$, we re-define*

$$\begin{aligned} \mathcal{C}_p(X) &\stackrel{def}{=} Tel\{\mathbf{C}_p(X), \{z_i\}\} \\ \mathcal{RC}_p(X) &\stackrel{def}{=} Tel\{\mathbf{RC}_p(X), \{r_i\}\} \\ \mathcal{DC}_p(X) &\stackrel{def}{=} Tel\{\mathbf{DC}_p(X), \{s_i\}\} \end{aligned}$$

where $\{z_i\}$, $\{r_i\}$ and $\{s_i\}$ are suitably chosen sequences in $\mathbf{C}_p(X)$, $\mathbf{RC}_p(X)$ and $\mathbf{DC}_p(X)$ respectively as in Proposition 2.2.1.

By Proposition 2.2.1, these spaces are homotopically equivalent to the cycle spaces defined as before in Section 1. As colimit spaces, they are geometrically simplier than their loop spaces counterparts. The importance of being colimit spaces is that they have the following property :

$$\begin{aligned} \pi_k(\mathcal{C}_p(X)) &= \lim_{\alpha\in\pi_o(\mathbf{C}_{p,\alpha}(X))} \pi_k(\mathbf{C}_{p,\alpha}(X)), \\ \pi_k(\mathcal{RC}_p(X)) &= \lim_{\alpha\in\pi_o(\mathbf{RC}_{p,\alpha}(X))} \pi_k(\mathbf{RC}_{p,\alpha}(X)), \\ \pi_k(\mathcal{DC}_p(X)) &= \lim_{\alpha\in\pi_o(\mathbf{DC}_{p,\alpha}(X))} \pi_k(\mathbf{DC}_{p,\alpha}(X)). \end{aligned}$$

In [LB2], Lawson studied the topological structure of the cycle space $\mathcal{C}_p(X)$ and proved a **'Complex Suspension Theorem'** by the constructions of 'holomorphic taffy' and 'magic fans'. We will apply these constructions to study real cycle spaces in Section 3.

We now discuss some basic examples of cycle spaces through the telescope construction.

2.4 Basic Examples of Cycle Spaces

Example 1. The case when $X = \mathbb{P}^n_{\mathbb{C}}$ and $p = n - 1$.

In this situation, it is well known that an effective divisor (real effective divisor respectively) of degree d is given by a homogeneous polynomial (with coefficients in $\mathbb{R}$ respectively) in $z_o, \ldots, z_n$ of degree d, uniquely determined up to a scalar multiple. Therefore,

$$\begin{aligned} \mathbf{C}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) &= \coprod_{d \geq o} \mathbb{P}_{\mathbb{C}}^{\binom{n+d}{d}-1} \\ \mathbf{RC}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) &= \coprod_{d \geq o} \mathbb{P}_{\mathbb{R}}^{\binom{n+d}{d}-1} \end{aligned}$$

and hence,

$$\pi_o(\mathbf{C}_{n-1}(\mathbb{P}^n_{\mathbb{C}})) = \mathbb{Z}^+$$

$$\pi_o(\mathbf{RC}_{n-1}(\mathbb{P}^n_{\mathbb{R}})) = \mathbb{Z}^+$$

are both generated by one generator. Therefore, by fixing a (real) hyperplane ℓ^{n-1} in $\mathbb{P}^n_{\mathbb{C}}$ as the generator, their telescopes can be constructed from the sequences of translations

$$\cdots \longrightarrow \mathbf{C}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) \stackrel{+\ell^{n-1}}{\longrightarrow} \mathbf{C}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathbf{RC}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) \stackrel{+\ell^{n-1}}{\longrightarrow} \mathbf{RC}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \cdots.$$

Therefore we have homotopy equivalences

$$\begin{aligned} \mathcal{C}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) &\sim \mathbb{Z} \times \mathbb{P}^\infty_{\mathbb{C}} \sim K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2) \\ \mathcal{RC}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) &\sim \mathbb{Z} \times \mathbb{P}^\infty_{\mathbb{R}} \sim K(\mathbb{Z}, 0) \times K(\mathbb{Z}_2, 1) \end{aligned}$$

It is also easy to see that the identity component of $\mathcal{DC}_{n-1}(\mathbb{P}^n_{\mathbb{C}})$ is the projectivization of the set of homogeneous polynomials which can be expressed as a product $p\bar{p}$ of homogeneous polynomials. A consequence of our suspension theorem is that this rather complicated polynomial space is homotopically quite simple. In fact, its path components are contractible. This follows from the fact that the suspension theorem implies that there is a homotopy equivalence

$$\mathcal{DC}_{n-1}(\mathbb{P}^n_{\mathbb{C}}) \sim \mathcal{DC}_o(\mathbb{P}^1_{\mathbb{C}}).$$

That $\mathcal{DC}_o(\mathbb{P}^1_{\mathbb{C}})_o$ is contractible follows from the discussion of the following example.

Example 2. The case when X is connected and $p = o$.

In this case,

$$\mathbf{C}_o(X) = \coprod_{d \geq o} SP^d(X)$$

$$\mathbf{RC}_o(X) = \coprod_{d \geq o} SP^d(X)^{\mathbb{Z}_2},$$

where $SP^d(X)$ is the d-fold symmetric product of X and $SP^d(X)^{\mathbb{Z}_2}$ is the fixed point set of the conjugation map. Since the set $\mathbf{DC}_{o,2d}(X)$ consists of the formal finite sums of points $\sum n_\alpha(v_\alpha + \bar{v}_\alpha)$, we also have

$$\mathbf{DC}_o(X) = \coprod_{d \geq o} SP^d(X/\mathbb{Z}_2).$$

Therefore

$$\mathcal{C}_o(X) \sim \mathbb{Z} \times SP(X)$$

$$\mathcal{RC}_o(X) \sim \mathbb{Z} \times SP(X)^{\mathbb{Z}_2}$$

$$\mathcal{DC}_o(X) \sim \mathbb{Z} \times SP(X/\mathbb{Z}_2),$$

where $SP(\cdot)$ is the infinite symmetric product functor and $SP(X)^{\mathbb{Z}_2}$ is the fixed point set of the conjugation action on $SP(X)$.

By taking coefficients in $\mathbb{Z}_2$, it is clear that for any connected projective variety X with connected real locus $X_{\mathbb{R}}$,

$$\begin{aligned}\mathcal{RC}_o(X)\otimes\mathbb{Z}_2 &\sim \mathbb{Z}_2 \times SP(X_{\mathbb{R}} \vee (X/X_{\mathbb{R}})/\mathbb{Z}_2), 2)\\ \mathcal{DC}_o(X)\otimes\mathbb{Z}_2 &\sim \mathbb{Z}_2 \times SP(X/\mathbb{Z}_2\ , 2)\end{aligned}$$

where $SP(\cdot, 2)$ is the infinite symmetric product space with coefficients in $\mathbb{Z}_2$.

Recall that the Dold-Thom Theorem states that :

Theorem 2.4.1 ([DT1,2]) *Let A be a connected finite complex. Then there are homotopy equivalences*

$$SP(A) \sim \prod_k K(\widetilde{H}_k(A,\mathbb{Z}), k)$$

$$SP(A,\mathbb{Z}_m) \sim \prod_k K(\widetilde{H}_k(A,\mathbb{Z}_m), k).$$

In particular, there are natural isomorphisms

$$\pi_*(SP(A)) \cong \widetilde{H}_*(A,\mathbb{Z}).$$

$$\pi_*(SP(A),\mathbb{Z}_m) \cong \widetilde{H}_*(A,\mathbb{Z}_m).$$

Hence we have isomorphisms

$$\begin{aligned}\pi_*(\mathcal{RC}_o(X)\otimes\mathbb{Z}_2) &\cong H_*(X_{\mathbb{R}} \vee ((X/\mathbb{Z}_2)/X_{\mathbb{R}}), \mathbb{Z}_2)\\ \pi_*(\mathcal{DC}_o(X)) &\cong H_*(X/\mathbb{Z}_2, \mathbb{Z})\\ \pi_*(\mathcal{DC}_o(X)\otimes\mathbb{Z}_2) &\cong H_*(X/\mathbb{Z}_2, \mathbb{Z}_2)\\ \pi_*(\mathcal{E}_o(X)) &\cong H_*(X_{\mathbb{R}}, \mathbb{Z}_2)\end{aligned}$$

and therefore Theorem 1.3.2 follows directly from Theorem 1.2.1 and 1.3.1.

3 Basic Properties of Cycle Spaces

In this section, we discuss some basic properties of cycle spaces. We first recall some basic facts from the theory of currents and analytic varieties. Then we recall the constructions of 'holomorphic taffy' and 'magic fans' from [LB2] and apply these constructions to study basic properties of real cycle spaces. For more details on geometric measure theory and complex varieties, see [F], [GH], [GR], [H], [HS], [LB3].

3.1 Algebraic Cycles and Integral Cycles

Let V be an irreducible subvariety of dimension p in $\mathbb{P}^n_{\mathbb{C}}$. Integration of 2p-forms on V defines a 2p-current without boundary on $\mathbb{P}^n_{\mathbb{C}}$, denoted by $[V]$. Moreover, the **mass** of $[V]$ defined with respect to the Fubini-Study metric on $\mathbb{P}^n_{\mathbb{C}}$ is given by

$$\mathsf{M}([V]) = \int_V \frac{1}{p!}\omega^p = degV \cdot \mathsf{M}([\mathbb{P}^n_{\mathbb{C}}])$$

where ω is the Kähler form on $\mathbb{P}^n_{\mathbb{C}}$.

This gives rise to an embedding of $\mathbf{C}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ into the set of **integral 2p-cycles** $\mathcal{Z}_{2p}(\mathbb{P}^n_{\mathbb{C}})$ on $\mathbb{P}^n_{\mathbb{C}}$ defined by $\sum n_\alpha V_\alpha \longrightarrow \sum n_\alpha [V_\alpha]$.

There are two topologies of interest on $\mathcal{Z}_{2p}(\mathbb{P}^n_{\mathbb{C}})$ arising from geometric measure theory, namely the **weak topology** and the **flat norm topology.** Recall that a sequence $\{c_i\}_{i=1}^{\infty}$ converges weakly to c in $\mathcal{Z}_{2p}(\mathbb{P}^n_{\mathbb{C}})$ if

$$\lim_{i\to\infty} c_i(\varphi) = c(\varphi)$$

for all 2p-form φ on $\mathbb{P}_{\mathbb{C}}^n$. The **Whitney flat norm** is defined by

$$\| c - c' \|_\flat \stackrel{def}{=} inf\{\mathbb{M}(c - c' - \partial U) + \mathbb{M}(U)\}$$

where the infimum is taken over all integral $(2p+1)$-currents U. The important work of Federer and Fleming ([FF]) shows that these two topologies agree on the subspace of integral cycles with bounded mass,

$$\mathcal{Z}_{k,\mu}(\mathbb{P}_{\mathbb{C}}^n) \stackrel{def}{=} \{c \in \mathcal{Z}_k(\mathbb{P}_{\mathbb{C}}^n) : \mathbb{M}(c) \leq \mu\},$$

which is compact in the weak topology (and hence also in flat norm topology).

$\mathbf{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$ is compact with the weak (flat norm) topology induced form this embedding. Moreover, the topology of $\mathbf{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$ as union of Chow varieties is also equivalent to the weak topology, or flat norm topology, on $\mathbf{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$. See the discussion in [LB2] for details.

Let $\mathcal{Z}_{2p}(X)$ denote the closed subspace of integral 2p-cycles supported on an algebraic subset $X \subset \mathbb{P}_{\mathbb{C}}^n$. Then we have an embedding

$$\mathbf{C}_p(X) \hookrightarrow \mathcal{Z}_{2p}(X)$$

where the induced weak topology(=flat norm topology) on $\mathbf{C}_p(X)$ is equivalent to the analytic topology on $\mathbf{C}_p(X)$ as disjoint union of Chow varieties.

The conjugation map $\mathbb{P}_{\mathbb{C}}^n \stackrel{\tau}{\longrightarrow} \mathbb{P}_{\mathbb{C}}^n$ induces a continuous involution on $\mathcal{Z}_{2p}(\mathbb{P}_{\mathbb{C}}^n)$ endowed with the weak topology :

$$\tau_\# : \mathcal{Z}_{2p}(\mathbb{P}_{\mathbb{C}}^n) \longrightarrow \mathcal{Z}_{2p}(\mathbb{P}_{\mathbb{C}}^n),$$

since $c_i \to c$ weakly in $\mathcal{Z}_{2p}(\mathbb{P}^n_{\mathbb{C}})$ implies that for any 2p-form φ,

$$\begin{aligned}\lim_{n\to\infty} \tau_{\#}(c_i)(\varphi) &= \lim_{n\to\infty} c_i(\tau^{\#}\varphi) \\ &= c(\tau^{\#}\varphi) \\ &= \tau_{\#}(c)(\varphi).\end{aligned}$$

Clearly, $\tau_{\#}$ preserves $\mathcal{Z}_{2p}(X)$ if X is τ-invariant. Moreover, for any effective algebraic p-cycle $\sum n_\alpha V_\alpha$, we have

$$\sum n_\alpha[\tau(V_\alpha)] = (-1)^p \tau_{\#}(\sum n_\alpha [V_\alpha])$$

as complex 2p-currents. Hence an effective algebraic p-cycle c on a real algebraic subset $X \subset \mathbb{P}^n_{\mathbb{C}}$ is real if and only if $(-1)^p\tau_{\#}(c) = c$ as complex current.

Lemma 3.1.1 $\mathbf{RC}_{p,d}(X)$ *and* $\mathbf{DC}_{p,d}(X)$ *are compact subsets of* $\mathcal{Z}_{2p,d}(X)$ *in the weak (flat norm) topology.*

3.2 Linear Projections and Cycle Spaces

Let $\pi : \mathbb{P}^n_{\mathbb{C}} - \ell_o^{n-p-1} \longrightarrow \ell_o^p$ be the linear projection associated to a pair of disjoint linear subspaces ℓ_o^{n-p-1}, ℓ_o^p of dimension n-p-1 and p respectively. Let μ_t be the holomorphic automorphism defined by scalar multiplication by $t \in \mathbb{C}^*$ along the fibers of the holomorphic vector bundle $\pi : \mathbb{P}^n_{\mathbb{C}} - \ell_o^{n-p-1} \longrightarrow \ell_o^p$. We then have an induced map

$$\mu_t : \mathbf{C}_{p,d}(\mathbb{P}^n_{\mathbb{C}} - \ell_o^{n-p-1}) \longrightarrow \mathbf{C}_{p,d}(\mathbb{P}^n_{\mathbb{C}} - \ell_o^{n-p-1})$$

with the property that $\mu_0(c) \stackrel{def}{=} \lim_{t\to o} \mu_t(c) = d[\ell_o^p]$

By general position argument, Lawson proved the following lemma in [LB 2].

Lemma 3.2.1 ([LB 2]) *For all positive integers n and p, the space $\mathbf{C}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ is simply connected. In particular, $\mathcal{C}_p(\mathbb{P}^n_{\mathbb{C}})$ is simply connected and the translation $\rho_c(c') = c + c'$ is a homotopy equivalence for any c in $\mathcal{C}_p(\mathbb{P}^n_{\mathbb{C}})$.*

As an immediate consequence, $\mathbf{DC}_{p,2d}(\mathbb{P}^n_{\mathbb{C}})$ is connected for all positive integer d.

Note that if both ℓ_o^{n-p-1}, ℓ_o^p are real linear subspaces with the induced real structures, then π commutes with the conjugation map on $\mathbb{P}^n_{\mathbb{C}}$ - ℓ_o^{n-p-1}. Hence we have :

Lemma 3.2.2 *Let μ_t be the map associated to a pair of disjoint real linear subspaces ℓ_o^{n-p-1}, ℓ_o^p in $\mathbb{P}^n_{\mathbb{C}}$ as above. Then, for any $t \in [0,1]$, μ_t preserves $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}} - \ell_o^{n-p-1})$.*

Lemma 3.2.3 *Let $\Phi : PGL(n+1,\mathbb{R}) \times \ell_o^q \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ be the map defined by*

$$\Phi([A],[z]) = [Az]$$

where $A \in GL(n+1,\mathbb{R}), [z] \in \ell_o^q \subset \mathbb{P}^n_{\mathbb{C}}$, where $\ell_o^q \subset \mathbb{P}^n_{\mathbb{C}}$ is a q-dimensional real linear subspace. Then

(1) $\Phi([A],[z]) \in \mathbb{P}^n_{\mathbb{R}}$ if and only if $[z] \in \ell_o^q \cap \mathbb{P}^n_{\mathbb{R}}$,

(2) $\Phi : PGL(n+1,\mathbb{R}) \times \ell_o^q \cap \mathbb{P}^n_{\mathbb{R}} \longrightarrow \mathbb{P}^n_{\mathbb{R}}$ is a submersion for all $o \leq q \leq n$,

(3) $\Phi : PGL(n+1,\mathbb{R}) \times (\ell_o^q - \ell_o^q \cap \mathbb{P}^n_{\mathbb{R}}) \longrightarrow \mathbb{P}^n_{\mathbb{C}} - \mathbb{P}^n_{\mathbb{R}}$ is a submersion for all $1 \leq q < n$.

Proof (1) is obvious and (2) follows from the fact that $PGL(n+1,\mathbb{R})$ acts transitively on $\mathbb{P}^n_{\mathbb{R}}$.

To show (3), without loss of generality, we can assume that

$$\ell^q_o = \{[z_o,\cdots,z_q,o,\cdots,o] \in \mathbb{P}^n_{\mathbb{C}}\} = \mathbb{P}^q_{\mathbb{C}}$$

and let $[z] \in \ell^q_o - \ell^q_o \cap \mathbb{P}^n_{\mathbb{R}}, [1,\xi_1,\cdots,\xi_n] \in \mathbb{P}^n_{\mathbb{C}} - \mathbb{P}^n_{\mathbb{R}}$ such that

$$\Phi([A],[z]) = [1,\xi_1,\cdots,\xi_n].$$

When $q > o$, since $A \in GL(n+1,\mathbb{R})$ acts transitively on the set of 2-planes in $\mathbb{R}^{n+1}$, hence we can choose $z' = a+ib$, with $a,b \in \mathbb{R}^{n+1} - \{o\}$ such that $Az' = (1,\xi_1,\cdots,\xi_n)$ and $[z'] = [z]$. Let

$$\Psi : \mathbb{P}^n_{\mathbb{C}} - \{[o,z_1,\cdots z_n]\} \longrightarrow \mathbb{C}^n$$

be the standard coordinate chart. For any tangent vector $x+iy \in \mathbb{C}^n$ at $(\xi_1,\cdots,\xi_n)$, where $x,y \in \mathbb{R}$, choose $B \in GL(n+1,\mathbb{R})$ such that $Bb = (o,y)$ if $y \neq o$ and $B = 0$ otherwise. Define a curve in $PGL(n+1,\mathbb{R}) \times (\mathbb{P}^q_{\mathbb{C}} - \mathbb{P}^q_{\mathbb{R}})$ by

$$\gamma(t) = ([QA+tB],[a+ib])$$

where $(c_o,\cdots,c_n) = Ba$ and

$$Q = \begin{pmatrix} 1-tc_o & 0 & \cdots & 0 \\ t(x_1-c_1) & & & \\ \vdots & & \mathrm{I} & \\ t(x_n-c_n) & & & \end{pmatrix}.$$

Then $\gamma(0) = ([A],[z])$, $\Phi(\gamma(0)) = [1,\xi_1,\cdots,\xi_n]$ and

$$\Psi \circ \Phi \circ \gamma(t) = (\xi_1,\cdots,\xi_n) + t(x+iy).$$

It is then straight-forward to check that $(\Psi \circ \Phi \circ \gamma)'(0) = x + iy$ and hence Φ is a submersion. ■

Proposition 3.2.1 *Let ℓ_o^p be a real linear subspace of dimension p in $\mathbb{P}_{\mathbb{C}}^n$ and let c be a real algebraic p-cycle in $\mathbf{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$. Then for almost all (n-p-1)-dimensional real linear subspace ℓ_o^{n-p-1} in $\mathbb{P}_{\mathbb{C}}^n$,*

$$\ell_o^{n-p-1} \cap (\ell_o^p \cup c) = \emptyset$$

Proof Note that $c \cap \mathbb{P}_{\mathbb{R}}^n$ has real dimension $\leq p$ and $c \cap \mathbb{P}_{\mathbb{C}}^n - \mathbb{P}_{\mathbb{R}}^n$ has real dimension $2p$. By the **Sard's theorem for family** (see [HL]) and the lemma above, we have

$$\ell_o^{n-p-1} \cap ((\ell_o^p \cup c) \cap \mathbb{P}_{\mathbb{R}}^n) = \emptyset$$

$$\ell_o^{n-p-1} \cap ((\ell_o^p - \ell_o^p \cap \mathbb{P}_{\mathbb{R}}^n) \cup (c \cap (\mathbb{P}_{\mathbb{C}}^n - \mathbb{P}_{\mathbb{R}}^n))) = \emptyset$$

for almost all real linear subspace ℓ_o^{n-p-1} of dimension $n - p - 1 \geq 1$.

The case for $p = n - 1$ is proved similarly by considering only the real locus of the support of c. ■

Corollary 3.2.1 $\mathbf{RC}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$ *is connected.*

Proof For any real cycle c in $\mathbf{RC}_{p,d}(\mathbb{P}_{\mathbb{C}}^n)$, choose a complementary real linear subspace ℓ_o^{n-p-1} such that

$$\ell_o^{n-p-1} \cap (\ell_o^p \cup c) = \emptyset.$$

Then the map μ_t associated to the linear projection for ℓ_o^{n-p-1} to ℓ_o^p definds a path joing c to $d[\ell_o^p]$. ■

Remark 3.2.1 Note that the $\mathbb{Z}_2$ **Euler characteristic** of the cycle space $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ can be determined rather easily, modulo the work in [LY]. In fact, by taking an equivariant triangulation of $\mathbf{C}_{p,d}(X)$ with $\mathbf{RC}_{p,d}(X)$ as a fixed subcomplex, it is not hard to see that

$$\chi(\mathbf{RC}_{p,d}(X)) \equiv \chi(\mathbf{C}_{p,d}(X)) \ (mod\ 2).$$

The Euler characteristic of $\mathbf{C}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ had been computed in [LY]. It is not clear to us whether one can in fact calculate the Euler characteristic of the real locus $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ without reducing coefficients to $\mathbb{Z}_2$.

3.3 Holomorphic Taffy for Real Cycle Spaces

The algebraic join operation can be described in terms of linear projections from linear subspaces. More precisely, let $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^m_{\mathbb{C}} \subset \mathbb{P}^{n+m+1}_{\mathbb{C}}$ be a pair of real linear subspaces. Then the linear projections

$$\pi_m : \mathbb{P}^{n+m+1}_{\mathbb{C}} - \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^m_{\mathbb{C}}$$

$$\pi_n : \mathbb{P}^{n+m+1}_{\mathbb{C}} - \mathbb{P}^m_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

have the structure of holomorphic vector bundles of rank $n+1$ and $m+1$ respectively, with the real structure induced from the conjugation map.

Given closed subsets $A \subset \mathbb{P}^n_{\mathbb{C}}$ and $B \subset \mathbb{P}^m_{\mathbb{C}}$, the algebraic join of A and B can be defined via π_m and π_n as follow :

$$A\#_{\mathbb{C}}B \overset{def}{=} \overline{\pi_n^{-1}(A) \cap \pi_m^{-1}(B)}.$$

In the case when $B = \mathbb{P}^m_{\mathbb{C}}$, we have the $(m+1)$-fold complex suspension of A :

$$\Sigma^{m+1} A = A\#_{\mathbb{C}}\mathbb{P}^m_{\mathbb{C}}$$

Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a real algebraic subset. Then we have the following suspension maps

$$\begin{array}{ccc}
\mathbf{C}_p(X) & \xrightarrow{\not\Sigma} & \mathbf{C}_{p+1}(\not\Sigma X) \\
\cup & & \cup \\
\mathbf{RC}_p(X) & \xrightarrow{\not\Sigma} & \mathbf{RC}_{p+1}(\not\Sigma X) \\
\cup & & \cup \\
\mathbf{DC}_p(X) & \xrightarrow{\not\Sigma} & \mathbf{DC}_{p+1}(\not\Sigma X)
\end{array}$$

Recall that we say $c \not\models X$ if the effective cycle has no irreducible component contained in X. Following the construction of [LB2], let

$$\begin{array}{rcl}
\Gamma_{p+1}(X) & \stackrel{def}{=} & \{c \in \mathbf{C}_{p+1}(\not\Sigma X) : c \not\models X\} \\
\Upsilon_{p+1}(X) & \stackrel{def}{=} & \{c \in \mathbf{RC}_{p+1}(\not\Sigma X) : c \not\models X\} \\
\Delta_{p+1}(X) & \stackrel{def}{=} & \{c \in \mathbf{DC}_{p+1}(\not\Sigma X) : c \not\models X\}.
\end{array}$$

Then the suspension maps factor into the following commutative diagram of inclusions :

$$\begin{array}{ccccc}
\mathbf{C}_p(X) & \xrightarrow{\not\Sigma} & \Gamma_{p+1}(X) & \subset & \mathbf{C}_{p+1}(\not\Sigma X) \\
\cup & & \cup & & \cup \\
\mathbf{RC}_p(X) & \xrightarrow{\not\Sigma} & \Upsilon_{p+1}(X) & \subset & \mathbf{RC}_{p+1}(\not\Sigma X) \\
\cup & & \cup & & \cup \\
\mathbf{DC}_p(X) & \xrightarrow{\not\Sigma} & \Delta_{p+1}(X) & \subset & \mathbf{DC}_{p+1}(\not\Sigma X)
\end{array}$$

Let $\mathbb{P}^n_{\mathbb{C}}$,$\mathbb{P}^o_{\mathbb{C}} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a pair of disjoint real linear subspaces. Up to a change of homogeneous coordinates by real linear forms, we may assume that $\mathbb{P}^n_{\mathbb{C}} = \{[o, z_1, \cdots, z_{n+1}] \in \mathbb{P}^{n+1}_{\mathbb{C}}\}$ and $\mathbb{P}^o_{\mathbb{C}} = \{[o, o, \cdots, o]\}$. For any $t \in \mathbb{C}^*$, define $\varphi_t : \mathbb{P}^{n+1}_{\mathbb{C}} \longrightarrow \mathbb{P}^{n+1}_{\mathbb{C}}$ by setting

$$\varphi_t([z_o, z_1, \cdots, z_{n+1}]) = [tz_o, z_1, \cdots, z_{n+1}].$$

Then the holomorphic automorphism $\varphi_t : \mathbb{P}^{n+1}_{\mathbb{C}} \longrightarrow \mathbb{P}^{n+1}_{\mathbb{C}}$ induces a holomorphic automorphism

$$\varphi_{t\#} : \mathbf{C}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}}) \longrightarrow \mathbf{C}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}}).$$

which preserves the subspaces $\mathbf{C}_{p+1}(\not{\Sigma} X)$ and $\Gamma_{p+1}(X)$. Furthermore, $\varphi_{t\#}$ is the identity map on $\not{\Sigma}(\mathbf{C}_p(X))$ for all $t \in \mathbb{C}^*$.

Theorem 3.3.1 ([LB2]) *For each $c \in \Gamma_{p+1}(X)$, there exists a limit*

$$\varphi_\infty(c) = \lim_{t \to \infty} \varphi_{t\#}(c) \in \not{\Sigma}(\mathbf{C}_p(X))$$

which is continuous in c and defines a retraction

$$\varphi_\infty : \Gamma_{p+1}(X) \longrightarrow \not{\Sigma}(\mathbf{C}_p(X)).$$

Furthermore, the extended map

$$\varphi : \Gamma_{p+1}(X) \times [1, \infty] \longrightarrow \Gamma_{p+1}(X)$$

is continuous and therefore φ_∞ is a deformation retraction.

As a consequence, $\mathbf{C}_p(X) \xrightarrow{\not{\Sigma}} \Gamma_{p+1}(X)$ is a homotopy equivalence.

Lemma 3.3.1 *$\varphi_{t\#}$ preserves $\mathbf{RC}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ and $\mathbf{DC}_{p+1}(\mathbb{P}^{n+1}_{\mathbb{C}})$ for all $t \in [1, \infty]$.*

Now by restricting the deformation retration in Theorem 2.2.1 to the spaces of real algebraic cycles, we have :

Corollary 3.3.1 *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a real algebraic subset. Then there are homotopy equivalences*

$$\mathbf{RC}_p(X) \xrightarrow{\not\Sigma} \Upsilon_{p+1}(X)$$

$$\mathbf{DC}_p(X) \xrightarrow{\not\Sigma} \Delta_{p+1}(X)$$

Following the telescope description of cycle spaces discussed in Section 2.3, let $\{r_i\}, \{s_i\}$ be the chosen sequences in $\mathbf{RC}_p(X)$, $\mathbf{DC}_p(X)$ respectively for construction of their telescopes. Choose corresponding sequences $\{\not\Sigma z_i\}, \{\not\Sigma t_i\}$ in $\Upsilon_{p+1}(X)$, $\Delta_{p+1}(X)$ respectively. Then by the Corollary above, we have

Proposition 3.3.1 *The suspension maps induce homotopy equivalences*

$$\begin{array}{ccc} \mathcal{RC}_p(X) & \xrightarrow{\not\Sigma} & \mathit{\Upsilon}_{p+1}(X) \\ \mathcal{DC}_p(X) & \xrightarrow{\not\Sigma} & \mathit{\Delta}_{p+1}(X) \end{array}$$

where $\mathit{\Upsilon}_{p+1}(X)$, $\mathit{\Delta}_{p+1}(X)$ are the telescopes associated to $\Upsilon_{p+1}(X)$, $\Delta_{p+1}(X)$ respectively.

Proposition 3.3.2 *The suspension maps induce homotopy equivalences*

$$\begin{array}{ccc} \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\widetilde{\not\Sigma}} & \tilde{\Upsilon}_{p+1}(X) \\ \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{\widetilde{\not\Sigma}} & \widetilde{\Delta}_{p+1}(X) \end{array}$$

where $\widetilde{\Upsilon}_{p+1}(X)$, $\widetilde{\Delta}_{p+1}(X)$ *are the universal groups associated to the abelian monoids* $\Upsilon_{p+1}(X)$, $\Delta_{p+1}(X)$ *respectively.*

Let $\mathbb{P}^n_{\mathbb{C}}$ be imposed with the Fubini-Study metric of constant curvature 1. Let $\mathbb{P}^m_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$ be a real linear subspace. Then for all $0 \le t < \pi$, the closed subset

$$\mathbb{P}^m_{\mathbb{C},t} \stackrel{def}{=} \{x \in \mathbb{P}^n_{\mathbb{C}} \ : dist(x, \mathbb{P}^n_{\mathbb{C}}) \le t\}$$

is invariant under the conjugation map on $\mathbb{P}^n_{\mathbb{C}}$. By using the linear flow

$$\varphi_{s\#} : \mathbf{C}_p(\mathbb{P}^m_{\mathbb{C},t}) \longrightarrow \mathbf{C}_p(\mathbb{P}^m_{\mathbb{C},t})$$

for $0 \le s \le 1$, it is easy to see that the inclusion map

$$\mathbf{C}_p(\mathbb{P}^m_{\mathbb{C}}) \hookrightarrow \mathbf{C}_p(\mathbb{P}^m_{\mathbb{C},t})$$

is a deformation retraction ([LM]) for $0 \le t < \pi$. Noticing that the linear flow preserves the real cycle spaces, we have.

Proposition 3.3.3 *For all* $0 \le t < \pi$, *the inclusion maps*

$$\begin{aligned} \mathbf{RC}_p(\mathbb{P}^m_{\mathbb{C}}) &\hookrightarrow \mathbf{RC}_p(\mathbb{P}^m_{\mathbb{C},t}) \\ \mathbf{DC}_p(\mathbb{P}^m_{\mathbb{C}}) &\hookrightarrow \mathbf{DC}_p(\mathbb{P}^m_{\mathbb{C},t}) \end{aligned}$$

are both deformation retractions.

In particular, when passing to their group completions, we have

Corollary 3.3.2 *The inclusion maps in Proposition 3.3.3 induce homo-*

topy equivalences

$$\begin{array}{ccc}
\mathcal{RC}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \mathcal{RC}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\mathcal{DC}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \mathcal{DC}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\mathcal{E}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \mathcal{E}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\widetilde{\mathcal{RC}}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \widetilde{\mathcal{RC}}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\widetilde{\mathcal{DC}}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \widetilde{\mathcal{DC}}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\tilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \tilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t}).
\end{array}$$

3.4 Magic Fans for Real Cycle Spaces

We now recall the construction of **'magic fans'** in [LB2]. Let $\mathbb{P}^n_{\mathbb{C}}$ be embedded in $\mathbb{P}^{n+1}_{\mathbb{C}}$ as a real linear subspace and let $x_\infty \in \mathbb{P}^{n+1}_{\mathbb{C}} - \mathbb{P}^n_{\mathbb{C}}$ be a real point. Let $\mathcal{V}_\infty = \mathbb{P}^{n+1}_{\mathbb{C}} - \{x_\infty\}$ and let

$$\pi_\infty : \mathcal{V}_\infty \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

be the linear projection. Then π_∞ commutes with the conjugation map on $\mathcal{V}_\infty$.

Let $Div'_{n,d} \stackrel{def}{=} \{D \in \mathbf{C}_{n,d}(\mathbb{P}^{n+1}_{\mathbb{C}}) : D \subset \mathcal{V}_\infty\}$. Then for any divisor $D \in Div'_{n,d}$ and any algebraic cycle $c = \sum n_\alpha V_\alpha$ in $\mathbf{C}_{p,d_o}(\mathbb{P}^n_{\mathbb{C}})$,

$$\varphi_D(c) \stackrel{def}{=} \sum n_\alpha D \cap \pi_\infty^{-1}(V_\alpha)$$

is a cycle of degree dd_o.(See [LB2]) Moreover $\tau_{\#}\varphi_D(c) = \tau_{\#}D \cap \pi_\infty^{-1}(\tau_{\#}c)$.

Let $tD \stackrel{def}{=} \mu_t(D)$ where μ_t is the scalar multiplication by $t \in \mathbb{C}^*$ along the fibers of the holomorphic vector bundle $\pi_\infty : \mathcal{V}_\infty \longrightarrow \mathbb{P}^n_{\mathbb{C}}$. Let $Div^{\mathbb{R}}_d \in Div'_{n,d}$ be the subspace of real divisors in $Div'_{n,d}$. Then we have :

Lemma 3.4.1 *For p, d_o and d, there are continuous maps*

$$d \cdot \mathbf{RC}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \times Div_d^{\mathbb{R}} \xrightarrow{\varphi} \mathbf{RC}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$
$$d \cdot \mathbf{DC}_{p,2d_o}(\mathbb{P}_{\mathbb{C}}^n) \times Div_d^{\mathbb{R}} \xrightarrow{\varphi} \mathbf{DC}_{p,2dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$

defined by

$$(d \cdot c, D) \longrightarrow \varphi_D(c).$$

Moreover, for any given $D \in Div_d^{\mathbb{R}}$, the families

$$d \cdot \mathbf{RC}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \xrightarrow{\varphi_{tD}} \mathbf{RC}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$
$$d \cdot \mathbf{DC}_{p,2d_o}(\mathbb{P}_{\mathbb{C}}^n) \xrightarrow{\varphi_{tD}} \mathbf{DC}_{p,2dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$

for $o \leq t \leq 1$ are homotopies of φ_D to the inclusion maps $d \cdot \mathbf{RC}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \subset \mathbf{RC}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$, and $d \cdot \mathbf{DC}_{p,2d_o}(\mathbb{P}_{\mathbb{C}}^n) \subset \mathbf{DC}_{p,2dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$.

Proof In [LB2], it was shown that the map

$$d \cdot \mathbf{C}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \times Div'_{n,d} \xrightarrow{\varphi} \mathbf{C}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$

defined by $(d \cdot c, D) \longrightarrow \varphi_D(c)$ is continuous and the family

$$d \cdot \mathbf{C}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \xrightarrow{\varphi_{tD}} \mathbf{C}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$

for $o \leq t \leq 1$ is a homotopy of φ_D to the inclusion map $d \cdot \mathbf{C}_{p,d_o}(\mathbb{P}_{\mathbb{C}}^n) \subset \mathbf{C}_{p,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$. Since π_∞ commutes with the conjugation map, it is clear that these maps φ, φ_{tD} restrict to maps on spaces of real cycles with the above properties in the lemma. ∎

Embed $\mathbb{P}_{\mathbb{C}}^{n+1} \subset \mathbb{P}_{\mathbb{C}}^{n+2}$ as a real linear subspace. Let x_1 be a real point on the real line $\overline{x_\infty, \mathbb{P}_{\mathbb{C}}^o}$ which is distinct from x_o and x_∞. Let

$$\pi_1 : \mathbb{P}_{\mathbb{C}}^{n+2} - \{x_1\} \longrightarrow \mathbb{P}_{\mathbb{C}}^{n+1}$$

be the corresponding linear projection.

For any $D \in Div'_{n+1,d}$ on $\mathbb{P}_{\mathbb{C}}^{n+2} - \{x_1\}$ such that $x_1 \notin \cup\{tD : o \leq t \leq 1\}$, there is a continuous maps

$$\Psi_{tD} : d \cdot \mathbf{C}_{p+1,d_o}(\mathbb{P}_{\mathbb{C}}^{n+1}) \longrightarrow \mathbf{C}_{p+1,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$

for all $o \leq t \leq 1$ defined by

$$\Psi_{tD}(dc) = \pi_1(\varphi_{tD}(c)).$$

Moreover, Ψ_{tD} is the identity map on $d \cdot \not\Sigma(\mathbf{C}_{p+1,d_o}(\mathbb{P}_{\mathbb{C}}^{n}))$.(See [LB2])

Lemma 3.4.2 *Let D be a real divisor in $Div_d^{\mathbb{R}}$. Then*

$$\Psi_{tD}(d \cdot \mathbf{RC}_{p+1,d_o}(\mathbb{P}_{\mathbb{C}}^{n+1})) \subset \mathbf{RC}_{p+1,dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1})$$
$$\Psi_{tD}(d \cdot \mathbf{DC}_{p+1,2d_o}(\mathbb{P}_{\mathbb{C}}^{n+1})) \subset \mathbf{DC}_{p+1,2dd_o}(\mathbb{P}_{\mathbb{C}}^{n+1}).$$

For any algebraic subset $X \in \mathbb{P}_{\mathbb{C}}^{n}$, Ψ_{tD} leaves the subspace $d \cdot \mathbf{C}_{p+1,d_o}(\not\Sigma X)$ invariant. In particular, $d \cdot \mathbf{RC}_{p+1,d_o}(\not\Sigma X)$ and $d \cdot \mathbf{DC}_{p+1,d_o}(\not\Sigma X)$ are also left invariant by Ψ_{tD} whenever X and D are real.

Let Div''_d denotes the set of all divisors of degree d on $\mathbb{P}_{\mathbb{C}}^{n+2}$ such that $x_1 \notin D$ and $x_1 \notin \cup\{tD : o \leq t \leq 1\}$. Denote the subset $\pi_\infty(D \cap \pi_1^{-1}(\mathbb{P}_{\mathbb{C}}^{n})) \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ by $\alpha(D)$. Then we have

Lemma 3.4.3 ([LB2]) *Fix $D \in Div''_d$ and let $V \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be an irreducible algebraic subvariety of dimension $p + 1$. Then*

$$\Psi_D(d \cdot V) \subset \mathbb{P}_{\mathbb{C}}^{n} \implies V \subset \alpha(D),$$

or equivalently,

$$V \not\subset \alpha(D) \implies \Psi_D(d \cdot V) \in \Gamma_{p+1}(\mathbb{P}_{\mathbb{C}}^{n})$$

Note that Lemma 2.9 shows that with suitable choice of D in $\mathbb{P}_{\mathbb{C}}^{n+2}$, one can lift a given irreducible subvariety $V \subset \mathbb{P}_{\mathbb{C}}^{n}$ to one 'transverse' to $\mathbb{P}_{\mathbb{C}}^{n}$. Furthermore, for any $c \in \mathcal{C}_p(\mathbb{P}_{\mathbb{C}}^{n+1})$, let

$$\mathcal{B}(c) = \{D \in Div_d'' : c \vDash \alpha(D)\},$$

where $c \vDash \alpha(D)$ means that c has some irreducible components contained in $\alpha(D)$. Then we have

Lemma 3.4.4 ([LB2]) $codim_{\mathbb{C}}\mathcal{B}(c) \geq \begin{pmatrix} p+d+1 \\ d \end{pmatrix}$.

In order to apply Lawson's method to the case of real cycle spaces, let $Div_d''^{\mathbb{R}}$ denote the set of real divisors in Div_d'' and let

$$\mathcal{R}(c) = \{D \in Div_d''^{\mathbb{R}} : c \vDash \alpha(D)\}$$

for any $c \in \mathcal{C}_p(\mathbb{P}_{\mathbb{C}}^{n+1})$. Then we have

Lemma 3.4.5 *For any* $c \in \mathcal{C}_p(\mathbb{P}_{\mathbb{C}}^{n+1})$,

$$codim_{\mathbb{R}}\mathcal{R}(c) \geq \begin{pmatrix} p+d+1 \\ d \end{pmatrix}$$

.

Proof Fix a real linear subspace ℓ_o^{p+1} in $\mathbb{P}_{\mathbb{C}}^{n+2}$, there is a real linear subspace ℓ_o^{n-p} in $\mathbb{P}_{\mathbb{C}}^{n+2}$ which is disjoint from $c \cup \tau(c) \cup \ell_o^{p+1}$. The linear projection from ℓ_o^{n-p} to ℓ_o^{p+1} then projects V on to ℓ_o^{p+1}. Without loss of generality, we may assume that ℓ_o^{p+1} is the real linear subspace defined by $z_{p+2} = \cdots = z_{n+2} = o$. Then the space of homogeneous polynomials in $z_o, \cdots, z_{p+1}$ with real coefficients is a real linear space of dimension

$\begin{pmatrix} p+d+1 \\ d \end{pmatrix}$ which defines non-zero real sections in $H^o(V, \mathcal{O}(d))$. Hence the lemma is proved. ■

4 Algebraic Suspension Theorems

The purpose of this section is to prove various suspension theorems described in Section 1.

4.1 The Main Algebraic Suspension Theorem

We now prove the main result :

Theorem 4.1.1 *For any real algebraic subset $X \subset \mathbb{P}^n_{\mathbb{C}}$, the suspension maps*

$$\begin{array}{ccc} \mathcal{RC}_p(X) & \xrightarrow{\not\Sigma} & \mathcal{RC}_{p+1}(\not\Sigma X) \\ \mathcal{DC}_p(X) & \xrightarrow{\not\Sigma} & \mathcal{DC}_{p+1}(\not\Sigma X) \end{array}$$

are homotopy equivalences for every dimension p. So also are the suspension maps

$$\begin{array}{ccc} \widetilde{\mathcal{RC}}_p(X) & \xrightarrow{\widetilde{\not\Sigma}} & \widetilde{\mathcal{RC}}_{p+1}(\not\Sigma X) \\ \widetilde{\mathcal{DC}}_p(X) & \xrightarrow{\widetilde{\not\Sigma}} & \widetilde{\mathcal{DC}}_{p+1}(\not\Sigma X). \end{array}$$

Consequently, Theorem 1.2.1 follows.

Recall that the suspension maps

$$\begin{array}{ccccc} \mathbf{RC}_p(X) & \xrightarrow{\not\Sigma} & \Upsilon_{p+1}(X) & \xrightarrow{i} & \mathbf{RC}_{p+1}(\not\Sigma X) \\ \mathbf{DC}_p(X) & \xrightarrow{\not\Sigma} & \Delta_{p+1}(X) & \xrightarrow{j} & \mathbf{DC}_{p+1}(\not\Sigma X) \end{array}$$

induce homotopy equivalences

$$\begin{array}{ccc} \mathcal{RC}_p(X) & \xrightarrow{\Sigma} & \Upsilon_{p+1}(X) \\ \mathcal{DC}_p(X) & \xrightarrow{\Sigma} & \Delta_{p+1}(X). \end{array}$$

Hence, to prove the first part of Theorem 4.1.1, it suffices to show that the inclusions i, j induce isomorphisms

$$\begin{array}{ccc} \pi_k(\Upsilon_{p+1}(X)) & \xrightarrow{j_*} & \pi_k(\mathcal{RC}_{p+1}(\Sigma X)) \\ \pi_k(\Delta_{p+1}(X)) & \xrightarrow{j_*} & \pi_k(\mathcal{DC}_{p+1}(\Sigma X)) \end{array}$$

for all positive integers k. This follows from the following claims :

Claim 1*For any maps*

$$f : S^k \longrightarrow \mathcal{RC}_{p+1}(\Sigma X),$$

$$g : S^k \longrightarrow \mathcal{DC}_{p+1}(\Sigma X),$$

there exist positive integers d_f, d_g *such that for all* $d > d_f$ *and* $d' > d_g$*,* $d \cdot f$ *is homotopic to a map*

$$\tilde{f} : S^k \longrightarrow \Upsilon_{p+1}(X)$$

and $d' \cdot g$ *is homotopic to a map*

$$\tilde{g} : S^k \longrightarrow \Delta_{p+1}(X).$$

Claim 2*For any maps*

$$f : (B^k, S^{k-1}) \longrightarrow (\mathcal{RC}_{p+1}(\Sigma X), \Upsilon_{p+1}(X)),$$

$$g : (B^k, S^{k-1}) \longrightarrow (\mathcal{DC}_{p+1}(\Sigma X), \Delta_{p+1}(X)),$$

there exist positive integer d_f, d_g such that for all $d > d_f$ and $d' > d_g$, $d \cdot f$ is homotopic to a map

$$\tilde{f} : (B^k, S^k) \longrightarrow (\varUpsilon_{p+1}(X), \varUpsilon_{p+1}(X))$$

and $d' \cdot g$ is homotopic to a map

$$\tilde{g} : (B^k, S^k) \longrightarrow (\varDelta_{p+1}(X), \varDelta_{p+1}(X)).$$

Note that for any map

$$f : S^k \longrightarrow M$$

to an abelian topological monoid, we have $[d \cdot f] = d[f]$ where $d \cdot f$ is the map $d \cdot f(x) = f(x) + f(x) + \cdots + f(x)$ (d times) and where $+$ is the addition in M.

Recall that a map $f : Z \longrightarrow Y$ between two triangulable spaces is called *regular* if for some triangulations on Z and Y, f is PL. Every continuous map between triangulable spaces is homotopic to a regular one. Therefore, without loss of generality, we may assume all the maps in Claims 1 and 2 are regular. Then the claims follow easily from the following two propositions.

Proposition 4.1.1 *Let*

$$f : S^k \longrightarrow \mathbf{RC}_{p+1}(\Sigma X)$$

$$g : S^k \longrightarrow \mathbf{DC}_{p+1}(\Sigma X)$$

be regular maps. Then for any positive integer d satisfying $\binom{p+d+1}{d} > k+1$, *there exists a divisor $D \in Div_d''^{\mathbb{R}}$ such that the homotopies*

$$f_t = \Psi_{tD}(d \cdot f),$$

$$g_t = \Psi_{tD}(d \cdot g)$$

satisfy the following properties :

$$f_o = d \cdot f, \quad f_t(S^k) \subset \Upsilon_{p+1}(X)$$
$$g_o = d \cdot g, \quad g_t(S^k) \subset \Delta_{p+1}(X)$$

for all $o < t \leq 1$.

Proof By Lemma 2.6, we have that $\Psi_{tD}(f(x))$ is a continuous family of real cycles of degree dd_o. Moreover, $\Psi_{tD}(f(x)) \in \Upsilon_{p+1}(\sum X)$ whenever $\alpha(D) \not\models f(x)$. Consider the set of divisor in $Div_d''^{\mathbb{R}}$

$$\mathcal{R}(f) = \bigcup_{\substack{o < t \leq 1 \\ x \in S^k}} t \cdot \mathcal{R}(f(x))$$

where $t \cdot \mathcal{R}(f(x)) = \{tD : D \in \mathcal{R}(c).\}$ Applying Lemma 3.4.5, we have

$$codim_{\mathbb{R}} \mathcal{R}(f) \geq \binom{p+d+1}{d} - k - 1.$$

Therefore there exists $D \in Div_d''^{\mathbb{R}} - \mathcal{R}(f)$ and the homotopy f_t satisfies the required properties. The proof for the homotopy g_t is the same. ■

By similarly argument, we have

Proposition 4.1.2 *Let*

$$f : (B^k, S^{k-1}) \longrightarrow (\mathbf{RC}_{p+1}(\Sigma X),\, \Upsilon_{p+1}(X))$$

$$g : (B^k, S^{k-1}) \longrightarrow (\mathbf{DC}_{p+1}(\Sigma X), \Delta_{p+1}(X))$$

be regular maps. Then for any positive integer d satisfying $\begin{pmatrix} p+d+1 \\ d \end{pmatrix}$ $> k+1$, *there exists a divisor $D \in Div_d''^{\mathbb{R}}$ such that the homotopies*

$$f_t = \Psi_{tD}(d \cdot f),$$

$$g_t = \Psi_{tD}(d \cdot g)$$

satisfy the following properties :

$$\begin{array}{ll} f_o = d \cdot f, & f_t(B^k, S^{k-1}) \subset (\Upsilon_{p+1}(X), \Upsilon_{p+1}(X)) \\ g_o = d \cdot g, & g_t(B^k, S^{k-1}) \subset (\Delta_{p+1}(X), \Delta_{p+1}(X)) \end{array}$$

for all $o < t \leq 1$.

In particular, we have

Corollary 4.1.1

$$\pi_o(\Upsilon_{p+1}(X)) \xrightarrow{i_*} \mathbf{RC}_{p+1}(X)$$

$$\pi_o(\Delta_{p+1}(X)) \xrightarrow{j_*} \mathbf{DC}_{p+1}(X)$$

are bijections.

Note that the homotopies in Propositions 4.1.1 and 4.2.2 preserves the based points chosen to construct $\Upsilon_{p+1}(X)$ and $\Delta_{p+1}(X)$.

Claims 1 and 2 then follow.

The second part of Theorem 4.1.1 can be proved in a similar manner. The only thing we should note is that the lifting map

$$\Psi_{tD}([(a,b)] = [(\Psi_{tD}(a'), \Psi_{tD}(b')]$$

is independent of the choice of representative (a', b') of $[(a,b)]$ and is continuous. This was proved in [LB2].

4.2 The Cases of Mod 2 Real Cycle Spaces

We provide here the proofs of suspension theorems 1.3.1, 1.3.2, 1.4.1.

Proof These theorems follow from Theorem 4.1.1 and the fibration results in Section 2.1 and the five lemma. For example, recall that by Corollary 2.1.1, we have a map of quasifibrations

$$\begin{array}{ccccccccc}
\longrightarrow & \mathcal{RC}_p(X) & \longrightarrow & \mathcal{RC}_p(X) & \longrightarrow & \mathcal{RC}_p(X)\otimes\mathbb{Z}_2 & \longrightarrow & \\
 & \not\Sigma\downarrow & & \not\Sigma\downarrow & & \not\Sigma\downarrow & & \\
\longrightarrow & \mathcal{RC}_{p+1}(\not\Sigma X) & \longrightarrow & \mathcal{RC}_{p+1}(\not\Sigma X) & \longrightarrow & \mathcal{RC}_{p+1}(\not\Sigma X)\otimes\mathbb{Z}_2 & \longrightarrow & .
\end{array}$$

By applying Theorem 1.1 and the five lemma, the suspension map

$$\mathcal{RC}_p(X)\otimes\mathbb{Z}_2 \xrightarrow{\not\Sigma} \mathcal{RC}_{p+1}(\not\Sigma X)\otimes\mathbb{Z}_2$$

is also a homotopy equivalence. The proofs for the remaining cases are similar. ■

In the case $X = \mathbb{P}^n_{\mathbb{C}}$, we can defined the cycle spaces $\mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})_o$ and $\mathcal{DC}_p(\mathbb{P}^n_{\mathbb{C}})_o$ as the weak limits of the following sequences of embeddings :

$$\cdots \longrightarrow \mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{+\ell_o} \mathbf{RC}_{p,d+1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbf{DC}_{p,2d}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{+2\ell_o} \mathbf{DC}_{p,2d+2}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \cdots.$$

Then the inclusion $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \widetilde{\mathcal{RC}}_p(\mathbb{P}^n_{\mathbb{C}})$ defined by $c \longrightarrow c - d\ell_o$ induces an inclusion

$$\mathcal{E}_p(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}).$$

Proposition 4.2.1 *The inclusion map* $\mathcal{E}_p(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}})$ *is a homotopy equivalence.*

Proof Since the algebraic suspension map commutes with the inclusion, it follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_p(\mathbb{P}^n_{\mathbb{C}}) & \longrightarrow & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}) \\ \Sigma\!\!\!/ \uparrow & & \Sigma\!\!\!/ \uparrow \\ \mathcal{E}_{p-1}(\mathbb{P}^{n-1}_{\mathbb{C}}) & \longrightarrow & \widetilde{\mathcal{E}}_{p-1}(\mathbb{P}^{n-1}_{\mathbb{C}}) \end{array}$$

that the theorem is true if it is true for $p = 0$. The case for when $p = 0$ was proved in [DT2]. ■

5 The Algebraic Join Operation on Real Cycle Spaces

In analogy with the work of Lawson and Michelsohn([LM]), there are interesting relations between the Stiefel-Whitney classes and the spaces of real algebraic cycles. In this section, we show that there is a 'Whitney characteristic map' from BO_q into $\lim_{n\to\infty} \mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_1 \sim K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, q)$, where $\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_1$ denotes the degree one component of $\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})$, which classifies the total Stiefel-Whitney class of the universal q-plane bundle on BO_q. The relations between the Whitney characteristic map and the algebraic join operation is also discussed here.

5.1 Stiefel-Whitney Classes and Cycle Spaces

Recall that by fixing a distinguished real linear subspace ℓ_o of dimension p in $\mathbb{P}^n_{\mathbb{C}}$, we may define $\mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})_o$ as the weak limit $\lim_d \mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ of the following sequence of embeddings :

$$\cdots \longrightarrow \mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{+\ell_o} \mathbf{RC}_{p,d+1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \cdots .$$

Noticing that the real Grassmannian of codimension q planes in $\mathbb{R}^{n+1}$ can be identified with the space of real effective cycles of codimension q and of degree one in $\mathbb{P}^n_{\mathbb{C}}$, we consider the composition of the maps

$$Gr^q(\mathbb{P}^n_{\mathbb{R}}) \cong \mathbf{RC}_{p,1}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})_1 \longrightarrow \mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_1 \sim \mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_o$$

where $p + q = n$. By fixing an infinite flag of real linear subspaces

$$\mathbb{R}^{q+1} \subset \mathbb{R}^{q+2} \subset \mathbb{R}^{q+3} \subset \cdots$$

and compatible splittings $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ for each n, we have inclusions

$$Gr^q(\mathbb{P}^n_{\mathbb{R}}) \longrightarrow Gr^q(\mathbb{P}^{n+1}_{\mathbb{R}})$$

defined by sending a real linear subspace V of codimension q in $\mathbb{R}^{n+1}$ to $V \oplus \mathbb{R}$. This is exactly the suspension map on real algebraic cycles of degree one and codimension q. Then we have the a commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^q_{\mathbb{R}} & \longrightarrow & \mathcal{E}^q(\mathbb{P}^q_{\mathbb{R}})_o \\
\not\Sigma \big\downarrow & & \not\Sigma \big\downarrow \\
Gr^q(\mathbb{P}^q_{\mathbb{R}}) & \longrightarrow & \mathcal{E}^q(\mathbb{P}^q_{\mathbb{R}})_o \\
\not\Sigma \big\downarrow & & \not\Sigma \big\downarrow \\
\vdots & & \vdots \\
\not\Sigma \big\downarrow & & \not\Sigma \big\downarrow \\
Gr^q(\mathbb{P}^n_{\mathbb{R}}) & \longrightarrow & \mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_o \\
\not\Sigma \big\downarrow & & \not\Sigma \big\downarrow \\
Gr^q(\mathbb{P}^{n+1}_{\mathbb{R}}) & \longrightarrow & \mathcal{E}^q(\mathbb{P}^{n+1}_{\mathbb{C}})_o \\
\not\Sigma \big\downarrow & & \not\Sigma \big\downarrow \\
\vdots & & \vdots
\end{array}$$

By Corollary 1.3.2, each $\not\Sigma$ on the right is a homotopy equivalence. Passing to the limit as $n \longrightarrow \infty$, we obtain a **Whitney Characteristic map**

$$BO_q \xrightarrow{w} K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, q).$$

Note that $K(\mathbb{Z}_2, k)$ is the classifying space for the cohomology functor $H^k(\cdot, \mathbb{Z}_2)$, i.e., there is a one-to-one correspondence

$$H^k(X, \mathbb{Z}_2) \cong [X, K(\mathbb{Z}_2, k)]$$

for each positive integer k and countable CW-complex X, where $[X, K(\mathbb{Z}_2, k)]$ denotes the homotopy classes of base-point preserving maps from X to $K(\mathbb{Z}_2, k)$. Consequently, each component of w represents a cohomology class in $H^*(BO_q, \mathbb{Z}_2)$.

Theorem 5.1.1 *The map w represents the total Stiefel-Whitney class of the universal q-plane bundle ξ_q over BO_q.*

Proof Fix real flags

$$\begin{array}{ccc} \mathbb{R}^q & \subset & \mathbb{R}^{q+1} \\ \cap & & \cap \\ \mathbb{R}^{q+1} & \subset & \mathbb{R}^{q+2} \\ \cap & & \cap \\ \vdots & & \vdots \end{array}$$

and consider the following induced commutative diagrams :

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{R}}^{q-1} & \longrightarrow & \mathcal{E}^{q-1}(\mathbb{P}_{\mathbb{C}}^{q-1})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ Gr^{q-1}(\mathbb{P}_{\mathbb{R}}^{q}) & \longrightarrow & \mathcal{E}^{q-1}(\mathbb{P}_{\mathbb{C}}^{q})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ \vdots & & \vdots \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ Gr^{q-1}(\mathbb{P}_{\mathbb{R}}^{n-1}) & \longrightarrow & \mathcal{E}^{q-1}(\mathbb{P}_{\mathbb{C}}^{n-1})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ \vdots & & \vdots \end{array} \qquad \begin{array}{ccc} \mathbb{P}_{\mathbb{R}}^{q} & \longrightarrow & \mathcal{E}^{q}(\mathbb{P}_{\mathbb{C}}^{q})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ Gr^{q}(\mathbb{P}_{\mathbb{R}}^{q+1}) & \longrightarrow & \mathcal{E}^{q}(\mathbb{P}_{\mathbb{C}}^{q+1})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ \vdots & & \vdots \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ Gr^{q}(\mathbb{P}_{\mathbb{R}}^{n}) & \longrightarrow & \mathcal{E}^{q}(\mathbb{P}_{\mathbb{C}}^{n})_o \\ \not\Sigma\downarrow & & \not\Sigma\downarrow \\ \vdots & & \vdots \end{array}$$

Stablizing the vertical maps by letting $n \longrightarrow \infty$, we have the following commutative diagram of maps

$$\begin{array}{ccccc}
\mathbb{P}_{\mathbb{R}}^{q-1} & \longrightarrow & BO_{q-1} & \longrightarrow & K(\mathbb{Z}_2,1) \times \cdots \times K(\mathbb{Z}_2, q-1) \\
\cap & & \cap & & \downarrow j \\
\mathbb{P}_{\mathbb{R}}^{q} & \longrightarrow & BO_q & \longrightarrow & K(\mathbb{Z}_2,1) \times \cdots \times K(\mathbb{Z}_2, q).
\end{array}$$

Note that $\mathcal{E}^q(\mathbb{P}_{\mathbb{C}}^q)_o \sim \mathrm{SP}(\mathbb{P}_{\mathbb{R}}^q, \mathbb{Z}_2)$ and the inclusion $\mathcal{E}^{q-1}(\mathbb{P}_{\mathbb{C}}^{q-1})_o \subset \mathcal{E}^q(\mathbb{P}_{\mathbb{C}}^q)_o$ corresponds to the inclusion $SP(\mathbb{P}_{\mathbb{R}}^{q-1}, \mathbb{Z}_2) \subset SP(\mathbb{P}_{\mathbb{R}}^q, \mathbb{Z}_2)$. Since $\sum$ is a homotopy equivalence, j is homotopic to the standard inclusion $j_o(x) = (x, x_o)$ as a factor. Therefore we have a map

$$S^q = \mathbb{P}_{\mathbb{R}}^q / \mathbb{P}_{\mathbb{R}}^{q-1} \hookrightarrow BO_q / BO_{q-1} \longrightarrow K(\mathbb{Z}_2, q)$$

which represents the generator of $\pi_q(K(\mathbb{Z}_2, q)) = \mathbb{Z}_2$ since, by [DT1,2],

$$\begin{aligned}
SP(\mathbb{P}_{\mathbb{R}}^q, \mathbb{Z}_2) & \longrightarrow SP(\mathbb{P}_{\mathbb{R}}^q, \mathbb{Z}_2) /\!\!/ SP(\mathbb{P}_{\mathbb{R}}^{q-1}, \mathbb{Z}_2) \\
& \cong SP(\mathbb{P}_{\mathbb{R}}^q / \mathbb{P}_{\mathbb{R}}^{q-1}, \mathbb{Z}_2) \\
& \cong SP(S^q, \mathbb{Z}_2) \\
& \sim K(\mathbb{Z}_2, q)
\end{aligned}$$

is a quasifibration and the canonical embedding $S^q \longrightarrow SP(S^q, \mathbb{Z}_2)$ represents the generator of $\pi_q(K(\mathbb{Z}_2, q))$.

Let $b_q \in H^q(BO_q, \mathbb{Z}_2), k = 1, \cdots, q$, be the cohomology class represented by the q-th component of the map

$$BO_q \xrightarrow{w = \beta_1 \times \cdots \times \beta_q} K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, q).$$

As we have shown above, the map β_q descends to a map on the quotient BO_q / BO_{q-1}, hence b_q vanishes when restricted to BO_{q-1}. Moreover, when restricted to $\mathbb{P}^q_{\mathbb{R}} \subset BO_q$, this map also descends to a map $\mathbb{P}^q_{\mathbb{R}}/\mathbb{P}^{q-1}_{\mathbb{R}}$ which represents the generator of $\pi_q(K(\mathbb{Z}_2, q))$. Therefore b_q is a non-zero cohomology class lying in the kernel of the homomorphism

$$H^*(BO_q, \mathbb{Z}_2) \longrightarrow H^*(BO_{q-1}, \mathbb{Z}_2)$$

induced by the inclusion $BO_{q-1} \subset BO_q$. It is well-known that $H^*(BO_q, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \cdots, w_q]$ is the $\mathbb{Z}_2$-polynomial ring generated by the Stiefel-Whitney classes of the universal q-plane bundle on BO_q, and the kernel of the homomorphism

$$H^*(BO_q, \mathbb{Z}_2) \longrightarrow H^*(BO_{q-1}, \mathbb{Z}_2)$$

is the ideal generated by w_q. Hence we conclude that $b_q = w_q$. Applying the argument inductively to $BO_{q-1}, BO_{q-2}, \cdots$, we have $b_k = w_k$ for $k = 1, \cdots, q$. ∎

5.2 Whitney Sum and The Algebraic Join Operation

Let $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^m_{\mathbb{C}}$ be embedded in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$ via the splitting $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1} = \mathbb{C}^{n+m+2}$ of their homogeneous coordinates. Also let ℓ_o, ℓ'_o be two fixed distinguished real linear subspaces in $\mathbb{P}^n_{\mathbb{C}}$, $\mathbb{P}^m_{\mathbb{C}}$ respectively.

The algebraic join operation

$$\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}}) \times \mathbf{RC}_{p',d'}(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\#_{\mathbb{C}}} \mathbf{RC}_{p+p'+1,dd'}(\mathbb{P}^{n+m+1}_{\mathbb{C}})$$

induces a pairing on degree one real cycles $\mathbf{RC}_{p,1}(\mathbb{P}^n_{\mathbb{C}}) = Gr^q(\mathbb{P}^n_{\mathbb{C}})$ where $p + q = n$. The pairing on degree one cycles is in fact given by taking the direct sum of linear subspaces

$$Gr^q(\mathbb{P}^n_{\mathbb{C}}) \times Gr^{q'}(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\oplus} Gr^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}).$$

By letting $n, m \longrightarrow \infty$, one has the classifying map

$$BO_q \times BO_{q'} \xrightarrow{\oplus} BO_{q+q'}$$

which classifies the Whitney sum operation on vector bundles. It has the property that $\oplus^*(\xi_{q+q'}) = \xi_q \oplus \xi_{q'}$, where ξ_q, $\xi_{q'}$ and $\xi_{q+q'}$ are the universal bundles over BO_q, $BO_{q'}$ and $BO_{q+q'}$ respectively. Theorem 5.1.1 shows that when $n \to \infty$, the map

$$Gr^q(\mathbb{P}^n_{\mathbb{R}}) \longrightarrow \mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})_1 \longrightarrow \mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}})_1$$

represents the Stiefel-Whitney classes. However the algebraic join operation on $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ does not extends to a pairing on $\mathcal{RC}_p(\mathbb{P}^n_{\mathbb{C}})$ directly. In order to study the relations between the Whitney sum operation and the algebraic join operation on cycle spaces, it is more convenient to look at the alternative naive cycle groups $\widetilde{\mathcal{RC}}_p(\mathbb{P}^n_{\mathbb{C}})$ and $\widetilde{\mathcal{DC}}_p(\mathbb{P}^n_{\mathbb{C}})$. Note that the algebraic join extends biadditively to a commutative diagram of pairings

$$\begin{array}{ccc}
\widetilde{\mathcal{RC}}_p(\mathbb{P}^n_{\mathbb{C}}) \times \widetilde{\mathcal{RC}}_{p'}(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{\#_{\mathbb{C}}} & \widetilde{\mathcal{RC}}_{p+p+1'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}) \\
\cup & & \cup \\
\widetilde{\mathcal{DC}}_p(\mathbb{P}^n_{\mathbb{C}}) \times \widetilde{\mathcal{DC}}_{p'}(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{\#_{\mathbb{C}}} & \widetilde{\mathcal{DC}}_{p+p+1'}(\mathbb{P}^{n+m+1}_{\mathbb{C}})
\end{array}$$

which descend to their smash products. Consequently, we have a pairing

$$\widetilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}) \times \widetilde{\mathcal{E}}^{q'}(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\#_{\mathbb{C}}} \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}),$$

where $p + q = n$, $p' + q' = m$, which also descends to the smash product

$$\widetilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}) \wedge \widetilde{\mathcal{E}}^{q'}(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\#_{\mathbb{C}}} \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}).$$

By Proposition 4.2.1, the map $\mathbf{RC}_{p,d}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow \widetilde{\mathcal{RC}}_p(\mathbb{P}^n_{\mathbb{C}})$ defined by $c \longrightarrow c - d\ell_o$ induces a homotopy equivalence

$$\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\sim} \tilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}).$$

Therefore we now consider the pairing

$$\mathcal{E}^q(\mathbb{P}^n_{\mathbb{C}}) \times \mathcal{E}^{q'}(\mathbb{P}^m_{\mathbb{C}}) \longrightarrow \tilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}}) \times \tilde{\mathcal{E}}^{q'}(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\#_{\mathbb{C}}} \tilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}})$$

defined by $(c, c') \longrightarrow (c - d\ell_o)\#_{\mathbb{C}}(c - d\ell_o)$. By the algebraic suspension theorem, we then have a pairing

$$(**) \qquad \prod^{q} \wedge \prod^{q'} \xrightarrow{\#_{\mathbb{C}}} \prod^{q+q'}$$

where $\prod^{k} \stackrel{def}{=} K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, k)$. To understand this pairing, it is sufficient to study it at the cohomology level.

Let τ_k denote the pull back to $\prod^{q}$ of the generator of $H^k(K(\mathbb{Z}_2, k), \mathbb{Z}_2) = \mathbb{Z}_2$ via the projection. Then we have :

Theorem 5.2.1 *The pairing $(**)$ satisfies the following property :*

$$\#_{\mathbb{C}}^*(\tau_k) = \sum_{\substack{r+s=k \\ r,s>o}} \tau_r \otimes \tau_s.$$

Let $\mathbb{P}^{m+1}_{\mathbb{C}} = \sum \mathbb{P}^m_{\mathbb{C}}$ be given the Fubini-Study metric. Consider the t-neighborhood $\mathbb{P}^m_{\mathbb{C},t}$ in $\mathbb{P}^n_{\mathbb{C}}$ as discussed in p.40. Recall that from Corollary 3.3.2, we have that the inclusion $\tilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C}}) \longrightarrow \tilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t})$ is a homotopy equivalence for all $o \leq t < \pi$.

Proposition 5.2.1 *For all $o \leq t < \pi$, the suspension homomorphism*

$$\sum : \widetilde{\mathcal{E}}^q(\mathbb{P}^m_{\mathbb{C},t}) \longrightarrow \widetilde{\mathcal{E}}^q(\mathbb{P}^{m+1}_{\mathbb{C},t})$$

is a homotopy equivalence.

Proof In the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{E}}^q(\mathbb{P}^m_{\mathbb{C},t}) & \xrightarrow{\sum} & \widetilde{\mathcal{E}}^q(\mathbb{P}^{m+1}_{\mathbb{C},t}) \\
\uparrow & & \uparrow \\
\widetilde{\mathcal{E}}^q(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{\sum} & \widetilde{\mathcal{E}}^q(\mathbb{P}^{m+1}_{\mathbb{C}}),
\end{array}$$

the vertical inclusion maps are deformation retractions and the suspension map $\widetilde{\mathcal{E}}^q(\mathbb{P}^m_{\mathbb{C}}) \xrightarrow{\sum^m} \widetilde{\mathcal{E}}^q(\mathbb{P}^{m+1}_{\mathbb{C}})$ is a homotopy equivalence. Therefore the suspension map

$$\sum : \widetilde{\mathcal{E}}^q(\mathbb{P}^m_{\mathbb{C},t}) \longrightarrow \widetilde{\mathcal{E}}^q(\mathbb{P}^{m+1}_{\mathbb{C},t})$$

is also a homotopy equivalence. ∎

Let $\mathbb{P}^n_{\mathbb{C}}$ be embedded in $\mathbb{P}^N_{\mathbb{C}}$ as a real linear subspace. By fixing $t < \pi$, we have the following commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C}}) & \longrightarrow & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}) & \xrightarrow{pr} & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}) /\!\!/ \widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C}}) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t}) & \longrightarrow & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}) & \xrightarrow{pr} & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C}}) /\!\!/ \widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t}) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t}) & \longrightarrow & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C},t}) & \xrightarrow{pr} & \widetilde{\mathcal{E}}_p(\mathbb{P}^n_{\mathbb{C},t}) /\!\!/ \widetilde{\mathcal{E}}_p(\mathbb{P}^m_{\mathbb{C},t})
\end{array}$$

where pr is the natural projection and where $\mathbb{P}^m_{\mathbb{C},t}$ in the third row is considered as a t-neighborhood of $\mathbb{P}^m_{\mathbb{C}}$ in $\mathbb{P}^N_{\mathbb{C}}$.

Proposition 5.2.2 *In the above commutative diagram, the vertical maps are homotopy equivalences and the horizontal maps are principal fibrations.*

Proof To prove that the horizontal maps are principal fibrations, we construct a local section for each of these projections following the inductive construction of Dold-Thom as in Section 2.1. That the vertical maps are homotopy equivalences then follow from Proposition 5.2.1 and the five lemma. ∎

Fix flags of real linear subspaces

$$\ell_o = \mathbb{P}^p_{\mathbb{C}} \subset \mathbb{P}^{p+1}_{\mathbb{C}} \subset \cdots \subset \mathbb{P}^n_{\mathbb{C}}$$

$$\ell'_o = \mathbb{P}^{p'}_{\mathbb{C}} \subset \mathbb{P}^{p'+1}_{\mathbb{C}} \subset \cdots \subset \mathbb{P}^m_{\mathbb{C}}$$

in $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^m_{\mathbb{C}}$ respectively. Then choose corresponding real flags in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$

$$\mathbb{P}^{p+p'+1}_{\mathbb{C}} \subset \mathbb{P}^{p+p'+2}_{\mathbb{C}} \subset \cdots \subset \mathbb{P}^{n+m+1}_{\mathbb{C}}$$

and an $\epsilon > 0$ such that, for all $r + s < k$,

$$\mathbb{P}^r_{\mathbb{C}} \#_{\mathbb{C}} \mathbb{P}^s_{\mathbb{C}} \subset \mathbb{P}^k_{\mathbb{C}\,\epsilon} \subset \mathbb{P}^{n+m+1}_{\mathbb{C}}.$$

For convenience, let $U(k) \overset{def}{=} \mathbb{P}^k_{\mathbb{C},\epsilon}$ be the ϵ-neighborhood of the corresponding real linear subspace $\mathbb{P}^k_{\mathbb{C}}$ in $\mathbb{P}^{n+m+1}_{\mathbb{C}}$. Then we have

$$\widetilde{\mathcal{E}}_p(\mathbb{P}^r_{\mathbb{C}}) \wedge \widetilde{\mathcal{E}}_{p'}(\mathbb{P}^s_{\mathbb{C}}) \xrightarrow{\#_{\mathbb{C}}} \widetilde{\mathcal{E}}_{p+p'+1}(U(k))$$

whenever $r + s < k$. By identifying cycle spaces in the principal fibration

$$\begin{array}{ccccc}
\widetilde{\mathcal{E}}^{q+q-1'}(U(n+m)) & \longrightarrow & \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}) & \xrightarrow{pr} & \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}) /\!\!/ \widetilde{\mathcal{E}}^{q+q'-1}(U(n+m)) \\
\wr & & \wr & & \wr \\
\prod^{q+q'-1} & \longrightarrow & \prod^{q+q'} & \xrightarrow{pr} & K_{q+q'}
\end{array}$$

with the corresponding products of Eilenberg-MacLane spaces, where we denote by $K_{q+q'}$ the spaces $K(\mathbb{Z}_2, q+q')$, we determine homotopically a projection map

$$\prod^{q+q'} \xrightarrow{pr} K_{q+q'}.$$

Remark 5.2.1 Similarly, one has a fibration

$$\prod^{r} \longrightarrow \prod^{q+q'} \longrightarrow K_{r+1} \times \cdots \times K_{q+q'}.$$

We now consider the following commutative diagram

$$\begin{array}{ccc} \prod^{q} \wedge \prod^{q'} & \xrightarrow{\#_{\mathbb{C}}} & \prod^{q+q'} \\ {\scriptstyle pr_1 \times pr_2}\big\downarrow & & \big\downarrow{\scriptstyle pr} \\ K_q \wedge K_{q'} & \xrightarrow{J} & K_{q+q'}. \end{array}$$

Let τ_k be the generator of the cohomology group $H^k(K_k, \mathbb{Z}_2)$. Then we have

Proposition 5.2.3 *The map J satisfies the following property :*

$$J^*(\tau_{q+q'}) = \tau_q \otimes \tau_{q'}.$$

Proof By [DT1,2] and the suspension theorem, we have that the map

$$\mathbb{P}^q_{\mathbb{R}}/\mathbb{P}^{q-1}_{\mathbb{R}} \hookrightarrow \tilde{\mathcal{E}}^q(\mathbb{P}^q_{\mathbb{C}})/\!\!/\tilde{\mathcal{E}}^{q-1}(\mathbb{P}^{q-1}_{\mathbb{C}}) \sim \tilde{\mathcal{E}}^q(\mathbb{P}^n_{\mathbb{C}})/\!\!/\tilde{\mathcal{E}}^{q-1}(\mathbb{P}^{n-1}_{\mathbb{C}}) \sim K_q$$

represents the generator of $\pi_q(K_q)$. Hence we have the commutative diagram

$$\begin{array}{ccc} K_q \wedge K_{q'} & \xrightarrow{J} & K_{q+q'} \\ \cup & & \cup \\ \mathbb{P}^q_{\mathbb{R}}/\mathbb{P}^{q-1}_{\mathbb{R}} \wedge \mathbb{P}^{q'}_{\mathbb{R}}/\mathbb{P}^{q'-1}_{\mathbb{R}} & \xrightarrow{\approx} & \mathbb{P}^{q+q'}_{\mathbb{R}}/\mathbb{P}^{q+q'-1}_{\mathbb{R}} \\ \| & & \| \\ S^q \wedge S^{q'} & \xrightarrow{\approx} & S^{q+q'} \end{array}$$

where $S^q \wedge S^{q'}$ generates $\pi_{q+q'}(K_{q+q'})$. The proposition then follows. ■

Proof of Theorem 5.2.1

Let SK_r^q be the r-skeleton of $\prod^q \sim K_1 \times \cdots K_q$. Since each K_l is $l-1$ connected, without loss of generality, we may assume that $SK_r^q \subset \prod^r \subset \prod^q$. Let SK_k denote the k-skeleton of $\prod^q \wedge \prod^{q'}$. Then we have

$$SK_k \subset \bigcup_{r+s\leq k} \prod^r \wedge \prod^s.$$

Moreover, we have the commutative diagram

$$\begin{tikzcd}
 & \prod^q \wedge \prod^{q'} \arrow[r, "\#_{\mathbf{c}}"] & \prod^{q+q'} \\
SK_k \arrow[r] & \bigcup_{r+s\leq k} \prod^r \wedge \prod^s \arrow[u] \arrow[r, "\#_{\mathbf{c}}"] & \widetilde{\mathcal{E}}^{q+q'}(U(k)) \arrow[u] \\
SK_{k-1} \arrow[r] \arrow[u, phantom, "\cup"] & \bigcup_{r+s\leq k-1} \prod^r \wedge \prod^s \arrow[u, phantom, "\cup"] \arrow[r, "\#_{\mathbf{c}}"] & \widetilde{\mathcal{E}}^{q+q'}(U(k-1)). \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Note that

$$SK_k/SK_{k-1} = \bigvee_{r+s=k} (SK_r^q/SK_{r-1}^q) \wedge (SK_s^{q'}/SK_{s-1}^{q'})$$

and that the class $\#_{\mathbf{c}}{}^*\tau_k$ is determined by the map

$$SK_k/SK_{k-1} \longrightarrow \bigcup_{r+s\leq k} \prod^r \wedge \prod^s \Big/ \bigcup_{r+s\leq k-1} \prod^r \wedge \prod^s \longrightarrow \widetilde{\mathcal{E}}^{q+q'}(U(k))/\!\!/\widetilde{\mathcal{E}}^{q+q'}(U(k-1)) \sim K_k.$$

Also note that this map is factored through the bouquet

$$\bigcup_{r+s\leq k} \prod^r \wedge \prod^s \Big/ \bigcup_{r+s\leq k-1} \prod^r \wedge \prod^s = \bigvee_{r+s=k} (\prod^r/\prod^{r-1}) \wedge (\prod^s/\prod^{s-1}).$$

It is easy to check that each map

$$\prod^{r} \wedge \prod^{s} \longrightarrow \widetilde{\mathcal{E}}^{q+q'}(U(k))/\!\!/\widetilde{\mathcal{E}}^{q+q'}(U(k-1)) \;\sim\; K_k$$

actually descends to the smash product $(\prod^{r}/\!\!/\prod^{r-1}) \wedge (\prod^{s}/\!\!/\prod^{s-1})$. Therefore, by Proposition 5.2.3, the theorem follows. ■

We now define

$$\mu : \widetilde{\mathcal{E}}^{q}(\mathbb{P}^n_{\mathbb{C}}) \times \widetilde{\mathcal{E}}^{q'}(\mathbb{P}^m_{\mathbb{C}}) \longrightarrow \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}})$$

by $\mu(c, c') = \#_{\mathbb{C}} + \ell_o \#_{\mathbb{C}} c' + c \#_{\mathbb{C}} \ell'_o$. Then we have the following commutative diagram

$$\begin{array}{ccc}
Gr^{q}(\mathbb{P}^n_{\mathbb{C}}) \times Gr^{q'}(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{\oplus} & Gr^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{E}}^{q}(\mathbb{P}^n_{\mathbb{C}}) \times \widetilde{\mathcal{E}}^{q'}(\mathbb{P}^m_{\mathbb{C}}) & \xrightarrow{\mu} & \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}}).
\end{array}$$

Similarly, we have

Theorem 5.2.2 *μ satisfies the following property :*

$$\mu^*(\tau_k) = \sum_{\substack{r+s=k \\ r,s \geq o}} \tau_r \otimes \tau_s.$$

Proof Since the map $\widetilde{\mathcal{E}}^{q}(\mathbb{P}^n_{\mathbb{C}}) \hookrightarrow \widetilde{\mathcal{E}}^{q+q'}(\mathbb{P}^{n+m+1}_{\mathbb{C}})$ given by $c \longrightarrow c\#_{\mathbb{C}}\ell'_o$ is homotopic to the inclusion $\prod^{q} \hookrightarrow \prod^{q+q'}$, the result follows from Theorem 5.2.1. and the fact the sum $\sum^{n+1} + \sum^{m+1}$ pulls τ_k back to $\tau_k \otimes 1 + 1 \otimes \tau_k$. ■

Let $\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}}) \stackrel{def}{=} \lim_{k\to\infty} \tilde{\mathcal{E}}^q(\sum^k \mathbb{P}^o_{\mathbb{C}})$. Then, by stablizing the commutative diagram above by letting n, $m' \longrightarrow \infty$, we have a commutative diagram

$$\begin{array}{ccc} BO_q \times BO_{q'} & \xrightarrow{\oplus} & BO_{q+q'} \\ \downarrow & & \downarrow \\ \mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}}) \times \mathfrak{E}^{q'}(\mathbb{P}^o_{\mathbb{C}}) & \xrightarrow{\mu} & \mathfrak{E}^{q+q'}(\mathbb{P}^o_{\mathbb{C}}). \end{array}$$

Thus Theorem 5.2.2 provides a cycle-theoretic proof of the Whitney sum formula

$$w(E \oplus E') = w(E) \cup w(E')$$

for the total Stiefel-Whitney class of real vector bundles.

5.3 Friedlander-Mazur Operations

Following the construction in [FM], we define a bigraded module associated to a real algebraic subset via the algebraic join operation on cycle spaces.

Recall that for a general real algebraic subset in $\mathbb{P}^n_{\mathbb{C}}$, we define

$$\mathfrak{E}^q(X) \stackrel{def}{=} \lim_{k\to\infty} \tilde{\mathcal{E}}^q(\textstyle\sum^k X)$$

and obtain a pairing

$$\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}}) \wedge \mathfrak{E}^{q'}(X) \xrightarrow{\#_{\mathbb{C}}} \mathfrak{E}^{q+q'}(X)$$

via the algebraic join operation. In particular, the algebraic join operation induces a pairing of the homotopy groups of these spaces :

$$\pi_i(\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}})) \otimes \pi_j(\mathfrak{E}^{q'}(X)) \xrightarrow{\#_{\mathbb{C}}} \pi_{i+j}(\mathfrak{E}^{q+q'}(X)).$$

When $X = \mathbb{P}^o_{\mathbb{C}}$, this pairing provides a ring structure on the bigraded group $\pi_*\mathfrak{E}^*(\mathbb{P}^o_{\mathbb{C}}) = \bigoplus \pi_i\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}})$. Hence $\pi_*\mathfrak{E}^*(X)$ admits the structure of a bigraded module over the bigraded ring $\pi_*\mathfrak{E}^*(\mathbb{P}^o_{\mathbb{C}})$.

Recall that we have

$$\pi_j\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}}) = \begin{cases} H_j(\mathbb{P}^q_{\mathbb{R}}, \mathbb{Z}_2) \cong \mathbb{Z}_2 & if\ 0 \le j \le q, \\ 0 & if\ r > q. \end{cases}$$

Therefore in order to determine the ring structure of the bigraded ring $\pi_*\mathfrak{E}^*(\mathbb{P}^o_{\mathbb{C}})$, it is sufficient to consider the generators of these homotopy groups. In fact, we have

Theorem 5.3.1 *The bigraded ring $\pi_*\mathfrak{E}^*(\mathbb{P}^o_{\mathbb{C}})$ is isomorphic to the polynomial algebra over $\mathbb{Z}_2$ on two generators :*

$$\pi_*\mathfrak{E}^*(\mathbb{P}^o_{\mathbb{C}}) \cong \mathbb{Z}_2[a, b],$$

where $a \in \pi_o\mathfrak{E}^1(\mathbb{P}^o_{\mathbb{C}})$ and $b \in \pi_1\mathfrak{E}^1(\mathbb{P}^o_{\mathbb{C}})$.

Proof For $0 \le i \le q$, let ξ^q_i denote the generator of the group $\pi_i\mathfrak{E}^q(\mathbb{P}^o_{\mathbb{C}})$. Let c denote a degree one cycle in $\widetilde{\mathcal{E}}_o(\mathbb{P}^s_{\mathbb{C}})$ which represents the generator ξ^s_o of $\pi_o(\widetilde{\mathcal{E}}_o(\mathbb{P}^s_{\mathbb{R}}))$.

Recall that $\widetilde{\mathcal{E}}_o(\mathbb{P}^q_{\mathbb{R}})_o \sim \prod^q$. Moreover, the inclusion $\prod^r \hookrightarrow \prod^q$ induces isomorphisms of their homotopy groups up to dimension r. Consider the commutative diagram

$$\begin{array}{ccc} \prod^q & \xrightarrow{\#_{\mathbb{C}} c} & \prod^{s+q} \\ \uparrow & & \uparrow \\ \prod^r & \xrightarrow{\#_{\mathbb{C}} c} & \prod^{s+r} \end{array}$$

where $\prod\limits^{r} \xrightarrow{\#_{\mathbb{C}}C} \prod\limits^{s+r}$ is homotopically equivalent to the inclusion map

$$SP(\mathbb{P}^r_{\mathbb{R}}, \mathbb{Z}_2) \longrightarrow SP(\mathbb{P}^{s+r}_{\mathbb{R}}, \mathbb{Z}_2)$$

induced by the linear embedding $\mathbb{P}^r_{\mathbb{R}} \hookrightarrow \mathbb{P}^{s+r}_{\mathbb{R}}$. Such a map sends the generator ξ^q_i to ξ^{s+q}_i. Hence we have $\xi^s_o \#_{\mathbb{C}} \xi^q_i = \xi^{s+q}_i$. Similarly, $\xi^q_i \#_{\mathbb{C}} \xi^s_o = \xi^{s+q}_i$.

For any $0 < i \leq s$, $0 < j \leq q$, consider the commutative diagram

$$\begin{tikzcd}
\prod\limits^{s} \wedge \prod\limits^{q} \arrow[r, "\#_{\mathbb{C}}"] & \prod\limits^{s+q} \\
\prod\limits^{i} \wedge \prod\limits^{j} \arrow[r, "\#_{\mathbb{C}}"] \arrow[u] \arrow[d, "pr \times pr"] & \prod\limits^{i+j} \arrow[u] \arrow[d, "pr"] \\
K_i \wedge K_j \arrow[r, "J"] & K_{i+j}.
\end{tikzcd}$$

From Proposition 5.2.3, the generators of K_i and K_j generate the $(i+j)$-th homotopy group of K_{i+j}. Hence we have

$$\xi^s_i \#_{\mathbb{C}} \xi^q_j = \xi^{s+q}_{i+j} = \xi^q_j \#_{\mathbb{C}} \xi^s_i.$$

By taking $a = \xi^1_o$ and $b = \xi^1_1$, the proposition follows. ∎

The generators a and b give rise to interesting operations on the bi-graded module $\pi_* \mathfrak{C}^*(X)$. Given an irreducible real algebraic subvariety $X \subset \mathbb{P}^n_{\mathbb{C}}$ with connected real locus $X_{\mathbb{R}}$, let $\not{\Sigma}_{\mathbb{R}} X$ denotes the real locus of $\not{\Sigma} X$. Then a and b provide the following interesting diagram of operations on the homotopy groups of cycles spaces $\mathfrak{C}^*(X)$:

$$
\begin{tikzcd}
\vdots & \vdots \arrow[r, dotted] & \cdots & \vdots \\
H_r(\not{\Sigma}_{\mathbb{R}}X, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] & H_{r+1}(\not{\Sigma}_{\mathbb{R}}X, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] \arrow[ur, "b \otimes \cdot"'] & & H_{r+n-q+1}(\not{\Sigma}_{\mathbb{R}}X, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] \\
H_r(X_{\mathbb{R}}, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] \arrow[ur, "b \otimes \cdot"'] & H_{r+1}(X_{\mathbb{R}}, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] & & H_{r+n-q}(X_{\mathbb{R}}, \mathbb{Z}_2) \arrow[u, "a \otimes \cdot"] \\
\vdots \arrow[u, "a \otimes \cdot"] & \vdots \arrow[u, "a \otimes \cdot"] & \cdots \arrow[ur, "b \otimes \cdot"'] & \\
\pi_r \mathfrak{E}^{q+1}(X) \arrow[u, "a \otimes \cdot"] & \pi_{r+1} \mathfrak{E}^{q+1}(X) \arrow[u, "a \otimes \cdot"] \arrow[ur, dotted, "b \otimes \cdot"'] & & \\
\pi_r \mathfrak{E}^{q}(X) \arrow[u, "a \otimes \cdot"] \arrow[ur, "b \otimes \cdot"'] & & &
\end{tikzcd}
$$

In view of the work of Friedlander and Mazur([FM]), these operations should have interesting relations with the 'Thom isomorphism' for homology of real algebraic varieties. The operation a induces an intriguing filtration on the $\mathbb{Z}_2$-homology of the real locus $X_{\mathbb{R}}$.

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