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Bounding topology by Ricci curvature in dimension three

A Dissertation Presented

by

Shun-hui Zhu

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Abstract of the Dissertation

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A general problem in the study of the relations between curvature and topology lies in understanding which results concerning sectional curvature continue to hold for Ricci curvature. Recently, J. Sha and D. Yang gave examples which show that certain results such as Gromov's estimate on the Betti numbers and Cheeger-Gromoll's Soul Theorem, can not be generalized to the case of Ricci curvature. In this dissertation, we prove several positive

results in this direction.

Our main result is related to the finiteness theorems of Cheeger and Grove-Petersen. These results say that the class of manifolds with a bound on sectional curvature (an absolute value bound in the former and a lower bound in the latter case), an upper bound on the diameter and a lower bound on the volume contains only finitely many diffeomorphism types. We tried to replace the lower bound on sectional curvature by a lower bound on Ricci curvature, and succeeded in proving a finiteness theorem in dimension three. In the mean time, we also prove a result in the open case for manifolds with nonnegative Ricci curvature. This serves as the local model for the class considered above.

Dedicated to my parents and Yun

Contents

Acknowledgments

This reminds me the annual Academy Award ceremonies where the winners, in apparent euphoria, thank those who helped along the way. For most of the audience, this is the boring part. But for the winners themselves, this is what brought them to that stage. Although, the award of a degree is perhaps not as grand as that of the Oscar, I am pleased to find that I have equally many, if not more, people to thank. As I list these names here, I am hoping that advice will still come, encouragement will continue to arrive and friendship will last forever.

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Chapter 1

Introduction

The goal of this dissertation is to establish some results which relate the topology of a Riemannian manifold to its Ricci curvature, a general problem of considerable current interest.

The past thirty years have witnessed very considerable achievments in the realm of Riemannian geometry. One of the central lines of development has centered around the attempt to generalize the $\frac{1}{4}$-Pinched Sphere Theorem. In particular, much of the work was about sectional curvature. In contrast, the progress in the study of Ricci curvature is a more recent event. Let us now briefly review the development relevent to our study and postpone stating our results until the end of this introduction.

The $\frac{1}{4}$-Pinched Sphere Theorem [Ra][Kl][Be] says if the sectional curvature of a manifold satisfies $\frac{1}{4} < K_M \leq 1$, then the manifold is homeomorphic to a sphere. Here a model space is needed. W. Ambrose [Am] made the first attempt to generalize this to a comparison theorem between arbitary two manifolds.

In his thesis [Ch1] (1967), J. Cheeger took a step further and proved the celebrated theorem now known as Cheeger's Finiteness Theorem.

Theorem (J. Cheeger[Ch2], S. Peters[Pe1]) *There are only finitely many diffeomorphism types in the class of n-dimensional Riemannian manifolds satisfying*

$$| K(M) | \leq \Lambda^2, \quad Diam(M) \leq D, \quad Vol(M) \geq V,$$

where $K(M)$ is the sectional curvature, $Diam(M)$, the diameter, and $Vol(M)$, the volume of M.

This indicates that, with a bound on sectional curvature, the diffeomorphism type of a manifold is controlled by its size, namely, a restriction on how big the manifold can be (by diameter) and how small it can get (by volume). The crucial idea of the proof is to show that under the given assumptions there is a lower bound on the injectivity radius, and hence a control on the local topology.

Twenty years later K. Grove, P. Petersen and J. Wu got essentially the same finiteness conclusion without assuming an upper bound on the sectional curvature. We will state their theorem later.

In the time between these two results, there were two developments directly related to Cheeger's Finiteness Theorem.

The first of these is a metric version of the theorem now called Cheeger-Gromov Compactness Theorem [GLP] [GW] [Pe2]. Roughly speaking it states that the bounds control not only the topology but also the metric of Riemannian manifolds. To obtain a statement which is optimally sharp, one must use harmonic coordinates instead of geodesic normal coordinates.

The second is the remarkable result of M. Gromov, estimating Betti numbers by sectional curvature and diameter.

Theorem (M. Gromov [Gr]) *For the class of n-dimensional Riemannian manifolds satisfying*

$$K(M) \geq -\Lambda^2, \qquad Diam(M) \leq D$$

there is a constant $C = C(n, \Lambda, D)$, such that

$$\sum b_i(M, \mathcal{F}) \leq \mathcal{C},$$

where $\mathcal{F}$ is an arbitary coeficient field and b_i is the i-th Betti number.

The lens spaces with constant curvature and the family of 7-dimensional examples of S. Aloff and N. Wallach [AW] show that a lower bound on sectional curvature and upper bound on diameter do not imply finiteness of homotopy types. Thus Gromov's result is the best possible, given his hypotheses. The principal technique employed in the proof was introduced by K.Grove and K. Shiohama in their proof of the Generalized Sphere Theorem [GS]. It involves a generalization of the isotopy lemma from Morse theory to (not everywhere smooth) distance functions on Riemannian manifolds. This is used in conjunction with Toponogov's Theorem on geodesic triangles.

Prior to the work of Grove-Shiohama, Morse theory, as applied to Riemannian geometry, had mainly been used in the study of variations of geodesics. Here the function is the length function, defined on the loop space. The distance function, however, is defined on a manifold itself and, through its Hessian, is directly related to the curvature.

After Gromov's work, several striking applications of this generalized Morse theory appeared. One of these (which was alluded to above) is the following,

Theorem (Grove-Petersen-Wu [GP] [GPW]) *There are only finitely many homotopy types in the class of n-dimensional Riemannian manifolds satisfying*

$$K_M \geq -\Lambda^2,\ Diam(M) \leq D,\ Vol(M) \geq V.$$

And if $n \neq 3,4$, the class contains only finitely many diffeomorphism types.

By comparison with Gromov's result, one sees that a lower bound on the volume is exactly the data needed to make the transition from bounding Betti numbers to bounding homotopy types. The remarkable feature of the theorem of Grove-Petersen-Wu is that for their class, there is no bound on the injectivity radius. This is illustrated by the exmples of cones with rounded tips. Roughly speaking, they showed that this example illustrates the worst that can happen. Thus instead of an apriori bound on the injectivity radius they got a bound on the so-called geometric contractibility radius (for the definition see Chapter 2). This amount of control on the local topology is enough to imply the finiteness of homotopy types.

In the three theorems cited above, Toponogov's Theorem was an essential tool (although E. Heintze and H. Karcher were eventually able to reprove Cheeger's estimate on the length of closed geodesics using only local geodesic variations). The Morse Theory of distance functions plays an indispensable role in the later two theorems, and the implementation of this theory relies heavily on Toponogov's Theorem. This powerful global theorem holds only for sectional curvature, there is no satisfactory generalization known for other curvatures, in particular, for Ricci curvature. As a consequence, the following question is extremely challenging.

Question. *Do the above three theorems hold if the bound on sectional curvature is replaced by that on Ricci curvature?*

To be more precise, what can we say about the following three classes of manifolds?

$$
\begin{aligned}
I \quad &= \left\{ M^n \mid Ric(M) \geq -H^2, \;\; Diam(M) \leq D \right\} \\
II \quad &= \left\{ M^n \mid Ric(M) \geq -H^2, \;\; Diam(M) \leq D, \;\; Vol(M) \geq V \right\} \\
III \quad &= \{ M^n \mid M \; open, \; Ric(M) \geq 0, \;\; Vol(B_P(r)) \geq cr^n \}
\end{aligned}
$$

Class III is closely related to class II. Essentially it serves as a *local model* for manifolds in class II. To see this, simply scale the metrics in class II by an arbitary large constant, then the curvature has a lower bound arbitary close to zero, the limit situation is the case of nonegative Ricci curvàture. The volume growth condition follows from standard comparison theorem (see Lemma 2.2 in Chapter 2), and notice that it is invariant under scaling.

The only tools presently at our disposal for the study of Ricci curvature are the Bishop-Gromov volume comparison theorem and the Laplacian comparison theorem. So far these theorems have yielded scant information about the behavior of geodesics, whereas, in the case of sectional curvature, controlling the behavior of geodesics is typically a powerful step towards controlling topology.

The difference between Ricci curvature and sectional curvature was first convincingly demonstrated by the ingenious examples of J. P. Sha and D.G. Yang. They used the techniques of a semi-local surgery and warping. Later, by a different method, M. Anderson got a related family of examples. Let us state it as the following.

Theorem (Sha-Yang [SY1] [SY2], Anderson [An]) *For any n, there are metrics on* $\sharp_{k=1}^{n} S^2 \times S^2$*, satisfying*

$$Ric > 0, \qquad Diam \leq 1.$$

These examples show that Gromov's result does not generalize to Ricci curvature.

We mention in passing that there are other instances of the success of warping techniques in constructing metrics of positive Ricci curvature; see the works of J. Nash [Na], L. Berard-Bergery [Bb] and G. Wei [We1]. Apart from families of examples constructed by algebraic geometric means (Gromoll-Meyer [GM], Anderson [An]), and by use of Riemannian submersions, warping is the only method which has proved successful in this context. In fact the use of Riemannian submersions is the only way known of constructing metrics with nonnegative sectional curvature.

By gluing together two copies of $T(S^2)$ with a slightly deformed Eguchi-Hanson metric, M. Anderson [An] also showed the following.

Theorem (M. Anderson [An]) *There are metrics on* $S^2 \times S^2$ *satisfying*

$$Ric > 0, \quad Diam \leq 5, \quad Vol \geq \frac{1}{5}$$

and with arbitary short closed geodesics.

This implies that a control of local topology in the form of a lower injectivity radius or geometric contractbility radius bound no longer holds for class *II*. As for class *III*, which tells us what to expect for the local structure of a manifold in class *II*, we have the examples of Eguchi-Hanson metric on $T(S^2)$, and the gravitational multi-instantons of Gibbons-Hawking and Hitchin. These are complete open Ricci flat manifolds whose metrics are asympototically locally Euclidean (ALE), namely, the metrics approach that of a cone over lens spaces near infinity. They show that the manifolds in class *II* locally can be as complicated as these ALE spaces. But let us notice that these examples are of finite topological type. And the metrics on $\sharp_{k=1}^{n} S^2 \times S^2$ in the previous theorem have volume roughly $\frac{1}{n}$, and hence no lower bound when $n \to \infty$. Let us also mention that using an infinite version of the gravitational instantons, Anderson-Kronheimer-Lebrun constructed four dimensional Ricci flat manifolds of infinite topological type [AKL]. but these examples have volume growth on the order of $r^{4-\epsilon}$. We thus have the following conjecture.

Conjecture. *The class II contains only finitely many homotopy (or diffeomorphism) types, and any manifold in class III has finite topological type.*

The first result that supports this conjecture is the following,

Theorem (M. Anderson [An]) *There are only finitely many possibilities for the fundamental groups of the manifolds in Class II.*

This shows the bounds in class II control the fundamental groups. The proof of this theorem uses a volume comparison argument to get a bound on the length of geodesic loops whose homotopy class is nontrivial to a high enough order. Then one applies a general theorem of M. Gromov [GLP] concerning the realization of generators and relations for the fundamental groups by loops (which have length comparable to the diameter).

With regard to homotopy types, let us notice that all the above examples, which demonstrate difficulties for proving the conjecture, are in dimensions four and higher.

Let us now consider dimesion three. Being an average of sectional curvature, Ricci curvature provides stronger constraint in lower dimensions than higher ones. In dimension 3, Ricci curvature is particularly strong. Indeed, in dimension 3, the Weyl conformal curvature tensor always vanishes. Hence the full curvature tensor can be recovered from Ricci curvature(this shows that a bound on $|Ric|$ in dimension three implies a bound on $|K_M|$, thus the nontrivial case is with only a lower bound on Ricci cuvature). This algebraic point allowed R. Hamilton to obtain via the "Ricci flow" method the spectacular result that a three manifold with positive Ricci curvature necessarily admits a metric with constant positive sectional curvature [Ha].

We mention that from the view point of geometry, Ricci curvature enters the variation formula for minimal hypersurfaces. Schoen and Yau [SCY] used the theory of minimal surfaces to prove that any open three manifold with positive Ricci curvature is diffeomorphic to $\mathbf{R}^3$.

Thus we might summarize the above paragraphs by saying that Ricci curvature is known to yield information about the fundamental groups and about minimal hypersurfaces. However, in dimension three, in order to control topology, essentially we have only these two cases to worry about. This suggests that a finiteness theorem for class *II* in dimension three might hold. Our main result confirms this.

Theorem 1.1 *There are only finitely many homotopy types for the class of three manifolds satisfying*

$$Ric \geq -H^2, \;\; Diam \leq D, \;\; Vol \geq V.$$

A closely related result is contained in the following noncompact (or local) version of Theorem 1.1, for the class *III*.

Theorem 1.2 *Let M^3 is a complete open three manifold satisfying*

$$Ric \geq 0, \quad Vol(B_p(r)) \geq cr^3.$$

Then M is contractible.

It is appropriate to note that the classification of complete open three manifolds with nonnegative Ricci is not yet finished. Let us point out that with an additional assumption on the sectional curvature, Anderson–Rodriguez [AR] and W. Shi [Sh] did give a classification. This kind of additional assumption on sectional curvature in studying Ricci curvature is also present in the works of Abrecsh–Gromoll [AG] and Z. Shen [Sn]. Roughly, for open manifolds, Ricci curvature yields the subharmonicity of Busemann functions and related estimates on distance functions. But to control the topology through the Morse theory of distance functions alluded to above, Toponogov's Theorem has to be used, and hence an assumption on the sectional curvature. These and other similar results in pure Riemannian geometry (apart from the Cheeger-Gromoll Splitting theorem, Gromov's precompactness theorem and the recent result of M. Anderson concerning compactness under Ricci curvature bounds) again illustrate the difficulty of dealing with Ricci curvature in the absence of other hypotheses on curvature. This is what makes Theorem 1.2 interesting.

In the rest of the dissertation, we give the proofs of the two results mentioned above.

Chapter 2

A finiteness theorem in dimension three

In this chapter, we prove Theorem 1.1 and Theorem 1.2. For the sake of clarity, we will first prove Theorem 1.2 . Our idea of the proof follows that of Schoen-Yau [SCY], except that in the case of the fundamental group, we will use a volume comparison argument instead of minimal surfaces. This argument has been used by M. Anderson in [An]. The proof of Theorem 1.1 runs along similar lines. But in this case, since we do not have the Cheeger-Gromoll Splitting Theorem and since essentially we are working on a noncomplete manifold, some technical difficulties have to be addressed. For the sake of exposition, in the first section, we give the proof of a topological lemma. This lemma will be needed in the proof of Proposition 2.1.

2.1 A topological lemma

In this section, we prove the following topological concerning fundamental groups of three dimensional manifolds. A basic reference on this subject is the book by J. Hempel [He].

Lemma 2.1 *Let $M \subset int(N)$ be two compact orientable three manifolds with nonempty boundary. If $\pi_2(M) \to \pi_2(N)$ is trivial, then $\pi_1(M)$ is torsion free.*

Proof. Let $M = \sharp_{i=1}^k M_i$ be a prime decomposition of M. Since $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k)$, a free product, we can assume that M itself is prime. Without loss of generality, we can assume $\pi_1(M) \neq \{e\}$. we first prove that $\pi_1(M)$ is infinite. In fact, if we denote by $\widehat{M}$ the manifold by capping off all two-spheres in ∂M by three balls, then we claim that $\widehat{M}$ is not closed. If it were, then ∂M would only consist of two-spheres. Since $\pi_2(M) \to \pi_2(N)$ is trivial, each S^2 in ∂M separates N. Moreover, at least one component of $N - S^2$ is compact and simply connected. In fact, if neither component were compact, it would follow from Poincarè duality that each such S^2 is nontrivial in

$\pi_2(N)$. If none of the compact components were simply connected, lifting to the universal covering space, duality would again imply that each such S^2 is nontrivial in $\pi_2(N)$. Hence we have that each S^2 in ∂M bounds a homotopy three ball in N. If one of such S^2 bounds a three ball containing M, then M is simply connected. This is a contradiction. So all 2-spheres in ∂M bound in the exterior. By adding these homotopy three balls to M, we get a closed three manifold embedded in a three manifold with nonempty boundary. This is impossible. So $\widehat{M}$ is not closed. By taking the double of $\widehat{M}$, we obtain a closed three manifold $\widehat{M} \bigcup_{\partial \widehat{M}} \widehat{M}$. Hence, $0 = \chi(\widehat{M} \bigcup_{\partial \widehat{M}} \widehat{M}) = \chi(\widehat{M}) + \chi(\widehat{M}) - \chi(\partial \widehat{M})$. So $\chi(\widehat{M}) = \frac{1}{2}\chi(\partial \widehat{M}) \leq 0$. On the other hand, $\chi(\widehat{M}) = 1 - b_1(\widehat{M}) + b_2(\widehat{M}) - b_3(\widehat{M})$, where b_i is the i-th Betti number. Since $\widehat{M}$ is not closed, we have $b_3(\widehat{M}) = 0$. Therefore $b_1(\widehat{M}) \geq 1 + b_2(\widehat{M}) \geq 1$. By Mayer-Vitoris sequence, we obtain $b_1(M) = b_1(\widehat{M}) \geq 1$. So $\pi_1(M)$ is infinite.

By the method of contradiction , we now prove $\pi_1(M)$ is torsion free. If not, let G be a finite subgroup of $\pi_1(M)$, and let M_1 be a covering space of M such that $p_*(\pi_1(M_1)) = G$. Using the same notation as before, we have $\pi_1(\widehat{M}_1) = G$. Let $\widetilde{M}_1$ be the universal covering space of $\widehat{M}_1$. Then $\pi_2(\widetilde{M}_1) = 0$ since $\widehat{M}_1$ is prime and orientable. We claim $\widetilde{M}_1$ is closed. Otherwise, by the Hurewicz theorem, $H_i(\widehat{M}_1) = H_i(\widetilde{M}_1) = 0$ for $i \geq 2$. Hence $H_i(G) = 0$ for $i \geq 2$. This is not possible since G is finite. Therefore $\widehat{M}_1$ is closed. Hence M_1 is compact and its boundary consists of two spheres. It then follows that $M_1 \xrightarrow{p} M$ is a finite covering. So $\pi_1(M)$ is finite. This is a contradiction. Thus $\pi_1(M)$ is torsion free. Q.E.D

2.2 Proof of the open case

In this section we give the proof of Theorem 1.2. Our argument follows closely that of Schoen-Yau [SCY]. The strategy is to prove that $\pi_1(M) = \pi_2(M) = 0$. Since M is open and of dimesion three, thus $H_k(M) = 0$, for all $K \geq 3$. By the Hurewicz Theorem, we have $\pi_k(M) = 0$ for all $k \geq 1$. Hence M is contractible by the Whitehead Theorem.

Let us first prove $\pi_2(M) = 0$. If $\pi_2(M) \neq 0$, then $\pi_2(\widetilde{M}) \neq 0$, where $\widetilde{M}$ is the universal covering space of M, the Sphere Theorem in three dimensional topology says that there exists an embedded S^2 in $\widetilde{M}$ which is not homotopically trivial. If $\widetilde{M} \setminus S^2$ were connected, we could take a loop in $\widetilde{M}$ intersecting S^2 at exactly one point. This loop could not be null-homotopic. This would contrdict $\pi_1(\widetilde{M}) = \{e\}$. Thus S^2 divides $\widetilde{M}$ into two connected components. By Van Kampen's theorem, each component is simply connected. If one of these were compact, then since S^2 is an trivial element in H_2 of the compact set, thus by the Hurewicz theorem it is trivial in π_2. This is a contradiction. Therefore, S^2 divides $\widetilde{M}$ into two noncompact components, now the Cheeger-Gromoll Splitting Theorem ([CG]) implies that $\widetilde{M}$ is a product of a line and a compact two manifold Σ. Let $T_r(\Sigma)$ be the r tubular neighbourhood of Σ of radius r. then $Vol(T_r(\Sigma)) = r \cdot Vol(\Sigma)$. It is easy to see that the volume growth condition in Theorem 1.2 is satisfied for any point. We can thus assume $p \in \Sigma$. Then it follows that $Vol(B_p(r)) \leq Vol(T_{r-Diam(\Sigma)}(\Sigma)) = Vol(\Sigma) \cdot (r - Diam(\Sigma)) \leq r^2$, for r big enough. This contradicts our assumption on the volume growth. Hence $\pi_2(M) = 0$.

Since $\dim(M) = 3$ and M is open, thus $H_k(M) = 0$ for $k \geq 3$. By the Hurewicz theorem, all higher homotopy groups of M vanish. Therefore M is a $K(\pi, 1)$ space, and $H^i(\pi_1(M)) = H^i(M) = 0$, for $i \geq 3$. Since infinitely many cohomology groups of a finite cyclic group are nonzero, hence $\pi_1(M)$ is torsion free.

We now prove that $\pi_1(M)$ is trivial. By passing to a covering space of M, we may assume $\pi_1(M) = \mathbf{Z}$. By using a volume comparison argument like the one in [An] we will show this is impossible. Fix a point $p \in M$ and $\tilde{p} \in \widetilde{M}$, such that $\pi(\tilde{p}) = p$, where π is the covering map. Let σ be a geodesic loop at p representing a generator for $\pi_1(M, p)$ and F be a fundamental domain of M containing $\tilde{p}$. Then it is obvious that

$$\bigcup_{k=1}^{N} [\sigma]^k (F \bigcap B_{\tilde{p}}^{\widetilde{M}}(r)) \subset B_{\tilde{p}}^{\widetilde{M}}(N \cdot L(\sigma) + r),$$

where $L(\sigma)$ is the length of σ. Notice that π is volume preserving when restricted to F. Then using $Vol([\sigma](F) \bigcap F) = 0$, we obtain,

$$\begin{aligned} N \cdot Vol(B_p(r)) &= N \cdot Vol(F \bigcap B_{\tilde{p}}^{\widetilde{M}}(r)) \\ &= Vol(\bigcup_{k=1}^{N} [\sigma]^k (F \bigcap B_{\tilde{p}}^{\widetilde{M}}(r))) \\ &\leq Vol(B_{\tilde{p}}^{\widetilde{M}}(N \cdot L(\sigma) + r)) \\ &\leq \frac{4}{3}\pi (N \cdot L(\sigma) + r)^3 \quad (\text{since} \quad Ric \geq 0). \end{aligned}$$

Choosing $N \geq [\frac{32\pi}{6c}]$, and $r \geq N \cdot L(\sigma)$, we have

$$Vol(B_p(r)) \leq \frac{4\pi}{3N} \cdot (2r)^3 \leq \frac{c}{2} r^3.$$

This is a contradiction. Thus, $\pi_1(M) = \{e\}$. Therefore all homotopy groups of M vanish. We hence conclude that M is contractible by the Whitehead theorem.

Q.E.D

2.3 Proof of the finiteness theorem

Let us denote by $\mathcal{M}(n)$ the class of n-dimensional manifolds satisfying the bounds: $Ric \geq -(n-1)H$, $Diam \leq D$, $Vol \geq V$. As pointed out in the introduction, the crucial step towards a finiteness theorem is to get a control of the local topology. For the class $\mathcal{M}(3)$, this takes the form of a lower bound on the geometric contractibility radius. By the examples of Sha-Yang and Anderson [An], such a bound does not exist for $\mathcal{M}(n)$ when $n > 3$.

We first define the geometric contractibility radius (of relative size R).

$$C_R(M) = \inf_{p \in M} \sup\{r \mid B_p(r) \text{ is contractible in } B_p(R \cdot r)\}.$$

The crucial step in proving Theorem 1.1 is the following proposition.

Proposition 2.1 *There exist constants R, r_0 depending only on H, D, V, such that*

$$C_R(M) \geq r_0$$

for any $M \in \mathcal{M}(3)$.

We begin the proof with a few lemmas. These lemmas hold for all dimensions. The restriction to dimsension three is only needed at the end of the proof.

Lemma 2.2 *There exist constants C_1, C_2 and d depending only on n, H, D, V, such that, for any $M \in \mathcal{M}(n)$, $p \in M$, we have,*

$$C_1 r^n \leq Vol(B_p(r)) \leq C_2 r^n, \quad 0 \leq r \leq D,$$

and

$$Diam_p(M) \geq d.$$

where $Diam_p(M) = \sup\{d(p,q) \mid q \in M\}$.

Proof. By the Bishop volume comparison theorem,

$$\begin{aligned} Vol(B_p(r)) &\leq Vol^H(B(r)) \\ &= \int_0^r \left(\frac{sinh\sqrt{H}t}{\sqrt{H}} \right)^{n-1} dt \\ &\leq C_2 r^n, \quad 0 \leq r \leq D, \end{aligned}$$

where $C_2 = \sup_{0\leq r\leq D} \frac{1}{r^n}\int_0^r \left(\frac{sinh\sqrt{H}t}{\sqrt{H}}\right)^{n-1} dt.$

Similarly, by the Bishop-Gromov relative volume comparison theorem, we obtain,

$$\begin{aligned} Vol(B_p(r)) &\geq \frac{Vol^H(B(r))}{Vol^H(B(D))}\cdot Vol(B_p(D)) \\ &\geq \frac{V}{Vol^H(B(D))}\cdot \int_0^r \left(\frac{sinh\sqrt{H}t}{\sqrt{H}}\right)^{n-1} dt \\ &\geq C_1 r^n, \end{aligned}$$

where $C_1 = \frac{V}{Vol^H(B(D))}\cdot \inf_{0\leq r\leq D} \frac{1}{r^n}\int_0^r \left(\frac{sinh\sqrt{H}t}{\sqrt{H}}\right)^{n-1} dt.$

For the diamter, since

$$V \leq Vol(M) \leq C_2(Diam_p(M))^n,$$

hence

$$Diam_p(M) \geq (\frac{V}{C_2})^{\frac{1}{n}}.$$

Q.E.D

Lemma 2.3 *There exist constants $R_1(n,H,D,V), r_1(n,H,D,V)$ such that for any $M^n \in \mathcal{M}(n,H,D,V), p \in M$ and $s \leq r_1$, $B_p(R_1\cdot s)\setminus B_p(s)$ has at most one component whose intersection with $\partial B_p(R_1\cdot s/3)$ is nonempty.*

Proof. We prove this by contradiction. Let C_1 and C_2 be two such components. We can assume without loss of generality that

$$Vol(C_1 \bigcap B_p(R\cdot s/3)) \leq Vol(C_2 \bigcap B_p(R\cdot s/3)).$$

Thus,

$$Vol(B_p(R\cdot s/3)\setminus B_p(s)) \le 2Vol(B_p(R\cdot s/3)\setminus (C_1\bigcap B_p(R\cdot s/3))).$$

Take $Q_1 \in C_1 \bigcap \partial B_p(R\cdot s/3)$. Since every minimal geodesic γ with $\gamma(l) \in B_p(R\cdot s/3)\setminus B_p(s)$ satisfies $l \le \frac{2}{3}R\cdot s$, and $\gamma(t)\in B_p(s)$ for some t satisfying $\frac{1}{3}R\cdot s - s \le t \le \frac{1}{3}R\cdot s + s$. Thus,

$$B_p(\frac{1}{3}R\cdot s)\setminus (C_1\bigcap B_p(\frac{1}{3}R\cdot s)) \subset T_{\frac{1}{3}R\cdot s-s,\frac{2}{3}R\cdot s}(Q_1),$$

where T_{r_1,r_2} is the annulus of radius r_1 and r_2. The triangle inequality implies,

$$T_{\frac{1}{3}R\cdot s-s,\frac{1}{3}R\cdot s+s}(Q_1) \subset B_p(3s).$$

Therefore,

$$\begin{aligned}
\frac{Vol(B_p(R\cdot s/3)\setminus B_p(s))}{Vol(B_p(3s))} &\le 2\frac{Vol(B_p(R\cdot s/3)\setminus (C_1\bigcap B_p(R\cdot s/3)))}{Vol(B_p(3s))}\\
&\le 2\frac{Vol(T_{\frac{1}{3}R\cdot s-s,\frac{2}{3}R\cdot s}(Q_1))}{Vol(T_{\frac{1}{3}R\cdot s-s,\frac{1}{3}R\cdot s+s}(Q_1))}\\
&\le 2\frac{Vol^H_{\frac{1}{3}R\cdot s-s,\frac{2}{3}R\cdot s}}{Vol^H_{\frac{1}{3}R\cdot s-s,\frac{1}{3}R\cdot s+s}}\\
&\le C_3(n,H,D)\cdot R,
\end{aligned}$$

where we have denoted by $Vol^H_{r_1,r_2}$ the volume of an annulus of radius r_1 and r_2 in the space of constant curvature $-(n-1)H$. Together with Lemma 2.2, the above implies,

$$\frac{C_1\cdot (R/3)^n - C_2}{C_2\cdot 3^n} \le C_3\cdot R.$$

This is impossible if we choose $R(n,H,D,V)$ big enough. In the proof, we also need that $s\cdot R\le d$. Thus $s \le \frac{d}{R_1} = r_1$. Q.E.D

Lemma 2.4 *There are constants R_2, r_2 and N depending only on n, H, D, V, such that for any $M \in \mathcal{M}(n), p \in M, r \leq r_2$, if $I : B_p(r) \to B_p(R \cdot r)$, then any subgroup G of $I_*(\pi_1(B_p(r)))$ satisfies,*

$$order(G) \leq N.$$

In particular, there is no element of infinite order in $I_*(\pi_1(B_p(r)))$ whenever $r \leq r_2$.

Proof. This is basicly the same as in [An] or as in the proof of Theorem 1.2. But let us point out that we are working with a metric ball, which is not complete. Hence its universal covering space with the pulled back metric is also not complete. Since we need to use the Bishop volume estimate for geodesic balls, we have to show it is still valid in this case. This turns out to be fairly easy in our situation since we are working with a relative version. Namely, although $\widetilde{B_p(r)}$ is not complete, in the universal covering space of a bigger ball $B_p(R_2 \cdot r)$, $B_{\tilde{p}}(Nr)$ ($N << R_2$) is a usual metric ball, and hence Bishop's volume estimate still holds. We will address the problem in the proof.

It is a well known fact that we can choose a set of generators $\{[\sigma_i]\}$ for $I_*(\pi_1(B_p(r)))$, such that length$(\sigma_i) \leq 2r$ and there is a bound on the number of generators [We2], say $k(n, D, V, H)$. Let V be the universal covering space of $B_p(R \cdot r)$ with the pulled back metric. Pick $\tilde{p} \in V$, such that $\pi(\tilde{p}) = p$. let F be a fundamental domain of the covering with $\tilde{p} \in F$. Denote $U(m) = \{$element of G of word length $\leq m\}$. It is easy to see that $\sharp U(m) \geq m$ unless $U(m) = G$. (This is because $\sharp U(m+1) > \sharp U(m)$ unless $U(m) = G$). Let $m_0 = [(\frac{3}{2})^n \cdot \frac{C_2}{C_1}] + 1$. Consider,

$$B = \bigcup_{g \in U(m_0)} g(F \bigcap \pi^{-1}(B_p(2m_0 r))).$$

Take any point $x \in B$, and a curve γ from x to $\tilde{p}$. Then $\pi(\gamma)$ is a curve from $\pi(x) \in B_p(2m_0 r)$ to p. Let σ be a minimal geodesic in the homotopy class of $\pi(\gamma)$ keeping the end points fixed. Then length$(\sigma) \leq m_0 \cdot \sup_i\{\text{length}(\sigma_i)\} + m_0 \cdot r \leq 3m_0 r$. If we choose $R_2 = 6m_0$, then σ is a smooth geodesic. Lift σ to V, we get a smooth geodesic from $\tilde{p}$ to x. What we have proved is that any point in B can be joined to $\tilde{p}$ by a smooth geodesic of length $\leq 3m_0 r$. It thus follows from the proof of Bishop volume comparison theorem that;

$$Vol(B) \leq Vol^H(3m_0 r).$$

If order$(G) > \sharp U(m_0)$, then,

$$\begin{aligned} m_0 Vol(B_p(2m_0 r)) &\leq \sharp U(m_0) Vol(F \bigcap \pi^{-1}(B_p(r))) \\ &= Vol(B) \leq Vol^H(3m_0 r). \end{aligned}$$

Therefore,

$$m_0 \leq \frac{Vol^H(3m_0 r)}{Vol(B_p(2m_0 r))}.$$

Let $r_2 = \frac{D}{6m_0}$. Then for any $r \leq r_2$, we have $3m_0 r < D$. It thus follows from Lemma 2.2 that

$$m_0 \leq \frac{C_2(3m_0 r)^n}{C_1(2m_0 r)^n} = (\frac{3}{2})^n \cdot \frac{C_2}{C_1}.$$

This contradicts the choice of m_0. Thus order$(G) \leq \sharp U(m_0) \leq k^{m_0} = N$.

Q.E.D

Lemma 2.5 *Let K be a compact Riemannian manifold and $\widetilde{K}$ a $k-$fold covering of K with the pulled back metric. Then*

$$Diam(\widetilde{K}) \leq 2k \cdot Diam(K).$$

Proof. We denote by Γ the group of deck transformations, $\sharp\Gamma = k$. Fix a point $p \in K$, and $\tilde{p} \in \widetilde{K}$ such that $\pi(\tilde{p}) = p$. Let F be the Dirichlet fundamental domain of the covering, that is,

$$F = \left\{ x \in \widetilde{K} \mid d(x, \tilde{p}) \leq d(\gamma x, \tilde{p}), \text{for any } \ \gamma \in \Gamma \right\}.$$

We first show that for any $x \in F$, $d(x, \tilde{p}) \leq \mathrm{Diam}(K)$. Indeed, let σ be a minimal geodesic from $\tilde{p}$ to x, with $\sigma(l) = x$. Then $\pi \circ \sigma$ is a curve from p to $\pi(x)$ with $\mathrm{length}(\pi \circ \sigma) = \mathrm{length}(\sigma) = l$. If $l > \mathrm{Diam}(K)$, there exists a curve α from p to $\pi(x)$ with length $l_1 \leq \mathrm{Diam}(K) < l$. Lift α to $\widetilde{K}$ with $\alpha(0) = \tilde{p}$. Then $d(\alpha(l_1), \tilde{p}) \leq l_1 < l$. But $\pi(\alpha(l_1)) = \pi(x)$, so $\alpha(l_1) = \gamma x$ for some $\gamma \in \Gamma$. This contradicts the definition of F. Hence $d(x, \tilde{p}) \leq \mathrm{Diam}(K)$.

Now for any two points x and y in $\widetilde{K}$, let γ be a curve connecting them. $\widetilde{K}$ is the union of k Dirichlet fundamental domains with centers at $\pi^{-1}(p)$. Since ∂F has measure zero, we can choose the curve γ such that it has the property that $\gamma \bigcap \partial F$ has no accumulation points. Thus $\gamma \bigcap \partial F$ is a finite set. Therefore for each fundamental domain F, we can pick the point where γ first enters F and the point where γ last leaves F, say at $\gamma(t_1)$ and $\gamma(t_2)$. We can replace the segment $\gamma([t_1, t_2])$ by a curve from $\gamma(t_1)$ to

$\tilde{p}$ and then from $\tilde{p}$ to $\gamma(t_2)$. The previous paragraph shows that we can choose this curve to have length $\leq 2\text{Diam}(K)$, Continue this process we get a curve from x to y which intersects each fundamental domain only once and inside each fundamental domain it has length at most $2\text{Diam}(K)$. Thus, $d(x, y) \leq 2k \cdot \text{Diam}(K)$. Therefore $\text{Diam}(\widetilde{K}) \leq 2k \cdot \text{Diam}(K)$.

Q.E.D

Remark. In the statement of Lemma 2.5, we assumed that K is a Riemannian manifold. From the proof we see that the same statement holds for a much larger class of objects. In particular, it holds for compact (smooth) metric balls $B_p(r) \subset (M, g)$. In the proof of Proposition 2.1, we will use lemma 2.5 in this form.

Proof of Proposition 2.1.

For any $M \in \mathcal{M}(3), p \in M$, consider the inclusion $I : B_p(r) \to B_p(R \cdot r)$. The precise value of R will be determined in the proof. Just as in the proof of Theorem 1.2, we first show that I induces trivial maps on π_2 and on π_1.

For this part, we have to distinguish between the orientable case and the nonorientable case. The arguments are along the same line with some difference in details. Let us briefly summarize it here. What we will actually show is that either $\pi_2(B_p(r)) = 0$ or a nontrivial element in $\pi_2(B_p(r))$, which is represented by an embedded S^2 or RP^2 according to orientability, divides $B_p(R \cdot r)$ into two parts, one part is compact and simply conneted. Thus, for the orientable case, this implies that I is trivial on π_2. For the nonorientable case, this implies that $\pi_2(B_p(r)) = 0$. Either case implies $\pi_1(B_p(r))$ is torsion free (the orientable case follows from Lemma 2.1 and the nonorientable case follows from group homology). Then the conclusion that I is trivial on π_1 follows immediately from Lemma 2.4.

In what follows, we first treat the orientable case.

If I is not trivial on π_2, by the sphere theorem in three dimensional topology, there is an (smoothly) embedded S^2 in $B_p(r)$, representing a nontrivial homotopy class in $B_p(R \cdot r)$. There are three possibilities we have to consider.

Case 1. S^2 does not separate $B_p(R \cdot r)$. From standard three dimesional topology (Lemma 3.8 in [He]), we have the decomposition $B_p(R \cdot r) = V_1 \sharp V_2$, where V_1 is a two sphere bundle over S^1. Hence there is an element $[\sigma] \in \pi_1(V_1)$ of infinity order and σ is contained in $B_p(R \cdot r)$. Since $B_p(R^2 \cdot r) = V_1 \sharp V_3$ for some manifold V_3, σ is also an element of infinite order in $B_p(R^2 \cdot r)$. This is impossible by Lemma 2.4. For this case we require that $R^2 \cdot r \leq d/2, R \geq R_2, R \cdot r \leq r_2$, where r_2, R_2, d are the constants in Lemma 2.2 and 2.4.

Case 2. S^2 separates $B_p(R \cdot r)$ into two components, both of them have nontrivial intersection with $\partial B_p(R \cdot r)$. This is impossible by Lemma 2.3.

For this to work we require that $R \geq 3R_1, r \leq r_1$, where R_1, r_1 are the constants in Lemma 2.3.

Case 3. S^2 separates $B_p(R \cdot r)$ into two connected components, one of them, M_1, has nontrivial intersection with $\partial B_p(R \cdot r)$, he other, M_2, is compact with $\partial(M_2) = S^2$. Hence $B_p(R \cdot r) = M_1 \sharp M_2$. Let us note that M_2 can not be simply connected. Otherwise the S^2 would be contractible, contradicting our assumption. Hence $\pi_1(M_2)$ is nontrivial and, because of the connected sum decomposition, the inclusion into $\pi_1(B_p(R \cdot r))$ is injective. Since the bigger ball $B_p(R^2 \cdot r)$ is also a connected sum of M_2 and another manifold, we conclude that $\pi_1(M_2)$ is also injectively included in $\pi_1(B_p(R^2 \cdot r))$. Notice that $M_2 \subset B_p(R \cdot r)$ (this is why we have to consider the bigger ball $B_p(R^2 \cdot r)$). By Lemma 2.4, the order of $\pi_1(M_2)$ is bounded by N. Consider the covering space K of $B_p(R^2 \cdot r)$ as follows. First take the universal covering space $\widetilde{M_2}$ of M_2, then glue $B_p(R^2 \cdot r) \setminus M_2$ to each lifting of S^2, denote the resulting space as K. Thus the deck transformation group of this covering is $\pi_1(M_2)$. It is obvious from this description that $\widetilde{M_2}$ seperates K into $\sharp\pi_1(M_2)$ components. Now by Lemma 2.5(see the remark after it), $\text{Diam}(T) \leq 2N \cdot \text{Diam}(M_1) \leq 2N \cdot 2Rr$. This is again impossible according to Lemma 2.3. For this part we need $R \geq 4N, R \geq R_2, R \cdot r \leq r_2$ and $R^2 \cdot r \leq d/2$.

Thus, we have proved that if $I : B_p(r) \to B_p(R \cdot r)$, then I_* is trivial on π_2 whenever $R \geq \max\{3R_1, R_2, 4N\}$ and $r \leq \min\{r_1, \frac{r_2}{R}, \frac{d}{2R^2}\}$.

We now show that I_* is trivial on π_1. This is now very easy. In fact, consider the inclusions $B_p(r) \subset B_p(R \cdot r) \subset B_p(R^2 \cdot r)$. From the previous paragraph, if we choose r smaller, say $r \le \min\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\}$, then the second inclusion $B_p(R \cdot r) \subset B_p(R^2 \cdot r)$ satisfies the condition above , hence this inclusion induces a trivial map on π_2. It now follows from Lemma 2.1 that $\pi_1(B_p(R \cdot r))$ is torsion free. Thus, if $I_* : B_p(r) \to B_p(R \cdot r)$ were not trivial on π_1, there would be an element of $\pi_1(B_p(r))$ which is nontrivial in $\pi_1(B_p(R \cdot r))$, hence is necessarily of infinite order in $\pi_1(B_p(R \cdot r))$ since the later is torsion free. This is impossible by Lemma 2.4. Therefore I_* is trivial on π_1.

We have thus proved that for the orientable case, if

$$R \ge \max\{3R_1, R_2, 4N\}, \quad r \le \min\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\},$$

then $I_* : B_p(r) \to B_p(R \cdot r)$ is trivial on π_1 and π_2.

Now we consider the case when M is not orientable. Consider,

$$B_p(r) \xrightarrow{i_1} B_p(R \cdot r) \xrightarrow{i_2} B_p(R^2 \cdot r).$$

We can assume at least one of the three sets are nonorientable. Otherwise we are in a situation we just dealed with. Furthermore, if $B_p(r)$ is orientable, we can consider the following inclusions,

$$B_p(r/R^2) \xrightarrow{j_1} B_p(r/R) \xrightarrow{j_2} B_p(r).$$

We are then in the orientable case. If this happens, we can just choose r smaller. This will not effect our result (we will take this into consideration when choosing R, r). Now we assume that $B_p(r)$ is nonorientable. Therefore all three sets involved are nonorientable.

we consider the first inclusion i_1. We will show $\pi_2(B_p(r)) = 0$. Let us point out here that for this to be true we need the nonorientability, since there are strong topological restrictions on nonorientable three manifolds. We again prove this by contradiction, along the same line as in the orientable case. If $\pi_2(B_p(r))$ is not trivial, then by the projective plane theorem (which is the nonorientable version of the sphere theorem, Theorem 4.12 in [He]), There is an embedded RP^2 in $B_p(r)$. Again, we need to consider three cases, each will lead to a contradiction.

Case 1. RP^2 does not seperate $B_p(R \cdot r)$. We consider the double covers of $B_p(r)$ and $B_p(R \cdot r)$ with the pulled back metric, denoted by $\widetilde{B_p(r)}$ and $\widetilde{B}_p(R \cdot r)$ respectively. Since the double cover of a nonorientable manifold can be constructed as the unit sphere bundle of the determinant bundle, there is a natural lift $\widetilde{i_1}$ of i_1, so that the following diagram commuts.

$$\begin{array}{ccc} \widetilde{B_p(r)} & \xrightarrow{\widetilde{i_1}} & \widetilde{B}_p(R \cdot r) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ B_p(r) & \xrightarrow{i_1} & B_p(R \cdot r) \end{array}$$

here $\widetilde{i_1}$ is again an inclusion. Now $\widetilde{B_p(r)}$ and $\widetilde{B}_p(R \cdot r)$ are subsets in the double cover of $\widetilde{M}$ which is orientable. Note that $\widetilde{B_p(r)} \subset B_{\widetilde{p}}^{\widetilde{M}}(4r)$, $\widetilde{B}_p(R \cdot r) \supset B_{\widetilde{p}}^{\widetilde{M}}(R \cdot r)$ and the previous argument showed that $B_{\widetilde{p}}^{\widetilde{M}}(4r) \to B_{\widetilde{p}}^{\widetilde{M}}(R \cdot r)$ induces trivial maps on π_1 and π_2. Thus $\widetilde{i_1}$ induces trivial maps on π_1 and π_2. Let $\pi_1{}^{-1}(RP^2) = S^2$. If this S^2 does not separate $\widetilde{B_p}(R \cdot r)$, then there is a closed curve in $\widetilde{B_p}(R \cdot r)$ intersecting S^2 at only one point. This implies from the Poincaré duality that S^2 is a nontrivial element in $\pi_2(\widetilde{B_p}(R \cdot r))$. This contradicts the fact that $\widetilde{i_1}$ induces a trivial map on π_2. If the S^2 separates $\widetilde{B_p}(R \cdot r)$, then both the two components necessarily have nontrivial intersections with $\partial \widetilde{B_p}(R \cdot r)$. In fact, if one component is compact with S^2 as its boundary(namely, does not intersect $\partial \widetilde{B_p}(R \cdot r)$), then projecting it down, we get $B_p(R \cdot r)$ as the union of a compact set and a noncompact set having RP^2 as the common boundary. This means that RP^2 separates $B_p(R \cdot r)$. This contradicts the assumption. Thus S^2 separates $B_p(\widetilde{R \cdot r})$ into two components both having nontrivial intersection with $\partial \widetilde{B_p}(R \cdot r)$. This is impossible by Lemma 2.3.

Case 2. RP^2 separates $B_p(R \cdot r)$ into two components both having nontrivial intersection with $\partial B_p(R \cdot r)$. This is impossible by Lemma 2.3.

Case 3. RP^2 separates $B_p(R \cdot r)$ into two components, one of them has nonempty intersection with $\partial B_p(R \cdot r)$, the other, denoted by V, is compact with boundary RP^2. We consider two cases separately.

The first case is when $\pi_1(V)$ is finite. Since V is nonorientable, it follows from the topology of three manifolds that ∂V consists of two RP^2's ([He] page 77(i)). This contradicts that $\partial V = RP^2$.

The second case is when $\pi_1(V)$ is infinite. As before, we have the following commuting diagram.

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\tilde{i}} & \widetilde{B_p}(R\cdot r) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ V & \xrightarrow{i_1} & B_p(R\cdot r) \end{array}$$

Then $\partial\widetilde{V} = S^2$ and this S^2 separates $\widetilde{B_p}(R\cdot r)$ into a compact $\widetilde{V}$ and a manifold K which has nontrivial intersection with $\partial\widetilde{B_p}(R\cdot r)$. Since $\widetilde{B_p}(R\cdot r) = \widetilde{V}\sharp K$, then $\widetilde{B_p}(R^2\cdot r) = \widetilde{V}\sharp K_1$ for some K_1. Thus, the inclusion $\pi_1(\widetilde{V}) \to \pi_1(B_p^{\widetilde{M}}(R^2\cdot r))$ is injective. Note that $\widetilde{V} \subset B_p^{\widetilde{M}}(2Rr)$ and by assumption $\pi_1(\widetilde{V})$ is infinite. This is impossible by lemma 2.4

We thus proved that if $\pi_2(B_p(r)) \neq 0$, then we will get a contradiction in all three cases. Hence $\pi_2(B_p(r)) = 0$. Therefore $B_p(r)$ is a $K(\pi,1)$ space. It follows that $\pi_1(B_p(r))$ is torsion free(see the argument in the proof of Theorem 1.2 on page 16). The same argument shows that $\pi_1(B_p(R\cdot r))$ is torsion free. Then $\pi_1(B_p(r)) \xrightarrow{(i_1)_*} \pi_1(B_p(R\cdot r))$ is a trivial map, otherwise it would contradict Lemma 2.4.

Let us summarize the nonorientable case. We have proved that if

$$R \geq 4\max\{3R_1, R_2, 4N\}, \quad r \leq \min\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\},$$

then either $B_p(r) \to B_p(R^2\cdot r)$ is trivial on π_1 and π_2(This happens when both balls are orientable or both are nonorientable), or $B_p(\frac{r}{R^2}) \to B_p(r)$ is trivial on π_1 and π_2. Thus the composition of the two inclusions,

$$B_p(\frac{r}{R^2}) \longrightarrow B_p(r) \longrightarrow B_p(R^2\cdot r)$$

always induces trivial maps on π_1 and π_2.

Thus, we have proved that, no matter M is orientable or not, $B_p(r) \xrightarrow{I} B_p(R \cdot r)$ induces trivial maps on π_1 and π_2 when

$$R \geq (4\max\{3R_1, R_2, 4N\})^4, \quad r \leq r_0 \leq \min\{\frac{r_1}{R}, \frac{r_2}{R^2}, \frac{d}{2R^3}\}.$$

We now show that for $r \leq \frac{r_0}{R}$, $B_p(r)$ is contractible in $B_p(R^2 \cdot r)$. In fact, consider the two inclusions,

$$B_p(r) \xrightarrow{i_1} B_p(R \cdot r) \xrightarrow{i_2} B_p(R^2 \cdot r).$$

From the condition on r, i_1 and i_2 both induce trivial maps on π_1 and π_2. Take a smoothing ρ_* of the distance function ρ, and consider a regular value c of ρ_* such that $\rho_*^{-1}([0, c]) \supset B_p(r)$. Then $\rho_*^{-1}([0, c])$ is a smooth three manifold with non-empty boundary. A well known theorem in Morse theory (Theorem 23.5 in [MC]) implies that $\rho_*^{-1}([0, c])$ has the homotopy type of a two-dimensional CW complex. Thus $B_p(r)$ also has the homotopy type of a two-dimensional CW complex. The same is true for $B_p(R \cdot r)$ and $B_p(R^2 \cdot r)$. Proposition 2.1 (with $R = [4\max\{3R_1, R_2, 4N\}]^8, r_0 = \min\{\frac{r_1}{R^2}, \frac{r_2}{R^3}, \frac{d}{2R^4}\}$) is an immediate consequence of the following lemma.

Q.E.D

Lemma 2.6 *Let X, Y, Z be two dimensional CW complexes and f, g continous maps,*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

such that f induces a trivial map on π_1 and g induces a trivial map on π_2. Then $g \circ f$ is homotopic to a constant map.

Proof. Since f is trivial on π_1, we have the lifting $\tilde{f}$ such that the following diagram commuts.

$$\begin{array}{ccccc} & & \widetilde{Y} & & \\ & \tilde{f} \nearrow & \downarrow \pi & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where $\widetilde{Y}$ is the universal covering space of Y. It thus suffices to prove that $g \circ \pi \circ \tilde{f}$ is homotopic to a constant map. Denote $\psi = g \circ \pi : \widetilde{Y} \to Z$. Then,

$$\psi_*(\pi_2(\widetilde{Y})) = g_* \circ \pi_*(\pi_2(\widetilde{Y})) = g_*(\pi_2(Y)) = e.$$

that is , ψ is trivial on π_2. We now show that ψ is homotopic to a constant map. Since $\widetilde{Y}$ is a two dimensional CW complex which is simply connected, by Corollary 3.6 on page 221 of [Wh], $\widetilde{Y}$ is homotopy equivalent to the wedge of S^2's, $\widetilde{Y} = S^2 \vee \cdots \vee S^2$. Each of these S^2 represents an element in $\pi_2(\widetilde{Y})$. Since ψ is trivial on $\pi_2(\widetilde{Y})$, it follows that ψ, when restricted on each S^2, is homotopic to a constant map. Therefore ψ is homotopic to a constant map. Hence $g \circ f$ is homotopic to a constant map.

Q.E.D

Proof of Theorem 1.1 The argument from Proposition 2.1 to Theorem

1.1 is somewhat formal. It is essentially the same for all types of finiteness theorems, namely, a center of mass argument. The observation is that $\mathcal{M}(n)$ is precompact with respect to the Hausdoff distance. And for two manifolds which are Hausdorff close, and geometrically contractible in the sense of Proposition 2.1, we can construct a map between them which is a homotopy equivalence. This can be easily seen from the point of view of obstruction theory. Proposition 2.1 garantees that there is no obstruction for extending maps. We can thus start constructing the map skeleton-wise. The detail is carried out by P. Petersen in [Pet]. This completes our proof of Theorem 1.1 Q.E.D

Remark. Proposition 2.1 in actually more than what we need to conclude Theorem 1.1. In fact, the statement that I is trivial on π_1 and π_2 is enough to imply Theorem 1.1. For this see P. Petersen [Pet].

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