Smooth Extensions for Finite CW Complexes and the Index Theory

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Guihua Gong

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Ronald Douglas, Professor of Mathematics
Dissertation Director

Chih-Han Sah, Professor of Mathematics
Chairman of Defense

John Spielberg, Assistant Professor of Mathematics

Johan Dupont, Professor of Mathematics
Outside Member
Aarhus University, Denmark

This dissertation is accepted by the Graduate School.

Alexander H. King

Graduate School

Abstract of Dissertation

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The C^*-algebra extensions of a topological space X can be made into an abelian group $\mathrm{Ext}(X)$ which is naturally equivalent to the K-homology group of odd dimension which has a close relation with index theory and is one of the starting points of KK theory.

The C_p-smoothness of an extension of a manifold was introduced by Douglas and is one source of the motivation of Connes' non-commutative geometry. In this thesis, we generalize the notion of C_p-smoothness to a finite CW complex and obtain necessary and sufficient conditions for an extension of a finite CW complex to be C_p-smooth modulo torsion.

Let X be a compact metrizable space and $\tau \in \mathrm{Ext}(X)$ is defined by a unital $*$ monomorphism $\tau : C(X) \to Q(H)$, where $Q(H)$ is Calkin algebra of the infinite dimensional complex separable Hilbert space H. For any n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$ which satisfies $|f_1(x)|^2 + |f_2(x)|^2 + \cdots + |f_n(x)|^2 \neq 0$ for all $x \in X$, we can study the index $(\tau(f_1), \tau(f_2), \cdots, \tau(f_n))$ associated to the n-tuple, where the index is Curto's index for the Fredholm n-tuple. It is easy to see that the index will be zero for an n-tuple of $(f_1, f_2, \cdots, f_n)$ whenever τ is a torsion element in $\mathrm{Ext}(X)$. In this thesis, we prove that the converse is true for X being finite CW complex.

To my wife Liangqing and my daughter Sherry

Contents

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Introduction

Let H be an infinite dimensional complex separable Hilbert space. By $L(H)$ and $K(H)$ we shall denote the C^*-algebras of bounded operators and compact operators respectively on H, and $Q(H)$ will denote the quotient $L(H)/K(H)$ with canonical surjection $\pi : L(H) \to Q(H)$.

Let X be a compact metrizable space . By $C(X)$ we shall denote the C^*-algebra of continuous complex valued functions on X. An extension $\tau \in \mathrm{Ext}(X)$ of the algebra $C(X)$ by $K(H)$ is defined by a unital $*$ monomorphism $\tau : C(X) \to Q(H)$ [9] .

For two extensions τ_1 and τ_2 of $C(X)$ by $K(H)$, we say τ_1 and τ_2 are equivalent if there exists a unitary operator U on H such that $\tau_1(f) = \pi(U^*)\tau_2(f)\pi(U)$. By $\mathrm{Ext}(X)$ we shall denote the collection of equivalence classes of extensions of $C(X)$ by $K(H)$.

Let τ_1 and τ_2 be $*$ monomorphisms from $C(X)$ into $Q(H)$ and $a_1 = [\tau_1]$ and $a_2 = [\tau_2]$ denote the elements of $\mathrm{Ext}(X)$ they determine. Further, let $\rho : Q(H) \oplus Q(H) \to Q(H)$, be the map determined by the diagram

$$\begin{array}{ccccc} \mathrm{L(H)}\oplus\mathrm{L(H)} & \rightarrow & \mathrm{L(H}\oplus\mathrm{H)} & \xrightarrow{\nu} & \mathrm{L(H)} \\ \downarrow \pi\oplus\pi & & & & \downarrow \pi \\ \mathrm{Q(H)}\oplus\mathrm{Q(H)} & & \xrightarrow{\rho} & & \mathrm{Q(H)} \end{array}$$

where ν is induced by any unitary between $H \oplus H$ and H. Now if $\tau : C(X) \to Q(H)$ is the map defined by

$$\tau(f) = \rho(\tau_1(f) \oplus \tau_2(f))$$

for f in $C(X)$, then we set $a_1 + a_2 = [\tau]$. One can verify that $[\tau]$ does not depend on the choice of ν.

An extension $\tau : C(X) \to Q(H)$ is said to be trivial if there exists a unital $*$ monomorphism $\sigma : C(X) \to L(H)$ such that $\tau = \pi \circ \sigma$. The basic fact of C^*-algebra extension theory is that $\mathrm{Ext}(X)$ is an abelian group with the equivalence class of the trivial extension as the unit.

We refer to [14] and [7] for the basic theory of C^*-algebra extensions.

$\mathrm{Ext}(X)$ was introduced by Brown, Douglas and Fillmore in order to classify essentially normal operators up to unitary equivalence modulo the compact operators. It can be proved that $\mathrm{Ext}(X)$ is isomorhpic to K-homology $K_1(X)$ (defined by using Spanier-Whitehead duality). But $\mathrm{Ext}(X)$ has a close relation with the index theory of elliptic operators [1], [14] and is a starting point of KK-theory. Furthermore, this kind of functor can be defined for "non-commutative spaces" which is useful to give a new invariant for group actions and foliations [12].

Let $Gl_n(\mathbb{C})$ and $U(n)$ be the topological groups of n×n complex invertible matrices and n×n unitary matrices respectively. $K^1(X)$ is defined to be the collection of homotopy equivalence classes of maps from X to $\cup_{n=1}^{\infty} Gl_n(\mathbb{C})$ or maps from X to $\cup_{n=1}^{\infty} U(n)$ (note $Gl_n(\mathbb{C})$ and $U(n)$ are homotopy equivalent). Therefore an element in $K^1(X)$ can be represented by a map $\theta : X \to Gl_n(\mathbb{C})$ or $\theta : X \to U(n)$ for n large enough.

Now we will establish the pairing between $\mathrm{Ext}(X)$ and $K^1(X)$. Let $\tau \in \mathrm{Ext}(X)$ and $\theta \in K^1(X)$. First we note that $\theta : X \to Gl_n(C)$ can be regarded as an invertible element in $C(X) \otimes M_n$. Therefore $(\tau \otimes 1_n)(\theta)$ is an invertible element in $Q(\underbrace{H \oplus H \oplus \cdots \oplus H}_{n-copies})$. The pairing of τ and θ is defined by

$$\langle \tau,\ \theta \rangle = index((\tau \otimes 1_n)(\theta)) \in \mathbb{Z}.$$

Before studying $\mathrm{Ext}(X)$ and $K^1(X)$, some topological results will be proved in Chapter 1 which will be used in later chapters and are interesting in their own right. In particular, we prove the following useful theorem which enables us to reduce some problems from the general case to the case of spheres.

Theorem 1.2. *Let X be a compact metrizable space. For any $\tau \in K^1(X)$, there exist maps $f_i : X \to S^{2n_i-1}$ $(i = 1, 2, \cdots, k)$ such that $m\tau = \sum_{i=1}^{k} f_i^* \theta_i$ for some integer $m \neq 0$ and $\theta_i \in K^1(S^{2n_i-1})$.*

In[13], R.E.Curto defined an index for a Fredholm n-tuple of almost commuting operators. And he associated a matrix $A_{(T_1,T_2,\cdots,T_n)}$ to every n-tuple of operators $(T_1, T_2, \cdots, T_n)$ such that

$$index(T_1, T_2, \cdots, T_n) = index(A_{(T_1,T_2,\cdots,T_n)}).$$

Similarly, we can associate an $A_{(f_1,f_2,\cdots,f_n)} \in K^1(X)$ to each n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$ with

$$|f_1(x)|^2 + |f_2(x)|^2 + \cdots + |f_n(x)|^2 \neq 0 \qquad (*)$$

for all $x \in X$, which satisfies

$$\langle \tau,\ A_{(f_1,f_2,\cdots,f_n)} \rangle = index(\tau(f_1), \tau(f_2), \cdots, \tau(f_n))$$

for any $\tau \in \mathrm{Ext}(X)$.

It is easy to see that $index(\tau(f_1), \tau(f_2), \cdots, \tau(f_n))$ will be zero for any n-tuple of $(f_1, f_2, \cdots, f_n)$ satisfying $(*)$ whenever τ is a torsion element in $\mathrm{Ext}(X)$.

In Chapter 2, we will prove the following theorem

Theorem 2.2. *Let X be a finite CW complex and $\tau \in Ext(X)$ such that for each n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$ satisfying $(*)$, we have*

$$index(\tau(f_1), \tau(f_2), \cdots, \tau(f_n)) = 0.$$

Then τ must be a torsion element in $Ext(X)$.

The above theorem is equivalent to : For any $\hat{\theta} \in K^1(X)$, there exists an n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$ satisfying $(*)$ such that $m\hat{\theta} = A_{(f_1,f_2,\cdots,f_n)}$ for some integer $m \neq 0$.

In Chapter 3, we study the C_p-smooth extensions of finite CW complexes and finite CW complex pairs. The following definition of C_p-smoothness for smooth manifolds can be found in [11].

Definition 0.1. *Let M be a smooth compact manifold (perhaps with boundary) and let $C^\infty(M)$ denote the $*$-algebra of all smooth functions on M. A $\tau \in Ext(M)$ is C_p-smooth if there exists a $*$-linear map $\rho : C^\infty(M) \to$*

$L(H)$ such that $\rho(ab) - \rho(a)\rho(b) \in C_p$ and $\pi \circ \rho = \tau|_{C^\infty(M)}$.

The notion of C_p-smoothness was introduced by Douglas and is one source of the motivation for Connes non-commutative geometry.

It was shown in [21], [15] that the C_1-smooth elements of $\mathrm{Ext}(X)$ come from the 1-skeleton of X modulo torsion. And also it was shown in [16] that each C_{n-1}-smooth element of $\mathrm{Ext}(S^{2n-1})$ is trivial. The natural problem is to classify C_p-smooth extensions modulo torsion for a general CW complex.

In Chapter 3, we generalize Definition 0.1 to a finite CW complex and obtain the following theorem which solves the above problem. Especially, the results in [21], [15] and [16] are direct consequences of our theorem.

Theorem 3.2 & Theorem 3.4. *Let X be a finite CW complex, X^k denote the k-skeleton of X, and $\tau \in Ext(X)$. Then there exists an integer $m_1 \neq 0$ such that $m_1\tau$ is C_n-smooth if and only if there exists an integer $m_2 \neq 0$ such that $m_2\tau \in i_*(Ext(X^{2n-1}))$, where $i_* : Ext(X^{2n-1}) \to Ext(X)$ is induced by the inclusion map $i : X^{2n-1} \to X$. Furthermore, if X is a smooth compact (2n-1)-manifold, then each element in Ext(X) is C_p-smooth when $p > n - \frac{1}{2}$.*

Theorem 3.3. *Let X be a finite CW complex, $\tau \in Ext(X) = K_1(X)$ and $ch : K_1(X) \otimes \mathbb{Q} \to H_{odd}(X, \mathbb{Q})$ be the Chern map, where $H_{odd}(X, \mathbb{Q})$ denotes the direct sum of all the ordinary homology groups of odd dimension with rational coefficients. Then there exists an integer $m \neq 0$ such that $m\tau$*

is C_n-smooth if and only if $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, \mathbb{Q})$.

More generally , we also obtain similar results for the relative extension theory of finite CW complex pairs.

In Chapter 4, we study p-summable Fredholm modules of $C^\infty(M)$, which can be thought of as elements of $K_0(M) = KK(C(M), \mathbb{C})$, and their Chern characters in the cyclic cohomology $H_\lambda^*(C^\infty(M))$, where $H_\lambda^*(C^\infty(M))$ is an analogue of deRham homology theory obtained by first using algebra language which then can be generalized to non-commutative algebras. (See [11] for details.) We will say more about $H_\lambda^*(C^\infty(M))$ in Chapter 4. In particular, we prove the following theorem.

Theorem 4.2. *If M is a compact smooth manifold without boundary and $\phi \in H_\lambda^k(C^\infty(M))$ (k even), then there exist (k+1)-summable Fredholm modules τ_i ($i = 1, 2, \cdots, n$) and complex numbers α_i ($i = 1, 2, \cdots, n$) such that $\sum_{i=1}^{n} \alpha_i ch^* \tau_i \sim \phi$ in $H_\lambda^*(C^\infty(M))$, where ch^* is Connes' Chern map.*

We would like to point out that A. Connes constructed the graded Chern characters

$$ch^* : \{(n+1)\text{-summable Fredholm modules}\} \longrightarrow H_\lambda^n(C^\infty(M))$$

in Section 2 of [11], where n is an even integer, and that he also proved that

$$ch^* \text{: \{finite summable Fredholm modules\}} \longrightarrow H^{even}(C^\infty(\mathrm{M}))$$

is surjective modulo torsion. Theorem 4.2 says that the Chern map is a graded surjection.

Some of the results in Chapter 1, 3 and 4 have been anounced in [18].

In Chapter 5, we will give some remarks for the case of non-commutative algebras and raise some open problems.

Chapter 1

Some Topological Results

In this Chapter, we will prove some topological results. Theorem 1.1 will be used in defining C_p-smooth extension for general finite CW complexes. Theorem 1.2 will be used in proving our main results in Chapter 2 and Chapter 3. Theorem 1.3 will be used in Chapter 3 and Chapter 4.

Theorem 1.1 *If X is a finite CW complex, then there exists a compact smooth manifold M (perhaps with boundary), and two maps $f : X \to M$ and $g : M \to X$ such that $g \circ f$ is homotopic to $id|_X$.*

Using the following Proposition [19], we can reduce the proof of this theorem to the case of X being a simplicial complex.

Proposition 1.1 *Every CW complex has the homotopy type of a simplicial complex.*

Before proving Theorem 1.1, we prove the following Lemma.

Lemma 1.1 *For any finite simplicial complex X, there exists an embedding $i : X \to R^n$ (for some n) and an open neighborhood U of $i(X)$ such that $i(X)$ is a retract of U.*

Proof: Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of X. We can define $i : X \to R^n$ to be the piecewise linear map determined by

$$i(v_k) = (\underbrace{0, \cdots, 0}_{k-1}, 1, \underbrace{0, \cdots, 0}_{n-k}) \in R^n.$$

Let $Y = \{(a_1, a_2, \cdots, a_n) \in R^n | -2 \leq a_k \leq 2 \text{ for all } k\}$. It is easy to see that Y is a closed neighborhood of $i(X)$ in R^n. We can triangulate Y so that $i(X)$ is a subcomplex of Y. The result of (1.6) on p.50 of [30] says that $(Y, i(X))$ is an NDR pair. This means that there is an open neighborhood U_1 of X in Y such that X is a retract of U_1. If U_1 is not open in R^n, we can replace U_1 by a smaller open neighborhood U of X.

Q.E.D.

Proof of Theorem 1.1: By Lemma 1.1, without loss of generality, we can assume X is a closed subset of R^n with an open neighborhood U such that X is a retract of U.

Define a map $u : R^n \to R$ by $u(x) = d(x, X)$ (for all $x \in R^n$) which is the distance between the point x and the closed subset X. Then there exists an $\epsilon > 0$ such that $\{x \mid u(x) < 3\epsilon\} \subset U$.

It is well known that there exists a smooth map $v : R^n \to R$ (See Proposition 17.8 on p.213 of [8]) with $|u(x) - v(x)| < \epsilon$ for all $x \in R^n$. By the Sard Theorem, there exists a regular value c of the map v in $(\epsilon, 2\epsilon)$. So if we choose $M = \{ x \mid v(x) \leq c\}$, then M is a compact smooth manifold with smooth boundary $\{ x \mid v(x) = c\}$. It is easy to see that $X \subset M \subset U$. So X is a retract of M. We can define the map f to be the inclusion from X to M, and the map g to be a retraction from M to X, thus completing the proof of Theorem 1.1.

Q.E.D.

We believe that Theorem 1.1 is a well known result in topology. We provide a proof here because we have been unable to find a precise reference for it.

Our next aim is to prove the following main theorem in this chapter.

Theorem 1.2 *Let X be a compact metrizable space. For any $\tau \in K^1(X)$, there exist maps $f_i : X \to S^{2n_i-1}$ $(i = 1, 2, \cdots, k)$ such that $m\tau = \sum_{i=1}^k f_i^*\theta_i$ for some integer $m \neq 0$ and $\theta_i \in K^1(S^{2n_i-1})$.*

The proof of Theorem 1.2 will be divided into several steps.

Lemma 1.2 *If Theorem 1.2 is true for the special case of $X = U(n)$, then the theorem is true for an arbitary compact metrizable space X.*

Proof: Assume the theorem is true for $U(n)$ and X is a compact metrizable space. Let $\tau \in K^1(X)$. Then τ can be realized as a map $f : X \to U(n)$ for n large enough. Let $\hat{\tau} \in K^1(U(n))$ be the element determined by the identity map from $U(n)$ to $U(n)$. Hence $\tau = f^*\hat{\tau}$. By our assumption $m\hat{\tau} = \sum_{i=1}^k f_i^*\theta_i$ for some maps $f_i : U(n) \to S^{2n_i-1}, \theta_i \in K^1(S^{2n_i-1})$ and integer $m \neq 0$. Therefore

$$m\tau = \sum_{i=1}^k (f \circ f_i)^*\theta_i.$$

This completes the proof of Lemma 1.2.

Q.E.D.

Lemma 1.3 *Theorem 1.2 is true for* $X = S^1 \times S^3 \times \cdots \times S^{2n-1}$.

Proof: Consider ordinary cohomology $H^*(X) = \oplus_i H^i(X)$. The Künneth formula yields:

$$H^*(X) = H^*(S^1) \otimes H^*(S^3) \otimes H^*(S^5) \otimes \cdots \otimes H^*(S^{2n-1}).$$

Let $\tau_1, \tau_3, \cdots, \tau_{2n-1}$ be generators of $H^1(S^1), H^3(S^3), \cdots, H^{2n-1}(S^{2n-1})$, respectively. Then $H^*(X)$ is generated by $1, \tau_1, \tau_3, \cdots, \tau_{2n-1}, \tau_1 \times \tau_3, \tau_1 \times \tau_5, \cdots, \tau_1 \times \tau_3 \times \tau_5 \times \cdots \times \tau_{2n-1}$ as a group. But each $\tau_{p_1} \times \tau_{p_2} \times \cdots \times \tau_{p_k}$ ($1 \leq p_i \leq 2n-1$ are odd numbers) is a generator of $H^{p_1+p_2+\cdots+p_k}(S^{p_1} \times S^{p_2} \times \cdots \times S^{p_k})$. Let f be the canonical map of degree 1 from $S^{p_1} \times S^{p_2} \times \cdots \times S^{p_k}$ to $S^{p_1+p_2+\cdots+p_k}$ (this map collapses $\bigcup_{i=1}^k S^{p_1} \times S^{p_2} \times \cdots \times S^{p_{i-1}} \times \{pt\} \times S^{p_{i+1}} \times \cdots \times S^{p_k}$ to one point, where $\{pt\}$ denotes a fixed point in S^{p_i}). Then f induces an isomorphism from $H^{p_1+p_2+\cdots+p_k}(S^{p_1+p_2+\cdots+p_k})$ to $H^{p_1+p_2+\cdots+p_k}(S^{p_1} \times S^{p_2} \times \cdots \times S^{p_k})$. This proves $\tau_{p_1} \times \tau_{p_2} \times \cdots \times \tau_{p_n} = f^*\theta$ for some $\theta \in H^{p_1+p_2+\cdots+p_k}(S^{p_1+p_2+\cdots+p_k})$. Therefore, for any element $\tau \in H^*(X)$ ($X = S^1 \times S^3 \times \cdots \times S^{2n-1}$) there exist maps $f_i : X \to S^{n_i}$ such that

$$\tau = \sum f_i^* \theta_i$$

for some $\theta_i \in H^*(S^{n_i})$.

By using bijectivity (up to a rational multiplier) and the naturality of the Chern map between $K^*(X)$ and $H^*(X)$, one verifies Lemma 1.3 easily.

Q.E.D.

By Lemma 1.2 and Lemma 1.3, we need only to reduce the case of $X = U(n)$ to the case of $X = S^1 \times S^3 \times \cdots \times S^{2n-1}$ for proving Theorem 1.2.

First, we recall some results on the topology of $U(n)$. Let $p_n : U(n) \to S^{2n-1}$ be the map which maps $A \in U(n)$ to the last column of A which can be thought of as an element in S^{2n-1}. Then we have the fibre bundle

$$U(n-1) \xrightarrow{i_n} U(n) \xrightarrow{p_n} S^{2n-1},$$

where i_n is the inclusion map which maps $B \in U(n-1)$ to $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \in U(n)$.

Suppose $m < n$, and consider the finite CW complex pair $(U(n), U(m))$, where $U(m)$ is embedded in $U(n)$ by the inclusion map which maps $B \in U(m)$ to $\begin{pmatrix} B & 0 \\ 0 & 1_{n-m} \end{pmatrix} \in U(n)$. The following Lemma is about the cell structure of the pair $(U(n), U(m))$.

Lemma 1.4 *There exists a finite CW complex X with $U(m)$ as the $2m$-skeleton of X and a homotopy equivalence $f : U(n) \to X$ such that $f|_{U(m)} =$ id.*

Proof: We need only to prove the case $n = m + 1$. Consider the fibre bundle

$$U(m) \to U(m+1) \to S^{2m+1}.$$

By Theorem 8.5 on p.187 of [30], we have $\pi_k(U(m+1), U(m)) = \pi_k(S^{2m+1})$ for all $k \geq 1$. Hence $\pi_k(U(m+1), U(m)) = 0$ whenever $0 \leq k \leq 2m$.

The Theorem 2.6 on p.219 of [30] implies that any n-connected CW pair (X, A) is weakly homotopy equivalent to a CW pair (Y, A) with A being the n-skeleton of Y. And Theorem 3.5 on p.220 of [30] says that weakly homotopy equivalence between two CW pairs is homotopy equivalence. Put these two theorems together to finish the proof of Lemma 1.4.

Q.E.D.

As in the proof of Lemma 1.4, if (X, A) is an n-connected CW pair, we can always assume that A is the n-skeleton of X. In particular, if X is an n-connected CW complex, we can assume that the n-skeleton of X is a set consisting of a single point. This argument will be used several times in this paper.

Lemma 1.5 $H^*(U(n)) = \mathbb{Z}\{x_1, x_3, \cdots, x_{2n-1}\}$ *is an exterior algebra with odd dimensional generators* $\{x_{2i-1}\}_{i=1}^{n}$. *In addition, the following statements are true :*

1. *If* $i : U(m) \to U(n)$ $(m < n)$ *is the inclusion, then* $i^* : H^*(U(n)) = \mathbb{Z}\{x_1, x_3, \cdots, x_{2n-1}\} \to H^*(U(m)) = \mathbb{Z}\{y_1, y_3, \cdots, y_{2m-1}\}$ *is defined by*

$$i^* x_{2i-1} = y_{2i-1}, \qquad \text{when } 0 \le i \le m$$

$$i^* x_{2i-1} = 0, \qquad \text{when } m+1 \le i \le n$$

for a proper choice of generators $\{x_{2i-1}\}_{i=1}^{n}$ *and* $\{y_{2i-1}\}_{i=1}^{m}$.

2. *If* $p_n : U(n) \to S^{2n-1}$ *is defined as before, then* p_n^* *maps the generator of* $H^{2n-1}(S^{2n-1})$ *to* $x_{2n-1} \in H^{2n-1}(U(n))$.

Proof: The first part of this lemma is a standard result in topology (see p.164 in [20]) which can be proved by using the spectral sequence of the fibre bundle $U(n-1) \to U(n) \to S^{2n-1}$. The additional parts (1) and (2) can be easily proved by using Lemma 1.4 and the above fibre bundle.

Q.E.D.

Let $\chi_k^{(n)}$ denote a map of degree k from S^{2n-1} to itself. Before proving the next lemma (which is a key lemma in proving Theorem 1.2), we state a result about homotopy operations.

Proposition 1.2 *For any integer k and any $[\alpha] \in \pi_i(S^{2n-1})$ represented by $\alpha : S^i \to S^{2n-1}$ we have $[\chi_{4k}^{(n)} \circ \alpha] = 4k[\alpha]$ in $\pi_i(S^{2n-1})$.*

This Proposition is a special case of Theorem 8.9 on p.537 of [30] which is proved by using the Whitehead product.

Lemma 1.6 *Let $(U(n), U(m))$ be as before $(m < n)$. Then there exists a $k \neq 0$ such that $\chi_k^{(m)} \circ p_m : U(m) \to S^{2m-1}$ can be extended to a map from $U(n)$ to S^{2m-1}.*

Proof: We use χ_k, p to denote $\chi_k^{(m)}$ and p_m respectively, for short. Let X be the space in Lemma 1.4, where we replace $U(n)$ by X. Let X^i denote the i-skeleton of X. Then $X^{2m} = U(m)$. We will prove by induction that there exists an integer $k_i \neq 0$ such that $\chi_{k_i} \circ p : U(m) \to S^{2m-1}$ can be extended to X^i. Assume that there exists an integer $k_i \neq 0$ such that $\chi_{k_i} \circ p$ can be extended to X^i $(i \geq 2m)$. We are going to prove that there exists an integer $k_{i+1} \neq 0$ such that $\chi_{k_{i+1}} \circ p$ can be extended to X^{i+1}. Because $i \geq 2m$, by Serre's Theorem, $\pi_i(S^{2m-1})$ is a finite group. Let N be the order of this group. Choose $k_{i+1} = 4Nk_i$. Then $\chi_{k_{i+1}} \circ p = \chi_{4N} \circ (\chi_{k_i} \circ p)$ can be extended to X^i (which we also denote by $\chi_{k_{i+1}} \circ p$) by the induction assumption.

Let e^{i+1} be an arbitary $(i+1)$-cell of X and $\theta : S^i \to \partial e^{i+1} \subset X^i$ be the attaching map of e^{i+1}. By obstruction theory, the map $\chi_{k_{i+1}} \circ p : X^i \to S^{2m-1}$ can be extended to e^{i+1} if and only if $(\chi_{k_{i+1}} \circ p) \circ \theta : S^i \to S^{2m-1}$ defines a trivial map. But by Proposition 1.2

$$\begin{aligned}
[(\chi_{k_{i+1}} \circ p) \circ \theta] &= [\chi_{4N} \circ (\chi_{k_i} \circ p) \circ \theta] \\
&= 4N[(\chi_{k_i} \circ p) \circ \theta] \\
&= 0
\end{aligned}$$

in $\pi_i(S^{2m-1})$. This completes the proof of Lemma 1.6.

Q.E.D.

Lemma 1.7 *There exists a map $u : U(n) \to S^1 \times S^3 \times \cdots \times S^{2n-1}$ such that for any $\tau \in H^*(U(n))$, there exists an integer $m \neq 0$ and $\theta \in H^*(S^1 \times S^3 \times \cdots \times S^{2n-1})$ with $m\tau = f^*\theta$.*

Proof: By Lemma 1.6, we have maps $u_i : U(n) \to S^{2i-1}$ $(i = 1, 2, \cdots, n)$ which extend some $\chi^{(i)}_{k_i} \circ p : U(i) \to S^{2i-1}$, where $k_i \neq 0$ $(i = 1, 2, \cdots, n)$ are integers.

Let θ_i be the generator of $H^{2i-1}(S^{2i-1})$ for each i and $x_1, x_3, \cdots, x_{2n-1}$ be generators of the ring $H^*(U(n))$ in each dimension. Then by Lemma 1.5 we have $u_i^*\theta = k_i x_{2i-1}$. It is easy to see that $u = u_1 \times u_2 \times \cdots \times u_n : U(n) \to S^1 \times S^3 \times \cdots \times S^{2n-1}$ is the map we want.

Q.E.D.

Now Theorem 1.2 just follows from Lemma 1.2, Lemma 1.3 and Lemma 1.7 where we use the Chern map again.

Corollary . *Let X be a compact metrizable space. For any $\tau \in H^{2n-1}(X)$, there exist an integer $m \neq 0$ and $f : X \to S^{2n-1}$ such that $m\tau = f^*\theta$, where θ is the generator of $H^{2n-1}(S^{2n-1})$.*

Proof: This is a direct consequence of Theorem 1.2 using the Chern map.

Q.E.D.

Remark 1 . The following result can be concluded from [23] and [10] : If X is a finite CW complex and $dimX < 2k - 1$, then for any $\tau \in H^k(X)$, there exist a map $f : X \to S^k$ and an integer $m \neq 0$ such that $m\tau = f^*\theta$. Our Corollary is a similar result which has no restriction on the dimension of X. And we should point out the following facts : (1) Theorem 1.2 and the Corollary are not true for $K^0(X)$ and $H^{2n}(X)$ (we give a counterexample below). (2) In Theorem 1.2 and the Corollary, it is essential to have a multiplier of τ. Generally, we cannot find f_i with $\tau = \sum_{i=1}^{n} f_i^*\theta_i$ or f with $\tau = f^*\theta$.

Remark 2 . By [20] (top line on p.165), $U(n)$ and $S^1 \times S^3 \times \cdots \times S^{2n-1}$ have the same rational homotopy type. But this is not enough to conclude the existence of the map u in Lemma 1.7

We give the following example which shows that Theorem 1.2 and its Corollary are not true for the even case.

Example . Let $g : S^3 \to S^2$ be the Hopf map and let $X = D^4 \cup_g S^2$ be the CW complex obtained by attaching a 4-dimensional disk to S^2 via g.

Let τ be the generator of $H^2(X) = \mathbb{Z}$. If $f : X \to S^2$ such that $m\tau = f^*\theta$ for some integer $m \neq 0$, where θ is the generator of $H^2(S^2)$, then $f|_{S^2}$ would be homotopic to $\chi_m^{(2)}$ which is a map of degree m from S^2 to itself. By p.227 - 228 of [8], the map :

$$S^3 \xrightarrow{g} S^2 \xrightarrow{f|_{S^2}} S^2$$

is the map with Hopf index m^2. Therefore it is nontrivial. But $f|_{S^2}$ can be extended to X. This is a contradiction.

We conclude this chapter with the following theorem which is a special case of the Theorem on p.210 line 7 of [27].

Theorem 1.3 *If (X, Y) is a finite CW complex pair and $\tau \in H_k(X, Y)$, then there exist a smooth compact oriented k-manifold M with boundary ∂M and a map $f : M \to X$ with $f(\partial M) \subset Y$ such that $m\tau = f^*\theta$ for some integer $m \neq 0$ and $\theta \in H_k(M, \partial M)$. In particular, if Y is empty, then M can be chosen as a smooth compact oriented manifold without boundary.*

Chapter 2

The Relation between K-Theory, Index Theory and Invertible n-Tuples of Functions

Throughout this chapter, X and Y will denote compact metrizable spaces. Let H, $L(H)$, $Q(H)$ and $\pi : L(H) \to Q(H)$ be as in the Introduction. Let $\tau : C(X) \to Q(H)$ be a faithful $*$ homomorphism which determines an element in Ext(X) (also denoted by τ). We can find a positive linear map (see [14]) $\rho : C(X) \to L(H)$ with $\pi \circ \rho = \tau$. For any n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$ which satisfies $|f_1(x)|^2 + |f_2(x)|^2 + \cdots + |f_n(x)|^2 \neq 0$ for all x, we can prove that $(\rho(f_1), \rho(f_2), \cdots, \rho(f_n))$ is a Fredholm essentially normal n-tuple of operators. Therefore we can associate an integer index$(\rho(f_1), \rho(f_2), \cdots, \rho(f_n))$ (in the sense of [13]) with this n-tuple. We will study the relation between index$(\rho(f_1), \rho(f_2), \cdots, \rho(f_n))$ and τ.

First we review some definitions and basic results in [13].

Definition . *Let $(T_1, T_2, \cdots, T_n)$ be an n-tuple of operators acting on H.*

1. *$(T_1, T_2, \cdots, T_n)$ is a commutative n-tuple, if $T_iT_j = T_jT_i$ for all $1 \leq i, j \leq n$.*
2. *$(T_1, T_2, \cdots, T_n)$ is an almost commutative n-tuple if $\pi(T_i)\pi(T_j) = \pi(T_j)\pi(T_i)$ or equivalently, if $T_iT_j - T_jT_i$ is a compact operator for all $1 \leq i, j \leq n$.*
3. *$(T_1, T_2, \cdots, T_n)$ is an essentially normal n-tuple if $\pi(T_i)\pi(T_j) = \pi(T_j)\pi(T_i)$ and $\pi(T_i)\pi(T_j^*) = \pi(T_j^*)\pi(T_i)$ for all $1 \leq i, j \leq n$.*

For any n-tuple of operators $(T_1, T_2, \cdots, T_n)$, we can associate a Koszul System to $(T_1, T_2, \cdots, T_n)$ as follows :

$$0 \longrightarrow H_n \xrightarrow{D_n} H_{n-1} \xrightarrow{D_{n-1}} \cdots \longrightarrow H_1 \xrightarrow{D_1} H_0 \longrightarrow 0 \qquad \text{(D)}$$

where $H_k = H \otimes \mathbb{C}^{\binom{n}{k}}$ ($\binom{n}{k}$ copies of H), and if $\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}\}$ are generators of $\mathbb{C}^{\binom{n}{k}}$, then

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \textstyle\sum_{i=1}^{k}(-1)^{i+1} T_{j_i} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_k}.$$

If $(T_1, T_2, \cdots, T_n)$ is a commutative n-tuple, then the Koszul system becomes a Koszul complex which means $D_kD_{k+1} = 0$. If $(T_1, T_2, \cdots, T_n)$ is an almost commutative n-tuple, then $D_k D_{k+1}$ is compact for all k. We can pass to the following complex

$$0 \longrightarrow Q_n \xrightarrow{d_n} Q_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \longrightarrow 0 \qquad \text{(d)}$$

where $Q_k = Q(H) \otimes \mathbb{C}^{\binom{n}{k}}$ ($\binom{n}{k}$ copies of Calkin algebra) and d_k is the matrix associated to $D_k \in L(H_k, H_{k-1})$ in the canonical way (i.e. the entries of d_k are the projections in Q(H) of the entries of D_k). Curto gave the following definition.

Definition . *An almost commutative n-tuple $(T_1, T_2, \cdots, T_n)$ is Fredholm if the complex (d) associated to the Koszul system (D) of $(T_1, T_2, \cdots, T_n)$ is exact. This is equivalent to the condition that $(\pi(T_1), \pi(T_2), \cdots, \pi(T_n))$ is nonsingular in the sense of [28].*

In order to study the index theory of a Fredholm n-tuple $(T_1, T_2, \cdots, T_n)$, Curto associated a $2^{n-1} \times 2^{n-1}$ matrix $A_{(T_1, T_2, \cdots, T_n)}$ as follows :

If $n = 2$, then

$$A_{(T_1, T_2)} = \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix}.$$

Generally, we can define $A_{(T_1, T_2, \cdots, T_n)}$ by

$$A_{(T_1,T_2,\cdots,T_n)} = \begin{pmatrix} A_{(T_1,T_2,\cdots,T_{n-1})} & \begin{pmatrix} T_n & & \\ & \ddots & \\ & & T_n \end{pmatrix}_{2^{n-2}\times 2^{n-2}} \\ -\begin{pmatrix} T_n^* & & \\ & \ddots & \\ & & T_n^* \end{pmatrix} & A^*_{(T_1,T_2,\cdots,T_{n-1})} \end{pmatrix}_{2^{n-1}\times 2^{n-1}} .$$

Furthermore, Curto proved that $(T_1, T_2, \cdots, T_n)$ is Fredholm if and only if $A_{(T_1,T_2,\cdots,T_n)}$ is Fredholm. And he defined

$$index(T_1, T_2, \cdots, T_n) = index A_{(T_1,T_2,\cdots,T_n)}.$$

It is easy to prove that the index of a Fredholm n-tuple is an invariant of compact pertubation and deformation. Therefore $index(T_1, T_2, \cdots, T_n)$ depends only on $(\pi(T_1), \pi(T_2), \cdots, \pi(T_n))$.

Let us go back to our topic. As at the begining of this chapter, let $(f_1, f_2, \cdots, f_n)$ be an n-tuple of functions on X. We will associate to $(f_1, f_2, \cdots, f_n)$ the $2^{n-1} \times 2^{n-1}$ matrix $A_{(f_1,f_2,\cdots,f_n)}$ (exactly the same as associating $A_{(T_1,T_2,\cdots,T_n)}$ to $(T_1, T_2, \cdots, T_n)$ above) by defining :

$$A_{(f_1,f_2,\cdots,f_n)} = \begin{pmatrix} f_1 & f_2 & f_3 & 0 & \cdots \\ -\bar{f}_2 & \bar{f}_1 & 0 & f_3 & \cdots \\ -\bar{f}_3 & 0 & \bar{f}_1 & -f_2 & \cdots \\ 0 & -\bar{f}_3 & \bar{f}_2 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{2^{n-1}\times 2^{n-1}} .$$

Then by Corollary 3.3 in [13], $A_{(f_1,f_2,\cdots,f_n)}(x)$ is an invertible matrix if and only if $|f_1(x)|^2 + |f_2(x)|^2 + \cdots + |f_n(x)|^2 \neq 0$.

If the n-tuple $(f_1, f_2, \cdots, f_n)$ satisfies $|f_1(x)|^2+|f_2(x)|^2+\cdots+|f_n(x)|^2 \neq 0$ for all $x \in X$, then we call $(f_1, f_2, \cdots, f_n)$ an invertible n-tuple. For any invertible n-tuple $(f_1, f_2, \cdots, f_n)$, the associated matrix $A_{(f_1,f_2,\cdots,f_n)}$ can be regarded as a map from X to $Gl_{2^{n-1}}(\mathbb{C})$. Therefore $A_{(f_1,f_2,\cdots,f_n)}$ defines an element in $K^1(X)$ (denoted by $A_{(f_1,f_2,\cdots,f_n)} \in K^1(X)$).

Let $\hat{K}^1(X)$ be the subgroup of $K^1(X)$ generated by all the elements of the form $A_{(f_1,f_2,\cdots,f_n)}$. We will prove the following Theorem.

Theorem 2.1 *For any $\tau \in K^1(X)$, there exists an integer $m \neq 0$ such that $m\tau \in \hat{K}^1(X)$.*

Lemma 2.1 $\hat{K}^1(S^{2n-1}) = K^1(S^{2n-1})$.

Proof : Let $S^{2n-1} = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \cdots + z_n^2 = 1\}$. Define $f_k : S^{2n-1} \to \mathbb{C}$ by

$$f_k(z_1, z_2, \cdots, z_n) = z_k$$

for $(z_1, z_2, \cdots, z_n) \in S^{2n-1}$. It is obvious that

$$|f_1(x)|^2 + |f_2(x)|^2 + \cdots + |f_n(x)|^2 \neq 0.$$

Therefore $A_{(f_1,f_2,\cdots,f_n)} \in \hat{K}^1(S^{2n-1})$. But it is easy to see that $A_{(f_1,f_2,\cdots,f_n)}$ is the element a_n defined on p.240, line 21 in [2] which is a generator of both $\pi_{2n-1}(Gl_{2^{n-1}}(\mathbb{C}))$ and $K^1(S^{2n-1})$, which completes the proof of Lemma 2.1.

Q.E.D.

Proof of Theorem 2.1 : Note the following fact : If $f : X \to Y$ is a map between compact metrizable spaces X and Y, then $f^*\hat{K}^1(Y) \subseteq \hat{K}^1(X)$. It is easy to see that Theorem 2.1 follows from Lemma 2.1 and Theorem 1.2.

Q.E.D.

As in the introduction, there is a pairing between $K^1(X)$ and $\mathrm{Ext}(X)$ defined by

$$\langle \theta,\ \tau \rangle = index(\tau \otimes 1_n)(\theta),$$

where $\theta \in K^1(X)$ is defined by $\theta : X \to Gl_n(\mathbb{C})$ and $\tau \in \mathrm{Ext}(X)$ is defined by $\tau : C(X) \to Q(H)$. If we take $\theta = A_{(f_1, f_2, \cdots, f_n)}$, then

$$\begin{aligned} \langle \theta,\ \tau \rangle &= index(\tau \otimes 1_{2^{n-1}})(A_{(f_1,f_2,\cdots,f_n)}) \\ &= index A_{(\tau(f_1),\tau(f_2),\cdots,\tau(f_n))} \\ &= index(\tau(f_1), \tau(f_2), \cdots, \tau(f_n)). \end{aligned}$$

The following result is a consequence of Theorem 2.1.

Theorem 2.2 *Let X be a finite CW complex and $\tau \in Ext(X)$. Then τ is a torsion element if and only if for any invertible n-tuple of functions $(f_1, f_2, \cdots, f_n) \in C(X)$, we have*

$$index(\tau(f_1), \tau(f_2), \cdots, \tau(f_n)) = 0.$$

Proof: The "only if" part is trivial. For the "if" part, by using Theorem 2.1, it is easy to prove that for any $\theta \in K^1(X)$, we have $\langle \theta,\ \tau \rangle = 0$. Therefore, τ is a torsion element (see p.41 in [14]).

Q.E.D.

Professor John Spielberg point out the following fact to the author: there are a compact metrizable space X and $\tau \in \mathrm{Ext}(X)$ such that $\langle \theta,\ \tau \rangle = 0$ for each $\theta \in K^1(X)$, but τ is not a torsion element. Therefore Theorem 2.2 is not true for arbitary compact metrizable space.

Chapter 3

Smooth Extensions for a Finite CW Complex

In this chapter, we study C_p-smoothness of extensions for finite CW complexes and CW complex pairs. The notion of C_p-smoothness was introduced by Douglas and studied in [21], [22], [15] and [16]. In this aspect, the two most important results are the following:

(1). C_1-smooth elements of $\mathrm{Ext}(X)$ come from the 1-skeleton of X modulo torsion [21], [15], when X is a simplicial complex.

(2). C_{n-1}-smooth elements of $\mathrm{Ext}(S^{2n-1})$ are trivial [16].

The natural problem is to characterize the C_p-smooth extensions for a general space X (e.g. CW complex). In this chapter, we first generalize the definition of C_p-smoothness (Definition 0.1 in the Introduction) to a finite CW complex. And we obtain necessary and sufficient conditions for an extension of a finite CW complex to be C_p-smooth modulo torsion. In particular, the above results (1) and (2) are direct consequences of our theorem. We also prove a similar result for a relative extension of a CW complex pair. Our results answer several open questions in [15] and [16]. Finally, we apply our theorem to operator theory and study an essentially normal n-tuple of operators $(T_1, T_2, \cdots, T_n)$ with the commutators $[T_i, T_j^*]$ and $[T_i, T_j]$ $(1 \leq i, j \leq n)$ in C_p-class.

Let H, $L(H)$, $K(H)$, $Q(H)$ and $\pi : L(H) \to Q(H)$ be as in the Introduction. Recall that if T is a compact operator on H, then there exists a complete orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ of H such that

$$T^*T\psi_n = \lambda_n^2\psi_n$$

where $\{\lambda_n\}_{n=1}^{+\infty}$ is a sequence of nonnegative real numbers with $\lambda_n \to 0$. We say T is in the Schatten – von Neuman p-class (denoted by $T \in C_p$), if $T \in K(H)$ and $\sum_{n=1}^{\infty} \lambda_n^p < +\infty$. It is obvious that $C_p \subset C_q$ when $p < q$. Therefore, in Definition 0.1 in the Introduction, if T is C_p-smooth, then T is C_q-smooth whenever $q > p$.

Theorem 1.1 in Chapter 1 is used to give the following definition of C_p-smooth for a finite CW complex.

Definition 3.1 *Let X be a finite CW complex, and M and f be as in Theorem 1.1. Then $\tau \in Ext(X)$ is C_p-smooth if $f_*\tau \in Ext(M)$ is C_p-smooth (see Definition 0.1).*

Theorem 3.1 *The definition of C_p-smoothness does not depend on the choice of M or on the maps f and g in Theorem 1.1.*

To prove Theorem 3.1, we need the following lemma :

Lemma 3.1 *Let M and N be compact smooth manifolds (perhaps with boundary), and let u be a continuous map between M and N. If $\tau \in$ Ext(M) is C_p-smooth, then $u_*\tau \in$ Ext(N) is C_p-smooth.*

Proof : By Proposition 17.8 on p.213 of [8], there exists a smooth map v between M and N such that u is homotopy equivalent to v. So we can assume u is a smooth map.

Let $\tau : C(M) \to Q(H)$ be a C_p-smooth extension. Then by Definition 0.1 there exists a positive linear $*$ map $\rho : C^\infty(M) \to L(H)$ such that $\rho \circ \pi = \tau$ and $\rho(fg) - \rho(f)\rho(g) \in C_p$ for all $f, g \in C^\infty(M)$.

Let us describe $u_*\tau \in \mathrm{Ext}(N)$. First we define $\tau_1 : C(N) \to Q(H)$ as the following :

$$\tau_1(f) = \tau(f \circ u)$$

for $f \in C(N)$. Then τ_1 is a $*$ homomorphism. But in general, τ_1 is not faithful. So we cannot say $u_*\tau$ is determined by τ_1. However, if $\tau_2 : C(N) \to Q(H)$ is a faithful $*$ homomorphism which can be lifted to a $*$ homomorphism $\rho_2 : C(N) \to L(H)$ (this means that τ_2 is trivial in $\mathrm{Ext}(N)$), then $u_*\tau$ is determined by $\tau_1 \oplus \tau_2 : C(N) \to Q(H \oplus H)$. Let $\rho_1 : C^\infty(N) \to L(H)$ be a positive linear map determined by

$$\rho_1(f) = \rho(f \circ u)$$

for $f \in C^\infty(N)$. It is easy to verify that $\rho_1 \oplus (\rho_2|_{C^\infty(N)}) : C^\infty(N) \to L(H \oplus H)$ satisfies the condition in Definition 0.1 which serves as the lifting for $\tau_1 \oplus \tau_2$. Therefore $u_*\tau$ is C_p-smooth.

Q.E.D.

Proof of Theorem 3.1. Let M_1 and f_1, g_1 be another choice of space and maps which satisfies Theorem 1.1. Because $g \circ f$ is homotopic to $id|_X$, f_1 is homotopic to $f_1 \circ g \circ f$. Therefore

$$f_{1_*}\tau = (f_1 \circ g)_*(f_*\tau).$$

By Lemma 3.1, if $f_*\tau$ is C_p-smooth, then $f_{1_*}\tau$ is C_p-smooth. The proof of the converse is exactly the same.

Q.E.D.

The following two corollaries are direct consequences of Lemma 3.1 and its proof.

Corollary 1. *In Definition 0.1, the notion of C_p-smoothness does not depend on the particular smooth structure associated to the manifold M.*

Proof. Take f to be the identity map on M in Lemma 3.1.

Q.E.D.

Corollary 2. *Let $f : X \to Y$ be a continuous map. Then f_* takes the C_p-smooth elements of Ext(X) to C_p-smooth elements of Ext(Y).*

Corollary 1 above answers the question on p.68 of [15].

Now let M be a smooth compact oriented manifold without boundary. We recall how to construct the element of $\mathrm{Ext}(M)$ from a self adjoint pseudo-differential operator A acting on the bundle E of M defined in [4]. Let E be a C^∞ vector bundle over M with a C^∞ Hermitian structure. Let $C^\infty(E)$ denote the vector space of all C^∞ sections of E. Choose a Riemannian metric for M and define an inner product $\langle\ ,\ \rangle$ for $C^\infty(E)$ by

$$\langle u,\ v\rangle = \textstyle\int_M \langle u(p), v(p)\rangle d\mu \qquad\qquad u, v \in C^\infty(E), p \in M,$$

where μ is the smooth measure on M determined by the Riemannian metric. The completion of $C^\infty(E)$ with respect to this inner product is the Hilbert space $L^2(E)$. Given $f \in C(M)$, $M_f : L^2(E) \to L^2(E)$ is the multiplication operator defined by

$$(M_f v)(p) = f(p)v(p) \qquad\qquad v \in L^2(E), p \in M, f \in C(M).$$

Let A be a self-adjoint pseudo-differential operator from $C^\infty(E)$ to $C^\infty(E)$. A can be viewed as a possibly unbounded self-adjoint operator on $L^2(E)$. Let P_A be the spectral projection of A for $[0, \infty)$. Then P_A is a pseudo-differential operator of order zero by Proposition 2.4 in [4]. Let H be the range of P_A. We can define an element $\tau_A \in \text{Ext}(M)$ associated to A by the following :

$$\tau_A(f) = \pi(P_A M_f P_A) \in Q(H)$$

for $f \in C(M)$. It is easy to check that τ_A is a $*$-homomorphism from $C(M)$ to $Q(H)$ (see Lemma 2.10 in [4]). Therefore τ_A determines an element in Ext(M). One of the main results in [4] is that all the elements of Ext(M) can be realized in the form τ_A for some A (Theorem 2 in [4]). We are going to use the above construction to prove the following theorem.

Theorem 3.2 *If M is an n-dimensional oriented compact smooth manifold without boundary, then all the elements of Ext(M) are C_p-smooth whenever $p > \frac{n}{2}$.*

Proof. We just need to verify that τ_A is C_p-smooth. Let $\rho_A : C^\infty(M) \to L(H)$ be the $*$ positive linear map defined by

$$\rho_A(f) = P_A M_f P_A \in L(H), \qquad f \in C^\infty(M).$$

Then it is obvious that $\tau|_{C^\infty(M)} = \pi \circ \rho_A$. To prove this theorem, we need only to prove

$$\rho_A(fg) - \rho_A(f)\rho_A(g) \in C_p \text{ when } p > \tfrac{n}{2} \text{ and } f, g \in C^\infty(M).$$

But

$$\begin{aligned}
& \rho_A(fg) - \rho_A(f)\rho_A(g) \\
= \;& P_A M_f M_g P_A - P_A M_f P_A P_A M_g P_A \\
= \;& P_A M_f M_g P_A - M_f P_A M_g P_A + M_f {P_A}^2 M_g P_A - P_A M_f P_A M_g P_A \\
= \;& [P_A, M_f] M_g P_A - [P_A, M_f] P_A M_g P_A \\
= \;& [P_A, M_f](M_g {P_A}^2 - P_A M_g P_A) \\
= \;& [P_A, M_f][M_g, P_A] P_A
\end{aligned}$$

According to Proposition 1 in Appendix 1 in [11], we need only to prove that $[P_A, M_f]$ and $[M_g, P_A]$ are in C_{2p}. By the argument in [4] we know that $[P_A, M_f]$ is a pseudo-differential operator of order -1. Therefore $[P_A, M_f]$ is a bounded operator from $H^0(E) = L^2(E)$ to $H^1(E)$ ($H^0(E)$ and $H^1(E)$ are Sobelev spaces). Please note that the embedding from $H^1(E)$ to $H^0(E)$ is in C_q when $q > \dim M$. Therefore $[P_A, M_f] \in C_{2p}$. It is the same to prove $[M_g, P_A] \in C_{2p}$. This completes the proof of Theorem 3.2.

Q.E.D.

Now we are going to prove our main results in this chapter.

Theorem 3.3 *Let X be a finite CW complex, $\tau \in Ext(X) = K_1(X)$ and $ch : K_1(X) \otimes \mathbb{Q} \to H_{odd}(X, \mathbb{Q})$ be the Chern map, where $H_{odd}(X, \mathbb{Q})$ denotes the direct sum of all the ordinary homology groups of odd dimension with rational coefficients. Then there exists an integer $m \neq 0$ such that $m\tau$ is C_n-smooth if and only if $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, \mathbb{Q})$.*

Proof. If $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, \mathbb{Q})$, then $ch\tau = \theta_1 + \theta_3 + \cdots + \theta_{2n-1}$, where $\theta_{2i-1} \in H_{2i-1}(X, \mathbb{Q})$. Without loss of generality, we can assume $ch\tau = \theta \in H_{2n-1}(X, \mathbb{Q})$. By Theorem 1.3 in Chapter 1, there exist a compact oriented smooth manifold M and a map $f : M \to X$ such that $\theta = f_*\tilde{\theta}$ where $\tilde{\theta} \in H_{2n-1}(M, \mathbb{Q})$. (Please note that we use rational coefficients here, so we do not need to multiply θ by an integer.) By surjectivity of the Chern map between $K_1(M) \otimes \mathbb{Q}$ and $H_{odd}(M, \mathbb{Q})$, there exist a $\tilde{\tau} \in \mathrm{Ext}(M) = K(M)$ and a rational number $\frac{p}{q}$ with $ch(\frac{p}{q}\tilde{\tau}) = \theta$. Therefore, $ch\tau = f_* ch(\frac{p}{q}\tilde{\tau}) = ch(f_*(\frac{p}{q}\tilde{\tau}))$. Using the injectivity of the Chern map,

$$\tau = \tfrac{p}{q}(f_*\tilde{\tau}) \qquad\qquad \text{in } K_1(X) \otimes \mathbb{Q}.$$

Hence, there exists an integer m_1 with

$$m_1\tau = \tfrac{m_1 p}{q}(f_*\tilde{\tau}) \qquad\qquad \text{in } K_1(X).$$

Therefore,

$$m_1 q\tau = m_1 p(f_*\tilde{\tau}) = f_*(m_1 p\tilde{\tau}).$$

Let $m = m_1 q$. From Theorem 3.2 and Corollary 2 of Theorem 3.1, we know that $m\tau$ is C_n-smooth. This completes the proof of the " if " part.

Suppose that there exists an integer $m \neq 0$ with $m\tau$ being C_n-smooth. We are going to prove $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, \mathbb{Q})$. If not, there exists a $\theta \in H^{2i-1}(X, \mathbb{Q})$ with $i > n$, such that the pairing $\langle ch\tau,\ \theta\rangle \neq 0$. By the Corollary of Theorem 1.2, there exist a map $f : X \to S^{2i-1}$ and $\tilde{\theta} \in H^{2i-1}(S^{2i-1})$ such that $\theta = f^* \frac{p}{q}\tilde{\theta}$ where $\frac{p}{q}$ is a rational number. By our assumption, $m\tau$ is C_n-smooth. Therefore $mf_*\tau = f_* m\tau$ is a C_n-smooth element in $\mathrm{Ext}(S^{2i-1})$. By Proposition 3 in [16], $mf_*\tau$ is the trivial element in $\mathrm{Ext}(S^{2i-1}) = \mathbb{Z}$. Therefore $f_*\tau = 0$. Hence

$$
\begin{aligned}
0 &= \langle chf_*\tau,\ \frac{p}{q}\tilde{\theta}\rangle \\
&= \langle f_* ch\tau,\ \frac{p}{q}\tilde{\theta}\rangle \\
&= \langle ch\tau,\ f^*(\frac{p}{q}\tilde{\theta})\rangle \\
&= \langle ch\tau,\ \theta\rangle \\
&\neq 0
\end{aligned}
$$

This contradiction completes the proof of the " only if " part of Theorem 3.3.

Q.E.D.

The following theorem is almost equivalent to the above results but is perhaps more useful in practice.

Theorem 3.4 *Let X be a finite CW complex, X^k denote the k-skeleton of X and $\tau \in Ext(X)$. Then there exists an integer $m_1 \neq 0$ such that $m_1\tau$ is C_n-smooth if and only if there exists an integer $m_2 \neq 0$ such that*

$m_2\tau \in i_*(ExtX^{2n-1})$, where $i_* : Ext(X^{2n-1}) \to Ext(X)$ is induced by the inclusion map $i : X^{2n-1} \to X$.

Proof : If there exists an integer $m_1 \neq 0$ with $m_1\tau$ being C_n-smooth, then by Theorem 3.3, $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, \mathbb{Q})$. Please note the following well known fact from homology theory, $i_* : H_k(X^n) \to H_k(X)$ is a bijection when $k < n$ and a surjection when $k = n$. It is routine to prove that there exists an integer $m_2 \neq 0$ with $m_2\tau \in i_*(X^{2n-1})$. The " if " part follows from Theorem 3.2, Theorem 3.3 and the bijectivity of the Chern map from $K_1(X) \otimes \mathbb{Q}$ to $H_{odd}(X) \otimes \mathbb{Q}$.

Q.E.D.

Corollary 1. *If X is a (2n-1)-connected finite CW complex, then all the C_n-smooth elements of Ext(X) are torsion elements.*

Proof : As in Chapter 1, we know X is homotopic to a CW complex with its 2n-1 skeleton being a single point. According to Corollary 2 of Theorem 3.1, we can assume that the 2n-1 skeleton of X is one point. Thus this Corollary follows from Theorem 3.4.

Q.E.D.

Corollary 2. *Let X be a (2n-1)-connected finite CW complex and $dimX \leq 2n+1$. Then all the C_n-smooth elements of Ext(X) are trivial.*

Proof : Let $\tau \in \mathrm{Ext}(X)$ be C_n-smooth. According to Corollary 1, there exists an integer $m \neq 0$ with $m\tau = 0$. Consider the relative homology exact sequence of (X, X^{2n}), where X^{2n} is the 2n-skeleton of X (see p.37 of [14] for the sequence).

$$\mathrm{Ext}(X^{2n}) \xrightarrow{i_*} \mathrm{Ext}(X) \xrightarrow{p_*} \mathrm{Ext}(X/X^{2n})$$

Since $dimX \leq 2n+1, X/X^{2n}$ is the space of the bouquet $S^{2n+1} \vee S^{2n+1} \vee \cdots \vee S^{2n+1}$ of several 2n+1 dimensional spheres. This implies that Ext(X/X^{2n}) is torsion free. Therefore $p_*\tau = 0$. So $\tau = i_*\theta$ for $\theta \in$ Ext(X^{2n}). But X is (2n-1)-connected, so we can assume that X^{2n-1} is one point. Hence X^{2n} is the bouquet of 2n-dimensional spheres. Therefore $K^1(X^{2n}) = 0$. This completes the proof of Corollary 2.

Q.E.D.

The main theorem of [16] (see Proposition 3 in that paper) is the special case of our Corollary 1 when we take X to be S^{2n+1}. And the two main theorems of [15] (see p.65 and p.66) are the special cases of our Corollary 1 and Corollary 2 respectively taking $n = 1$. But we should point out that Proposition 3 in [16] is used in proving our main results and that the original proof of the theorem in [15] inspired our proof of Corollary 2.

Now, we will briefly discuss the C_p-smoothness for a relative extension of a finite CW complex pair.

Definition 3.2 *Let (X, Y) be a relative finite CW complex pair. Then $\tau \in$ Ext(X, Y) is said to be C_p-smooth if the image of τ under the canonical isomorphism from Ext(X, Y) to Ext(X/Y) is C_p-smooth.*

One can prove the following theorem.

Theorem 3.5 *Let $\tau \in$ Ext(X, Y). Then the following are equivalent :*

1. There exists an integer $m \neq 0$ such that $m\tau$ is C_n-smooth.

2. $ch\tau \in \oplus_{k=1}^{n} H_{2k-1}(X, Y, \mathbb{Q})$, where $H_i(X, Y, \mathbb{Q})$ denotes the relative homology group of the CW complex pair with rational coefficients.

Before we end this chapter, we give an application of our theorem to operator theory.

Let X be a finite simplicial complex embedded in R^n. We say that X is smoothly embedded in R^n if we can find a closed neighborhood U with smooth boundary, such that X is a retract of U and the retraction $r : U \to X$ is homotopic to a map $f : U \to X$ with $i \circ f : U \to X \to U$ being smooth. We can prove that any 1-dimensional simplicial complex can be smoothly embedded in R^n. And while I believe that this is true for any dimension, I have been unable to prove it.

We give the following theorem as an application of our results in this chapter.

Theorem 3.6 *Let $(T_1, T_2, \cdots, T_n)$ be an essentially normal n-tuple of operators and $X \subset \mathbb{C}^n$ be the essential spectrum of the n-tuple. Then the following statements are true :*

(1) If X is a closed m-dimensional smooth oriented manifold embedded in $\mathbb{C}^n$, then for any $p > \frac{m}{2}$, there exists an n-tuple of compact operators $(K_1, K_2, \cdots, K_n)$ with $[T_i + K_i, T_j^* + K_j^*] \in C_p$.

(2) If X is a m-dimensional simplicial complex smoothly embedded in $\mathbb{C}^n$, then for some n-tuple $(S_1, S_2, \cdots, S_n) =$ $(\underbrace{T_1 \oplus T_1 \oplus \cdots \oplus T_1}_{k-copies}, \underbrace{T_2 \oplus T_2 \oplus \cdots \oplus T_2}_{k-copies}, \cdots, \underbrace{T_n \oplus T_n \oplus \cdots \oplus T_n}_{k-copies})$ (k is an integer), there exists an n-tuple of compact operators acting on $\underbrace{H \oplus H \oplus \cdots \oplus H}_{k-copies}$ such that $[S_i + K_i, S_j + K_j] \in C_{[\frac{m}{2}]+1}$ and $[S_i + K_i, S_j^* + K_j^*] \in C_{[\frac{m}{2}]+1}$.

Proof : (1) follows from Theorem 3.2 and (2) follows from Theorem

3.4 and the Corollary 2 of Theorem 3.1.

Q.E.D.

If we take $X = S^{2n-1}$, then we answer the question on p.109 of [16]. In particular, we have the following fact : If $(T_{z_1}, T_{z_2}, \cdots, T_{z_n})$ is the n-tuple of Toeplitz operators on $H^2(\partial B_n)$, then there exist n compact operators $(K_1, K_2, \cdots, K_n)$ such that $[T_{z_i} + K_i, T_{z_j} + K_j] \in C_p$ when $p > n - \frac{1}{2}$. There doesn't seem to be any direct proof of this fact.

Chapter 4

Summable Fredholm Modules of $C^\infty(M)$ for a Compact Smooth Manifold M

Corresponding to $K_1(X) = \mathrm{Ext}(X)$ [9] [14], in the even case, Kasparov [25] proved that each element of $K_0(X)$ can be realized as a Fredholm module of $C(X)$ and therefore $K_0(X) = KK(C(X), \mathbb{C})$. We refer to [7] for the general theory of K-homology and KK-groups.

Let M be a compact smooth manifold, and $C^\infty(M)$ denote the algebra of smooth functions on M. The notion of a p-summable Fredholm module of $C^\infty(M)$, which can be thought of as an element in $K_0(M) = KK(C(X), \mathbb{C})$, is the even analogy of C_p-smooth extension and it was introduced by Connes [11].

In this chapter, we study the p-summable Fredholm modules of $C^\infty(M)$

and their Chern characters in $H^*_\lambda(C^\infty(M))$, where $H^*_\lambda(C^\infty(M))$ is the cyclic cohomology of $C^\infty(M)$. Furthermore, we also study the case of a compact manifold with boundary.

First we briefly recall some definitions and basic results. Let X be a compact metrizable space, and $\Gamma(X)$ denote the collection of triples (H, σ, F) which are called the Fredholm modules of $C(X)$, where,

(1) $H = H_0 \oplus H_1$ is a $\mathbb{Z}_2$ graded Hilbert space with a grading operator $\mathcal{E}$, $\mathcal{E}\xi = (-1)^{deg\xi}\xi$ for all $\xi \in H_0$ or $\xi \in H_1$;

(2) $\sigma = \sigma_0 \oplus \sigma_1$, and $\sigma_i : C(X) \to L(H_i)$ is a continuous $*$ homomorphism;

(3) $F \in L(H)$, $F^2 = I$, $F\mathcal{E} = -\mathcal{E}F$, and for any $f \in C(X)$, one has

$$F\sigma(f) - \sigma(f)F \in K(H).$$

$K_0(X)$ is defined to be $\Gamma(X)$ modulo certain equivalence relations (for details see § 2 of [5] or Chapter 5 of [14]).

If M is a compact smooth manifold, then a p-summable Fredholm module of $C^\infty(M)$ is an element of $\Gamma(M)$ which satisfies the following stronger condition:

(3') $F \in L(H)$, $F^2 = I$, $F\mathcal{E} = -\mathcal{E}F$, and for any $f \in C^\infty(M)$, one has

$$F\sigma(f) - \sigma(f)F \in C_p.$$

In [11], Connes defined the cyclic cohomology $H^n_\lambda(\mathcal{A})$ of an algebra $\mathcal{A}$ over $\mathbb{C}$ as follows. Let $C^n_\lambda(\mathcal{A})$ be the set of n+1 linear functions on $\mathcal{A}$ which satisfy :

$$\tau(a^1, a^2, \cdots, a^n, a^0) = (-1)^n \tau(a^0, a^1, a^2, \cdots, a^n), \forall a^0, a^1, a^2, \cdots, a^n \in \mathcal{A}.$$

Define $b : C^n_\lambda(\mathcal{A}) \to C^{n+1}_\lambda(\mathcal{A})$ by

$$\begin{aligned}(b\tau)(a^0, a^1, a^2, \cdots, a^{n+1}) &= \tau(a^0 a^1, a^2, \cdots, a^{n+1}) \\ &+ \sum_{i=1}^{n}(-1)^i \tau(a^0, \cdots, a^i a^{i+1}, \cdots, a^{n+1}) \\ &+ (-1)^{n+1}\tau(a^{n+1}a^0, \cdots, a^n)\end{aligned}$$

One can verify $b^2 = 0$. $H^n_\lambda(\mathcal{A})$ is defined to be

$$\frac{Ker\{b : C^n_\lambda(\mathcal{A}) \to C^{n+1}_\lambda(\mathcal{A})\}}{Im\{b : C^{n-1}_\lambda(\mathcal{A}) \to C^n_\lambda(\mathcal{A})\}}.$$

Connes defined a useful map $S : H^n_\lambda(\mathcal{A}) \to H^{n+2}_\lambda(\mathcal{A})$, and defined $H^{even}(\mathcal{A})$ to be the inductive limit of the groups $H^{2n}_\lambda(\mathcal{A})$ under the map $S : H^{2n}_\lambda(\mathcal{A}) \to H^{2n+2}_\lambda(\mathcal{A})$, or equivalently, the quotient of $\oplus_{n=1}^{\infty} H^{2n}_\lambda(\mathcal{A})$ by the equivalence relation $\phi \sim S\phi$. $H^{odd}(\mathcal{A})$ is defined in same way.

In §2 of [Connes], Connes also constructed the graded Chern map

$$ch^* : \{\text{(n+1)-summable Fredholm modules}\} \to H^n_\lambda(C^\infty(M)),$$

where n is an even integer. And he proved that

$$ch^* : \{\text{finite summable Fredholm modules}\} \to H^{even}(C^\infty(M))$$

is a surjection up to complex multipliers. Our main result in this chapter says that the Chern map is a graded surjection.

Theorem 4.1 *Let M be an n-dimensional compact oriented smooth manifold (n even). Then all the elements of $K_0(M)$ can be realized as (n+1)-summable Fredholm modules.*

Proof : In § 5 of [4], Baum-Douglas proved that each element τ in $K_0(M)$ can be represented by a first order elliptic pseudo-differential operator. Then by § 6 of [11], we know that τ can be realized as an (n+1)-summable Fredholm module. This completes the proof.

Q.E.D.

Before proving our main result, we give the following two lemmas.

Lemma 4.1 *If X is a finite CW complex and $\tau \in H_k(X, \mathbb{C})$, then there exist compact oriented smooth k-manifolds M_i $(i = 1, 2, \cdots, n)$ without boundary and maps $f_i : M_i \to X$ such that $\tau = \sum_{i=1}^{n} f_{i*}\theta_i$ for some $\theta_i \in H_k(M_i, \mathbb{C})$, where $H_k(X, \mathbb{C})$ denotes the homology of X with complex coefficients.*

This Lemma is another version of Theorem 1.3. We omit the proof here.

Please note that one should use several manifolds rather than a single manifold (in Theorem 1.3) because of the complex coefficients.

Lemma 4.2 *If M, N are compact smooth manifolds and $f : M \to N$ is a continuous map, then f_* maps p-summable elements of $K_0(M)$ to p-summable elements of $K_0(N)$.*

The proof of this Lemma is exactly the same as that of Lemma 3.1.

Theorem 4.2 *If M is a compact smooth manifold without boundary and $\phi \in H^k_\lambda(C^\infty(M))$ (k even), then there exist (k+1)-summable Fredholm modules τ_i $(i = 1, 2, \cdots, n)$ and complex numbers α_i $(i = 1, 2, \cdots, n)$ such that $\sum_{i=1}^{n} \alpha_i ch^* \tau_i \sim \phi$ in $H^*(C^\infty(M))$, where ch^* is Connes' Chern map.*

Proof : Let $\phi \in H^k_\lambda(C^\infty(M))$. According to (1) and (2) of Theorem 4.6 in [11], we know that ϕ corresponds to $\phi_k + \phi_{k-2} + \cdots + \phi_0$ under a map $h : H^{even}(C^\infty(M)) \to \oplus_j H_{2j}(M, \mathbb{C})$, where $\phi_{k-j} \in H_{k-j}(M, \mathbb{C})$. As pointed out by Connes in [11], the composition of the following two maps

$$K_0(M) \otimes \mathbb{C} \xrightarrow{ch^*} H^{even}(C^\infty(M)) \xrightarrow{h} \oplus_j H_{2j}(M, \mathbb{C})$$

is the usual Chern map from $K_0(M) \otimes \mathbb{C}$ to $\oplus_j H_{2j}(M, \mathbb{C})$ (denoted by ch), where ch^* is induced by Connes' Chern map $ch^* : K_0(M) \to H^{even}(C^\infty(M))$. Therefore the composition of the following sequence of maps

$$H^{even}(C^\infty(M)) \xrightarrow{h} \oplus_j H_{2j}(M, \mathbb{C}) \xrightarrow{ch^{-1}} K_0(M, \mathbb{C}) \xrightarrow{ch^*} H^{even}(C^\infty(M))$$

is the identity map. From the above fact, we need only to prove that for each ϕ_{k-j}, there exist k+1-summable Fredholm modules $\tau_i \in K_0(M)$ $(i = 1, 2, \cdots, n)$ and $\alpha_i \in \mathbb{C}$ $(i = 1, 2, \cdots, n)$ such that $ch^{-1}\phi_{k-j} = \sum_{i=1}^n \alpha_i \tau_i$ as elements of $K_0(M) \otimes \mathbb{C}$.

Without loss of generality, we just need to prove this for ϕ_k.

Using Lemma 4.1, we have $\phi_k = \sum_{i=1}^n f_{i*}\theta_i$, where $\theta_i \in H_k(M_i, \mathbb{C})$ and M_i is a connected compact oriented smooth k-manifold. Since $H_k(M_i) \otimes \mathbb{C} \cong H_k(M_i, \mathbb{C}) = \mathbb{C}$ as well as $H_k(M_i, \mathbb{Z}) = \mathbb{Z}$, there exists a $\beta_i \in \mathbb{C} \setminus \{0\}$ such that $\beta_i\theta_i$ corresponds to an integer under the isomorphism between $H_i(M, \mathbb{C})$ and $\mathbb{C}$. This means that $\beta_i\theta_i$ can be expressed as an element of $H_k(M_i, \mathbb{Z})$. Therefore $ch^{-1}\beta_i\theta_i \in K_0(M) \otimes \mathbb{C}$ can be chosen as an element in $K_0(M_i)$. Hence $ch^{-1}\beta_i\theta_i$ can be represented by a (k+1)-summable Fredholm module according to Theorem 4.1. Lemma 4.2 says that $f_{i*}ch^{-1}\beta_i\theta_i \in K_0(M)$ can be represented by a k+1-summable Fredholm module. But

$$ch^{-1}\phi_k = \sum_{i=1}^{n} \frac{1}{\beta_i}\tau_i$$

as the element in $K_0(M) \otimes \mathbb{C}$, where $\tau_i = f_{i*}ch^{-1}\beta_i\theta_i$. This completes the proof.

Q.E.D.

Now, we characterize the p-summable Fredholm module of $C_c^\infty(M)$ for M being the interior of a compact oriented smooth manifold $\overline{M}$ with boundary ∂M, where $C_c^\infty(M)$ denotes the smooth functions on M with compact support. Let $C_0(\overline{M})$ be the set of the continuous functions on $\overline{M}$ vanishing on ∂M and $\Gamma(M, \partial M)$ be the collection of Fredholm modules of $C_0(\overline{M})$ (To define the Fredholm modules of $C_0(\overline{M})$, one need only to replace $C(X)$ by $C_0(\overline{M})$ in the definition of $\Gamma(X)$.) A p-summable Fredholm module of $C_c^\infty(M)$ is an element of $\Gamma(M, \partial M)$ which satisfies the stronger condition (3'), where we replace $C^\infty(M)$ by $C_c^\infty(M)$.

Lemma 4.3 *For each $\tau \in K_0(\overline{M}, \partial M)$, there exists a Fredholm module (H, σ, F) of $C_0(\overline{M})$ which represents τ.*

Proof : The lemma follows from the fact : The inclusion map from $C_0(\overline{M})$ to $C_0(\overline{M})^+$ induces a surjection from $KK(C_0(\overline{M})^+, \mathbb{C})$ to $KK(C_0(\overline{M}), \mathbb{C})$, where $C_0(\overline{M})^+$ is the algebra of $C_0(\overline{M})$ adjoint a unit.

Q.E.D.

If $\tau \in K_0(\overline{M}, \partial M)$ can be represented by a p-summable Fredholm module, we say that τ is a p-summable element.

In the old preprint of [6], the authers gave some estimates which can be used to prove that all the elements of $K_0(\overline{M}, \partial M)$ are n+1-summable when $n = dimM$.

Lemma 4.4 *If $f : \overline{M} \to \overline{N}$ is a continuous map between oriented compact smooth manifolds with boundaries such that $f(\partial M) \subset \partial N$, then f_* maps p-summable elements of $K_0(\overline{M}, \partial M)$ to p-summable elements of $K_0(\overline{N}, \partial N)$.*

Proof : Note the following fact which can be proved by using collared neighborhoods of boundaries : f is homotopic to a smooth map $g : M \to N$ with $g|_{\partial M} \subset \partial N$. We reduce the proof to the case of f being smooth. The rest of the proof is similar to Lemma 3.1.

Q.E.D.

From homology theory we know that $K_0(\overline{M}, \partial M)$ is isomorphic to $K_0(\overline{M}/\partial M, \{pt\})$ which is the reduced K-homology of $\overline{M}/\partial M$. If f is a map from $\overline{M}/\partial M$ to X, then f induces a map

$$f_* : K_0(\overline{M}, \partial M) \to K_0(X).$$

Lemma 4.5 *Let $\overline{M}$ be an oriented compact smooth manifold with boundary ∂M and N be an oriented compact smooth manifold without boundary. If $f : \overline{M}/\partial M \to N$ is a continuous map, then f_* maps p-summable elements of $K_0(\overline{M}, \partial M)$ to p-summable elements of $K_0(N)$.*

Proof : Note that f is homotopic to a map g which is smooth out to ∂M and take a neighborhood of ∂M in $\overline{M}/\partial M$ to a single point in N, where ∂M is thought of as a point in $\overline{M}/\partial M$. The Lemma will follow.

Q.E.D.

Theorem 4.3 *Let $\tau \in K_0(\overline{M}, \partial M)$ and p be an even integer. Then the following conditions are equivalent :*

(1). There exists an integer $m \neq 0$ *such that* $m\tau$ *is (p+1)-summable.*

(2). $ch\tau \in \oplus_{k=1}^{\frac{p}{2}} H_{2k}(\overline{M}, \partial M, \mathbb{Q})$*, where* $H_{2k}(\overline{M}, \partial M, \mathbb{Q})$ *denotes the relative homology group of the pair* $(M, \partial M)$ *with rational coefficients.*

Proof : (2) $\Longrightarrow$ (1) follows from Theorem 1.3 , Lemma 4.4 and the argument before Lemma 4.4.

(1) $\Longrightarrow$ (2) : If $ch\tau \notin \oplus_{k=1}^{\frac{p}{2}} H_{2k}(\overline{M}, \partial M, \mathbb{Q})$, then there exists $\theta \in K^0(\overline{M}, \partial M)$ with $\langle \theta, \tau \rangle \neq 0$ and $ch^*\theta \in H^{2i}(\overline{M}, \partial M, \mathbb{Q})$ with $i > k$. But θ can be realized as a pull-back of an element $\hat{\theta} \in K^0(BU(n))$ via a map f from $\overline{M}/\partial M$ to the oriented compact smooth manifold $BU(n)$, where $BU(n)$ is the classifying space of complex bundles. Because $m\tau$ is (p+1)-summable, $mf_*\tau$ is (p+1)-summable. By Theorem 46 in [11]

$$chf_*\tau \in \oplus_{k=1}^{\frac{p}{2}} H_{2k}(BU(n), \mathbb{Q}).$$

But $ch^*\hat{\theta} \in H^{2i}(BU(n), \mathbb{Q})$. Therefore $\langle \hat{\theta},\ f_*\tau \rangle = 0$.

On the other hand,

$$\langle \hat{\theta},\ f_*\tau \rangle = \langle f^*\hat{\theta},\ \tau \rangle = \langle \theta,\ \tau \rangle \neq 0.$$

This contradiction completes the proof of the Theorem.

Q.E.D.

Chapter 5

Open Problems

In this chapter, we consider some open problems and speculate on what their solution might be .

1. In Theorem 3.3 and Theorem 3.4, we classify the C_p-smooth extensions modulo torsion for p an integer. Can one classify the C_p-smooth extensions modulo torsion for arbitary p?

If $p \in (n - \frac{1}{2}, n]$ for an integer n, the problem can be solved. Actually Theorem 3.3 and Theorem 3.4 are still true if we replace C_n-smooth by C_p-smooth for $p \in (n - \frac{1}{2}, n]$. Because $C_p \subset C_n$ when $p \in (n - \frac{1}{2}, n]$, we only need to prove the " if " part of the theorem for this case. But the " if " part is true since Theorem 3.2 holds not only for $p = n$ but also for $p > n - \frac{1}{2} = \frac{2n-1}{2}$ if we assume that M is a 2n$-$1 dimensional manifold.

If $p \in (n, n + \frac{1}{2}]$ for an integer n, the problem is still open. We would like to point out that the problem will be solved if one can prove the following conjecture.

Conjecture 1 *All the $C_{n-\frac{1}{2}}$-smooth extensions of S^{2n-1} are trivial.*

This conjecture is a refinement of Proposition 3 of [16].

2. In this paper, we only classify C_n-smooth extensions modulo torsion. But we believe the following is true.

Conjecture 2 *Let X be a finite CW complex and X^k be the k-skeleton of X. Then $\tau \in Ext(X)$ is C_n-smooth if and only if $\tau \in i_*(Ext(X))$, where $i_* : Ext(X^{2n-1}) \to Ext(X)$ is induced by the inclusion map $i : X^{2n-1} \to X$.*

This conjecture is a refinement of our Theorem 3.4 and a generalization of the conjecture on p.67 in [15] to the case of higher dimensions.

3. In Theorem 3.6, we need the condition that X is a simplicial complex. If we drop this condition, a reasonable guess would be the following statement:

Let $(T_1, T_2, \cdots, T_n)$ be an essentially normal n-tuple of operators. Then there exists an n-tuple of compact operators $(K_1, K_2, \cdots, K_n)$ such that $[T_i + K_i, T_j + K_j] \in C_n$ and $[T_i + K_i, T_j^* + K_j^*] \in C_n$ for all $1 \leq i, j \leq n$.

But the following example is a counterexample.

Example Let $H = \oplus_{i=1}^{\infty} H_i$, and let U_i be the unilateral shift on H_i and $S_i = \alpha_i(U_i - 1)$, where $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence of positive numbers with

$\alpha_i \to 0$. Let $T = \oplus_{i=1}^{\infty} S_i$ act on $H = \oplus_{i=1}^{\infty} H_i$. Then T is an essentially normal operator with essential spectrum X defined by

$$X = \cup_{n=1}^{\infty} \{(x, y) | (x + \alpha_n)^2 + y^2 = \alpha_n^2\} \subseteq R^2 \cong \mathbb{C}.$$

If we let $\alpha_n = n^{-\frac{1}{2}}$, one can prove that $[T + K, T^* + K^*] \notin C_1$ for any compact perturbation $T + K$ of T by using the theorem on p.64 of [15]. And if we let $\alpha_n = n^{-\frac{1}{2p}}$, then one can easily prove $[T, T^*] \notin C_p$. It seems to be true that $[T+K, T^*+K^*] \notin C_p$ for any compact perturbation $T+K$ of T.

4. One can study the C_p-smooth elements of $K^1(\mathcal{A})$ for a noncommutative C^*-algebra $\mathcal{A}$. This problem involves making a suitable choice of a dense subalgebra of $\mathcal{A}$. It will be a start if one can prove the following result corresponding to our Lemma 3.1 : If u is a continuous $*$-homomorphism between $\mathcal{A}$ and $\mathcal{B}$, then u^* maps the C_p-smooth elements of $K^1(\mathcal{B})$ to the C_p-smooth elements of $K^1(\mathcal{A})$. One way to prove this result is to find a $*$-homomorphism v between $\mathcal{A}$ and $\mathcal{B}$ such that v maps the dense subalgebra of $\mathcal{A}$ to the dense subalgebra of $\mathcal{B}$ and v is homotopy equivalent to u.

I believe that the Exel-Loring filtration of $K^1(\mathcal{A})$ [17] should be useful in giving some necessary conditions for an element of $K^1(\mathcal{A})$ to be C_p-smooth. In particular, the following conjecture should be true for some algebras with nice dense subalgebras :

Conjecture 3 *Let $\tau \in K^1(\mathcal{A})$. Then there exists $m \neq 0$ with $m\tau$ C_n-smooth if and only if $\langle \tau,\ \theta \rangle = 0$ for all $\theta \in F_{2n+1}K^1(\mathcal{A})$, where $F_{2n+1}K^1(\mathcal{A})$ is the $(2n+1)^{th}$ filtration of $K^1(\mathcal{A})$ of Exel-Loring.*

One way to approach the conjecture is to reduce the case of a non-commutative algebra to the case of a commutative algebra. If $\tau \in K^1(\mathcal{A})$ and there exists a $*$-homomorphism $f : \mathcal{A} \to C(M)$ with $\tau = f^*\theta$ for some $\theta \in K^1(C(M)) = \mathrm{Ext}(M)$, then we can conclude that τ is C_p-smooth from the fact that θ is C_p-smooth for a suitable choice of a dense subalgebra of $\mathcal{A}$. One can use such an argument to obtain some results along this line.

5. Another problem is to find the "correct" smooth subalgebra of the cross product algebra $C(M) \rtimes G$, where M is a compact smooth manifold and G is a Lie group acting on M (when G is discrete and the action is proper, Baum and Connes have given a nice smooth subalgebra [3]). Furthermore, one could construct nontrivial p-summable Fredholm modules over the smooth subalgebra and find the relation between the set of p-summable Fredholm modules over the smooth subalgebra and the cyclic cohomology of this subalgebra. If G is trivial, we have found the explicit relation in Theorem 4.2.

6. Finally, one can study the p-summable Fredholm modules for bivariant K-homology $KK(C(M), C(N))$ and the relation of the p-summable Fredholm modules with their bivariant Chern character. Work in this direction should be related to that of [29], [26] and [24].

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