Examples of Symplectic Structures On Fiber Bundles

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Abstract of the Dissertation

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In this dissertation we investigate the uniqueness of a symplectic form constructed on the total space of a fiber bundle. We first show that any symplectic form on a product of two compact Riemann surfaces compatible with the topological splitting is isotopic to a split symplectic form. But such uniqueness need not hold in higher dimensions, for we also produce an example of a symplectic form on a 6-dimensional product manifold $X = X' \times X''$ compatible with the topological splitting but not isotopic to any symplectic form split across X' and X''. This result is proven by studying an example of McDuff, and in particular by examining spaces of pseudo-holomorphic spheres in $X' \times X''$ associated to the symplectic form.

In the general case we consider a symplectic fibration $\pi : E \longrightarrow B$ with fiber F, where (F, ω_F) and (B, ω_B) are closed symplectic manifolds. We produce an example of such a fibration and cohomologous 2-forms β and β' on E such that $\beta + \pi^*\omega_B$ and $\beta' + \pi^*\omega_B$ are non-isotopic symplectic

forms on E, both compatible with the fibration. To construct the example we generalize a result of Sternberg to show how a family of connections on a principal bundle $P \longrightarrow B$ can be used to build an extension β of the symplectic forms on the fibers, and also prove a nondegeneracy result to guarantee that $\beta + \pi^*\omega_B$ is nondegenerate and so symplectic. The proof of the non-uniqueness result again uses the example of McDuff and the same techniques of examining spaces of pseudo-holomorphic spheres.

For my parents,

Harry and Vivienne Kasper

Contents

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Chapter 1

Introduction

A symplectic form ω on a manifold M determines two basic pieces of topological data: a deRham cohomology class $[\omega] \in H^2(M)$, and a homotopy class of almost-complex structures on M. By work of Gromov [6] it has long been known that if the manifold M is open then such topological data can always be realized by some symplectic form. If, however, M is closed (compact without boundary) it is not known if such an existence result holds. So at this point we are held to the problem of constructing, by various techniques, examples of symplectic forms on closed manifolds to see what phenomena can occur. On the other hand, there is a basic method (Moser's Theorem) for understanding problems of uniqueness of symplectic forms on closed manifolds. The particular problems we study in this dissertation involve uniqueness of a symplectic form built on the total space of a fiber bundle, given symplectic forms on the base and the fiber.

In Chapter 3 we discuss the easiest example of a fiber bundle, the Cartesian product of two manifolds. If (M_1, ω_1) and (M_2, ω_2) are two symplectic manifolds, then there is a obvious split symplectic form $\omega_1 \oplus \omega_2$ on the product manifold $M_1 \times M_2$; in this case we need make no choices to construct the form on the total space. Split symplectic forms are always compatible with the topological splitting $M_1 \times M_2$, meaning the symplectic form splits in cohomology and restricts symplectically to slices. The question we deal with here is whether a symplectic form on $M_1 \times M_2$ compatible with the topological splitting is isotopic to a split symplectic form. On compact 4-manifolds this is the case.

Theorem 3.1.1 *Let Σ_1 and Σ_2 be compact orientable surfaces and ω a symplectic form on $\Sigma_1 \times \Sigma_2$ compatible with the topological splitting, with $[\omega] = a \oplus b$. Then for any symplectic forms ω_1 on Σ_1 with $[\omega_1] = a$ and ω_2 on Σ_2 with $[\omega_2] = b$, ω is isotopic to $\omega_1 \oplus \omega_2$.*

The main result of Chapter 3 is an example of a 6-dimensional product manifold on which there is a symplectic form compatible with the topological splitting, but that is not isotopic to any split symplectic form.

Theorem 3.1.2 *There exists a symplectic form τ_1 on a product of closed symplectic manifolds $X' \times X''$ such that τ_1 is compatible with the topological splitting but is not isotopic to any symplectic form split across X' and X''.*

Theorem 3.1.2 is proven in Section 3.3 by studying an example of McDuff (reproduced in Section 3.2) of two symplectic forms which determine the same topological data but that are not isotopic. The proofs of both these results follow the same outline. To each of two different symplectic forms on the same manifold we associate spaces of pseudo-holomorphic spheres, and on each space of spheres we define maps to S^2. These maps have different generalized Hopf invariants, which are invariants of a compact bordism class of the maps. If the symplectic forms were isotopic, one could use a (weak) isotopy of the symplectic forms together with a Compactness Theorem for spaces of pseudo-holomorphic spheres to construct a bordism of the above maps to S^2, producing a contradiction.

In Chapter 4 we discuss the general case of a compact fiber bundle; let (F, ω_F) and (B, ω_B) be closed symplectic manifolds, and consider a fiber bundle $\pi : E \longrightarrow B$ with fiber F, and such that the structure group of the bundle preserves ω_F. A symplectic form ω_E on E is compatible with the fibration π if ω_E restricts on each fiber to be ω_F. By a result of Thurston's, the necessary and sufficient condition for finding a compatible symplectic form on E is the existence of a cohomology class in $H^2(E)$ which restricts on each fiber to $[\omega_F]$. To show this, one first constructs a closed 2-form β on E that restricts on each fiber to be ω_F (there are many choices of such an extension); then using the compactness, it is not hard to show that $\beta + K\pi^*\omega_B$ is a compatible symplectic form on E for $K \in \mathbb{R}$ large. A basic uniqueness question is how the isotopy class of the symplectic form so built depends on the choice of the extension β. Our main result is the following.

Theorem 4.1.4 *There exists a compact symplectic fibration $\pi : E \longrightarrow X$ over a compact symplectic manifold (X, ω_X) and two closed cohomologous 2-forms β_0 and β_1 on E such that*

$$\beta_0 + \pi^*\omega_X \quad \text{and} \quad \beta_1 + \pi^*\omega_X$$

are symplectic forms on E, both compatible with the fibration π, but which are not isotopic.

The proof of this Theorem is given in Chapter 5. The construction of the example is based on a generalization (given in Section 4.2) of a method due to Sternberg for building a closed extension β using a connection on a principal bundle. The actual proof of Theorem 4.1.4 follows the same outline as the work in McDuff's Example, and explicitly uses the symplectic forms constructed by her.

To begin with, in Chapter 2 we review basic facts of symplectic geometry and of pseudo-holomorphic curves in symplectic manifolds used to prove our results.

Chapter 2

Preliminaries

2.1 Basic Symplectic Geometry

For basic references on symplectic geometry we refer the reader to Arnold [1], Liebermann and Marle [9], and Weinstein [19].

A *symplectic manifold* is a pair (M, ω) where M is a smooth 2n-manifold and ω is a smooth 2-form on M which is closed ($d\omega = 0$) and nondegenerate; here nondegenerate means the top power

$$\omega^n = \omega \wedge \cdots \wedge \omega$$

never vanishes and so is a volume form on M. Hence all symplectic manifolds have a natural orientation and a natural volume. The basic example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$ where

$$\omega_0 = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Theorem 2.1.1 (Darboux) *About every point in a symplectic manifold (M^{2n}, ω) there exist local coordinates $(x_1, \ldots, x_{2n})$ such that*

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Thus the only local invariant of a symplectic manifold is its dimension. We say a smooth map between symplectic manifolds

$$F : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$$

is *symplectic* if $F^*\omega_2 = \omega_1$. A symplectic diffeomorphism we call a *symplectomorphism.*

Since a symplectic form ω on M is closed it determines a deRham cohomology class $[\omega] \in H^2(M)$. Two symplectic forms ω_0 and ω_1 on M are *weakly isotopic* if they can be joined by a smooth 1-parameter family of symplectic forms ω_t, $0 \leq t \leq 1$, with constant cohomology class. We say ω_0 and ω_1 are *strongly isotopic* if there is an isotopy of M (a smooth 1-parameter family of diffeomorphisms g_t of M with $g_0 = \mathrm{Id}$) such that $g_1^*\omega_1 = \omega_0$. Clearly strongly isotopic implies weakly isotopic.

Theorem 2.1.2 (Moser) *Let M be a compact manifold and ω_0 and ω_1 be weakly isotopic symplectic forms on M. Then ω_0 and ω_1 are strongly isotopic.*

Thus on compact manifolds it is natural to consider uniqueness of symplectic forms up to isotopy. Since all manifolds we deal with will be closed (compact without boundary) we shall simply say two symplectic forms on M are *isotopic*, and equally consider a (weak) isotopy of symplectic forms and an underlying isotopy of the manifold.

An *almost-complex structure* on a manifold M is a bundle automorphism

$$J : TM \longrightarrow TM$$

with $J^2 = -\mathrm{Id}$. Since the symplectic group $Sp(2n, \mathbf{R})$ deformation retracts to the unitary group $U(n)$ every symplectic manifold possesses an almost-complex structure, determined by the symplectic form up to homotopy. We can always choose such an almost-complex structure J to *tame* ω, meaning

$$\omega(v, Jv) > 0 \qquad\qquad \forall v \in T_xM,\ v \neq 0.$$

The space of almost-complex structures on M that tame ω is contractible.

The original definition of nondegeneracy of a 2-form ω is equivalent to the requirement that the map

$$\begin{array}{ccc} TM & \longrightarrow & T^*M \\ v & \longmapsto & i(v)\omega \end{array}$$

is a bundle isomorphism, where $i(v)\omega$ is contraction of v with ω. So any real-valued function f on (M,ω) determines a vector field V_f on M by $i(V_f)\omega = df$; we call V_f the *Hamiltonian vector field* associated to the (Hamiltonian) function f. Since ω is closed, it follows by the Weyl identity $\mathcal{L}_V\omega = d(i(V)\omega) + i(V)d\omega$ that $\mathcal{L}_{V_f}\omega = 0$, so every Hamiltonian vector field is a *symplectic vector field* (the flow generated by the vector field preserves the symplectic form). We will denote by $\mathcal{X}(M,\omega)$ the symplectic vector fields on (M,ω). If a symplectic vector field has compact support, we can integrate it to get a 1-parameter group of symplectomorphisms of (M,ω); thus every symplectic manifold has many automorphisms.

A (left) action of a connected Lie group G on a symplectic manifold (M,ω) is *symplectic* if we are given a smooth embedding of G into the group Diff(M,ω) of symplectic diffeomorphisms of (M,ω); i.e., if $\forall g \in G$ the map

$$\begin{array}{ccc} M & \longrightarrow & M \\ x & \longmapsto & g \cdot x \end{array}$$

is a symplectomorphism of M. There is then an induced Lie algebra homomorphism

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{X}(M,\omega) \\ \xi & \longmapsto & \frac{d}{dt}\left(exp(-t\xi)\cdot x\right)\big|_{t=0} \end{array}$$

where $\mathcal{G}$ is the Lie algebra of G. A symplectic action of G is *strongly Hamiltonian* if this Lie algebra homomorphism lifts as Lie algebra homomorphisms

$$\begin{tikzcd}
\mathcal{C}^\infty(M) \arrow[rr] & & \mathcal{X}(M,\omega) \\
& & \mathcal{G} \arrow[u] \arrow[ull]
\end{tikzcd}$$

so that the diagram commutes. The Lie bracket on $\mathcal{C}^\infty(M)$ is the Poisson bracket $\{f,g\} = \omega(V_g, V_f)$. For each $\xi \in \mathcal{G}$ let $\xi_M \in \mathcal{X}(M,\omega)$ be the symplectic vector field on M corresponding to ξ, and let $H_\xi \in \mathcal{C}^\infty(M)$ be the Hamiltonian function corresponding to ξ. Thus $i(\xi_M)\omega = dH_\xi$.

Given such a lifting there is then a *moment map*

$$\mu : M \longrightarrow \mathcal{G}^*$$

defined by

$$\langle \xi, \mu(z) \rangle = H_\xi(z) \qquad \forall z \in M\,, \xi \in \mathcal{G}^* \tag{2.1.1}$$

where $\langle \cdot, \cdot \rangle$ is the pairing of $\mathcal{G}$ and $\mathcal{G}^*$. Since G is connected the moment map μ is *equivariant*

$$\mu(g \cdot z) = Ad_g^* \mu(z) \qquad \forall g \in G\,, z \in M \tag{2.1.2}$$

where Ad^* is the coadjoint representation of G on $\mathcal{G}^*$ defined by

$$\langle \xi, Ad_g^* \mu \rangle = \langle Ad_{g^{-1}} \xi, \mu \rangle \qquad \forall \xi \in \mathcal{G}\,, \mu \in \mathcal{G}^*\,, g \in G \tag{2.1.3}$$

and Ad is the standard adjoint representation of G on $\mathcal{G}$.

2.2 Pseudo-holomorphic Curves

For standard references to this material we refer the reader to Gromov [5], McDuff [10,11] and Wolfson [22]. Our presentation here is that of McDuff [11].

Let (M, J) be an almost-complex manifold and let (S^2, i) be the 2-sphere with its standard complex structure; a map

$$f : (S^2, i) \longrightarrow (M, J)$$

is said to be a *(rational) pseudo-holomorphic curve* if its derivative is complex linear

$$f_* \circ i = J \circ f_* .$$

The image of such a map we call a J-holomorphic sphere. These curves were introduced into symplectic geometry by Gromov with the idea that, since every symplectic manifold (M, ω) has almost-complex structures, one can study the J-holomorphic curves for a J which tames ω to try to obtain information about the symplectic structure ω. In our case this involves defining invariants associated to certain spaces of J-holomorphic spheres. Since there is a contractible space of almost-complex structures taming ω one needs to understand these invariants as J, and hence the J-spheres, varies in the moduli space.

Let (M, ω) be a closed symplectic manifold. Choose integers s and p large so that the Sobolev space $W_{s,p}(S^2, M)$ of maps $f : S^2 \longrightarrow M$ whose s^{th}-derivative is in L^p makes sense; we want the maps to be at least continuous. Fix a class $A \in H_2(M; \mathbf{Z})$ that is not a multiple class, meaning $A \neq mB$ for $B \in H_2(M; \mathbf{Z})$. Let

$$\mathcal{F}_A = \left\{ f \in W_{s,p}(S^2, M) \mid f \text{ represents class } A \right\}.$$

Let $\mathcal{J} = \mathcal{J}(\omega)$ be the Frechet space of C^∞-smooth almost-complex structures J on M which tame ω. As noted by McDuff [11, Section 2] and proven by Floer [2, Section 5], we may always work with C^∞-smooth almost-complex structures, as follows: For each $J \in \mathcal{J}$ one can define a Hilbert space $\mathcal{J}' = \mathcal{J}'(J)$ of C^∞-smooth perturbations of J such that the L^2-closure of $\mathcal{J}'$ contains an open neighborhood of J in $\mathcal{J}$. In particular $\mathcal{J}$ equals the union of all such $\mathcal{J}'$. Let

$$\mathcal{M}_A(\mathcal{J}') = \{(f, J) \in \mathcal{F}_A \times \mathcal{J}' \mid f \text{ is } J\text{–holomorphic}\}.$$

Theorem 2.2.1 *$\mathcal{M}_A(\mathcal{J}')$ is a C^∞-smooth oriented Banach manifold and the projection $P_A = P_A(\mathcal{J}') : \mathcal{M}_A(\mathcal{J}') \longrightarrow \mathcal{J}'$ is Fredholm with*

$$\operatorname{Index} P_A = 2(n + c(A))$$

where $\dim M = 2n$ *and* $c = c_1(TM, J)$.

By elliptic regularity if J is C^∞ and f is J-holomorphic then f is C^∞. Since we shall always work with C^∞-smooth almost-complex structures, all maps and spaces we deal with will be C^∞-smooth.

A $J \in \mathcal{J}$ is called *regular* if it is a regular value of some Fredholm projection $P_A(\mathcal{J}')$. By standard Fredholm theory the set of regular values is of second category in $\mathcal{J}'$, and hence is dense in $\mathcal{J}$. Let

$$M_p(A, J) = P_A^{-1}(J)$$

be the set of parametrized J-holomorphic A-curves.

Theorem 2.2.2 *For a dense set of regular values $J \in \mathcal{J}$, $M_p(A, J)$ is a C^∞-smooth oriented manifold of dimension $2(n + c(A))$, if it is nonempty. If α is a path between two regular J_1 and J_2, then α may be slightly perturbed relative to the endpoints so that $M_p(A, \alpha) = P_A^{-1}(\alpha)$ is a C^∞-smooth noncompact oriented cobordism of $M_p(A, J_0)$ and $M_p(A, J_1)$.*

For rational pseudo-holomorphic curves we can locate a regular value of P_A as follows. If J is integrable and $f : (S^2, i) \longrightarrow (M, J)$ is J-holomorphic then $f^*(TM)$ is a holomorphic vector-bundle over S^2; by a theorem of Grothendieck $f^*(TM)$ splits uniquely (up to ordering) as a sum of holomorphic line bundles

$$f^*(TM) = L_1 \oplus L_2 \oplus \ldots \oplus L_k.$$

Theorem 2.2.3 *If J is integrable and for all J-holomorphic A-curves $f : (S^2, i) \longrightarrow (M, J)$ every summand of $f^*(TM)$ has $c_1(L_j)(S^2) > -2$, then J is a regular value of $P_A(\mathcal{J}')$, for any $\mathcal{J}'$ containing J.*

Since the Lie group $G = PSL(2, \mathbb{C})$ of automorphisms of (S^2, i) is a noncompact manifold of 6 real dimensions, if $M_p(A, J)$ is nonempty then it cannot be compact. However, in some cases one can show the quotient space of J-holomorphic A-spheres $M_A(J) = M_p(A, J)/G$ is compact.

Theorem 2.2.4 *If $M_A(J) = M_p(A, J)/G$ is not compact then there exists a continuous map $S^2 \longrightarrow M$ representing a homology class $B \in H_2(M; \mathbb{Z})$ such that*

$$0 < \omega(B) < \omega(A) .$$

We say $A \in H_2(M; \mathbb{Z})$ is *ω-simple* if there does not exist a $B \in H_2(M; \mathbb{Z})$ with $0 < \omega(B) < \omega(A)$. Thus if A is ω-simple then $M_A(J)$ is compact for all $J \in \mathcal{J}(\omega)$.

Remark 2.2.5 If A is ω-simple and if J is a smooth regular value for P_A, then $M_p(A, J) \times_G S^2$ is a smooth compact oriented manifold of dimension $2(n + c(A)) + 2 - 6$, and the evaluation map

$$\begin{aligned} ev_A(J) : M_p(A, J) \times_G S^2 &\longrightarrow M \\ (f, z) &\longmapsto f(z) \end{aligned}$$

is a smooth map.

Remark 2.2.6 Theorem 2.2.4 remains true if one replace $M_A(J)$ by

$$M_A(\alpha) = M_p(A, \alpha)/G = P_A^{-1}(\alpha)/G$$

where α is a subset of $\mathcal{J}(\omega)$ which is compact in the C^∞-topology. Moreover, for $\{\omega_t \mid 0 \leq t \leq 1\}$ a weak isotopy of symplectic forms and $\alpha = \{J_t\}$ a smooth path of almost-complex structures with $J_t \in \mathcal{J}(\omega_t)$, if A is ω_0-simple then $M_A(\alpha)$ is also compact.

To see this last statement, we note that A is ω_t-simple for all t, and that the condition for J to tame ω_t is open. One now applies the first statement and uses the compactness of the path α.

Finally, in a 4-dimensional manifold one can obtain extra information about pseudo-holomorphic spheres by looking at their intersections. For the following two theorems we refer to [13].

Theorem 2.2.7 *Two distinct J-holomorphic spheres C and C' in an almost-complex 4-manifold (M, J) have only a finite number of intersection points, and each point of intersection contributes a positive number to the algebraic intersection number $C \cdot C'$. Moreover, a point of intersection contributes $+1$ if and only if the curves intersect transversally at that point.*

Theorem 2.2.8 *A J-holomorphic sphere C in an almost-complex 4-manifold (M, J) is embedded if and only if*

$$c(C) = C \cdot C + 2$$

where $c = c_1(TM, J)$.

Chapter 3

Symplectic Structures on Product Manifolds

3.1 Splitting Symplectic Structures

If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds then there is an obvious symplectic form on the Cartesian product manifold $M_1 \times M_2$,

$$\omega_1 \oplus \omega_2 \stackrel{\text{def}}{=} \pi_1^* \omega_1 + \pi_2^* \omega_2$$

where π_1 and π_2 are the projections of $M_1 \times M_2$ onto the first and second factors. We say a symplectic form ω on $X = M_1 \times M_2$ *splits* with respect to M_1 and M_2 if there exist symplectic forms ω_1 on M_1 and ω_2 on M_2 such that $\omega = \omega_1 \oplus \omega_2$.

Given a manifold X written as a topological product $X = M_1 \times M_2$ and ω a symplectic form on X, we say ω is *compatible with the topological splitting* if

ω splits in cohomology, meaning there exist classes $a \in H^2(M_1)$ and $b \in H^2(M_2)$ such that $[\omega] = a \oplus b$; and

> ω *restricts symplectically to slices*, meaning that ω restricts to some symplectic form on each submanifold of type $M_1 \times \{p_2\}$ and of type $\{p_1\} \times M_2$.

Note that a split symplectic form on $X = M_1 \times M_2$ is compatible with the topological splitting. It is also clear that the converse need not hold; e.g., consider $\sigma_1 \oplus \sigma_2 + df_1 \wedge df_2$ on $S^2 \times S^2$, where f_1 and f_2 are functions on the first and second factors respectively. However, if ω is isotopic to a split symplectic form then the cohomology condition certainly holds. The purpose of this chapter is to study when a symplectic form compatible with a topological splitting is isotopic to a split symplectic form. Our first result is that on compact 4-manifolds this is always the case.

Theorem 3.1.1 *Let Σ_1 and Σ_2 be compact orientable surfaces and ω a symplectic form on $\Sigma_1 \times \Sigma_2$ compatible with the topological splitting, with $[\omega] = a \oplus b$. Then for any symplectic forms ω_1 on Σ_1 with $[\omega_1] = a$ and ω_2 on Σ_2 with $[\omega_2] = b$, ω is isotopic to $\omega_1 \oplus \omega_2$.*

proof:

Let ω_1 be a symplectic form on Σ_1 with $[\omega_1] = a$ and ω_2 be a symplectic form on Σ_2 with $[\omega_2] = b$. Then $[\omega] = [\omega_1] \oplus [\omega_2] = [\omega_1 \oplus \omega_2]$, so ω and $\omega_1 \oplus \omega_2$ determine the same orientation on $\Sigma_1 \times \Sigma_2$. Also, there exists an exact 2-form $d\alpha$ on $\Sigma_1 \times \Sigma_2$ such that $\omega = \omega_1 \oplus \omega_2 + d\alpha$. Let

$$\rho_t = \omega_1 \oplus \omega_2 + t\, d\alpha \qquad 0 \leq t \leq 1$$

so $\rho_0 = \omega_1 \oplus \omega_2$, $\rho_1 = \omega$ and $[\rho_t]$ is constant in $H^2(\Sigma_1 \times \Sigma_2)$. By Moser's Theorem it suffices to show that ρ_t is nondegenerate (and so symplectic) for all $0 \leq t \leq 1$. This we do at each point of $\Sigma_1 \times \Sigma_2$. The idea is that since we are in dimension 4, the hypotheses on ω give us lower bounds on its coefficients.

Let $(p,q) \in \Sigma_1 \times \Sigma_2$; by Darboux's Theorem there exist local coordinates (x_1, x_2) about $p \in \Sigma_1$ and (x_3, x_4) about $q \in \Sigma_2$ such that

$$\omega_1 = dx_1 \wedge dx_2 \qquad \text{and} \qquad \omega_2 = dx_3 \wedge dx_4.$$

Then (x_1, x_2, x_3, x_4) are local coordinates about $(p,q) \in \Sigma_1 \times \Sigma_2$, and so near (p,q)

$$\begin{aligned}
\omega &= \omega_1 \oplus \omega_2 + d\alpha \\
&= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + f_1\, dx_1 \wedge dx_2 + f_2\, dx_1 \wedge dx_3 + f_3\, dx_1 \wedge dx_4 \\
&\quad + f_4\, dx_2 \wedge dx_3 + f_5\, dx_2 \wedge dx_4 + f_6\, dx_3 \wedge dx_4 \\
&= (1+f_1)\, dx_1 \wedge dx_2 + (1+f_6)\, dx_3 \wedge dx_4 + f_2\, dx_1 \wedge dx_3 + f_3\, dx_1 \wedge dx_4 \\
&\quad + f_4\, dx_2 \wedge dx_3 + f_5\, dx_2 \wedge dx_4
\end{aligned}$$

for some functions $f_1, \ldots, f_6$ near (p,q). Since ω restricted to slices is symplectic and

$$\omega\big|_{\Sigma_1 \times \{q\}} = (1+f_1) dx_1 \wedge dx_2\big|_{\Sigma_1 \times \{q\}}$$

the function $1 + f_1$ never vanishes. Moreover, $\omega\big|_{\Sigma_1 \times \{q\}}$ and ω_1 determine the same orientation on Σ_1, and so

$$1 + f_1 > 0. \tag{3.1.1}$$

Similarly

$$1 + f_6 > 0. \tag{3.1.2}$$

A simple calculation shows

$$\omega^2 = 2(1 + f_1 + f_6 + f_1 f_6 + f_3 f_4 - f_2 f_5)\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

and since ω and $\omega_1 \oplus \omega_2$ determine the same orientation, we have

$$1 + f_1 + f_6 + f_1 f_6 + f_3 f_4 - f_2 f_5 > 0. \tag{3.1.3}$$

Let $A = f_1 + f_6$ and $B = f_1 f_6 + f_3 f_4 - f_2 f_5$; then by (3.1.1) and (3.1.2)

$$A > -2$$

and by (3.1.3)

$$B > -1 - A.$$

Finally, the same calculations show that near (p, q)

$$\begin{aligned}(\rho_t)^2 &= (\omega_1 + \omega_2 + t\, d\alpha)^2 \\ &= 2(1 + tA + t^2 B) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.\end{aligned}$$

To show nondegeneracy of ρ_t at (p, q) it suffices to show the function $(1 + tA + t^2 B)$ is positive for all $0 \leq t \leq 1$. But

$$\begin{aligned}1 + tA + t^2 B &> 1 + tA + t^2(-1 - A) \\ &= 1 - t^2 + (t - t^2)A \\ &> 1 - t^2 + (t - t^2)(-2) \\ &= 1 - 2t + t^2 \\ &= (1 - t)^2 \\ &> 0 \qquad \forall\, 0 \leq t < 1\end{aligned}$$

And at $t = 1$, $1 + tA + t^2 B = 1 + A + B > 0$ since $\rho_1 = \omega$ is symplectic .

qed

Thus on a product of compact orientable surfaces a symplectic form compatible with the topological splitting is determined, up to isotopy, by its cohomology class. The main result of this chapter is that a symplectic form compatible with a topological splitting need not be isotopic to a split symplectic form.

Theorem 3.1.2 *There exists a symplectic form τ_1 on a product of closed symplectic manifolds $X' \times X''$ such that τ_1 is compatible with the topological splitting but is not isotopic to any symplectic form split across X' and X''.*

The proof of Theorem 3.1.2 is given in Section 3.3, by examining a specific symplectic form constructed by McDuff on the 6-dimensional manifold $S^2 \times S^2 \times T^2$.

We should compare these results to splitting theorems in other geometries. It is well-known that a Riemannian metric on a product manifold need not be isometric to a split metric $g_1 \oplus g_2$; in fact, since curvature is a local obstruction such splittings are very special. On the other hand, by Moser's Theorem for volume forms [15] every volume form Ω on a product of compact orientable manifolds $M_1 \times M_2$ is isotopic to a split volume form $\pi_1^*\Omega_1 \wedge \pi_2^*\Omega_2$, where Ω_1 and Ω_2 are volume forms on M_1 and M_2, respectively. To apply Moser's Theorem one chooses Ω_1 and Ω_2 by scaling volumes and choosing orientations so that $[\Omega] = [\pi_1^*\Omega_1 \wedge \pi_2^*\Omega_2]$. Then since the space of top-dimensional forms on a manifold is 1-dimensional

$$\Omega_t = (1-t)\,\Omega + t\,\pi_1^*\Omega_1 \wedge \pi_2^*\Omega_2 \qquad 0 \le t \le 1$$

is a 1-parameter family of volume forms with constant cohomology class.

3.2 An Example of McDuff

In this section we explain an example, due to McDuff, of two symplectic forms on $S^2 \times S^2 \times T^2$ that determine the same cohomology class (and homotopy class of almost-complex structures) but that are not isotopic. One of these forms is the standard one, and the other is the one used to prove Theorem 3.1.2. Since this example has not appeared explicitly in the literature we reproduce the complete proof. This construction was motivated by an observation of Gromov [5] that there exists a symplectic form ρ on $S^2 \times S^2$ such that there is a noncontractible loop in $\mathrm{Diff}(S^2 \times S^2, \rho)$. The example is also the basis for the construction in McDuff [10], and is in fact a simpler version of that one.

Let $\sigma_1 = \sigma_2$ both denote the standard symplectic form on S^2 and σ_3 the standard symplectic form on T^2, all of total area 1. Here we consider T^2 as the usual quotient of the unit square $\{(s,t) \mid 0 \le s, t \le 1\}$. Let

$$X = S^2 \times S^2 \times T^2$$

and let

$$\tau_0 = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \tag{3.2.1}$$

be the standard symplectic form on X. Let $\theta : X \longrightarrow X$ be the diffeomorphism

$$\theta(z, w, s, t) = (z, \psi_{z,t}(w), s, t) \tag{3.2.2}$$

where $\psi_{z,t} : S^2 \longrightarrow S^2$ is rotation of S^2 by angle $2\pi t$ about the axis through z, and let

$$\tau_1 = \theta^* \tau_0. \tag{3.2.3}$$

These are the two symplectic forms we are interested in.

By checking on the generators of $H_2(X; \mathbb{Z})$

$$\begin{aligned} A &= [S^2 \times w_0 \times s_0 \times t_0] \\ B &= [z_0 \times S^2 \times s_0 \times t_0] \\ C &= [z_0 \times w_0 \times T^2] \end{aligned}$$

it is easy to see that $\theta^* = Id$ on $H^2(X)$. In particular

$$[\tau_0] = [\tau_1] = [\sigma_1] \oplus [\sigma_2] \oplus [\sigma_3].$$

The main goal of this section is to prove the following.

Theorem 3.2.1 (McDuff) *τ_1 is not isotopic to τ_0.*

We first want to construct two spaces Γ_0 and Γ_1 of rational pseudo-holomorphic spheres in X associated to τ_0 and τ_1, respectively. Let J_0 be the standard (split, integrable) complex structure on X, so $J_0 \in \mathcal{J}(\tau_0)$ (i.e., J_0 tames τ_0). Recall $M_p(A, J_0)$ is the space of parametrized J_0-holomorphic A-curves. Also let

$$J_1 = \theta^* J_0 \stackrel{def}{=} (\theta^{-1})_* \circ J_0 \circ \theta_* \quad \in \mathcal{J}(\tau_1).$$

J_1 is also integrable, and θ is a biholomorphism of the complex manifolds (X, J_0) and (X, J_1).

Lemma 3.2.2 *(i) The $f \in M_p(A, J_0)$ are exactly the maps*

$$f(z) = (g(z), pt., pt., pt.) \text{ where } g \in PSL(2, \mathbb{C}).$$

(ii) The $f' \in M_p(A, J_1)$ are exactly the maps

$$f' = \theta^{-1} \circ f \text{ where } f \in M_p(A, J_0).$$

proof:

Since J_0 is split and integrable, the projections onto each factor S^2, S^2 and T^2 are holomorphic. If $f \in M_p(A, J_0)$ then $\pi_1 \circ f : S^2 \longrightarrow S^2$ is holomorphic, and since f represents class A, $\pi_1 \circ f$ represents a generator of $H_2(S^2; \mathbb{Z})$ and so has degree 1. Thus $\pi_1 \circ f \in PSL(2, \mathbb{C})$. Similarly $\pi_2 \circ f$ and $\pi_3 \circ f$ are holomorphic and represent class 0 in $H_2(S^2; \mathbb{Z})$ and $H_2(T^2; \mathbb{Z})$, respectively, and so are constant.

To see (ii), if f is J_0-holomorphic then

$$\theta_*^{-1} \circ f_* \circ i = \theta_*^{-1} \circ J_0 \circ f_* = J_1 \circ \theta_*^{-1} \circ f_*$$

and so $f' = \theta^{-1} \circ f$ is J_1-holomorphic. Similarly, if f' is J_1-holomorphic, then $f = \theta \circ f'$ is J_0-holomorphic.

qed

Corollary 3.2.3 *Up to parametrization, there exists a unique J_0-holomorphic A-curve and a unique J_1-holomorphic A-curve through each point of X.*

We now claim that $c = c_1(TX, J_0) = c_1(TX, J_1) = 2[\sigma_1] \oplus 2[\sigma_2]$ and so $c(A) = 2$. To see this, note that (TX, J_0) splits as holomorphic line bundles

$$TX = TS^2 \oplus TS^2 \oplus TT^2.$$

Then $c_1(TX, J_0) = c_1(TS^2, i) \oplus c_1(TS^2, i) \oplus c_1(TT^2, i) = 2[\sigma_1] \oplus 2[\sigma_2]$. And since θ is a biholomorphism of (X, J_0) and (X, J_1) and is the identity on $H^2(X)$, $c_1(TX, J_0) = c_1(TX, J_1)$. Let $G = PSL(2, \mathbb{C})$.

Proposition 3.2.4 *J_0 and J_1 are regular values for the projection P_A, for each $i = 0, 1$ $M_p(A, J_i) \times_G S^2$ is a smooth compact oriented 6-manifold, and the evaluation maps*

$$\begin{array}{rcl} ev_A(J_i) : M_p(A, J_i) \times_G S^2 & \longrightarrow & X \\ (f, z) & \longmapsto & f(z) \end{array}$$

are diffeomorphisms.

proof:

To see that J_0 and J_1 are regular values, we apply Theorem 2.2.3. Since (TX, J_0) splits as holomorphic line bundles and $f \in M_p(A, J_0)$ has the form $f(z) = (g(z), pt., pt., pt.)$,

$$f^*(TX, J_0) = TS^2 \oplus L_2 \oplus L_3$$

where L_2 and L_3 are trivial holomorphic line bundles over S^2. Now $c_1(L_2) = c_1(L_3) = 0$ while $c_1(TS^2)(S^2) = 2$. Thus J_0 is regular. For $f' \in M_p(A, J_1)$, $f' = \theta^{-1} \circ f$ with $f \in M_p(A, J_0)$, and so

$$\begin{aligned} (f')^*(TX, J_1) &= f^*(\theta^{-1})^*(TX, J_1) \\ &= f^*(TX, J_0) \\ &= TS^2 \oplus L_2 \oplus L_3 \end{aligned}$$

as above, and so J_1 is regular.

By our area normalizations, τ_0 and τ_1 evaluate on each of the generators of $H_2(X;\mathbf{Z})$ to be 1. Then by the Compactness Theorem (Remark 2.2.5) $M_p(A,J_0)\times_G S^2$ and $M_p(A,J_1)\times_G S^2$ are smooth compact oriented 6-manifolds, and the evaluation maps are smooth. By Corollary 3.2.3 these maps are surjective.

We need to show the maps have full rank. There is a smooth section of the fibration $M_p(A,J_0)\times S^2 \longrightarrow M_p(A,J_0)\times_G S^2$ with image

$$\Sigma_0 = \{(f_{w_1,s_1,t_1}, z)\,|\, w_1 \in S^2,\ s_1,t_1 \in S^1,\ z\in S^2\}$$

where $f_{w_1,s_1,t_1} \in M_p(A,J_0)$ is $f_{w_1,s_1,t_1}(z) = (z,w_1,s_1,t_1)$. Considering $ev_A(J_0)$ as defined on Σ_0,

$$ev_A(J_0)(f_{w_1,s_1,t_1}, z) = (z, w_1, s_1, t_1)$$

which clearly has full-rank. Similarly one considers $ev_A(J_1)$ as defined on the image of the section

$$\Sigma_1 = \{(f'_{w_1,s_1,t_1}, z)\,|\, w_1 \in S^2,\ s_1,t_1 \in S^1,\ z\in S^2\}$$

where $f'_{w_1,s_1,t_1} \in M_p(A,J_0)$ is $f'_{w_1,s_1,t_1}(z) = (z, \psi_{z,-t_1}(w_1), s_1, t_1)$.

qed

The above results can be repeated for the class $B = [z_0 \times S^2 \times s_0 \times t_0]$ to give the following, which we will use in Section 3.3.

Proposition 3.2.5 *J_0 and J_1 are regular values for the projection P_B, for each $i=0,1$ $M_p(B,J_i)\times_G S^2$ is a smooth compact oriented 6-manifold, and the evaluation maps*

$$ev_B(J_i) : M_p(B,J_i)\times_G S^2 \longrightarrow X$$

are diffeomorphisms.

Using the diagonal action of G on $S^2 \times S^2$, Proposition 3.2.4 shows that $M_p(A, J_0) \times_G (S^2 \times S^2)$ and $M_p(A, J_1) \times_G (S^2 \times S^2)$ are smooth compact oriented manifolds of dimension 8, and the evaluation maps

$$\begin{array}{rcl} ev_i = ev_A(J_i) : M_p(A, J_i) \times_G (S^2 \times S^2) & \longrightarrow & X \\ (f, z, z') & \longmapsto & f(z) \end{array}$$

are submersions for each $i = 0, 1$. Note that we evaluate on the first factor of $S^2 \times S^2$.

Fix distinguished points $w_0 \in S^2$ and $s_0 \in S^1$ and let

$$Z = \{(w_0, w_0, s_0, u) \,|\, u \in S^1\}$$

which is a smooth S^1 embedded in X. Then ev_0 and ev_1 are transverse to Z; let

$$\begin{array}{rcl} \Gamma_0 = ev_0^{-1}(Z) & \cong & S^1 \times S^2 \\ \Gamma_1 = ev_1^{-1}(Z) & \cong & S^1 \times S^2. \end{array}$$

To see that Γ_0 and Γ_1 are each diffeomorphic to $S^1 \times S^2$, consider ev_0 as defined on $\Sigma_0 \times S^2 \subset M_p(A, J_0) \times S^2 \times S^2$, where Σ_0 is as in the proof of Proposition 3.2.4. Then

$$ev_0^{-1}(Z) = \{(f_u, z') \,|\, u \in S^1 \,,\, z' \in S^2\} \cong S^1 \times S^2$$

where $f_u \in M_p(A, J_0)$ is $f_u(z) = (z, w_0, s_0, u)$. Similarly,

$$ev_1^{-1}(Z) = \{(f'_u, z') \,|\, u \in S^1 \,,\, z' \in S^2\} \cong S^1 \times S^2$$

where $f'_u \in M_p(A, J_1)$ is $f'_u(z) = (z, \psi_{z,-u}(w_0), s_0, u)$. Γ_0 and Γ_1 are the spaces of J_0- (resp. J_1-) holomorphic A-spheres which intersect Z. To prove that τ_1 is not isotopic to τ_0, we will define an invariant on each of these spaces of spheres, calculate these invariants are different, and finally argue that if τ_1 were isotopic to τ_0 the invariants must be equal, producing a contradiction.

Consider the map $\varphi_0 : \Gamma_0 \longrightarrow S^2$ given as the composition

$$\begin{array}{ccccc} \Gamma_0 & \longrightarrow & X & \longrightarrow & S^2 \\ (f,z,z') & \longmapsto & f(z') & \longmapsto & \pi_2(f(z')) \end{array}$$

where now we evaluate $f \in M_p(A, J_0)$ on the second factor of $S^2 \times S^2$, and then project onto the second factor of X. By Lemma 3.2.2 each J_0-holomorphic A-sphere which intersects Z has constant second factor in X, so $\varphi_0 \equiv$ constant. Similarly consider $\varphi_1 : \Gamma_1 \longrightarrow S^2$ given as the composition

$$\begin{array}{ccccc} \Gamma_1 & \longrightarrow & X & \longrightarrow & S^2 \\ (f',z,z') & \longmapsto & f'(z') & \longmapsto & \pi_2(f'(z')) \end{array}$$

again evaluating on the second factor of $S^2 \times S^2$ and then projecting. Then as a map from $S^1 \times S^2$ to S^2

$$\varphi_1(u,z) = \psi_{z,-u}(w_0) \qquad u \in S^1,\, z \in S^2. \tag{3.2.4}$$

We note that as maps from $S^1 \times S^2$ to S^2 both φ_0 and φ_1 are zero on $H_2(S^1 \times S^2; \mathbb{Z})$. This is trivial for φ_0 the constant map; for φ_1, a generator of $H_2(S^1 \times S^2; \mathbb{Z})$ is $[\{0\} \times S^2]$, and

$$(\varphi_1)_*[\{0\} \times S^2] = [\{\psi_{z,0}(w_0)\,|z \in S^2\}] = [w_0] = 0$$

since $\psi_{z,0} = Id$. In [10, Section 5], McDuff defines a *generalized Hopf invariant* $\chi(\varphi)$ for any map $\varphi : S^1 \times S^2 \longrightarrow S^2$ which is zero on $H_2(S^1 \times S^2; \mathbb{Z})$ – such maps factor through S^3 – and shows $\chi(\varphi)$ is an oriented bordism invariant for bordisms (W, Φ) with $\Phi_*(H_2(W; \mathbb{Z})) = 0$. Moreover, she shows that the maps

$$\begin{array}{ccccc} S^1 \times S^2 & \longrightarrow & S^2 & & \\ (t,z) & \longmapsto & \psi_{z,rt}(w_0) & & r \in \mathbb{Z} \end{array}$$

have generalized Hopf invariant $\chi = r$. Thus $\chi(\varphi_0) = 0$ and $\chi(\varphi_1) = -1$, and so there does *not* exist a compact oriented manifold W with boundary $\Gamma_0 \amalg -\Gamma_1$ and a map $\Phi : W \longrightarrow S^2$ with $\Phi|_{\Gamma_0} = \varphi_0$ and $\Phi|_{\Gamma_1} = \varphi_1$ and such that Φ_* is zero on $H_2(W; \mathbf{Z})$.

We now show that if τ_0 is isotopic to τ_1 then such a bordism must exist; i.e., that $\chi(\varphi_0) = \chi(\varphi_1)$. So suppose τ_t, $0 \leq t \leq 1$, is a smooth 1-parameter family of symplectic forms on X with constant cohomology class. Then there exists a path $\alpha = \{J_t \,|\, 0 \leq t \leq 1\}$ of smooth almost-complex structures from J_0 to J_1, where each $J_t \in \mathcal{J}(\tau_t)$. By Theorem 2.2.2 we may assume that α is transverse to the Fredholm projection operator P_A. Let $M_p(A, \alpha) = P_A^{-1}(\alpha)$; then, as in Remark 2.2.5, $M_p(A, \alpha) \times_G (S^2 \times S^2)$ is a smooth oriented manifold of dimension 9. Moreover, since A is τ_t-simple for all t, by Remark 2.2.6 it is compact. Note this is where we use the existence of the isotopy τ_t.

The boundary of $M_p(A, \alpha) \times_G (S^2 \times S^2)$ is

$$M_p(A, J_0) \times_G (S^2 \times S^2) \;\amalg\; -M_p(A, J_1) \times_G (S^2 \times S^2).$$

We may perturb (modulo the boundary) the evaluation map

$$\begin{array}{rcl} ev_\alpha = ev_A(\alpha) : M_p(A, \alpha) \times_G (S^2 \times S^2) & \longrightarrow & X \\ (f, z, z') & \longmapsto & f(z) \end{array}$$

to a map $\widetilde{ev_\alpha}$ transverse to Z, so that $W = \widetilde{ev_\alpha}^{-1}(Z)$ is a smooth compact oriented 4-manifold, with boundary $\partial W = \Gamma_0 \amalg -\Gamma_1$. Let $\Phi : W \longrightarrow S^2$ be given as the composition

$$\begin{array}{ccccc} W & \longrightarrow & X & \longrightarrow & S^2 \\ (f, z, z') & \longmapsto & f(z') & \longmapsto & \pi_2(f(z')) \end{array}$$

and so $\Phi|_{\Gamma_0} = \varphi_0$ and $\Phi|_{\Gamma_1} = \varphi_1$.

Lemma 3.2.6 $\Phi_* = 0$ *on* $H_2(W)$.

proof:

To do this, we show we can be a bit more careful in the construction of W. The evaluation map $ev_\alpha : M_p(A,\alpha) \times_G (S^2 \times S^2) \longrightarrow X$ factors through the fibration $S^2 \longrightarrow M_p(A,\alpha) \times_G (S^2 \times S^2) \longrightarrow M_p(A,\alpha) \times_G S^2$, where the fibers are the second factor of $S^2 \times S^2$. Thus we may choose our perturbation $\widetilde{ev_\alpha}$ to be independent of the second factor of $S^2 \times S^2$, so that $W = \widetilde{ev_\alpha}^{-1}(Z)$ is an oriented bundle with fibers S^2 over a compact oriented surface-with-boundary $\overline{W} \subset M_p(A,\alpha) \times_G S^2$ and with structure group G. Here the fibers correspond to the second factor of $S^2 \times S^2$, while a point of $\overline{W}$ represents a parametrized J_t-holomorphic A-sphere which intersects Z. By restricting to the components of $\overline{W}$ with nonempty boundary, if necessary, $H^2(\overline{W}) = 0$.

By the Gysin sequence

$$\longrightarrow H^2(\overline{W}) \xrightarrow{\pi^*} H^2(W) \xrightarrow{\pi_*} H^0(\overline{W}) \xrightarrow{\wedge e} H^3(\overline{W}) \longrightarrow$$

we see that integration along the fiber $\pi_* : H^2(W) \longrightarrow H^0(\overline{W})$ is an isomorphism, and moreover $H^2(W) \cong H^0(\overline{W}) \otimes H^2(S^2)$. Thus it suffices to show $\pi_*\Phi^*[\sigma_2] \in H^0(\overline{W})$ is zero, where $[\sigma_2]$ is the generator of $H^2(S^2)$. Let $f \in M_p(A,\alpha)$ be such that f intersects Z, so f represents a 0-cycle of $\overline{W}$. Then

$$\begin{aligned}
(\pi_*\Phi^*[\sigma_2])[f] &= (\Phi^*[\sigma_2])([f] \otimes [S^2] \\
&= [\sigma_2](\Phi_*([f] \otimes [S^2])) \\
&= [\sigma_2](A) \\
&= 0
\end{aligned}$$

qed

Thus (W, Φ) is a bordism of the required type between (Γ_0, φ_0) and (Γ_1, φ_1), and so $\chi(\varphi_0) = \chi(\varphi_1)$. This is a contradiction, and completes the proof of Theorem 3.2.1.

The key point of this proof is the area normalizations of the symplectic form τ_0, which allows us to apply the Compactness Theorem (Remark 2.2.6) to construct the (compact) cobordism W. If the area of τ_0 on any factor of X is increased then the Compactness Theorem no longer applies in this manner, and in fact these perturbed symplectic forms can be isotopic. The following results are from [10, section 3].

For $\lambda > 1$ consider the symplectic form $(\lambda\sigma_1) \oplus \sigma_2$ on $S^2 \times S^2$. Define a loop of diffeomorphisms of $S^2 \times S^2$ by

$$\theta_t(z, w) = (z, \psi_{z,t}(w)) \qquad 0 \leq t \leq 1.$$

Lemma 3.2.7 ([10, lemma 3.1]) *For each $\lambda > 1$, $\{\theta_t\}$ is isotopic to a loop of diffeomorphisms which preserve $(\lambda\sigma_1) \oplus \sigma_2$.*

Now on $X = S^2 \times S^2 \times T^2$ consider, for each $\lambda > 1$, the symplectic forms

$$\tau_0^\lambda = (\lambda\sigma_1) \oplus \sigma_2 \oplus \sigma_3 \quad \text{and} \quad \tau_1^\lambda = \theta^*(\tau_0^\lambda). \tag{3.2.5}$$

Proposition 3.2.8 *For each $\lambda > 1$, τ_0^λ is isotopic to τ_1^λ.*

proof:

See [10, Corollary 3.2]. Let $\{\theta'_t\}$ be a loop of $((\lambda\sigma_1) \oplus \sigma_2)$-preserving diffeomorphisms of $S^2 \times S^2$ which is isotopic to $\{\theta_t\}$, and define a diffeomorphism θ' of X by $\theta'(z, w, s, t) = (\theta'_t(z, w), s, t)$. Then there is an exact 1-form α on $S^2 \times S^2$ such that $\tau_1^\lambda = (\theta')^*(\tau_0^\lambda) = \tau_0^\lambda + dt\wedge\alpha$. Apply Moser's Theorem to the path $\tau_0^\lambda + r(dt\wedge\alpha)$, $0 \leq r \leq 1$, to get a diffeomorphism θ'' isotopic to θ' and such that θ'' preserves τ_0^λ. Then $\tau_0^\lambda = (\theta'')^*\tau_0^\lambda$ is isotopic to $(\theta)^*\tau_0^\lambda = \tau_1^\lambda$.

qed

Corollary 3.2.9 *There is a smooth path of symplectic forms joining* τ_0 *and* τ_1.

Note, however, no such path can have constant cohomology class. In any event, there is a smooth path of almost-complex structures J_t joining J_0 and J_1. Thus τ_0 and τ_1 define the same homotopy class of almost-complex structures on TX.

3.3 Proof of Theorem 3.1.2

Let $X' = S^2 \times S^2$ and $X'' = T^2$ and let τ_1 be the symplectic form on $X = X' \times X''$ defined by equation (3.2.3).

Proposition 3.3.1 τ_1 *is compatible with the topological splitting* $X' \times X''$.

proof:

Since $[\tau_1] = [\sigma_1] \oplus [\sigma_2] \oplus [\sigma_3]$ and $H^2(X) \cong H^2(X') \oplus H^2(X'')$, τ_1 splits in cohomology.

For slices of the form $X' \times (s_0, t_0)$, consider the composition

$$X' \overset{j}{\hookrightarrow} X \xrightarrow{\theta} X \xrightarrow{\pi_1 \times \pi_2} S^2 \times S^2$$

with $j(z, w) = (z, w, s_0, t_0)$. Then $(\pi_1 \times \pi_2) \circ \theta \circ j$ is a diffeomorphism from X' to $S^2 \times S^2$. Since $\mathrm{Image}(\theta \circ j)_* \subseteq T(S^2 \times S^2) \oplus 0$,

$$j^*\tau_1 = j^*\theta^*(\sigma_1 \oplus \sigma_2) = j^*\theta^*(\pi_1 \times \pi_2)^*(\sigma_1 \oplus \sigma_2)$$

and so $j^*\tau_1$ is symplectic on X'.

For slices of the form $(z_0, w_0) \times X''$, consider the composition

$$X'' \overset{j}{\hookrightarrow} X \xrightarrow{\theta} X$$

with $j(s,t) = (z_0, w_0, s, t)$. Then $(\theta \circ j)(s,t) = (z_0, \psi_{z_0,t}(w_0), s, t)$ and so for $v \in T_{(s,t)}X''$, $(\theta \circ j)_*(v) = v_1 \oplus v$ where $v_1 \in \xi$ and ξ is a 1-dimensional subspace of $T_{(z_0,w_0)}X'$. Then for $v, w \in T_{(s,t)}X''$

$$\begin{aligned}
(j^*\tau_1)(v,w) &= ((\theta \circ j)^*\tau_0)(v,w) \\
&= \tau_0(v_1 \oplus v, w_1 \oplus w) \\
&= \sigma_2(v_1, w_1) + \sigma_3(v,w) \\
&= \sigma_3(v,w)
\end{aligned}$$

where $\sigma_2(v_1, w_1) = 0$ since ξ is 1-dimensional.

qed

To prove Theorem 3.1.2, we suppose there exist symplectic forms ρ on X' and σ on X'' such that τ_1 is isotopic to $\rho \oplus \sigma$. Note we must have $[\rho] = [\sigma_1] \oplus [\sigma_2]$ and $[\sigma] = [\sigma_3]$. Thus we may assume τ_1 is isotopic to $\rho \oplus \sigma_3$. Our original idea for this proof was to argue, using a result of Gromov's (c.f. Lemma 3.3.3) that ρ must be isotopic to a split symplectic form on X', for then $\rho \oplus \sigma_3$ (and so τ_1) would be isotopic to τ_0, contradicting Theorem 3.2.1. One can show that ρ is diffeomorphic to a split symplectic form, but it is not clear this diffeomorphism is isotopic to the identity. Instead we follow the outline of Section 3.2, and again define two spaces of pseudo-holomorphic spheres associated to τ_1 and $\rho \oplus \sigma_3$, respectively, and for maps from these spaces get generalized Hopf invariants. Since τ_1 and $\rho \oplus \sigma_3$ are assumed to be isotopic, these Hopf invariants must be equal; but we can then calculate directly they are different, producing a contradiction.

Let $J_2 = (h_1)_* J_1$ where h_1 is the diffeomorphism of X (generated by the isotopy) such that $h_1^*(\rho \oplus \sigma_3) = \tau_1$. Then $J_2 \in \mathcal{J}(\rho \oplus \sigma_3)$, and since J_2 is diffeomorphic to J_0 by a diffeomorphism which is the identity on $H_2(X; \mathbb{Z})$, by the proofs of Propositions 3.2.4 and 3.2.5 we have

Lemma 3.3.2 *J_2 is a regular value for P_A and for P_B, and the evaluation maps*

$$\begin{aligned} ev_A(J_2) : M_p(A, J_2) \times_G S^2 &\longrightarrow X \\ ev_B(J_2) : M_p(B, J_2) \times_G S^2 &\longrightarrow X \end{aligned}$$

are diffeomorphisms.

In particular, both maps have degree 1.

Choose $J_3' \in \mathcal{J}(\rho)$ an almost-complex structure on X' and let $J_3'' \in \mathcal{J}(\sigma_3)$ be the standard complex structure on $X'' = T^2$; then

$$J_3 = J_3' \oplus J_3'' \in \mathcal{J}(\rho \oplus \sigma_3)$$

is an almost-complex structure on $X = X' \times X''$. If $J \in \mathcal{J}(\rho \oplus \sigma_3)$ is a regular value for both P_A and P_B then by Theorem 2.2.2 and Remark 2.2.6 the evaluation maps

$$\begin{aligned} ev_A(J) : M_p(A, J) \times_G S^2 &\longrightarrow X \\ ev_B(J) : M_p(B, J) \times_G S^2 &\longrightarrow X \end{aligned}$$

also have degree 1 since degree is a bordism invariant. Since regular values are dense in $\mathcal{J}(\rho \oplus \sigma_3)$ it follows that these maps are surjective for any $J \in \mathcal{J}(\rho \oplus \sigma_3)$, and in particular for $J = J_3$. Thus there exists a J_3-holomorphic A-sphere and a J_3-holomorphic B-sphere through each point of X. Also note that since J_3 is split each J_3-holomorphic A-curve and B-curve is constant in the X''-factor. Thus each J_3-holomorphic A-sphere and B-sphere project onto a J_3'-holomorphic A'-sphere and B'-sphere in X', where

$$\begin{aligned} A' &= [S^2 \times w_0] \in H_2(X'; \mathbb{Z}) \\ B' &= [z_0 \times S^2] \in H_2(X'; \mathbb{Z}). \end{aligned}$$

In particular there is a J_3'-holomorphic A'-sphere and B'-sphere through each point of X'. Since $\rho(A') = \rho(B') = 1$, it follows by the Compactness Theorem (Theorem 2.2.4) that regular values of $P_{A'}$ and $P_{B'}$ are open and dense in $\mathcal{J}(\rho)$, and so we may assume that J_3' is a regular value for both $P_{A'}$ and $P_{B'}$. Thus $M_p(A', J_3') \times_G S^2$ and $M_p(B', J_3') \times_G S^2$ are compact oriented manifolds of dimension 4. Since $A' \cdot B' = 1$ and $A' \cdot A' = B' \cdot B' = 0$, by Theorems 2.2.7 and 2.2.8 each J_3'-holomorphic A'-sphere and B'-sphere is embedded, and each A'-sphere and B'-sphere intersect transversally at exactly one point. As observed by Gromov [5, 2.4.A_1'], in this situation one can show that

Lemma 3.3.3 *There exists a symplectomorphism F_2 from (X', ρ) to $(S^2 \times S^2, \sigma_1 \oplus \sigma_2)$.*

proof:

We first define a diffeomorphism $F_1 : X' \longrightarrow S^2 \times S^2$ which takes the J_3'-holomorphic A'-spheres to $S^2 \times pt.$ and the J_3'-holomorphic B'-spheres to $pt. \times S^2$, as follows. The evaluation maps

$$ev_{A'} = ev_{A'}(J_3') : M_p(A', J_3') \times_G S^2 \longrightarrow X'$$
$$ev_{B'} = ev_{B'}(J_3') : M_p(B', J_3') \times_G S^2 \longrightarrow X'$$

are surjective since there is an A'-sphere and a B'-sphere through each point of X'; and since $A' \cdot A' = B' \cdot B' = 0$ both maps have degree 1. The argument of [12, lemma 3.5] shows they have full-rank, and so are diffeomorphisms. If Δ is the diagonal in $X' \times X'$, then $(ev_{A'} \times ev_{B'})^{-1}(\Delta)$ is a smooth submanifold of

$$(M_p(A', J_3') \times_G S^2) \times (M_p(B', J_3') \times_G S^2) = \\ (M_p(A', J_3') \times M_p(B', J_3')) \times_{G \times G} (S^2 \times S^2).$$

This last fibers over $S^2 \times S^2$, via projection. We claim that $(ev_{A'} \times ev_{B'})^{-1}(\Delta)$ is a smooth section of this fibration. It projects onto and one-to-one $S^2 \times S^2$

since there is a J_3'-holomorphic A'- and B'-curve through each point of $X' \cong \Delta$, and since these curves are unique up to parametrization. And it is transverse to the fibers since the A'- and B'-spheres intersect transversally. Thus we may define F_1 to be the composition

$$\begin{aligned} X' \overset{diag}{\hookrightarrow} X' \times X' &\longrightarrow (M_p(A', J_3') \times_G S^2) \times (M_p(B', J_3') \times_G S^2) \\ &= (M_p(A', J_3') \times M_p(B', J_3')) \times_{G\times G} (S^2 \times S^2) \longrightarrow S^2 \times S^2. \end{aligned}$$

By construction F_1 takes the J_3'-holomorphic A'-spheres to $S^2 \times pt.$ and the J_3'-holomorphic B'-spheres to $pt. \times S^2$, and it is pseudo-holomorphic with respect to J_3' on X' and $i \oplus i$ on $S^2 \times S^2$.

Now $(F_1^{-1})^*\rho$ is a symplectic form on $S^2 \times S^2$ with $[(F_1^{-1})^*\rho] = [\sigma_1 \oplus \sigma_2]$, and moreover $(F_1^{-1})^*\rho$ restricts symplectically to slices by construction of F_1. Thus by Theorem 3.1.1 $(F_1^{-1})^*\rho$ is isotopic to $\sigma_1 \oplus \sigma_2$. So we may isotope F_1 to get a diffeomorphism $F_2 : X' \longrightarrow S^2 \times S^2$ such that $F_2^*(\sigma_1 \oplus \sigma_2) = \rho$.

qed

Now define a diffeomorphism

$$F : X' \times X'' \longrightarrow X = S^2 \times S^2 \times T^2$$

by $F(p, q) = (F_2(p), q)$, and let $J_4 = F^* J_0$. Then J_4 is integrable, $J_4 \in \mathcal{J}(\rho \oplus \sigma_3)$, and is split as $J_4 = J_4' \oplus J_4''$. By construction F is a biholomorphism of $(X' \times X'', J_4)$ and (X, J_0). Arguing as in Proposition 3.2.4, J_4 is a regular value for P_A, $M_p(A, J_4) \times_G S^2$ is a smooth compact oriented 6-manifold, and the evaluation map

$$ev_A(J_4) : M_p(A, J_4) \times_G S^2 \longrightarrow X' \times X''$$

is a diffeomorphism. As before, we consider the evaluation map

$$\begin{aligned} ev_4 : M_p(A, J_4) \times_G (S^2 \times S^2) &\longrightarrow X' \times X'' \\ (f'', z, z') &\longmapsto f''(z) \end{aligned}$$

This map is a submersion, and so transverse to Z; let

$$\Gamma_4 = ev_4^{-1}(Z) \cong S^1 \times S^2$$

which is the space of pseudo-holomorphic spheres associated to $\rho \oplus \sigma_3$ we are interested in. Also let $\varphi_4 : \Gamma_4 \longrightarrow S^2$ be given as the composition

$$\begin{array}{ccccc} \Gamma_4 & \longrightarrow & X' \times X'' & \longrightarrow & S^2 \\ (f'', z, z') & \longmapsto & f''(z') & \longmapsto & \pi_2(f''(z')) \end{array}$$

where π_2 is projection onto the second factor of $X' = S^2 \times S^2$.

Since we are assuming τ_1 and $\rho \oplus \sigma_3$ are isotopic and J_1 and J_4 are both regular for P_A, the same argument as at the end of the proof of Theorem 3.2.1 shows that φ_1 and φ_4 are bordant via a bordism (W, Φ) which is zero on $H_2(W; \mathrm{Z})$, and so

$$\chi(\varphi_4) = \chi(\varphi_1) = -1$$

where $\varphi_1 : \Gamma_1 \longrightarrow S^2$ is given by equation (3.2.4).

However, one can calculate directly that $\chi(\varphi_4) = 0$, producing the desired contradiction to finish the proof of Theorem 3.1.2.

Lemma 3.3.4 $\chi(\varphi_4) = 0$.

proof:

As in the proof of Proposition 3.2.4, we identify $M_p(A, J_4) \times_G (S^2 \times S^2)$ with

$$\{(f''_{w_1,s_1,t_1}, z, z') \,|\, w_1 \in S^2,\ s_1, t_1 \in S^1,\ z, z' \in S^2\}$$

where $f''_{w_1,s_1,t_1}(z) = F^{-1}(z, w_1, s_1, t_1)$. Then

$$\Gamma_4 = \{(f''_u, z') \,|\, u \in S^1,\ z' \in S^2\} \cong S^1 \times S^2$$

where $f''_u(z) = F^{-1}(z, w_0, s_0, u)$, and

$$\varphi_4(u, z') = \pi_2(F_2^{-1}(z', w_0)).$$

Note that φ_4 is independent of u. Moreover, for fixed u, as a map from S^2 to S^2 it represents class 0 in $H_2(S^2;\mathbf{Z})$ since F_2^{-1} takes $[S^2 \times pt.]$ to a sphere of class A', but we are projecting onto the second factor of X'. Thus φ_4 is homotopic to a constant map, and so $\chi(\varphi_4) = 0$.

qed

This completes the proof of Theorem 3.1.2.

Chapter 4

Symplectic Structures on Fiber Bundles

4.1 The Construction of Sternberg

Starting with two closed manifolds F and B one can build more general topological types of closed manifolds, other than the obvious product, by constructing a fiber bundle E over B with fiber F and structure group G. If F and B are symplectic manifolds then in many cases the total space E also possesses a symplectic structure.

Following Gotay et.al. [3] we say a differentiable fibration

$$\pi : E \longrightarrow B$$

is a *symplectic fibration* if the canonical fiber F is a symplectic manifold with symplectic form ω_F and if the structure group of the bundle preserves ω_F. Thus on each fiber $\pi^{-1}(b)$ there is a well-defined symplectic form ω_b and $(\pi^{-1}(b), \omega_b)$ is identified with (F, ω_F) in a natural way. We say a closed 2-form β on E is *compatible* with a symplectic fibration if β restricts on each fiber $\pi^{-1}(b)$ to be ω_b.

If ω_E is a symplectic form on E compatible with π then there is a global cohomology class $[\omega_E] \in H^2(E)$ which restricts on each fiber to the class $[\omega_F] \in H^2(F)$. By a result of Thurston [18], if the base and fiber are compact symplectic manifolds then the existence of such a global class is sufficient to guarantee the existence of some symplectic form on E compatible with the symplectic fibration.

Theorem 4.1.1 (Thurston) *Suppose $\pi : E \longrightarrow B$ is a symplectic fibration where the base B and the fiber F are compact symplectic manifolds. If there exists a class $b \in H^2(E)$ such that b restricts on each fiber to be $[\omega_F] \in H^2(F)$ then there exists a symplectic form on E compatible with π.*

To prove this theorem, Thurston first uses a partition-of-unity argument to show the existence of a closed 2-form β on E (in class b) compatible with π. Note there are many such choices; e.g., if β is one such extension and γ is any closed 2-form on the base B then $\beta + \pi^*\gamma$ is another extension. Using the compactness of E it is then not hard to show that there exists a $K_0 \in \mathbb{R}$ such that for all $K \geq K_0$

$$\beta + K\pi^*\omega_B$$

is a nondegenerate, and so symplectic, form on E. The lower bound K_0 depends on the choice of extension β.

To understand which cohomology classes in $H^2(E)$ may be realized by a symplectic form built by Thurston's construction, one needs to understand the choices of closed 2-forms compatible with the fibration and the (non)degeneracy in nonvertical directions of such a choice; all these questions seem very difficult. Some aspects of these problems are discussed by Weinstein [20,21], and Guillemin, Lerman and Sternberg [7]. In order to get a better grip on the closed extension β one can use a technique first given by Sternberg [17] which uses a choice of connection to construct such an extension. Here are some details.

Let G be a Lie group, $P \longrightarrow B$ be a principal G-bundle, and (F, ω_F) a symplectic manifold on which there is a strongly Hamiltonian action of G (c.f. Section 2.1) with moment map $\mu : F \longrightarrow \mathcal{G}^*$. Let $E = P \times_G F$ be the associated bundle over B with fiber F and structure group G; since G acts on F by symplectomorphisms the fibration $\pi : E \longrightarrow B$ is symplectic. Let η be a $\mathcal{G}$-valued 1-form on P defining an Ehresmann connection on $P \longrightarrow B$. If $\langle \cdot, \cdot \rangle$ denotes the pairing of $\mathcal{G}$ and $\mathcal{G}^*$, then using the natural pull-backs, $d\langle \eta, \mu \rangle + \omega_F$ is a real-valued closed 2-form on $P \times F$.

Theorem 4.1.2 (Sternberg) *$d\langle \eta, \mu \rangle + \omega_F$ descends to the quotient $E = P \times_G F$ to give a closed 2-form β on E which is compatible with the fibration π.*

This construction depends only on the choice of the connection on $P \longrightarrow B$. The connection η defines a splitting of TE into vertical and horizontal sub-bundles, which are orthogonal with respect to the extension β. Moreover, β resticted to a horizontal space of TE pulls-back under the quotient map to the 2-form $\langle d\eta, \mu \rangle$ restricted to a corresponding horizontal space of TP; this is essentially the curvature of the connection η. Thus if the connection η is *fat* (c.f. [21]) on the image of the momentum map μ, then β is nondegenerate on all of TE, and so is a symplectic form on E.

After existence, there is also the question of the uniqueness (up to isotopy) of the symplectic forms on E built by Thurston's construction. (There is also the more general question of uniqueness of a symplectic form on E compatible with a symplectic fibration π.) In particular, how does the isotopy class of such a symplectic form depend on the choice of the extension β? An easy application of Moser's Theorem is the following.

Proposition 4.1.3 (Guillemin-Lerman-Sternberg [7])
Let $\pi : E \longrightarrow B$ be a compact symplectic fibration over a compact symplectic manifold (B, ω_B). Suppose β and β' are two closed cohomologous 2-forms on E compatible with the symplectic fibration. Then there exists a $K_1 \in \mathbf{R}$ such that for all $K \geq K_1$

$$\beta + K\pi^*\omega_B \qquad \text{and} \qquad \beta' + K\pi^*\omega_B$$

are isotopic symplectic forms on E, both compatible with π.

Note again that the bound K_1 depends on the choice of β and β', and is a priori much larger than the bounds insuring the nondegeneracy of $\beta + K\pi^*\omega_B$ and $\beta' + K\pi^*\omega_B$.

The main result of the remainder of this dissertation is that if one only chooses K large enough so that $\beta + K\pi^*\omega_B$ and $\beta' + K\pi^*\omega_B$ are symplectic, we may get two symplectic forms on E which are cohomologous and compatible with the symplectic fibration, but which are *not* isotopic. The proof of the following theorem is the subject of Chapter 5.

Theorem 4.1.4 *There exists a compact symplectic fibration $\pi : E \longrightarrow X$ over a compact symplectic manifold (X, ω_X) and two closed cohomologous 2-forms β_0 and β_1 on E such that*

$$\beta_0 + \pi^*\omega_X \quad \text{and} \quad \beta_1 + \pi^*\omega_X$$

are symplectic forms on E, both compatible with the fibration π, but which are not isotopic.

4.2 A Generalized Construction

In this section we generalize Sternberg's construction by showing how a family of connections on a principal bundle $P \longrightarrow B$ parametrized by a symplectic manifold (F, ω_F) may be used to construct a closed 2-form β on the associated bundle $E = P \times_G F$ compatible with the symplectic fibration $E \longrightarrow B$. We will use this construction and a nondegeneracy result proven in Section 4.3 to build the example claimed by Theorem 4.1.4.

Let G be a connected Lie group and $\pi_P : P \longrightarrow B$ be a principal G-bundle over B. Let (F, ω_F) be a symplectic manifold with a strongly Hamiltonian (left) action of G on (F, ω_F), and let $\mu : F \longrightarrow \mathcal{G}^*$ be the equivariant moment map of this action. For each $\xi \in \mathcal{G}$ let ξ_F denote the induced symplectic vector field on F, and ξ_P the fundamental vertical vector field on P generated by right-multiplication on P by $exp(-t\xi)$.

Let $E = P \times_G F$ be the associated fiber bundle over B with fiber F and structure group G, where the diagonal action of G on $P \times F$ is $g \cdot (p, z) = (p \cdot g^{-1}, g \cdot z)$ and where

$$Q : P \times F \longrightarrow E = P \times_G F$$

is the quotient map. Since G acts on F by symplectomorphisms we may consider ω_F to be a symplectic form on each fiber of $\pi : E \longrightarrow B$, and π is a symplectic fibration.

Define a *family of Ehresmann connections on $P \longrightarrow B$ smoothly parameterized by F* to be a $\mathcal{G}$-valued 1-form N on $P \times F$ satisfying

$$\forall z \in F \ \eta_z = i_z^* N \text{ is a connection 1-form on } P \longrightarrow B \tag{4.2.1}$$

where $\mathrm{i}_z : P \longrightarrow P \times F$ is the inclusion $i_z(p) = (p, z)$; and also satisfying

$$0 \oplus TF \subset \ker N \tag{4.2.2}$$

using the natural splitting $T(P \times F) \cong TP \oplus TF$.

Via pull-back we consider ω_F as a closed 2-form on $P \times F$ and μ as a $\mathcal{G}^*$-valued function on $P \times F$. Then by the dual pairing of $\mathcal{G}$ and $\mathcal{G}^*$, $d\langle N, \mu\rangle + \omega_F$ is a closed 2-form on $P \times F$. Our goal is to prove the following.

Theorem 4.2.1 *If N is constant along the orbits of G on F, then there exists a closed 2-form β on E with*

$$Q^*\beta = d\langle N, \mu\rangle + \omega_F \tag{4.2.3}$$

and such that β is compatible with the symplectic fibration $\pi : P \longrightarrow F$.

The proof of this theorem mimics the proof of Sternberg's result, with a bit more care in the calculations. First, for each $\xi \in \mathcal{G}$ let

$$\xi_Q = \xi_P \oplus \xi_F;$$

the ξ_Q are the vector fields on $P \times F$ generating the diagonal quotient action of G. Also, we will say a vector $v \in T(P \times F)$ is of type TP (respectively type TF) if $v \in TP \oplus 0$ (resp. $0 \oplus TF$) with respect to the natural splitting $T(P \times F) \cong TP \oplus TF$.

Lemma 4.2.2 *(i) $N(\xi_Q) = \xi$*

(ii) $i(\xi_Q)dN = ad_\xi N$, where ad_ξ acts on values of N.

proof:

(i) is easy, for

$$N(\xi_Q) = N(\xi_P \oplus \xi_F) = N(\xi_P) + N(\xi_F) = N(\xi_P)$$

where the last equality holds since N vanishes on vectors of type TF. Then since ξ_P is of type TP and independent of $z \in F$, $N(\xi_P) = \eta_z(\xi_P) = \xi$ by (4.2.1).

For (ii), at each point $(p,z) \in P \times F$,

$$
\begin{aligned}
i(\xi_Q)dN &= \mathcal{L}_{\xi_Q}N - d(i(\xi_Q)N) \\
&= \mathcal{L}_{\xi_Q}N - d(\xi) \\
&= \mathcal{L}_{\xi_Q}N \\
&= \mathcal{L}_{\xi_P}N + \mathcal{L}_{\xi_F}N \\
&= \mathcal{L}_{\xi_P}N \\
&= ad_\xi N
\end{aligned}
$$

where $\mathcal{L}_{\xi_F}N = 0$ since N is constant along the orbits of G on F, and the last equality holds since ξ_P is of type TP.

qed

Lemma 4.2.3 $i(\xi_Q)(d\langle N, \mu\rangle + \omega_F) = 0$

proof:

$$
\begin{aligned}
&i(\xi_Q)(d\langle N, \mu\rangle + \omega_F) = \\
&\qquad i(\xi_Q)(\langle dN, \mu\rangle - \langle N \overset{\wedge}{,} d\mu\rangle + \omega_F) = \\
&\qquad \langle i(\xi_Q)dN, \mu\rangle - \langle i(\xi_Q)N, d\mu\rangle + \langle N, i(\xi_Q)d\mu\rangle + i(\xi_Q)\omega_F = \\
&\qquad \langle ad_\xi N, \mu\rangle - \langle \xi, d\mu\rangle + \langle N, i(\xi_F)d\mu\rangle + i(\xi_F)\omega_F = \\
&\qquad \langle ad_\xi N, \mu\rangle + \langle N, ad^*_\xi \mu\rangle = \\
&\qquad \langle ad_\xi N, \mu\rangle - \langle ad_\xi N, \mu\rangle = 0
\end{aligned}
$$

where $\langle \xi, d\mu\rangle = i(\xi_F)\omega_F$, $i(\xi_F)d\mu = ad^*_\xi\mu$, and $\langle N, ad^*_\xi\mu\rangle = -\langle ad_\xi N, \mu\rangle$ follow from differentiating (2.1.1), (2.1.2), and (2.1.3) repectively.

qed

By the above lemma $d\langle N, \mu\rangle + \omega_F$ vanishes on the vector fields ξ_Q generating the diagonal quotient action on $P \times F$. Since this form is closed, by the Weyl identity $\mathcal{L}_{\xi_Q}(d\langle N, \mu\rangle + \omega_F) = 0$ and so it is invariant under the diagonal quotient action. Thus there is a 2-form β on E with $Q^*\beta = d\langle N, \mu\rangle + \omega_F$; β must be closed since Q^* is injective.

It remains to show that β is compatible with the symplectic fibration. Let $V = \ker \pi_*$ be the vertical sub-bundle of TE. Recall each $\eta_z = i_z^* N$ defines a horizontal sub-bundle $\ker \eta_z \subset TP$.

Lemma 4.2.4 *(i)* $Q_*\big|_{TF} : TF \longrightarrow V$ *is an isomorphism of bundles over* B.
(ii) There exists a well-defined sub-bundle H *of* TE *such that* $TE = V \oplus H$, *defined by*

$$H_{Q(p,z)} = Q_*(\ker N_{(p,z)} \cap T_p P).$$

proof:
We have the commutative diagram

$$\begin{tikzcd} P \times F \arrow[r, "Q"] \arrow[d, "\pi_P"'] & E \arrow[d, "\pi"] \\ B \arrow[r, "Id"] & B \end{tikzcd}$$

Since TF and V have the same rank and by construction $Q_*(TF) \subseteq V$, to prove (i) it suffices to show $\ker Q_*\big|_{TF} = \ker Q_* \cap TF = 0$. But this is obvious since every ξ_Q has type $TP \oplus TF$.

For (ii), to see that H is well-defined we need to show that if $Q(p, z) = Q(p', z')$ then $Q_*(\ker N_{(p,z)} \cap T_p P) = Q_*(\ker N_{(p',z')} \cap T_{p'} P)$. Since $Q(p, z) = Q(p', z')$, then $(p', z') = g \cdot (p, z) = (p \cdot g^{-1}, g \cdot z)$ for some $g \in G$. By (4.2.1) and since N is constant along the orbits of G on F,

$$\ker N_{(p',z')} \cap T_{p'}P = (R_{g^{-1}} \times L_g)_*(\ker N_{(p,z)} \cap T_pP).$$

Since $Q \circ (R_{g^{-1}} \times L_g) = Q$, H is well-defined.

Now note that

$$\begin{aligned} \pi_* H_{Q(p,z)} &= \pi_* Q_*(\ker N_{(p,z)} \cap T_pP) \\ &= (\pi_P)_*(\ker N_{(p,z)} \cap T_pP) \\ &= (\pi_P)_*(\ker \eta_z|_p) \\ &= T_{\pi_P(p)}B \end{aligned}$$

and so H is horizontal in TE.

qed

Lemma 4.2.5 *β is compatible with π.*

proof:

It suffices to show that $Q^*(\beta|_V) = \omega_F$. Let $v_1, v_2 \in V_{Q(p,z)}$; by the previous lemma there exist $v_1', v_2' \in T_zF$ with $Q_*v_1' = v_1$ and $Q_*v_2' = v_2$. Then

$$\begin{aligned} \beta(v_1, v_2) &= (d\langle N, \mu\rangle + \omega_F)(v_1', v_2') \\ &= \langle dN, \mu\rangle(v_1', v_2') - \langle N \wedge d\mu\rangle(v_1', v_2') + \omega_F(v_1', v_2') \end{aligned}$$

The second term vanishes since v_1' and v_2' are both of type TF. We may extend v_1' and v_2' as vector fields on $T(P \times F) = TP \oplus TF$ in such a way that they are purely of type TF and independent of the point in P; thus $[v_1', v_2']$ is also of type TF and so the first term also vanishes.

qed

This completes the proof of Theorem 4.2.1.

There is also an interpretation of the restriction of the extension β to the horizontal sub-bundle H of TE.

Lemma 4.2.6

$$Q^*(\beta|_H) = \langle dN, \mu\rangle\big|_{\ker N \cap TP}$$

proof:

Let $w_1, w_2 \in H_{Q(p,z)}$; by Lemma 4.2.4 there are $w_1', w_2' \in \ker N_{(p,z)} \cap T_pP$ with $Q_*w_1' = w_1$ and $Q_*w_2' = w_2$. Then

$$\begin{aligned} \beta(w_1, w_2) &= \langle dN, \mu\rangle(w_1', w_2') - \langle N \overset{\wedge}{;} d\mu\rangle(w_1', w_2') + \omega_F(w_1'{,}'_2) \\ &= \langle dN(w_1', w_2'), \mu\rangle \end{aligned}$$

where the second and third terms vanish by type.

qed

At a point $(p, z) \in P \times F$

$$\langle dN, \mu\rangle\big|_{\ker N \cap TP} = \langle d\eta_z\big|_{\ker \eta_z}, \mu\rangle$$

and $d\eta_z\big|_{\ker \eta_z}$ is the curvature of the connection 1-form η_z on $P \longrightarrow B$. Thus $\beta|_H$ is essentially the curvatures of the family of connections N. However, in this case nondegeneracy on the horizontal sub-bundle $H \subset TE$ (fatness of the family of connections) no longer immediately implies nondegeneracy of the form β on all of TE. The problem is that the horizontal and vertical sub-bundles of TE are not necessarily orthogonal with respect to the form β. In particular, if $v \in V$ and $w \in H$ then

$$\begin{aligned} \beta(v, w) &= \langle dN, \mu\rangle(v', w') - \langle N \overset{\wedge}{;} d\mu\rangle(v', w') + \omega_F(v', w') \\ &= \langle dN(v', w'), \mu\rangle \end{aligned}$$

and it need not be true that this last term vanishes.

4.3 A Result on Nondegeneracy

Here we consider the case of a principal $U(1)$-bundle $\pi_P : P \longrightarrow B$ and a family of connection 1-forms $\{\eta_t\}$ on π_P smoothly parametrized by the image of $\mu : F \longrightarrow \mathcal{G}^*$, where μ is the moment map for a strongly Hamiltonian action of $U(1)$ on a symplectic manifold (F, ω_F). Since $U(1)$ is abelian μ is invariant under the action of $U(1)$ of F. Define a 1-form N on $P \times F$ with values in the Lie algebra of $U(1)$ by

$$N(v) = \eta_\mu((\pi_1)_* v) \qquad v \in T_{(p,z)}(P \times F) \tag{4.3.1}$$

where $\mu = \mu(z)$ and π_1 is projection of $P \times F$ onto the first factor. Then N satisfies (4.2.1) and (4.2.2), and is constant along the orbits of G on F. By Theorem 4.2.1 there exists a closed 2-form β on the associated bundle $E = P \times_{U(1)} F$ compatible with π, defined by equation (4.2.3).

Picking a basis for the Lie algebra of $U(1)$, we consider each η_t and μ as real-valued. Since $U(1)$ is abelian, for each fixed t, $d\eta_t$ is η_t-horizontal and thus the curvature 2-form of the connection 1-form η_t. Moreover, each $d\eta_t = \pi_P^* \gamma_t$ where each γ_t is a closed 2-form on B.

Proposition 4.3.1 *Suppose there is a closed 2-form ω_B on B so that for each value $t \in Im\,\mu$, $\omega_B + t \cdot \gamma_t$ is a nondegenerate (and so symplectic) form on B. Then $\beta + \pi^* \omega_B$ is a symplectic form on E compatible with the symplectic fibration $\pi : E \longrightarrow B$.*

Note that here we have a condition which is checked on the base B. Moreover, there is some freedom in this condition, since the image of μ can be scaled by scaling the choice of basis for the Lie algebra of $U(1)$; in addition μ (the Hamiltonian function of this action) is well-defined only up to a constant. The key idea of the proof is that since $U(1)$ is 1-dimensional there is only one term of type $TP \oplus TF$ in $Q^*\beta$, which we can control (see Lemma 4.3.2).

To begin the proof, recall that β on $E = P \times_G F$ is defined by $Q^*\beta = d(N \cdot \mu) + \omega_F$, where N and μ are considered as real-valued so the dual-pairing is multiplication. Write

$$dN = d\eta_\mu - \dot{\eta_\mu} \wedge d\mu$$

where $d\eta_\mu$ is the exterior derivative of η_μ on P for a fixed value of μ, $\dot{\eta_\mu}$ is differentiation of (the coefficients of) η_μ with respect to the parameter, and the second term follows by the chain rule. Thus

$$\begin{aligned}
Q^*\beta &= d(N \cdot \mu) + \omega_F \\
&= dN \cdot \mu - N \wedge d\mu + \omega_F \\
Q^*\beta &= \mu\, d\eta_\mu - \mu\, \dot{\eta_\mu} \wedge d\mu - \eta_\mu \wedge d\mu + \omega_F
\end{aligned} \qquad (4.3.2)$$

considering all forms and functions as on $P \times F$.

Nondegeneracy is a pointwise condition, so it suffices to show that at each point $e = Q(p, z) \in E$

$$(\beta + \pi^*\omega_B)^n \neq 0$$

where $\dim B = 2k$, $\dim F = 2m$ and $n = k + m$.

Let $v_1, \ldots, v_{2k}$ be a basis for H_e and $v_{2k+1}, \ldots, v_{2n}$ be a basis for V_e, so together they form a basis for T_eE. By Lemma 4.2.4 there is a basis $v'_1, \ldots, v'_{2k}$ of $\ker \eta_\mu \cap T_pP$ and a basis $v'_{2k+1}, \ldots, v'_{2n}$ of T_zF such that $Q_*(v'_j) = v_j\ \forall j = 1, \ldots, 2n$. Then

$$\begin{aligned}
&(\beta + \pi^*\omega_B)^n(v_1, \ldots, v_{2n}) = \\
&\qquad (Q^*\beta + Q^*\pi^*\omega_B)^n(v'_1, \ldots, v'_{2n}) = \\
&\qquad (\mu\, d\eta_\mu - \mu\, \dot{\eta_\mu} \wedge d\mu - \eta_\mu \wedge d\mu + \omega_F + \pi_P^*\omega_B)^n(v'_1, \ldots, v'_{2n}) = \\
&\qquad (\mu\, d\eta_\mu - \mu\, \dot{\eta_\mu} \wedge d\mu + \omega_F + \pi_P^*\omega_B)^n(v'_1, \ldots, v'_{2n})
\end{aligned}$$

since $v'_1, \ldots, v'_{2n}$ are all in $\ker \eta_\mu$. Now

$$\begin{aligned}
&(\mu\, d\eta_\mu - \mu\, \dot{\eta_\mu} \wedge d\mu + \omega_F + \pi_P^*\omega_B)^n = \\
&(\mu\, d\eta_\mu + \omega_F + \pi_P^*\omega_B)^n - n(\mu\, d\eta_\mu + \omega_F + \pi_P^*\omega_B)^{n-1} \wedge \dot{\eta_\mu} \wedge d\mu
\end{aligned}$$

with no other terms since $(\dot{\eta_\mu} \wedge d\mu)^2 = 0$.

Lemma 4.3.2 $(\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1} \wedge \dot{\eta_\mu} \wedge d\mu(v_1', \ldots, v_{2n}') = 0.$

proof:

First note that $d\mu(v) = 0\ \forall v \in TP$ and $\dot{\eta_\mu}(v) = 0\ \forall v \in TF$, where the second follows since η_μ vanishes on vectors in TF and we have only differentiated coefficients. Let K be the set of shuffle permutations of $S_{(2n-2)+2}$ (c.f. [16]), so a permutation $\sigma \in K$ if and only if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(2n-2) \text{ and } \sigma(2n-1) < \sigma(2n)$$

Then

$$\begin{aligned}
&(\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1} \wedge \dot{\eta_\mu} \wedge d\mu(v_1', \ldots, v_{2n}') = \\
&\sum_{\sigma \in K} (\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1} \otimes \dot{\eta_\mu} \wedge d\mu(v_1', \ldots, v_{2n}') = \\
&\sum_{\sigma \in K} (\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1}(v_1', \ldots, v_{2n-2}') \cdot \dot{\eta_\mu} \wedge d\mu(v_{2n-1}', v_{2n}')
\end{aligned}$$

Now $\dot{\eta_\mu} \wedge d\mu(v_{2n-1}', v_{2n}') \neq 0$ if and only if the set $\{v_{2n-1}', v_{2n}'\}$ consists of exactly one vector of type TP and one vector of type TP. In this case the set $\{v_1', \ldots, v_{2n-2}'\}$ consists of an odd number of vectors of type TP and an odd number of vectors of type TF. Writing out

$$(\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1} = \sum_{j=0}^{n-1} const(\mu\, d\eta_\mu + \pi_P^* \omega_B)^{n-1-j} \wedge \omega_F^j$$

and noting that $(\mu\, d\eta_\mu + \pi_P^* \omega_B)^{n-1-j}$ vanishes on vectors of type TF and ω_F^j vanishes on vectors of type TP, it follows that

$$\dot{\eta_\mu} \wedge d\mu(v_{2n-1}', v_{2n}') \neq 0 \iff (\mu\, d\eta_\mu + \omega_F + \pi_P^* \omega_B)^{n-1}(v_1', \ldots, v_{2n-2}') = 0.$$

qed

Thus

$$\begin{aligned}(\beta + \pi^*\omega_B)^n(v_1,\ldots,v_{2n}) &= (\mu\, d\eta_\mu + \omega_F + \pi_P^*\omega_B)^n(v_1',\ldots,v_{2n}') \\ &= \sum_{j=0}^{n} const(\mu\, d\eta_\mu + \pi_P^*\omega_B)^{n-j} \wedge \omega_F^j(v_1',\ldots,v_{2n}')\end{aligned}$$

Again, since $(\mu\, d\eta_\mu + \pi_P^*\omega_B)^{n-j}$ vanishes on vectors of type TF and ω_F^j vanishes on vectors of type TP, the only nonzero term of this last is

$$\begin{aligned}&(\mu\, d\eta_\mu + \pi_P^*\omega_B)^k \wedge \omega_F^m(v_1',\ldots,v_{2n}') = \\ &\qquad (\mu\, d\eta_\mu + \pi_P^*\omega_B)^k(v_1',\ldots,v_{2k}') \cdot \omega_F^m(v_{2k+1}',\ldots,v_{2n}')\end{aligned}$$

Since $v_{2k+1}',\ldots,v_{2n}'$ is a basis for T_zF and ω_F is symplectic

$$\omega_F^m(v_{2k+1}',\ldots,v_{2n}') \neq 0.$$

Similarly, since $v_1',\ldots,v_{2k}'$ is a basis for the horizontal space $\ker\eta_\mu \cap T_pP$, then $(\pi_P)_*v_1',\ldots,(\pi_P)_*v_k'$ is a basis for $T_{\pi_P(p)}P$ and so

$$(\mu\, d\eta_\mu + \pi_P^*\omega_B)^k(v_1',\ldots,v_{2k}') = (\mu\,\gamma_\mu + \omega_B)^k((\pi_P)_*v_1',\ldots,(\pi_P)_*v_k') \neq 0$$

since $(\mu\,\gamma_\mu + \omega_B)$ is nondegenerate by hypothesis. Thus $(\beta + \pi^*\omega_B)^n \neq 0$.

This completes the proof of Proposition 4.3.1.

Note that for higher dimensional Lie groups dN and μ take values in $\mathcal{G}$ and so the term $\langle dN, \mu\rangle$ contributes more than one term of cross-type.

Chapter 5

Proof of Theorem 4.1.4

5.1 The Construction

Let $\sigma_1 = \sigma_2$ both denote the standard symplectic form on S^2 and σ_3 the standard symplectic form on T^2, all of total area 1. Define two 1-parameter families of symplectic forms on $X = S^2 \times S^2 \times T^2$ as follows. For each $-1 \leq t \leq 0$ let

$$\begin{aligned} \omega_t &= ((2+t)\sigma_1) \oplus \sigma_2 \oplus \sigma_3 \\ \omega_t'' &= \theta^* \omega_t \end{aligned}$$

where θ is the diffeomorphism of X defined by equation (3.2.2). By Theorem 3.2.1 ω_{-1} is not isotopic to ω''_{-1}, and by Proposition 3.2.8 each ω''_t is isotopic to ω_t for $t > -1$. Change ω''_t near $t = 0$ to obtain two smooth 1-parameter families of symplectic forms ω_t and ω'_t on X with $[\omega_t] = [\omega'_t]$ for each $-1 \leq t \leq 0$, $\omega_t = \omega'_t$ for t near 0, and ω_{-1} *not* isotopic to ω'_{-1}. We will also consider

$$\omega_X = \omega_0 = (2\sigma_1) \oplus \sigma_2 \oplus \sigma_3$$

to be a distinguished symplectic form on X.

Let $\pi_P : P \longrightarrow B$ be the principal $U(1)$-bundle over X with first Chern class $c_1 = [\sigma_1]$. We use our families of symplectic forms to define two smooth 1-parameter families of Ehresmann connection 1-forms on P as follows. For each $-1 \leq t \leq 0$ let $\gamma_t = \sigma_1$, and so $\omega_t = \omega_X + t\gamma_t$. Since P is a $U(1)$-bundle, there exists a connection 1-form $\eta_t \equiv \eta_0$ on P such that $d\eta_t = \pi_P^*\gamma_t = \pi_P^*\sigma_1$, where by picking a basis for the Lie algebra of $U(1)$ we treat the η_t as real-valued. Similarly for each $-1 \leq t \leq 0$ define a closed 2-form γ_t' on X by $\omega_t' = \omega_X + t\gamma_t'$, which is well-defined at $t = 0$ since $\omega_t' = \omega_t$ near $t = 0$. Then $[\gamma_t'] = [\gamma_t] = [\sigma_1] = c_1(P)$, and so again there exist connection 1-forms η_t' on P depending smoothly on t such that $d\eta_t' = \pi_P^*\gamma_t'$.

Let $F = S^2 \cong \mathbf{CP}_1$ and let ω_F be the standard symplectic form on F of total area 1. There is an action of $U(1) = \{e^{2\pi i s} \mid 0 \leq s \leq 1\}$ on F given by

$$e^{2\pi i s} \cdot [z_1, z_2] = [z_1, e^{2\pi i s} z_2]$$

where $[z_1, z_2]$ are homogeneous coordinates on $\mathbf{CP}_1$. This action is symplectic and in fact strongly Hamiltonian since $U(1)$ is abelian and F is simply-connected. If $\xi = 2\pi i \frac{\partial}{\partial s}$ is the basis for the Lie algebra, then the moment map (the Hamiltonian function generating ξ_F) is

$$\mu([z_1, z_2]) = -\frac{\|z_1\|^2}{\|z_1\|^2 + \|z_2\|^2}$$

Note μ is invariant under the action on F, and has a maximum of 0 at $[0,1]$ and a minimum of -1 at $[1,0]$.

Since the $U(1)$-action on F is symplectic the associated bundle

$$E = P \times_{U(1)} F$$

is a symplectic fibration over X with each fiber naturally identified with (F, ω_F). Now define two 1-parameter families of Ehresmann connections N and N' on P, as in equation (4.3.1), by

$$\begin{aligned} N(v) &= \eta_\mu((\pi_1)_* v) \\ N'(v) &= \eta'_\mu((\pi_1)_* v) \end{aligned}$$

for $v \in T_{(p,z)}(P \times F)$ where $\mu = \mu(z)$. By Theorem 4.2.1 there exist closed 2-forms β and β' on E compatible with the symplectic fibration $\pi : E \longrightarrow X$, where

$$\begin{aligned} Q^*\beta &= d(\eta_\mu \cdot \mu) + \omega_F \\ Q^*\beta' &= d(\eta'_\mu \cdot \mu) + \omega_F \end{aligned}$$

Here and in the following we are using the notation and conventions of Section 4.3. By the construction of the connection 1-forms and by Proposition 4.3.1

$$\begin{aligned} \omega_E &= \beta + \pi^*\omega_B \\ \omega'_E &= \beta' + \pi^*\omega_B \end{aligned}$$

are symplectic forms on E compatible with π.

To prove Theorem 4.1.4 we will show that $[\beta] = [\beta'] \in H^2(E)$, and (in the next section) that ω_E and ω'_E are not isotopic symplectic forms. In the remainder of this section we prove some preliminary lemmas.

E is identified in the obvious way with $\mathsf{P}(\mathsf{L} \oplus \mathbb{C})$, the projectivization of the rank-2 complex vector-bundle $\mathsf{L} \oplus \mathbb{C}$ where L is the complex line-bundle over X associated to P and $\mathbb{C}$ is the trivial line-bundle. There is a section $s = s_\infty$ of $E \cong \mathsf{P}(\mathsf{L} \oplus \mathbb{C})$ with image

$$\Sigma = \Sigma_\infty = [\mathsf{L}, 0] = Q(P \times [1, 0])$$

where $[\mathsf{L}, 0]$ is the line in each fiber of $\mathsf{L} \oplus \mathbb{C}$ determined by L and the zero section of $\mathbb{C}$. Since $\mathsf{P}(\mathsf{L} \oplus \mathbb{C}) \cong \mathsf{P}(\mathbb{C} \oplus \mathsf{L}^*)$, Σ has normal bundle

$$\nu = \nu_\Sigma \cong (\pi|_\Sigma)^*(\mathsf{L}^*)$$

where L^* is the dual line bundle of L. If we consider L as a holomorphic line-bundle over (X, J_0), E inherits a holomorphic structure (see below);

then s is the rigid section of π, meaning it has no nontrivial holomorphic sections, since $c_1(\nu) = -(\pi|_\Sigma)^*[\sigma_1]$.

By the Leray-Hirsch Theorem $H^2(E) \cong H^2(X) \oplus H^2(F)$ and so $H_2(E; \mathbb{Z})$ is generated by the classes

$$\begin{aligned} A^* &= s_* A \\ B^* &= s_* B \\ C^* &= s_* C \\ D^* &= [fiber] \end{aligned}$$

where A, B, and C are the classes generating $H_2(X; \mathbb{Z})$.

By Lemma 4.2.4 the families of connections N and N' define horizontal sub-bundles of TE, where

$$\begin{aligned} H_{Q(p,z)} &= Q_*(\ker \eta_{\mu(z)} \oplus 0) \\ H'_{Q(p,z)} &= Q_*(\ker \eta'_{\mu(z)} \oplus 0) \end{aligned}$$

We first notice that $H|_\Sigma = H'|_\Sigma = T\Sigma$ as sub-bundles of TE. To see that $H|_\Sigma = T\Sigma$ it suffices to show that $H|_\Sigma \subseteq T\Sigma$. But

$$T\Sigma = Q_*(T(P \times [1,0])) = Q_*(TP \oplus 0)$$

and

$$H|_\Sigma = Q_*(\ker \eta_{-1} \oplus 0) \subseteq Q_*(TP \oplus 0)$$

since $\mu([1,0]) = -1$. Similarly $H'|_\Sigma = T\Sigma$.

Lemma 5.1.1 $\omega_E|_{T\Sigma} = \pi^* \omega_{-1}|_{T\Sigma}$ *and* $\omega'_E|_{T\Sigma} = \pi^* \omega'_{-1}|_{T\Sigma}$

proof:

Let $v, w \in T_{Q(p,[1,0])}$ and pick $v', w' \in \ker \eta_{-1}$ such that $Q_* v' = v$ and $Q_* w' = w$. Then calculating as in the proof of Proposition 4.3.1,

$$\omega_E(v, w) = Q^* \omega_E(v', w')$$

$$
\begin{aligned}
&= (Q^*\beta + Q^*\pi^*\omega_B)(v', w') \\
&= (d(N \cdot \mu)\big|_{\mu=-1} + \omega_F + \pi_P^*\omega_B)(v', w') \\
&= (-d\eta_{-1} + \dot{\eta}_{-1} \wedge d\mu - \eta_{-1} \wedge d\mu + \omega_F + \pi_P^*\omega_B)(v', w') \\
&= (-d\eta_{-1} + \pi_P^*\omega_B)(v', w')
\end{aligned}
$$

where the second, third and fourth terms vanish since v' and w' are of type TP. Thus

$$
\begin{aligned}
\omega_E(v, w) &= (-\pi_P^*\gamma_{-1} + \pi_P^*\omega_B)(v', w') \\
&= (\pi_P^*\omega_{-1})(v', w') \\
&= (\pi^*\omega_{-1})(v, w).
\end{aligned}
$$

Similarly $\omega'_E\big|_{T\Sigma} = \pi^*\omega'_{-1}\big|_{T\Sigma}$.

qed

In particular, β and β' are distinct closed 2-forms on E.

Proposition 5.1.2 $[\beta] = [\beta'] \in H^2(E)$.

proof:

We check this on the generators of $H_2(E; \mathbf{Z})$. Since β and β' are compatible with the fibration π, $Q^*(\beta\big|_V) = Q^*(\beta'\big|_V) = \omega_F$ where V is the vertical sub-bundle of TE. Thus $[\beta](D^*) = [\omega_F]([F]) = [\beta'](D^*)$. By the proof of the above lemma $\beta\big|_{T\Sigma} = \pi^*(-\gamma_{-1})$ and $\beta'\big|_{T\Sigma} = \pi^*(-\gamma'_{-1})$. Then $[\beta](A^*) = [\beta](s^*A) = [-\gamma_{-1}](A) = [-\gamma'_{-1}](A) = [\beta'](A^*)$, and similarly on the other generators.

qed

Lemma 5.1.3 *The splitting $TE|_\Sigma = V|_\Sigma \oplus T\Sigma$ is orthogonal with respect to both ω_E and ω'_E.*

proof:

Let $v \in V|_\Sigma$ and $w \in T\Sigma$, and pick $v' \in T_{[1,0]}F$ and $w' \in \ker \eta_{-1}$ with $Q_*(v') = v$ and $Q_*(w') = w$. Then as in Lemma 5.1.1

$$\omega_E(v,w) = (-d\eta_{-1} + \dot{\eta}_{-1} \wedge d\mu - \eta_{-1} \wedge d\mu + \omega_F + \pi_P^* \omega_B)(v', w') = 0$$

where the first, fourth and fifth terms vanish by type, while the second and third terms are zero since $d\mu|_{[1,0]} = 0$ since μ is a minimum at $[1, 0]$.

qed

Thus we may identify $V|_\Sigma$ with the (symplectic) normal bundle ν of Σ in E. Since $s^*(\nu) \cong \mathsf{L}^*$, $s^*(V|_\Sigma) \cong \mathsf{L}^*$.

Finally, we define two almost-complex structures J_E and J'_E on E, as follows. Since $c_1(\mathsf{L}^*) = -[\sigma_1] \in H^{1,1}_{J_0}(X)$, we may consider L^* as a holomorphic line-bundle over X, and so using $\pi|_\Sigma$ make $V|_\Sigma$ into a holomorphic line-bundle over Σ, where the complex structure on Σ is $(\pi|_\Sigma)^* J_0$. Identifying a neighborhood of Σ in E with $\nu \cong V|_\Sigma$, we get an integrable complex structure J_E on a neighborhood of Σ such that $J_E|_{T\Sigma} = (\pi|_\Sigma)^* J_0$ and $(J_E)|_\Sigma$ is split with respect to $V|_\Sigma \oplus T\Sigma$. Now $J_E|_{T\Sigma}$ tames $\omega_E|_{T\Sigma}$ since J_0 tames ω_{-1} and $J_E|_V$ tames $\omega_E|_V$ at a point of Σ since V is a complex line-bundle and ω_E and J_E determine the same orientation on $V|_\Sigma$. Thus by Lemma 5.1.3 J_E tames ω_E along Σ and so in a neighborhood of Σ. We may then extend J_E to an almost-complex structure on all of E such that J_E tames ω_E.

To define J'_E, recall $\theta : (X, J_0) \longrightarrow (X, J_1)$ is a biholomorphism; then $(\theta^{-1})^*(\mathsf{L}^*)$ is a holomorphic line-bundle over (X, J_1), and since θ is the identity on $H^2(X; \mathbb{Z})$, $s^*(V|_\Sigma) \cong (\theta^{-1})^*(\mathsf{L}^*)$. Pull-back the complex structure on $(\theta^{-1})^*(\mathsf{L}^*)$ by $\pi|_\Sigma$ to make $V|_\Sigma$ into a holomorphic line-bundle over

$(\Sigma, (\pi|_\Sigma)^* J_1)$. Then arguing as above we build an almost-complex structure J'_E on E such that J'_E tames ω'_E, is integrable near Σ, $J'_E|_{T\Sigma} = (\pi|_\Sigma)^* J_1$ and $J'_E|_\Sigma$ is split with respect to $V|_\Sigma \oplus T\Sigma$.

5.2 The Proof

By Lemma 5.1.1, ω_E and ω'_E restrict on $T\Sigma$ to look like τ_0 and τ_1, repectively. This does not immediately imply that ω_E and ω'_E are not isotopic via an isotopy of E. The idea of the proof is to repeat the arguments of Section 3.2 by constructing spaces of pseudo-holomorphic spheres Γ and Γ' associated to ω_E and ω'_E. The key step is to use the following lemma to identify these spaces with the spaces Γ_0 and Γ_1 of Section 3.2; thus a (weak) isotopy from ω_E to ω'_E can be used to construct a bordism of (Γ_0, φ_0) and (Γ_1, φ_1), producing a contradiction.

Lemma 5.2.1 *(i) The $f \in M_p(A^*, J_E)$ are exactly the maps $f = s \circ f_0$ where $f_0 \in M_p(A, J_0)$.*
(ii) The $f' \in M_p(A^*, J'_E)$ are exactly the maps $f = s \circ f_1$ where $f_1 \in M_p(A, J_1)$.

proof:

First note that $A^* \cdot [\Sigma] = -1$. To see this, recall that as complex line-bundles $\nu \cong (\pi|_\Sigma)^*(\mathsf{L}^*)$. Then

$$A^* \cdot [\Sigma] = c_1(\nu_\Sigma)(A^*) = c_1(\mathsf{L}^*)(A) = -[\sigma_1](A) = -1.$$

In particular, any $f \in M_p(A^*, J_E)$ must intersect Σ.

Since s is J_0-J_E-holomorphic, $f = s \circ f_0 \in M_p(A^*, J_E)$ if $f_0 \in M_p(A, J_0)$. Conversely, let $f \in M_p(A^*, J_E)$. We claim that $\operatorname{Image} f \subset \Sigma$; one argues as in [13]. Let $p \in \operatorname{Image} f \cap \Sigma$. Since J_E is integrable near Σ we can pick holomorphic coordinates (z_1, z_2, z_3, z_4) centered at p so that $\Sigma = \{z_1 = 0\}$.

Then $f = (f_1, f_2, f_3, f_4)$ where each f_i is a holomorphic map from $\mathbb{C}$ to $\mathbb{C}$. If p is not an isolated point of intersection then $f_1 \equiv 0$ near p; since Image f is connected, Image $f \subset \Sigma$. So we may assume all intersections of Image f and Σ are isolated. We will show each intersection point p contributes a positive integer to the intersection number $A^* \cdot [\Sigma]$ (see Theorem 2.2.7), contradicting the fact that $A^* \cdot [\Sigma] = -1$.

Since f_1 is holomorphic there exists a $k \geq 1$ such that

$$\begin{aligned} f_1(z) &= a_k z^k + a_{k+1} z^{k+1} + \ldots \\ &= z^k (a_k + a_{k+1} z + \ldots) \end{aligned}$$

with $a_k \neq 0$. Then for $\delta > 0$ very small, $f_\delta = (f_1 + \delta, f_2, f_3, f_4)$ is a small perturbation of f near p which is holomorphic and intersects Σ transversally exactly k times near p. Moreover, all these intersections are positive since f_δ and Σ are J_E-holomorphic.

Thus if $f \in M_p(A^*, J_E)$ then Image $f \subset \Sigma$, and since $\pi|_\Sigma$ is J_E-J_0-holomorphic

$$\pi|_\Sigma \circ f \in M_p(A, J_0)$$

But $\pi|_\Sigma$ is an inverse of s. The argument for (ii) is identical.

qed

Lemma 5.2.2 *J_E and J'_E are regular values for the Fredholm projection operator P_{A^*}.*

proof:

We apply Theorem 2.2.3. Let $f \in M_p(A^*, J_E)$, so $f = s \circ f_0$ where $f_0 \in M_p(A, J_0)$. Since $TE|_\Sigma$ splits holomorphically with respect to J_E as $V_\Sigma \oplus T\Sigma$ and Image $f \subset \Sigma$,

$$f^*(TE, J_E) = f_0^*(s^* V|_\Sigma \oplus s^* T\Sigma)$$

$$\begin{aligned} &= f_0^*(\mathsf{L}^* \oplus TX) \\ &= f_0^*(\mathsf{L}^*) \oplus TS^2 \oplus L_2 \oplus L_3 \end{aligned}$$

where as in the proof of Proposition 3.2.4 L_2 and L_3 are trivial line-bundles over S^2. Then $c_1(TS^2)(S^2) = 2$ and $c_1(L_2)(S^2) = c_1(L_3)(S^2) = 0$, while

$$c_1(f_0^*\mathsf{L}^*)(S^2) = f_0^* c_1(\mathsf{L}^*)(S^2) = f_0^*(-[\sigma_1])(S^2) = -[\sigma_1](A) = -1$$

and so J_E is regular. Similarly J'_E is regular.

qed

Since A^* is simple for both ω_E and ω'_E, by Remark 2.2.5

$$M_p(A^*, J_E) \times_G (S^2 \times S^2) \quad \text{and} \quad M_p(A^*, J'_E) \times_G (S^2 \times S^2)$$

are smooth compact oriented 8-manifolds, and the evaluation maps

$$\begin{aligned} ev = ev(A^*, J_E) : M_p(A^*, J_E) \times_G (S^2 \times S^2) &\longrightarrow E \\ ev' = ev(A^*, J'_E) : M_p(A^*, J'_E) \times_G (S^2 \times S^2) &\longrightarrow E \end{aligned}$$

are smooth, where as before we evaluate on the first factor of $S^2 \times S^2$. Moreover, by Lemma 5.2.1 we may smoothly identify $M_p(A^*, J_E) \times_G (S^2 \times S^2)$ and $M_p(A^*, J'_E) \times_G (S^2 \times S^2)$ with $M_p(A, J_0) \times_G (S^2 \times S^2)$ and $M_p(A, J_1) \times_G (S^2 \times S^2)$, respectively. Using these identifications

$$ev = s \circ ev_0 \quad \text{and} \quad ev' = s \circ ev_1$$

Thus by Proposition 3.2.4 and Lemma 5.2.1, ev and ev' are submersions onto $\Sigma \subset E$.

Let $Z^* = \pi^{-1}(Z)$, which is a smooth submanifold of E of codimension 3 and which is transverse to Σ, where recall $Z = \{(w_0, w_0, s_0, u) \mid u \in S^1\}$. Then

$$\Gamma = ev^{-1}(Z^*) \quad \text{and} \quad \Gamma' = (ev')^{-1}(Z^*)$$

are smooth compact oriented 3-manifolds. Using the identifications above

$$\begin{aligned} \Gamma &\cong \Gamma_0 \cong S^2 \times S^1 \\ \Gamma' &\cong \Gamma_1 \cong S^2 \times S^1 \end{aligned}$$

Define $\varphi : \Gamma \longrightarrow S^2$ as the composition

$$\Gamma \longrightarrow E \longrightarrow X \longrightarrow S^2$$

which is evaluation on the second factor of $S^2 \times S^2$, followed by π, and finally by projection of $X = S^2 \times S^2 \times T^2$ onto the second factor. Then

$$\varphi \cong \varphi_0 : \Gamma \cong \Gamma_0 \longrightarrow S^2$$

and so the generalized Hopf invariant of φ is $\chi(\varphi) = 0$. Similarly define $\varphi' : \Gamma' \longrightarrow S^2$ as the composition

$$\Gamma' \longrightarrow E \longrightarrow X \longrightarrow S^2$$

and so

$$\varphi' \cong \varphi_1 : \Gamma' \cong \Gamma_1 \longrightarrow S^2$$

has $\chi(\varphi') = -1$. Hence, as in Section 3.2, the maps φ and φ' cannot be bordant.

Once again the point of the proof is now to show that if ω_E and ω'_E are isotopic then a bordism of (Γ, φ) and (Γ', φ') must exist. So let ρ_t, $0 \leq t \leq 1$ be a (weak) isotopy from ω_E to ω'_E. Since J_E and J'_E are regular values of P_{A^*} there is a smooth path of almost-complex structures $\alpha = \{J_t\}$ from J_E to J'_E such that each J_t tames ρ_t, and such that the path α is transverse to P_{A^*}. And since each ρ_t evaluates on the generators of $H_2(E; \mathbb{Z})$ to be 1, the Compactness Theorem (Remark 2.2.6) holds, and so $M_p(A^*, \alpha) \times_G (S^2 \times S^2)$ is a smooth compact oriented manifold of dimension 9.

Consider $ev_\alpha : M_p(A^*, \alpha) \times_G (S^2 \times S^2) \longrightarrow E$, evaluating on the first factor of $S^2 \times S^2$. Then ev_α is transverse along the boundary to Z^*, and so we may perturb ev_α away from the boundary to a map

$$\widetilde{ev_\alpha} : M_p(A^*, \alpha) \times_G (S^2 \times S^2) \longrightarrow E$$

such that $\widetilde{ev_\alpha}$ is transverse to Z^*. Let $W = \widetilde{ev_\alpha}^{-1}(Z^*)$, which is a smooth compact oriented 4-manifold, with boundary $\partial W = \Gamma \amalg -\Gamma'$. Define $\Phi : W \longrightarrow S^2$ as the composition

$$W \longrightarrow E \longrightarrow X \longrightarrow S^2$$

where the first map is evaluation on the second factor of $S^2 \times S^2$. Then Φ restricts on the boundary components of W to be φ and φ', and by the proof of Lemma 3.2.6 $\Phi^*(H^2(S^2)) = 0$. Thus (W, Φ) is a bordism of (Γ, φ) and (Γ', φ') with $\Phi_*(H_2(W; \mathbf{Z})) = 0$, contradicting the fact that $\chi(\varphi) \neq \chi(\varphi')$. Thus ω_E cannot be isotopic to ω'_E.

This completes the proof of Theorem 4.1.4.

Note that in constructing the cobordism W it is not necessary to show that the curves in $M_p(A^*, \alpha)$ lie inside Σ, since we need only construct some appropriate bordism between Γ and Γ'.

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