Adiabatic limits, Non-multiplicativity of the Signature and the Leray Spectral Sequence

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Abstract of the Dissertation

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In this thesis, we first prove an adiabatic limit formula for the η-invariant of a Dirac type operator, which generalizes the recent work of J.-M. Bismut and J. Cheeger. By their work, one is essentially reduced to the study of the large time behavior of the heat kernel. This involves careful analysis of degenerate elliptic operators, which is fashioned after the very recent work of R. Mazzeo and R. Melrose. The analysis enables us to apply perturbation theory to obtain detailed informations about the spectrum of the Dirac operator in the adiabatic limit. A new contribution arises in the adiabatic

limit formula, in the form of a global term coming from the (asymptotically) very small eigenvalues.

We then proceed to show that, for signature operators, these very small eigenvalues have a purely topological significance. In fact, we show that the Leray spectral sequence can be recast in terms of the very small eigenvalues. This is reminiscent of the Hodge theory. Hence the term Hodge-Leray theory. A consequence of the Hodge-Leray theory is a refined adiabatic limit formula for signature operators where the global term is identified with a topological invariant, the signature of a certain bilinear form arising from the Leray spectral sequence.

As an interesting application, we give intrinsic characterizations of the non-multiplicativity of signature. Appendix B contains a proof of a special case of Bismut-Cheeger's Families Index Theorem for manifolds with boundary, which is used in the applications.

Contents

To my parents and Guofang.

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Introduction

In this thesis, we study the limiting behavior of the η-invariant of Atiyah-Patodi-Singer for Dirac operators on the total space of a fibration, when the metric along base direction is multiplied by a factor x^{-2} and $x \to 0$. This operation of blowing up the metric is called passing to the *adiabatic limit.*

The original motivation comes from a paper of E. Witten, [W], where he considers a family of Dirac operators acting on an even dimensional manifold. The parameter space of his family is a circle and thus, the (odd dimensional) total space of his family is the total space of fibration over the circle. Witten gives an argument relating the holonomy of the determinant line bundle of his family to the adiabatic limit of the η-invariant of the Dirac operator on the total space.

Witten's result was proved rigorously in [BF] and [C2]. The emphasis in [BF] was on the superconnection formalism of Quillin, [Q], in relation to the proof given in [B] of the local index theorem for families. In [C2], two proofs were given, one based on Duhamel's principle and second exploited the connection with previous work on conical singularities; see [C1], [C2]. For the case of the signature operator, an expression equivalent to that considered by Witten had arisen in [C1], when considering the variation of the η-invariant, for a space with isolated conical singularities. Finally, it was emphasized in [C2] that for a fibration of compact manifolds, the η-invariant of the total space can be viewed as a renormalized difference of η-invariants, with coefficients in a infinite dimensional bundle whose fiber is a space of sections along the fiber of the fibration.

Recently J.-M. Bismut and J. Cheeger, [BC1], [BC2], extend the results of these papers to the case in which the base of the fibration is a compact spin manifold of arbitrary dimension. What they found is that the adiabatic limit of the η-invariant of a Dirac operator on the total space is expressible in terms of a canonically constructed differential form, $\tilde{\eta}$, on the base, which can be viewed as a higher dimensional analogue of the η-invariant. What is more, this $\tilde{\eta}$ is exactly the boundary correction term in the Families Index Theorem for manifolds with boundary, [BC3].

In their work, they discussed in detail the case when the operators along the fibres are always invertible. Their treatment of the general case (i.e., the kernels of these operators have constant dimension) was less complete and explicit although it did suffice for various applications. For example, one can recover the work of Atiyah-Donnelly-Singer, [ADS], on the Hirzebruch conjecture, and prove results closely related to those of W. Müeller, [Mü], and M. Stern, [S].

We study questions related to the noninvertible case, which is essential when we consider signature operators because then the kernel is always nonempty and is given by cohomology. Here some new phenomena occur. Besides the semi-local contribution to the adiabatic limit formula involving the $\tilde{\eta}$-form, and an expected twisted η-invariant on the base, there is a global contribution from the (asymptotically) very small eigenvalues (Cf. below). It turns out that in the case of signature operator, these eigenvalues have a purely topological significance. This is our second main result, the Hodge-Leray theory of the very small eigenvalues, which is a refinemnet of the very recent work R. Mazzeo and R. Melrose, [MM]. In particular, it enables us to identify the global contribution in the adiabatic limit formula with a topological invariant coming from the Leray spectral sequence of the fibration.

As an interesting application we give intrinsic characterizations of the non-multiplicativity of signature (see below), one in terms of the Leray spectral sequence and the other the adiabatic limit.

Precisely, let us consider a fibration of closed manifolds

$$Y \to M^{2k-1} \xrightarrow{\pi} B. \qquad (.1)$$

Equip M with a submersion metric g_M,

$$g_M = \pi^* g_B + g_Y,$$

where g_B is the metric on B and g_Y annihilates the orthogonal compliment of the tangent space to the fibres. Blowing up the metric in the horizontal direction by a factor x^{-2} gives us a family of metrics g_x,

$$g_x = x^{-2} \pi^* g_B + g_Y.$$

We assume that 1) M is spin so that we can consider the Dirac operator

D_x associated with g_x; 2) The bundle of vertical spaces $T^V M$ is also spin so that we have the family of Dirac operators D_Y along the fibers.

It follows that the base space B is also spin and the Dirac operator D_B on the base is well-defined. When $\ker D_Y$ is a vector bundle on B, one can consider the twisted Dirac operator $D_B \otimes \ker D_Y$. Here the connection on $\ker D_Y$ is the projection of a unitary connection on the infinite dimensional bundle of smooth spinor sections along the fibres (see Appendix A for details; Cf. also [BC2]). More generally, let ξ be a hermitian vector bundle with a unitary connection on it, we can then consider the above operators twisted by ξ. We denote these by the same notations. Let $\eta(D_x)$ be the η-invariant of D_x. The following result is proved in this thesis.

Theorem 0.1 (Adiabatic limit formula) *Assume that* $\ker D_Y$ *is a vector bundle on* B. *Assume further that* $\dim(\ker D_x)$ *stabilizes after some small* x. *Then* $\lim_{x\to 0} \eta(D_x)$ *exists in* $\mathbb{R}$ *and*

$$\lim_{x\to 0} \eta(D_x) = 2\int \hat{A}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \eta(D_B \otimes \ker D_Y) + \lim_{x\to 0} \sum_{\lambda_0, \lambda_1 = 0} sgn \lambda_x, \quad (.2)$$

where R^B *is the curvature tensor of* g_B.

Here in the first term the $\tilde{\eta}$-from is defined in the same way as in [BC2] (see Appendix A). As we observed in Appendex A, it is well-defined without any assumptions. The last term in the above formula will be explained in a moment. But first we would like to mention that in the invertible case where the spectrum of D_x is uniformly bounded away from 0, the last two terms drop out (see below), and the above formula reduces to Bismut-Cheeger's adiabatic limit formula [BC2]. One of the essential difficulties in the noninvertible case is that there are infinitely many eigenvalues of D_x

approaching 0. However, one still has a nice description of them. Thus in Chapter 2 we will show that (Theorem 1.1) the spectrum of D_x can be divided into three parts: those uniformly bounded away from 0, those decaying linearly in x (the small eigenvalues), and those, finitely many in number, decaying at least quadratically (the very small ones). Roughly speaking, the twisted η term in (.2) comes from the small eigenvalues, and the last term from the very small ones (the *finite* sum of the signs of them).

Whereas asking $\dim(\ker D_x)$ to stabilize is a bit unnatural, it warrants the existence of the adiabatic limit of the η-invariant, which does not happen in general. On the other hand, the limit of the reduced η-invariant, $\bar{\eta}$, always exists in $\mathbb{R}/\mathbb{Z}$ ([C2], [BC2]). Hence, we have the following modulo $\mathbb{Z}$ counterpart of Theorem 0.1.

Theorem 0.1′ *Assume that* $\ker D_Y$ *is a vector bundle on* B. *Then*

$$\lim_{x\to 0}\bar{\eta}(D_x) \equiv \int \hat{A}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \bar{\eta}(D_B \otimes \ker D_Y) + \frac{1}{2}h \mod \mathbb{Z}, \qquad (.3)$$

where $\frac{1}{2}h \equiv \frac{1}{2}\dim(\ker D_x)$ *mod* $\mathbb{Z}$ *is a spin cobordism invariant, [AS, V].*

An important operator satisfying the hypothesis of Theorem 0.1 is the so-called signature operator A, whose kernel space is identified by the well-known Hodge theory with the cohomology. In this case, as we have indicated, the global contribution turns out to be a topological invariant, a consequence of the following interesting result, to which we refer as the Hodge-Leray theory of the (asymptotically) very small eigenvalues.

Theorem 0.2 *Let* E_r*=the limit of space spanned by* λ_x*-eigenforms associated to eigenvalues* λ_x *such that* λ_x *is* $O(x^r)$ $(r \geq 2)$ *in the adiabatic* *limit, then* $(E_r, x^{-r}d)$ *forms a spectral sequence which is isomorphic to the Leray spectral sequence of the fibration. Moreover, the* $*$ *map induced by the metric* g_x *gives rise to the duality map.*

In [MM], the Leray spectral sequence is related to the asymptotic solutions of the Laplacian in the adiabatic limit. Theorem 0.2 is partly motivated by this result. The analysis involved here is in fact fashioned after [MM].

Theorem 0.2 gives us a refined adiabatic limit formula in the case of the signature operator. Let (E_r, d_r) $(r \geq 2)$ be the E_r-term of the Leray spectral sequence of the fibration $Y \to M^{4k-1} \to B$. The orientation gives a natural basis ξ_2 on E_2 (in the sense of Chern-Hirzebruch-Serre [CHS], see Section 4.4) which then induces a basis ξ_r on E_r for each $r > 2$. Consider the pairing

$$\begin{aligned} <,>_r: \qquad E_r^p \otimes E_r^q &\longrightarrow \mathbb{R}, \\ \varphi \otimes \psi &\longrightarrow (\varphi \cdot d_r\psi, \xi_r). \end{aligned}$$

It can be verified that $<,>_r$ is symmetric when restricted to E_r^{2k-1}. Therefore it gives rise to a symmetric matrix whose signature we will denote by τ_r. Define $\tau = \sum_{r \geq 2} \tau_r$.

Theorem 0.3 *Suppose the fibration* $Y \to M^{4k-1} \to B$ *is oriented (see Section 4.1), then*

$$\lim_{x \to 0} \eta(A_x) = 2 \int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + 2\tau,$$

where A_B *denotes the signature operator on* B *and* A_Y *the family of signature operators along* Y*.*

Theorem 0.3 combined with the Families Index Theorem for manifolds with boundary [BC3] has the following interesting application. Consider an oriented fibration $Z \to N^{4k} \to B$ such that B is closed but $\partial Z = Y$ and $\partial N = M$. Associated with it is the boundary fibration $Y \to M^{4k-1} \to B$. Let $Sign^Z$ denote the signature bundle of fibres on B. This is a virtual bundle, thus can be considered as a $\mathbb{Z}_2$-graded bundle. It has a natural flat structure. Therefore we can define $sign(B, Sign^Z)$, the signature of B with coeffiencts in the $\mathbb{Z}_2$-graded flat bundle $Sign^Z$ (see [L]). Let

$$\Delta \stackrel{def}{=} sign(B, Sign^Z) - sign(N).$$

This is clearly a topological invariant of the fibration $Z \to N \to B$. Moreover Δ depends only on the boundary fibration (extended Novikov additivity, see [BC4]). We call Δ the *non-multiplicativity of signature.* It measures the deviation of signature for manifolds with boundary from being multiplicative. In the case of an oriented fibration of *closed* manifolds $Z \to N \to B$, when $\pi_1(B)$ acts trivially on $H^*(Z)$, the Chern-Hirzebruch-Serre Theorem, [CHS], says

$$signN = signB\, signZ$$

In general, signature for closed oriented manifolds is multiplicative in the sense that

$$signN = sign(B, Sign^Z).$$

This follows from the Atiyah-Singer index theorems and the signature theorem for twisted coefficients; see [AS], [L]. For manifolds with boundary, Bismut-Cheeger noted in [BC4] that the index theory is asymptotically multiplicative in the invertible case. Here we have

Theorem 0.4 (intrinsic characterization of the non-multiplicativity) *The topological invariant τ, which is defined from the closed fibration $Y \to M \to B$, intrinsically characterizes the non-multiplicativity of signature. That is, whenever there exists another fibration of manifolds with boundary $Z \to N^n \to B$ such that $\partial Z = Y$, $\partial N = M$, then $\tau = \Delta$.*

Thus we characterize the non-multiplicativity of signature intrinsically in terms of the topological data of the boundary fibration. One can also give an intrinsic characterization in terms of the analytical data.

Theorem 0.4′ *Let*

V_0= limit of space of A_x-harmonic forms on M in the adiabatic limit,
$V_\pm$=limit of space of λ_x-eigenforms such that $\lambda_x > 0$ (respectively < 0) and λ_x is $O(x^2)$.
Then $V_+ \oplus V_- \oplus V_0 = H^*(B, \mathcal{H}^*(Y))$, and $\tau' \stackrel{def}{=} \dim V_+ - \dim V_- = \tau$. Therefore $\tau' = \Delta$ whenever the fibration $Y \to M \to B$ bounds $Z \to X \to B$.

The advantage of the adiabatic limit formulation of the intrinsic non-multiplicativity is that it allows an extension to the general Dirac operators, which we omit here.

Before we present the organization of this thesis, let us point out that a natural question arises from Theorem 0.1′, that is, whether the assumption that $\ker D_Y$ gives a vector bundle on B is really necessary. In fact one can make sense of all the terms involved in (.3) without making this assumption. Thus, as we mentioned earlier, the first term is always defined, and so is the last term. As for the twisted η term, if $\dim B$ is even it drops out, while for $\dim B$ odd, one can replace $\ker D_Y$ by $Ind\, D_Y^+$, which is always defined. However, the analysis breaks down here.

The thesis is organized as follows. In Chapter 1, with the work of [BC2] as our starting point, we give a formal argument leading to Theorem 0.1. Then a complete proof is given modulo Theorem 1.1 (the asymptotic behavior of $spec(D_x)$) and Proposition 1.2 (the uniform asymptotic expansion). These are subsequently established in Chapter 2 and Chapter 3, using respectively Melrose's theory on the degenerate elliptic operators [MM] together with perturbation theory [K], and the finite propagation speed technique of [CGT]. In Chapter 4, we discuss the Hodge-Leray theory and prove the refined adiabatic limit formula for signature operators. We then proceed to give the intrinsic characterizations of non-multiplicativity of the signature. There are two appendices. The first one, included for the convenience of the readers, contains algebraic and geometric preliminaries as well as a review of the work of [BC2]. In the second appendix, we give a proof of a special case of Bismut-Cheeger's Families Index Theorem, which is used in the application.

Chapter 1

Adiabatic limit of η-invariant

By the work of Bismut-Cheeger the study of the adiabatic limit of the η-invariant of D_x is essentially reduced to the study of the large time behavior of the heat kernel of D_x. In this chapter, we analyze this large time behavior via Theorem 1.1, which gives detailed information about the spectrum of D_x in the adiabatic limit. Theorem 1.1 will be proven in the next chapter. We begin, in Section 1, with the statement of a result from [BC2] and Theorem 1.1. Then we present a formal computation, indicating some ideas behind the proof of Theorem 0.1. The justification for it is given in the following sections.

1.1 Formal computation and outline of its proof

For the clarity of the presentation, we develop all the preliminaries and backgrouds in Appendix A. We will use freely the notions and notations there.

By virture of the local regularity result of [BF] (Cf. Appendix A), the η-invariant of D_x has a heat kernel representation:

$$\eta(D_x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-1/2} tr(D_x e^{-tD_x^2})\, dt.$$

This involves the contribution from the heat kernel for all the time. For the finite time, by exploitation of the so-called Getzler's transformation, Bismut-Cheeger showed that $tr(D_x e^{-tD_x^2})$ converges pointwisely uniformly (to certain expression; see Appendix A or [BC2] for detail). It can also be shown that in the invertible case, $spec(D_x)$ being uniformly bounded away from 0, the large time contribution is negligible. In general, it is a direct consequence of the results in [BC2] that

Proposition 1.1 *Without the assumption that D_Y is invertible, one can still find a small positive number α such that*

$$\lim_{x\to 0} \eta(D_x) = 2\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \lim_{x\to 0} \frac{1}{\sqrt{\pi}} \int_{x^{-\alpha}}^{\infty} t^{-1/2} tr(D_x e^{-D_x^2 t})\, dt, \qquad (1.1)$$

provided either one of the limits exists.

Here the second term is the large time contribution alluded to previously. The above discussion indicates that the information about the spectrum of

D_x in the adiabatic limit is crucial in understanding the large time behavior of the heat kernel. In this respect, we have the following result, the proof of which is defered to Chapter 2.

Theorem 1.1 *For $x > 0$ the eigenvalues of D_x depend analytically on x. Thus there are (countably many) analytic functions λ_x which describe the spectrum of D_x (i.e. if we let $\lambda_{x,j}$, $j = 1, 2, \ldots$ be the collection of these analytic functions, for any fixed $x > 0$, $\{\lambda_{x,j}\}_{j=1}^{\infty}$ is the (unordered) spectrum of D_x). Moreover,*

*A) (**asymptotic behavior**) there exists a positive constant λ_0 such that either λ_x is uniformly bounded away from 0 by λ_0,*

$$|\lambda_x| \geq \lambda_0 > 0,$$

or λ_x has a complete asymptotic expansion as $x \to 0$,

$$\lambda_x \sim \lambda_1 x + \lambda_2 x^2 + \cdots,$$

where $\lambda_1 \in spec(D_B \otimes \ker D_Y)$. In the latter case, the correspondence $\lambda_x \leftrightarrow \lambda_1$ is one-to-one;

*B) (**uniform remainder estimate**) if λ_x corresponds to λ_1 and $\lambda_1 \neq 0$, then*

$$\lambda_x = \lambda_1 x + x^2 C(x) \lambda_1^2, \tag{1.2}$$

with $|C(x)| \leq C$ uniformly bounded;

*C) (**finiteness**) for any $K > 0$,*

$$\#\{\lambda_x \mid \lambda_x \sim \lambda_1 x + \cdots \text{ and } |\lambda_1| \leq K\} < +\infty.$$

In particular, the number of eigenvalues with $\lambda_1 = 0$ (those of at least quadratic decay) is finite.

Granted this, the generalized adiabatic limit formula can be seen formally as follows. From Proposition 1.1

$$\lim_{x\to 0}\eta(D_x) = 2\int_B \hat{A}(\frac{R^B}{2\pi})\wedge\tilde{\eta} + \lim_{x\to 0}\frac{1}{\sqrt{\pi}}\int_{x^{-\alpha}}^{\infty} t^{-1/2}tr(D_x e^{-D_x^2 t})\,dt.$$

And by making a substitution (or rescaling)

$$\lim_{x\to 0}\int_{Cx^{-\alpha}}^{+\infty} t^{-1/2}tr(D_x e^{-tD_x^2})dt = \lim_{x\to 0}\int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}tr(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2})dt \quad (1.3)$$

$$\begin{aligned}
&= \lim_{x\to 0}\Big(\int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\sum_{|\lambda_x|\geq\lambda_0}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt \\
&\quad + \int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\sum_{\substack{\lambda_x\sim\lambda_1 x+\cdots\\ \lambda_1\neq 0}}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt \\
&\quad + \int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\sum_{\lambda_x\sim\lambda_2 x^2+\cdots}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt\Big) \\
&= \int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\sum_{|\lambda_x|\geq\lambda_0}\lim_{x\to 0}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt \\
&\quad + \int_{0}^{+\infty} t^{-1/2}\sum_{\substack{\lambda_x\sim\lambda_1 x+\cdots\\ \lambda_1\neq 0}}\lim_{x\to 0}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt \\
&\quad + \lim_{x\to 0}\Big(\int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\sum_{\lambda_x\sim\lambda_2 x^2+\cdots}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}dt\Big) \\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Here we are ignoring technical problems arising from infinite sums.

In calculating I, note that for $t\in[Cx^{2-\alpha},+\infty)$ and $|\lambda_x|\geq\lambda_0$, $t(\frac{1}{x}\lambda_x)^2\geq C\lambda_0^2 x^{-\alpha}$. Therefore, $\lim_{x\to 0}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}=0$.

For II, we have $\frac{1}{x}\lambda_x\to\lambda_1$ for some $\lambda_1\in spec(D_B\otimes\ker D_Y)$. Thus $\lim_{x\to 0}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}=\lambda_1 e^{-t\lambda_1^2}$.

Finally, for III, we use Mellin's formula to get $\int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2}\frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}\,dt =$ $sgn(\frac{1}{x}\lambda_x)\int_{Cx^{-\alpha}\lambda_x^2}^{+\infty} u^{-1/2}e^{-u}\,du$. Hence,

$$\lim_{x\to 0}\int_{Cx^{-\alpha}}^{+\infty} t^{-1/2}tr(D_x e^{-tD_x^2})dt =$$

$$0+\int_0^{+\infty} t^{-1/2}tr[(D_B\otimes\ker D_Y)e^{-t(D_B\otimes\ker D_Y)^2})dt+\lim_{x\to 0}\sqrt{\pi}\sum_{\lambda_x\sim\lambda_2 x^2+\cdots} sgn\lambda_x,$$

and consequently,

$$\lim_{x\to 0}\eta(D_x)=\int \hat{A}\wedge\tilde{\eta}+\eta(D_B\otimes\ker D_Y)+\lim_{x\to 0}\sum_{\lambda_x\sim\lambda_2 x^2+\cdots} sgn\lambda_x.$$

The above formal calculation suggests the following.

1) Those eigenvalues which are uniformly bounded away from 0 do not contribute.

2) Those eigenvalues which decay exactly linearly in x give rise to the twisted η-invariant on the base.

3) The finiteness result in Theorem 1.1 justifies the calculation of the contribution coming from those eigenvalues which decay at least quadratically in x.

The justification of 1) involves estimates from the finite propagation speed technique [CGT], while that of 2) is a consequence of Theorem 1.1. One can in fact combine 1) and 2), reformulating them as a relation between the heat kernel on the total space under the metric shrinking of the fibers and the heat kernel on the base. This is done in the next section.

Remark Notice that after rescaling, the time interval for integration in (1.3) is $[Cx^{2-\alpha},\infty)$. So there arises the question of small time convergence. This will be dealt with in Section 3.

1.2 Large time behavior of heat kernels

Motivated by the above formal calculation, we rescale the (large) time interval $[Cx^{-\alpha}, \infty)$ back to $[Cx^{2-\alpha}, \infty)$,

$$\int_{Cx^{-\alpha}}^{+\infty} t^{-1/2} tr(D_x e^{-tD_x^2})\, dt = \int_{Cx^{2-\alpha}}^{+\infty} t^{-1/2} tr(\frac{1}{x} D_x e^{-t(\frac{1}{x}D_x)^2})\, dt.$$

This rescaling corresponds to the metric rescaling

$$x^2 g_x = \pi^* g_B + x^2 g_Y,$$

which shrinks the metric along the fibers. The first step towards the proving of the adiabatic limit formula is the following result, which appears to have an interest of its own.

Proposition 1.2 *Let $D_0 = D_B \otimes \ker D_Y$. There exists $N > 0$ sufficiently large such that*

$$|tr'(\frac{1}{x} D_x e^{-t(\frac{1}{x}D_x)^2}) - tr(D_0 e^{-tD_0^2})| \leq \frac{Cx}{t^N}, \tag{1.4}$$

where tr' indicates taking trace over those eigenvalues which decay at most linearly in x.

Taking into account the rest of the eigenvalues (i.e., those decaying quadratically) by using Theorem 1.1, we can rewrite the result in Proposition 1.2 in a nicer, although weaker, form,

$$|tr(\frac{1}{x} D_x e^{-t(\frac{1}{x}D_x)^2}) - tr(D_0 e^{-tD_0^2})| \leq (\frac{C}{t^N} + C')x.$$

Roughly speaking, this formula says that when the total space collapses to the base (in the sense of Cheeger-Gromov, [CG]) its heat kernels converge to the corresponding heat kernels on the base.

The main technical difficulty in the proof of Proposition 1.2 lies in establishing the dividing line between the large eigenvalues (those do not contribute) and the small ones (those give rise to the twisted η term). This is determined by the uniform remainder estimate of Theorem 1.1. It turns out that a workable dividing line is given by $\lambda_0 x^a$ for any $0 < a < 1$. We take $a = 1/2$.

Proof. One has

$$tr'(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2}) = \sum{}' \frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2},$$

where the summation $\sum'$ runs over all eigenvalues of D_x which decay at most linearly. Also

$$tr(D_0 e^{-tD_0^2}) = \sum \lambda_1 e^{-t\lambda_1^2},$$

where the summation runs over all (nonzero) eigenvalues of D_0. Thus

$$tr'(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2}) - tr(D_0 e^{-tD_0^2}) = \sum{}' \frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2} - \sum \lambda_1 e^{-t\lambda_1^2}.$$

We now divide our summations into

1) Large eigenvalues.

Define

$$f_x(\lambda) = \begin{cases} \lambda e^{-t\lambda^2} & \text{if } |\lambda| \geq \lambda_0/\sqrt{x} \\ 0 & \text{otherwise} \end{cases}, \quad h_x(\lambda) = \begin{cases} \lambda^2 e^{-t\lambda^2} & \text{if } |\lambda| \geq \lambda_0/\sqrt{x} \\ 0 & \text{otherwise} \end{cases}.$$

Then the operators $f_x(\frac{1}{x}D_x)$ and $h_x(\frac{1}{x}D_x)$ can be defined by the spectral theorem (see the appendix at the end of this chapter), moreover, by the spectral mapping theorem,

$$|tr(f_x(\frac{1}{x}D_x))| = |\sum_{|\lambda_x/x| \geq \lambda_0/\sqrt{x}} \frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}|$$

$$\leq \sum_{|\lambda_x/x|\geq\lambda_0/\sqrt{x}} |\frac{1}{x}\lambda_x|e^{-t(\frac{1}{x}\lambda_x)^2}$$
$$\leq \sum_{|\lambda_x/x|\geq\lambda_0/\sqrt{x}} (\frac{1}{x}\lambda_x)^2 e^{-t(\frac{1}{x}\lambda_x)^2} = tr(h_x(\frac{1}{x}D_x).$$

Choose a nonnegative function $\varphi_x(\lambda) \in C^\infty(-\infty,+\infty)$ such that $\varphi_x(\lambda) \equiv 0$ for $|\lambda| \leq \lambda_0/2\sqrt{x}$ and $\varphi_x(\lambda) \equiv 1$ for $|\lambda| \geq \lambda_0/\sqrt{x}$. Further $\|\varphi_x\|_{C^2} \leq C$ uniformly. Set

$$H = H_x(\lambda) = \lambda^2 e^{-t\lambda^2}\varphi_x(\lambda).$$

Clearly we have

$$tr(h_x(\frac{1}{x}D_x)) \leq tr(H_x(\frac{1}{x}D_x)).$$

Let $k_{H_x} = k_{H_x(\frac{1}{x}D_x)}$ denote the (Schwartzian) kernel of the operator $H_x(\frac{1}{x}D_x)$ (with respect to the metric $x^2 g_x$). We want to estimate k_{H_x} by the finite propagation speed technique of Cheeger-Gromov-Taylor [CGT] (see also the appendix at the end of this chapter).

First of all, note that $\frac{1}{x}D_x$ corresponds to the metric $x^2 g_x$. The sectional curvature of this metric are bounded by $C\frac{1}{x^2}$, and the injectivity radius is bounded from below by Cx (Cf. [C2]). Therefore, by (1.17), one has the pointwise estimates

$$|k_{H_x}| \leq \frac{C}{x^{n+1}} \sum_{k=0}^{n+1} \int_0^{+\infty} |\hat{H}^{(k)}(s)|\, ds, \tag{1.5}$$

where $n = \dim M$. Thus it suffices to estimate $\int_0^{+\infty} |\hat{H}^{(k)}(s)|\, ds$. To do this, write

$$\int_0^{+\infty} |\hat{H}^{(k)}(s)|\, ds = \int_0^{+\infty} \frac{1}{(1+s^2)}(1+s^2)|\hat{H}^{(k)}(s)|\, ds. \tag{1.6}$$

We will show that

$$|(1+s^2)\hat{H}^{(k)}(s)|\,ds \leq \frac{C(k)}{t^{(n+5)/2}}e^{-t\lambda_0/4x}.$$

By calculus and the basic properties of the Fourier transform, we have

$$\begin{aligned}
(1+s^2)\hat{H}^{(k)}(s) &= \hat{H}^{(k)}(s) + s^2\hat{H}^{(k)}(s) \\
&= \hat{H}^{(k)}(s) + (s^2\hat{H}(s))^{(k)} - 2k(s\hat{H})^{(k-1)} - (k-1)^2\hat{H}^{(k-1)}(s) \\
&= \hat{H}^{(k)}(s) + (\widehat{H''}(s))^{(k)} - 2k(\widehat{H'})^{(k-1)} - (k-1)^2\hat{H}^{(k-1)}(s). \quad (1.7)
\end{aligned}$$

Now

$$\begin{aligned}
|\hat{H}^{(k)}(s)| &= |\int e^{-is\lambda}(-i\lambda)^k H(\lambda)d\lambda| \\
&\leq C\int_{|\lambda|\geq\lambda_0/2\sqrt{x}} |\lambda^{k+2}e^{-t\lambda^2}|\,d\lambda \\
&\leq 2\int_{\lambda_0/2\sqrt{x}}^{\infty} \frac{C(k)}{t^{(k+1)/2}}\lambda e^{-t\lambda^2/2}d\lambda = \frac{C(k)}{t^{(k+1)/2}}e^{-t\lambda_0^2/4x}. \quad (1.8)
\end{aligned}$$

The last step comes from the following elementary inequality

$$\lambda^{2j}e^{-t\lambda^2} \leq t^{-j}C(j)e^{-t\lambda^2/2}. \quad (1.9)$$

One can apply the same argument as above to estimate $(\widehat{H'})^{(k-1)}$, and $(\widehat{H''})^{(k)}$. Plug (1.8) and the corresponding estimates for $(\widehat{H'})^{(k-1)}$ and $(\widehat{H''})^{(k)}$ in (1.7), and together with (1.5), (1.6), one finds

$$|k_{H_x(\frac{1}{x}D_x)}| \leq \frac{C}{x^n t^{(n+5)/2}}e^{-t\lambda_0^2/4x}.$$

Thus one can integrate to obtain

$$tr(H_x(\frac{1}{x}D_x) \leq \frac{C}{x^p t^{(n+5)/2}}e^{-t\lambda_0^2/4x},$$

where p denotes the dimension of the base B. Consequently,

$$|\sum_{|\lambda_x|>\lambda_0} \frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2}| \leq \frac{C}{x^p t^{(n+5)/2}} e^{-t\lambda_0^2/4x}. \tag{1.10}$$

Remark This estimate shows that those eigenvalues λ_x such that $|\lambda_x| \geq \lambda_0\sqrt{x}$ do not contribute in the limit. The choice of $\sqrt{x}$ is not essential here, see the explanation immediately after Proposition 1.2. In fact, if we choose x^β instead and β sufficiently small (e.g. $2\beta < \alpha$), we can use the argument from [C2] (see also [BC2]) to give a simpler proof of this fact.

2) Small eigenvalues.

If $|\lambda_x| < \lambda_0$, then by Theorem 1.1, $\frac{1}{x}\lambda_x \to \lambda_1$ for some $\lambda_1 \in spec(D_0)$. Since we are taking tr', we can assume $\lambda_1 \neq 0$. Now

$$\begin{aligned} &|\sum_{|\lambda_x|<\lambda_0\sqrt{x}} {}' \frac{1}{x}\lambda_x e^{-t(\frac{1}{x}\lambda_x)^2} - \sum \lambda_1 e^{-t\lambda_1^2}| \\ &\leq |\sum_{|\lambda_x|<\lambda_0\sqrt{x}} {}'(\frac{1}{x}\lambda_x - \lambda_1) e^{-t(\frac{1}{x}\lambda_x)^2}| + |\sum_{|\lambda_x|<\lambda_0\sqrt{x}} {}'\lambda_1(e^{-t(\frac{1}{x}\lambda_x)^2} - e^{-t\lambda_1^2})| \\ &+|\sum_{|\lambda_1|\geq\frac{\lambda_0}{2\sqrt{x}}} \lambda_1 e^{-t\lambda_1^2}| = \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{1.11}$$

By the same argument as we show (1.10) (in a simpler situation), it is easy to see that

$$\mathrm{III} \leq \frac{C}{t^{p/2}} e^{-t\lambda_0^2/4x}.$$

To estimate I and II, we have to use the uniform remainder estimate in Theorem 1.1. Note that the remainder there is bounded by a uniform constant *times the square of* λ_1, which can be arbitrarily large. To cope with this problem, we now use the fact that the summations are over $|\lambda_x| < \lambda_0\sqrt{x}$

and the estimates (2.25), (2.26) in the proof of Theorem 1.1 to show that λ_1 can not grow too fast relative to x.

Since $\frac{1}{x}\lambda_x \to \lambda_1$, by taking $x \to 0$ in (2.25) where $\bar{\lambda}_x$ obeys (2.26), we deduce

$$|\lambda_1 - \lambda_0'| \leq C.$$

Now $|\lambda_x| < \lambda_0\sqrt{x}$ together with (2.26) and (2.25) yields

$$|\lambda_0'| \leq |\bar{\lambda}_x| + Cx \leq |\frac{1}{x}\lambda_x| + 2C \leq \frac{\lambda_0}{\sqrt{x}} + 2C \leq \frac{2\lambda_0}{\sqrt{x}}.$$

Combining the two gives

$$|\lambda_1| \leq \frac{3\lambda_0}{\sqrt{x}}.$$

Hence λ_1 grows no faster than $x^{-1/2}$. It follows from this and the uniform remainder estimate that

$$(\frac{1}{x}\lambda_x)^2 = \lambda_1^2(1 + xC(x)\lambda_1)^2 \geq \frac{\lambda_1^2}{4},$$

provided $x \leq \frac{1}{36\lambda_0^2C^2}$. This together with (1.2) implies

$$\begin{aligned} \mathrm{I} &\leq \sum_{\lambda_1 \neq 0} xC\lambda_1^2 e^{-t(\frac{1}{x}\lambda_x)^2} \leq xC\sum \lambda_1^2 e^{-t\lambda_1^2/4} \\ &\leq \frac{Cx}{t^N}. \end{aligned}$$

The last step is by virtue of (1.9) and the standard heat kernel estimates (Cf., say, [CGT]).

For II, notice that by using the inequality $|e^{-\lambda} - 1| \leq |\lambda|e^{|\lambda|}$ (which follows by the standard expansion).

$$\begin{aligned} |e^{-t(\frac{1}{x}\lambda_x)^2} - e^{-t\lambda_1^2}| &= |e^{-t\lambda_1^2}(e^{-t[(\frac{1}{x}\lambda_x)^2 - \lambda_1^2]} - 1)| \\ &\leq tC\lambda_1^2 x e^{-t\lambda_1^2(1 - x|\lambda_1|\,|C(x)|)^2}. \end{aligned}$$

Therefore,

$$\mathrm{II} \leq Cxt \sum |\lambda_1|^3 e^{-t\lambda_1^2/4} \leq \frac{Cx}{t^N}.$$

These estimates combined with (1.10) give us the estimate (1.4).

Q.E.D.

Remark The above method allows a more general statement. For example, if $f \in \mathcal{S}([0,+\infty))$ is in the Schwartz class, then we can show by the same argument (with slight adaptation)

$$|tr'(f(\frac{1}{x}D_x)) - tr(f(D_0))| \leq \frac{Cx}{t^N}.$$

An interesting example of such f would be $e^{-t\lambda^2}$, which gives rise to the heat kernel. Thus

$$|tr'(e^{-t(\frac{1}{x}D_x)^2}) - tr(e^{-tD_0^2})| \leq \frac{Cx}{t^N}.$$

An immediate consequence of Proposition 1.2 is

Corollary 1.1 *There exists $\beta > 0$ sufficiently small such that*

$$\lim_{x\to 0} \frac{1}{\sqrt{\pi}} \int_{Cx^{-2+\beta}}^{+\infty} t^{-1/2} tr(D_x e^{-tD_x^2}) dt = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-1/2} tr(D_0 e^{-tD_0^2}) dt + \lim_{x\to 0} \sum_{\lambda_x \sim \lambda_2 x^2 + \cdots} sgn \lambda_x, \quad (1.12)$$

if either one of the limits exists in $\mathbb{R}$.

Proof. If one makes a change of variables in the lefthand side of (1.12), one finds

$$\begin{aligned}
&\lim_{x\to 0}\frac{1}{\sqrt{\pi}}\int_{Cx^{-2+\beta}}^{+\infty} t^{-1/2}tr(D_x e^{-tD_x^2})dt \\
&= \lim_{x\to 0}\frac{1}{\sqrt{\pi}}\int_{Cx^{\beta}}^{+\infty} t^{-1/2}tr(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2})dt \\
&= \lim_{x\to 0}\frac{1}{\sqrt{\pi}}\int_{Cx^{\beta}}^{+\infty} t^{-1/2}tr'(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2})dt + \lim_{x\to 0}\sum_{\lambda_x\sim\lambda_2 x^2+\cdots} sgn\lambda_x.
\end{aligned}$$

In the last step above, we have used Mellin's formula and the fact that the number of eigenvalues which decay at least quadratically is finite (Cf. Section 1). Now take $0 < \beta < 1/2N$ and apply Proposition 1.2. We obtain Corollary 1.1.

1.3 Small time convergence

In the proceeding section large time behavior of the heat kernel is analyzed by virtue of our detailed knowledge of the (small) eigenvalues. As a result we have obtained (1.12), which furnished a justification for 1) and 2) of Section 1, but only for the time interval $[x^{-2+\beta}, \infty)$, where β is a sufficiently small positive number. Thus we are left with the task of showing that the remaining piece (the small time contribution after the rescaling) is negligible. This is done by extending an argument in [BC2], where they encountered a similar task. Here an important step is to establish a uniform asymptotic expansion.

Proposition 1.3 *One has the following uniform pointwise asymptotic expansion,*

$$tr(D_x e^{-tD_x^2}) = \sum_{i=-n}^{N-1} a_i(t)(tx^2)^{i/2} + O((tx^2)^{N/2}), \tag{1.13}$$

where $a_i(t)$'s are bounded for $t \geq 1$, and so is $O(\cdot)$.

The proof of this proposition will be given in Chapter 3. Here we demonstrate that Proposition 1.2 and Proposition 1.3 combined imply the small time convergence.

Proposition 1.4 *For any $\beta > 0$ small, $2 > \alpha > 0$,*

$$\lim_{x\to 0} \int_{x^{-\alpha}}^{x^{-2+\beta}} t^{-1/2} tr(D_x e^{-tD_x^2}) dt = 0. \tag{1.14}$$

Proof. We first rescale the time interval, i.e., make a change of variables in the lefthand side of (1.14). Thus

$$\lim_{x\to 0} \int_{x^{-\alpha}}^{x^{-2+\beta}} t^{-1/2} tr(D_x e^{-tD_x^2}) dt = \lim_{x\to 0} \int_{x^{2-\alpha}}^{x^{\beta}} t^{-1/2} tr(\frac{1}{x} D_x e^{-t(\frac{1}{x}D_x)^2}) dt.$$

Now we apply the uniform asymptotic expansion (1.13) to $tr(D_x e^{-t(\frac{1}{x}D_x)^2})$ with t/x^2 as the new time parameter. Thus

$$tr(D_x e^{-t(\frac{1}{x}D_x)^2}) = \sum_{i=-n}^{N'-1} a_i(t/x^2) t^{i/2} + O(t^{N'/2}).$$

To eliminate the negative powers of t in the above asymptotic expansion, we put $u = \frac{t}{x^2} \geq 1$, and rewrite it as

$$tr(D_x e^{-uD_x^2}) = \sum_{i=-n}^{N'-1} a_i(u) u^{i/2} x^i + O(u^{N'/2} x^{N'}). \tag{1.15}$$

From the proof of Theorem A.3, we know that for fixed u, the lefthand side of (1.15) approaches a finite limit as $x \to 0$. Therefore

$$a_i = 0, \qquad \text{if } i < 0,$$

and thus

$$tr(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2}) = \sum_{i=0}^{N'-1} a_i(t/x^2)t^{i/2}x^{-1} + O(t^{N'/2}x^{-1}). \qquad (1.16)$$

This is the cancellation in t parameter. To obtain cancellation in x parameter we use the estimate (1.4). Take $M \geq N+1, N' \geq 2M+1$ and set $x = t^M \to 0$. By the same reasoning as above (together with Theorem A.1) one finds

$$a_i(t^{1-2M})t^{-M+\frac{i}{2}} = O(1).$$

Or

$$a_i(t^{-1}) = O(t^{\frac{M}{2M-1}-\frac{i}{2(2M-1)}}).$$

Pluging in (1.16) we finally arrive at

$$tr(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2}) = \sum_{i=0}^{N'-1} a_i' \, t^{(i-1)\frac{M}{2M-1}} x^{-\frac{i-1}{2M-1}} + O(t^{N'/2}/x^{-1}),$$

where a_i' are bounded functions. Consequently

$$\begin{aligned}
|\int_{x^{2-\alpha}}^{x^\beta} t^{-1/2} tr(\frac{1}{x}D_x e^{-t(\frac{1}{x}D_x)^2})dt| &\leq Cx^{1/(2M-1)} \int_{x^{2-\alpha}}^{x^\beta} t^{-1/2-M/(2M-1)}\, dt \\
&+ \sum_{i=1}^{N'-1} Cx^{(i-1)(\alpha M-1)/(2M-1)} \int_{x^{2-\alpha}}^{x^\beta} t^{-1/2}\, dt \\
&+ C(x^\beta - x^{2-\alpha})x^{(N'-1)\alpha/2-1} \\
&\leq Cx^{\alpha/(4M-2)} + \sum_{i=1}^{N'-1} Cx^{(i-1)(\alpha M-1)/(2M-1)}(x^{\beta/2} - x^{1-\alpha/2}) \\
&+ C(x^\beta - x^{2-\alpha})x^{(N'-1)\alpha/2-1}.
\end{aligned}$$

Taking $M \geq \alpha^{-1}$ and $N' \geq 2\alpha^{-1} + 1$ gives us (1.14) Q.E.D.

As a consequence of Proposition 1.1, Corollary 1.1 and Proposition 1.3, Theorem 0.1 follows.

Appendix to Chapter 1: Finite propagation speed technique

Consider a complete Riemannian manifold M. The Laplacian Δ is a non-negative essentially self-adjoint operator. Thus functions $f(\sqrt{\Delta})$ can be defined by the spectral theorem for unbounded self-adjoint operators, according to the prescription

$$f(\sqrt{\Delta}) = \int_0^\infty f(\lambda) dE_\lambda,$$

where dE_λ is the projection valued measure associated with $\sqrt{\Delta}$. The following theorem of Cheeger-Gromov-Taylor, [CGT], gives explicit bounds on the kernel $k_{f(\lambda)}$.

Theorem 1.2 (Cheeger-Gromov-Taylor) *Let M^n be complete and $H \leq K_M \leq K$. Fix $x_1, X_2 \in M^n$. Let $d = \rho(x_1, x_2)$ the distance between x_1 and x_2 and N be an integer such that $4N > n$. Assume that f is a function with the property that up to N-th derivatives of its Fourier transform are integrable, $\hat{f}^{(k)} \in L^1(0, \infty), 0 \leq k \leq N$. Let $r_1, r_2 \leq min(inj_M, |H|^{-1/2})$,* *then*

$$|k_f(x_1,x_2)| \leq \frac{c(n)^2}{\pi} \sum_{0\leq i,j\leq N} r_1^{2i-n/2} r_2^{2j-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}^{(2i+2j)}(s)|ds. \quad (1.17)$$

Moreover one also has the estimats for the derivatives,

$$|k_f(x_1,x_2)|_{\infty,l_1,l_2} \leq \frac{c(n)^2}{\pi} \sum_{\substack{0\leq i\leq N+[l_1/2]+1 \\ 0\leq j\leq N+[l_2/2]+1}} r_1^{2i-n/2} r_2^{2j-n/2} \int_{d-r_1-r_2}^{\infty} |\hat{f}^{(2i+2j)}(s)|ds, \quad (1.18)$$

where $|\cdot|_{\infty,l_1,l_2}$ *denotes the sup norm of up to* l_1*-th derivatives on the first variable and* l_2*-th on the second.*

Proof. We give the argument in [CGT]. First we construct a nice parametrix for Δ. To do this we first assume that $r_j = 1$, $|K_M| \leq 1$.

Let ϕ be a smooth function supported on $[0,1/2]$ with $\phi|_{[0,1/4]} \equiv 1, |\phi'| < 5$, and set $\phi_\epsilon = \phi(r/\epsilon)$. Let

$$P = \phi_\epsilon(r)\frac{r^{2-n}}{\alpha(n-1)} \; (P = \phi_\epsilon \frac{\log r}{2\pi},\; n=2).$$

Finally, let $\Delta_x(P(x_0,x)) = Q$. Then on $B_1(x_0)$

$$|Q(r)| \leq C_\epsilon r^{2-n}.$$

If g is a smooth function on $B_1(x_0)$, then

$$\int_{B_1(x_0)} P(x_0,x)\Delta g(x)dx = g(x_0) - \int_{B_1(x_0)} Q(x)g(x)dx. \quad (1.19)$$

Fix ϵ sufficiently small so that the N^2 compositions below are defined and apply the standard iteration argument. Thus if we write (1.19) as

$$P\Delta = I - Q,$$

and set

$$\mathcal{P} = (I + Q + \cdots + Q^{N-1})P, \ N = [\frac{n}{4}] + 1,$$
$$\mathcal{Q} = Q^N.$$

Then it follows easily that

$$\mathcal{P}^N(\Delta^N g) + \mathcal{P}^{N-1}\mathcal{Q}(\Delta^{N-1}g) + \cdots + \mathcal{P}\mathcal{Q}(\Delta g) + \mathcal{Q}g = g. \tag{1.20}$$

The kernels on the left hand are continuous and their sup norms are bounded since the metric is boundedly related to the Euclidean metric in normal coordinates, and thus

$$|g(x_0)| \le C(n)(\|g\|_{B_1(x_0)} + \|\Delta g\|_{B_1(x_0)} + \cdots + \|\Delta^N g\|_{B_1(x_0)}).$$

Now we only assume that $|K_M| \le K$, and

$$r \le \min(|K|^{-1/2}, i(x_0)), \ N = [\frac{n}{4}] + 1.$$

Via scaling, we have

$$|g(x_0)| \le C(n)(r^{-n/2}\|g\|_{B_1(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_1(x_0)}).$$

Next, consider $\bigtriangledown g$. Note that in (1.19), if we take $4N > n+1$, then

$$\bigtriangledown g = \bigtriangledown\mathcal{P}^N(\Delta^N g) + \bigtriangledown\mathcal{P}^{N-1}\mathcal{Q}(\Delta^{N-1}g) + \cdots + \bigtriangledown\mathcal{P}\mathcal{Q}(\Delta g) + \bigtriangledown(\mathcal{Q}g),$$

and again the kernels on the righthand side are continuous and their sup norms are bounded. Thus

$$|g(x_0)|_{\infty,1} \le C(n)(r^{-n/2}\|g\|_{B_r(x_0)} + r^{-n/2+2}\|\Delta g\|_{B_r(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_r(x_0)}),$$

provided $4N > n+1$. In general, we have

$$|g(x_0)|_{\infty,l} \leq C(n)(r^{-n/2}\|g\|_{B_r(x_0)} + \cdots + r^{2N-n/2}\|\Delta^N g\|_{B_r(x_0)}),$$

provided $4N > n+l$.

To obtain the corresponding L^2-estimates, we exploit the unit propagation speed of the wave operator $e^{is\sqrt{\Delta}}$ to get (see [CGT] for details)

$$\|f(\sqrt{\Delta})u\|_{M\setminus B_R(x_2)} \leq \|u\|\frac{1}{\pi}\int_{R-r}^{\infty} |\hat{f}(s)|\, ds,$$

provided $\operatorname{supp} u \subset B_r(x_2)$, $r < R$. Intuitively, this is because it takes a definite amount of time for the effect of a source to be felt in a region at a distance. By the same token

$$\|\Delta^k f(\sqrt{\Delta})\Delta^l u\|_{M\setminus B_R(x_2)} \leq \|u\|\frac{1}{\pi}\int_{R-r}^{\infty} |\hat{f}^{(2k+2l)}(s)|\, ds.$$

To obtain the desired estimate for $k_f(x_1, x_2)$, let

$$g = \Delta_2^j k_f(x_1, x_2)(u(x_2)) = \int_M \Delta_2^j k_f(x_1, x_2)u(x_2)dx_2,$$

for u supported on $B_{r_2}(x_2)$. Apply the elliptic estimate and L^2-estimates above,

$$\begin{aligned}
|\Delta_2^j k_f(x_1, x_2)(u)| &\leq C(n)\sum_{i=0}^{N} r_1^{2i-n/2}\|\Delta_1^i \Delta_2^j k_f(u)\|_{B_{r_1}(x_1)} \\
&= C(n)\sum_{i=0}^{N} r_1^{2i-n/2}\|\Delta_1^i k_f \Delta_2^j(u)\|_{B_{r_1}(x_1)} \\
&\leq \frac{C(n)}{\pi}\|u\|_{B_{r_2}(x_2)}\sum_{i=0}^{N} r_1^{2i-n/2}\int_{d-r_1-r_2}^{\infty} |\hat{f}^{(2i+2j)}(s)|\, ds.
\end{aligned}$$

Since u is arbitary, we have, for all x_1

$$\|\Delta_2^j k_f(x_1, x_2)\|_{B_{r_2}(x_2)} \leq \frac{C(n)}{\pi}\sum_{i=0}^{N} r_1^{2i-n/2}\int_{d-r_1-r_2}^{\infty} |\hat{f}^{2i+2j}(s)|ds$$

Now apply the elliptic estimates again, we obtain (1.17). The estimate for the derivative can be obtained similarly. Q.E.D

Chapter 2

Asymptotic behavior of spectrum

This chapter is devoted to the proof of Theorem 1.1, which gives the asymptotic behavior of the spectrum of D_x as $x \downarrow 0$. Recall that D_x is associated with the metric which is rescaled in the base direction by x^{-2}. From the local point of view, the rescaling is making the operator D_x better since the local geometry is simplifying. And this is the reason that for the finite time behavior of the heat kernel one can work effectively with the so called Getzler's transformation. However, from the global point of view, the elliptic operator D_x becomes degenerate as $x \downarrow 0$. This degeneracy of the elliptic operator D_x is the essential difficulty here. The length of this chapter largely comes out of dealing with it.

In Section 2.1, we utilize the theory developed by Melrose et.al. to treat D_x. We show that as L^2-operators the resolvent of $\frac{1}{x}D_x$ depends smoothly on the parameter x down to $x = 0$. This result, combined with the regular perturbation theory, gives us the asymptotic behavior of the small eigenvalues at $x = 0$, in Section 2.2. The finiteness result, which is a consequence of ellipticity, is also discussed there. The uniform remainder estimate is established by a deformation argument where the local geometry of the fibration plays a definite role.

2.1 Analysis of degenerate elliptic operators

We shall study the degenerate elliptic operator D_x by constructing a parametrix which is uniform down to $x = 0$. The construction relies on a calculus of pseudodifferential operators well adapted to the type of the degeneracy exhibited by D_x. A salient feature of this calculus is that the residual operators of the (first) symbol filtration are not compact operators. In fact the novelty here is to find the right tools to probe the residual operators further.

To begin with, recall that any differential operator L on a smooth manifold X, $L \in \mathrm{Diff}(X)$, is locally the sum of products of vector fields. Correspondingly, natural subrings of $\mathrm{Diff}(X)$ may be defined as generated in this sense by certain geometrically natural subalgebras of the Lie algebra of all smooth vector fields. From (A.14) of Appendix A, we know that D_x is in the subring of $\mathrm{Diff}(\mathbb{R}_+ \times M)$, generated by vector fields of the type (2.1) below, whose elements (when they degenerate) degenerate in a *uniform* fashion at the boundary of $\mathbb{R}_+ \times M$. In this context, there is still a good notion of ellipticity, and a class of pseudodifferential operators exists which contains parametrices with compact remainder for elliptic elements in this subring.

The general program of study of this types of pseudodifferential operators originates from R. Melrose, [M3], and further developed in [M1], [M2], and [MM]. In [MM], the Laplacian in the adiabatic limit is analyzed in detail, which provides the model for the first part of our analysis of D_x.

Consider the fibration $Y \to M^n \xrightarrow{\pi} B^p$. To incorporate the parameter x, let us consider the product manifold $X = [0,\infty) \times M$. On X, we have the space $\mathcal{V}$ of C^∞ vector fields which are tangent to the fibers, M, of the product structure and which are as well tangent to the fibers of the fibration, π, above $M_0 = \{x = 0\}$. In fact $\mathcal{V}$ is a locally free sheaf of finite rank, therefore (C^∞) sections of a vector bundle, which we denote by ${}^{\mathcal{V}}TM$. In local coordinates $y_1, \cdots, y_p, z_1, \cdots, z_{n-p}$ on M where the $y's$ give the coordinates on B, a basis of local generating sections of ${}^{\mathcal{V}}TM$ is given by

$$x\partial_{y_1},\ \cdots,\ x\partial_{y_p},\ \partial_{z_1},\ \cdots,\ \partial_{z_{n-p}}. \tag{2.1}$$

The dual, ${}^{\mathcal{V}}T^*M$, to ${}^{\mathcal{V}}TM$ plays an important role below. The dual basis to (2.1) is then

$$x^{-1}dy_1,\ \cdots,\ x^{-1}dy_p,\ dz_1,\ \cdots,\ dz_{n-p}. \tag{2.2}$$

There is a natural splitting of ${}^{\mathcal{V}}T^*M$ at M_0,

$${}^{\mathcal{V}}T^*M = T^*Y \oplus x^{-1}T^*B. \tag{2.3}$$

The family of metrics g_x on M lifts to a non-degenerate inner product on ${}^{\mathcal{V}}TM$ and dually on ${}^{\mathcal{V}}T^*M$.

Notice that away from $x = 0$ the bundle ${}^{\mathcal{V}}TM$ (${}^{\mathcal{V}}T^*M$) is isomorphic to the lift to X of the tangent bundle TM (the cotangent bundle T^*M) but not naturally so. Therefore the significance of this bundle is at $x = 0$.

From (A.14) we see that our operator

$$D_x : C^\infty(X; F(M)) \longrightarrow C^\infty(X; F(M)) \tag{2.4}$$

is a $\mathcal{V}$-differential operator (i.e. generated by $\mathcal{V}$, see Section A.4),

$$D_x \in \mathrm{Diff}^1_{\mathcal{V}}(X; F(M)). \tag{2.5}$$

Away from $x = 0$ this corresponds to nothing but a change of basis. Thus again its importance is at $x = 0$.

With the notations out of our way, we can now state the main result of this section (Compare [MM]).

Theorem 2.1 *1) Let $\lambda_x \in C^\infty([0,\infty))$. Then there is a parametrix G for $D_x - \lambda_x$,*

$$(D_x - \lambda_x)G = Id + F,$$

where $F \in x^{-1}C^\infty(Z; End\, F(M))$. Moreover, G is uniform in x in the sense that

$$G: \ C^\infty(X, F(M)) \to x^{-1}C^\infty(X, F(M)).$$

2) Assume that $\ker D_Y$ *gives a vector bundle on B. Let $P = P_{\ker D_Y}$ be the orthogonal projection onto* $\ker D_Y$*, and set $D_0 = P(\sum_\alpha f_\alpha \nabla^u_{f_\alpha})P = D_B \otimes \ker D_Y$ (see (A.14) of section A.3), the Dirac operator on B coupled* *to the vector bundle* $\ker D_Y$. *If* $\lambda \notin spec(D_0)$, *then* $(\frac{1}{x}D_x - \lambda)^{-1}$ *exists for small* x *and*

$$(\frac{1}{x}D_x - \lambda)^{-1} = x(D_x - \lambda x)^{-1} : L^2(M; F(M)) \longrightarrow L^2(M; F(M))$$

is C^∞ *in* x *down to* $x = 0$.

Remark Let H be a Hilbert space. A family of bounded linear operators on H can be thought of as a map

$$\mathbb{R} \longrightarrow L(H),$$

where $L(H)$ denotes the Banach space of bounded linear operators on H. The usual definition of derivative extends to this case, thus the smoothness. Consider a map

$$[0,1] \longrightarrow L(H).$$

When we say that it is C^∞ down to 0 we mean that all its derivatives are uniformly bounded near 0 (or equivalently it extends to C^∞ map in a neighborhood of 0).

We divide the proof of this theorem into several lemmas. The construction of parametrices is approached through their Schwartz kernels. Although these are distributions on $Z = \mathbb{R}_+ \times M \times M$, they live more naturally on a slightly larger manifold, obtained by "blowing up" Z along the fibre diagonal of $M \times M$ in the boundary of Z. This process of blowing up, on which the microlocalization takes place, is the essential point here. Our operator, when lifted to the blown up manifold, becomes transversally elliptic. Thus in the first step of the construction, we can use the standard elliptic theory. The resulting error term, while smooth on the blown up, is not smooth on Z, however. We remove the singular part by killing the Taylor coefficients of the error term at the points being blown up. This is a global construction on a Euclidean bundle over the fibers which distinguishes the fiber-harmonic spinors at the boundary and involves the inversion of certain model operator on the Euclidean space.

The existence of the resolvent follows from this construction and an asymptotic analysis, which is defered to Section 4.4. The uniformity of the constructed parametrix and the smooth dependence of the resolvent are obtained by showing the tangential and normal regularity separately. While the former is often a consequence of the construction, the latter is considerably harder, involving detailed analysis and various estimates.

We begin with the notion of blow up. Let

$$Q = \{(0, p, p'); \pi_L(p) = \pi_R(p')\} \subset Z = [0, \infty) \times M \times M$$

where π_L and π_R are the left and right fibrations. The *blow up* of Z along the submanifold Q is (Cf. [M1], [M2])

$$Z_Q = SNQ \sqcup [Z \backslash Q],$$

where SNQ is the inward-pointing unit spherical bundle of Q in Z. As a set it is given by replacing Q by SNQ. It comes equipped with the 'blow-down' map

$$\pi_Q : Z_Q \to Z,$$

which is the identity away from the 'front face' SNQ, which we will denote $\mathrm{ff}(Z_Q)$. The blown up space Z_Q has a unique C^∞ structure such that π_Q

is C^∞, is a diffeomorphism from $Z_Q \backslash \mathrm{ff}(Z_Q)$ to $Z \backslash Q$ and has rank $\dim B + 2\dim Y + 1$ at $\mathrm{ff}(Z_Q)$.

The front face $\mathrm{ff}(Z_Q)$ is fibred by hemispheres $\pi_Q^{-1}(q)$, $q \in Q$. In fact the projection of Q down to the right factor of M in Z shows that $\mathrm{ff}(Z_Q)$ fibres over M with fibres $Y \times S^p_+$, where S^p_+ is the p-dimensional hemisphere,

$$Y \times S^p_+ \to \mathrm{ff}(Z_Q) \to M.$$

The hemisphere S^p_+ are parameterized non-singularly by the component ω of the polar coordinates around Q,

$$R = (x^2 + |y - y'|^2)^{1/2},\ \omega = (\frac{x}{R}, \frac{y - y'}{R}),\ y,\ z,\ z'. \tag{2.6}$$

More usefully for computations one can introduce the projective coordinates

$$x,\ u = \frac{y - y'}{x},\ y,\ z,\ z', \tag{2.7}$$

valid in the interior of the front face. Since u takes values in $\mathbb{R}^p$, the interior of each fibre S^p_+ of the front face has a *Euclidean structure.*

The significance of Z_Q is mainly related to the following result.

Lemma 2.1 *The closure, $\Delta_{\mathcal{Y}}$, in Z_Q of the submanifold $\{(x,\ p,\ p);\ x > 0,\ p \in M\}$ is an embedded submanifold which meets the boundary of Z_Q only in the interior of ff(Z_Q) and does so transversally, and D_x lifts to a differential operator with smooth coefficients on Z_Q which is transversally elliptic with respect to $\Delta_{\mathcal{Y}}$. Moreover, this lift is tangent to the fibres $Y \times \mathbb{R}^p$ of the interior of ff(Z_Q) with its restriction to a fibre $D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}$.*

Proof. Check in local coordinates; Cf. [MM].

1) Symbolic Construction.

This is the usual symbolic calculus, carried out uniformly here on the blown-up Z_Q of Z.

Lemma 2.2 *There exists $G_1 \in I^{-1}(Z_Q, \Delta_\mathcal{V}; End\,F(M))$ which vanishes in a neighborhood of $\partial Z_Q / ff(Z_Q)$ such that*

$$(D_x - \lambda_x)G - Id = F_1 \in C^\infty(Z_Q, End\,F(M)).$$

Proof. In our notations, we are suppressing the density factor, which we choose to be the Riemannian density $dg_x = x^{-p}\,dg$ (or its lift). Lemma 2.1 above gives the lift of our operator D_x. As for the Schwartz kernel of the identity operator, on Z it is a Dirac delta section over the diagonal,

$$\mathrm{Id} = x^p\, h\, \delta(y - y')\, \delta(z - z'),$$

where h is an isomorphism on the bundle, and the factors of x comes from the choice of density. Lifted to Z_Q, the factors of x just compensate for the homogeneity of the delta function,

$$\mathrm{Id} = h\, \delta(z - z')\, \delta(u).$$

From Lemma 2.1 we know that D_x, therefore $D_x - \lambda_x$, is transversally elliptic to $\Delta_\mathcal{V}$, i.e., its symbol restricted to the conormal bundle of $\Delta_\mathcal{V}$, $N(\Delta_\mathcal{V})$, is invertible. Now repeated application of Theorem A.6 and Theorem A.5 (equivalent to the recursive procedure in the standard symbolic calculus) finishes the proof. The extra vanishing condition is trivial. Q.E.D.

For the simplicity of notation, in what follows, we will use "Hom" to denote "End F(M)" and "F" denote "F(M)". (One should not confuse the spinor bundle F with the error term F's).

2) Model Problem.

It is important to note that the error F_1 is not a smoothing operator on M, although it is smoothing for $x > 0$ and depends on x smoothly down to $x = 0$. In order to obtain a compact error, one has to kill all the Taylor coefficients of F_1 at $\mathrm{ff}(Z_Q)$ so that F_1 projects downstairs as a smooth function on Z. This is essentially reduced to a model problem on Euclidean space as we shall see.

Lemma 2.3 *There exists* $G_2 \in \rho^{p-1}C^\infty(Z_Q, Hom) + \log\rho\, C^\infty(Z; Hom)$ *such that*

$$(D_x - \lambda_x)G_2 - F_1 = F_2' + \log\rho \cdot F_2''$$

where $F_2', F_2'' \in C^\infty(Z; Hom)$ *and* $F_2'' \cong 0$ *in the sense of Taylor series at* Q*.*

Proof. In terms of the polar coordinates (2.6), we can take R to be the defining function of the front face, and $\rho = \omega_0$ that of the non-front face of Z_Q. Note that

$$x = \rho R.$$

Expand F_1 in the direction of R:

$$F_1 \sim \sum_{i=0}^{\infty} F_{1,i}(\omega, y, z, z')\; R^i,$$

where $F_{1,i} \in C^\infty(S^p_+ \times U \times Y \times Y, F)$ are the Taylor coefficients, and U is a trivializing neighborhood on B. It turns out to be advantageous to rewrite this expansion in terms x (see below),

$$F_1 \sim \sum_{i=0}^{\infty}(\rho^{-i}F_{1,i})\ x^i.$$

The desired kernel will be constructed in the form

$$G_2 \sim \sum_{i=0}^{\infty} x^i\, e_i(\omega, y, z, z') + \log\rho\ e',$$

with $e_i \in \rho^{p-1-i}C^\infty(S_+ \times U \times Y \times Y, F)$, and $e' \in C^\infty(Z, F)$, by solving recursively for the e_i's. The logarithmic term will arise naturally in this recursion.

Writing the Taylor series in terms of x introduces extra singularities in the e_i's, but has the advantage that a corresponding series for $(D_x - \lambda_x)G_2$ is easily obtained:

$$(D_x - \lambda_x)G_2 \sim \sum x^i\,(D_x - \lambda_x)e_i + (D_x - \lambda_x)[\log\rho\ e'].$$

Now in the projective coordinates (2.7),

$$D_x = (D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}) + xB_{(1)}(x, y, z, \partial_z, \partial_u, \partial_y),$$

with

$$B_{(1)} \sim \sum_{j\geq 1} x^{j-1}B_j(y, z, \partial_z, \partial_u, \partial_y).$$

It follows that $D_x - \lambda_x$ has the same form. Now to solve

$$(D_x - \lambda_x)G_2 \sim F_1,$$

we equate their coefficients, obtaining

$$(D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p})e_i = -\sum_{j=0}^{i-1} B_{i-j}e_j + F_{1,i}. \qquad (2.8)$$

To invert the operator on the lefthand side of (2.8), we treat the harmonic and non-harmonic parts on Y separately. Thus, in terms of the splitting

$$C^\infty(\mathrm{ff}(Z_Q), F) = \ker D_Y \oplus \mathrm{Image}\, D_Y,$$

$D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}$ splits accordingly as

$$\begin{pmatrix} 1 \otimes D_{\mathbb{R}^p} & 0 \\ 0 & D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p} \end{pmatrix}.$$

The second piece can be more readily inverted. Hence we will first concentrate on the first one. This is the Dirac operator on the flat $\mathbb{R}^p$. However, some care must be taken because of the extra singularities in e_i's mentioned earlier. We observe that the function $\frac{1}{\rho} = \frac{1}{\omega_0}$ is of precisely linear growth in $|u|$, since

$$\omega_0 u = \frac{x}{R}\frac{y - y'}{x} = \omega'.$$

Therefore, the setting for our model problem is

$$D_{\mathbb{R}^p} v = f \quad \text{on } \mathbb{R}^p$$

for $f \in C^\infty(\mathbb{R}^p, F)$ having a growth condition at infinity. We need to solve v with certain control over its growth. This can be done by introducing suitable spaces. Following [MM], we define for $l \in \mathbb{Z}$

$$\hat{\mathcal{J}}_l = \{\hat{f} \in C^\infty(\mathbb{R}^p - \{0\}, F); \hat{f}(\eta) = |\eta|^{-2q} g(\eta) \text{ for some } q \in \mathbb{N} \text{ and } |\hat{f}(\eta)| \le C|\eta|^{-l}\}.$$

There is a regularization map (see [MM]),

$$e: \ \hat{\mathcal{J}}_l \to \mathcal{S}'(\mathbb{R}^p, F); \quad e(\hat{f})|_{\eta \neq 0} = \hat{f}.$$

Now define $\mathcal{J}_l \subset \mathcal{S}'(\mathbb{R}^p, F)$ by

$$g \in \mathcal{J}_l \Longleftrightarrow \hat{g} = e(\hat{f}) \text{ for some} \hat{f} \in \hat{\mathcal{J}}_l.$$

One checks that

$$D_{\mathbb{R}^p} : \mathcal{J}_l \to \mathcal{J}_{l-1}$$

is always an isomorphism. Further any $f \in \mathcal{J}_l$ is smooth on $\mathbb{R}^p$ with a complete asymptotic expansion at infinity of the form (Cf. [MM])

$$f(u) \sim \sum_{j \geq 0} |u|^{l-p-j} f_j(\theta) + \log |u| \, p(u),$$

where $p(u)$ is a polynomial of degree $\leq l-p$. Since $F_{1,0}$ has compact support on $u \in \mathbb{R}^p$, $F_{1,0} \in \mathcal{J}_0$. From (2.8), one has

$$(D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p}) e_0 = F_{1,0}. \tag{2.9}$$

Let $\mathcal{J}_l^{\perp}$ be the part of $\mathcal{J}_l$ that is orthogonal to $\ker D_Y$. Then it is not hard to see that

$$D_Y \otimes 1 + 1 \otimes D_{\mathbb{R}^p} : \mathcal{J}_l^{\perp} \to \mathcal{J}_l^{\perp}$$

is an isomorphism. From these discussions we see that one can solve (2.9) with $e_0 \in \mathcal{J}_1 + \mathcal{J}_0^{\perp}$. Solving (2.8) inductively gives

$$e_i \in \mathcal{J}_{i+1} + \mathcal{P}_{i+1-p},$$

where $\mathcal{P}_{i+1-p}$ denotes the space of polynomials of degree $\leq i+1-p$.

In this way we construct a formal power series. Notice that $u = \frac{y-y'}{x}$ and this means that the series for e' (corresponding to the $\mathcal{P}$ terms) is a Taylor series at Q on Z. Using Borel's lemma to sum this to a smooth function on Z and the other series as a smooth function on Z_Q gives G_2 as desired. Q.E.D.

3) Uniformity.

We shall slightly modify G_2 before examining its regularity as an operator. Write $G_2 = g_2' + \log\rho \cdot g_2''$, then

$$\begin{aligned}(D_x - \lambda_x)G_2 &= (D_x - \lambda_x)g_2' + (D_x - \lambda_x)(\log\rho \cdot g_2'') \\ &= \tilde{g}_2' + \log\rho \cdot (D_x - \lambda_x)g_2''.\end{aligned}$$

This implies that $(D_x - \lambda_x)g_2'' \cong 0$ at Q. Now let

$$\begin{aligned}\tilde{G} &= G_1 - (G_2 - \log x \cdot g_2'') = G_1 - g_2' + \log R \cdot g_2'' \\ &\in I_0^{-1}(Z_Q, \Delta_v; Hom) + \rho^{p-1}C^\infty(Z_Q, Hom) + \log R\, C^\infty(Z; Hom).\end{aligned}$$

Then $\tilde{G}$ is a true parametrix,

$$(D_x - \lambda_x)\tilde{G} = Id + F_2$$

where $F_2 \in x^{p-1}C^\infty(Z, Hom)$. Taking into account the singular density, one has $F_2 : C^\infty(X, F) \to x^{-1}C^\infty(X, F)$. We prove the same for $\tilde{G}$.

Lemma 2.4 *$\tilde{G}$ is a uniform parametrix for $D_x - \lambda_x$,*

$$\tilde{G} : C^\infty(X; F) \longrightarrow x^{-1}C^\infty(X; F).$$

Proof. For every $u \in C^\infty(X; F)$, we shall prove that $x\tilde{G}u \in C^\infty(X, F)$ by showing the tangential and normal regularity at the boundary separately. The tangential regularity is in the form of conormal regularity (see Section A.4),

$$\tilde{G} : \ C^\infty(X; F) \to \mathcal{A}(X; F).$$

That is, for some s, $V_1 \cdots V_r(\tilde{G}u)$ is in the Sobolev space $H^s(X,F)$ for all V_i's in $\mathcal{V}_b(X)$, the space of vector fields on X tangent to ∂X at ∂X. The point here is that there is a lifting map (see [MM] for details)

$$l_L : \mathcal{V}_b(X) \to \mathcal{V}_b(Z_Q)$$

such that $l_L(V)$ is also tangent to $\Delta_{\mathcal{V}}$. The integration by parts then gives

$$V(\tilde{G}u) = (l_L(V)\tilde{G})u + \sum \tilde{G}(W'u),$$

where W' is the transpose of the projection of $l_L(V)$ onto the right factor M in Z. Note that because of the special type of the vector fields involved, there is no boundary contribution. Now $\tilde{G} \in \mathcal{G}$ implies that $l_L(V)\tilde{G} \in \mathcal{G}$. Hence it suffices to note that any operator with kernel in $\mathcal{G}$ is bounded from $C^\infty(X;F)$ into some fixed Sobolev space.

To show the normal regularity of $\tilde{G}$, one investigates the holomorphic properties of its Mellin transform

$$\frac{x^t}{\Gamma(t)} x\tilde{G}\, dx\, dg_x\, dg \in \dot{C}^{-\infty}(Z; Hom). \tag{2.10}$$

We shall show that as a supported distributional density on Z, it extends to an entire function of t. To see this we lift it to Z_Q, noting that $x = \rho R$ and

$$x\, dx\, dg_x\, dg = x^{1-p} R^p\, dR\, d\omega\, dy\, dz\, dz' = R\rho^{1-p}\nu,$$

with $\nu = dR\, d\omega\, dy\, dz\, dz'$ a smooth density on Z_Q. Thus the lift is of the form

$$\frac{R^t}{\Gamma(t)} \rho^t R \rho^{1-p} \tilde{G}\nu \in \dot{C}^{-\infty}(Z_Q; Hom). \tag{2.11}$$

Recall that $\tilde{G} = G_1 - g_2' + \log R\, g_2''$. G_1 will contribute an entire term to (2.11), since it is C^∞ up to $R = 0$ and on its support $\rho \neq 0$. For g_2', $\rho^{1-p} g_2'$ is C^∞, and hence we conclude that

$$\frac{R^t}{\Gamma(t)} \frac{\rho^t}{\Gamma(t)} R\rho^{1-p} g_2' \nu$$

is entire. The presence of the factor $\log R$ introduces the possibility of simple poles at the negative integers of

$$\frac{R^t}{\Gamma(t)} \frac{\rho^t}{\Gamma(t)} \log R \; R\rho^{1-p} g_2'' \nu.$$

Combining these two statements we see that the distribution (2.11) is meromorphic in $\mathbb{C}$ with at most double poles at $-\mathbb{N}$.

Consider the form of the residues. Since $\frac{\rho^t}{\Gamma(t)}$ is entire and R^t is entire away from the front face, where $\rho^{1-p}\tilde{G}$ is also smooth, the support of the residue must be contained in the front face. Moreover all these residues must be smooth in z, z', y as distributions in $(R,\ \omega)$. Projecting down to Z gives (2.10), which is therefore at worst meromorphic with residues supported in Q. To remove the poles we need only subtract a distribution of the form

$$L = (\log x)^2 f_2(x, y, z, z', y - y') + (\log x) f_1(x, y, z, z', y - y'),$$

where f_1 and f_2 are smooth in x and are chosen to have the correct Taylor series at $x = 0$ to reproduce the residues. Now applying $D_x - \lambda_x$ we find

$$(D_x - \lambda_x)\tilde{G} = (D_x - \lambda_x)(\tilde{G} - L) + (D_x - \lambda_x)L \ \in \ C^\infty(Z, Hom).$$

Since $\frac{x^t}{\Gamma(t)} x(\tilde{G} - L)\, dx\, dg_x\, dg$ is entire by construction the same is true of $\frac{x^t}{\Gamma(t)} x(D_x - \lambda_x)(\tilde{G} - L)\, dx\, dg_x\, dg$. This implies that

$$(D_x - \lambda_x) f_i \cong 0 \ \ (i = 1, 2) \tag{2.12}$$

in Taylor series at $x = 0$. However, from the support and regularity conditions on the kernels noted above, the operators defined by f_i are of the form

$$\sum_{j=0}^{\infty} x^j P_j(z, z', y, \partial_y).$$

That is, they are differential operators in y with coefficients which are smoothing operators in z and smooth in y. Then (2.12) implies that the ranges of these two formal power series operators lie in the null space of $D_x - \lambda_x$ (in the sense of formal power series). By the lemma below, the latter space has finite dimensional coefficients (see 4) below). Hence f_i vanishes as Taylor series in x. By definition this means (2.10) is entire. This is the normal regularity of $\tilde{G}$.

By Theorem A.7 and Proposition A.4, we then have

$$u \in C^\infty(X, F) \Longrightarrow x\tilde{G}u \in \mathcal{A}(X, F) \cap \mathcal{A}'(X, F) = C^\infty(X, F).$$

Q.E.D.

4) Asymptotic Calculus.

In the proof of Lemma 2.4, we made use of the fact that the kernel space of $D_x - \lambda_x$ in the sense of power series has finite dimensional coefficients. To make this precise, let us introduce the space of Laurent series [MM],

$$\mathcal{L}(X, F) = \{u \in C^\infty(X, F), x^q u \in C^\infty(X, F), \text{for some } q \in Z\}/\dot{C}^\infty(X, F),$$

where $\dot{C}^\infty(X, F)$ is the space of C^∞ sections vanishing to all orders at the boundary. $\mathcal{L}(X, F)$ is a module over the ring of formal Laurent series

$$\mathcal{L} = \mathcal{L}([0, \infty))$$

in the variable x. Our operator $D_x - \lambda_x$ acts on $\mathcal{L}(X, F)$ in the obvious way.

Lemma 2.5 *Assume that* $\ker D_Y$ *gives a vector bundle on* B. *Let* $N \subset \mathcal{L}(X, F)$ *be the null space of* $D_x - \lambda_x$ *acting on the Laurent series. Then* N *is finite dimensional as an* $\mathcal{L}$*-module. Further,*

a) if $\lambda_x \sim x\lambda_1 + x^2\lambda_2 + \cdots$ *and* $\lambda_1 \notin spec(D_0)$ *then* $N = 0$ *and* $D_x - \lambda_x$ *is an isomorphism on* $\mathcal{L}(X, F)$*;*

b) if λ_x *is real-valued, then* $D_x - \lambda_x$ *is an isomorphism on the orthogonal complement of* N *with repect to the* $\mathcal{L}$*-inner product naturally induced by* g_x.

The proof of this lemma is defered to Section 4.4.

An immediate consequence of this lemma is the existence of the resolvent.

Proposition 2.1 *If* $\lambda \notin spec(D_0)$, *then the resolvent* $(\frac{1}{x}D_x - \lambda)^{-1}$ *exists for* x *sufficiently small. Further, it lies in a certain space* $x\mathcal{G}$ *defined below.*

Proof. Take $\lambda_x = \lambda x$. By previous discussions, there is a

$$\tilde{G} \in I_0^{-1}(Z_Q, \Delta_V; Hom) + \rho^{p-1}C^\infty(Z_Q; Hom) + \log R \cdot x^{p-1}C^\infty(Z, Hom)$$

such that

$$(D_x - \lambda x)\tilde{G} = Id + F_2$$

where $F_2 \in x^{p-1}C^\infty(Z; Hom)$. However, we can modify $\tilde{G}$ so that $F_2 \in x^pC^\infty(Z; Hom)$. This is because the (leading) coefficient of x^{p-1} in F_2 must arise directly from the coefficient of x^{p-1} in $\tilde{G}_0 - G_1$, at least away from Q where this is C^∞. Thus it must be in the range of D_Y. In particular, we can remove it by adding to $\tilde{G}$ a term in $x^{p-1}C^\infty(Z;Hom)$. Therefore $F_2 : C^\infty(X,F) \to C^\infty(X,F)$. In fact, $F_2 : C^\infty(\mathbb{R}_+, C^{-\infty}(M,F)) \to C^\infty(X,F)$. Let $u \in C^\infty(X,F)$. Then $F_2 u \in C^\infty(X,F) \xrightarrow{r} \mathcal{L}(X,F)$. Since $\lambda \notin spec(D_0)$, by Lemma 2.5 (Cf. also its proof), there exists $H(u) \in x^{-1}r(C^\infty(X,F)) \subset \mathcal{L}(X,F)$ so that $F_2 u = (D_x - \lambda x)H(u)$ in the sense of formal series. Let H_i be the coefficients of x^i in H $(i \geq -1)$. Then $H_i : C^\infty(X,F) \to C^\infty(M,F)$. In fact, $H_i(C^{-\infty}(M,F)) \subset C^\infty(M,F)$. This implies that H_i is an integral operator with kernel $H_i \in C^\infty(M \times M, Hom)$. Using Borel's lemma to sum up the formal series H then provides us a $G_3 \in x^{-1}C^\infty(Z;Hom)$, or $x^{p-1}C^\infty(Z;Hom)$ with respect to the singular density, such that

$$(D_x - \lambda x)(G_3 u) \cong F_2 u \text{ at } x = 0,$$

for every $u \in C^\infty(X,F)$. Letting $G' = \tilde{G} - G_3$, one has

$$(D_x - \lambda x)G' = Id + F$$

with $F \cong 0$ at $x = 0$. Clearly G' has the same regularity as $\tilde{G}$,

$$G' : C^\infty(X;F) \longrightarrow x^{-1}C^\infty(X;F).$$

Note that $(Id + F)^{-1} = Id + \tilde{F}$ for small x and $\tilde{F} \cong 0$ at $x = 0$. Putting $G = G'(Id + F)^{-1} = G' + G'\tilde{F}$, it satisfies

$$(D_x - \lambda x)G = Id,$$

i.e. $G = (D_x - \lambda x)^{-1} \in \mathcal{G}$ where

$$\begin{aligned}\mathcal{G} \;=\; & I_0^{-1}(Z_Q, \Delta_{\mathcal{V}}; Hom) + \rho^{p-1}C^\infty(Z_Q; Hom) + x^{p-1}\log R \cdot C^\infty(Z, Hom) \\ & + x^{p-1}C^\infty(Z, Hom).\end{aligned}$$

And $xG = (\frac{1}{x}D_x - \lambda)^{-1} \in x\mathcal{G}$. Q.E.D.

Thus, the resolvent of our operator will lie in the space $x\mathcal{G}$, for which we have precise description. We proceed next to study the mapping properties of the elements in this space.

5) L^2-boundedness of $x\mathcal{G}$.

We shall show that each of

$$xI_0^{-1}(Z_Q, \Delta_\mathcal{V}; Hom), \qquad x\rho^{p-1}C^\infty(Z_Q; Hom),$$
$$x^p \log R \cdot C^\infty(Z, Hom), \qquad x^p C^\infty(Z, Hom)$$

defines L^2-bounded operators on $L^2(M, F)$. This is clear for $x^p C^\infty(Z, Hom)$. Now we treat

a) $x\rho^{p-1}C^\infty(Z_Q; Hom)$.

Since we are using the singular measure $dg_x = x^{-p} dy dz$, if G is a Schwartz kernel, then

$$\begin{aligned}(G\phi)(x,y,z) &= \int_M G(x,y,z,y',z')\phi(y',z')dg_x \\ &= \int_M x^{-p} G\phi dy'\, dz'.\end{aligned}$$

Now if $G \in x\rho^{p-1}C^\infty(Z_Q; Hom)$, $G' \overset{def}{=} x^{-p}G \in x^{-p+1}\rho^{p-1}C^\infty(Z_Q; Hom) = R^{1-p}C^\infty(Z_Q; Hom)$. Therefore it suffices to show for those supported near the front face. By using the Schwarz inequality,

$$\begin{aligned}\int |(G\phi)(x,y,z)|^2 dy\, dz &= \int |\int G'(x,y,z,y',z')\phi(y',z')dy'\, dz'|^2 dy\, dz \\ &\leq \int (\int |G'(x,y,z,y',z')| dy'\, dz') \times \qquad (2.13) \\ &\quad (\int |G'(x,y,z,y',z')||\phi(y',z')|^2 dy'\, dz') dy\, dz.\end{aligned}$$

Let

$$|G'|(R,\omega) = \sup_{z,y',z'} |R^{p-1}G'(R,\omega,z,y',z')|$$

in terms of the polar coordinates (2.6) on Z_Q. This is a smooth function of R and ω. Therefore

$$\begin{aligned}\int |G'(0,y,z,y',z')| dy'\, dz' &= \int |G'(R,\omega,z,y',z)| R^{p-1}\, dR\, d\omega \\ &\leq \int |G'|(R,\omega) dR\, d\omega = C' < \infty.\end{aligned}$$

Plug this into (2.13) to get

$$\begin{aligned}\int |(G\phi)(0,y,z)|^2 dydz &\leq C' \int\int |G'(0,y,z,y',z')||\phi(y',z')|^2 dy'\, dz'\, dy\, dz \\ &= C' \int (\int |G'(0,y,z,y',z')|\, dy\, dz)|\phi(y',z')|^2 dy'\, dz' \\ &\leq C'^2 \int |\phi(y',z')|^2 dy'\, dz'.\end{aligned}$$

This is the desired L^2-boundedness of $x\rho^{p-1}C^\infty(Z_Q; Hom)$. From the proof, we see that the singularity here comes from the factor R^{1-p}, which is cancelled in the polar coordinates. The argument clearly carries over for $x^p \log R \cdot C^\infty(Z, Hom)$.

b) $xI_0^{-1}(Z_Q, \Delta_V; Hom)$.

One reduces it to the L^2-boundedness of

$$xI_0^{-\infty}(Z_Q, \Delta_V; Hom) \subset xC^\infty(Z_Q, \Delta_V; Hom).$$

Then, since elements of $xI_0^{-\infty}(Z_Q, \Delta_V; Hom)$ vanish on the non-front faces of the boundary, where $\rho = 0$, the same argument as above goes through for $xI_0^{-\infty}(Z_Q, \Delta_V; Hom)$. For the reduction we need to establish the following lemma.

Lemma 2.6 *If* $A \in I^0(Z_Q, \Delta_\mathcal{V}; Hom)$, *and* $\nu \in C^\infty(M, \Omega M)$ *is a positive density on* M, *then there exists a constant* $C > 0$ *and* $B \in I^0(Z_Q, \Delta_\mathcal{V}; Hom)$, *self-adjoint with respect to* ν *such that*

$$A^*A = -B^2 + C + R, \qquad R \in I^{-\infty}(Z_Q, \Delta_\mathcal{V}; Hom), \tag{2.14}$$

where A^* *is taken with repect to* ν.

Before we go into the proof of the lemma, we explain a little on the notations. Note that (see [M1])

$$I^0(Z_Q, \Delta_\mathcal{V}; Hom) \subset C^{-\infty}(Z_Q; Hom) = C^{-\infty}(Z; Hom).$$

Thus every $G \in I^0(Z_Q, \Delta_\mathcal{V}; Hom)$ defines

$$\begin{array}{cccc} G: & C^\infty(X, F) & \longrightarrow & C^\infty(X, F) \\ & \cup & & \cup \\ & C^\infty(\{x\} \times M, F) & \longrightarrow & C^\infty(\{x\} \times M, F). \end{array}$$

The element ν gives a positive density on every $\{x\} \times M$, with respect to which we can define

$$G^* : C^\infty(\{x\} \times M, F) \longrightarrow C^\infty(\{x\} \times M, F).$$

One checks that G^* is still in $I^0(Z_Q, \Delta_\mathcal{V}; Hom)$ and its symbol behaves like the usual symbols. (This is nothing but simultaneously taking adjoints of the operators for each parameter.)

Proof of Lemma 2.6. This follows from the formal properties of the symbolic calculus. Recall the exact sequence of the symbol map

$$0 \to I^{r-1}(Z_Q, \Delta_\mathcal{V}; Hom) \to I^r(Z_Q, \Delta_\mathcal{V}; Hom) \overset{\sigma_r}{\to} S^r(N^*(\Delta_\mathcal{V}), Hom) \to 0 \tag{2.15}$$

where $N^*(\cdot)$ denotes the conormal bundle, see Theorem A.5. Here we are suppressing the density factor.

Any representative $a \in S^0(N^*(\Delta_\mathcal{V}), Hom)$ of $\sigma_0(A)$ is bounded, so there exists a constant C such that $b_0^2 = C - |a|^2$ for $b_0 \in S^0(N^*(\Delta_\mathcal{V}), Hom)$ real. Using the surjectivity of the symbol map, we can choose $B_0' \in I^0(Z_Q, \Delta_\mathcal{V}; Hom)$ with $\sigma_0(B_0') \equiv b_0$, then $B_0 = \frac{1}{2}(B_0' + (B_0')^*)$ is self-adjoint and from the symbol calculus

$$R_1 \overset{def}{=} C - A^*A - B_0^2 \in I^{-1}.$$

Continuing by induction, suppose $B_i \in I^{-i}$ has been chosen for $i = 0, 1, \dots, N-1$ so that

$$R_N \overset{def}{=} C - A^*A - (B_0 + \cdots + B_{N-1})^2 \in I^{-N}.$$

The next term B_N should therefore satisfy

$$R_N \equiv 2B_0B_N \mod I^{-N-1}.$$

To do this just choose $\sigma_{-N}(B_N') = \sigma_{-N}(R_N)/2b_0$ and $B_N = \frac{1}{2}(B_N' + (B_N')^*)$. Finally choose $B = \frac{1}{2}(B' + (B')^*)$ where

$$B' \sim \sum_{j=0}^{\infty} B_j \in I^0$$

and we are done. Q.E.D.

With this lemma at our disposal, we can set out to show the L^2-boundedness of $xI_0^{-1}(Z_Q, \Delta_\mathcal{V}; Hom)$. Take $\chi \in C^\infty(Z_Q)$ such that $\chi \equiv 1$ on the support of the given element A of xI_0^{-1} and χ vanishes on the non-front faces of the boundary. Then by (2.14)

$$\|A\phi\|^2 = (\phi, \chi A^*A\chi\phi)$$

$$\begin{aligned} &= -\|\chi B\phi\|^2 + C\|\chi\phi\|^2 + (\phi, \chi R\phi) \\ &\leq (C + C')\|\phi\|^2, \qquad \text{if } \|\chi R\phi\| \leq C'\|\phi\|. \end{aligned}$$

This reduces the L^2-boundedness of xI_0^{-1} to that of $xI_0^{-\infty}$.

6) Smooth Dependence of the Resolvent.

By the principle of uniform boundedness, it suffices to show that for $\phi, \psi \in L^2(M, F)$, $G \in x\mathcal{G}$, $F(x) = < G\phi, \psi > \in C^\infty([0, +\infty))$. Here the inner product $< , >$ on $L^2(M, F)$ is induced by g_M. First let us note that $F(x) \in L_{loc}([0, +\infty))$. In fact, one has

$$\int |F(x)|\ |\chi(x)| dx \leq C(\chi)\|\phi\|_2\|\psi\|_2 \tag{2.16}$$

where $C(\chi)$ is a constant depending only on χ, a compactly supported smooth function on $[0, \infty)$. To see this, write out

$$F(x) = \int G(x, y, z, y', z')\phi(y', z')\psi(y, z) dg'_x\, dg.$$

Then

$$\int |F(x)|\ |\chi(x)| dx \leq \int |\chi|\ |G|\ |\phi|\ |\psi| dx\, dg'_x\, dg.$$

In polar coordinates we have

$$G dx\, dg'_x\, dg = x^{-p} G R^p dR\, d\omega\, dy\, dz\, dz' = \rho^{-p} G dR\, d\omega\, dy\, dz\, dz'.$$

Since

$$\begin{aligned} G \in x\mathcal{G} &= xI_0^{-1}(Z_Q, \Delta_\mathcal{V}; Hom) + x\rho^{p-1}C^\infty(Z_Q; Hom) \\ &\quad + x^p \log R \cdot C^\infty(Z, Hom) + x^p C^\infty(Z, Hom), \end{aligned}$$

one finds

$$\rho^{-p}G \;\in\; x\rho^{-p}I_0^{-1}(Z_Q,\Delta_{\mathcal{V}};Hom) + R\,C^\infty(Z_Q;Hom)$$
$$+R^p \log R \cdot C^\infty(Z,Hom) + R^p C^\infty(Z,Hom).$$

Therefore we obtain (2.16), using, in addition, Lemma 2.6. We now show the tangential regularity.

Lemma 2.7 *$F(x)$ is conormal to $\{x=0\}$,*

$$F(x) \in \mathcal{A}([0,+\infty)).$$

Proof. To show this, we have to check that $(x\partial_x)^k F(x) \in L_{loc}([0,+\infty))$ for any integer k. For $k=0$, this is (2.8). For $k=1$, note that $x\partial_x$ lifts trivially to $\mathcal{V}_b(Z_Q)$, tangent to $\Delta_{\mathcal{V}}$. Since $\mathcal{G}$ (thus $x\mathcal{G}$) is invariant under the action of this type of vector fields of $\mathcal{V}_b(Z_Q)$, $(x\partial_x)G$ is also in $x\mathcal{G}$. Hence

$$(x\partial_x)F(x) =< (x\partial_x)G\varphi, \psi > \in L_{loc}([0,+\infty)).$$

Repeating this argument finishes the proof. Q.E.D.

The next lemma gives the normal regularity.

Lemma 2.8 *The Mellin transform of $F(x)$,*

$$\frac{x^{t-1}}{\Gamma(t)}F(x) \in \dot{C}^{-\infty}([0,+\infty)),$$

extends to an entire function in $t \in \mathbb{C}$, i.e. $F(x) \in \mathcal{A}'([0,+\infty))$.

Proof. Note that $\frac{x^{t-1}}{\Gamma(t)}F(x) =< (\frac{x^{t-1}}{\Gamma(t)}G)\phi, \psi >$. Hence $\frac{x^{t-1}}{\Gamma(t)}F(x)$ is holomorphic on the half plane Re $t \geq 1$. Now if $\phi \in C^\infty(M,F)$, then Lemma 2.4 implies $G\phi \in C^\infty(Z,F)$. Consequently $\frac{x^{t-1}}{\Gamma(t)}F(x)$ is entire.

For $\phi \in L^2(M,F)$, choose $\phi_\delta \in C^\infty(M,F)$ such that $\|\phi_\delta - \phi\|_2 \to 0$ as $\delta \to 0$. Set

$$F_\delta(x) = < G\phi_\delta, \psi > .$$

We claim that $\{\frac{x^{t-1}}{\Gamma(t)} F_\delta(x)\}_\delta$ is a *normal family* of entire functions. Granted this, the limit of the normal family $\{\frac{x^{t-1}}{\Gamma(t)} F_\delta(x)\}_\delta$ (which exists by the normality and the principle of uniform boundedness, i.e. the Banach-Steinhaus Theorem) will also be an entire function with values in $\dot{C}^{-\infty}([0,+\infty))$. On the other hand, one has for Re $t > 1$, $\frac{x^{t-1}}{\Gamma(t)} F_\delta(x) \to \frac{x^{t-1}}{\Gamma(t)} F(x)$ weakly by (2.8). This proves the lemma.

It remains to establish the claim. We shall show that the family is locally uniformly bounded. For this note that for Re $t > 1$, and $\chi \in C^\infty([0,\infty))$ compactly supported,

$$\begin{aligned} \left|\int_0^\infty \frac{x^{t-1}}{\Gamma(t)} F_\delta(x)\chi(x)\,dx\right| &\leq \frac{C(\chi)a(\chi)^{\mathrm{Re}\,t-1}}{|\Gamma(t)|}\int |F_\delta|\,|\chi|\,dx \\ &\leq \frac{C(\chi)a(\chi)^{\mathrm{Re}\,t-1}}{|\Gamma(t)|}\|\phi\|_2\|\psi\|_2, \end{aligned}$$

where the last inequality comes from (2.16) (and $\|\phi_\delta\|_2 \leq 2\|\phi\|$). Since $\gamma(t) \overset{def}{=} \frac{x^{t-1}}{\Gamma(t)}$ satisfies the functional equation

$$\frac{d}{dx}\gamma(t) = \gamma(t-1), \tag{2.17}$$

using integration by parts, we have for Re $t > -k+1$,

$$\left|\int_0^\infty \frac{x^{t-1}}{\Gamma(t)} F(x)\chi(x)\,dx\right| \leq \frac{C(\chi)a(\chi)^{\mathrm{Re}\,t+k-1}}{|\Gamma(t+k)|}\sum_{i=0}^{k}\|\partial_x^i F_\delta\|_{L^1_{loc}}. \tag{2.18}$$

We want to estimate $\|\partial_x^k F_\delta\|_{L^1_{loc}}$ in terms of $\|\phi\|_2$ and $\|\psi\|_2$. Without loss of generality let us assume that F's and F_δ's have compact support. Define

$$F_M(t) \overset{def}{=} \int_0^\infty x^{t-1}F(x)\,dx, \quad F_\gamma(t) = \int_0^\infty \frac{x^{t-1}}{\Gamma(t)}F(x)\,dx.$$

Then

$$F_\gamma(t) = \frac{F_M(t)}{\Gamma(t)}. \tag{2.19}$$

The following are the basic properties of $F_M(t)$.

a) $(\partial_x F)_M(t) = -(t-1)F_M(t-1)$.

b) $(x\partial_x F)_M(t) = -tF_M(t)$.

c) (Plancherel's identity) $< F, \bar{F'} >= \int_{\mathrm{Re}\, t=0} F_M(t)\bar{F}'_M(\bar{t})\, dt$.

Verifying a) and b) is just computations, while c) follows from Plancherel's identity for the Fourier transform, when we make the substitution $x = e^{-iu}$.

Now $F(x) \in C^\infty([0,+\infty))$ implies that $F_\gamma(t)$ is entire. It follows from (2.19) that $F_M(t)$ is then meromorphic with at most simple pole at negative integers. If, in addition, $F(x)$ vanishes near $x = 0$, $F_M(t)$ will then be entire. This can be seen trivially from the definition.

Take $\varphi(x) \in C^\infty([0,+\infty))$ identically 1 on $[1,+\infty)$ and 0 on $[0,1/2]$, and put $\varphi_\epsilon(x) = \varphi(x/\epsilon)$. Apply Plancherel's identity to $\varphi_\epsilon(x)\partial_x^k F_\delta(x)$ to obtain

$$\int_0^\infty \varphi_\epsilon^2(x)|\partial_x^k F_\delta(x)|^2\, dx = \int_{\mathrm{Re}\, t=0} (\varphi_\epsilon \partial_x^k F_\delta)_M(t)\overline{(\varphi_\epsilon \partial_x^k F_\delta)_M(\bar{t})} dt.$$

By the above discussion, $(\varphi_\epsilon \partial_x^k F_\delta)_M(t)\overline{(\varphi_\epsilon \partial_x^k F_\delta)_M(\bar{t})}$ is entire, which allows us to shift the line of integration $\mathrm{Re}\, t = 0$ to any other such line. Hence

$$\int_0^\infty \varphi_\epsilon^2(x)|\partial_x^k F_\delta(x)|^2\, dx = \int_{\mathrm{Re}\, t=k+3} (\varphi_\epsilon \partial_x^k F_\delta)_M(t)\overline{(\varphi_\epsilon \partial_x^k F_\delta)_M(\bar{t})}\, dt.$$

Letting $\epsilon \to 0$ and noting that for $\mathrm{Re}\, t > 0$, $(\varphi_\epsilon \partial_x^k F_\delta)_M(t) \to (\partial_x^k F_\delta)_M(t)$, we deduce

$$\begin{aligned}\int_0^\infty |\partial_x^k F_\delta(x)|^2 dx &= \int_{\mathrm{Re}\, t=k+3} (\partial_x^k F_\delta)_M(t)\overline{(\partial_x^k F_\delta)_M(\bar{t})}\, dt \qquad (2.20)\\ &= \int_{\mathrm{Re}\, t=k+3} (t-1)^2 \cdots (t-k-1)^2 (F_\delta)_M(t-k-1)\overline{(F_\delta)_M(\bar{s}-k-1)}\, dt.\end{aligned}$$

To estimate $(F_\delta)_M(t)$, we use b) above. Thus for any large N,

$$|(F_\delta)_M(t)| \leq C(1+|t|)^{-N} \sum_{j=0}^{N} |((x\partial_x)^j F_\delta)_M(t)|.$$

By the invariance of $x\mathcal{G}$ under the action of $x\partial_x$, $(x\partial_x)^j F_\delta$ obeys an estimate of the type (2.16), which yields

$$|((x\partial_x)^j F_\delta)_M(t)| \leq C a^{\mathrm{Re}\, t-1} \|\phi\|_2 \, \|\psi\|_2,$$

provided $\mathrm{Re}\, t > 1$. Combining these two, we finally arrive at

$$|(F_\delta)_M(t)| \leq C a^{\mathrm{Re}\, t-1} (1+|t|)^{-N} \|\phi\|_2 \|\psi\|_2. \tag{2.21}$$

The claim follows from (2.18), (2.20), and (2.21) by taking $N > k+1$.

Q.E.D.

Proof of Theorem 2.1. The first statement follows from 1), 2), 3). The second statement is in 4). The smooth dependence of the resolvent on the parameter is a consequence of Lemma 2.7, 2.8 and the following fact (see Theorem A.7),

$$\mathcal{A}([0,+\infty)) \cap \mathcal{A}'([0,+\infty)) = C^\infty([0,+\infty)).$$

Q.E.D.

2.2 Asymtotic behavior of spectrum

The proceding analysis of the resolvent of $\frac{1}{x} D_x$ bears important implications for the spectrum of D_x in the adiabatic limit. However, before one can apply regular perturbation theory to obtain the asymptotic behavior of $spec(D_x)$, one needs certain finiteness results. To this end we now take a closer look at our operator by exploring the local geometry of the fibration.

In terms of the decomposition

$$L^2(M,F) = Image\, P^{\perp} \oplus Image\, P,$$

where $P = P_{\ker D_Y}$, our operator can be written in the matrix form

$$\frac{1}{x} D_x = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix},$$

where (see (A.14))

$$\begin{aligned} A_1 &= x^{-1} D_Y + P^{\perp} \tilde{D}_B P^{\perp} + x P^{\perp} \frac{T}{4} P^{\perp}, \\ A_2 &= P^{\perp} \tilde{D}_B P + x P^{\perp} \frac{T}{4} P, \\ A_3 &= D_0 + x P \frac{T}{4} P. \end{aligned}$$

Lemma 2.9 *a) There exists a constant* $\bar{\lambda}_0 > 0$ *such that spec* $(|A_1|)$ *is contained in the half line* $[\frac{\bar{\lambda}_0}{x}, +\infty)$.
b) A_2 *(consequently* A_2^**) is bounded on* $L^2(M,F)$.

Proof. a) is essentially Proposition 4.41 in [BC2], see also the outline of the proof for Theorem A.3.

b) First note that the commutator $[D_Y, \tilde{D}_B] \stackrel{def}{=} D_Y \tilde{D}_B + \tilde{D}_B D_Y$ is a first order differential operator which acts fiberwise (Cf. [p.51, BC2]). On the other hand $P^{\perp}[D_Y, \tilde{D}_B]P = P^{\perp} D_Y \tilde{D}_B P$ is fiberwisely of finite rank. Therefore $P^{\perp} D_Y \tilde{D}_B P = (P^{\perp} D_Y P^{\perp}) P^{\perp} \tilde{D}_B P$ is bounded on $L^2(M,F)$. But

$(P^\perp D_Y P^\perp)^{-1}$ is also bounded on $L^2(M,F)$. It follows that $P^\perp \tilde{D}_B P$, consequently $A_2 = P^\perp \tilde{D}_B P + x P^\perp \frac{T}{4} P$, is bounded. Q.E.D.

To have a first picture of $spec(D_x)$, we deform $\frac{1}{x} D_x$ to an operator whose spectrum we know much more about. For this purpose, the following lemma is quite useful.

Lemma 2.10 *If $T(\epsilon)$ is an analytic family of self-adjoint operators on a Hilbert space. Then its eigenvalue $\lambda(\epsilon)$ (which can be arranged to be analytic in ϵ) satisfies*

$$\lambda'(\epsilon) = (T'(\epsilon)\phi(\epsilon), \phi(\epsilon)) \tag{2.22}$$

where $\phi(\epsilon)$ is a normalized eigenvector associated with the eigenvalue $\lambda(\epsilon)$. In particular,

$$|\lambda'(\epsilon)| \leq \|T'(\epsilon)\|.$$

For the proof, see [K, p.391].

Lemma 2.11 *Consider the operator*

$$T = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}.$$

Its eigenvalues $\bar{\lambda}_x$ are analytic in x for $x > 0$. Furthermore for the positive constant $\bar{\lambda}_0$ of Lemma 2.9, either

$$|\bar{\lambda}_x| \geq \frac{\bar{\lambda}_0}{x}, \tag{2.23}$$

or $\bar{\lambda}_x$ is analytic at $x = 0$,

$$\bar{\lambda}_x = \lambda'_0 + \lambda'_1 x + \cdots, \text{ with } \lambda'_0 \in spec(D_0). \tag{2.24}$$

Moreover for each $\bar{\lambda}_x$ there corresponds an eigenvalue $\frac{1}{x}\lambda_x$ of $\frac{1}{x}D_x$ so that

$$|\frac{1}{x}\lambda_x - \bar{\lambda}_x| \leq C \tag{2.25}$$

for a uniform constant C.

Proof. The eigenvalues of T are either eigenvalues of A_1 or that of A_3. Accordingly, we call them of *type* I or II. By the previous lemma, the eigenvalues $\bar{\lambda}_x$ of type I, i.e. of A_1 satisfy

$$|\bar{\lambda}_x| \geq \frac{\bar{\lambda}_0}{x}.$$

On the other hand $A_3 = D_0 + xP\frac{T}{4}P$ is an entire family of self-adjoint operators (in the sense of Kato). Therefore its eigenvalues $\bar{\lambda}_x$ (type II) depend analytically on x and we have (2.24). In this case, we also have a uniform remainder estimate

$$\bar{\lambda}_x = \lambda_0' + xC(x), \tag{2.26}$$

with $|C(x)| \leq C$ uniformly. This can be shown by the argument below, using the deformation $A_3(\lambda) = D_0 + \lambda xP\frac{T}{4}P$ and the fact that $\|P\frac{T}{4}P\| \leq C$.

Now consider the deformation

$$T(\lambda) = \begin{pmatrix} A_1 & \lambda A_2 \\ \lambda A_2^* & A_3 \end{pmatrix},$$

we have $T(0) = T$ and $T(1) = \frac{1}{x}D_x$. For each fixed $x > 0$, $T(\lambda)$ is an entire family of self-adjoint operators. Therefore its eigenvalues depend analytically on λ. Thus for each $\bar{\lambda}_x$ of $T(0)$, there corresponds a $\frac{1}{x}\lambda_x$ of $T(1)$. Moreover (see Lemma 2.10)

$$|\frac{1}{x}\lambda_x - \bar{\lambda}_x| \leq \int_0^1 \|T'(\lambda)\| d\lambda \leq C, \tag{2.27}$$

where the last inequality follows from b) of Lemma 2.9.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 For $x > 0$, D_x is what is called in [K] the analytic family of type (B) of self-adjoint operators. The first statement follows immediately. Thus the main point here is the study of the behavior of λ_x as $x \to 0$. Let us begin with

a) Regular Perturbation.

From Theorem 2.1, one knows that for $\lambda \notin spec(D_0)$,

$$(\frac{1}{x}D_x - \lambda)^{-1} : L^2(M,F) \longrightarrow L^2(M,F)$$

is C^∞. It follows that for any $\lambda_1 \in spec(D_0)$, there is $\epsilon_0 > 0$ such that

$$P(x) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_1|=\epsilon_0}(\frac{1}{x}D_x - \lambda)^{-1}d\lambda$$

is well-defined and C^∞ in x as a family of orthogonal projections. The following lemma says that they are unitarily equivalent to each other.

Lemma 2.12 *Let P and Q be (orthogonal) projections on a Hilbert space with $\|P - Q\| < 1$, then*

$$W = [1-(P-Q)^2]^{-1/2}[PQ + (1-P)(1-Q)]$$

is well-defined and invertible (unitary). Furthermore,

$$W^{-1} = [(1-Q)(1-P) + QP][1-(P-Q)^2]^{-1/2},$$

$$W^{-1}PW = Q.$$

For the proof of this lemma see [p.72, RS].

Lemma 2.12 applied to the family $P(x)$ produces a smooth family of unitary operators $U(x)$ such that

$$U(x)^{-1}P(x)U(x) = P(0) \qquad (x \text{ small}).$$

Moreover, $U(x)^{-1}(\frac{1}{x}D_x - \lambda_1 - \epsilon_0)^{-1}U(x)$ is a C^∞ family on $Image\, P(0)$:

$$\begin{array}{ccc} Image\, P(x) & \stackrel{(\frac{1}{x}D_x - \lambda_1 - \epsilon_0)^{-1}}{\longrightarrow} & Image\, P(x) \\ \uparrow U(x) & & \uparrow U(x) \\ Image\, P(0) & \longrightarrow & Image\, P(0). \end{array}$$

To apply regular purturbation theory one need to know that $\dim(Image\, P(0)) < +\infty$. This is established below.

b) Finiteness.

To finish the proof of our theorem, we note that, being a smooth family of projections,

$$rank\, P(x) = rank\, P(0).$$

Further, by the discreteness of $spec(D_x)$ and functional analysis, for $x > 0$,

$$P(x) = \bigoplus_{|\frac{1}{x}\lambda_x - \lambda_1| < \epsilon_0} P_{\lambda_x}, \tag{2.28}$$

where P_{λ_x} is the orthogonal projection onto the λ_x-eigenspace. In other words, $rank\, P(x)$ equals to the number of λ_x (counted with multiplicity) so that $\frac{1}{x}\lambda_x$ lies in an ϵ_0-ball around λ_1. For x small such λ_x could only correspond to the eigenvalues $\bar{\lambda}_x$ of A_3, which obeys

$$\bar{\lambda}_x = \lambda_0' + \lambda_1' x + \cdots$$

with $\lambda'_0 \in spec(D_0)$. This, together with (2.27) and (2.26), implies that $rank\, P(x)$ is bounded from above by the number of eigenvalues of D_0 lying in a $(2C + \epsilon_0)$-ball around λ_1, counted with multiplicity. There are only finitely many of them, yielding

$$rank\, P(x) < +\infty,$$

i.e.

$$\dim(Image\, P(0)) < +\infty.$$

Thus $U(x)^{-1}(\frac{1}{x}D_x - \lambda_1 - \epsilon_0)^{-1}U(x)$ is a smooth family on a finite dimensional vector space. By standard functional analysis so is its inverse $U(x)^{-1}(\frac{1}{x}D_x - \lambda_1 - \epsilon_0)U(x) = U(x)^{-1}(\frac{1}{x}D_x)U(x) - \lambda_1 - \epsilon_0$. One can then apply the following lemma (Cf. [Sn], Lemma 5.2), showing the existence of complete asymptotic expansions for those λ_x with $\frac{1}{x}\lambda_x$ lying near an eigenvalue of D_0.

Lemma 2.13 *Let $C(x)$ be a family of symmetric matrices whose element have complete asymptotic expansion in x as $x \to 0$. Then the eigenvalues of $C(x)$ have complete asymptotic expansions.*

Proof. For any n, we can write $C(x) = A_n(x) + B_n(x)$ where $A_n(x)$ is polynomial in x hence analytic and $B_n(x) = O(x^n)$. Since all matrices are symmetric, the difference of eigenvalues of $C(x)$ and those of $A_n(x)$ is $O(x^n)$ and by standard theory (on regular perturbation, [K], [RS]) those of A_n are analytic. Q.E.D.

To show the alternative for the eigenvalues of D_x, we use (2.27). Thus if $\bar{\lambda}_x$ is of type I so that $x|\bar{\lambda}_x| \geq \bar{\lambda}_0$, we have

$$|\lambda_x| \geq x|\bar{\lambda}_x| - Cx \geq \frac{\bar{\lambda}_0}{2} \stackrel{def}{=} \lambda_0,$$

provided $x \leq \frac{\bar{\lambda}_0}{2C}$. On the other hand, if $\bar{\lambda}_x$ is of type II, we have

$$|\frac{1}{x}\lambda_x| \leq |\bar{\lambda}_x| + C \leq \lambda_0' + 2C$$

by (2.26). In particular $\{\frac{1}{x}\lambda_x\}$ is uniformly bounded. They must converge as $x \to 0$ to an eigenvalue of D_0. Otherwise, a subsequence of $\{\frac{1}{x}\lambda_x\}$ converges to $\lambda_1 \notin spec(D_0)$. This implies that the resolvent $(\frac{1}{x}D_x - \lambda_1)^{-1}$ can not exist no matter how small x is, in contradiction to Theorem 2.1.

It remains to prove (1.2). Choose $\lambda \notin spec(D_0)$ real and consider

$$S(\epsilon) = \begin{pmatrix} A_1 - \lambda & \epsilon A_2 \\ \epsilon A_2^* & A_3 - \lambda \end{pmatrix}.$$

Set $T(\epsilon) = S(\epsilon)^{-1}$. One has

$$\frac{d}{d\epsilon}T(\epsilon) = S(\epsilon)^{-1}\frac{dS(\epsilon)}{d\epsilon}S(\epsilon)^{-1} = T(\epsilon)\begin{pmatrix} 0 & A_2 \\ A_2^* & 0 \end{pmatrix}T(\epsilon).$$

By straightforward calculation, one finds

$$T(\epsilon) = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix},$$

where

$$\begin{aligned} T_1 &= (A_1 - \lambda - \epsilon^2 A_2(A_3 - \lambda)^{-1}A_2^*)^{-1}, \\ T_2 &= -\epsilon(A_1 - \lambda)^{-1}A_2(A_3 - \lambda - \epsilon^2 A_2^*(A_1 - \lambda)^{-1}A_2)^{-1}, \\ T_3 &= (A_3 - \lambda - \epsilon^2 A_2^*(A_1 - \lambda)^{-1}A_2)^{-1}. \end{aligned}$$

From Lemma 2.9, we have

$$\|T_1\| \leq Cx, \ \|T_2\| \leq Cx, \ \|T_3\| \leq C.$$

It follows then

$$\|\frac{d}{d\epsilon}T(\epsilon)\| \leq Cx.$$

This, together with (2.22), implies $|\lambda'(\epsilon)| \leq Cx$. But $\lambda(1) = (\frac{\lambda_x}{x} - \lambda)^{-1}$ and $\lambda(0) = (\bar{\lambda}_x - \lambda)^{-1}$ where $\bar{\lambda}_x$ is the same as in Lemma 2.11. Thus

$$|(\frac{\lambda_x}{x} - \lambda)^{-1} - (\bar{\lambda}_x - \lambda)^{-1}| \leq Cx,$$

or

$$|\frac{\lambda_x}{x} - \bar{\lambda}_x| \leq Cx|\frac{\lambda_x}{x} - \lambda|\,|\bar{\lambda}_x - \lambda|. \tag{2.29}$$

Since $\lambda_x/x \rightarrow \lambda_1$, we must have $\bar{\lambda}_x = \lambda_1 + C(x)x$, where $|C(x)| \leq C$. Pluging in (2.29), and using the estimate (2.25) of Lemma 2.11, one obtains

$$|\frac{\lambda_x}{x} - \lambda_1| \leq Cx\lambda_1^2.$$

This is (1.2). Q.E.D.

Chapter 3

The uniform asymptotic expansion

In what follows we will also use the notation $D_x e^{-tD_x^2}$ to denote its own kernel, taken with respect to the metric g. Therefore $tr(D_x e^{-tD_x^2})$ could mean either the integrated trace or the pointwise trace, depending on the situation. We shall make it clear in the context.

Proposition 3.1 *One has the following pointwise uniform asymptotic expansion (compare [BC2]),*

$$tr(D_x e^{-tD_x^2}) = \sum_{i=-n}^{N-1} a_i(t)(tx^2)^{i/2} + O((tx^2)^{N/2}), \tag{3.1}$$

where $a_i(t)$'s are bounded for $t \geq 1$, and so is $O(\cdot)$.

Proof. We first localize the problem, i.e., we transplant the problem to a trivial fibration whose base space is an Euclidean space. Then we construct a parametrix by taking the product of the heat kernels on the base and along the fibre, and then applying Duhamel's principle. The asymptotic expansion comes from the standard heat expansion.

3.1 Localization

Fix a point $y \in B$, let $U = B_{r_0}(y)$ in the metric g_B for a fixed small number $r_0 \geq 0$. Denote by $K_x(t, y, z, y', z')$ be the kernel of $D_x e^{-tD_x}$ with respect to the metric $g = \pi^* g_B + g_Y$.

Lemma 3.1 *$K_x(t, y, z, y', z')$ decays exponentially as $x \to 0$ for those $y' \notin U$, i.e., there exists positive constants C and C' such that*

$$|K_x(t, y, z, y', z')| \leq C x^{-p} e^{-C'\overline{y,y'}^2/tx^2}. \tag{3.2}$$

One has the same bounds for the derivatives in y, z, y', z'.

Proof. This follows essentially from the theorem of Cheeger-Gromov-Taylor. Let $\tilde{K}_x(t, y, z, y', z')$ be the kernel of $D_x e^{-tD_x}$ with respect to the metric g_x, then

$$K_x(t, y, z, y', z') = x^{-p} \tilde{K}_x(t, y, z, y', z').$$

By (1.17)

$$|\tilde{K}_x(t, y, z, y', z')| \leq \frac{C(n)}{\pi^2} \sum_{0 \leq i,j \leq N} \int_{d_x - r}^{\infty} |\hat{f}^{(2i+2j)}(s)| ds$$

where $r = \min(r_0/3, inj(M, g))$, $f(\lambda) = \lambda e^{-t\lambda^2} \stackrel{def}{=} \lambda g(\lambda)$, and $d_x =$ *the distance in g_x between (y, z) and (y', z')* $\geq \overline{y,y'}/x \geq \overline{y,y'}/2x + r$ for x small. Note that (see [CGT, p.28])

$$|\hat{f}^{(k)}(s)| = |\hat{g}^{(k+1)}(s)| \leq \frac{C(k)}{t^{(k+1)/2}} t^{-1/2} e^{-s^2/8t}.$$

Therefore,

$$\begin{aligned}|\tilde{K}_x(t,y,z,y',z')| &\leq \frac{C}{\sqrt{t}}\int_{\overline{y,y'}/2x}^{+\infty} t^{-1/2}e^{-s^2/8t}dt \\ &\leq \frac{C}{\sqrt{t}}e^{-\overline{y,y'}^2/32tx^2}.\end{aligned}$$

This implies (3.2). The estimates for the derivatives are similar.

Let r_0 be small so that U is a trivializing neighborhood of π. Consider now the trivial fibration

$$Y \longrightarrow \bar{M} = Y \times \mathbb{R}^p \longrightarrow \mathbb{R}^p,$$

$$\bar{g} = \bar{\pi}^* g_{\mathbb{R}^p} + g_Y,$$

where $g_{\mathbb{R}^p}$ coincides with g_B in a neighborhood of 0 and is flat outside a compact domain. g_Y coincides with $g_{Y,B}$ in a neighborhood of 0 and is constant outside the compact domain. Correspondingly, we have $\bar{D}_x$. Let $\bar{K}_x(t,y,z,y',z')$ be the kernel of $\bar{D}_x e^{-tD_x^2}$ with respect to $\bar{g}$. The following lemma says that instead of considering D_x, we can just as well consider $\bar{D}_x$.

Lemma 3.2 *K_x and $\bar{K}_x$ differ by an exponentially decaying terms,*

$$|K_x(t,y,z,y',z') - \bar{K}_x(t,y,z,y',z')| \leq Ctx^{-n}e^{-C'/tx^2}.$$

Proof. Basically it is because that inside a small neighborhood the two operators coincide while outside the neighborhood the heat kernels decay exponentially. Let G_x and $\bar{G}_x$ be the kernels of e^{-tD_x} and $e^{-t\bar{D}_x}$. Then

$$K_x = D_x G_x, \qquad \bar{K}_x = \bar{D}_x \bar{G}_x.$$

Choose a smooth function $\phi = \phi(y, y')$ such that $\phi(y, y') \equiv 1$ when $y' \in U/3$ and $\phi(y, y') \equiv 0$ when $y' \notin U$. Consider

$$\begin{aligned}
& G_x(t, y, z, y'', z'')\phi(y, y'') - \bar{G}_x(t, y, z, y'', z'')\phi(y, y'') \\
= \; & \int_0^t \frac{\partial}{\partial s} \int G_x(s, y, z, y', z')\bar{G}_x(t - s, y', z', y'', z'')\phi(y, y')\phi(y', y'')\, dy'\, dz'\, ds \\
= \; & -\int_0^t \int G_x(s, y, z, y', z')(D_x - \bar{D}_x)\bar{G}_x(t - s, y', z', y'', z'') \times \\
& \qquad \phi(y, y')\, \phi(y', y'')\, dy'\, dz'\, ds.
\end{aligned}$$

Since $D_x = \bar{D}_x$ in U, one has

$$\begin{aligned}
& K_x(t, y, z, y'', z'')\phi(y, y'') - \bar{K}_x(t, y, z, y'', z'')\phi(y, y'') \\
& = -\int_0^t \int D_x G_x(s, y, z, y', z')(D_x - \bar{D}_x)\bar{G}_x(t - s, y', z', y'', z'') \times \\
& \qquad \phi(y, y')\, \phi(y, y'')\, dy'\, dz'\, ds \\
& = -\int_0^t \int_{\overline{y', y} \geq r_0} (D_x - \bar{D}_x) D_x G_x(s, y, z, y', z')\bar{G}_x(t - s, y', z', y'', z'') \times \\
& \qquad \phi(y, y')\, \phi(y, y'')\, dy'\, dz'\, ds.
\end{aligned}$$

Apply the previous estimates on the derivatives of G_x and using the fact that as an L^2 operator $\bar{G}_x$ has norm bounded by 1, we find

$$|K_x(t, y, z, y'', z'')\phi(y, y'') - \bar{K}_x(t, y, z, y'', z'')\phi(y, y'')| \leq Ctx^{-p}e^{-C'/tx^2}.$$

In particular one has

$$|K_x(t, y, z, y'', z'') - \bar{K}_x(t, y, z, y'', z'')| \leq Ctx^{-n}e^{-C'/tx^2}$$

when $\overline{y, y''} \leq r_0/3$. For those y, y'' such that $\overline{y, y''} \geq r_0/3$ each of K_x and $\bar{K}_x$ are exponentially decaying by (3.2). Combining the two finishes the proof. Q.E.D.

3.2 A "rough" parametrix

By the above result we can replace the original fibration by the trivial one

$$Y \longrightarrow \bar{M} = Y \times \mathbb{R}^p \longrightarrow \mathbb{R}^p.$$

We now construct a parametrix for our operator living on the trivial fibration. For the simplicity of notations we will suppress the "bar's" below.

Recall that the tangent bundle of M splits as the Whitney sum of its vertical space and its horizontal space by the connection,

$$TM = T^V \oplus T^H,$$

and the horizontal lift gives an isomorphism

$$T_y\mathbb{R}^p \xrightarrow{\simeq} T^H_{(y,z)} \tag{3.3}$$

for any $z \in \pi^{-1}(y)$. Also we have by (A.14),

$$D_x = D_Y + x\tilde{D}_B + x^2\frac{T}{4}, \tag{3.4}$$

where D_Y is the family of the signature operators along the fibres and $\tilde{D}_B$ the signature operator on $\mathbb{R}^p$ lifted via the isomorphism (A.16). Taking the square of (3.4) we have

$$D_x^2 = D_Y^2 + x[D_Y, \tilde{D}_B] + x^2\tilde{D}_B^2 + x^2[D_Y, \frac{T}{4}] + x^3[\tilde{D}_B, \frac{T}{4}] + x^4(\frac{T}{4})^2.$$

Recall the basic fact that $[D_Y, \tilde{D}_B]$ is a first order differential operator acting fibrewise. Thus we can construct the heat kernel $F_y(t, x, z, z')$ of $D_Y^2 + x[D_Y, \tilde{D}_B]$. Here the subscript indicates its dependence on the base variable. Let $E(t,y,y')$ be the heat kernel of $e^{-t\tilde{D}_B^2}$, originally living on $\mathbb{R}^p$, and lifted to M via the horizontal lift. Consider

$$K_x(t,y,z,y',z') = F_y(t,x,z,z')E(tx^2,y,y'),$$

$$G_x(t,y,z,y',z') = (\partial/\partial t + D_x^2)K_x(t,y,z,y',z').$$

Note that here, and all throughout, we use *y's as the coordinates of the base,* $\mathbb{R}^p$*, and z's as the coordinates of the fibre,* Y. One has

Lemma 3.3 *K_x is a "rough" parametrix for $\partial/\partial t + D_x^2$, i.e. it satisfies the estimate*

$$|(\partial/\partial t + D_x^2)K_x| \leq C(tx^2)^{-n/2-1/2}x^2$$

for $t \geq 1$.

Proof. We calculate,

$$\begin{aligned} G_x &= (\partial/\partial t F_y)E + F_y(\partial/\partial t E) + ((D_Y^2 + x[D_Y,\tilde{D}_B])F_yE + x^2(\tilde{D}_B^2F_y)E \\ &+ x^2(\tilde{D}'_BF_y)(\tilde{D}''_BE) + x^2F_y(\tilde{D}_B^2E) + x^2F_y(RE) + x^2([D_Y,\frac{T}{4}]F_y)E \\ &+ x^2F_y(\tilde{D}'''_BE) + x^3([\tilde{D}_B,\frac{T}{4}]F_y)E + x^3F_y([\tilde{D}_B,\frac{T}{4}]E) + x^4(\frac{T}{4})^2F_yE \\ &= x^2(\tilde{D}_B^2F_y)E + x^2(\tilde{D}'_BF_y)(\tilde{D}''_BE) + x^2F_y(RE) + x^2([D_Y,\frac{T}{4}]F_y)E \\ &+ x^2F_y(\tilde{D}'''_BE) + x^3([\tilde{D}_B,\frac{T}{4}]F_y)E + x^3F_y([\tilde{D}_B,\frac{T}{4}]E) + x^4(\frac{T}{4})^2F_yE \quad (3.5) \end{aligned}$$

Here $\tilde{D}'_B$ etc. are first order differential operators acting horizontally, and the curvature factor R comes from the use of the Lichnerowicz formula.

To estimate (3.5), we introduce the following notation. We use $|\ |_\infty$ to denote the sup norm, and $|\ |_{\infty,i}$ the sup norm of all the derivatives up to i-th order. Here we make the convention that derivatives are taken with respect to the space variables but not the time variable. Thus $|F_{y'}(t,x,z,z')|_{\infty,i_1,i_2,i_3}$ denotes the sup norm of F together with its derivatives with respect to y' up to i_1-th order, z up to i_2-th order, z' up to i_3-th order.

First we show that for $t \geq 1$,

$$|F_{y'}(t,x,z,z')|_{\infty,i_1,i_2,i_3} \leq C(i_1,i_2,i_3).$$

In fact, for $i_1 = 0$, this is the standard heat kernel estimate and follows from, for example, [CGT]. For arbitrary i_1, note that, since $\dim\ker D_Y$ is assumed to be a (finite) constant and $D_Y^2 + x[D_Y, \tilde{D}_B]$ is a small perturbation of D_Y^2, it follows that all its eigenvalues are bounded away from 0 by a positive constant λ_0 except a finite number of them, which decay in x. Since the number of those is $\dim\ker D_Y$, it follows from Theorem 1.1 and a minimax argument that such eigenvalues must be decaying quadratically. Let P be the projection on the space spanned by the eigenspaces corresponding to such eigenvalues. P is a smoothing operator which depends smoothly also on the base variable. Let $F_y^0(t,x,z,z')$ denote the Schwartz kernel of PF_yP. $F_y^0(t,x,z,z')$ obeys

$$|F_y^0(t,x,z,z')|_{\infty,i_1,i_2,i_3} \leq C(i_1,i_2,i_3). \tag{3.6}$$

Put $\underline{D}_Y = D_Y^2 + x[D_Y, \tilde{D}_B] - P$. Then its eigenvalues are bounded away from 0 by λ_0. Therefore if $\bar{F}_y(t,x,z,z')$ denotes the heat kernel of $\underline{D}_Y$, for $t \geq 1$,

$$|\bar{F}_y(t,x,z,z')|_{\infty,0,i_2,i_3} \leq C(i_2,i_3)e^{-t\lambda_0}.$$

Now by Duhamel's principle,

$$\frac{\partial}{\partial y}\bar{F}_y(t,x,z,z') = -\int_0^t \int_Y \bar{F}_y(s,x,z,z'')\frac{\partial \underline{D}_Y}{\partial y}\bar{F}_y(t-s,x,z'',z')\,dz''\,dt.$$

To estimate this convolution, we consider the time intervals $s \le t/2$ and $s \ge t/2$ separately. If $s \le t/2$, then $t - s \ge t/2$. Thinking $\bar{F}_y(s,x,z,z'')$ as the kernel of an integral operator on $L^2(Y)$ whose norm is bounded by 1, we see that by the elliptic estimate,

$$\begin{aligned}
&|\int_Y \bar{F}_y(s,x,z,z'')\frac{\partial \underline{D}_Y}{\partial y}\bar{F}_y(t-s,x,z'',z')\,dz''|_{\infty,0,i_2,i_3} \\
&\le \|\underline{D}_{Y,1}^{i_2}\underline{D}_{Y,2}^{i_3}\frac{\partial \underline{D}_Y}{\partial y}\bar{F}_y(t-s,x,z'',z')\|_{L^2(Y)} \\
&\le C|\bar{F}_y(t-s,x,z'',z')|_{\infty,0,2i_2+2,2i_3} \le C(i_2,i_3)e^{-t\lambda_0}.
\end{aligned}$$

On the other hand, if $s \ge t/2$, note that

$$\int_Y \bar{F}_y(s)\frac{\partial \underline{D}_Y}{\partial y}\bar{F}_y(t-s)\,dz'' = \int_Y (\frac{\partial \underline{D}_Y}{\partial y})^* \bar{F}_y(s)\bar{F}_y(t-s)\,dz''.$$

Thus letting $\bar{F}_y(t-s,x,z'',z')$ play the role of $\bar{F}_y(s,x,z,z'')$, we obtain the same estimate. Hence

$$|\frac{\partial}{\partial y}\bar{F}_y(t,x,z,z')|_{\infty,0,i_2,i_3} \le C(i_2,i_3)te^{-t\lambda_0}.$$

Or

$$|\bar{F}_y(t,x,z,z')|_{\infty,1,i_2,i_3} \le C(i_2,i_3).$$

Proceed inductively, one proves that

$$|\bar{F}_{y'}(t,x,z,z')|_{\infty,i_1,i_2,i_3} \le C(i_1,i_2,i_3).$$

Combined with (3.6), this gives the estimate we want.

Secondly, for tx^2 small, note the standard heat kernel estimate, which follows from the finite propagation speed technique [CGT],

$$|E(tx^2, y, y')|_{\infty, i_1, i_2} \leq C(i_1, i_2)(tx^2)^{-n/2-i_1/2-i_2/2}.$$

From these discussions we see by (3.5)

$$|G_x| \leq C(tx^2)^{-n/2-1/2}x^2, \qquad \text{for } t \geq 1. \tag{3.7}$$

Q.E.D.

Denote

$$G_x \# G_x = -\int_0^t \int G_x(s, y, z, y', z') G_x(t-s, y', z', y'', z'')\, dy'\, dz'\, ds.$$

In general denote

$$G_x^m = \underbrace{G_x \# \cdots \# G_x}_{m+1}.$$

A version of Duhamel's principle says

$$e^{-tD_x^2} = \sum_{i=0}^{m-1} (-1)^{i+1} G_x^i \# K_x + G_x^m \# e^{-tD_x^2}. \tag{3.8}$$

To apply this to get the asymptotic expansion, we need estimates on the m-fold convolutions G_x^m.

Lemma 3.4 *G_x^m improves with m,*

$$|G_x^m| \leq C(m)(tx^2)^{-n/2+(m-1)/2}x^2. \tag{3.9}$$

Proof. Let us begin with $G_x \# G_x$. We divide the integral over $[0, t]$ into two, one over $[0, t/2]$ and the other $[t/2, t]$,

$$\begin{aligned} G_x \# G_x &= -\int_0^{t/2} \int G_x(s, y, z, y', z') G_x(t-s, y', z', y'', z'')\, dy'\, dz'\, ds \\ &\quad - \int_{t/2}^t \int G_x(s, y, z, y', z') G_x(t-s, y', z', y'', z'')\, dy'\, dz'\, ds \\ &\overset{def}{=} \mathrm{I} + \mathrm{II}. \end{aligned}$$

By (3.5) and integration by parts we can rewrite I as

$$I = -\int_0^{t/2}\int K_x(s,y,z,y',z')G'_x(t-s,y',z',y'',z'')\,dy'\,dz'\,ds, \qquad (3.10)$$

where by the same consideration as in showing (3.7)

$$|G'_x| \le C(tx^2)^{-n/2-1}x^4.$$

As an integral operator on the L^2 space the norm of K_x is bounded by 1. Therefore,

$$\begin{aligned} |I| &\le t/2\|G'_x(t-s,\cdot,\cdot,y'',z'')\|_2 \\ &\le C(tx^2)^{-n/2}x^2. \end{aligned}$$

The estimate for II is similar. In this way one obtains

$$|G_x\#G_x| \le C(tx^2)^{-n/2+1}.$$

In estimating the m-fold convolution one divides the time interval into m subintervals, on each of them one of the m-tuplet $t_0-t_1,\cdots,t_{m-2}-t_m$ is greater than or equal to t/m. One lets the kernels corresponding to the other time parameters play the role of K (after integration by parts), and proceeds as for $G_x\#G_x$. Q.E.D.

Now by the standard heat asymptotic expansion and the above estimate we arrive at the uniform asymptotic expansion as claimed.

Chapter 4

Hodge-Leray theory

In Chapter 1, we proved the adiabatic limit formula (.2) with the global contribution expressed in terms of the very small eigenvalues. Here, confining ourself to signature operators, we identify this global contribution with a topological invariant constructed from the Leray spectral sequence of the fibration. In fact one can recover the Leray spectral sequence from the eigenspaces associated to these eigenvalues. This result, termed the Hodge-Leray theory for the apparent analogy with the Hodge theory, is partly motivated by [MM], where they relate the Leray spectral sequence to the asymptotic solutions.

Included in Section 4.1 is a brief review of signature operators and their relation to Dirac operators. Section 4.2 contains our main results in this chapter, Hodge-Leray theorem, which states that the space of eigenforms corresponding to the eigenvalues which decay polynomially of degree at least r is isomorphic to the E_r term of the Leray spectral sequence, and the adiabatic limit formula of η-invariant for signature operator. First we recall the general theory of spectral sequence. Then the space of eigenforms corresponding to asymptotically the same eigenvalues is considered and smooth parameter-depending basis constructed by the regular perturbation theory along with Theorem 2.1. From these we construct the spaces used in the Hodge-Leray theorem. The Hodge-Leray theorem is shown by introducing an auxiliary complex which incorporates the asymptotic properties. Section 4.3 is devoted to the applications of our adiabatic limit formula. First we introduce the non-multiplicativity of signature and we interpret the the works of Chern-Hirzebruch-Serre and Atiyah-Singer in this context. Then we give intrinsic characterizations by virtue of Bismut-Cheeger's Families Index Theorem for manifolds with boundary and the adiabatic limit formula of η-invariant. In the last section we will discuss the formal Hodge decompsition on the space of Laurent series, which is used in Section 2.1 and the proof of the Hodge-Leray Theorem.

4.1 Signature operator

Let M^n be a compact Riemannian manifold. Consider the bundle of differential forms $\Lambda^* M = \Lambda^* T^* M$. The exterior derivative d acts on it:

$$d: \; C^\infty(\Lambda^* M) \longrightarrow C^\infty(\Lambda^* M).$$

Let δ be the formal adjoint of d, then $d + \delta$ is a first order, self-adjoint, elliptic differential operator on $\Lambda^* M$ such that

$$(d+\delta)^2 = \Delta,$$

the Hodge Laplacian. The well-known Hodge theorem [dR] is

Theorem 4.1 (Hodge) *The kernel of Δ can be identified with $H^*(M)$, the cohomology of M. In fact, in every de Rham class, there is a unigue harmonic representative.*

Now assume that M is orientable. The orientation together with the metric gives an (oriented) volume form *dvol* on M. The Hodge $*$ operator is defined such that

$$\alpha \wedge *\beta = (\alpha, \beta)\, dvol$$

holds for $\alpha,\ \beta \in \Lambda^* M$, where (α, β) is the induced metric on $\Lambda^* M$. One checks that $\delta|_{\Lambda^p} = (-1)^{np+n+1} * d*$. Further, $*^2|_{\Lambda^p} = (-1)^{np+p}$.

Assume in addition that n is even. Then $\tau|_{\Lambda^p} \overset{def}{=} i^p *$ defines an involution on $\Lambda^* M$, which therefore gives rise to a splitting

$$\Lambda^* M = \Lambda^*_+ \oplus \Lambda^*_-.$$

Since $d + \delta$ actually anti-commutes with τ, in terms of this splitting,

$$d + \delta = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix},$$

where

$$A : \ C^\infty(\Lambda^*_+) \longrightarrow C^\infty(\Lambda^*_-)$$

is called the signature operator on the even dimensional manifold M. The terminology comes from the basic fact [AS] that the index of A gives the signature of M.

If $\dim M = n$ is odd, such decomposition no longer exists. However, one sets $\tau|_{\Lambda^p} = i^{p+(n-1)/2}*$ and

$$A = \tau(d+\delta) : \; C^\infty(\Lambda^* M) \longrightarrow C^\infty(\Lambda^* M).$$

An important fact is that in this case A is the tangential part of the signature operator on an $(n+1)$-dimensional manifold which bounds M. To be more precise, let N^{n+1} be a compact Riemannian manifold with $\partial N = M$ and assume that the metric on N is of product type near the boundary. Then

$$A_N = \sigma(\partial_u + A_M),$$

where σ is a bundle isomorphism, ∂_u is the differentiation along the normal direction of the boundary, and the equality is to be interpreted with the identification (Cf. [APS])

$$\Lambda^*_+(N) \cong \Lambda^*(M).$$

The Atiyah-Patodi-Singer theorem in this case yields

$$sign(N) = \int_N \mathcal{L}(\frac{R^N}{2\pi}) - \frac{1}{2}\eta(A_M). \tag{4.1}$$

Signature operators are Dirac type operators (at least locally so). In fact, one has the bundle isomorphisms (Cf. [Gi]),

$$\Lambda^* M \cong F(M) \otimes F(M), \quad \Lambda^*_\pm M \cong F_\pm(M) \otimes F(M).$$

Under these isomorphisms,

$$A = D_+ \otimes F(M) \qquad \text{if } n \text{ is even, or}$$
$$A = D \otimes F(M \times \mathbb{R}) \quad \text{if } n \text{ is odd.}$$

Suppose now that M is an oriented odd dimensional manifold which is fibered by (A.9). Here we assume the base B, and the vertical bundle $T^V M$ are both oriented. (In this case we say the fibration is oriented.) By the Hodge theorem, it is easy to see that all the assumptions in Theorem 0.1 are satisfied by A_x. Therefore the following is an immediate consequence of Theorem 0.1.

Corollary 4.1 *The adiabatic limit of the η-invariant of signature operator always exists. Moreover,*

$$\lim_{x\to 0} \eta(A_x) = 2\int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + \lim_{x\to 0} \sum_{\lambda_0,\lambda_1=0} sgn\lambda_x.$$

For the rest of this chapter, we will develop a tool to identify the last term here with a topological invariant.

4.2 Hodge-Leray theory of the very small eigenvalues

In this section we shall prove Theorem 0.2. We will use Serre's construction of the Leray spectral sequence. To relate our space E_r to the E_r-term of the Leray spectral sequence, we introduce another complex which incorporates the asymptotic properties in its filtration. We first recall the general theory of spectral sequence.

Let (C, d) be a filtered differential complex with filtration $\{F^p(C)\}_{p\in Z}$, then we can construct differential complexes (E_i, d_i) as follows.
First, set

$$Z_i^p = F^p(C) \cap d^{-1}(F^{p+i}(C)), \qquad -\infty < p < \infty,\ 0 \leq i < \infty,$$

$$D_i^p = F^p(C) \cap d(F^{p-i}(C)), \qquad -\infty < p < \infty,\ 0 \leq i < \infty,$$

$$Z_\infty^p = F^p(C) \cap \ker d, \qquad D_\infty^p = F^p(C) \cap d(C).$$

Clearly

$$D_0^p \subset D_1^p \subset \cdots \subset D_\infty^p \subset Z_\infty^p \subset \cdots \subset Z_1^p \subset Z_0^p.$$

Define

$$E_i^p = Zi^p/(Z_{i-1}^{p+1} + D_{i-1}^p), \qquad 1 \leq i \leq \infty,$$

$$E_0^p = Z_0^p/F^{p+1}(C) + d(F^{p+1}(C)) = F^p(C)/F^{p+1}(C).$$

The differential d naturally induces a linear map $d_i^p : E_i^p \longrightarrow E_i^{p+i}$ such that $d_i^{p+i} \circ d_i^p = 0$.

Let $E_i = \oplus_p E_i^p,\ d_i = \oplus_p d_i^p$. The following proposition is a basic fact of spectral sequence [GHV].

Proposition 4.1 *The projection $Z_i^p \longrightarrow E_i^p$ restricts to Z_{i+1}^p to give a surjective linear map $Z_{i+1}^p \longrightarrow H^p(E_i, d_i)$ which in turn induces an isomorphism*

$$E_{i+1}^p \xrightarrow{\simeq} H^p(E_i, d_i).$$

In general, a sequence of differential complexes (E_i, d_i) is called a *spectral sequence* if $E_{i+1} \cong H(E_i, d_i)$, and E_i is usually refered as the E_i-term of the spectral sequence. Thus a filtered differential complex gives rise to a spectral sequence.

Later we will need to compare several spectral sequences. For that purpose the following basic principle of comparison will be quite useful [GHV].

Proposition 4.2 *Suppose (C, d) and (C', d') are two filtered complexes and there exists a filtration preserving homomorphism $C \longrightarrow C'$ which also intertwines the differentials. If the homomorphism induces an isomorphism between the E_{r_0}-terms of the two spectral sequences, then it induces isomorphisms between all E_r-terms for $r \geq r_0$.*

We now recall Serre's filtration. Consider the fibration $Y \to M \xrightarrow{\pi} B$ and basic complex $(C^\infty(\Lambda^* M), d)$. Serre's filtration is defined as follows

$$\omega^l \in F^i \Longleftrightarrow < \omega^l(p), \tau_1(p) \wedge \cdots \wedge \tau_l(p) >= 0$$

whenever $l - i + 1$ of the tangent vectors $\tau_j(p)$ are virtical.

In our case, we are given a splitting $TM = T^H M \oplus T^V M \cong TB \oplus TY$ thus $T^*M = T^*B \oplus (TY)^*$ and $\Lambda^* M = \Lambda^* B \otimes \Lambda^* Y$

Formally, we can use $a(y, z)\, dy^\alpha \wedge dz^\beta$ (y local coordinates of B, z local coordinates of Y) to indicate such a splitting. Then Serre's filtration is just

$$F^i = \{a(y, z)\, dy^\alpha \wedge dz^\beta : |\alpha| \geq i\}.$$

It is well-known that Serre's filtration gives rise to the Leray spectral sequence of the fibration, $(\bar{E}_i, \bar{d}_i)$.

Consider now complex $(\mathcal{L}(X,{}^{\mathcal{V}}\Lambda^*), d)$. It comes equipped with a natural filtration $F^i = \{\sum_{j\geq i} a_j x^j\}$. By the general theory quoted above, we can construct spectral sequence $(\tilde{E}_i, \tilde{d}_i) = (\oplus_p \tilde{E}_i^p, \oplus_p \tilde{d}_i^p)$. Note that $x^q : F^i \cong F^{i+q}$ induces $\tilde{E}_i^p \cong \tilde{E}_i^{p+q}$. Further the diagram

$$\begin{array}{ccc} \tilde{E}_i^p & \xrightarrow{\tilde{d}_i^p} & \tilde{E}_i^{p+i} \\ \downarrow x^q & & \downarrow x^q \\ \tilde{E}_i^{p+q} & \xrightarrow{\tilde{d}_i^{p+q}} & \tilde{E}_i^{p+q+i} \end{array}$$

commutes, $x^q \tilde{d}_i^p = \tilde{d}_i^{p+q} x^q$. This implies that $\tilde{E}_i \cong \tilde{E}_i^0 \otimes \mathcal{L}(x)$ (with the natural gradation induced by that of $\mathcal{L}(x)$) and $\tilde{d}_i^p \cong x^p \tilde{d}_i^0 \otimes 1$.

Explicit calculation showes

$$\tilde{E}_0 \cong C^\infty(\Lambda^* B \otimes C^\infty(\Lambda^* Z)) \otimes \mathcal{L}(x),$$

$$\tilde{E}_1 \cong C^\infty(\Lambda^* B \otimes \mathcal{H}^*(Z)) \otimes \mathcal{L}(x),$$

$$\tilde{E}_2 \cong H^*(B, \mathcal{H}^*(Z)) \otimes \mathcal{L}(x).$$

By the finite dimensionality of $\tilde{E}_2$, this spectral sequence must degenerate after finite steps.

Remark These quotient modules ($\mathcal{L}(x)$-modules) can actually be viewed as vector spaces.

To introduce the spaces considered in the Hodge-Leray theorem (Theorem 0.2) we proceed to construct nice bases for certain spaces. These bases will also be used in proving the Hodge-Leray theorem. Let us first introduce some notations.

Let Λ_k denote k-tuplets of real numbers

$$\Lambda_k = \{\lambda_0, \lambda_1, \ldots, \lambda_{k-1}\}$$

and similarly we use Λ to denote infinite sequence of real numbers

$$\Lambda = \{\lambda_0, \lambda_1, \ldots\}.$$

Also, let λ'_x be a function of x which has complete asymptotic expansion at $x = 0$. When we say $\lambda'_x \in \Lambda_k$ we mean that the first k coefficients in the asymptotic expansion of λ'_x at $x = 0$ is prescribed by Λ_k

$$\lambda'_x \sim \lambda_0 + \lambda_1 x + \cdots + \lambda_{k-1} x^{k-1} + \cdots.$$

Similarly $\lambda'_x \in \Lambda$ means all the coefficients are prescribed by Λ. Now for the eigenvalues λ_x of A_x set $\lambda'_x = \lambda_x / x$ and define

$$G_{\Lambda_k} = \bigoplus_{\lambda'_x \in \Lambda_k} E(\lambda_x), \qquad G_\Lambda = \bigoplus_{\lambda'_x \in \Lambda} E(\lambda_x), \tag{4.2}$$

where $E(\lambda_x)$ is the eigenspace associated to λ_x. Thus G_{Λ_k} is the direct sum of the eigenspaces corresponding to the eigenvalues whose first coefficient in the asymptotic expansion is 0 and whose next k coefficients are prescribed by Λ_k, and G_Λ is the direct sum of the eigenspaces corresponding to the eigenvalues whose first coeffient is 0 and whose other coefficients are all prescribed by Λ. Obviously

$$G_\Lambda \subset G_{\Lambda_k} \subset G_{\Lambda_{k-1}}.$$

Proposition 4.3 *Let λ_x be an eigenvalue of A_x such that $\lambda_x \sim \lambda_0 x + \lambda_1 x^2 + \cdots$ and let Λ be the infinite sequence given by the coefficients of λ_x. Then*

G_Λ is a family of finite dimensional vector space which depends smoothly on x down to $x = 0$. That is, there is an orthonormal basis for G_Λ which is smooth in x down to $x = 0$.

Proof. Same analysis as in the proof of Theorem 1.1 reduces it to the finite dimensional case. Therefore it suffices to prove the following lemma

Lemma 4.1 *If $T(x)$ is a finite dimensional family of self-adjoint operators which depends smoothly on x. Then $G_{\Lambda_k} = \bigoplus_{\lambda_x \in \Lambda} E(\lambda_x)$ depends smoothly on x for all $k = 1, 2, \ldots, \infty$. ($\Lambda_\infty = \Lambda$)*

Proof. Note the slight modification in the definition of G_{Λ_k}. The idea is to show by induction that the orthogonal projection $Q_k(x)$ on G_{Λ_k} depends smoothly on x. Then by Lemma 2.12 we have a smooth family of unitary operators $V_k(x)$ such that

$$V_k(x)^{-1} Q_k(x) V_k(x) = Q_k(0).$$

Therefore the desired basis can be gotten by taking the $V_k(x)$-image of an orthonormal basis for Image $Q_k(0)$.

By the discreteness of spec$(T(0))$ there exists an $\epsilon > 0$ such that λ_0 is the only point of $spec(T(0))$ in $|\lambda - \lambda_0| < \epsilon$. By the first resolvent formula and the smoothness of $T(x)$

$$Q(x) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \epsilon} (T(x) - \lambda)^{-1} d\lambda$$

is well defined for small x and is smooth. One easily sees that $Q(x) = Q_1(x)$.

Now assume that $Q_{k-1}(x)$ is smooth, then the family

$$S(x) \stackrel{def}{=} V_k(x)^{-1} T(x) V_k(x)$$

is smooth on Image $Q_{k-1}(0)$ and the eigenvalues of $S(x)$ are the eigenvalues of $T(x)$ having the first $k-1$ coefficients of their asymptotic expansions prescribed by Λ_{k-1}. Set

$$R(x) = \frac{S(x) - (\lambda_0 + \lambda_1 x + \cdots + \lambda_{k-1}x^{k-1})}{x^k}.$$

This is a smooth family of self-adjoint operators and as above one finds

$$Q_k(x) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_k|=\epsilon}(R(x)-\lambda)^{-1}d\lambda.$$

This showes that $Q_k(x)$ is also smooth and the induction is completed.

Since we are in a finite dimensional space the induction process must stop in finite steps, proving it for $k=\infty$

Now we can proceed to prove the Hodge-Leray theorem of the very small eigenvalues. Consider $\Lambda_r^0 = \{\underbrace{0,0,\ldots,0}_{r-1}\}$, $r \geq 2$. Then $G_{\Lambda_r^0}$ is the direct sum of the eigenspaces associated to eigenvalues which decay at least like x^r. From the previous discussion one knows that $G_{\Lambda_r^0}$ is a smooth family of finite dimensional vector spaces, thus it makes sense to consider

$$E_r = \lim_{x\to 0} G_{\Lambda_r^0}$$

for $r \geq 2$.

Theorem 4.2 (Hodge-Leray theorem) $(E_r, x^{-r}d)$ *forms a spectral sequence which is isomorphic to the Leray spectral sequence of the fibration. Moreover, the* $*$ *map induced by the metric* g_x *gives rise to the duality map.*

Proof. We will prove $E_r \cong \bar{E}_r$ by showing that both of them are isomorphic to $\tilde{E}_r^0$.

a) $\bar{E}_r \cong \tilde{E}_r^0$ for all $r \geq 0$.

Consider the natural inclusion:

$$C^\infty(\Lambda^* M) \hookrightarrow \mathcal{L}(X, {}^\mathcal{V}\Lambda^*),$$

$$a(y,z)\, dy^\alpha\, dz^\beta \to x^{|\alpha|} a(y,z)(\frac{dy}{x})^\alpha\, dz^\beta.$$

This is clearly filtration preserving. Thus it induces homomorphisms $\imath_r^p$: $\bar{E}_r^p \to \tilde{E}_r^p$. Define

$$\imath_r : \bar{E}_r = \bigoplus_{p=0}^{n} \bar{E}_r^p \xrightarrow{\oplus \imath_r^p} \bigoplus_{p=0}^{n} \tilde{E}_r^p \xrightarrow{\sum_p x^{-p}} E_r^0,$$

i.e. $\imath_r = \sum_{p=0}^n x^{-p} \imath_r^p$ ($p = \dim B$).

We show that $\imath_r$ is a homomorphism which preserves the differentials.

In fact we know that

$$\begin{array}{ccc} \bar{E}_r^p & \xrightarrow{\bar{d}_r^p} & \bar{E}_r^{p+r} \\ \downarrow \imath_r^p & & \downarrow \imath_r^{p+r} \\ \tilde{E}_r^p & \xrightarrow{\tilde{d}_r^p} & \tilde{E}_r^{p+r} \end{array}$$

commutes. Hence $\imath_r^{p+r} \bar{d}_r^p = \tilde{d}_r^p \imath_r^p$ and $x^q \tilde{d}_i^p = \tilde{d}_i^{p+q} x^q$. Therefore

$$x^{-p-r} \imath_r^{p+r} \bar{d}_r^p = x^{-p-r} \tilde{d}_r^p \imath_r^p = \tilde{d}_r^0 x^{-p} \imath_r^p.$$

This shows

$$\begin{array}{ccc} \bar{E}_r & \xrightarrow{\bar{d}} & \bar{E}_r \\ \downarrow \imath_r & & \downarrow \imath_r \\ \tilde{E}_r^0 & \xrightarrow{\tilde{d}} & \tilde{E}_r^0 \end{array}$$

commutes, i.e. $\imath_r \bar{d} = \tilde{d} \imath_r$.

Furthermore, the explicit calculation (Cf. p. 81) showes that $\imath_r$ induces isomorphism for $r \leq 2$, therefore it induces isomorphisms on all r.

b) $E_r \cong \tilde{E}_r^0$, for $r \geq 2$.

Recall that the scaled metric g_x induces a bilinear form

$$\mathcal{L}(X, {}^{\mathcal{V}}\Lambda^*) \times \mathcal{L}(X, {}^{\mathcal{V}}\Lambda^*) \xrightarrow{<,>} \mathcal{L}(x)$$

and $< d\cdot, \cdot > = < \cdot, \delta\cdot >$. This is the same as saying that δ is the adjoint of d with respect to g_x for every x. (Here of course we are suppressing the x-dependence of δ.)

Our operator A_x is, up to sign, $*(d+\delta)$. Therefore if $A_x\varphi_x = \lambda_x\varphi_x$, then

$$\|d\varphi_x\|^2 + \|\delta\varphi_x\|^2 = |\lambda_x|^2\|\varphi_x\|^2. \tag{4.3}$$

For simplicity we will use the shorthand notation $z \sim x^r$ for $z \in F^r(\mathcal{L}(X, {}^{\mathcal{V}}\Lambda^*))$. One has (Cf. (4.3))

$$(d+\delta)z \sim x^r \Longleftrightarrow dz \sim x^r,\ \delta z \sim x^r. \tag{4.4}$$

Basic to our proof is the observation that $(E_r, x^{-r}d)$ does form a spectral sequence, though it is not clear if it comes from a filtered complex. To see this, let us first show that $x^{-r}d$ maps E_r into E_r. In fact, by the definition of E_r and (4.3), (4.4), it suffices to show that d leaves each eigenspaces invariant. This can be seen in the following way. Clearly d maps the 0-eigenspace to 0 (by, say, (4.3)). Now let $\lambda_x \neq 0$ be an eigenvalue of A_x and φ_x a corresponding eigenform, $A_x\varphi_x = \lambda_x\varphi_x$. But

$$A_x = \pm * d \pm d * .$$

Thus

$$\begin{aligned} \varphi_x &= *d(\pm\varphi_x/\lambda_x) + d*(\pm\varphi_x/\lambda_x) \\ &\stackrel{def}{=} *d\psi_x + d*\psi'_x. \end{aligned}$$

Since $*d\psi_x$ $(d * \psi'_x)$ is coexact (exact) and A_x takes coexact (exact) forms to coexact (exact) forms, one must have

$$\begin{aligned} A_x(*d\psi_x) &= \lambda_x(*d\psi_x), \\ A_x(d * \psi'_x) &= \lambda_x(d * \psi'_x). \end{aligned} \tag{4.5}$$

That is, the eigenspace corresponding to a non-zero eigenvalue splits into the direct sum of coexact and exact eigenspaces. Obviously d takes exact forms to zero. And (4.6) is the same as

$$*d(*d\psi_x) = \lambda_x(*d\psi_x).$$

Taking d of this equation one finds

$$d * d(*d\psi_x) = \lambda_x d(*d\psi_x), \quad \text{or} \quad d * (d * d\psi_x) = \lambda_x(d * d\psi_x).$$

This proves the invariance of the λ_x-eigenspace under d.

To see $E_{r+1} \cong H(E_r, x^{-r}d)$, note that by the finite dimensional analog of the Hodge theory

$$H(E_r, x^{-r}d) \cong \ker(x^{-r}d) \cap \ker(x^{-r}\delta).$$

Together with (4.3), this implies $\lambda_x = O(x^{r+1})$ and the converse is easier.

Let $h_r = \dim E_r$, $h = \dim E_\infty = \dim H^*(M)$. By Proposition 4.3 we can choose

$$\varphi_1, \ldots, \varphi_h, \ldots, \varphi_{h_2} \tag{4.6}$$

such that $\varphi_1, \ldots, \varphi_{h_r}$ form a smooth basis for E_r, all r. From (4.4) it is clear that (4.6) induces a linear map

$$E_r \longrightarrow \tilde{E}^0_r.$$

It is a homomorphism between the two differential complex because the two differentials are formally identical. We will show that it is an isomorphism by showing that it is an isomorphism for $r = 2$.

1-1 : If $z \in E_r$ s.t. $z = 0$ in $\tilde{E}_r^0$, then $z = z_0 + dz_1$ with $z_0 \sim x$ and $dz_0 \sim x^r$, $z_1 \sim x^{-r+1}$, $dz_1 \sim x^0$. But then $\delta dz_1 = \delta(z - z_0) \sim x$, therefore $dz_1 \sim x$ and thus $z \sim x$. i.e. $z = 0$ in E_r.

onto : Let $z \in \tilde{E}_2^0$, then $z \sim x^0$, $dz \sim x^2$. Consider $z' = z - \sum_{i=1}^{h_2} < \varphi_i, z > \varphi_i$, $z' \perp E_2$ and $dz' \sim x^2$. By the formal Hodge decomposition (Lemma 2.5) $z' = (d + \delta)z_1$, $z_1 \sim x^{-1}$. Now $dz' = d\delta z_1 \sim x^2 \Rightarrow \delta z_1 \sim x^2$, i.e. $\delta z_1 \in Z_1^1$. Also $dz_1 \in D_1^0$. Hence $z' = 0$ in $\tilde{E}_r^0$. This show that the map is onto for $r = 2$. Q.E.D.

4.3 Adiabatic limit of η-invariant and non-multiplicativity of signature

In Chapter 1, we proved (.2), the adiabatic limit formula. There the global contribution is found to be $\sum_{\lambda_0, \lambda_1 = 0} \lim sgn \lambda_x$. In the light of the Hodge-Leray theorem proved above, it should be possible to characterize this term topologically. This is indeed the case here. To have a topological counterpart of this term, let us first discuss some algebraic aspects of the Leray spectral sequence.

Let $Y \longrightarrow M^n \longrightarrow B^p$ be a fibration of closed manifolds. Consider the

Leray spectral sequence (E_r, d_r). One has

$$E_2 = H^*(B, \mathcal{H}^*(Y)), \; E_\infty = H^*(M).$$

As a matter of fact (E_r, d_r) has fine algebraic structures as shown in the following theorem [F].

Theorem 4.3 *1) There is a bigrading on E_r: $E_r = \bigoplus_{i,j} E_r^{i,j}$, and*

$$d_r : \; E_r^{i,j} \longrightarrow E_r^{i+r, j-r+1}.$$

Set $E_r^k = \bigoplus_{i+j=k} E_r^{i,j}$. Thus $d_r : \; E_r^k \longrightarrow E_r^{k+1}$.
2) For all $r \geq 2$, the group E_r may be endowed with a natural multiplicative structure, i.e., there is a mapping

$$\begin{aligned} E_r^{i,j} \otimes E_r^{k,l} &\longrightarrow E_r^{i+k, j+l}, \\ a \otimes b &\longrightarrow a \cdot b, \end{aligned}$$

which defines the associative and distributive multiplication consistent with the action of differentials,

$$d(a \cdot b) = da \cdot b + (-1)^k a \cdot db, \; \textit{if } a \in E_r^k.$$

The multiplication in E_r induces that in E_{r+1}. Moreover, the multiplication in E_2 and E_∞ coincides with their natural ring structures.

Now assume that our fibration is oriented in the sense that both B and the vertical bundle $T^V M$ are oriented. The orientation of $T^V M$ gives a trivialization of the flat line bundle $\mathcal{H}^{n-p}(Y)$ on B (note that $n-p = \dim Y$). Together with the orientation of B one finds

$$E_2^n = H^p(B, \mathcal{H}^{n-p}(Y)) \cong H^p(B) \cong \mathbb{R}.$$

Thus the multiplicative structure on E_2 gives rise to a Poincaré pairing

$$E_2^k \otimes E_2^{n-k} \longrightarrow \mathbb{R},$$

i.e., E_2 is a so called Poincaré ring with differentiation, defined as follows [CHS].

A graded ring $A = \bigoplus_{i=0}^n A_i$ is called a *Poincaré ring* if $\dim A_n = 1$ and the multiplication is graded in the sense that

$$xy = (-1)^{ij} yx \qquad \text{if} \qquad x \in A_i, y \in A_j.$$

Further the bilinear pairing

$$A_i \otimes A_{n-i} \longrightarrow A_n \cong \mathbb{R}$$

is a Poincaré pairing (i.e. nondegenerate).

A differentiation in a Poincaré ring is a differentiation d of degree $+1$ such that $dA_{n-1} = 0$ and d is also an anti-derivation,

$$d(xy) = (dx)y + (-1)^i x(dy), \qquad \text{if} \qquad x \in A_i.$$

It is proved in [CHS] that the derived ring of a Poincaré ring with differentiation is again a Poincaré ring. Thus all E_r for $r \geq 2$ are Poincaré rings, and if we denote ξ_2 the natural base for E_2^n induced by the orientations, then it induces a natural base ξ_r for each E_2^r $(r \geq 2)$ (Cf. [CHS]). Now consider the bilinear pairing:

$$\tau_r : \qquad E_r \otimes E_r \longrightarrow \mathbb{R}$$
$$\tau_r(\varphi \otimes \psi) = <\varphi \cdot d_r\psi, \xi_r> \stackrel{\text{def}}{=} <\varphi, d_r\psi>$$

It is easy to verify that

$$< \varphi, d_r\psi >= (-1)^{(deg\varphi+1)(deg\psi+1)} < \psi, d_r\varphi > .$$

In our situation $n = 4m - 1$. This implies that τ_r restricted to E_r^{2m-1} is symmetric. By viewing τ_r as a symmetric matrix one can take its signature, $sign\,\tau_r$. Now define

$$\tau = \sum_{r\geq 2} sign\,\tau_r,$$

This is clearly a topological invariant.

Theorem 4.4 (Adiabatic limit formula) *We have the equality*

$$\tau = \frac{1}{2} \sum_{\lambda_0,\lambda_1=0} \lim_{x\to 0} sgn\lambda_x.$$

Consequently the following adiabatic limit formula holds,

$$\lim_{x\to 0} \eta(A_x) = 2\int_B \mathcal{L}(\frac{R^B}{2\pi}) \wedge \tilde{\eta} + \eta(A_B \otimes \ker A_Y) + 2\tau.$$

Proof. It suffices to to show the first equality (see Corollary 4.1). To do this, let us consider

1) Partial-symmetry of eigenvalues of A_x.

It is enough to consider $A = *d \pm d* : \Omega(M) \longrightarrow \Omega(M)$. Since $* : \Omega^{even} \longrightarrow \Omega^{odd}$ is an isomorphism, A is just two copies of its restriction to Ω^{odd}. By Hodge decomposition

$$\Omega^{odd} = \mathcal{H}^{odd} \oplus d\Omega^{even} \oplus \delta\Omega^{even}$$

Observe that the operator A annihilates $\mathcal{H}^{odd}$, coincides up to sign with $d*$ on $d\Omega^{even}$, and with $*d$ on $\delta\Omega^{even}$.

Now $d* : d\Omega^{2p} \longrightarrow d\Omega^{4k-2p-2}, *d : \delta\Omega^{2p} \longrightarrow \delta\Omega^{4m-2p}$ and $2p \neq 4m - 2p - 2$ $2p = 4m - 2p$ iff $p = m$. Hence we can decompose

$$A = A_0 \oplus A_1 \oplus A_2,$$

where A_0 is the zero operator on $\mathcal{H}^{odd}$ and $A_1 = *d$ on $\delta\Omega^{2m}$ and A_2 is an operator of the form

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix},$$

therefore having symmetric spectrum.

This shows that

$$\sum_{\lambda_0, \lambda_1 = 0} \frac{1}{2} \lim_{x \to 0} sgn\lambda_x = \sum_{\lambda_0, \lambda_1 = 0}{}' \lim_{x \to 0} sgn\lambda_x,$$

where $\sum'$ indicates that the summation is over all λ which are the eigenvalues of $*d$ on Ω^{2m-1}

2) Since $E_2 = H^*(B, \mathcal{H}^*(Y))$, it inherits an inner product $(\cdot, \cdot)$, which then induces an inner product on each E_r. Define

$$< \varphi, d_r\psi >= (\varphi, *_r d_r \varphi),$$

then as symmetric mappings the eigenvalues of τ_r are the same as those of $*_r d_r$. By Theorem 0.2,

$$d_r \sim x^{-r} d, \qquad *_r \sim *.$$

We see immediately that the first statement of Theorem 4.4 holds. The adiabatic limit formula follows then from (.2). Q.E.D.

As an interesting application we now give intrinsic characterizations of the non-multiplicativity of signature for manifolds with boundary.

Chern-Hirzebruch-Serre first studied the multiplicative behavior of signature for closed manifolds [CHS]. They showed that for a fibration $Z \longrightarrow N^n \longrightarrow B$, we have the multiplicativity

$$sign N = sign B \cdot sign Z \tag{4.7}$$

provided Z, B are closed manifolds and $\pi_1(B)$ acts trivially on $H^*(Z)$. Later Atiyah [A] gave an example showing that this naive sense of multiplicativity does not hold in general. However, he observed that, by the Hirzebruch signature theorem and Atiyah-Singer Family Index Theorem [AS], one does have a generalized multiplicativity,

$$sign N = \int_B \mathcal{L}(B) \wedge ch(Sign^Z), \tag{4.8}$$

where $\mathcal{L}(B)$ is a characteristic class which may substitute for the Hirzebruch L-genus, $Sign^Z$ is the signature bundle of Z over B and $ch(Sign^Z)$ is its Chern character. When $\pi_1(B)$ acts trivially on $H^*(Z)$, this reduces to (4.7). See Introduction for a purely topological formulation of this generalized multiplicativity.

This generalized multiplicativity has a Leray spectral sequence interpretation. Consider the pairing

$$\begin{aligned} E_r^i \otimes E_r^{n-i} &\longrightarrow \mathbb{R}, \\ \varphi \otimes \psi &\longrightarrow < \varphi \cdot \psi, \xi_r >, \end{aligned}$$

where E_r is the Leray spectral sequence of the fibration and ξ_r is defined in the same way as before. When $n = 4k$, the above pairing restricts to a symmetric pairing on E_r^{2k}, whose signature will be denoted by σ_r. Then (4.8) is equivalent to

$$\sigma_2 = \sigma_\infty.$$

This follows from [CHS] and the signature theorem for twisted coefficients [L], [AS].

However, for manifolds with boundary, the generalized multiplicativity fails, as can be seen in the following example.

Example. (Disk bundles) Let $V^2 \longrightarrow B^2$ be an oriented 2-plane bundle over an oriented surface. Denote Φ the Thom class and $\chi(V)$ the Euler number of V. Consider its disk bundle $\mathbb{D}^2 \longrightarrow D(V) \longrightarrow B^2$ with its sphere bundle $S(V)$. By the Thom isomorphism

$$H^{*-2}(B) \xrightarrow{\cup\Phi} H^*(D(V), S(V)),$$

one easily finds

$$sign(D(V)) = sgn\chi(V).$$

But $sign(B) = sign(\mathbb{D}^2) = 0$. In particular, if V is the Hopf bundle, $sign(D(V)) = 1 \neq 0$.

On the other hand, note that by the extended Novikov additivity [BC4] and (4.8), the difference, called the *non-multiplicativity of signature* (compare also Introduction),

$$\Delta \overset{def}{=} \int_B \mathcal{L}(B) \wedge ch(Sign^Z) - signN$$

depends only on the associated boundary fibration $Y \longrightarrow M \longrightarrow B$, where $Y = \partial Z, M = \partial N$. We assume here B is closed, and we will assume below that $\dim B$ is even.

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depends only on the associated boundary fibration $Y \longrightarrow M \longrightarrow B$, where $Y = \partial Z, M = \partial N$. We assume here B is closed, and we will assume below that $\dim B$ is even.

Theorem 4.5 *The topological invariant τ, which is defined from the closed fibration $Y \longrightarrow M \longrightarrow B$, intrinsically characterize the non-multiplicativity of signature. That is, whenever there exists another fibration of manifolds with boundary $Z \longrightarrow N^n \longrightarrow B$ such that $\partial Z = Y$, $\partial N = M$, then $\tau = \Delta$.*

Proof. We start with (4.1), Atiyah-Patodi-Singer's signature formula for manifold with boundary, and take the adiabatic limit of both sides. The lefthand side remains since signature is a topological invariant. The first term on the righthand side yields, by Lemma A.3,

$$\int_N \mathcal{L}(\frac{R^B}{2\pi i}) \wedge \mathcal{L}(\frac{R^Z}{2\pi i}) = \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i})),$$

while the second term is the adiabatic limit formula (Theorem 4.4). Therefore we obtain

$$\begin{aligned} sign(N) &= \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i})) - \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge \tilde{\eta} - \frac{1}{2}\eta(A_B \otimes \ker A_Y) - \tau \\ &= \int_B \mathcal{L}(\frac{R^B}{2\pi i}) \wedge (\int_Z \mathcal{L}(\frac{R^Z}{2\pi i}) - \tilde{\eta}) - \tau, \end{aligned}$$

where the η-term drops out because $\dim B$ is even. On the other hand, Bismut-Cheeger's Families Index Theorem states that $\int_Z \mathcal{L}(\frac{R^Z}{2\pi i}) - \tilde{\eta}$ is a representative for the chern character of the signature bundle $Sign^Z$. Combining these two finishes the proof. Q.E.D.

Thus we characterize the non-multiplicativity of signature intrinsically in terms of the topological data of the boundary fibration. One can also give an intrinsic characterization in terms of the analytical data.

Theorem 0.4$'$ *Let*

V_0= limit of space of A_x-harmonic forms on M in the adiabatic limit,

$V_\pm$=limit of space of λ_x-eigenforms such that $\lambda_x > 0$ (respectively < 0) and λ_x is $O(x^2)$.
Then $V_+ \oplus V_- \oplus V_0 = H^*(B, \mathcal{H}(Y))$, and $\tau' \stackrel{def}{=} \dim V_+ - \dim V_- = \tau$. Therefore $\tau' = \Delta$ whenever the fibration $Y \to M \to B$ bounds $Z \to X \to B$.

Remark One can also study the non-multiplicativity in the "dual" case, i.e. for fibration $Z \to N \to B$ where the fibre Z is closed but the base B has boundary. Theorem 4.5 holds in this case as well. Instead of the Families Index Theorem for manifolds with boundary, a transgression formula for $\tilde{\eta}$ is used.

4.4 Formal Hodge decomposition and asymptotic analysis

In this section, we discuss the analysis of formal asymptotic series and prove the formal Hodge decomposition (Lemma 2.5). Consider the manifold $X = [0,\infty) \times M$ with its boundary $M_0 = \{x = 0\}$ and the "compressed form bundle" ${}^{\mathcal{V}}\Lambda^*$ (see [MM] and Section 2.1). Formally a smooth section of ${}^{\mathcal{V}}\Lambda^*$ is a linear combination of the form

$$a(x,y,z)dz^\alpha \wedge (\frac{dy}{x})^\beta$$

with $a(x,y,z)$ a smooth function.

The metric g_x gives rise to a well-defined non-degenerate inner product $< \cdot, \cdot >$ on ${}^{\mathcal{V}}\Lambda^*$ which in turn gives us a Hodge $*$ operator

$$*: \quad {}^{\mathcal{V}}\Lambda^* \longrightarrow {}^{\mathcal{V}}\Lambda^*. \tag{4.9}$$

This is nothing but a uniform construction of the $*$ operators on M for the parameter x down to $x = 0$.

Since $d \in \mathrm{Diff}^1_{\mathcal{V}}(X;^{\mathcal{V}}\Lambda^*)$ its formal adjoint δ with respect to $< \cdot, \cdot >$ is also a $\mathcal{V}$-differential operator and $A_x = *(d+\delta)$ up to a sign.The operator A_x can be restricted to $C^\infty(M_0,^{\mathcal{V}}\Lambda^*)$, the resulting operator $I(A_x)$, called the *indicial operator* of A_x (Cf. [MM]), is just A_Y.

$$I(A_x)u_0 = (A_x u)|_{x=o},$$

where $u \in C^\infty(X,^{\mathcal{V}}\Lambda^*)$ such that $u|_{x=0} = u_0$.

Consider now $A_x - \lambda_x$ where λ_x is a smooth function of x and vanishes at $x = 0$. Thus asymptotically $\lambda_x = x\lambda_1 + x^2\lambda_2 + \cdots$. Then $I(A_x - \lambda_x) = A_Y$ again and its kernel is the fiber harmonic forms ${}^{\mathcal{V}}E_1$, which can be viewed as a smooth bundle on the base B. Recall that $A_0 = PA_xP$ is the signature operator on B coupled to this bundle.

Let

$$E_2^{\lambda_1} = \{v \in C^\infty(M_0,^{\mathcal{V}}\Lambda^*),\ v \in C^\infty(B,^{\mathcal{V}}E_1) \text{ and } (A_0 - \lambda_1)v = 0\}.$$

Clearly $E_2^{\lambda_1}$ is finite dimensional.

Lemma 4.2 *The space $E_2^{\lambda_1} \subset C^\infty(M_0,^{\mathcal{V}}\Lambda^*)$ consists precisely of the boundary values of ${}^{\mathcal{V}}\Lambda^*$-forms which are "$\lambda_x$-harmonic" up to second order error:*

$$\begin{aligned} E_2^{\lambda_1} &= \{u_0 \in C^\infty(M_0,^{\mathcal{V}}\Lambda^*);\ \exists u \in C^\infty(X,^{\mathcal{V}}\Lambda^*) \\ &\quad \textit{with } u|_{M_0} = u_0 \textit{ and } (A_x - \lambda_x)u \in x^2 C^\infty(X,^{\mathcal{V}}\Lambda^*)\}. \end{aligned}$$

Proof. We mimick the proof of (49) in [MM]. Essentially by definition the space ${}^{\mathcal{V}}E_1$ is characterized as the boundary values of harmonic ${}^{\mathcal{V}}\Lambda^*$-forms modulo first order error. Thus for any u with boundary value $u_0 \in^{\mathcal{V}}$

E_1, we must have $A_x u|_{M_0} = 0$. Thus

$$A_x u = xv, \; v \in C^\infty(X, {}^\nu \Lambda^*) \text{ and } A_0 u = P(v).$$

Hence

$$(A_x - \lambda_x)u = x(v - \lambda_1 u) + x^2 w, \; w \in C^\infty(X, {}^\nu \Lambda^*).$$

By the Hodge Theorem on the fibre, we can choose $u_1 \in C^\infty(X, {}^\nu \Lambda^*)$ such that restricted to boundary,

$$v - Pv = A_Y u_1 = (A_x - \lambda_x)u_1 + xu_1'.$$

Therefore,

$$(A_x - \lambda_x)(u - xu_1) = x(A_0 - \lambda_1)u_0 + x^2 w_1'.$$

Therefore if $u_0 \in E_2^{\lambda_1}$, $u - xu_1$ is λ_x-harmonic up to second order error. Conversely, if $u \in C^\infty(X, {}^\nu \Lambda^*)$, $u|_{M_0} = u_0 \in^\nu E_1$, and $(A_x - \lambda_x)u \in x^2 C^\infty(X, {}^\nu \Lambda^*)$. Then

$$x(v - \lambda_1 u) \in x^2 C^\infty(X, {}^\nu \Lambda^*),$$

i.e.

$$(v - \lambda_1 u)|_{M_0} = 0, \quad or \quad v|_{M_0} \in^\nu E_1.$$

From above then

$$(A_x - \lambda_x)u = x(A_0 - \lambda_1)u_0 + x^2 w_1',$$

hence $(A_0 - \lambda_1)u_0 = 0$. Q.E.D.

To show Lemma 2.5 consider the space of Laurent series introduced in Section 2.1, $\mathcal{L}(X,^{\mathcal{V}}\Lambda^*)$. This is a module over the formal Laurent series ring $\mathcal{L} = \mathcal{L}(x)$. By choosing an extension

$$C^\infty(M_0,^{\mathcal{V}}\Lambda^*) \longrightarrow C^\infty(X,^{\mathcal{V}}\Lambda^*) \longrightarrow \mathcal{L}(X,^{\mathcal{V}}\Lambda^*),$$

we can identify $\mathcal{L}(x)\cdot E_2^{\lambda_1}$ as a subspace of $\mathcal{L}(X,^{\mathcal{V}}\Lambda^*)$. This subspace is finite dimensional over $\mathcal{L}(x)$ so has a basis $e_1,\ldots,e_L$. the inner product $<,>_G$ gives a bilinear form

$$\mathcal{L}(X,^{\mathcal{V}}\Lambda^*) \times \mathcal{L}(X,^{\mathcal{V}}\Lambda^*) \longrightarrow \mathcal{L}(x).$$

Using this to orthonormalize the basis, we obtain a projection operator onto a complement to $\mathcal{L}(x)\cdot^{\mathcal{V}} E_2^{\lambda_1}$:

$$\pi_2^\perp : \mathcal{L}(X,^{\mathcal{V}}\Lambda^*) \longrightarrow \mathcal{L}(X,^{\mathcal{V}}\Lambda^*),$$

$$\pi_2^\perp(f) = f - \sum_{j=1}^{L} < f, e_j >_G e_j.$$

Now our operator $A_x - \lambda_x$ also extends. We have

Lemma 4.3 *The operator* $P_2 = \pi_2^\perp(A_x - \lambda_x)\pi_2^\perp$ *is an isomorphism on* $\pi_2^\perp\mathcal{L}(X,^{\mathcal{V}}\Lambda^*)$.

Proof. We first show the surjectivity by an inductive argument. Since our operator has the decomposition

$$A_x = A_Y + xA_B + x^2V,$$

the basic idea is to invert each term succesively, using respectively the Hodge theory (or ellipticity) along fibers and on the base. Thus if $f \in$

$\pi_2^\perp \mathcal{L}(X, {}^\mathcal{V}\Lambda^*)$ then the leading term $x^l f_l, l \in \mathbb{Z}$ must have coefficient $f_l \perp^\mathcal{V} E_2^{\lambda_1}$. By the Hodge theorem along fibers we can choose $u_l \in C^\infty(M_0, {}^\mathcal{V}\Lambda^*)$ so that

$$f_l = A_Y u_l + P f_l.$$

Therefore $f' = (A_x - \lambda_x)(x^l u_l) - f$ has leading term $x^l P f_l$. Now by the ellipticity of A_0 and the assumption that $f_l \perp^\mathcal{V} E_2^{\lambda_1}$ one has $P f_l = A_0 u_{l-1}$ for some $u_{l-1} \in^\mathcal{V} E_1$. It follows that

$$\pi_2^\perp (A_x - \lambda_x)(x^l u_l - x^{l-1} u_{l-1}) - f = O(x^{l+1}).$$

Let $u' = \pi_2^\perp (x^l u_l - x^{l-1} u_{l-1})$, then $(x^l u_l - x^{l-1} u_{l-1} - u') = \pi_2 (x^l u_l - x^{l-1} u_{l-1})$ and thus by the above lemma (and the definition of π_2) we have

$$(A_x - \lambda_x)(x^l u_l - x^{l-1} u_{l-1} - u') = O(x^{l+1}).$$

Hence

$$\pi_2^\perp (A_x - \lambda_x) \pi_2^\perp u' = O(x^{l+1}).$$

Proceeding inductively in the Laurent series we can construct u such that $\pi_2^\perp (A_x - \lambda_x) \pi_2^\perp u = f$.

To show the injectivity let $f \in \pi_2^\perp \mathcal{L}(X, {}^\mathcal{V}\Lambda^*)$ such that $\pi_2^\perp (A_x - \lambda_x) \pi_2^\perp f = 0$, i.e., $(A_x - \lambda_x) f \in \mathcal{L} \cdot E_2^{\lambda_1}$. By the surjectivity proved above one can find a $u \in \pi_2^\perp \mathcal{L}(X, {}^\mathcal{V}\Lambda^*)$ so that $f = (A_x - \lambda_x) u$. Thus

$$< (A_x - \lambda_x)^2 u, u >= 0.$$

This implies that the leading term of f must vanish, proving the injectivity.

Proof of Lemma 2.5. Certainly N is finite dimensional $\mathcal{L}$-module since π_2 is injective on it. Let B be the inverse of $\pi_2^\perp (A_x - \lambda_x) \pi_2^\perp$ from the previous lemma. The projection of the matrix equation

$$\begin{pmatrix} \pi_2(A_x-\lambda_x)\pi_2 & \pi_2(A_x-\lambda_x)\pi_2^\perp \\ \pi_2^\perp(A_x-\lambda_x)\pi_2 & \pi_2^\perp(A_x-\lambda_x)\pi_2^\perp \end{pmatrix}\begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} f' \\ f'' \end{pmatrix}$$

onto the image of π_2 is then

$$\begin{aligned} Qv &= (\pi_2(A_x-\lambda_x)\pi_2 - \pi_2(A_x-\lambda_x)\pi_2^\perp B\pi_2^\perp(A_x-\lambda_x)\pi_2)v \\ &= g = f' - \pi_2(A_x-\lambda_x)\pi_2^\perp Bf''. \end{aligned}$$

That is, if (u', u'') satisfies the first equation then $v = u'$ satisfies the second and conversely if v satisfies the second equation then

$$\begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} v \\ Bf'' - B(\pi_2^\perp(A_x-\lambda_x)\pi_2)v \end{pmatrix}.$$

The null space of this self-adjoint operator Q is $\pi_2 N$ and by standard argument it is an isomorphism on the orthocomplement in $\pi_2\mathcal{L}(X,^{\mathcal{V}}\Lambda^*)$. This showes that $A_x - \lambda_x$ is an isomorphism on $N^\perp$.

Appendix A

Bismut-Cheeger's adiabatic limit formula

This appendix contains the algebraic and geometric preliminaries, which are mostly taken from [BC2] and [B]. Results that are basic to our discussions, in particular, recent results of Bismut-Cheeger on the adiabatic limit of η-invariants will also be reviewed. Finally, for use in Chapter 2, we include some facts about the analysis on the manifolds with boundary.

A.1 Z_2-gradings, Clifford algebras and superconnections

A vector space V is called $\mathbb{Z}_2$-*graded* if it comes equipped with a direct sum decomposition $V = V_0 \oplus V_1$. This decomposition determines and is determined by an involution ρ

$$\rho|_{V_j} \stackrel{def}{=} (-1)^j. \tag{A.1}$$

A $\mathbb{Z}_2$-graded vector space $A = A_0 \oplus A_1$, which is also an algebra, is called a $\mathbb{Z}_2$-graded algebra if $A_j A_{j'} \subset A_{j+j'}$. We will use subscripts on elements to denote their grading, e.g. $a_j \in A_j$. All such subscripts take values in $\mathbb{Z}_2$. The supercommutator of $a_j,\ a_{j'}$ is defined as

$$[a_j, a_{j'}] = a_j a_{j'} - (-1)^{jj'} a_{j'} a_j. \tag{A.2}$$

If V is a $\mathbb{Z}_2$-graded vector space, then $End(V)$ is naturally a $\mathbb{Z}_2$-graded algebra, where $End_0(V)$ consists of endomorphisms commuting with the involution ρ and $End_1(V)$ consists of those anti-commuting with ρ. A $\mathbb{Z}_2$-graded representation of a $\mathbb{Z}_2$-graded algebra A on a $\mathbb{Z}_2$-graded vector space V is a grading-preserving homomorphism $\phi : \ A \longrightarrow End(V)$, i.e., $\phi(A_j) \subset End_j(V)$.

Given A, V, ϕ as above then

$$tr_s(a) = tr(\rho\phi(a)) \tag{A.3}$$

defines a *supertrace* on A. This is a linear map whose kernel contains all supercommutators.

If A, B are $\mathbb{Z}_2$-graded algebras, their $\mathbb{Z}_2$-graded tensor product $A\hat{\otimes}B$ is naturally isomorphic as a vector space to $A \otimes B$. The multiplication on $A\hat{\otimes}B$ satisfies

$$(a\hat{\otimes}b_j)(c_{j'}\hat{\otimes}d) = (-1)^{jj'} ac_{j'}\hat{\otimes}b_j d.$$

$A\hat{\otimes}B$ inherits a natural $\mathbb{Z}_2$-grading, determined by $\rho_A \otimes \rho_B$.

The following constructions will be quite useful. Let $\phi,\ \psi$ be representations of $A,\ B$ on $X,\ Y$. Let $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ define the grading on $\mathbb{C}^2$ and let $J,\ K \in End_1(\mathbb{C}^2)$ denote the involutions,

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $JK = -KJ$. If the grading on $X \otimes Y \otimes \mathbb{C}^2$ is determined by the involution $Id \otimes Id \otimes iJK$, then

$$a_j \hat{\otimes} b_{j'} \longrightarrow \phi(a_j) \otimes \psi(b_{j'}) \otimes J^j K^{j'} \tag{A.4}$$

defines a $\mathbb{Z}_2$-graded representation of $A \hat{\otimes} B$.

Let $\phi,\ \psi$ be as above. In addition, let ψ be $\mathbb{Z}_2$-graded and ρ be as in (A.1) for Y. Then

$$a_j \hat{\otimes} b \longrightarrow \phi(a_j) \otimes \rho^j \psi(b) \tag{A.5}$$

defines a representation of $A \hat{\otimes} B$. If ϕ is also $\mathbb{Z}_2$-graded, this representation is $\mathbb{Z}_2$-graded for the natural $\mathbb{Z}_2$-grading of $X \otimes Y$.

Let V^n be an Euclidean space. The complex *Clifford algebra* $\mathbb{C}l(V^n)$ is generated by an orthonormal basis $\{e_i\}$ of V satisfying the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

As a vector space $\mathbb{C}l(V^n)$ can be identified with the complex exterior algebra. The Clifford multiplication is then exterior multiplication minus interior multiplication. The elements $e_\alpha = e_{i_1} \cdots e_{i_j}$, where $\alpha = (i_1, \cdots, i_j)$, $i_1 < \cdots < i_j$, form a basis for $\mathbb{C}l(V^n)$. The subspaces $\mathbb{C}l_0(V^n)$, $\mathbb{C}l_1(V^n)$ spanned by those e_α with $|\alpha|$ even, respectively odd give $\mathbb{C}l(V^n)$ the structure of a $\mathbb{Z}_2$- graded algebra. If V is oriented, the element

$$\tau = \begin{cases} i^k e_1 \cdots e_{2k-1} & \text{if } n = 2k-1 \\ i^k e_1 \cdots e_{2k} & \text{if } n = 2k \end{cases} \tag{A.6}$$

is independent of the choice of $\{e_i\}$ and satisfies $\tau^2 = 1$.

For $n = 2k$, up to isomorphism, $\mathbb{C}l(V^n)$ has a unique irreducible module F, which has dimension 2^k and is $\mathbb{Z}_2$-graded (by $\phi(\tau)$). In fact $\mathbb{C}l(V^n) = End(F)$. We will follow the standard convention and write $F_+ \oplus F_-$ for $F_0 \oplus F_1$.

If $n = 2k-1$, $\mathbb{C}l(V^n)$ has two inequivalent irreducible modules, each of dimension 2^{k-1}. For arbitrary n,

$$e_i \longrightarrow e_i e_{n+1}$$

defines an isomorphism, $\mathbb{C}l(V^n) \cong \mathbb{C}l_0(V^n \oplus \mathbb{R})$. Thus we can regard F_+, F_- for $V^n \oplus \mathbb{R}$ as (inequivalent) modules of $\mathbb{C}l(V^n)$ (and $\tau \to \pm Id_{F_\pm}$). For V^{2k-1} oriented, the notation $tr(a)$ refers to the representation F_+.

Lemma A.1 *1) If $n = 2k$ is even then*

$$tr_s(e_\alpha) = \begin{cases} 0 & \textit{if } \alpha \neq (1, \cdots, 2k) \\ i^{-k} 2^k & \textit{if } \alpha = (1, \cdots, 2k). \end{cases} \tag{A.7}$$

2) If $n = 2k-1$ is odd and $|\alpha| \geq 1$,

$$tr(e_\alpha) = \begin{cases} 0 & \textit{if } \alpha \neq (1, \cdots, 2k-1) \\ i^{-k} 2^{k-1} & \textit{if } \alpha = (1, \cdots, 2k-1). \end{cases} \tag{A.8}$$

Lemma A.2 *The irreducible modules of* $\mathbb{C}l(V^n \oplus W^m)$ *can be obtained from the irreducible modules* $F(V^n)$, $F(W^m)$ *for* $\mathbb{C}l(V^n)$, $\mathbb{C}l(W^m)$ *as follows,*
1) For n, m *both odd, the module* $F(V^n) \otimes F(W^m) \otimes \mathbb{C}^2$ *defined by (A.4) is irreducible.*
2) For m *even, the module* $F(V^n) \otimes F(W^m)$ *defined by (A.5) is irreducile.*

The proof of these is fairly easy, and we omit it (see [BC2]).

Finally, we note the effect of scaling the inner product $< , >$ on V. For any inner product, $\mathbb{C}l(V^n)$ coincides as a vector space with $\Lambda^*(V) \otimes \mathbb{C}$. Fix an inner product, $< , >$ and let $\mathbb{C}l_x(V^n)$ denote $\Lambda^*(V)$ with Clifford multiplication coming from $x^{-2} < , >$. Then the automorphism of $\Lambda^*(V) \otimes \mathbb{C}$ induced by $xv \to v$, provides a natural isomorphism $\mathbb{C}l_x(V^n) \cong \mathbb{C}l(V^n)$. It also provides a natural isomorphism between the orthonormal frames $\{xe_i\}$ for $x^{-2} < , >$ and $\{e_i\}$ for $< , >$. Thus, although there is no canonical choice for the space $F(V)$ for $< , >$, any fixed choice also provides an irreducible module for $\mathbb{C}l_x(V^n)$ via the above isomorphism.

In the sequel, if M is a Riemannian spin manifold, we will always assume that the space of spinors has been chosen independent of the scaling parameter of the metric. As a consequence, the action of $xe_i \in \mathbb{C}l_x(T_pM)$ on F_p is independent of x and the Dirac operator D_x corresponding to $x^{-2} < , >$ is xD, where D is the Dirac operator for $< , >$.

A vector bundle E over a manifold M is called $\mathbb{Z}_2$-*graded* (see [Q]) if it comes equipped with a Whitney sum decomposition $E = E_0 \oplus E_1$. Consider

$$\Omega(M,E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M,E),$$

the algebra of E-valued smooth differential forms with complex coefficients, where $\Omega(M) = \oplus \Omega^p(M)$ is the algebra of smooth differential forms with complex coefficients and $\Omega^0(M,E)$ is the space of smooth sections of E. This is a $\Omega(M)$-module. It has a grading with respect to $\mathbb{Z} \times \mathbb{Z}_2$. However, we are primarily concerned with its total $\mathbb{Z}_2$-grading, and so regard it as a $\mathbb{Z}_2$-graded module over $\Omega(M)$.

We will be working in the $\mathbb{Z}_2$-graded algebra of endomorphism valued forms

$$\Omega(M, End(E)) = \Omega(M) \hat{\otimes}_{\Omega^0(M)} \Omega^0(M, End(E)),$$

and we shall write ωX instead of $\omega \hat{\otimes} X$. This algebra operates on $\Omega(M,E)$ on the left:

$$(\omega X)(\eta e) = (-1)^{(deg X)(deg \eta)} \omega \wedge \eta X(e).$$

The following is easily verified (Cf. [Q]).

Proposition A.1 *In the above way the algebra $\Omega(M, End(E))$ can be identified with the algebra of operators on $\Omega(M,E)$ which are $\Omega(M)$-linear in the sense that the even and odd components satisfy*

$$T(\omega\alpha) = (-1)^{(deg T)(deg \omega)} \omega T(\alpha).$$

After D. Quillen, [Q], we define a *superconnection* on E to be an operator D on $\Omega(M,E)$ of odd degree satisfying the derivation property

$$D(\omega\alpha) = (d\omega)\alpha + (-1)^{deg \omega} \omega D\alpha$$

for $\omega \in \Omega(M)$ and $\alpha \in \Omega(M, E)$. For example, a connection in E preserving the grading, when extended to an operator on $\Omega(M, E)$ in the usuall way, determines a superconnection. Using Proposition A.1 sees that the difference of two superconnections is an odd element of $\Omega(M, End(E))$.

If D is a superconnection and $\phi \in \Omega(M, End(E))$, then the supercommutator $[D, \phi]$ as an operator on $\Omega(M, E)$ is linear over $\Omega(M)$, hence an element of $\Omega(M, End(E))$ by Proposition A.1.

Proposition A.2 *One has*

$$d(tr_s \phi) = tr_s[D, \phi].$$

Proof. see [Q].

In the sequel, we want to consider infinite dimensional analogues of vector bundles and superconnections. In principle, results discussed above continue to hold, although some care must be taken in their application.

A.2 Elementary geometry of fibrations

Let

$$Y \to M \xrightarrow{\pi} B \tag{A.9}$$

be a fibration of smooth manifolds. A connection determines a splitting of the tangent bundle of M into its vertical subbundle and horizontal subbundle,

$$TM = T^V M \oplus T^H M.$$

Let P^H, P^V denote the projections on T^H, T^V relative to this splitting.

If U is a locally defined vector field on B, we denote by $\tilde{U}$ its horizontal lift. The integrability tensor (or curvature) of $T^H M$ is the 2-form $\mathcal{R}$ on $T^H M$ with values in $T^V M$ defined by

$$\mathcal{R}(\tilde{U},\tilde{V}) = -[\tilde{U},\tilde{V}] + \widetilde{[U,V]} = -P^V[\tilde{U},\tilde{V}].$$

Now equip M with a submersion metric g which respects the above splitting,

$$g = \pi^* g_B + g_Y.$$

Let ∇^B be the Levi-Civita connection of TB. ∇^B lifts to a so called Euclidean connection on $T^H M$, which we still denote by ∇^B,

$$\begin{aligned} \nabla^B_{\tilde{U}}\tilde{V} &= \nabla^B_U V, \\ \nabla^B_W \tilde{V} &= [W,\tilde{V}] \quad \text{if } W \text{ is vertical.} \end{aligned}$$

Also let ∇^Y be the connection on $T^V M$ defined by the projection of the Levi-Civita connection ∇^L on TM. Then $\nabla = \nabla^Y \oplus \nabla^B$ defines a connection on TM which preserves the metric g. If we let T be the torsion tensor of this connection and S the difference tensor of ∇^L and ∇,

$$S = \nabla^L - \nabla.$$

One has

1) T takes values in $T^V M$ and it vanishes on $T^V M$.

2) For W_1, W_2 vertical

$$< S(W_1)W_2, \tilde{U}_1 > = \; - < S(W_1)\tilde{U}_1, W_2 > = \frac{1}{2} < A_{\tilde{U}_1}(W_1), W_2 >,$$

where A is the second fundamental form of the fibers.

3)

$$\begin{aligned} < S(\tilde{U}_1)W_1, \tilde{U}_2 > &= - < S(\tilde{U}_1)\tilde{U}_2, W_1 > = < S(W_1)\tilde{U}_1, \tilde{U}_2 > \\ &= \frac{1}{2} < \mathcal{R}(\tilde{U}_1, \tilde{U}_2), W_1 >, \end{aligned}$$

and all other components of $< S(\cdot)\cdot, \cdot >$ vanish.

It follows easily from these that if $\nabla^{L,x}$ is the Levi-Civita connection of $g_x = x^{-2}\pi^* g_B + g_Y$ and $S^x = \nabla^{L,x} - \nabla$, then

$$P^H S^x = x^2 P^H S, \quad P^V S^x = P^V S,$$

and hence the limit,

$$\lim_{x \to 0} \nabla^{L,x} = \nabla^L - P^H S = \nabla + P^V S$$

exists and is in uppertriangular form (with respect to the splitting) with the diagonal entry ∇. Consequently, its curvature is also in uppertriangular form with the diagonal entry $R = \nabla^2$. From this we deduce

Lemma A.3 *Let P be an $O(n)$-invariant polynomial on the Lie algebra $o(n)$, and R^x, $R = R^Y \oplus R^B$ the curvatures of $\nabla^{L,x}$, ∇ respectively. Then*

$$\lim_{x \to 0} P(R^x/2\pi i) = P(R/2\pi i) = P(R^Y/2\pi i)P(R^B/2\pi i).$$

Let γ be an arbitary vector bundle over M. Associated to γ is a vector bundle $\tilde{\gamma}$ over B whose (infinite dimensional) fiber $\tilde{\gamma}_p$ is the space of the smooth sections of $\gamma|_{\pi^{-1}(p)}$. There is an obvious functorial isomorphism between the space of smooth sections of γ and the space of sections of $\tilde{\gamma}$ which are smooth in the appropriate sense.

If γ has a connection ∇^{γ}, then we define the associated connection $\nabla^{\tilde{\gamma}}$ (sometime abbreviated $\tilde{\nabla}$) on $\tilde{\gamma}$ by

$$\nabla^{\tilde{\gamma}}_{U}\,\tilde{s} = \nabla^{\gamma}_{\tilde{U}} s, \tag{A.10}$$

where $\tilde{s}$ denotes the section s of γ, regarded as a section of $\tilde{\gamma}$. Clearly, the curvature $R^{\tilde{\gamma}}$ is given by

$$R^{\tilde{\gamma}}(U_1, U_2) = R^{\gamma}(\tilde{U}_1, \tilde{U}_2) - \nabla^{\gamma}_{\mathcal{R}(\tilde{U}_1, \tilde{U}_2)} \tag{A.11}$$

If γ has a hermitian inner product $<\ ,\ >$, then an inner product can be defined on $\tilde{\gamma}$ via $<\ ,\ >$ and dz, the smoothly varying volume forms on the fibers induced by g_Y,

$$(\tilde{s}_1, \tilde{s}_2)_p = \int_{\pi^{-1}(p)} < s_1, s_2 > dz.$$

Even if ∇^{γ} is unitary, $\nabla^{\tilde{\gamma}}$ need not preserve this inner product. For this, one can correct it by subtracting a term involving the mean curvature of the fiber Y. In fact, $\nabla^{\tilde{\gamma},u}_{U} = \nabla^{\tilde{\gamma}}_{U} - \frac{1}{4}\sum_i < A_{\tilde{U}}(e_i), e_i >$ is unitary.

A.3 Bismut-Cheeger's adiabatic limit formula

Suppose that M is a compact connected Riemannian manifold, which is oriented and spin (abbreviated spin manifold). Thus, the $SO(n)$-bundle of oriented orthonormal frames in TM is covered by a $Spin(n)$-bundle Q, so that the covering map restricts fiberwisely to the covering projection $Spin(n) \to SO(n)$. Since $Spin(n)$ is canonically embedded in $\mathbb{C}l(\mathbb{R}^n)$, the irreducible module F of $\mathbb{C}l(\mathbb{R}^n)$ gives rise to a hermitian vector bundle on M, also denoted by F (or $F(M)$ when there is a need to indicate the base manifold). The Clifford multiplication on F by a tangent vector is well-defined:

$$TM \longrightarrow Hom(F, F).$$

Since $SO(n)$ and $Spin(n)$ have the same Lie algebra, the Levi-Civita connection $\bigtriangledown$ of M lifts to Q, thus inducing a connection on F, again denoted by $\bigtriangledown$ (or sometimes $\bigtriangledown^F$). Let $\{e_i\}$ be a local orthonormal frame on M, the relation between the two connections is

$$\bigtriangledown_X f = \frac{1}{4}\sum_{i,j} < \bigtriangledown_X e_i, e_j > e_i e_j \cdot f, \qquad \text{(A.12)}$$

where f is a section of F and "$\cdot$" indicates Clifford multiplication.

The *Dirac operator* D acting on F is defined by

$$C^\infty(M, F) \to C^\infty(M, T^*M \otimes F) \to C^\infty(M, TM \otimes F) \to C^\infty(M, F),$$

with the first arrow given by the connection and the second the metric, and the third the Clifford multiplication. Locally,

$$D = e_i \bigtriangledown_{e_i} .$$

This is a first order self-adjoint elliptic differential operator whose symbol is given by the Clifford multiplication. Similarly, if ξ is a hermitian vector bundle over M with a unitary connection $\bigtriangledown^\xi$, one can define a Dirac type operator on $F \otimes \xi$,

$$D \otimes \xi = e_i(\bigtriangledown \otimes \bigtriangledown^\xi)_{e_i} = e_i \bigtriangledown_{e_i} \otimes 1 + e_i \otimes \bigtriangledown^\xi_{e_i}.$$

We call $D \otimes \xi$ the Dirac operator *coupled to the connection* $\bigtriangledown^\xi$ (or twisted Dirac operator). For the simplicity of notation, we will sometimes denote

$D \otimes \xi$ by D alone. This notion generalizes suitably to superconnections [BC2]. Thus suppose $\xi = \xi_+ \oplus \xi_-$ is $\mathbb{Z}_2$-graded with a unitary connection ∇^ξ preserving the grading. Then ∇^ξ extends to a supperconnection and any other supperconnection is of the form $\nabla^\xi + V$ with V an odd element of $\Lambda^*(M)\hat{\otimes}End(\xi)$. Write $V = \sum_{j=0}^n V_j$ as a sum of j-forms with values in $\mathrm{End}_{j+1}(\xi)$. We define the operator $\underline{V}$ on $F(M) \otimes \xi$ by

$$\underline{V} = \sum e_{i_1} \cdots e_{i_j} V_j(e_{i_1}, \ldots, e_{i_j}).$$

If we assume that V_j takes values in self-adjoint (respectively skew-adjoint) elements of $\mathrm{End}_{j+1}(\xi)$ for j even (respectively j odd), we can again form the self-adjoint operator

$$D \otimes \xi + \underline{V},$$

which we call the Dirac operator coupled to the supperconnection (∇^ξ, V).

The *η-function* of the Dirac type operator D is defined for Re $s \gg 0$ as

$$\eta(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} tr(De^{-tD^2}) dt.$$

In terms of the eigenvalues λ of D, $\eta(s)$ can be written as

$$\eta(s) = \sum_{\lambda \neq 0} \frac{sgn\lambda}{|\lambda|^s}.$$

The parity of $\dim M$ plays a role here. We note that when $\dim M$ is even, $\eta(s)$ vanishes identically. In fact, τ of (A.6) now gives a well-defined parallel section of End F, anti-commuting with D. It follows that pointwisely

$$tr(De^{-tD^2}) = tr(\tau De^{-tD^2}\tau) = -tr(De^{-tD^2}).$$

Thus for an even dimensional spin manifold, $tr(De^{-tD^2})$ is identically zero (pointwisely), consequently so is $\eta(s)$. On the other hand, for an odd dimensional spin manifold, we have the following local regularity result due to J.-M. Bismut and D. Freed, [BF].

Theorem A.1 *Pointwisely there is a uniform asymptotic expansion,*

$$tr(De^{-tD^2})(p,p) = \sum_{i=0}^{N} b_{\frac{1}{2}+i}(p)t^{\frac{1}{2}+i} + O(t^{\frac{1}{2}+N+1}), \qquad \text{(A.13)}$$

consequently $\eta(s)$ is holomorphic in Re $s > -2$.

Of particular interest is the value of $\eta(s)$ at $s = 0$, which is called the *η-invariant* of D, denoted $\eta(D)$. Its significance can be seen from the following result, [APS].

Theorem A.2 (Atiyah-Patodi-Singer theorem) *Let N be an even dimensional compact spin manifold with boundary M, ξ be a hermitian bundle on N with unitary connection ∇^ξ and curvature L^ξ. Assume that the metric on N and the connection on ξ are of product type near the boundary. Then*

$$ind(D_N^+) = \int_N \hat{A}(\frac{R^N}{2\pi i}) \wedge tr(e^{-\frac{L^\xi}{2\pi i}}) - \frac{\eta(D_M) + \dim(\ker D_M)}{2},$$

where D_N is the Dirac operator on N together with the Atiyah-Patodi-Singer boundary condition (Cf. Appendix B).

From here we define the reduced η-invariant

$$\bar{\eta}(D_M) = \frac{\eta(D_M) + \dim(\ker D_M)}{2}.$$

This is a spectral invariant and in general, it does not depend continuously on the metric. In the process of deformation, it has simple discontinuities caused by the crossing of zero spectrum. When a negative eigenvalue reaches zero, $\bar{\eta}$ will jump by 1, and when a positive eigenvalue reaches zero, $\bar{\eta}$ will jump by -1. Since $\bar{\eta}$ has only integer jumps, its modulo $\mathbb{Z}$ reduction will depend differentiably on the metric.

Now assume that M is the total space of a fibration (A.9). The rescaled metric g_x gives rise to a family of Dirac operators D_x. The limiting behaviour of $\bar{\eta}(D_x)$ (or $\eta(D_x)$) is refered to as the adiabatic limit. As for the existence of such limit, generally we have (Cf. [C2], [BC2])

Proposition A.3 *As a $\mathbb{R}/\mathbb{Z}$-valued function, the limit* $\lim_{x\to 0}\bar{\eta}(D_x)$ *exists.*

However, we are more interested in the question of when this limit actually exists in $\mathbb{R}$, and if it exists, how it relates to the other index-theoretic quantities. This is studied in the recent work of Bismut-Cheeger [BC2] in case D_Y is invertible, which we now summarize briefly.

Assume from now on that $\dim M$ is odd. Then from Lemma A.2, we have

$$F(M) = \pi^* F(B) \otimes F(Y).$$

Let ξ be a hermitian bundle on M with unitary connection ∇^ξ and curvature L^ξ and let ∇^u denote the connection on $F(M)\otimes\xi$ defined by

$$\nabla^u = \nabla^{F(B)} - \frac{1}{2}\sum < S(e_i)e_i, \cdot >,$$

where $\{e_i\}$ is a local orthonormal basis for $T^V M$. The Dirac operator along the fiber D_Y is defined as $D_Y = e_i \nabla_{e_i}$ where ∇ is the Euclidean connection on $T^V M$ defined in Section A.2. We have the following nice formula for D_x.

Lemma A.4 *If $\{f_\alpha\}$ is an orthonormal basis for TB, then*

$$D_x = x \sum_\alpha f_\alpha \nabla^u_{f_\alpha} + D_Y - \frac{x^2}{4} \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta). \tag{A.14}$$

For simplicity, we will denote $\tilde{D}_B = \sum_\alpha f_\alpha \nabla^u_{f_\alpha}$, $T = \sum_{\alpha \leq \beta} f_\alpha f_\beta T(f_\alpha, f_\beta)$.

Proof. D_x is defined via the lift of the Levi-Civita connection of g_x. (A.14) follows from a straightforward computation by virtue of (A.12) and the results of Section A.2 (see also [BC2]).

We now recall the definition of the Levi-Civita superconnection introduced by Bismut in [B]. The smooth sections of $F(Y) \otimes \xi$ can be viewed as the smooth sections of a vector bundle over B, as observed at the end of Section A.2. Denote this bundle by $H_\infty = H^+_\infty \oplus H^+_\infty$. The connection ∇^u induces a connection on H_∞, $\tilde{\nabla}^u$. This is a unitary connection on H_∞ with respect to the natural metric defined by the L^2-metric on the fibers of (A.9). If we wish to regard the operator D_x as acting on sections of $F(B) \otimes H_\infty$, we simply write $\tilde{\nabla}^u$ for ∇^u in (A.14). The Levi-Civita superconnection B_t on H_∞ is the superconnection

$$B_t = \tilde{\nabla}^u + t^{1/2} D_Y - \frac{c(T)}{4t^{1/2}}, \tag{A.15}$$

where $c(T) = \sum_{\alpha \leq \beta} dy^\alpha \, dy^\beta T(f_\alpha, f_\beta)$, and dy^α denotes the 1-form dual to f_α. It follows from (A.14) that $\frac{1}{x} D_x$ is the Dirac operator coupled to the Levi-Civita superconnection $B_{x^{-2}}$.

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The asymptotics of kernels associated to the Levi-Civita superconnection exhibit some remakable cancellations. The first one is expressed in the local index theorem for families, [B], [BF], which says that when $\dim Y = 2l$ is even

$$\lim_{t\to 0} tr_s(e^{-B_t^2}) = \frac{1}{(2\pi i)^l}\int_Y \hat{A}(iR^Y)tr(e^{-L^\xi}), \tag{A.16}$$

or when $\dim Y = 2l-1$ is odd

$$\lim_{t\to 0} tr^{odd}(e^{-B_t^2}) = \frac{1}{(2\pi i)^l}\int_Y \hat{A}(iR^Y)tr(e^{-L^\xi}), \tag{A.17}$$

where tr^{odd} indicates taking the odd form part of tr.

Essential to our discussion are the other two cancellation results, [BC2], [BGS], which state that when $\dim Y = 2l$ is even

$$tr_s[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{1/2}) \text{ as } t \to 0, \tag{A.18}$$

or when $\dim Y = 2l-1$ is odd

$$tr^{even}[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{1/2}) \text{ as } t \to 0, \tag{A.19}$$

where tr^{even} indicates taking the even form part of tr. We now show that the expressions in (A.18), (A.19) are also well behaved for the large time. We claim that for $\dim Y = 2l$,

$$tr_s[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{-1}) \text{ as } t \to \infty, \tag{A.20}$$

and for $\dim Y = 2l-1$,

$$tr^{even}[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] = O(t^{-1}) \text{ as } t \to \infty. \tag{A.21}$$

In fact, let us define $\varphi_t: \ \Lambda^*(B) \to \Lambda^*(B)$ by

$$\varphi_t|_{\Lambda^1(B)} = \frac{1}{\sqrt{t}}.$$

Then the key observation is

$$tr^{even}[(D^Y+\frac{c(T)}{4t})e^{-(\tilde{\nabla}^u+\sqrt{t}D^Y-\frac{c(T)}{4\sqrt{t}})^2}] = \varphi_t(tr^{even}[(D^Y+\frac{c(T)}{4})e^{-t(\tilde{\nabla}^u+D^Y-\frac{c(T)}{4})^2}]),$$

and

$$tr_s[(D^Y+\frac{c(T)}{4t})e^{-(\tilde{\nabla}^u+\sqrt{t}D^Y-\frac{c(T)}{4\sqrt{t}})^2}] = \varphi_t(tr_s[(D^Y+\frac{c(T)}{4})e^{-t(\tilde{\nabla}^u+D^Y-\frac{c(T)}{4})^2}]).$$

For t large, $(D^Y+\frac{c(T)}{4})e^{-t(\tilde{\nabla}^u+D^Y-\frac{c(T)}{4})^2}$ is uniformly bounded. Thus, except the 0-form component, $\varphi_t(tr^{even}[(D^Y+\frac{c(T)}{4})e^{-t(\tilde{\nabla}^u+D^Y-\frac{c(T)}{4})^2}])$ decays like t^{-1}. But its 0-form component is just $tr(D^Y e^{-t(D^Y)^2})$ which decays exponentially.

As for the other parity, the same reasoning shows that we just have to worry about the 1-form component, which is by Duhamel's principle

$$\sqrt{t}\, tr_s(D^Y \tilde{\nabla}^u D^Y e^{-t(D^Y)^2}) = \sqrt{t}\, tr_s[(\tilde{\nabla}^u D^Y)D^Y e^{-t(D^Y)^2}]$$

exponentially decaying as well.

By virtue of (A.18 - A.21) we now define a differential form on B, the $\hat{\eta}$ form

$$\hat{\eta} = \begin{cases} \frac{1}{\sqrt{\pi}}\int_0^\infty tr_s[(D_Y+\frac{c(T)}{4t})e^{-B_t^2}]\frac{dt}{2t^{1/2}} & \text{if } \dim Y = 2l \\ \\ \frac{1}{\sqrt{\pi}}\int_0^\infty tr^{even}[(D_Y+\frac{c(T)}{4t})e^{-B_t^2}]\frac{dt}{2t^{1/2}} & \text{if } \dim Y = 2l-1 \end{cases}.$$

For example, the first integral is convergent at 0 because of (A.18), and convergent at ∞ because of (A.20).

The $\hat{\eta}$ should be viewed as a higher dimensional analogue of the η-invariant. In fact, when $\dim Y = 2l - 1$ is odd, its 0-form component is exactly the η- invariant of the Dirac operator along the fibre. We also point out that when $\dim Y = 2l$ the 1-form componet of $\hat{\eta}$ represents the Quillen connection of the determinant line bundle $\det D_Y$, which is interpreted by Witten as the *covariant anomaly*, [W]; see also [BF], [C2], and [F]. We now normalize $\hat{\eta}$ by defining

$$\tilde{\eta} = \begin{cases} \sum \frac{1}{(2\pi i)^j}[\hat{\eta}]_{2j-1} & \text{if } \dim Y = 2l \\ \sum \frac{1}{(2\pi i)^j}[\hat{\eta}]_{2j} & \text{if } \dim Y = 2l-1 \end{cases} .$$

Here we decompose the odd (respectively even) form $\hat{\eta}$ into its homogeneous components $[\hat{\eta}]_{2j-1}$ (respectively $[\hat{\eta}]_{2j}$).

Theorem A.3 (Bismut-Cheeger) *Assume that D_Y, the Dirac operator along the fiber, is always invertible, then the limit $\lim_{x\to 0}\bar{\eta}(D_x) = \lim_{x\to 0}\frac{1}{2}\eta(D_x)$ exists in $\mathbb{R}$ and*

$$\lim_{x\to 0}\bar{\eta}(D_x) = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \tilde{\eta}.$$

Moreover,

$$d\tilde{\eta} = \int_Y \hat{A}(\frac{R^Z}{2\pi}) \wedge tr(e^{-L^t/2\pi i}).$$

We present an outline of the proof given in [BC2], emphasising on the ideas and techniques we will be using. One is refered to the original paper for further details.

Outline of Proof. As we noticed above, the parity of $\dim B$ (or equivalently, that of $\dim Y$, since we now always assume that $\dim M$ is odd) makes only a slight difference in the discussions, which are otherwise parallel. Thus we assume the $\dim B$ is odd.

1) There is a uniform lower bound for the smallest eigenvalue of D_x. Set $E_x = \sum_\alpha f_\alpha(\nabla^u_{f_\alpha} - \frac{x}{8} f_\beta T(f_\alpha, f_\beta))$ then (A.14) implies

$$D_x^2 = (D_Y)^2 + x[D_Y, E_x] + x^2 E_x^2.$$

By the results of Section A.2, one easily sees that the operator $[D_Y, E_x]$ is a first order differential operator which acts fibrewise. Thus if x is small enough

$$D_x^2 \geq (D_Y)^2 + x[D_Y, E_x] \geq \frac{1}{2}(D_Y)^2,$$

where the last inequality follows from Garding's inequality for D_Y and the assumption that D_Y is always invertible.

2) For $t \in (0, T]$, we have uniform convergence,

$$\lim_{x \to 0} tr(D_x e^{-D_x^2 t}) = \frac{\sqrt{\pi}}{(2\pi i)^k} \int_B \hat{A}(iR^B) tr^{even}((D_Y + \frac{c(T)}{4t}) e^{-B_t^2}) + O(x(1+T^N)) \tag{A.22}$$

for some N.

The main technical tool here is the so called Getzler's transformation [G], [BC2]. We first introduce an auxiliary Grassman variable, z (i.e. $z^2 = 0$), which is *odd*; see [BC2] for the reason. Once z has been introduced, instead of considering $tr(D_x e^{-D_x^2 t})$, we can consider the part of $tr(e^{-D_x^2 + z D_x \sqrt{t}})$ which involves z, denoted by $tr_z(e^{-D_x^2 + z D_x \sqrt{t}})$. In fact, an application of Duhamel's principle yields

$$e^{-D_x^2 + z D_x \sqrt{t}} = e^{-D_x^2 t} + z\sqrt{t} D_x e^{-D_x^2 t}.$$

Thus,

$$tr_z(e^{-D_x^2+zD_x\sqrt{t}}) = z\sqrt{t}\,tr(D_x e^{-D_x^2 t}).$$

To define Getzler's transformation, one first has to localize the problem. The estimates of [C2], Section 3 and a straightforward generalization of the arguments of [C2], Section 4 show that we can pretend that our base is $\mathbb{R}^{2l-1}$ with a metric flat outside a compact set, the error term being exponentially decaying in x as $x \to 0$. Similarly, we can assume that the bundle is isometrically a product on that region.

By definition, Getzler's transformation, $G_{\delta^{1/2}}$, of a function (usually the square) of a Dirac operator means that we conjugate this operator by the coordinate change in $\mathbb{R}^{2l-1}$,

$$y^\alpha \to \delta^{1/2} y^\alpha,$$

and *make the replacement*

$$f_\alpha \to \delta^{-1/2} f_\alpha$$

in the Clifford variables and *change the Clifford multiplication*, $\cdot$, to $\cdot_\delta$, which satisfies

$$\delta^{-1/2} f_\alpha \cdot_\delta \delta^{-1/2} f_\beta + \delta^{-1/2} f_\beta \cdot_\delta \delta^{-1/2} f_\alpha = -2\delta_{\alpha\beta}.$$

The main algebraic ingredient in the application of Getzler's transformation is a Lichnerowicz type formula, which contains no singular terms of δ, Cf. [G], [BC2], and [D]. In [G], the usual Lichnerowicz formula implies that Getzler's transformation of the square of a Dirac operator converges to a Hermite operator (or harmonic oscillator in Physicist's term), whose heat kernel is explicit. For $D_x^2 t - zD_x\sqrt{t}$, one does have a Lichnerowicz type formula; see (4.53) of [BC2]. However, it contains a singular term. What is remarkable is that this can be removed by a simple algebraic manipulation. Thus, one first conjugates the operator $D_x^2 t - zD_x\sqrt{t}$ by

$$e^{\frac{zy^\alpha f_\alpha}{2(tx^2)^{1/2}}} = 1 + \frac{zy^\alpha f_\alpha}{2(tx^2)^{1/2}},$$

and then applies Getzler's transformation $G_{(tx^2)^{1/2}}$. The formula (4.68) of [BC2] says that

$$\lim_{x\to 0} G_{(tx^2)^{1/2}}[e^{\frac{zy^\alpha f_\alpha}{2(tx^2)^{1/2}}}(D_x^2 t - zD_x\sqrt{t})e^{-\frac{zy^\alpha f_\alpha}{2(tx^2)^{1/2}}}] = \mathcal{H} + B_t^2 - z(\sqrt{t}D_Y + \frac{c(T)}{4\sqrt{t}}),$$

where $\mathcal{H}$ is the generalized Hermite operator on $\mathbb{R}^{2l-1}$ which was considered by Getzler in [G]. Now the same arguments as in [G] show that

$$\begin{aligned} tr_z(e^{-D_x^2 t + zD_x\sqrt{t}}) &\to tr_z[e^{-\mathcal{H} - B_t^2 + z\sqrt{t}(D_Y + \frac{c(T)}{4t})}] \\ &= \frac{\sqrt{\pi}}{(2\pi i)^l}\hat{A}(iR^B) \wedge z\sqrt{t}\, tr_s[(D_Y + \frac{c(T)}{4t})e^{-B_t^2}] \end{aligned}$$

uniformly, for $t \in [\delta,\ T]$. Or

$$\frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) \to \frac{1}{(2\pi i)^k}\int_B \hat{A}(iR^B) tr_s((D_Y + \frac{c(T)}{4t})e^{-B_t^2})\frac{1}{2t^{1/2}}. \quad \text{(A.23)}$$

To see the convergence is actually uniform for $t \in [0, T]$, it suffices to show that the coefficient, c_x, in the small time expansion,

$$\frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) = c_x + O(t),$$

approaches a limit and that the remainder term remains uniformly bounded by ct, independent of x. To this end, note that

$$\begin{aligned} z\frac{1}{2\sqrt{\pi t}} tr(D_x e^{-D_x^2 t}) &= \frac{1}{2\sqrt{\pi} t^{3/2}} tr_z(e^{-t(D_x^2 - zD_x)}) \\ &= \frac{1}{2\sqrt{\pi} t^{3/2}} tr_z e^{-t\mathcal{L}_x}, \end{aligned} \quad \text{(A.24)}$$

where $\mathcal{L}_x = G_x[e^{\frac{zy^\alpha f_\alpha}{2x}}(D_x^2 t - zD_x\sqrt{t})e^{-\frac{zy^\alpha f_\alpha}{2x}}]$ is Getzler's transformation of $D_x^2 - zD_x$, except that we desingularize it by first conjugating the operator. By (4.68) of [BC2] (taking $u = 1,\ \epsilon^{1/2} = x$), $\mathcal{L}_x$ is smooth family of elliptic operators down to $x = 0$. (In fact, $\mathcal{L}_0 = \mathcal{H} + B_1^2 - z(D_Y + \frac{c(T)}{4})$). Since the asymptotic expansion of $tr(e^{-t\mathcal{L}_x})$ at the origin depends only on the local symbol of $\mathcal{L}_x$, it follows that the coefficients in this expasion converge as $x \to 0$, to those in the expansion for $tr(e^{-t\mathcal{L}_0})$ and that the remainder terms are bounded by ct independent of x. Puting this together with (A.23) and (A.24) we see that $c_x \to c_0$ and that the remainder is bounded by ct independent of x.

Remark The above argument contains a proof of (A.18) and (A.19); see [BC2].

To have a remainder estimate for (A.23), notice that a straightforward generalization of the argument of [C2] shows that, for fixed t, there is an expansion

$$tr(D_x e^{-tD_x^2}) \sim \sum_{i=0}^{2k-1} a_i\, x^{2i-(2k-1)}, \tag{A.25}$$

whose coefficients and remainder term are bounded by T^N for say $t \in [1, T]$ and some N. Then (A.23) implies that (A.25) does not contain singular terms. The desired estimate follows.

3) Large time contribution is negligible as $x \to 0$.

This is shown by using 1) and the finite propagation speed technique [CGT].

From the proof we see that

Corollary A.1 *Without the assumption that D_Y is invertible, one can still find a small positive number α such that*

$$\lim_{x\to 0}\frac{1}{2}\eta(D_x)=\int_B \hat{A}(\frac{R^B}{2\pi})\wedge\tilde{\eta}+\lim_{x\to 0}\frac{1}{2\sqrt{\pi}}\int_{x^{-\alpha}}^{\infty}t^{-1/2}tr(D_x e^{-D_x^2 t})dt, \quad (A.26)$$

provided either one of the limits exists.

A.4 Analysis on manifolds with boundary

There are two ways to describe pseudo-differential operators, the symbol description and the Schwartzian kernel description. The two are related by the Fourier transform. The advantage of the Schwartzian kernel description over the symbol description is that it allows a precise description of the singularities of the Schwartzian kernel of pseudo-differential operators. This will be important for the pseudo-differential operators on manifolds with boundary. The Schwartzian kernel of a pseudo-differential operator gives rise to a *conormal distribution* [Hor] [Me].

Let X^n be a C^∞ manifold, E a C^∞ vector bundle on X, and $Y^k \subset X$ a (closed) submanifold. Let $\mathcal{V}=\mathcal{V}(Y)$ denote the space of vector fields on X which are tangent to Y at Y. The space of $\mathcal{V}$-differential operators, $\mathrm{Diff}_{\mathcal{V}}(X,E)$, is generated by $\mathcal{V}$, i.e. elements of $\mathrm{Diff}_{\mathcal{V}}(X,E)$ are sums of products of elements in $\mathcal{V}$. The space $I(X,Y;E)$ of distribution sections of E, conormal with respect to Y, is defined to be the set of distribution sections $u\in\mathcal{D}'(X,E)$ such that $V_1\cdots V_k u\in H^s(X)$, $V_i\in\mathcal{V}(Y)$ for some real number $s=s(u)$ and all k. Here $H^s(X)$ denotes the Sobolev space on X. Thus $I(X,Y;E)$ are distributions which have fixed regularities under the action of $\mathcal{V}$-differential operators. The following theorem (see [Hor]) says that such a distribution is locally defined by a symbol via the oscillatory integral.

Theorem A.4 *$u \in I(X,Y;E)$ iff $\phi_j u \in I(X,Y;E)$ for every ϕ_j in a partition of unity on X. If X is an open set in $\mathbb{R}^n$ and Y is defined by $x' = (x_1, \cdots, x_k) = 0$ while $E = X \times \mathbb{C}^N$, then any $u \in I^m(X,Y;E)$ with compact support is of the form*

$$u(x) = \int e^{i<x',\xi'>} a(x'', \xi') d\xi',$$

where $a \in S(\mathbb{R}^{n-k} \times \mathbb{R}^k; \mathbb{C}^N)$, the space of symbols. Conversely, every u of this form is in $I(X,Y;E)$.

The order of symbol gives a filtration on the space of symbols which in term gives a filtration on the space of conormal distribution. The significance of this filtration lies in the following [Hor]

Theorem A.5 *We have the following short exact sequence*

$$0 \to I^{m-1}(X,Y;\Omega_X^{1/2} \otimes E) \to I^m(X,Y;\Omega_X^{1/2} \otimes E) \xrightarrow{\sigma} S^m(N(Y), \Omega_{N(Y)}^{1/2} \otimes E) \to 0,$$

where $N(Y)$ is the normal bundle of Y in X and $\Omega^{1/2}$ denotes the $\frac{1}{2}$-density bundle. σ is called the principal symbol map.

The space of conormal distributions behaves nicely under the action of pseido-differential operators (Cf. [Hor]).

Theorem A.6 *If $u \in I^m(X,Y;E)$, and $P \in \Psi^{m'}(X,Y;F)$ is properly supported, then $Pu \in I^{m+m'}(X,Y;F)$ and the principal symbol is that of u multiplied by the restriction to $N(Y)$ of the principal symbol of P.*

Finally, we note that conormal distributions have singularities only on the submanifold Y and then only along the normal directions (this can be made precise by the notion of wave front set of a distribution).

Now suppose X is a C^∞-manifold with boundary. The boundary ∂X is therefore a closed submanifold of X. In this case, the $\mathcal{V}$-differential operators are called the totally characterictic differential operators. Regarding to the behavior at the boundary, one has three types of distributions. Thus $C^{-\infty}(\overset{\circ}{X})$ are those which have no control at the boundary at all. In general, we wish to consider only the distributions having "slow growth" at the boundary. The space of extendible distributions is defined as the dual of the space of smooth functions on X vanishing to infinite order at the boundary,

$$C^{-\infty}(X) = (\dot{C}^\infty(X))'$$

and the space of supported distributions is defined as the dual of the space of smooth functions on X (up to boundary),

$$\dot{C}^{-\infty}(X) = (C^\infty(X))'.$$

The space of almost regular functions

$$\mathcal{A}(X) = I(X \cup X, \partial X)$$

and its supported counterpart

$$\dot{\mathcal{A}}(X) = I(X \cup X, \partial X) \cap \dot{C}^{-\infty}(X)$$

play an important role here. Roughly speaking, $\mathcal{A}(X)$ is the space of functions smooth in the interior of X and along ∂X, singular only along the normal direction of ∂X. We mention that the relation of $\mathcal{A}(X)$ to totally characteristic operators is similar to the relation of $C^\infty(X)$ to the pseudo-differential operators when X is a compact manifold.

In dealing with regularity on manifold with boundary, of equal importance is the dual of $\dot{\mathcal{A}}(X)$, denoted $\mathcal{A}'(X)$. Intuitively, $\mathcal{A}'(X)$ consists of distributions having normal derivatives on ∂X of all orders, thus suggesting the following [M2]

Theorem A.7 *On any C^∞-manifold X with boundary,*

$$\mathcal{A}'(X) \cap \mathcal{A}(X) = C^\infty(X).$$

A more workable description of the space $\mathcal{A}'(X)$ is furnished by the Mellin transform. For simplicity, we assume that $X = [0, \infty)$. The Mellin transform of a smooth function u on $[0, \infty)$ is defined as

$$u_M(z) = \int_0^\infty x^{z-1} u(x) dx$$

if Re $z > 0$. This can be extended to distributions ($C^{-\infty}$ or $\dot{C}^{-\infty}$) by duality. The normalized Mellin tranform is just $\frac{u_M}{\Gamma(z)}$. In general, they are defined only for Re z large.

Proposition A.4 *$\mathcal{A}'(X)$ can be characterized as those distributions u whose normalized Mellin transform $\frac{u_M}{\Gamma(z)} \in \dot{C}^{-\infty}(X)$ extends to an entire function of z.*

See [M1] for its proof.

Appendix B

A proof of the Families Index Theorem for manifolds with boundary

The Families Index Theorem for manifolds with boundary is proved in [BC2] by the cone method. There the singularity is essential. Here, with additional assumption, we give a proof along the line of Atiyah-Patodi-Singer [APS]. The Atiyah-Patodi-Singer boundary condition plays an important role.

To be more precise, suppose we are given the following data:

a) A fibration of compact manifolds $Z^{2k} \longrightarrow N \longrightarrow B$ with base B closed and fibre Z^{2k} having dimension $2k$ and nonempty boundary $\partial Z = Y$. Associated is then the boundary fibration $Y \longrightarrow M \longrightarrow B$ where $M = \partial N$.

b) A splitting $TN = T^H N \oplus T^V N$ with $T^V N$ the vertical subspace on which a metric g^Z is given such that g^Z becomes a product metric near ∂N. As in [B], we can define connection ∇^V on $T^V N$ with curvature R^Z (see also Section 1.2).

c) A hermitian bundle ξ with metric and compatible connection whose curvature is L^ξ. These are of product type near ∂N.

With these data (and additional spin conditions if necessary), we can consider D^Z, the family of Dirac operators along Z. Let D_Y denote its tangential part near the boundary. We give a proof of the Families Index Theorem for manifolds with boundary in the following case.

Theorem B.1 *Assume that* $\ker D_Y$ *is a well-defined element in* $K^0(B)$. *Further, assume the boundary fibration has compact holonomy. Then* $Ind(D_+^Z, P_+)$, *the index bundle of* D_+^Z *with APS boundary condition [APS], is a well-defined element in* $K^0(B)$ *and its chern character is represented in cohomology by*

$$\int_Z \hat{A}(\frac{R^Z}{2\pi}) \wedge tr(e^{iL^\xi/2\pi}) - \bar{\eta} - \frac{1}{2} ch(\ker D_Y).$$

When B is a single point, $\hat{\eta}$ reduces to the usual η-invariant and the above recovers the Atiyah-Patodi-Singer theorem [APS].

Proof. The outline of proof is

1) We show it is possible to define a superconnection for our family of the APS boundary value problems and we establish a heat equation formula for $ch(Ind(D_+^Z, P_+))$.

2) In order to have local convergence in the interior, we have to use Levi-Civita superconnection of [B]. (See also [Q].) But for the computation near the boundary, it is much easier to use the unitary superconnection of [BF], [B]. We patch them on the cylinder.

3) To calculate the contribution from the mixed term, we let the cylinder become longer and longer and bring in the adiabatic calculation in the spirit of [C] and [BC2].

4) It is much easier to deal with numbers than with forms. We couple our forms with $\hat{A}$-genus and employ the meromorphic continuation method of [APS]. To get back to differential forms, we use a method of Bismut (in the second proof of [B]).

1) APS boundary problem and heat equation formula for the index

We first consider general first order differential operators of the split type. Suppose that

$$A: \qquad C^\infty(Z,E) \longrightarrow C^\infty(Z,F)$$

is a first order elliptic differential operator such that in a tubular neighborhood of the boundary $\partial Z = Y$ it is of the form

$$A = \sigma(\partial_u + B), \tag{B.1}$$

where σ is a bundle isomorphism given by the symbol of A in the direction du, and $B: \ C^\infty(Y,E) \longrightarrow C^\infty(Y,E)$ is a self-adjoint first order elliptic differential operator on Y, called the *tangential part* of A. Consider the spectral projection $P_\pm$ of B onto the eigenspaces with non-negative respectively negative eigenvalues. The Atiyah-Patodi-Singer boundary condition (abbreviated APS boundary condition) for the operator A is $P_+(\phi|_Y) = 0$.

In [APS], they showed that the following are true.

a) Let $C^\infty(Z, E; P_+)$ denote the smooth sections ϕ of E such that $P_+(\phi|_Y) = 0$. Then there is a linear operator $Q : C^\infty(Z, F) \longrightarrow C^\infty(Z, E; P_+)$ which is a parametrix for $A : C^\infty(Z, E; P_+) \longrightarrow C^\infty(Z, F)$. i.e., both $AQ-1, QA-1$ are smoothing operators.

b) Q extends to a continuous operator $H^{l-1} \to H^l$ for $l \geq 1$. Therefore

$$A : C^\infty(Z, E; P_+) \longrightarrow C^\infty(Z, F)$$

and

$$A_l : H^l(Z, E; P_+) \longrightarrow H^{l-1}(Z, F) \quad \text{for } l \geq 1$$

are Fredholm operators with the same null-space.

c) Denote $\mathcal{D} = (A, P_+)$. When $Z = \mathbb{R}_+ \times Y$ is a cylinder, the heat kernels $e^{-t\mathcal{D}^*\mathcal{D}}$, $e^{-t\mathcal{D}\mathcal{D}^*}$ are explicit in terms of the spectral resolution $\{\lambda, \varphi_\lambda\}$ of B:

$$\begin{aligned}
e^{-t\mathcal{D}^*\mathcal{D}} &= \sum_{\lambda \geq 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{\exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t})\} \varphi_\lambda(x) \otimes \varphi_\lambda(y) \\
&+ \sum_{\lambda < 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{\exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t}) \\
&+ \lambda e^{-\lambda(u+v)} \operatorname{erfc}(\frac{u+v}{2\sqrt{t}} - \lambda\sqrt{t})\} \varphi_\lambda(x) \otimes \varphi_\lambda(y), \\
e^{-t\mathcal{D}\mathcal{D}^*} &= \sum_{\lambda < 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{\exp(\frac{-(u-v)^2}{4t}) - \exp(\frac{-(u+v)^2}{4t})\} \varphi_\lambda(x) \otimes \varphi_\lambda(y) \\
&+ \sum_{\lambda \geq 0} \frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}} \{\exp(\frac{-(u-v)^2}{4t}) + \exp(\frac{-(u+v)^2}{4t}) \\
&- \lambda e^{+\lambda(u+v)} \operatorname{erfc}(\frac{u+v}{2\sqrt{t}} + \lambda\sqrt{t})\} \varphi_\lambda(x) \otimes \varphi_\lambda(y),
\end{aligned}$$

where erfc is the complimentary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du.$$

Now suppose that Z is parametrized by a closed manifold B, i.e., Z is a typical fiber of the fibration (A.9). Correspondingly, we assume that a smooth family A_y of the form (B.1) has been given on Z, where $y \in B$ (we sometimes suppress the subscript). We want to study the parameter dependence of the APS boundary problems. We begin with that of the APS boundary conditions.

Lemma B.1 *Assume that* $\dim\ker B_y$ *is constant, where* B_y *is the tangential part of* A_y *(Cf. (B.1)). Then* P_+ *is a smooth family of pseudodifferential operators of order 0. Consequently,* $\mathcal{D} = (A, P_+)$ *is a smooth family of APS boundary value problems. The same for* $\mathcal{D}^* = (A^*, P_-)$.

Proof. Since B_y is a smooth family of self-adjoint elliptic differential operators, by the compactness of B and the assumption, we can find $\delta > 0$, such that for all $y \in B$,

$$spec(B_y) \cap \{|z| \leq \delta\} = \{0\}.$$

Consider the Dunford-Schwartz integral

$$B_+^{-1} = \frac{1}{2\pi i} \int_{\Gamma_+} \lambda^{-1}(B_y - \lambda)^{-1} d\lambda,$$

where Γ_+ is the counterclockwise contour:

$$\arg\lambda = \pm\pi/4 \text{ if } |\lambda| \geq \delta, \text{ or } |\lambda| = \delta \text{ if } \pi/4 \leq \arg\lambda \leq 7\pi/4.$$

By the standard theory, B_+^{-1} is a smooth family of pseudodifferential operators of order -1. Consequently, $P_+ = BB_+^{-1}$ is a smooth family of pseudodifferential operators of order 0. Q.E.D.

Next we address the existence of the index bundle.

Lemma B.2 *With the same hypothesis as above, $H^l(Z, E; P_+)$, $H^{l-1}(Z, F)$ are smooth family of Hilbert spaces and*

$$\mathcal{D}_y : H^l(Z, E; P_+) \to H^{l-1}(Z, F)$$

is a smooth family of Fredholm operators. Thus $\mathcal{D}$ has a well-defined index bundle as an element of $K^0(B)$.

Proof. The statement about $H^{l-1}(Z, F)$ is trivial. To prove that of $H^l(Z, E; P_+)$, we fix a $y_0 \in B$ and construct a smooth family of invertible operators

$$\tilde{U}(y) : H^l(Z_y, E; P_+^y) \to H^l(Z_{y_0}, E; P_+^{y_0}).$$

In fact, since P_+^y is a smooth family of projections, by Lemma 2.12, there exists a smooth family $U(y)$ of invertible operators (on $L^2(y)$) such that

$$U(y) P_+^y U(y)^{-1} = P_+^{y_0}. \tag{B.2}$$

By the explicit formula there, one sees that $U(y)$ is actually pseudodifferential of order 0. Now we want to extend $U(y)$ to $\tilde{U}(y) : H^l(Z_y) \to H^l(Z_{y_0})$. Then it is clear that from (B.2) $\tilde{U}(y)$ restricts to

$$H^l(Z_y, E; P_+^y) \to H^l(Z_{y_0}, E; P_+^{y_0}).$$

To extend $U(y)$, one first deform $U(y)$ to the identity operator and then extend $U(y)$ to be the deformation on the cylinder and identity in the interior. Q.E.D.

By a) and b), $\mathcal{D}$ and $\mathcal{D}^*$ are elliptic boundary problems. Set

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix}.$$

We construct the heat kernel of $\mathcal{A}^2$,

$$e^{-t\mathcal{A}^2} = \begin{pmatrix} e^{-t\mathcal{D}^*\mathcal{D}} & 0 \\ 0 & e^{-t\mathcal{D}\mathcal{D}^*} \end{pmatrix}.$$

Proposition B.1 $e^{-t\mathcal{A}^2}$ *is a smooth family of heat kernels parametrized by* B.

Proof. We first study the resolvent $(\mathcal{A}^2 - \lambda)^{-1}$, showing that it depends smoothly on $y \in B$ as L^2-operators. To do this, we will construct a nice parametrix by patching together the interior parametrix with the boundary parametrix. Let Q_2 be the Green's operatores for $(\mathcal{A}^2 - \lambda)$ on the double of Z. Then it follows from the standard theory that Q_2 as a pseudodifferential operator of order -2 depends smoothly on $y \in B$. We now study the corresponding operator Q_1 for $(\mathcal{A}^2 - \lambda)$ on the half cylinder $Y \times \mathbb{R}_+$.

To solve $(\mathcal{A}^2 - \lambda)f = g$, we expand f and g in terms of the spectral resolution $\{\lambda_1,\ \phi_{\lambda_1}\}$ of B:

$$f(z,u) = \sum f_{\lambda_1}(u)\phi_{\lambda_1}(z), \quad g(z,u) = \sum g_{\lambda_1}(u)\phi_{\lambda_1}(z).$$

We must now solve

$$\begin{cases} -\frac{d^2 f_{\lambda_1}}{du^2} + (\lambda_1^2 - \lambda) f_{\lambda_1} = g_{\lambda_1} \\ f_{\lambda_1}(0) = 0 \text{ if } \lambda_1 \geq 0 \text{ or } \frac{df_{\lambda_1}}{du}(0) + \lambda_1 f_{\lambda_1}(0) = 0 \text{ if } \lambda_1 < 0 \end{cases}.$$

We take the explicit solution

$$f_{\lambda_1}(u) = -\frac{1}{2\bar{\lambda}}[\int_0^u g(v) e^{\bar{\lambda}(v-u)}\, dv + \int_u^{+\infty} g(v) e^{\bar{\lambda}(v-u)}\, dv] + C,$$

where $\bar{\lambda} = \sqrt{\lambda_1^2 - \lambda}$ with $\mathrm{Re}\bar{\lambda} > 0$ and

$$C = \begin{cases} \frac{1}{2\bar{\lambda}} \int_0^{+\infty} g(v) e^{-\bar{\lambda} v} dv \text{ if } \lambda_1 \geq 0 \\ (\frac{1}{\bar{\lambda}} + \frac{1}{2\lambda_1}) \int_0^{+\infty} g(v) e^{-\bar{\lambda} v} dv \text{ if } \lambda_1 < 0 \end{cases}.$$

It follows that $Q_1 = Q_1' + Q_1''$, where Q_1' is given by convolution in the u-variable with

$$P(t) = \frac{\epsilon(t)}{2\sqrt{B^2 - \lambda}} e^{-t\sqrt{B^2-\lambda}} - \epsilon(-t) \frac{e^{t\sqrt{B^2-\lambda}}}{2\sqrt{B^2 - \lambda}},$$

where $\epsilon(t)$ is the characteristic function of the nonnegative line, and Q_1'' has as kernel that of

$$\epsilon(t) \frac{e^{-t\sqrt{B^2-\lambda}}}{\sqrt{B^2 - \lambda}} P_+ + \epsilon(t)(\frac{e^{-t\sqrt{B^2-\lambda}}}{2\sqrt{B^2 - \lambda}} + \frac{e^{-t\sqrt{B^2-\lambda}}}{2B}) P_-.$$

Hence, Q_1 as L^2-operators depends smoothly on $y \in B$. Moreover, the kernel of Q_1 is smooth in all variables away from the diagonal and the boundary.

Now let ϕ's and ψ's be as [APS] (see also p.141). Put $Q = \phi_1 Q_1 \psi_1 + \phi_2 Q_2 \psi_2$. Then Q is a smooth family of L^2-operators such that $(\mathcal{A}^2 - \lambda)Q - \mathrm{Id}$ and $Q(\mathcal{A}^2 - \lambda) - \mathrm{Id}$ are smooth family of smoothing operators. This implies that $(\mathcal{A}^2 - \lambda)^{-1}$ is smooth in $y \in B$ as L^2-operators. Consequently, so is

$$e^{-t\mathcal{A}^2} = \frac{1}{2\pi i} \int_{\Gamma_+} e^{-t\lambda} (\mathcal{A}^2 - \lambda)^{-1}\, d\lambda.$$

To show that $e^{-t\mathcal{A}^2}$ is smooth as smoothing operators, we need the following lemma.

Lemma B.3 *Let $K: B \to \mathcal{L}(L^2(Z), L^2(Z))$ be a smooth family of L^2-operators. Suppose that for all $y \in B$, $K(y)$ is actually a smoothing operator with kernel $K(y, z, z')$. If, in addition, we have the uniform estimates*

$$|\partial_z^\alpha \partial^\beta z' K(y, z, z')| \leq C(\alpha, \beta),$$

then $K(y, z, z')$ is also smooth in y. i.e., K is actually a smooth family of smoothing operators.

By the construction as in [APS], one can check that $e^{-t\mathcal{A}^2}$ satisfies the hypothesis above, yielding the desired result.

We now restrict our attention to the family of Dirac operators. Let H^∞, $H^\infty_\pm$ be C^∞-sections of $F(TZ) \otimes \xi$, $F_\pm(TZ) \otimes \xi$. We regard H^∞, $H^\infty_\pm$ as C^∞-sections on B of infnite dimensional vector bundles. Recall that one can define a connection $\tilde{\nabla}$ on $H^\infty_\pm$ via the horizontal lift

$$\tilde{\nabla}_X h \stackrel{def}{=} \nabla_{X^H} h$$

If the holonomy of the fibration is compact, then

$$[\tilde{\nabla}, \mathcal{A}] = 0 \tag{B.3}$$

and $\tilde{\nabla} + \sqrt{t}\mathcal{A}$ on $H^\infty \otimes \Lambda^* TB$ is reduced to the unitary superconnection $\tilde{\nabla}^u + \sqrt{t}\mathcal{A}$. By (B.3), $(\tilde{\nabla} + \sqrt{t}\mathcal{A})^2 = \tilde{\nabla}^2 + t\mathcal{A}^2$ and $\tilde{\nabla}^2 = \tilde{R}$ is a 2-form valued first order differential operator acting fibrewise. By the nilpotency of the exterior algebra, $e^{-\tilde{R}}$ is well-defined and is a form-valued differential operator acting fibrewise. Thus the superconnection heat kernel can be defined as

$$e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A})^2} = e^{-\tilde{R}}(e^{-t\mathcal{A}^2}). \tag{B.4}$$

However, we only assume that the holonomy of the boundary fibration is compact, or the holonomy of the fibration is compact near the boundary. We have

Lemma B.4 *The superconnection heat kernel* $e^{-(\tilde{\nabla}^u+\sqrt{t}\mathcal{A})^2}$ *exists for small* t *and its supertrace* $tr_s e^{-(\tilde{\nabla}^u+\sqrt{t}\mathcal{A})^2}$ *is a smooth differential form of even degree which is closed and represents the renormalized chern character of the index bundle* $\overline{ch}(Ind\mathcal{D})$.

Notation $\overline{ch}(\xi) = tr(e^{-L^\xi}$ if L^ξ is the curvature operator of ξ. Thus $\overline{ch}$ differs from ch by the factor $\frac{1}{2\pi i}$.

Proof. Since the holonomy of the boundary fibration is compact, we can use (B.4) to define the superconnection heat kernel on the half cylinder while in the interior we use the superconnection heat kernel on the double. The two are patched together by the standard procedure, producing a parametrix with an error term exponentially small as $t \to 0$. The first claim follows then from Duhamel's principle. The proof of the second statement is essentially the same as in [B].

For later use, we shall make some computations for our superconnection heat kernels on the half cylinder, $N = M \times \mathbb{R}_+$. Locally we use the coordinates (y, z, u). Denote the pointwise kernel of $e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A})^2}$ by

$P_t(y,z,u,y',z',u')$ and set

$$K(t) = \int_0^\infty \int_Y tr_s(P_t)\, dz\, du.$$

Then by c) on p.132 , we find

$$K(t) = -\sum \frac{sgn\lambda}{2} \mathrm{erfc}(|\lambda|\sqrt{t}) \int_Y e^{-\tilde{R}} \varphi_\lambda \cdot \varphi_\lambda \, dz.$$

Thus

$$\begin{aligned} K'(t) &= \frac{1}{\sqrt{4\pi t}} \sum \lambda e^{-\lambda^2 t} \int_Y e^{-\tilde{R}} \varphi_\lambda \cdot \varphi_\lambda \, dz \\ &= \frac{1}{\sqrt{4\pi t}} tr(D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}). \end{aligned}$$

Note that

$$K(t) \to -\frac{1}{2}\overline{ch}(\ker D_Y) \quad \text{as } t \text{ goes to infinity.}$$

Moreover,

$$K(t) + \frac{1}{2}\overline{ch}(\ker D_Y) \to 0 \ \text{ exponentially.}$$

One deduce via integration by part,

Lemma B.5 *We have*

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty t^{s-\frac{1}{2}} tr(D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}) dt = -s \int_0^\infty t^{s-1}[K(t) + \frac{1}{2}\overline{ch}(\ker D_Y)] dt. \tag{B.5}$$

2) Levi-Civita superconnection.

From the manifold N, one can construct a closed manifold $X = N \cup_{\partial N} N$ by the doubling procedure. We shall use $\mathcal{A}$ again to denote the operators obtained by the doubling construction. On a closed manifold, of great interest is Bismut's Levi-Civita superconnection

$$\tilde{\nabla} + \sqrt{t}\mathcal{A} - \frac{c(T)}{4\sqrt{t}}.$$

The following cancellation result is proved in [B].

Theorem B.2 *The pointwise supertrace $tr_s e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{c(T)}{4\sqrt{t}})^2}$ of the heat kernel of the Levi-Civita superconnection has the following asymptotic expansion*

$$tr_s e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{c(T)}{4\sqrt{t}})^2} = \sum_{k=0}^{N} a_k t^{k/2} + O(t^{(N+1)/2}),$$

where $a_0 = \hat{A}(iR^Z) \wedge tr(e^{-L^\xi})$.

Remark Note that $(\tilde{\nabla} + \sqrt{t}\mathcal{A} - \frac{c(T)}{4\sqrt{t}})^2 = t(\frac{1}{\sqrt{t}}\tilde{\nabla} + \mathcal{A} - \frac{c(T)}{4t})^2$. If we let φ_t be the homomorphism from $\Lambda(B)$ into itself which sends a one-form ω to $\frac{1}{\sqrt{t}}\omega$, then

$$tr_s e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{c(T)}{4\sqrt{t}})^2} = \varphi_t(tr_s e^{-(\tilde{\nabla}+\mathcal{A}+\frac{c(T)}{4})^2}).$$

From this and the standard elliptic theory, one easily sees the existence of an asymptotic expansion. It also follows that the coefficients in the asymptotic expansion are local.

Thus the Levi-Civita superconnection yields local convergence, which in general does not happen to the unitary superconnection. However, to incorporate the boundary condition, we have to use the unitary superconnection. We now patch them up near the boundary.

Let $\varphi(u)$ be a smooth function on $[0,1]$ such that it is identically 1 on $[0,a]$ and 0 on $[1-a,1]$, where a is a small positive number. Consider the superconnection

$$\mathcal{B} = \tilde{\nabla} + \sqrt{t}\mathcal{A} - \frac{\varphi(u)T}{4\sqrt{t}}.$$

The same consideration as in showing Lemma B.4 shows that the superconnection heat kernel for $\mathcal{B}$ also exists for small t.

Lemma B.6 *$tr_s e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{\varphi(u)T}{4\sqrt{t}})^2}$ is also a representation of $\overline{ch}(Ind\mathcal{D})$. In fact, we have*

$$\frac{\partial}{\partial s} tr_s e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{s\varphi(u)T}{4\sqrt{t}})^2} = d(-tr_s[\frac{\varphi(u)T}{4\sqrt{t}} e^{-(\tilde{\nabla}+\sqrt{t}\mathcal{A}-\frac{s\varphi(u)T}{4\sqrt{t}})^2}]).$$

Now let e_1 be the heat kernel of $\tilde{\nabla} + \sqrt{t}\mathcal{A}$ on the cylinder constructed earlier, e_2 the heat kernel of $\mathcal{B}$ on the double of Z. The parametrix R for $e^{-\mathcal{B}^2}$, the superconnection heat kernel with the APS boundary condition, can be constructed by patching together e_1 with e_2. More precisely, let $\rho(b,c)$ denote an increasing smooth function of the real variable u, such that $\rho = 0$ for $u \leq b$ and $\rho = 1$ for $u \geq c$, and define smooth functions $\phi_1, \phi_2, \psi_1, \psi_2$ by

$$\phi_2 = \rho(1/4, 1/2), \qquad \psi_2 = \rho(1/2, 3/4),$$
$$\phi_1 = 1 - \rho(3/4, 1), \quad \psi_1 = 1 - \psi_2.$$

Note that $\phi_i = 1$ on the support of ψ_i. We regard these functions of u as functions on the cylinder $M \times [0,1]$ and then extend them to N in the obvious way. Finally we put

$$R = \phi_1 e_1 \psi_1 + \phi_2 e_2 \psi_2.$$

One checks that

$$e^{-B^2} - R = O(e^{-C/t}). \tag{B.6}$$

From this and the above lemma, we have

$$\begin{aligned}\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \overline{ch}(Ind\mathcal{D}) &= \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_0^1 \int_Y (tr_s P_t)(y,z,u)\phi_1(u)\,dz\,du \\ &+ \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_Z (tr_s e_2)\phi_2 + O(e^{-C/t}).\end{aligned}$$

Because of c) on p.132, it follows that, in the first term, we can replace ϕ_1 by 1 and $\int_0^1$ by $\int_0^\infty$ so that we end up with $\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge K(t)$. On the other hand

$$tr_s e_2 = \sum_{k=-n}^{N} a_k t^{k/2} + O(t^{(N+1)/2}).$$

Thus,

$$\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge K(t) \sim \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \overline{ch}(Ind\mathcal{D}) - \sum_{k\geq -n} t^{k/2} \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \int_Z a_k. \tag{B.7}$$

3) adiabatic limit.

Let us consider the differential form of even degree (Cf. (B.3))

$$\hat{\eta}_1(s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{s-\frac{1}{2}} tr(D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2})\,dt.$$

By virtue of (B.3), this integral is convergent for Re $s \gg 0$. Standard method showes that $\hat{\eta}_1(s)$ has a meromorphic extension to the s-plane. It is possible that $s = 0$ might be a simple pole for $\hat{\eta}_1(s)$. We thus define $\hat{\eta}_1(0)$ to be the finite part of $\hat{\eta}_1(s)$ at $s = 0$:

$$\hat{\eta}_1(0) = (s\hat{\eta}_1(s))'|_{s=0} = (\hat{\eta}_1(s) - \frac{Res_{s=0}\hat{\eta}_1(s)}{s})|_{s=0}.$$

The foregoing discussion together with (B.5) implies that $\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \hat{\eta}_1(s)$ is actually regular at $s = 0$ and its value at $s = 0$ is given by

$$\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \hat{\eta}_1(0) = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge [\int_Z a_0 - \frac{1}{2}\overline{ch}(\ker D_Y) + \overline{ch}(Ind\mathcal{D})],$$

or

$$\int_B \hat{A}(\frac{R^B}{2\pi}) \wedge \overline{ch}(Ind\mathcal{D}) = \int_B \hat{A}(\frac{R^B}{2\pi}) \wedge [\int_Z a_0 - \frac{1}{2}\overline{ch}(\ker D_Y) + \hat{\eta}_1(0)]. \quad \text{(B.8)}$$

Note that if we replace D_+^Z by $D_+^Z \otimes \pi^*\zeta$, where ζ is a vector bundle on B, we have the same formula (B.8) except that every integrand be wedged by $ch(\zeta)$. Since $ch: K^0(B) \to H^{2*}(B)$ is a rational isomorphism, we have the following consequences:

1) $\mathrm{Res}_{s=0}\hat{\eta}_1(s)$ is exact. This justifies the way we plug in $s = 0$ in the above formulas.

2) $\overline{ch}(Ind\mathcal{D})$ is represented by

$$\int_Z a_0 - \frac{1}{2}\overline{ch}(\ker D_Y) - \hat{\eta}_1(0).$$

Now by the remark following Theorem B.2, we have

$$a_0|_{Z-Y\times[0,1]} = \hat{A}(iR^Z) \wedge tr(e^{-L^t}).$$

Thus

$$\int_Z a_0 = \int_{Z-Y\times[0,1]} \hat{A}(iR^Z) \wedge tr(e^{-L^t}) - \frac{1}{2}\overline{ch}(\ker D_Y) - \hat{\eta}_1(0) + \int_{Y\times[0,1]} a_0.$$

It remains to identify $\int_{Y\times[0,1]} a_0$. We shall calculate it by the adiabatic limit method. The starting point is that instead of using metric $du^2 + g$ on the cylinder, we use $\delta^{-2}du^2 + g$. Then the whole discussion goes through and in fact we observe that

1) The index bundle $Ind\mathcal{D}$ remains unchanged. This follows from the charaterization of $\ker\mathcal{D}$ and $\ker\mathcal{D}^*$ in [APS] as the L^2-solutions repectively extended L^2-solutions on the elongation manifold.

2) In (B.6), $O(\cdot)$ is uniform in δ.

Therefore we can take the limit $\delta \to 0$, obtaining a representative for $\overline{ch}(Ind\mathcal{D})$ where a_0 is replaced by $\lim_{\delta\to 0} a_0$. Recall that a_0 is the constant term in the asymptotic expansion of $tr_s e^{-\mathcal{B}^2}$ and we can assume that $\mathcal{B}$ lives in $M \times \mathbb{R}$ since we are only interested in a_0 restricted to the cylinder. From Section A.1, we know that $F(M\times\mathbb{R}) = F(M)\otimes\mathbb{C}^2$ with grading given by the involution $\rho = \mathrm{Id}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover $X \in TM$ acts like

$$X \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whereas $\frac{\partial}{\partial u} \in T\mathbb{R}$ acts like

$$\mathrm{Id}\otimes\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Put $e_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In what follows we will abuse notations and suppress the "$\otimes$" sign. One has

$$\mathcal{B} = \tilde{\nabla} + \sqrt{t}\partial_u e_0 + \sqrt{t}D_Y\sigma - \frac{\varphi(u)c(T)}{4\sqrt{t}}\sigma. \tag{B.9}$$

At any fixed point $u_0 \in \mathbb{R}$, consider the coordinate change

$$v = \frac{u - u_0}{\delta}.$$

Under this new coordinate, the rescaled metric becomes

$$dv^2 + g.$$

Therefore, if we let $\mathcal{B}_\delta$ denote the superconnection in the rescaled metric corresponding to (B.9), then

$$\mathcal{B}_\delta = \tilde{\nabla} + \sqrt{t}\partial_v e_0 + \sqrt{t} D_Y \sigma - \frac{\varphi(u_0 + \delta v)c(T)}{4\sqrt{t}}\sigma.$$

Since this is a smooth family of superconnections, for the pointwise supertrace, one has

$$\begin{aligned}
\lim_{\delta \to 0} tr_s e^{-\mathcal{B}_\delta^2}|_{M \times \{u_0\}} &= tr_s e^{-(\tilde{\nabla} + \sqrt{t}\partial_v e_0 + \sqrt{t} D_Y \sigma - \frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2} \\
&= tr_s [e^{-(\tilde{\nabla} + \sqrt{t} D_Y \sigma - \frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2} e^{-t\partial_v^2 e_0^2}] \equiv 0.
\end{aligned}$$

This is because the term of which we are taking supertrace does not involve e_0. By the property of the supertrace (Cf. Lemma A.1), its supertrace vanishes.

Since in the adiabatic limit, the volume grows like δ^{-1}, we need to compute $\lim_{\delta \to 0} \frac{d}{d\delta} tr_s e^{-\mathcal{B}_\delta^2}$. This can be done by Duhamel's principle:

$$\frac{d}{d\delta} tr_s e^{-\mathcal{B}_\delta^2}|_{\delta=0, v=0} = -tr_s(e^{-\mathcal{B}_0^2} \# \frac{d}{d\delta}(\mathcal{B}_\delta)^2|_{\delta=0,v=0} e^{-\mathcal{B}_0^2}).$$

Computation shows

$$\frac{d}{d\delta}(\mathcal{B}_\delta)^2|_{\delta=0,v=0} = \frac{\varphi'(u_0)c(T)}{4} e_0 \sigma.$$

Thus

$$e^{-\mathcal{B}_0^2}\#\frac{d}{d\delta}(\mathcal{B}_\delta)^2|_{\delta=0,v=0}e^{-\mathcal{B}_0^2} = -\frac{\varphi'(u_0)c(T)}{4}e_0\sigma\frac{1}{\sqrt{4\pi t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y\sigma-\frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2}.$$

To compute its supertrace, note that the Clifford element must be saturate, i.e., if we let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of TM, then the supertrace of $F(M\times\mathbb{R})$ is nontrivial only on the element

$$e_0(e_1\sigma)(e_2\sigma)\cdots(e_n\sigma),$$

which involves an odd number (n) of σ's. From this fact, and the fact that the trace on $F(M)$ is nontrivial only on Id and $e_1\cdots e_n$, we see that

$$tr_s[-\frac{\varphi'(u_0)}{4\sqrt{4\pi t}}e_0T\sigma e^{-(\tilde{\nabla}+\sqrt{t}D_Y\sigma-\frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2}] = \frac{1}{2\sqrt{\pi}}tr^{even}[\frac{c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2}].$$

Hence,

$$\frac{d}{d\delta}tr_s e^{-\mathcal{B}_\delta^2}|_{\delta=0,v=0} = -\frac{1}{2\sqrt{\pi}}tr^{even}[\frac{c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2}].$$

Moreover, the argument in [C2] can easily be adapted in this simple situation to show that

$$tr_s e^{-\mathcal{B}_\delta^2}|_{M\times\{u_0\}} = -\delta\frac{1}{2\sqrt{\pi}}tr^{even}[\frac{c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{\varphi(u_0)c(T)}{4\sqrt{t}}\sigma)^2}] + o(\delta^{3/2}).$$

This implies that

$$\begin{aligned}\lim_{\delta\to 0}\int_{Y\times[0,1]} tr_s e^{-\mathcal{B}_\delta^2} &= -\frac{1}{2\sqrt{\pi}}\int_0^1\int_Y tr^{even}(\frac{\varphi'(u)c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{\varphi(u)c(T)}{4\sqrt{t}})^2}\,dz\,du\\ &= -\frac{1}{2\sqrt{\pi}}\int_0^1\int_Y tr^{even}(\frac{c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{sc(T)}{4\sqrt{t}})^2}\,dz\,ds.\end{aligned}$$

Consequently, if we expand

$$\frac{1}{2\sqrt{\pi}}\int_0^1\int_Y tr^{even}(\frac{c(T)}{2\sqrt{t}}e^{-(\tilde{\nabla}+\sqrt{t}D_Y-\frac{sc(T)}{4\sqrt{t}})^2}\,dz\,ds = \sum_{k=-n}^{N} C_k t^{k/2}+O(t(N+1)/2),$$

then

$$\lim_{\delta\to 0}\int_{Y\times[0,1]} a_0 = -C_0.$$

This shows that $\overline{ch}(Ind\mathcal{D})$ is represented by

$$\int_{Z-Y\times[0,1]} \hat{A}(iR^Z)\wedge tr(e^{-L^t}) - \frac{1}{2}\overline{ch}(\ker D_Y) - \hat{\eta}_1(0) - C_0.$$

To finish the proof of Theorem B.1, we just have to show

$$\hat{\eta} = \hat{\eta}_1(0) + C_0. \tag{B.10}$$

This can be seen as follows.

For t small and u large, consider

$$I^u(s) = \frac{1}{2\sqrt{\pi}}\int_t^u t^{-1/2} tr^{even}[(D_Y + \frac{s\,c(T)}{4\sqrt{t}})e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}]\,dt.$$

Differentiate with respect to s, one finds

$$\frac{dI^u}{ds} = \frac{1}{2\sqrt{\pi}}\int_t^u t^{-1/2} tr^{even}[\frac{c(T)}{4t}e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}]\,dt$$
$$+\frac{1}{2\sqrt{\pi}}\int_t^u t^{-1/2} tr^{even}[(D_Y + \frac{s\,c(T)}{4\sqrt{t}})2(\tilde{\nabla} + \sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})\frac{c(T)}{4t}e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}]\,dt.$$

On the other hand

$$\frac{d}{dt}e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2} = \frac{-1}{\sqrt{t}}(D_Y + \frac{s\,c(T)}{4\sqrt{t}})(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}.$$

Therefore,

$$tr^{even}[(D_Y + \frac{s\,c(T)}{4\sqrt{t}})2(\tilde{\nabla} + \sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})\frac{c(T)}{4t}e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}]$$
$$= -tr^{even}[\frac{c(T)}{2}\frac{d}{dt}e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}].$$

Integration by part gives

$$\frac{dI^u}{ds} = \frac{1}{2\sqrt{\pi}} tr^{even}[\frac{c(T)}{4\sqrt{t}} e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}] - \frac{1}{2\sqrt{\pi}} tr^{even}[\frac{c(T)}{4\sqrt{u}} e^{-(\tilde{\nabla}+\sqrt{u}D_Y - \frac{s\,c(T)}{4\sqrt{u}})^2}].$$

Or

$$\begin{aligned} I^u(1) - I^u(0) &= \frac{1}{2\sqrt{\pi}} \int_0^1 tr^{even}[\frac{c(T)}{4\sqrt{t}} e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{s\,c(T)}{4\sqrt{t}})^2}]\, ds \\ &\quad - \frac{1}{2\sqrt{\pi u}} \int_0^1 tr^{even}[\frac{c(T)}{4} e^{-(\tilde{\nabla}+\sqrt{u}D_Y - \frac{s\,c(T)}{4\sqrt{u}})^2}]\, ds. \qquad (B.11) \end{aligned}$$

Note that when $u \to +\infty$,

$$I^u(1) \to I(1) = \frac{1}{2\sqrt{\pi}} \int_t^{+\infty} t^{-1/2} tr^{even}[(D_Y + \frac{c(T)}{4t}) e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{c(T)}{4\sqrt{t}})^2}]\, dt$$

which is convergent by virtue of the cancellation result (A.21). Similarly

$$I^u(0) \to I(0) = \frac{1}{2\sqrt{\pi}} \int_t^{+\infty} t^{-1/2} tr[D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}]\, dt,$$

which is convergent because of (B.3). On the other hand, for u large, $tr[\frac{c(T)}{4} e^{-(\tilde{\nabla}+\sqrt{u}D_Y - \frac{s\,c(T)}{4\sqrt{u}})^2}]$ is uniformly bounded (Cf. Remark after Theorem B.2). Therefore, taking $u \to +\infty$ in the formula (B.11), one obtains

$$I(1) - I(0) = \frac{1}{2\sqrt{\pi}} \int_0^1 tr^{even}[\frac{c(T)}{2\sqrt{t}} e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{c(T)}{4\sqrt{t}})^2}]\, dt. \qquad (B.12)$$

Now when $t \to 0$,

$$I(1) \to \hat{\eta} = \int_0^{+\infty} t^{-1/2} tr^{even}[(D_Y + \frac{c(T)}{4t}) e^{-(\tilde{\nabla}+\sqrt{t}D_Y - \frac{c(T)}{4\sqrt{t}})^2}]\, dt,$$

which is convergent because of the cancellation result (A.19). Unfortunately, we do not have a corresponding statement for the unitary superconnection, therefore the same thing does not happen to $I(0)$. However, we claim that

$$\hat{\eta}_1(0) = \text{the finite part of } I(0) \text{ when } t \to 0.$$

To see this, recall that for Re$s \gg 0$, we have

$$\hat{\eta}_1(s) = \int_0^{+\infty} t^{s-1/2} tr[D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}]\, dt + \int_0^t t^{s-1/2} tr[D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}]\, dt.$$

For t small, one has the asymptotic expansion

$$tr[D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}] = \sum_{k=-n}^{N} a_k t^{k/2} + O(t^{(N+1)/2}).$$

Plug this in the formula for $\hat{\eta}_1(s)$, we obtain

$$\begin{aligned} \hat{\eta}_1(s) \;&=\; \int_0^{+\infty} t^{s-1/2} tr[D_Y e^{-(\tilde{\nabla}+\sqrt{t}D_Y)^2}]\, dt + \sum_{k=-n}^{N} \frac{a_k}{s+k/2+1/2} t^{s+k/2+1/2} \\ &\quad + O(t^{s+1+N/2}). \end{aligned}$$

This is how we see the existence of meromorphic extension of $\hat{\eta}_1(s)$. In fact, we can see more. It follows that

$$\hat{\eta}_1(0) = I(0) + \sum_{k=-n, k\neq -1}^{N} \frac{2a_k}{k+1} t^{\frac{k+1}{2}} + O(t^{N/2+1}).$$

Our claim follows as well.

Armed with this and formula (B.12), we easily arrive at (B.10). QED.

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