Homotopy groups of cycle spaces

A Dissertation Presented

by

Paulo Lima Filho

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Graduate School

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Preface

In his foundational paper [23], B. Lawson established a remarkable homotopy equivalence:

$$\Sigma\!\!\!/ : \mathcal{C}_p(X) \to \mathcal{C}_{p+1}(\Sigma\!\!\!/ X),$$

between the space of algebraic p-cycles on a projective algebraic variety X and the space of (p+1)-cycles on the projective cone $\Sigma\!\!\!/ X$ over X. This theorem is called "the Complex Suspension Theorem". It was implicit in his considerations and results, and more explicit in his subsequent paper with M.-L. Michelsohn [24], that the homotopy groups of such cycle spaces provide an interesting invariant for the variety X. The techniques used in his paper are a beautiful combination of complex geometry, geometric measure theory and homotopy theory. Despite that, Lawson devised the algebraic character of the results just proven, and E. Friedlander [11] and [12] took up, successfully, the difficult task of extending Lawson's results to a broader algebraic context. In his paper [11], Friedlander introduces the terminology "Lawson homology" for the bigraded groups

$$L_pH_{i+2p}(X) \stackrel{def}{=} \pi_i(\mathcal{C}_p(X)),$$

and describes some of their functorial properties. Concomitantly, E. Friedlander and B. Mazur [14] explored further those invariants, providing the bigraded Lawson homology with operations and filtrations. In particular, they showed it to carry the natural structure of a bigraded module over the polynomial ring in two variables $\mathbb{Z}[h, s]$. The action of the generator $s \in \mathbb{Z}[h, s]$ on Lawson homology sends $L_pH_{i+2p}(X)$ into $L_{p-1}H_{i+2p}(X)$, and its p-fold composition yields a map

$$s^p : L_pH_{i+2p}(X) \to L_0H_{i+2p}(X) \equiv H_{i+2p}(X; \mathbb{Z}).$$

We call this map the "generalized cycle map".

Taking into consideration the newborn character of the theory, we provide in Chapter 1 a detailed account of the results mentioned above. Essentially, all the basic results needed subsequently are presented in this chapter.

It is a natural question to raise, when developing a homology-like theory for a category of topological spaces), which of the Eilenberg-Steenrod axioms it satisfies. We will pursue this categorical framework for presenting the results contained in this work.

In Chapter 2 we give two possible definitions for the relative Lawson homology $L_pH_{i+2p}(X, Y)$ of a pair $X \subset Y$ of algebraic sets. One of the definitions, despite its simplicity, just works for algebraic sets over the complex numbers. The other definition uses certain machinery from homotopy theory, and has the advantage that almost from its mere definition we can derive the existence of long exact sequences for the Lawson homology of a pair. The equivalence of the two approaches is shown in 2.1, using a result (2.1.7,personal communication) of B. Lawson. For sake of completeness and also for future references we have also included a proof of this result in Chapter 2, §1.

Before we go further in this categorical approach, we make a detour, in §2 of Chapter 2, to analyze a particular feature of the theory, namely the "generalized cycle map". It is again a general philosophy, see [18], that a homology (or cohomology) theory derived from algebraic cycles on algebraic varieties should carry a "cycle map", which should be a natural transformation into ordinary homology (or cohomology). By ordinary we mean any canonical theory - e.g. singular homology, Cech cohomology, étale homology etc. - which suits the context. This occurs, for example, with the classical Chow groups of algebraic cycles modulo algebraic equivalence.

Recall that we do have a "generalized cycle map" s^p obtained by iterating the s-map p-times. However, it is not *a priori* clear that this cycle map is a natural transformation, for the s-map itself seems to depend on the polarization of X. Here occurs one of these frequent and interesting phenomena where the unified character of Mathematics is manifest. We first have a canonical map e from the (naïve) cycle spaces $\tilde{\mathcal{C}}_p(X)$ into the space $\mathcal{Z}_{2p}(X)$ of integral cycles with the flat-norm topology (see 2.2). Then, there is a beautiful isomorphism

$$\mathcal{A} : \pi_i(\mathcal{Z}_k(X)) \to H_{i+k}(X; \mathbb{Z})$$

between the homotopy groups of the space of integral cycles in X onto the singular homology of X, established by F. Almgren in his Ph.D. Thesis, [2].

In §2 of Chapter 2 we prove the following:

Theorem 2.2.3 : *For an algebraic set X, the map $\mathcal{A} \circ e_*$ coincides with the "generalized cycle map" $s^{(p)} : L_pH_{i+2p}(X) \to H_{i+2p}(X, \mathbb{Z})$, where $\mathcal{A}$ is Almgren's isomorphism and e_* is the map induced by e on homotopy.*

In particular one obtains the desired functorial properties, cf **Corollary 2.2.**

In Chapter 3 we go back to the categorical track and prove the main computational tool of the Thesis, namely the "excision type result". In §1 we define the notion of relatively isomorphic pairs of algebraic sets. Namely, we say that two pairs $X' \subset X$ and $Y' \subset Y$ of algebraic sets are relatively isomorphic if the quasiprojective sets $X \setminus X'$ and $Y \setminus Y'$ are isomorphic. Then we prove our major result:

Theorem 3.1.7 : *Let (X, X') and (Y, Y') be relatively isomorphic pairs of algebraic sets. Then, any given relative isomorphism $\Psi : (X, X') \leftrightarrow (Y, Y')$ induces an isomorphism of topological groups:*

$$\Psi_* : \tilde{\mathcal{C}}_p(X, X') \xrightarrow{\cong} \tilde{\mathcal{C}}_p(Y, Y'),$$

for all $p \geq 0$.

The combination of exact sequences, cycle maps and excision, enables us to compute Lawson homology in several basic cases, as, for example, products of projective spaces or hyperquadrics.

In §3 we study the projectivization of vector bundles and prove the following result, which has close resemblance with Bott periodicity:

Theorem 3.3.1 : *Let $E \to X$ be a very ample vector bundle over a projective algebraic variety. Then, the following homotopy equivalence holds:*

$$\tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X)) \cong \tilde{\mathcal{C}}_p(X) \times \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E^*)),$$

for all $p \geq 1$.

Corollary 3.3 : *For any algebraic vector bundle over a projective variety X there exists an integer $m_0 \geq 0$ such that for $m \geq m_0$ we have:*

$$\tilde{\mathcal{C}}_p(\mathbb{P}(E \oplus H^{\otimes m})) \cong \tilde{\mathcal{C}}_p(X) \times \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E))$$

for all p, where H is the hyperplane bundle over X.

In the fourth and last chapter we make a brief investigation on a question that arises naturally from the examples computed in §2 of Chapter 3: *For which spaces X is the cycle map $s^{(p)} :\to H_{i+2p}(X;\mathbb{Z})$ an isomorphism for all p and i ?* The class of spaces which have this property will be called the class $\mathcal{L}$. In §2 of Chapter 4 we show that this class is closed under "cellular algebraic extensions" (see Definition 4.2). In particular we show that the spaces having a cellular decomposition in the sense of Fulton [15] belong to the class $\mathcal{L}$. These properties show that the class $\mathcal{L}$ is richly endowed, for it contains, for example, the generalized flag varieties G/P and products of them, where G is a semi-simple linear algebraic group and P is a parabolic subgroup. This is shown is §3. In particular one obtains that the compact hermitian symmetric spaces belong to the class $\mathcal{L}$.

Let us make a final philosophical remark. The examples computed in this thesis corroborate (the necessity of) the existence of a "dual" cohomology theory (with respect to some natural pairing) carrying a well behaved cycle map, and fitting into the context of motivic cohomology. In this context, the examples above will be computed by general principles, see [18]. Such dual theory is being developed by Lawson and Friedlander.

We finally say that we have tried to use the simplest terminology we could, so as to avoid unnecessary technicalities.

To my son
Fernando.

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Chapter 1

An overview

1.1 Background material

Here we provide basic definitions. We work over the complex numbers, and the definitions and results presented will be restricted to this category, unless otherwise stated. We refer to [23], [16], [28] and [17] as the main basic references.

In our treatment an *algebraic set* X is always a closed algebraic subset of some projective space $\mathbb{P}^N$, provided with an embedding $j : X \hookrightarrow \mathbb{P}^N$. An algebraic set X is a *variety* if it is irreducible.

Definition 1.1.1 *An (effective)* algebraic cycle of dimension p *in the pro-*

jective space $\mathbb{P}^N$ *is a finite formal sum* $\sigma = \sum_\lambda n_\lambda V_\lambda$, *where the* n_λ*'s are positive integers and the* V_λ*'s are (irreducible) subvarieties of dimension* p *in* $\mathbb{P}^N$. *We call those cycles simply* p-cycles. *Recall that the* degree deg(V) *of an irreducible subvariety* $V \subseteq \mathbb{P}^N$ *of dimension* p *is the number of points in the intersection of* V *with a generic* $N-p$ *dimensional linear subspace of* $\mathbb{P}^N$. *For a p-cycle* $\sigma = \sum_\lambda n_\lambda V_\lambda$ *in* $\mathbb{P}^N$ *we define its* degree *as* $deg(\sigma) = \sum_\lambda n_\lambda deg(V_\lambda)$. *The* support *of* σ *is the algebraic subset* $\bigcup_\lambda V_\lambda$ *of* $\mathbb{P}^N$.

It is a standard fact that the set of p-cycles of a fixed degree d in $\mathbb{P}^N$ can be given the structure of an algebraic set, which we denote by $\mathbf{C}_{p,d}(\mathbb{P}^N)$. See [30] and [29] for details. In case $j : X \hookrightarrow \mathbb{P}^N$ is an algebraic subset of $\mathbb{P}^N$, the subset $\mathbf{C}_{p,d}(X,j) \subset \mathbf{C}_{p,d}(\mathbb{P}^N)$ consisting of those cycles whose support is contained in X has a structure of algebraic set which makes it an algebraic subset of $\mathbf{C}_{p,d}(\mathbb{P}^N)$. The set $\mathbf{C}_{p,d}(X,j)$ is called the *Chow bunch of (effective) p-cycles of degree d on X*. The algebraic structure of $\mathbf{C}_{p,d}(X,j)$ depends on the embedding $j : X \hookrightarrow \mathbb{P}^N$. See [29] and [11].

Definition 1.1.2 *The set*

$$\mathbf{C}_p(X,j) = \coprod_{d\geq 0} \mathbf{C}_{p,d}(X,j) = \{\underline{0}\} \coprod \{\coprod_{d>0} \mathbf{C}_{p,d}(X,j)\}$$

of all effective p-cycles in X, together with an isolated point $\underline{0}$ ("the cycle of degree zero") is an abelian topological monoid when taken with the natural (disjoint union) topology. We call $\mathbf{C}_p(X,j)$ the Chow monoid of effective p-cycles in X.

The Chow monoid $\mathbf{C}_p(X,j)$ can also be written as

$$\mathbf{C}_p(X,j) = \coprod_{\alpha \in \mathcal{A}_p} \mathbf{C}_\alpha(p,X)$$

where $\mathcal{A}_p \stackrel{def}{=} \pi_0(\mathbf{C}_p(X,j))$ is the discrete monoid of connected components of $\mathbf{C}_p(X,j)$. It is shown that $\mathcal{A}_p$ is also the monoid of effective p-cycles modulo effective algebraic equivalence, cf [11]. The (naïve) group completion of $\mathcal{A}_p$ is the classical Chow group of p-cycles modulo algebraic equivalence. See [11] for a proof of this fact and [15] for further information about the classical Chow groups.

The center of our investigation is certain invariants for algebraic sets $j : X \hookrightarrow \mathbb{P}^N$ obtained essentially from the homotopy of the Chow monoids $\mathbf{C}_p(X,j)$. Those invariants were implicitly defined in the foundational paper of Lawson [23]. Afterwards, while setting a systematic treatment to the subject, Friedlander, in [11], called those invariants *Lawson homology.*

Up to this point, all that was said carries over to the case of varieties over algebraically closed fields of arbitrary characteristic p. In order to proceed we need to use certain functors which are commonplace in algebraic topology and algebraic geometry. Using Friedlander's terminology we first apply a "topologizing" functor to the components of $\mathbf{C}_p(X,j)$. This gives a topological space with multiplication. Then we apply a "homotopy theoretic group-completion" functor following the previous one and obtain the desired "cycle space". More precisely:

Definition 1.1.3 *Let $(-)^{an}$ be the functor*

$$(-)^{an} : \left\{ \begin{array}{c} \textit{algebraic sets} \\ \textit{over } \mathbb{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{topological} \\ \textit{spaces} \end{array} \right\}$$

which takes an algebraic set over $\mathbb{C}$ to its underlying set with the analytic topology. Let

$$B : \left\{ \begin{array}{c} \textit{abelian topological} \\ \textit{monoids} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{abelian topological} \\ \textit{monoids} \end{array} \right\}$$

be the "classifying space" functor or "delooping" functor as in [26]. For an algebraic variety X over $\mathbb{C}$, we define the space of p-cycles in X, *$\mathcal{C}_p(X)$, as*

$$\mathcal{C}_p(X) \stackrel{def}{=} \Omega B(\mathbf{C}_p(X,j)^{an}) = \Omega B\left(\coprod_{d\geq 0}(\mathbf{C}_{p,d}(X,j))^{an}\right),$$

for some embedding $j : X \hookrightarrow \mathbb{P}^N$.

In other words, $\mathcal{C}_p(X)$ is the space of loops on the classifying space of the Chow monoid of p-cycles on X. As the notation indicates, the homeomorphism type of $\mathcal{C}_p(X)$ is independent of the embedding $j : X \hookrightarrow \mathbb{P}^N$, cf. [11].

Remark 1.1.4

1. We recall here that, for any abelian topological monoid M, its classifying space BM is again an abelian topological monoid, which is $(k+1)$-connected if M is k-connected. (See [26]). Hence, the space ΩBM of loops in BM carries a natural structure of abelian topological monoid, given by the pointwise addition of loops. There is a canonical morphism of topological monoids (continuous homomorphism)

$$i : M \to \Omega BM$$

whose effect in homology is to "group complete" the action of the monoid $\pi_0 M$ on the Pontrjagin ring $H_*(M; \mathbb{Z})$. In particular, if M is a group-like monoid (i.e., translations are homotopy equivalences, and hence $\pi_0 M$ is a group) then i is a homology equivalence. Since topological monoids (H-spaces in general) are simple spaces, in the sense that the fundamental group acts trivially on the higher homotopy groups, standard techniques in homotopy theory show that i is a homotopy equivalence if and only if it is a homology equivalence.

2. Since ΩBM is already group-like (for $\pi_0\Omega BM \cong \pi_1 BM$ is a group), the above comments show that

$$\Omega BM \overset{h.eq.}{\cong} \Omega B(\Omega BM) \overset{h.eq.}{\cong} \Omega^2 B^2 M \overset{h.eq.}{\cong} \Omega^3 B^3 M \ldots.$$

It follows that ΩBM is an infinite loop space, and in being so, it carries lots of interesting properties as those, for example, described in [25] and [1]. The functor $\Omega B(-)$ is the "homotopy theoretic group completion" functor. See [33], [4], [27] and [26] for further properties of this functor

3. As a particular case of item 1, we obtain a canonical morphism of topological monoids

$$i : \mathbf{C}_p(X,j)^{an} \longrightarrow \mathcal{C}_p(X) \overset{def}{=} \Omega B(\mathbf{C}_p(X,j)^{an}) = \Omega B(\coprod_{d\geq 0}(\mathbf{C}_{p,d}(X,j))^{an})$$

group-completing the action of the monoid $\mathcal{A}_p \overset{def}{=} \pi_0(\mathbf{C}_p(X,j))$ on the Pontrjagin ring $H_*(\mathbf{C}_p(X,j)^{an}, \mathbb{Z})$.

4. Since we will always be working with algebraic varieties over $\mathbb{C}$, we omit the notation $(-)^{an}$ whenever no confusion is likely to arise.

5. Taken with the analytic topology, the algebraic sets $\mathbf{C}_{p,d}(X,j)$ are finite CW-complexes. If $Y \subset X$ is a closed algebraic subset, then

$\mathbf{C}_{p,d}(X,j)$ can be triangulated so that $\mathbf{C}_{p,d}(Y,j')$ has the structure of a subcomplex of $\mathbf{C}_{p,d}(X,j)$, where j' is the composition $j' = j \circ i$ of the inclusion $i : Y \hookrightarrow X$ with the embedding $j : X \hookrightarrow \mathbb{P}^N$.

In the case of varieties over a field of characteristic p, there is an analogous definition where the "topologization" functor is defined via étale topology. See [11] and [3] for details.

At this point we are able to define the desired invariants. Those invariants were introduced and denoted *Lawson homology* by E. Friedlander in [11], after the work of B. Lawson [23].

Definition 1.1.5 *Let X be an algebraic set and choose p to be an integer, $0 \leq p \leq dim(X)$. Define the* Lawson homology $L_pH_{i+2p}(X)$ *of X by*

$$L_pH_{i+2p}(X) \stackrel{def}{=} \pi_i(\mathcal{C}_p(X)).$$

Remark 1.1.6 Notice that $\mathbf{C}_0(X,j)$ is equal to $\coprod_{d\geq 0} SP^d(X)$, where $SP^d(X)$ is the *d-fold* symmetric product of X. We will see in the alternative description of the cycle spaces below, that

$$\tilde{\mathcal{C}}_0(X) \overset{h.eq.}{\cong} SP^\infty(X),$$

with $SP^\infty(X)$ being the infinite symmetric product of X. Therefore the Dold-Thom theorem gives an isomorphism

$$L_0H_i \cong H_i(X, \mathbb{Z}).$$

Alternative description of the cycle spaces and of Lawson homology

Write $\mathbf{C}_p(X,j) = \coprod_{\alpha \in \mathcal{A}_p} \mathbf{C}_\alpha(p, X)$ and choose one element $z_\alpha \in \mathbf{C}_{p,\alpha}(X)$ for each $\alpha \in \mathcal{A}_p$. Translation by the elements z_α are algebraic embeddings of $\mathbf{C}_p(X,j)$ into itself, sending $\mathbf{C}_{p,\beta}(X)$ into $\mathbf{C}_{p,\alpha+\beta}(X)$. In this way we can consider $\mathcal{A}_p$ as an indexing category for $\mathbf{C}_p(X,j)$, with

$$Mor(\mathbf{C}_{p,\alpha}(X), \mathbf{C}_{p,\beta}(X)) = \{\lambda \in \mathcal{A}_p : \lambda + \alpha = \beta\},$$

where

$$\lambda : \mathbf{C}_{p,\alpha}(X) \longrightarrow \mathbf{C}_{p,\beta}(X)$$

is obtained by sending $\sigma \in \mathbf{C}_{p,\alpha}(X)$ to $\sigma + z_\lambda \in \mathbf{C}_{p,\beta}(X)$. The following diagram commutes up to homotopy, cf. [11]:

$$\begin{array}{ccc} \mathbf{C}_{p,\alpha}(X) & \xrightarrow{\lambda} & \mathbf{C}_{p,\alpha+\lambda}(X) \\ \mu \downarrow & & \downarrow \mu \\ \mathbf{C}_{p,\alpha+\mu}(X) & \xrightarrow[\lambda]{} & \mathbf{C}_{p,\alpha+\lambda+\mu}(X) \end{array}$$

which allows one to construct a homotopy limit called "Friedlander completion" by Lawson [23], as follows: (cf. [11])

Choose a sequence $\Lambda = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$ of elements in $\mathcal{A}_p$, so that each $\alpha \in \mathcal{A}_p$ appears infinitely often in Λ. Now, using the sequence of maps $\{\alpha_n : \mathbf{C}_p(X, j) \to \mathbf{C}_p(X, j)\}$ one can form the mapping telescope $Tel(\mathbf{C}_p(X, j))$ and obtain the following (cf. Friedlander [11]):

Proposition 1.1.7 *Let $j : X \hookrightarrow \mathbb{P}^N$ and $\Lambda = \{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in \mathcal{A}_p$, be as above. Then the canonical map*

$$i : \mathbf{C}_p(X, j) \to \mathcal{C}_p(X) = \Omega B(\mathbf{C}_p(X, j)^{an})$$

factors up to homotopy through a homology equivalence, and therefore, a homotopy equivalence

$$\tilde{\imath} : Tel(\mathbf{C}_p(X, j), \Lambda) \to \mathcal{C}_p(X).$$

We observe that the proposition asserts the existence of a homology equivalence between $Tel(\mathbf{C}_p(X, j), \Lambda)$ and $\mathcal{C}_p(X)$. The last conclusion follows after one proves that $Tel(\mathbf{C}_p(X, j), \Lambda)$ is a simple space, together with the arguments of Remark 1.1.4 .

In view of the above result we can use $Tel(\mathbf{C}_p(X,j),\Lambda)$ as another definition of $\mathcal{C}_p(X)$. We point out here that the homotopy equivalence of the proposition is only valid for varieties defined over the complex numbers. Also, the Lawson homology can be alternatively defined as

$$L_pH_{i+2p}(X) = \pi_i(Tel(\mathbf{C}_p(X,j),\Lambda)) = \varinjlim_{\alpha\in\Lambda} \pi_i(\mathbf{C}_{p,\alpha}(X)).$$

We conclude this section with a last remark:

Remark 1.1.8 As abelian topological monoids, the cycle spaces $\mathcal{C}_p(X)$ are products of Eilenberg-MacLane spaces, [33]. Therefore they are determined, up to homotopy, by the Lawson homology. In other words, we have a homotopy equivalence

$$\mathcal{C}_p(X) \cong \prod_i K(L_pH_{i+2p}(X), i).$$

1.2 Basic Results

Here we describe the results obtained by Lawson [23], Friedlander [11], Lawson and Michelson [24] and Friedlander and Mazur [14] which will be needed later on. For sake of brevity and clearness of exposition we are altering the chronological order of the results presented, observing that the work of Lawson [23] was the precursor of all that follows.

Definition 1.2.1 *Let* $X = \coprod_\alpha X_\alpha$ *and* $Y = \coprod_\beta Y_\beta$ *be disjoint unions of algebraic sets (not necessarily finite unions) taken with the disjoint union topology (of the Zariski topology of their components). We say that a continuous map* $f : X \to Y$ *is a* morphism *of* X *into* Y *if the restriction of* f *to any component* X_α *is a morphism of algebraic sets. A proper morphism* $f : X \to Y$ *is a* birrational, bicontinuous *morphism if it is a set theoretic bijection and for every* $y \in Y$ *the induced map on residue fields* $\mathbb{C}(y) \to \mathbb{C}(f^{-1}(y))$ *is an isomorphism. A* rational continuous map $f : X \to Y$ *is a correspondence, i.e., a pair* $\{g : Z \to X,\ h : Z \to Y\}$ *in which* g *is a birational, bicontinuous morphism. Here we follow Friedlander's [11] terminology closely.*

We see that a birational, bicontinuous morphism $f : X \to Y$, with X and Y as in the definition above, induces birational equivalences (in the sense of [17]) between the irreducible components of X and Y. Furthermore, taking X and Y with the analytic topology, we see that f induces an homeomorphism between $(X)^{an}$ and $(Y)^{an}$, whose restriction to an irreducible component of X is a homeomorphism onto a corresponding component of Y. Observe that a rational continuous map $f : X \to Y$ induces a continuous map $\overline{f} : (X)^{an} \to (Y)^{an}$.

With the notions just introduced, it now makes sense to talk about rational continuous maps between Chow monoids. The first "functorial" property of the Lawson homology comes from the following proposition, cf. [11].

Proposition 1.2.2 *Let $j : X \hookrightarrow \mathbb{P}^N$, $j' : Y \hookrightarrow \mathbb{P}^M$ and $j'' : W \hookrightarrow \mathbb{P}^L$ be algebraic sets.*

(a) *For any morphism $f : X \to Y$ and integer $0 \leq p \leq dimX$, there exists a rational continuous map*

$$f_{\#} : \mathbf{C}_p(X, j) \to \mathbf{C}_p(Y, j')$$

defined by

$$f_{\#}(\sum_i n_i V_i) = \sum_i n_i deg(V_i/f(V_i)) f(V_i).$$

The map $f_{\#}$ is a morphism of abelian topological monoids in the analytic topology and induces a morphism f_* on Lawson homology for all $i \geq 0$

$$f_* : L_pH_{i+2p}(X) \to L_pH_{i+2p}(Y).$$

(b) *For any flat morphism $f : W \to X$ of relative dimension $r \geq 0$, and any integer p, $0 \leq p \leq dimX$, there exists a rational continuous map*

$$f^{\#} : \mathbf{C}_p(X, j) \to \mathbf{C}_{p+r}(W, j'')$$

which is a morphism of topological monoids in the analytic topology. Furthermore, the map $f^{\#}$ induces a morphism on Lawson homology, for all $i \geq 0$:

$$f^* : L_pH_{i+2p}(X) \to L_{(p+r)}H_{i+2(p+r)}(W).$$

Remark 1.2.3

1. Recall that for a morphism of algebraic sets $f : X \to Y$, and for a subvariety $V \subset X$, $\deg(V/f(V))$ is defined as

$$\deg(V/f(V)) = \begin{cases} 0, & \text{if } \dim V > \dim f(V) \\ [\mathbb{C}(V) : \mathbb{C}(f(V))], & \text{if } \dim V = \dim f(V), \end{cases}$$

where $\mathbb{C}(V)$, $\mathbb{C}(f(V))$ are the functions fields of V and $f(V)$ respectively. In other words, $\deg(V/f(V))$ is the number of sheets of V as a branched covering of $f(V)$.

2. A *flat* morphism $f : W \to X$ of relative dimension r is a morphism satisfying:

   (a) If $U \subset W$, $U' \subset X$ are affine open sets such that $f(U) \subset$

U', then the induced map f^* on the ring of regular functions $f^* : \mathbb{C}[U'] \to \mathbb{C}[U]$ makes $\mathbb{C}[U]$ into a flat $\mathbb{C}[U']$-module.

(b) For any subvariety $V' \subset X$ and any irreducible component V of $f^{-1}(V')$ one has $\dim V = \dim V' + r$.

Examples of flat morphisms of relative dimension r are: the projection of a vector or projective bundle of rank r to its base; the projection of a cartesian product $X \times Y$ to its first factor, where Y is a pure-dimensional algebraic set. See [17] and [15] for examples and further properties.

From 1.2.2 one obtains that Lawson homology is a covariant functor from the category of algebraic sets and morphisms to the category of bigraded groups; also, it is a contravariant functor from the category of algebraic varieties and flat morphisms (with relative dimension) to bigraded groups.

Definition 1.2.4 *Let* $i : X \hookrightarrow \mathbb{P}^N$ *and* $j : Y \hookrightarrow \mathbb{P}^M$ *be algebraic sets. Embed* $\mathbb{P}^N$ *and* $\mathbb{P}^M$ *linearly in* $\mathbb{P}^{N+M+1}$ *as two disjoint linear subspaces. Define the* complex join $i\#j : X\#Y \hookrightarrow \mathbb{P}^{N+M+1}$ *of* X *and* Y *as the algebraic subset of* $\mathbb{P}^{N+M+1}$ *obtained as the union of all projective lines joining points of* X *to points of* Y *in* $\mathbb{P}^{N+M+1}$. *In the particular case where* Y *is* *a point* $\mathbb{P}^0 \in \mathbb{P}^{N+1}$ *not lying in* $\mathbb{P}^N$, *the complex join* $\mathbb{P}^0 \# X$ *of* $\mathbb{P}^0 \in \mathbb{P}^{N+1}$ *with* $X \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ *is called the* complex suspension *of* X *and is denoted* $\not{\Sigma} X$. *The* m-fold complex suspension *of* X, $\not{\Sigma}^m X$, *is*

$$\underbrace{\not{\Sigma}(\not{\Sigma}(\ldots(\not{\Sigma} X)\ldots))}_{m-times}.$$

Observe that the complex suspension can also be seen as the Thom space of the hyperplane bundle $\mathcal{O}(1)$ *over* X, *and its structure as an algebraic set does not depend on the point* $\mathbb{P}^0 \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$.

It is easy to see that if V is a subvariety of $X \stackrel{i}{\hookrightarrow} \mathbb{P}^N$ having dimension p and degree d, and if W is a subvariety of $Y \stackrel{j}{\hookrightarrow} \mathbb{P}^M$ with dimension q and degree e (in other words, $V \in \mathbf{C}_{p,d}(X,i)$ and $W \in \mathbf{C}_{q,e}(Y,j)$), then the join $V \# W$ is a subvariety of $X \# Y$ having dimension $p+q+1$ and degree $d \cdot e$, i.e., $V \# W \in \mathbf{C}_{p+q+1,d\cdot e}(X \# Y, i \# j)$.

Notice that the m-fold suspension $\not{\Sigma}^m X$ of X can also be viewed as the join $\mathbb{P}^{m-1} \# X$ of $\mathbb{P}^{m-1}$ with X. From the above we conclude that the m-fold suspension takes irreducible cycles in $\mathbf{C}_{p,d}(X,i)$ to irreducible cycles in $\mathbf{C}_{p+m,d}(\not{\Sigma}^m X, \not{\Sigma}^m i)$. Actually the join operation can be extended linearly to the cycle spaces as follows: (cf. Friedlander [11])

Proposition 1.2.5 *Let* $i : X \hookrightarrow \mathbb{P}^N, j : Y \hookrightarrow \mathbb{P}^M$ *be algebraic sets. The*

the external join induces a rational continuous map

$$\mathbf{C}_{r,d}(X,i) \times \mathbf{C}_{s,e}(Y,j) \longrightarrow \mathbf{C}_{r+s+1,d\cdot e}(X\#Y, i\#j)$$

for any $r \leq dimX, s \leq dimY$, d *and* e. *Up to birational, bicontinuous equivalence, this pairing is independent of the embeddings* i *and* j. *These rational continuous maps induce a bi-additive rational continuous map*

$$\# : \mathbf{C}_r(X,i) \times \mathbf{C}_s(Y,j) \longrightarrow \mathbf{C}_{r+s+1}(X\#Y, i\#j)$$

which sends $\mathbf{C}_r(X,i) \times \{0\}$ *and* $\{0\} \times \mathbf{C}_r(Y,j)$ *to* $\{0\} \in \mathbf{C}_{r+s+1}(X\#Y, i\#j)$.

If we consider $\mathbb{P}^{m-1}$ as a cycle in $\mathbf{C}_{m-1,1}(\mathbb{P}^{m-1}, id)$ we obtain an algebraic map

$$\Sigma\!\!\!/^m X : \mathbf{C}_p(X,j) \longrightarrow \mathbf{C}_{p+m}(\Sigma\!\!\!/^m X, \Sigma\!\!\!/^m j)$$

which is defined in such a way that it sends a cycle $\sigma = \sum_\lambda n_\lambda V_\lambda$ in $\mathbf{C}_{p,d}(X,j)$ to $\Sigma\!\!\!/^m \sigma \equiv \sum_\lambda n_\lambda (\Sigma\!\!\!/^m V_\lambda) \equiv \sum_\lambda n_\lambda (\mathbb{P}^{m-1} \# V_\lambda)$. The map induced on the cycle spaces (by functoriality)

$$\Sigma\!\!\!/^m : \mathcal{C}_p(X) \to \mathcal{C}_{p+m}(\Sigma\!\!\!/^m X)$$

is remarkably well behaved and satisfies the following theorem, which is the foundation stone of the theory, proven by Lawson in [23]:

Theorem 1.2.6 (The Complex Suspension Theorem) *The m-fold complex suspension* $\Sigma\!\!\!/^m : \mathcal{C}_p(X) \to \mathcal{C}_{p+m}(\Sigma\!\!\!/^m X)$ *is a homotopy equivalence for every integer* p, *with* $0 \leq p \leq dimX$ *and every positive integer* m.

Equivalently:

Corollary 1.2.7 *The m-fold complex suspension Σ^m induces an isomorphism*

$$\Sigma^m{}_* : L_pH_{i+2p}(X) \to L_{(p+m)}H_{i+2(p+m)}(\Sigma^m X)$$

for every $i \geq 0$.

This powerful theorem was later extended by Friedlander [11] to a broader algebraic context. Furthermore, as shown by Friedlander and Mazur [14], it enables one to provide the Lawson homology with lots of extra structure, as we briefly outline below.

The cycle spaces of the complex projective space $\mathbb{P}^t$ are completely determined by the Complex Suspension Theorem. Namely

$$\mathcal{C}_p(\mathbb{P}^t) \cong \mathcal{C}_0(\mathbb{P}^{t-p}) \cong K(\mathbb{Z},2) \times \ldots \times K(\mathbb{Z},2(t-p)),$$

for all $0 \leq p \leq \dim X$, the later equivalence being a consequence of the Dold-Thom theorem. Equivalently, one has the isomorphisms

$$L_pH_{i+2p}(\mathbb{P}^t) \cong L_0H_i(\mathbb{P}^{t-p},\mathbb{Z}) \cong H_i(\mathbb{P}^{t-p},\mathbb{Z}).$$

Let us introduce a bit of notation before we proceed.

Definition 1.2.8 *For an algebraic set $j : X \hookrightarrow \mathbb{P}^n$, and integers $i \geq 0, p$,* *define* $L_pH_{i+2p}(\Sigma^\infty X)$ *as*

$$L_pH_{i+2p}(\Sigma^\infty X) = \varinjlim_{s\to\infty} L_{(p+s)}H_{i+2(p+s)}(\Sigma^s X),$$

where the limit is taken with respect to the maps induced by the complex suspension. Actually one has

$$L_pH_{i+2p}(\Sigma^\infty X) = \begin{cases} L_pH_{i+2p}(X), & \text{if } p \geq 0 \\ L_0H_i(\Sigma^{|p|}X), & \text{if } p < 0, \end{cases}$$

by the Complex Suspension Theorem (1.2.6). It follows from the Thom isomorphism theorem that $L_pH_{i+2p}(\Sigma^\infty X) = H_{i-|p|}(X, \mathbb{Z})$ *for* $p < 0$ *and* $i - |p| \geq 0$.

Now as a corollary of 1.2.5 and functoriality of the group completion, one sees that the join operation descends to the smash product of the cycle spaces involved. In other words, the join # induces a pairing of infinite loop spaces:

$$\# : \mathcal{C}_p(X) \wedge \mathcal{C}_s(Y) \to \mathcal{C}_{p+s+1}(X\#Y) \tag{1.1}$$

and therefore a pairing in Lawson homology:

$$L_pH_{i+2p}(X) \otimes L_sH_{j+2s}(Y) \to L_{(p+s+1)}H_{i+j+2(p+s+1)}(X\#Y). \tag{1.2}$$

Specializing to the case $X = \mathbb{P}^{t+p}$ one gets a pairing

$$L_pH_{i+2p}(\mathbb{P}^{t+p}) \otimes L_sH_{j+2s}(Y) \to L_{(p+s+1)}H_{i+j+2(p+s+1)}(\mathbb{P}^{t+p}\#Y). \tag{1.3}$$

Since $\mathbb{P}^{t+p}\#Y = \Sigma^{t+p+1}Y$, the complex suspension theorem (1.2.6) implies that the above pairing is equivalent to

$$L_0H_i(\mathbb{P}^t) \otimes L_sH_{j+2s}(\Sigma^\infty Y) \to L_{(s-t)}H_{i+j+2(s-t)}(\Sigma^\infty Y). \qquad (1.4)$$

Definition 1.2.9 *Define the bigraded group $\mathcal{R}^{an}$ to be*

$$\mathcal{R}^{an} \stackrel{def}{=} \bigoplus_{0\le i\le t} L_0H_{2i}(\mathbb{P}^t) \equiv \bigoplus_{0\le i\le t} \pi_{2i}(\mathcal{C}_0(\mathbb{P}^t)).$$

Proposition 1.2.10 (Friedlander-Mazur,[11]) *The bigraded group $\mathcal{R}^{an}$ inherits the structure of a graded ring from the pairing*

$$L_0H_{2i}(\mathbb{P}^t) \otimes L_0H_{2j}(\mathbb{P}^p) \to L_1H_{2(i+j)+2}(\mathbb{P}^t\#\mathbb{P}^p) \cong L_0H_{2(i+j)}(\mathbb{P}^{t+p}).$$

As graded ring, $\mathcal{R}^{an}$ is a polynomial algebra over $\mathbb{Z}$ on two generators

$$\mathcal{R}^{an} \cong \mathbb{Z}[h, s],$$

where $h \in L_0H_0(\mathbb{P}^1)$ and $s \in L_0H_2(\mathbb{P}^1)$.

Using the pairing 1.4 one obtains the following

Corollary 1.2.11 *Let $j : X \hookrightarrow \mathbb{P}^N$ be an algebraic set. The bigraded* *group* $\bigoplus_{p,i} L_pH_{i+2p}(\Sigma^\infty X)$ *is a graded module over* $\mathcal{R}^{an}$, *with action by the generators* $h, s \in \mathcal{R}^{an}$ *taking the form*

$$h : L_pH_{i+2p}(\Sigma^\infty X) \to L_{p-1}H_{i+2p-2}(\Sigma^\infty X)$$

$$s : L_pH_{i+2p}(\Sigma^\infty X) \to L_{p-1}H_{i+2p}(\Sigma^\infty X).$$

In the next remark we briefly explain how the generators h and s act on $\bigoplus_{i,p} L_pH_{i+2p}(\Sigma^\infty X)$.

Remark 1.2.12 The generator $h \in L_0H_0(\mathbb{P}^1) \cong H_0(\mathbb{P}^1, \mathbb{Z})$ is taken to be the class of a point $\mathbb{P}^0$ in $\mathbb{P}^1$, and the generator $s \in L_0H_2(\mathbb{P}^1) \cong H_2(\mathbb{P}^1, \mathbb{Z})$ is the fundamental class of $\mathbb{P}^1$.

The action of $h \in L_0H_0(\mathbb{P}^1)$

$$h : L_pH_{i+2p}(\Sigma^\infty X) \to L_{(p-1)}H_{i+2(p-1)}(\Sigma^\infty X)$$

comes from the map

$$h_{\mathbb{P}^0} : \mathcal{C}_p(X) \to \mathcal{C}_{p-1}(X)$$

induced by the pairing 1.1 of infinite loop spaces

$$\mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(X) \to \mathcal{C}_{p+1}(\Sigma^2 X) \cong \mathcal{C}_{p-1}(X),$$

derived from Proposition 1.2.5. In other words, h turns out to be equal to $h_{\mathbb{P}^0 *} : \pi_i(\mathcal{C}_p(X)) \to \pi_i(\mathcal{C}_{p-1}(X))$. The map $h_{\mathbb{P}^0} : \mathcal{C}_p(X) \to \mathcal{C}_{p-1}(X)$ is essentially given by taking a "cycle" in $\mathcal{C}_p(X)$ and "joining" it to the point $\mathbb{P}^0 \in \mathbb{P}^1$, obtaining a cycle $\Sigma\!\!\!/\sigma$ in $\mathcal{C}_{p+1}(\mathbb{P}^1 \# X) \equiv \mathcal{C}_{p+1}(\Sigma\!\!\!/^2 X)$ and then applying the Complex Suspension Theorem 1.2.6: $\mathcal{C}_{p+1}(\Sigma\!\!\!/^2 X) \cong \mathcal{C}_{p-1}(X)$. Another way of defining $h_{\mathbb{P}^0}$ is to consider the map $i_\star : \mathcal{C}_p(X) \to \mathcal{C}_p(\Sigma\!\!\!/ X)$ induced by the inclusion $i : X \hookrightarrow \Sigma\!\!\!/ X$ and then compose it with the isomorphism $\mathcal{C}_p(\Sigma\!\!\!/ X) \cong \mathcal{C}_{p-1}(X)$.

The action of $s \in L_0 H_2(\mathbb{P}^1)$

$$s : L_p H_{i+2p}(\Sigma\!\!\!/^\infty X) \to L_{(p-1)} H_{i+2(p-1)}(\Sigma\!\!\!/^\infty X)$$

is induced by the pairing 1.1

$$\mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(X) \to \mathcal{C}_{p+1}(\mathbb{P}^1 \# X) \equiv \mathcal{C}_{p+1}(\Sigma\!\!\!/^2 X) \cong \mathcal{C}_{p-1}(X)$$

restricted to

$$S^2 \wedge \mathcal{C}_p(X) \to \mathcal{C}_{p-1}(X),$$

using the natural inclusion $S^2 \equiv \mathbb{P}^1 \hookrightarrow \mathcal{C}_0(\mathbb{P}^1)$. Actually the pairing $S^2 \wedge \mathcal{C}_p(X) \to \mathcal{C}_{p-1}(X)$ is given by taking $(\mathbb{P}^0, \sigma) \in S^2 \times \mathcal{C}_p(X)$ to $h_{\mathbb{P}^0}(\sigma) \in \mathcal{C}_{p-1}(X)$.

Definition 1.2.13 *The p-fold composition of s for $p \geq 0$,*

$$s^p : L_p H_{i+2p}(X) \to L_0 H_{i+2p}(\Sigma\!\!\!/^\infty X) \equiv H_{i+2p}(X, \mathbb{Z})$$

is called the (generalized) cycle map. *It can be seen, (cf. [11]) that in the case* $i = 0$, *this is the classical cycle map, see [15], which takes* p*-cycles modulo algebraic equivalence* $(\cong L_pH_{2p}(X))$ *to* $H_{2p}(X;\mathbb{Z})$.

In the next chapter we will provide another description of this generalized cycle map, as well as discuss some of its properties.

Chapter 2

The relative theory

2.1 Relative Lawson homology

We start by introducing another object in the theory, which arises naturally when working over the complex numbers, namely, the (topological) naïve group-completion of the Chow monoid. More precisely:

Definition 2.1.1 *For an algebraic set $j : X \hookrightarrow \mathbb{P}^N$, let $\tilde{\mathcal{C}}_p(X)$ denote the free abelian group generated by the p-dimensional subvarieties of X. Endow $\tilde{\mathcal{C}}_p(X)$ with the topology induced by the quotient map:*

$$\begin{aligned} p : \mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j) &\rightarrow \tilde{\mathcal{C}}_p(X) \\ (\sigma,\tau) &\longmapsto \sigma - \tau. \end{aligned} \tag{2.1}$$

This makes $\tilde{\mathcal{C}}_p(X)$ into an abelian topological group.

Remark 2.1.2 Observe that there is a natural embedding $i : \mathbf{C}_p(X, j) \hookrightarrow \tilde{\mathcal{C}}_p(X)$ given by $\sigma \mapsto \sigma - \underline{o}$, where $\underline{o}$ is the identity element of the monoid $\mathbf{C}_p(X, j)$. This embedding provides $\tilde{\mathcal{C}}_p(X)$ with the following "universal property": For any morphism $f : \mathbf{C}_p(X, j) \to G$ of topological monoids from $\mathbf{C}_p(X, j)$ to an abelian topological group, there exists a morphism $\tilde{f} : \tilde{\mathcal{C}}_p(X) \to G$ of topological groups making the following diagram commute:

$$\begin{array}{ccc} \mathbf{C}_p(X,j) & \xrightarrow{\quad i \quad} & \tilde{\mathcal{C}}_p(X) \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle \tilde{f}} & \\ G & & \end{array}$$

To see this, take any continuous homomorphism $f : \mathbf{C}_p(X, j) \to G$, where G is a topological group, and consider the composition of the map $\delta : G \times G \to G$, sending (g, h) to $g - h$, with $f \times f : G \times G \to G \times G$. It is easy to see that $\delta \circ (f \times f)$ factors through p, yielding a group homomorphism $\tilde{f} : \tilde{\mathcal{C}}_p(X) \to G$, which is automatically continuous, since p is a proclusion.

Recall, from [23] that there is an embedding $\mathbf{C}_p(X,j) \hookrightarrow \mathcal{Z}_{2p}(X)$, where $\mathcal{Z}_{2p}(X)$ is the group of integral 2p-cycles in X with the flat-norm topology. The universal property mentioned in the remark above factors this embedding through an embedding $\tilde{\mathcal{C}}_p(X) \hookrightarrow \mathcal{Z}_{2p}(X)$. In particular one sees that $\tilde{\mathcal{C}}_p(X)$ is Hausdorff. Since $\mathbf{C}_p(X,j)$ is compactly generated and locally compact, Theorem 4.3 of [32] implies that $\mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j)$ is also compactly generated. This fact and the Hausdorff property of $\tilde{\mathcal{C}}_p(X)$ make $\tilde{\mathcal{C}}_p(X)$ compactly generated with the topology given by p, as observed in 2.6 of [32]. More precisely, one can see that $\tilde{\mathcal{C}}_p(X)$ has the "weak" topology generated by the compact subsets

$$F_{\alpha,\beta} \stackrel{def}{=} p(\mathbf{C}_{p,\alpha}(X) \times \mathbf{C}_{p,\beta}(X)),$$

where $\mathbf{C}_{p,\alpha}(X)$ and $\mathbf{C}_{p,\beta}(X)$ are connected components of the Chow monoid $\mathbf{C}_p(X,j)$, and p is the quotient map 2.1. For more details about the basics on the topology of $\tilde{\mathcal{C}}_p(X)$, look at Lemma 3.1.4.

Notice that there is a natural morphism (in the homotopy category) from the cycle space $\mathcal{C}_p(X)$ into the group $\tilde{\mathcal{C}}_p(X)$, defined as follows:

By functoriality of the topological bar construction, (cf. [26]), the inclusion map $i : \mathbf{C}_p(X,j) \hookrightarrow \tilde{\mathcal{C}}_p(X)$ induces a morphism of classifying monoids

$$B(i) : B(\mathbf{C}_p(X,j)) \to B(\tilde{\mathcal{C}}_p(X)),$$

which, in turn, induces a morphism

$$\tilde{\imath} : \Omega B(\mathbf{C}_p(X,j)^{an}) \to \Omega B(\tilde{\mathcal{C}}_p(X)).$$

The first monoid is, by definition, the cycle space $\mathcal{C}_p(X)$ and the second one (which is actually a group) is homotopy equivalent to $\tilde{\mathcal{C}}_p(X)$, according to Remark 1.1.4 . An important feature of this group is the following result proven by Friedlander and Lawson [13]

Theorem 2.1.3 *The natural map*

$$\tilde{\imath} : \mathcal{C}_p(X) \to \tilde{\mathcal{C}}_p(X)$$

is a homotopy equivalence.

In particular one sees that there is a homotopy equivalence between the identity component of $\mathcal{C}_p(X)$ and that of $\tilde{\mathcal{C}}_p(X)$. Also this provides an alternative definition for Lawson homology, namely, we may define

$$L_pH_{i+2p}(X) \stackrel{def}{=} \pi_i(\tilde{\mathcal{C}}_p(X), e). \tag{2.2}$$

At this stage we are able to define the relative Lawson homology, which can be done in two different ways:

Definition 2.1.4

(a) *Let $Y \subset X$ be a closed algebraic subset of the algebraic set $j : X \hookrightarrow \mathbb{P}^N$. Naturally, $\tilde{\mathcal{C}}_p(Y)$ is a closed subgroup of the topological group $\tilde{\mathcal{C}}_p(X)$.*

Define the (relative) Lawson homology of the pair (X,Y) as

$$L_pH_{i+2p}(X,Y) \stackrel{def}{=} \pi_i(\tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(Y)),$$

where $\tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(Y)$ denotes the quotient topological group.

Notice that this definition only works for algebraic sets over $\mathbb{C}$. A modified definition which has its counterpart for fields of arbitrary characteristic is as follows:

(b) *For an algebraic subset $Y \subset X$, consider the monoid (see Remark 1.1.4) $\mathcal{C}_p(Y)$ as a transformation monoid acting on $\mathcal{C}_p(X)$ by translations. Let*

$$\mathcal{C}_p(X,Y) \stackrel{def}{=} B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *)$$

be the homotopy quotient of $\mathcal{C}_p(X)$ and $\mathcal{C}_p(Y)$ obtained via the triple bar construction, as in [26]. Define the relative Lawson homology of the pair (X,Y) as

$$L_pH_{i+2p}(X,Y) \stackrel{def}{=} \pi_i(B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *)).$$

From this second definition one easily obtains the following

Proposition 2.1.5 *There exists a long exact sequence for the Lawson homology of a pair of algebraic sets (X,Y). Namely, the following sequence* *is exact:*

$$\ldots \to L_pH_{i+1+2p}(X,Y) \xrightarrow{\delta} L_pH_{i+2p}(Y) \xrightarrow{i} L_pH_{i+2p}(X) \to \ldots$$

Proof

The proposition is a consequence of the following lemma:

Lemma 2.1.6 *The sequence of maps*

$$\mathcal{C}_p(Y) \to \mathcal{C}_p(X) \to \mathcal{C}_p(X,Y)$$

is a quasifibration.

Proof

By definition, $\mathcal{C}_p(Y) = \Omega B(\mathbf{C}_p(Y,j')^{an})$, where j' is the composition $Y \hookrightarrow X \overset{j}{\hookrightarrow} \mathbb{P}^n$. Since $B(\mathbf{C}_p(Y,j'))$ is connected, we can consider $\mathcal{C}_p(Y)$ as a group-like transformation monoid (see [33]) acting on $\mathcal{C}_p(X)$ and on itself by translations. (Actually we just need a group-like H-space, which is suitable for the l-adic theory). It follows from Proposition 7.9 of [26] that we have a quasifibration sequence

$$\mathcal{C}_p(Y) \xrightarrow{i} \mathcal{C}_p(X) \xrightarrow{\tau} B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *) \xrightarrow{q} B(\mathcal{C}_p(Y)),$$

where i is the map induced by the inclusion, and τ sends $\sigma \in \mathcal{C}_p(X)$ to $|\sigma, \underline{o}, *| \in B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *)$. The lemma now follows from the definitions.

□

The unifying element in the two approaches is the following result proven by Lawson [21]

Proposition 2.1.7 *For a pair of algebraic sets* (X,Y) *the quotient map* $\tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(Y)$ *admits a local cross-section. In particular one has a principal fibration:*

$$\tilde{\mathcal{C}}_p(Y) \xrightarrow{i} \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X,Y) \equiv \tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(Y).$$

Corollary 2.1.8 *The natural maps*

$$\mathcal{C}_p(X) \to \tilde{\mathcal{C}}_p(X) \quad \textit{and} \quad \mathcal{C}_p(Y) \to \tilde{\mathcal{C}}_p(Y)$$

induce an equivalence of quasifibrations:

$$\begin{array}{ccccc} \mathcal{C}_p(Y) & \longrightarrow & \mathcal{C}_p(X) & \longrightarrow & \mathcal{C}_p(X,Y) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{C}}_p(Y) & \longrightarrow & \tilde{\mathcal{C}}_p(X) & \longrightarrow & \tilde{\mathcal{C}}_p(X,Y). \end{array}$$

Proof

By functoriality of the bar construction one has a map of quasifibrations

$$\begin{array}{ccccc} \mathcal{C}_p(Y) & \longrightarrow & \mathcal{C}_p(X) & \longrightarrow & B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{C}}_p(Y) & \longrightarrow & \tilde{\mathcal{C}}_p(X) & \longrightarrow & B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *), \end{array}$$

induced by the maps $\mathcal{C}_p(X) \to \tilde{\mathcal{C}}_p(X)$ and $\mathcal{C}_p(Y) \to \tilde{\mathcal{C}}_p(Y)$. Since those maps are homotopy equivalences so is $B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *) \to B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *)$, cf. Proposition 7.3(ii) of [26]. Therefore we have an equivalence of quasifibrations. On the other hand, the quotient map $\tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X, Y)$ induces a map

$$B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *) \to \tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(Y)$$

compatible with the inclusion $\tilde{\mathcal{C}}_p(X) \to B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *)$, and therefore we have another map of quasifibrations:

$$\begin{array}{ccccc} \tilde{\mathcal{C}}_p(Y) & \longrightarrow & \tilde{\mathcal{C}}_p(X) & \longrightarrow & \tilde{\mathcal{C}}_p(X, Y) \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{C}}_p(Y) & \longrightarrow & \tilde{\mathcal{C}}_p(X) & \longrightarrow & B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *). \end{array}$$

The same argument as before shows that $B(\tilde{\mathcal{C}}_p(X), \tilde{\mathcal{C}}_p(Y), *) \to \tilde{\mathcal{C}}_p(X, Y)$ is a homotopy equivalence. This proves the corollary. $\square$

After the previous discussion we use both definitions of Lawson homology interchangeably, whichever is more convenient in the context it is being used.

As a final remark we observe that a given morphism of pairs $f:(X,Y)\to(X',Y')$, i.e., an algebraic map $f:X\to X'$ taking Y into Y', induces, by functoriality, maps

$$f_*: B(\mathcal{C}_p(X),\mathcal{C}_p(Y),*)\to B(\mathcal{C}_p(X'),\mathcal{C}_p(Y'),*)$$

and

$$f_*:\tilde{\mathcal{C}}_p(X,Y)\to\tilde{\mathcal{C}}_p(X',Y').$$

In particular f induces a morphism of long exact sequences in Lawson homology.

2.2 More about the cycle map

In this section we show that the action of $s\in\mathcal{R}^{an}$ on Lawson homology (as described in Section 1.2) extends to an action on the relative Lawson homology, inducing a morphism of long exact sequences for pairs. We also analyze in more detail the "generalized cycle map" $s^p: L_pH_{i+2p}(X)\to L_0H_{2p+i}(X)$. However, before we proceed we need the following remark, from [23].

Remark 2.2.1 Consider a pair of algebraic sets $Y\subset X$, $X\overset{j}{\hookrightarrow}\mathbb{P}^N$. Since the suspension $\Sigma\!\!\!/ : \mathbf{C}_p(X,j)\to\mathbf{C}_{p+1}(\Sigma\!\!\!/ X,\Sigma\!\!\!/ j)$ restricts to $\Sigma\!\!\!/ :\mathbf{C}_p(Y,j')\to\mathbf{C}_{p+1}(\Sigma\!\!\!/ Y,\Sigma\!\!\!/ j')$, it therefore induces a morphism of pairs of monoids

$$\Sigma\!\!\!\!/ : (\mathbf{C}_p(X,j), \mathbf{C}_p(Y,j')) \to (\mathbf{C}_{p+1}(\Sigma\!\!\!\!/ X, \Sigma\!\!\!\!/ j), \mathbf{C}_{p+1}(\Sigma\!\!\!\!/ Y, \Sigma\!\!\!\!/ j')).$$

By functoriality of the triple bar construction, cf. [26], $\Sigma\!\!\!\!/$ induces a morphism of relative cycle spaces

$$\Sigma\!\!\!\!/ : \mathcal{C}_p(X,Y) \to \mathcal{C}_{p+1}(\Sigma\!\!\!\!/ X, \Sigma\!\!\!\!/ Y)$$

and hence a morphism on relative Lawson homology:

$$\Sigma\!\!\!\!/_* : L_pH_{i+2p}(X,Y) \to L_{(p+1)}H_{i+2(p+1)}(\Sigma\!\!\!\!/ X, \Sigma\!\!\!\!/ Y).$$

The later fits into a morphism of long exact sequences:

$$\begin{array}{ccccc}
\ldots L_pH_{i+2p}(X) & \longrightarrow & L_pH_{i+2p}(X,Y) & \longrightarrow & L_pH_{i+2p-1}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\ldots L_{(p+1)}H_{i+2(p+1)}(\Sigma\!\!\!\!/ X) & \longrightarrow & L_{(p+1)}H_{i+2(p+1)}(\Sigma\!\!\!\!/ X, \Sigma\!\!\!\!/ Y) & \longrightarrow & L_{p+1}H_{2p+i+1}(\Sigma\!\!\!\!/ Y)
\end{array}$$

Using the five lemma and the complex suspension Theorem 1.2.6 one concludes that the map $\Sigma\!\!\!\!/ : \mathcal{C}_p(X,Y) \to \mathcal{C}_{p+1}(\Sigma\!\!\!\!/ X, \Sigma\!\!\!\!/ Y)$ is actually a homotopy equivalence.

In a similar fashion to the above remark, let us consider a pair of algebraic sets $Y \subset X \stackrel{j}{\hookrightarrow} \mathbb{P}^N$. Observe that the join map $\#_X : \mathcal{C}_0(\mathbb{P}^1) \times \mathcal{C}_p(X) \to \mathcal{C}_{p+1}(\mathbb{P}^1 \# X)$, of Section 1.2, restricts to the join $\#_Y : \mathcal{C}_0(\mathbb{P}^1) \times \mathcal{C}_p(Y) \to \mathcal{C}_{p+1}(\mathbb{P}^1 \# Y)$. Those are morphisms of topological monoids, which make, for every $x \in \mathcal{C}_0(\mathbb{P}^1)$, the map

$$\begin{aligned} \#_{X,x} : \mathcal{C}_p(X) &\to \mathcal{C}_{p+1}(\mathbb{P}^1 \# X) \\ \sigma &\mapsto x \# \sigma \end{aligned}$$

equivariant with respect to the actions of $\mathcal{C}_p(Y)$ and $\mathcal{C}_{p+1}(\mathbb{P}^1\#Y)$ on $\mathcal{C}_p(X)$ and $\mathcal{C}_{p+1}(\mathbb{P}^1\#X)$, respectively. It follows, cf. [26] §7, that the maps $\#_{X,x}$, for $x \in \mathcal{C}_0(\mathbb{P}^1)$, induce maps

$$\#_{X,Y,x} : B(\mathcal{C}_p(X), \mathcal{C}_p(Y), *) \to B(\mathcal{C}_{p+1}(\mathbb{P}^1\#X), \mathcal{C}_{p+1}(\mathbb{P}^1\#Y), *),$$

and therefore a map

$$\#_{X,Y} : \mathcal{C}_0(\mathbb{P}^1) \times \mathcal{C}_p(X,Y) \to \mathcal{C}_p(\mathbb{P}^1\#X, \mathbb{P}^1\#Y).$$

Since $\#_X$ descends to the smash product $\mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(X)$, so it does in the level of simplicial bar construction, and therefore we obtain (after geometric realization) a map

$$\#_{X,Y} : \mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(X,Y) \to \mathcal{C}_p(\mathbb{P}^1\#X, \mathbb{P}^1\#Y).$$

The above mentioned invariance of $\#_X$, together with the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(X) & \xrightarrow{\#_X} & \mathcal{C}_{p+1}(\mathbb{P}^1\#X) \\ {\scriptstyle id\wedge i}\uparrow & & \uparrow{\scriptstyle id\wedge i} \\ \mathcal{C}_0(\mathbb{P}^1) \wedge \mathcal{C}_p(Y) & \xrightarrow[\#_Y]{} & \mathcal{C}_{p+1}(\mathbb{P}^1\#Y) \end{array}$$

shows that we actually have a morphism of quasifibrations

$$\begin{array}{ccccc}
\mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(Y) & \longrightarrow & \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X) & \longrightarrow & \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X,Y)\\
\downarrow \#_Y & & \downarrow \#_X & & \downarrow \#_{X,Y}\\
\mathcal{C}_{p+1}(\mathbb{P}^1\#Y) & \longrightarrow & \mathcal{C}_p(\mathbb{P}^1\#X) & \longrightarrow & \mathcal{C}_{p+1}(\mathbb{P}^1\#X,\mathbb{P}^1\#Y).
\end{array}$$

As in Chapter 1, one can use the map $\#_{X,Y}$ to define naturally a map from $\pi_i(\mathcal{C}_p(X,Y))$ to $\pi_{i+2}(\mathcal{C}_{p+1}(\mathbb{P}^1\#X,\mathbb{P}^1\#Y))$. The latter group is isomorphic to $\pi_{i+2}(\mathcal{C}_{p-1}(X,Y)) \equiv L_{p-1}H_{2p+i}(X,Y)$ via the double complex suspension Σ^2, according to Remark 2.2.1 . The final output of all this is the relative *s-map*

$$s_{X,Y} : L_pH_{i+2p}(X,Y) \to L_{p-1}H_{2p+i}(X,Y).$$

Summarizing, we have the following Proposition-definition:

Proposition 2.2.2 *Let $Y \subset X \overset{j}{\hookrightarrow} \mathbb{P}^N$ be a pair of algebraic sets. The join operations*

$$\#_X : \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X) \to \mathcal{C}_{p+1}(\mathbb{P}^1\#X)$$

and

$$\#_Y : \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(Y) \to \mathcal{C}_{p+1}(\mathbb{P}^1\#Y)$$

induce a "relative" join operation

$$\#_{X,Y} : \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X,Y) \to \mathcal{C}_{p+1}(\mathbb{P}^1\#X,\mathbb{P}^1\#Y),$$

which fits into a morphism of quasifibrations

$$\begin{array}{ccccc} \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(Y) & \longrightarrow & \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X) & \longrightarrow & \mathcal{C}_0(\mathbb{P}^1)\wedge\mathcal{C}_p(X,Y) \\ \downarrow{\scriptstyle \#_Y} & & \downarrow{\scriptstyle \#_X} & & \downarrow{\scriptstyle \#_{X,Y}} \\ \mathcal{C}_{p+1}(\mathbb{P}^1\#Y) & \longrightarrow & \mathcal{C}_p(\mathbb{P}^1\#X) & \longrightarrow & \mathcal{C}_{p+1}(\mathbb{P}^1\#X,\mathbb{P}^1\#Y). \end{array}$$

Naturally, one can define a relative s-map

$$s_{X,Y} : L_pH_{i+2p}(X,Y) \to L_{p-1}H_{i+2p}(X,Y)$$

and obtain a morphism of long exact sequences, for the Lawson homology of a pair:

$$\begin{array}{ccccccc} \ldots L_pH_{i+2p}(Y) & \longrightarrow & L_pH_{i+2p}(X) & \longrightarrow & L_pH_{i+2p}(X,Y) & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ \ldots L_{p-1}H_{i+2p}(Y) & \longrightarrow & L_{p-1}H_{i+2p}(X) & \longrightarrow & L_{p-1}H_{i+2p}(X,Y) & \longrightarrow & \ldots \end{array}$$

Let us go back to Remark 1.2.12 . There we explain how the generators h and s of $\mathcal{R}^{an}$ act on Lawson homology, in particular we discuss the s map. The discussion was made in terms of the homotopy-theoretic group-completion. The same arguments can be carried out, when working over $\mathbb{C}$, in terms of the naïve group completion of the Chow monoid $\mathbf{C}_p(X,j)$, taken with the quotient topology, as in Section 2.1. This will shed more light in the complex case and yield another interpretation of the "generalized cycle map" of Chapter 1.

More concretely, let $X \overset{j}{\hookrightarrow} \mathbb{P}^N$ be an algebraic set and let $\tilde{\mathcal{C}}_p(X)$ be the naïve group completion of the Chow monoid $\mathbf{C}_p(X,j)$ taken with the quotient topology, an in 2.1. Recall that the rational continuous map

$$\# : \mathbf{C}_0(\mathbb{P}^1, id) \times \mathbf{C}_p(X, j) \to \mathbf{C}_{p+1}(\mathbb{P}^1 \# X, i\#j)$$

of Proposition 1.2.5 is biadditive. Therefore, it yields a morphism of topological groups

$$\tilde{\#} : \tilde{\mathcal{C}}_0(\mathbb{P}^1) \times \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_{p+1}(\mathbb{P}^1 \# X),$$

by the universal property of the naïve group completion shown in Remark 2.1.2 . As before this map descends to

$$\tilde{\#} : \tilde{\mathcal{C}}_0(\mathbb{P}^1) \wedge \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_{p+1}(\mathbb{P}^1 \# X).$$

In his paper [2], F. Almgren established a remarkable isomorphism

$$\mathcal{A} : \pi_i(\mathcal{Z}_{2p}(X), \underline{o}) \xrightarrow{\cong} H_{i+2p}(X, \mathbb{Z}),$$

where X is any compact, Lipschitz, neighborhood retract in some $\mathbb{R}^M$ and $\mathcal{Z}_{2p}(X)$ is the space of integral cycles with the flat-norm topology. See [9], [20], [2] for further definitions. On the other hand, as observed in Remark 2.1.2, in case $X \overset{j}{\hookrightarrow} \mathbb{P}^N$ is an algebraic set, there is an embedding $\tilde{\mathcal{C}}_p(X) \overset{e}{\hookrightarrow} \mathcal{Z}_{2p}(X)$. Composing $\mathcal{A}$ with the map e_*, induced by e in homotopy, yields a map

$$\mathcal{A} \circ e_* : \pi_i(\tilde{\mathcal{C}}_p(X)) \equiv L_pH_{i+2p}(X) \to H_{i+2p}(X, \mathbb{Z}).$$

Theorem 2.2.3 *For an algebraic set X, the above map $\mathcal{A} \circ e_*$ coincides with the "generalized cycle map" $s^p : L_pH_{i+2p}(X) \to H_{i+2p}(X, \mathbb{Z})$.*

Proof

We use induction on the dimension of the cycles. For cycles of dimension zero, $\tilde{\mathcal{C}}_0(Y) \equiv \mathcal{Z}_0(Y)$ for any algebraic set Y. Furthermore, the cycle map from $\pi_i(\tilde{\mathcal{C}}_0(Y)) \equiv \pi_i(\mathcal{Z}_0(Y))$ into $H_i(Y;\mathbb{Z})$ is actually the Dold-Thom isomorphism [8], as pointed out in [14] and [2].

Let us assume, now, that for any algebraic set Y the cycle map

$$s^r : L_rH_{i+2r}(Y) \to H_{i+2r}(Y;\mathbb{Z})$$

is equal to the composition $\mathcal{A} \circ e_*$, where e is the inclusion $\tilde{\mathcal{C}}_r(Y) \hookrightarrow \mathcal{Z}_{2r}(Y)$ and

$$\mathcal{A} : \pi_i(\mathcal{Z}_{2r}(Y)) \to H_{i+2r}(Y;\mathbb{Z})$$

is Almgren's isomorphism, for all r, $r \leq r_0$ and any $i \geq 0$.

In order to conclude the proof, we need to show the following three steps:

Step 1 The complex suspension map actually extends to a continuous homomorphism

$$\Sigma\!\!\!/ : I_k(Y) \to I_{k+2}(\Sigma\!\!\!/ Y)$$

for all $k \geq 0$, where $I_k(Y)$ denotes the group of integral currents taken with the flat norm topology. This homomorphism commutes with the boundary operator, giving a map of chain complexes, between the complexes of integral currents of X and $\Sigma\!\!\!/ X$, respectively. In particular, it gives a morphism of topological groups

$$\Sigma\!\!\!/ : \mathcal{Z}_{2r}(Y) \to \mathcal{Z}_{2r+2}(\Sigma\!\!\!/ Y)$$

extending $\Sigma\!\!\!/ : \tilde{\mathcal{C}}_r(Y) \to \tilde{\mathcal{C}}_{r+1}(\Sigma\!\!\!/ Y)$.

Step 2 The homomorphism $\Sigma\!\!\!/_*$ induced in homotopy by $\Sigma\!\!\!/ : \mathcal{Z}_{2r}(Y) \to \mathcal{Z}_{2r+2}(\Sigma\!\!\!/ Y)$ corresponds to the Thom isomorphism via Almgren's isomorphism. In other words, the following diagram commutes:

$$\begin{array}{ccc} \pi_i(\mathcal{Z}_{2r}(Y) & \xrightarrow{\Sigma\!\!\!/_*} & \pi_i(\mathcal{Z}_{2r+2}(\Sigma\!\!\!/ Y) \\ \mathcal{A}\downarrow & & \downarrow\mathcal{A} \\ H_{i+2r}(Y;\mathbb{Z}) & \xrightarrow[\tau]{} & H_{i+2r+2}(\Sigma\!\!\!/ Y;\mathbb{Z}) \end{array}$$

Diag. 1

where $\tau : H_{i+2r}(Y;\mathbb{Z}) \to H_{i+2r+2}(\Sigma\!\!\!/ Y;\mathbb{Z})$ is the Thom isomorphism.

Step 3 The join map $\# : \tilde{\mathcal{C}}_0(\mathbb{P}^1) \wedge \tilde{\mathcal{C}}_r(Y) \to \tilde{\mathcal{C}}_{r+1}(\mathbb{P}^1 \# Y)$ extends to a map

$$\# : \mathcal{Z}_0(\mathbb{P}^1) \wedge \mathcal{Z}_{2r}(Y) \to \mathcal{Z}_{2r+2}(\mathbb{P}^1 \# Y),$$

which naturally induces a homomorphism

$$\#_\star : \pi_k(\mathcal{Z}_{2r}(Y)) \to \pi_{k+2}(\mathcal{Z}_{2r+2}(\mathbb{P}^1 \# Y)).$$

Identifying $\mathbb{P}^1 \# Y$ with $\Sigma^2 Y$, we get a commutative diagram

$$\begin{array}{ccc}
\pi_k(\mathcal{Z}_{2r}(Y)) & \xrightarrow{\mathcal{A}} & H_{k+2r}(Y) \\
\Big\downarrow \#_\star & & \Big\downarrow \tau_Y \\
 & & H_{k+2r+2}(\Sigma Y) \\
 & & \Big\downarrow \tau_{\Sigma Y} \\
\pi_{k+2}(\mathcal{Z}_{2r+2}(\Sigma^2 Y)) & \xrightarrow[\mathcal{A}]{} & H_{k+2r+4}(\Sigma^2 Y)
\end{array}$$

Diag. 2

where the $\mathcal{A}$'s are Almgren's maps and the τ's are Thom's morphisms.

Let us complete our induction argument before we proceed to the proof of the three steps above. Recall that the map

$$s : \pi_i(\tilde{\mathcal{C}}_{r_0+1}(X)) \to \pi_{i+2}(\tilde{\mathcal{C}}_{r_0}(X))$$

is the composition of

$$\#_\star : \pi_i(\tilde{\mathcal{C}}_{r_0+1}(X)) \to \pi_{i+2}(\tilde{\mathcal{C}}_{r_0+2}(\mathbb{P}^1 \# X))$$

with the inverses of the isomorphisms

$$\not\Sigma_1 : \pi_{i+2}(\tilde{\mathcal{C}}_{r_0}(X)) \to \pi_{i+2}(\tilde{\mathcal{C}}_{r_0+1}(\not\Sigma X))$$

and

$$\not\Sigma_2 : \pi_{i+2}(\tilde{\mathcal{C}}_{r_0+1}(\not\Sigma X)) \to \pi_{i+2}(\tilde{\mathcal{C}}_{r_0+1}(\not\Sigma^2 X)).$$

Gathering all the above information together one obtains the following situation: (see *Diag. 3* below)

By induction, the composition $\mathcal{A}_1 \circ e_1$ (in the bottom line of the diagram) is simply the cycle map $s^{r_0} : L_{r_0}H_{i+2+2r_0}(X) \to H_{i+2+2r_0}(X;\mathbb{Z})$. By definition, $s^{r_0+1} : L_{r_0+1}H_{i+2+2r_0}(X) \to H_{i+2+2r_0}(X;\mathbb{Z})$ is equal to the composition $s^{r_0} \circ s$ and hence:

$$\begin{aligned} s^{r_0+1} &= s^{r_0} \circ s = \mathcal{A}_1 \circ e_1 \circ (\not\Sigma_1)^{-1} \circ (\not\Sigma_2)^{-1} \circ \#_\star \\ &= \mathcal{A}_1 \circ (\not\Sigma'_1)^{-1} \circ e_2 \circ (\not\Sigma_2)^{-1} \circ \#_\star \\ &= \tau_1^{-1} \circ \tau_2^{-1} \circ \mathcal{A}_3 \circ \#'_\star \circ e_4 \\ &= \mathcal{A}_4 \circ e_4. \end{aligned}$$

$$
\begin{array}{ccccc}
\pi_i(\tilde{\mathcal{C}}_{r_0+1}(X)) & \xrightarrow{e_4} & \pi_i(\mathcal{Z}_{2r_0+2}(X)) & & \\
\downarrow \#_\star & & \downarrow \#'_\star & & \\
\pi_{i+2}(\tilde{\mathcal{C}}_{r_0+2}(\Sigma^2 X)) & \xrightarrow{e_3} & \pi_{i+2}(\mathcal{Z}_{2r_0+4}(\Sigma^2 X)) & \xrightarrow{\mathcal{A}_3} & H_{i+2r_0+6}(\Sigma^2 X) \\
\uparrow \Sigma_2 & & \uparrow \Sigma'_2 & & \uparrow \tau_2 \\
\pi_{i+2}(\tilde{\mathcal{C}}_{r_0+1}(\Sigma X)) & \xrightarrow{e_2} & \pi_{i+2}(\mathcal{Z}_{2r_0+2}(\Sigma X)) & \xrightarrow{\mathcal{A}_2} & H_{i+2r_0+4}(\Sigma X) \\
\uparrow \Sigma_1 & & \uparrow \Sigma'_1 & & \uparrow \tau_1 \\
\pi_{i+2}(\tilde{\mathcal{C}}_{r_0}(X)) & \xrightarrow{e_1} & \pi_{i+2}(\mathcal{Z}_{2r_0}(X)) & \xrightarrow{\mathcal{A}_1} & H_{i+2r_0+2}(X)
\end{array}
\qquad s^{r_0}: \pi_i(\tilde{\mathcal{C}}_{r_0+1}(X)) \to \pi_{i+2}(\tilde{\mathcal{C}}_{r_0}(X)), \quad \mathcal{A}_4: \pi_i(\mathcal{Z}_{2r_0+2}(X)) \to H_{i+2r_0+2}(X)
$$

Diag. 3

Now we proceed to prove Steps 1, 2 and 3 above in order to conclude the proof of the theorem.

Proof of Step 1.

<u>Assertion :</u> *Let $X \hookrightarrow \mathbb{P}^n$ be an algebraic set. We extend the 'complex join' map of 1.2.5 to a continuous homomorphism*

$$\# : I_r(\mathbb{P}^1) \times I_k(X) \to I_{r+k+2}(\mathbb{P}^1 \# X),$$

where $I_*(-)$ denotes the space of integral currents with the flat norm topology. This map satisfies the following properties:

- *There is a constant $\gamma_{k,r}$ such that*

$$M(\sigma \# \tau) \leq \gamma_{k,r} M(\gamma) M(\tau), \tag{2.3}$$

where M denotes the mass norm,and

$$\partial(\sigma \# \tau) = (\partial\sigma) \# \tau + (-1)^r \sigma \# (\partial\tau). \tag{2.4}$$

In particular the join of two cycles still is a cycle. Observe that taking $\sigma = \mathbb{P}^0 \subset \mathbb{P}^1$ one obtains:

<u>*Corollary:*</u> The complex suspension $\not\Sigma$ gives a map

$$\not\Sigma : I_k(X) \to I_{k+2}(\not\Sigma X)$$

satisfying:

- *There is a constant* γ_k *such that for every* $\sigma \in I_k(X)$ *we have*

$$M(\not\Sigma\sigma) \leq \gamma_k M(\sigma), \tag{2.5}$$

  and

$$\partial(\not\Sigma\sigma) = (-1)^k \not\Sigma(\partial\sigma). \tag{2.6}$$

Therefore, $\not\Sigma$ yields a chain map

$$\not\Sigma : I_*(X) \to I_{*+2}(\not\Sigma X)$$

of degree 2 between the complexes of integral currents of X and $\not\Sigma X$, respectively.

We just need to verify the above assertions for the case $X = \mathbb{P}^n$, for (cf. [9]) we can find a constant ρ so that

$$M_{\mathbb{P}^n}(T) \leq M_X(T) \leq \rho M_{\mathbb{P}^n}(T),$$

for any integral current T.

Now, consider the projective bundle $E = \mathbb{P}(\pi_1^* H \oplus \pi_2^* H) \xrightarrow{p} \mathbb{P}^r \times \mathbb{P}^n$, where π_1 and π_2 are the canonical projections and the H's are hyperplanes the bundles. The total space E can also be obtained by blowing-up $\mathbb{P}^{r+n+1}$ along two disjoint linear subspaces, $\mathbb{P}^r$ and $\mathbb{P}^n$. Let $b : E \to \mathbb{P}^{r+n+1}$ denote the blow-down map.

One can see clearly that if V and W are subvarieties of $\mathbb{P}^r$ and $\mathbb{P}^n$, respectively, then $b(p^{-1}(V \times W)) = V\#W \subset \mathbb{P}^{r+n+1}$, where $V\#W$ is the projective join of V with W.

Now we claim that the bundle map p provides a pull back map

$$p^{\#} : I_k(\mathbb{P}^r \times \mathbb{P}^n) \to I_{k+2}(E)$$

satisfying

$$p^{\#} \circ \partial T = \partial \circ p^{\#} T \tag{2.7}$$

and

$$M(p^{\#} T) \leq a_k M(T) \tag{2.8}$$

for $T \in I_k(\mathbb{P}^r \times \mathbb{P}^n)$, where a_k is a constant depending on k.

With this in hand, define

$$\# : I_p(\mathbb{P}^r) \times I_q(\mathbb{P}^n) \to I_{p+k+2}(\mathbb{P}^{r+n+1})$$

$$(\sigma, \tau) \mapsto b_{\#}(p^{\#}(\sigma \times \tau)).$$

Here $\sigma \times \tau$ denotes the cartesian product of currents, as in [9], §4.1.8. .

We then have

$$\begin{aligned} M(\sigma\#\tau) = \quad & M(b_{\#}(p^{\#}(\sigma \times \tau)) \quad \leq \lambda^{p+k+2} M(p^{\#}(\sigma \times \tau)) \\ \leq \quad & \lambda^{p+k+2} a_k M(\sigma \times \tau) \quad \leq \lambda^{p+k+2} a_k c M(\sigma) M(\tau), \end{aligned}$$

where $\lambda = Lip(b)$ is the Lipschitz constant of b, and c is obtained as in [9] §4.1.8.. Setting $\gamma_{p,k} = \lambda^{p+k+1} a_k c$ we have the desired inequality:

$$M(\sigma\#\tau) \leq \gamma_{p,k} M(\sigma) M(\tau).$$

The identity for the boundary of the join of two currents follows from equation 2.7 above and the corresponding identity for the product of currents, cf. [9] §4.1.8 .

To prove the claim above, we just observe that it follows from the compactness of the spaces involved, the local triviality of the bundle and the corresponding properties for the trivial bundle. See Brothers [7] for more details.

DIGRESSION:

Here we make a brief incursion into some of the ideas in Almgren's paper [2], preserving his notation as much as possible. First of all, a definition:

<u>*Definition:*</u> • For each $n = 0, 1, 2, \ldots$, let $\mathcal{I}(1, n)$ be the cell complex of the unit interval $I = [0,1]$ whose 1-cells are the subintervals $[0, 1\cdot 2^{-n}], [1\cdot 2^{-n}, 2\cdot 2^{-n}], \ldots [(2^n - 1)\cdot 2^{-n}, 1]$, and whose 0-cells are the endpoints, $[0], [1\cdot 2^{-n}], [2\cdot 2^{-n}], \ldots, [1]$. One has the usual boundary homomorphism

$$d : \mathcal{I}(1,n) \to \mathcal{I}(1,n)$$

$$d([a,b]) = [b] - [a] \quad \text{for each 1-cell}[a,b]$$

$$d([a]) = 0 \qquad \text{for each 0-cell}[a].$$

• For each $m = 1, 2, 3, \ldots$ and each $n = 0, 1, 2, \ldots$

$$\mathcal{I}(m,n) = \mathcal{I}(1,n) \otimes \ldots \otimes \mathcal{I}(1,n) \quad (m \ times)$$

is a cell complex on I^m, where $\alpha = \alpha_1 \ldots \alpha_m \in \mathcal{I}(m,n)$ is a p-cell and $\dim(\alpha) = p$ if and only if for each $i = 1, \ldots, m$, α_i is a cell in $\mathcal{I}(1,n)$ and $\sum_{i=1}^{m} \dim(\alpha_i) = p$. Correspondingly, $\mathcal{I}(m,n)_p$ is the direct summand of $\mathcal{I}(m,n)$ generated by cells of dimension p. The boundary homomorphism d is given on each cell by

$$d(\alpha) = d(\alpha_1 \otimes \ldots \otimes \alpha_i \ldots \otimes \alpha_m) = \sum_{i=1}^{m} (-1)^{\sigma(i)} \alpha_1 \otimes \ldots \otimes d\alpha_i \ldots \otimes \alpha_m$$

$$\sigma(i) = \sum_{j<i} \dim(\alpha_j).$$

A cell β is a *face* of a cell α if and only if for each $i = 1, \ldots, m$, either $\beta_i = \alpha_i$ or β_i is an endpoint of α_i. The *vertex set* of α consists of all 0-dimensional faces of α.

With this notation in hand, let us recall some standard facts in Geometric Measure Theory. The following result can be found in [10] §6.1:

- *For each CLNR (compact Lipschitz neighborhhod retract) $A \in \mathbb{R}^n$ there are numbers $\nu_A^1 > 0$ and $\nu_A^2 < \infty$ such that, if $T \in \mathcal{Z}_k(A)$, $k > 0$, and $M(T) < \nu_A^1$, then there exists $S \in I_{k+1})(A)$ with*

$$\partial S = T \text{ and } M(S) \leq \nu_A^2 M(T)^{1+\frac{1}{k}}. \tag{2.9}$$

If T and S are as in 2.9, then S is called an M-isoperimetric choice for T. Due to compactness properties of integral currents with the flat norm and semicontinuity of the mass-norm one can make an isoperimetric choice S, for T as above, with the following additional property:

$$M(S) = inf\{M(Q) : Q \in I_{k+1}(A) \text{ and } \partial Q = T\} \tag{2.10}$$

An isoperimetric choice with this additional property is called a *mass minimizing* choice.

As an easy corollary of the above result we obtain the following: (cf. [2] §1.13)

- *For each CLNR A and each positive integer p, there is a number*

$\nu = \nu(p, A) > 0$ *such that if* $T_i \in I_k(A)$, $k > 0$, *for* $i = 1, 2, \ldots, p$, *and*

$$\sum_{i=0}^{p} \partial T_i = 0 \tag{2.11}$$

$$sup\{M(T_i) : i = 1, 2, \ldots, p\} \leq \nu \tag{2.12}$$

then, there is $S \in I_{k+1}(A)$ *which is an* M*-isoperimetric choice for* $(\sum_{i=0}^{p} T_i)$ *(there is even a mass minimizing isoperimetric choice) with*

$$M(S) \leq sup\{M(T_i) : i = 1, 2, \ldots, p\}.$$

Recall that the flat norm $F_A(T)$ of an integral current $T \in I_k(A)$, for a CLNR A is defined as

$$F_A(T) = inf\{M(T + \partial S) + M(S) : S \in I_{k+1}(A)\} \tag{2.13}$$

As an outcome of the above results we also obtain the following corollary

- *For each CLNR A there is a number* $\nu_A^3 > 0$ *such that if* $T \in \mathcal{Z}_k(A)$ *and* $F_A(T) < \nu_A^3$, *then*

  $$F_A(T) = inf\{M(Q) : Q \in I_{k+1}(A)\ \partial Q = T\}. \tag{2.14}$$

  Furthermore, for some $S \in I_{k+1}(A)$, $\partial S = T$ *and* $M(S) = F_A(T)$. If S is as above it is called a F_A*-isoparametric choice* for T.

Combining all the results above we obtain the following Theorem which is one of the main tools of Almgren's constructions in [2].

Theorem 2.2.4 *For any CLNR A there exists a positive number ν_A with the following property: Let*

$$f : \mathcal{I}(m,0)_0 \to \mathcal{Z}_k(A)$$

be any homomorphism satisfying:

$$F_A(f(\alpha), f(\beta)) < \nu_A$$

whenever α and β are 0-cells in the vertex set of some m-cell in $\mathcal{I}(m,0)$. Then one can find a chain map

$$\phi_A^f : \mathcal{I}(m,0) \to I_*(A) \tag{2.15}$$

of degree k, such that

1. $\phi_{|\mathcal{I}(m,0)_0} = f$;
2. *For each 1-cell $\alpha \in \mathcal{I}(m,0)$, $\phi_A^f(\alpha)$ is an F_A-isoperimetric choice for $\phi_A^f(d\alpha)$;*
3. *For each p-cell $\alpha \in \mathcal{I}(m,0)$, $p > 1$ ϕ_A^f is an M-isoperimetric choice for $\phi_A^f(d\alpha)$ as in 2.11 (with $p \geq 2^m$);*
4. *If $\Theta = sup\{F_A(f(\alpha), f(\beta)) : \alpha$ and β are 0-cells lying in the vertex set of some m-cell in $\mathcal{I}(m,0)\}$, then for each p-cell $\alpha \in \mathcal{I}(m,0), p > 1$,*

$$M(\phi_A^f(\alpha)) \leq \rho\Theta,$$

where $\rho > 1$ is a constant satisfying

$$F_{\mathbb{R}^n}(T) \leq F_A(T) \leq \rho F_{\mathbb{R}^n}(T)$$

for all $T \in I_*(A)$;

5. *If ϕ' is another chain map satisfying (1), (2), (3) and (4), then ϕ' is chain homotopic with ϕ^f_A.*

Remark 2.2.5 In his original construction, Almgren used F_A and M-minimizing isoperimetric choices. However one can see that any choice satisfying the first four conditions of the theorem above actually yields the chain homotopy of item 5. in the Theorem.

Now, let us establish the link between what was said and what we are aiming at:

Given a continuous map $f : I^m \to \mathcal{Z}_k(X)$, and a constant $\lambda > 0$, define $N_f(\lambda) > 0$ (using a standard argument with the Lebesgue number of a finite cover of A) so that

$$F_X(f(\alpha), f(\beta)) < \lambda$$

whenever $dist(\alpha, \beta) < 2^{-N_f(\lambda)}$. Now, given f, λ and $N_f(\lambda)$ as above, subdivide I^m so as to obtain the complex $\mathcal{I}(m, n)$, as in <u>*Definition*</u> above, with

$n > N_f(\lambda)$. In particular

$$F_X(f(\alpha), f(\beta)) < \lambda$$

for α, β being 0-cells of same m-cell C in $\mathcal{I}(m,n)$. Therefore, for λ suitably chosen, we can apply Theorem 2.2.4 to $f_{|C}$ ($C \cong I^m$), and in doing so we get chain maps of degree k:

$$\phi_X^{f,n} : \mathcal{I}(m,n)_* \to I_{*+k}(X)$$

for $n \geq N_f(\lambda)$. Observe that two such maps satisfying conditions (1), (2), (3) and (4) are chain homotopic. This construction has the following properties:

P1. Given $f, f' : (I^m, \dot{I}^m) \to (\mathcal{Z}_k(X), 0)$, homotopic maps, then $\phi_X^{f,n}$ and $\phi_X^{f',n}$ are chain homotopic maps.

P2. There is a sequence of chain maps and chain homotopies connecting the various $\phi_X^{f,n}$, for $n > N_f(\nu)$, with ν as in Theorem 2.2.4.

These two properties allow one to (well-)define the Almgren map:

Definition 2.2.6 *Define*

$$\mathcal{A}([f]) \stackrel{def}{=} [\sum_i \phi_X^f(\alpha_i)] \in H_{k+m}(X; \mathbb{Z})$$

where ϕ_X^f *is any* $\phi_X^{f,n}$ *with* $n > N_f(\nu)$, *and the* α_i*'s are the m-cells of* $\mathcal{I}(m,n)$.

The construction actually shows that we do have a well-defined homology class.

Proof of Step 2 (Analysis of the complex suspension)

Our purpose is to prove the commutativity of *Diag. 1* above. As we saw in **Step 1**, the complex suspension $\Sigma\!\!\!/$ gives a chain map of degree 2 between the complexes of integral currents of X and $\Sigma\!\!\!/ X$, respectively. It is clear that the map induced in homology by this chain map is the Thom isomorphism, i.e., for a class $[\sigma] \in H_k(X)$, we have that $[\Sigma\!\!\!/\sigma] = \tau([\sigma])$, where τ denotes the Thom isomorphism.

Now, choose a representative $f : (I^m, \overset{\bullet}{I}{}^m) \to \mathcal{Z}_k(X)$ for a class $[f] \in \pi_m(\mathcal{Z}_k(X))$, and let $\Sigma\!\!\!/ f$ denote the corresponding representative for $\Sigma\!\!\!/_*[f]$.

Let

- Let ν^1, ν^2, ν^3 and ν be the constants ν_X^1, ν_X^2, ν_X^3 and $\nu_X$as in 2.9, 2.14 and 2.11, respectively, where $\nu = \nu(N(m)+2, X)$ with $N(m)$ being the total number of cells in $\mathcal{I}(m, 0)$. Let ρ be as in Theorem 2.2.4, item 4.
- Let ν'^1, ν'^2, ν'^3, ν' and ρ' be the analogous constants for $\Sigma\!\!\!/ X$.
- Let γ_k be so that

$$M(\Sigma\!\!\!/\sigma) \leq \gamma_k M(\sigma),$$

for $\sigma \in I_k(X)$.

Finally, choose $\delta > 0$ satisfying:

$$\delta < min\{1, \nu_X, \nu_{\not\Sigma X}\}, \tag{2.16}$$

$$\delta \cdot (1 + \gamma_{k+1}) < \nu'^3, \tag{2.17}$$

$$max\{\rho, \rho'\}(1 + \gamma_{p+k+1} + N(m))\delta < \nu'^3, \;\; p = 0, \ldots, m. \tag{2.18}$$

where ν_X and $\nu_{\not\Sigma X}$ are as in Theorem 2.2.4.

Now, fixing any $n > max\{N_f(\delta), N_{\not\Sigma f}(\delta)\}$, we obtain Almgren's chain maps

$$\phi_X^f : \mathcal{I}(m,n) \rightarrow I_{*+k}(X)$$

and

$$\phi_{\not\Sigma X}^{\not\Sigma f} : \mathcal{I}(m,n) \rightarrow I_{*+k+2}(\not\Sigma X)$$

of degrees k and $k+2$, respectively. Define:

$$\Theta = sup\{F_X(f(\alpha) - f(\beta)) : \alpha,\ \beta \in \mathcal{I}(m,n)_0\},$$

and

$$\Theta' = sup\{F_{\not\Sigma X}(\not\Sigma f(\alpha) - \not\Sigma f(\beta)) : \alpha,\ \beta \in \mathcal{I}(m,n)_0\}.$$

Suppose, for convenience, that the chain maps above also satisfy M- and F-minimizing conditions, as in Almgren's original constructions. From now on we omit the subscripts from the chain maps.

Define a new chain map

$$\Psi : \mathcal{I}(m,n) \rightarrow I_{*+k+2}(\not\Sigma X)$$

of degree $k+2$, as the composition $\not{P} \circ \phi^f$. This chain map has the following properties:

a. If α is a 0-cell in $\mathcal{I}(m,n)$, then

$$\begin{aligned}\Psi(\alpha) = &\ \not{P}(\phi^f(\alpha)) \\ = &\ \not{P}(f(\alpha)) \quad = (\not{P}f)(\alpha) \\ = &\ \phi^{\not{P}f}(\alpha) \quad ,\end{aligned}$$

since $\phi^f_{|\mathcal{I}(m,n)_0} \equiv f$ by definition.

b. If α is a 1-cell in $\mathcal{I}(m,n)$, then

$$\begin{aligned}\partial\Psi(\alpha) = &\ \partial \circ \not{P} \circ \phi^f(\alpha) \\ = &\ \not{P} \circ \phi^f(d\alpha) \quad = \not{P}\phi^f(\alpha_1 - \alpha_2) \\ = &\ \not{P}(f(\alpha_1 - \alpha_2)) \quad = \not{P}f(\alpha_1) - \not{P}f(\alpha_2) \\ = &\ \phi^{\not{P}f}(d\alpha) \\ = &\ \partial\phi^{\not{P}f}(\alpha),\end{aligned}$$

where we write $d\alpha = \alpha_1 - \alpha_0$ with α_1 and α_0 0-cells in $\mathcal{I}(m,n)$. Also

$$\begin{aligned}M(\Psi(\alpha)) = &\ M(\not{P}\phi^f(\alpha)) \\ \leq &\ \gamma_{k+1}M(\phi^f(\alpha)) \quad = \gamma_{k+1}F_X(f(\alpha_1) - f(\alpha_0)) \\ \leq &\ \gamma_{k+1}\Theta \\ \leq &\ \gamma_{k+1}\delta,\end{aligned}$$

where the second equality comes from the F_X-minimizing choice for ϕ^f when restricted to the 1-cells of $\mathcal{I}(m,n)$, and the last inequality comes from the choice of $n > N_f(\delta)$.

c. Let α now be a p-cell in $\mathcal{I}(m,n)$ with $p > 1$, and recall that $\phi^f(\alpha)$ was chosen so as to satisfy the conditions of Theorem 2.2.4. In particular, we have the inequality $M(\phi^f(\alpha)) \leq \rho\Theta$, and hence

$$\begin{aligned} M(\Psi(\alpha)) = \quad & M(\yen\phi^f(\alpha)) \\ \leq \quad & \gamma_{p+k} M(\phi^f(\alpha)) \;\; \leq \gamma_{p+k}\rho\Theta \\ \leq \quad & \gamma_{p+k}\rho\delta. \end{aligned}$$

Now, let us define, inductively, homomorphisms $K_i : \mathcal{I}(m,n)_i \to I_{i+k+3}(\yen X)$ as follows:

For $i = 0$ let K_0 be the zero homomorphism.

Let α be a 1-cell in $\mathcal{I}(m,n)_1$. From item **b.** above, we have that $\partial\Psi(\alpha) = \partial\phi^{\yen f}$. Also

$$\begin{aligned} M(\phi^{\yen f}(\alpha) - \Psi(\alpha)) \leq \; & M(\phi^{\yen f}(\alpha)) + M(\Psi(\alpha)) \\ \leq \; & M(\phi^{\yen f}(\alpha)) + \gamma_{k+1} M(\phi^f(\alpha)) \\ = \; & F_{\yen X}(\yen f(\alpha_1) - \yen f(\alpha_0)) + \gamma_{k+1} F_X(\alpha_1 - \alpha_0) \\ \leq \; & \delta + \gamma_{k+1}\delta \\ = \; & (1 + \gamma_{k+1})\delta \\ < \; & \nu'^3. \end{aligned}$$

Our hypothesis on δ assures the existence of an M-minimizing isoperimetric choice $S \in I_{k+4}(\not\Sigma X)$ for $\phi^{\not\Sigma f}(\alpha) - \Psi(\alpha)$ satisfying the conditions of 2.11. Define $K_1(\alpha) = S$ and notice that

$$\phi^{\not\Sigma f} - \Psi = \partial \circ K_1 + K_0 \circ d$$

and

$$M(K_1(\alpha)) \leq sup\{M(\phi^{\not\Sigma f}(\alpha)), M(\Psi(\alpha)\},$$

the latter inequality coming from 2.11.

Suppose, inductively, that we have defined homomorphisms K_p for $p \leq p_0$, satisfying

$$\phi^{\not\Sigma f} - \Psi = \partial \circ K_p + K_{p-1} \circ d$$

and

$$M(K_p(\alpha)) \leq sup\{M(\phi^{\not\Sigma f}(\alpha)), M(\Psi(\alpha)\},$$

on p-cells. Let α be a $(p_0 + 1)$-cell in $\mathcal{I}(m, n)$ and set $T = \phi^{\not\Sigma f}(\alpha) - \Psi(\alpha) - K_{p_0}(d\alpha)$. Now, observe that $d\alpha = \sum_i \alpha_i$, where the α_i's are p_0-cells not exceeding $N(m)$ in number. Hence we have

$$M(K_{p_0}(d\alpha)) \leq \sum_i M(\alpha_i) \leq N(m)\, max\{M(K_{p_0}(\alpha_i))\}.$$

It is immediate from its construction that $\partial T = 0$, and,

$$M(T) \leq M(\phi^{\not\Sigma f}(\alpha)) + M(\Psi(\alpha)) + M(K_{p_0}(d\alpha))$$

$$\leq \rho'\Theta' + \gamma_{p_0+k+1}\rho\Theta + N(m)\,max\{M(K_{p_0}(\alpha_i))\}$$

$$\leq \rho'\Theta' + \gamma_{p_0+k+1}\rho\Theta + N(m)\,max\{\rho\Theta, \rho'\Theta'\}$$

$$\leq max\{\rho, \rho'\}(1 + \gamma_{p_0+k+1} + N(m))\delta < \nu'^3.$$

Our choice of δ again implies that we can make an isoperimetric choice $S \in I_{k+p_0+4}(\not{\Sigma}X)$ for T, as in 2.11, and define $K_{p_0+1}(\alpha) = S$. We now see that the maps K_* provide a chain homotopy between $\phi^{\not{\Sigma}f}$ and Ψ. Finally, let $\{\alpha_i\}$ be the m-cells of $\mathcal{I}(m,n)$. Since $\phi^{\not{\Sigma}f}$ and Ψ are chain homotopic, we have equality of homology classes:

$$[\Psi(\sum_i \alpha_i)] = [\phi^{\not{\Sigma}f}(\sum_i \alpha_i)].$$

However, by definition, the first class is

$$[\not{\Sigma}(\sum_i \phi^f(\alpha_i))] = \tau[\sum_i \phi^f(\alpha_i)] = \tau \circ \mathcal{A}[f],$$

and the second one is

$$[\phi^{\not{\Sigma}f}(\sum_i \alpha_i)] = \mathcal{A}([\not{\Sigma}f]) = \mathcal{A} \circ \not{\Sigma}_*[f].$$

This concludes the proof of **Step 2.**

Proof of Step 3.

Choose $f : (I^m, \overset{\bullet}{I}^m) \to \mathcal{Z}_k(X)$ representing a class $[f]$ in $\pi_m(\mathcal{Z}_k(X))$, and let $\#f : (I^{m+2}, \overset{\bullet}{I}^{m+2}) \to \mathcal{Z}_{k+2}(\mathbb{P}^1\#X)$ be the corresponding representative for $\#_*[f]$ in $\pi_{m+2}(\mathcal{Z}_{k+2}(\mathbb{P}^1\#X)$. Choose $n > max\{N_f(\delta), N_{\#f}(\delta)\}$, for δ suitably chosen. As in **Step 2.**, let

$$\phi^f : \mathcal{I}(m,n) \to I_{*+k}(X)$$

and

$$\phi^{\#f} : \mathcal{I}(m+2,n) \to I_{*+k+2}(\mathbb{P}^1\#X)$$

be the associated Almgren's chain maps. Define a new chain map

$$\Psi : \mathcal{I}(m+2,n) \to I_{*+k+2}(\mathbb{P}^1\#X)$$

by sending a p-cell $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_m \otimes \alpha_{m+1} \otimes \alpha_{m+2}$ to

$$\Psi(\alpha) = \phi^f(\alpha')\#\alpha'',$$

where $\alpha' = \alpha_1 \otimes \ldots \otimes \alpha_m$, $\alpha'' = \alpha_{m+1} \otimes \alpha_{m+2}$, and we are identifying $\alpha_{m+1}\otimes\alpha_{m+2}$ with an integral cycle of dimension d, supported in $\mathbb{P}^1 \equiv I^2/\overset{\bullet}{I}{}^2$. Our map Ψ satisfies the following properties:

a. We have

$$\begin{aligned}\partial\Psi(\alpha) &= \partial(\phi^f(\alpha')\#\alpha'') \\ &= \partial(\phi^f(\alpha'))\#\alpha'' + \phi^f(\alpha')\#(\partial\alpha'') , \\ &= \Psi(d\alpha)\end{aligned}$$

in other words, Ψ is a chain map.

b. Let $\alpha = \alpha' \otimes \alpha''$ be a 0-cell, and hence α'' is identified with a point $\mathbb{P}^0 \in \mathbb{P}^1$. Therefore:

$$\begin{aligned}\Psi(\alpha) = \;& \phi^f(\alpha')\#\alpha'' \;= \phi^f(\alpha')\#\mathbb{P}^0 \\ = \;& f(\alpha')\#\mathbb{P}^0 \quad \#f(\alpha' \otimes \alpha'') \\ = \;& \phi^{\#f}(\alpha).\end{aligned}$$

c. If α is a 1-cell, then either α' or α'' is a point, and hence, either

$$\partial\alpha = -\alpha' \otimes (\partial\alpha'')$$

or

$$\partial\alpha = (\partial\alpha') \otimes \alpha''.$$

Therefore, in the first case we have:

$$\begin{aligned}\partial\Psi(\alpha' \otimes \alpha'') \;&= \phi^f(d\alpha')\#\alpha'' \\ &= (f(\alpha'_1) - f(\alpha'_0))\#\alpha'' \\ &= f(\alpha'_1)\#\alpha'' - f(\alpha'_0)\#\alpha'' \\ &= \#f(\alpha'_1 \otimes \alpha'') - \#f(\alpha'_0 \otimes \alpha'') \\ &= \#f(d\alpha' \otimes \alpha'') \\ &= \partial\phi^{\#f}(\alpha' \otimes \alpha'').\end{aligned}$$

Similarly, in the second case:

$$\begin{aligned}
\partial\Psi(\alpha'\otimes\alpha'') &= \phi^{f}(\alpha')\#d\alpha'' \\
&= \phi^{f}(\alpha')\#(\alpha''_1-\alpha''_0) \\
&= f(\alpha')\#\alpha''_1 - f(\alpha')\#\alpha''_0 \\
&= \#f(\alpha'\otimes\alpha''_1) - \#f(\alpha'\otimes\alpha''_0) \\
&= \phi^{\#f}(d(\alpha'\otimes\alpha'')) \\
&= \partial\phi^{\#f}(\alpha'\otimes\alpha'').
\end{aligned}$$

Therefore, $\partial\Psi(\alpha) = \partial\phi^{\#f}(\alpha)$ for all 1-chains.

Furthermore, we obtain inequalities, e.g., in the first case:

$$M(\Psi(\alpha)-\phi^{\#f}(\alpha)) \leq M(\Psi(\alpha)) + M(\phi^{\#f}(\alpha))$$

$$\leq M(\phi^{\#f}(\alpha')\#\alpha'') + \rho'\Theta'$$

$$= M(\Sigma_{\gamma}\phi^{\#f}(\alpha')) + \rho'\Theta'$$

$$\leq \gamma M(\phi^{\#f}(\alpha')) + \rho'\Theta'$$

$$= \gamma F(\phi^{\#f}(d\alpha')) + \rho'\Theta'$$

$$\leq \gamma\rho\Theta + \rho'\Theta'$$

$$\leq (\gamma\rho+\rho')\delta.$$

In the second case we obtain similar inequality.

d. For p-cells $\alpha = \alpha' \otimes \alpha''$, we first note that we may bound, a priori, the mass $M(\alpha'')$ in $\mathbb{P}^1$ by a constant, say C, since the currents α'' correspond to canonical subdivisions of the unit square. Now, given a p-cell, $p > 1$, we have:

$$
\begin{aligned}
M(\Psi(\alpha) - \phi^{\#f}(\alpha)) &\leq M(\Psi(\alpha)) + M(\phi^{\#f}(\alpha)) \\
&\leq M(\phi^{\#f}(\alpha')\#\alpha'') + \rho'\Theta' \\
&\leq \gamma M(\phi^{\#f}(\alpha'))M(\alpha'') + \rho'\Theta' \\
&\leq \gamma C\rho\Theta + \rho'\Theta' \\
&\leq (\gamma C\rho + \rho')\delta.
\end{aligned}
$$

With the above properties in hand, after a suitable choice of δ, we can construct a chain homotopy between Ψ and $\phi^{\#f}$ in the same way we did in **Step 2.**

Finally, let $\{\alpha_{i,j} = \alpha'_i \otimes \alpha''_j\}$ be the m-cells of $\mathcal{I}(m+2, n)$. By definition,

$$
\begin{aligned}
[\phi^{\#f}(\sum_{i,j} \alpha_{i,j})] &= [\sum_i \phi^{\#f}(\alpha'_i)\# \sum_j \alpha''_j] \\
&= [\phi^{\#f}(\sum_i \alpha'_i)\#\mathbb{P}^1] = [\not\Sigma\not\Sigma(\phi^{\#f}(\sum_i \alpha'_i)) \\
&= \tau \circ \tau[\phi^{\#f}(\sum_i \alpha'_i)] \\
&= \tau \circ \tau \circ \mathcal{A}[f],
\end{aligned}
$$

and this concludes the proof of **Step 3.** . □

Corollary 2.2.7 *Let $f : X \to Y$ be a morphism of algebraic sets. Then the morphisms induced by f on Lawson homology and on singular homology, commute with the "generalized cycle maps" s_X^p, s_Y^p of X and Y respectively. In other words, for every r and $i \geq 0$, the following diagram commutes:*

$$\begin{array}{ccc} L_pH_{i+2p}(X) & \xrightarrow{f_*} & L_pH_{i+2p}(Y) \\ {\scriptstyle s_X^p}\downarrow & & \downarrow{\scriptstyle s_Y^p} \\ L_0H_{i+2p}(X) & \xrightarrow[f_*]{} & L_0H_{i+2p}(Y). \end{array}$$

Proof

The corollary follows from the corresponding property for continuous maps between CLNR's (compact Lipschitz neighborhood retracts) and the Almgren's map, cf. [2]. □

Remark 2.2.8 The above corollary is simply the statement that the generalized cycle map is a natural transformation between Lawson and singular homology. The same techniques used above can also prove the naturality of the relative cycle map 2.2.2 with respect to morphisms of pairs $f : (X, X') \to (Y, Y')$.

Chapter 3

Excision

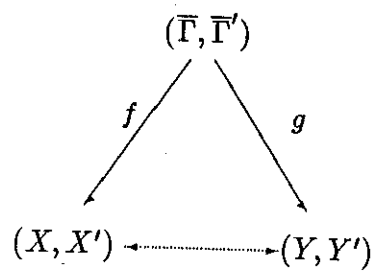
3.1 An "excision type" result

Here we present a result relating relative isomorphisms of pairs of algebraic sets and the relative Lawson homology. This will turn out to be one of the main computational tools of the present work.

Definition 3.1.1 *A* relative isomorphism $\Psi : (X, X') \overset{\cong}{\leftrightarrow} (Y, Y')$ *between two pairs of algebraic sets is an algebraic isomorphism between the quasiprojective sets* $X \setminus X'$ *and* $Y \setminus Y'$*. We say that two pairs of algebraic sets are* relatively isomorphic *if there is a relative isomorphism between them.*

Observe that in case X and Y are irreducible sets, this notion coincides with the usual notion of birational equivalence, with a little extra information.

Remark 3.1.2 Given a relative isomorphism of pairs $\Psi : (X, X') \overset{\cong}{\leftrightarrow} (Y, Y')$, consider the subset $\Gamma = \{(x, \Psi(x)) : x \in X \setminus X'\}$ of $X \times Y$. Let $\overline{\Gamma}$ be the closure of Γ in $X \times Y$, and let $f : \overline{\Gamma} \to X$, $g : \overline{\Gamma} \to Y$ be the restrictions to $\overline{\Gamma}$ of the projections onto the first and second factors of $X \times Y$, respectively. Define $\overline{\Gamma}' \subset \overline{\Gamma}$ to be $\overline{\Gamma} \setminus \Gamma$. Then we see that the relative isomorphism of pairs $\Psi : (X, X') \overset{\cong}{\leftrightarrow} (Y, Y')$ fits into a "correspondence of pairs"



with actual morphisms of pairs $f : (\overline{\Gamma}, \overline{\Gamma}') \to (X, X')$ and $g : (\overline{\Gamma}, \overline{\Gamma}') \to (Y, Y')$ realizing relative isomorphisms between $(\overline{\Gamma}, \overline{\Gamma}')$ and (X, X'), (Y, Y') respectively.

Definition 3.1.3 *Let (X, X') be a pair of algebraic sets, $X' \subset X \hookrightarrow \mathbb{P}^N$. We adopt the following notation:*

- $\Upsilon_{p,d}(X,X') \stackrel{def}{=} \{\sigma \in \mathbf{C}_{p,d}(X,j) : \sigma \textit{ has no component in } X'\}$;
- $\Upsilon_{p,\leq D}(X,X') \stackrel{def}{=} \bigcup_{d\leq D} \Upsilon_{p,d}(X,X')$;
- $\Upsilon_{p}(X,X') \stackrel{def}{=} \bigcup_{D} \Upsilon_{p,\leq D}(X,X')$;
- $\mathbf{C}_{p,\leq D}(X,j) \stackrel{def}{=} \bigcup_{d\leq D} \mathbf{C}_{p,d}(X,j)$;
- $\mathcal{K}_D(X) \stackrel{def}{=} \mathbf{C}_{p,\leq D}(X,j) \times \mathbf{C}_{p,\leq D}(X,j) \subset \mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j)$.

Notice that $\Upsilon_p(X,X')$ is a submonoid of $\mathbf{C}_p(X,j)$ and that $C_{p,\leq D}(X,j)$ is an algebraic set. Using the canonical projections $p : \mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j) \to \tilde{\mathcal{C}}_p(X)$ and $\pi : \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(X') \equiv \tilde{\mathcal{C}}_p(X,X')$ we finally define the sets:

- $\tilde{\mathcal{K}}_D(X) \stackrel{def}{=} p(\mathcal{K}_D(X)) \subset \tilde{\mathcal{C}}_p(X)$;
- $\tilde{\Upsilon}_{p\leq D}(X,X') \stackrel{def}{=} p(\Upsilon_{p,\leq D}(X,X') \times \Upsilon_{p,\leq D}(X,X')) \subset \tilde{\mathcal{C}}_p(X)$;
- $\tilde{\Upsilon}_{p}(X,X') \stackrel{def}{=} \bigcup_{D} \tilde{\Upsilon}_{p\leq D}(X,X')$;
- $Q_D(X,X') \stackrel{def}{=} \pi(\tilde{\mathcal{K}}_D(X))$.

Here, the sets $\{\tilde{\mathcal{K}}_D(X)\}_{D=1}^{\infty}$ and $\{Q_D(X,X')\}_{D=1}^{\infty}$ form a filtering family of compact subsets of $\tilde{\mathcal{C}}_p(X)$ and $\tilde{\mathcal{C}}_p(X,X')$, respectively. Also notice that

$\tilde{\Upsilon}_p(X,X')$ is the subgroup of $\tilde{\mathcal{C}}_p(X)$ obtained by (naïvely) group-completing $\tilde{\Upsilon}_p(X,X')$. Summarizing we have the following commutative diagram:

$$\begin{array}{ccccc}
\Upsilon_{p,\leq D}(X,X') \times \Upsilon_{p,\leq D}(X,X') & \longrightarrow & \mathcal{K}_D(X) & \longrightarrow & \mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j) \\
\downarrow p & & \downarrow p & & \downarrow p \\
\tilde{\Upsilon}_{p\leq D}(X,X') & \longrightarrow & \tilde{\mathcal{K}}_D(X) & \longrightarrow & \tilde{\mathcal{C}}_p(X) \\
 & & \downarrow \pi & & \downarrow \pi \\
 & & Q_D(X,X') & \longrightarrow & \tilde{\mathcal{C}}_p(X,X'),
\end{array}$$

where the horizontal arrows are inclusions and the vertical ones are proclusions.

From the above picture we draw the following

Lemma 3.1.4 *Let (X,X') be a pair of algebraic sets. Then:*

(a) *The topology of $\mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j)$ is the weak topology induced by the filtering family of compact sets $\mathcal{K}_1(X) \subset \mathcal{K}_2(X) \subset \ldots \mathcal{K}_D(X) \subset \ldots$. The same holds for $\tilde{\mathcal{C}}_p(X)$ and $\tilde{\mathcal{C}}_p(X,X')$ with respect to the families $\tilde{\mathcal{K}}_1(X) \subset \tilde{\mathcal{K}}_2(X) \subset \ldots$ and $Q_1(X,X') \subset Q_2(X,X') \subset \ldots$, respectively.*

(b) *The composition $\tilde{\Upsilon}_p(X,X') \hookrightarrow \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X,X')$ is an abstract group isomorphism which takes $\tilde{\Upsilon}_{p\leq D}(X,X')$ onto $Q_D(X,X')$.*

Proof

(a) By definition $\mathbf{C}_p(X,j) \equiv \coprod_{d\geq 0} \mathbf{C}_{p,d}(X,j)$ has the disjoint union topology of the algebraic sets $\mathbf{C}_{p,d}(X,j)$, with their analytic topology. Since

$C_{p,\leq D}(X,j)$ is the disjoint union of the compact sets $\mathbf{C}_{p,d}(X,j)$, $d \leq D$, it is clear that the intersection of a subset F of $\mathbf{C}_p(d,j)$ with $C_{p,\leq D}(X,j)$ is closed if and only if its intersection with each $\mathbf{C}_{p,d}(X,j)$ is closed for all $d \leq D$. Hence the weak topology given by the filtration $C_{p,\leq 1}(X,j) \subseteq C_{p,\leq 2}(X,j) \subseteq \ldots$ of $\mathbf{C}_p(X,j)$ coincides with its standard topology. From this we conclude that $\mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j)$ has the topology given by the filtration by the $\mathcal{K}_D(X)$'s.

Now, since $p : \mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j) \to \tilde{\mathcal{C}}_p(X)$ is, by definition, a proclusion and $\tilde{\mathcal{C}}_p(X)$ is Hausdorff (see Remark 2.1.2) we know that $\tilde{\mathcal{C}}_p(X)$ has compactly generated topology. Let F be a subset of $\tilde{\mathcal{C}}_p(X)$ with the property that $F \cap \tilde{\mathcal{K}}_D(X)$ is closed for all D. Therefore $p^{-1}(F \cap \tilde{\mathcal{K}}_D(X))$ is closed, which is equivalent to $p^{-1}(F \cap \tilde{\mathcal{K}}_D(X)) \cap \mathcal{K}_{D'}(X)$ being closed for all D, D'. In particular

$$p^{-1}(F \cap \tilde{\mathcal{K}}_D(X)) \cap \mathcal{K}_D(X) \equiv p^{-1}(F) \cap \mathcal{K}_D(X)$$

is closed for all D, and hence $p^{-1}(F)$ is closed in $\mathbf{C}_p(X,j) \times \mathbf{C}_p(X,j)$. Recalling that p is a proclusion we obtain that F is closed in $\tilde{\mathcal{C}}_p(X)$, in other words, that the topology of $\tilde{\mathcal{C}}_p(X)$ is generated by the filtration given by $\tilde{\mathcal{K}}_1(X) \subset \tilde{\mathcal{K}}_2(X) \subset \ldots$.

A similar proof applies to $\tilde{\mathcal{C}}_p(X, X')$.

(b) Both maps $\tilde{\Upsilon}_p(X, X') \hookrightarrow \tilde{\mathcal{C}}_p(X)$ and $\tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X, X')$ are homomorphisms of groups, hence so is their composition, furthermore, the first map is an inclusion. Recall that the kernel of $\tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(X,X')$ is $\tilde{\mathcal{C}}_p(X')$ which, in turn, intersects (the image of) $\tilde{\Upsilon}_p(X,X')$ only at the identity of the group, since the elements of $\tilde{\Upsilon}_p(X,X')$ have no components contained in X'. This makes the map $\tilde{\Upsilon}_p(X,X') \to \tilde{\mathcal{C}}_p(X,X')$ injective.

Now, choose an element $[\sigma] \in \tilde{\mathcal{C}}_p(X,X')$. It has a representative $\sigma \in \tilde{\mathcal{C}}_p(X)$ with no components contained in X', in other words, $\sigma \in \tilde{\Upsilon}_p(X,X')$, which shows that the map is also surjective.

The last assertion follows from the definition of the objects involved. □

The following proposition is the essential tool for the main result of this Chapter:

Proposition 3.1.5 *Given a morphism of algebraic pairs $f : (X,X') \to (Y,Y')$ which induces a relative isomorphism between (X,X') and (Y,Y'), we have that the induced morphism $f_\# : \mathbf{C}_p(X,j) \to \mathbf{C}_p(Y,i)$ of Chow monoids restricts to a homomorphism of submonoids*

$$f_\# : \Upsilon_p(X,X') \to \Upsilon_p(Y,Y').$$

Furthermore, for every $d > 0$, there exists $D > 0$ such that

$$f_\#^{-1}(\Upsilon_{p,\leq d}(Y,Y')) \subseteq \Upsilon_{p,\leq D}(X,X'),$$

for all $p \geq 0$.

Proof

Take a p-dimensional subvariety $V \subset X$ not contained in X'. Since $f_{|X \setminus X'} : X \setminus X' \overset{\cong}{\to} Y \setminus Y'$ is an isomorphism, we see that $f(V)$ is a p-dimensional subvariety of Y not contained in Y'. Also we have that $\deg(V/f(V)) = 1$, since $f_{|X \setminus X'}$ restricts to an isomorphism between $V \setminus (V \cap X')$ and $f(V) \setminus (f(V) \cap Y')$. Therefore

$$f_{\#}(V) \overset{def}{=} \deg(V/f(V)) \cdot f(V) = f(V) \in \Upsilon_p(Y, Y').$$

Extending the result by linearity to all of $\Upsilon_p(X, X')$ we obtain the first assertion of the Proposition.

In order to prove the remaining part of the Proposition we need the following technical

Lemma 3.1.6 *Given $f : (X, X') \to (Y, Y')$ as in the Proposition 3.1.5, there can not exist a sequence $\{V_n\}_{n=1}^{\infty}$ of p-dimensional subvarieties of X satisfying:*

(a) limsup $deg(V_n) = \infty$;

(b) $\deg f_{\#}(V_n) \leq M$, *for some constant M, for all n;*

(c) $V_n \not\subset X'$, *for all n.*

Let us finish the proof of the Proposition 3.1.5 before we prove the Lemma. Assume the Lemma is true and suppose the Proposition does not hold. In other words, there exists $d > 0$ such that

$$f_{\#}^{-1}(\Upsilon_{p,\leq d}(Y,Y')) \not\subseteq \Upsilon_{p,\leq n}(X,X')$$

for all $n > 0$. This allows us to select a sequence of p-cycles $\{\sigma_n\}_{n=1}^{\infty}$ contained in $\Upsilon_p(X,X')$ satisfying:

$$\deg \sigma_n \geq n \quad \textit{and} \quad \deg f_{\#}\sigma_n \leq d,$$

for all n. Write $\sigma_n = \sum_{i=1}^{l_n} k_n^i V_i^n$, where the V_i^n's are irreducible subvarieties of X not contained in X'. As we pointed out at the beginning of the proof, we have $\deg(V_i^n/f(V_i^n)) = 1$, and hence

$$f_{\#}(\sigma_n) \stackrel{def}{=} \sum_{i=1}^{l_n} k_n^i \cdot \deg(V_i^n/f(V_i^n)) \cdot f(V_i^n) = \sum_{i=1}^{l_n} k_n^i \cdot f(V_i^n).$$

Suppose that $\deg(V_i^n) \leq B$ for all n and $1 \leq i \leq l_n$, where B is some constant. In this case one has

$$n \leq \deg \sigma_n = \sum_{i=1}^{l_n} k_n^i \cdot \deg(V_i^n) \leq \sum_{i=1}^{l_n} k_n^i \cdot B$$

and hence

$$\sum_{i=1}^{l_n} k_n^i \geq \frac{1}{B} \cdot n$$

for all n. On the other hand, by hypothesis

$$M \geq \deg f_{\#}(\sigma_n) = \sum_{i=1}^{l_n} k_n^i \cdot \deg(f(V_i^n)) \geq \sum_{i=1}^{l_n} k_n^i \geq \frac{1}{B} \cdot n,$$

which is a contradiction. Therefore no such constant B can exist. In other words

$$\text{limsup deg}(V_i^n) = \infty$$

and $\text{deg}f_{\#}(V_i^n) \leq \text{deg}f_{\#}(\sigma_n) \leq M$, which contradicts the Lemma, and the Proposition is proven. $\square$

Let us proceed to the proof of the lemma:

Proof *(of Lemma 3.1.6)* It suffices to assume that X and Y are irreducible, for if $\{V_n\}$ is an infinite sequence of irreducible subvarieties of X (not contained in X') we can extract a subsequence $\{V_{n_i}\}$ so that all V_{n_i}'s are contained in a unique irreducible component X_1 of X. Define $X_1' = X \cap X'$ and observe that $f(X_1)$ must be contained in an irreducible component Y_1 of Y. Define $Y_1' = Y_1 \cap Y'$, in doing so we obtain a morphism of pairs $f : (X_1, X_1') \to (Y_1, Y_1')$. Since $f_{|X \setminus X'}$ is an isomorphism of quasiprojective sets, its restriction to $X_1 \setminus X_1' \subset X \setminus X'$ establishes an isomorphism between $X_1 \setminus X_1'$ and $Y_1 \setminus Y_1'$, because the restriction of $(f_{|X\setminus X'})^{-1}$ to $Y_1 \setminus Y_1'$ sends $Y_1 \setminus Y_1'$ into some irreducible quasiprojective subvariety of $X \setminus X'$ which the, turns out to be $X_1 \setminus X_1'$. Therefore $f : (X_1, X_1') \to (Y_1, Y_1')$ is a relative isomorphism and the sequence $\{V_{n_i}\}$ satisfies the hypothesis of the lemma with X_1 and Y_1 irreducible.

We now use induction to prove the lemma with X and Y irreducible.

For $p = 0$ it is immediate, since a 0-dimensional variety is a point which has degree 1 always.

Consider the case $p = 1$. Take a sequence of irreducible curves $V_n \subset X$ satisfying the hypothesis of the lemma. We may assume $\deg V_n \geq n$. Observe that the set $V'_n \stackrel{def}{=} V_n \cap X'$ is finite (or empty), since V_n is irreducible. Now use the following facts (see e.g. [16], page 174)

- The generic hyperplane intersection of an irreducible subvariety of $\mathbb{P}^N$ of dimension ≥ 2 is irreducible.
- The generic hyperplane intersects the curve V_n transversally and misses the finite subset $V'_n \subset V_n$,

to obtain (by Baire category arguments) a hyperplane $H \subset \mathbb{P}^N$ satisfying:

(a) H is transversal to V_n, for all n;

(b) $H \cap V'_n \equiv H \cap V_n \cap X' = \emptyset$, for all n;

(c) $H \cap X$ is irreducible and is not contained in X'.

Now, we first observe that (by definition), the cardinality of the intersection of V_n with H is its degree, and hence,

$$\#(H \cap V_n) \stackrel{def}{=} \deg(V_n) \geq n,$$

by hypothesis. On the other hand, since $H \cap X$ is irreducible and not contained in X', we have

$$f_{\#}(H \cap X) = f(H \cap X),$$

as observed at the beginning of the proof of the Proposition 3.1.5. Call $D = f(H \cap X) \subset Y \subset \mathbb{P}^M$. As a subvariety of $\mathbb{P}^M$, D is a set-theoretic intersection of a finite number of (irreducible) hypersurfaces $H_1, \ldots, H_k \subset \mathbb{P}^M$. Consequently,

$$f(H \cap V_n) = f(H \cap X \cap V_n) \subseteq f(H \cap X) \cap f(V_n) = D \cap f(V_n).$$

Since $f(V_n) \not\subset D$, there must be one hypersurface H_{j_0} (among $H_1, \ldots, H_k$) not containing $f(V_n)$. From this we get:

$$\#(D \cap f(V_n)) \leq \#(H_{j_0} \cap f(V_n)) \leq \sum_{x \in H_{j_0} \cap f(V_n)} i(H_{j_0}, f(V_n); x)$$

$$= \deg(H_{j_0}) \cdot \deg(f(V_n) \leq \sup\{\deg(H_j)\} \cdot \deg(f(V_n),$$

where $i(H_{j_0}, f(V_n); x)$ denotes the multiplicity of the intersection of H_{j_0} and $f(V_n)$ along x. Again by hypothesis $\deg f_{\#}(V_n) = \deg f(V_n) \leq M$, and $\#f(V_n \cap H) = \#(V_n \cap H)$, and hence:

$$n \leq \deg(V_n \cap H) = \#f(V_n \cap H) \leq \#(D \cap f(V_n)) \leq S \cdot M,$$

where $S = \sup\{\deg(H_j)\}$. This is s contradiction.

Suppose, by induction, that the lemma is true for subvarieties of dimension $\leq p-1$, $p \geq 2$. Let $\{V_n\}$ be a sequence of p-dimensional subvarieties satisfying the hypothesis of the lemma, and suppose that $\deg V_n \geq n$. Using the same general position arguments as before we can choose a generic hyperplane $H \subset \mathbb{P}^N$ so that:

(a') $H \cap V_n$ is an irreducible (p-1)-dimensional subvariety of $\mathbb{P}^N$;

(b') $H \cap V_n \not\subset X'$;

(c') $H \cap X$ is also irreducible and $H \cap X \not\subset X'$.

Define $D = f_{\#}(H \cap X) = f(H \cap X)$, and let $H_1, H_2, \ldots H_k$ be irreducible hypersurfaces in $\mathbb{P}^M$ whose set theoretic intersection is D. Since

$$f(V_n \cap H) \subset f(V_n) \cap f(H \cap X) = f(V_n) \cap D = f(V_n) \cap H_1 \cap \ldots \cap H_k$$

and $f(V_n \cap H)$ is irreducible, we know that $f(V_n \cap H)$ is an irreducible component of the intersection $f(V_n) \cap H_{j_0}$, for some j_0 such that $f(V_n) \not\subset H_{j_0}$. Write $f(V_n) \cap H_{j_0} = \bigcup_r Z_r$, Z_r irreducible. Therefore

$$\deg f(V_n \cap H) \leq \sum_r i(H_{j_0}, f(V_n); Z_r) \cdot \deg Z_r$$

$$= \deg H_{j_0} \cdot \deg f(V_n) \leq \sup\{\deg H_j\} \cdot M.$$

However $\deg(V_n \cap H) = \deg V_n \geq n$, which contradicts the induction hypothesis, and proves the lemma. □

We now prove the MAIN result of the Chapter, essentially as a corollary of the previous results.

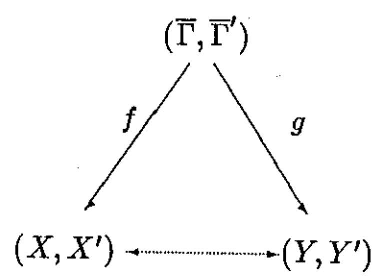
Theorem 3.1.7 *Let (X, X') and (Y, Y') be relatively isomorphic pairs of algebraic sets. Then, any given relative isomorphism $\Psi : (X, X') \leftrightarrow (Y, Y')$ induces an isomorphism of topological groups:*

$$\Psi_\star : \tilde{\mathcal{C}}_p(X, X') \xrightarrow{\cong} \tilde{\mathcal{C}}_p(Y, Y'),$$

for all $p \geq 0$.

Proof

Consider the correspondence of pairs described in Remark 3.1.2:



built from the relative isomorphism Ψ. Here we have two morphisms of pairs f and g which induce relative isomorphisms between $(\overline{\Gamma}, \overline{\Gamma}')$ and (X, X'), (Y, Y') respectively. Therefore, it suffices to show the theorem when Ψ is actually a morphism of pairs, for we can take $g_\star \circ f_\star^{-1}$ as the definition of $\Psi_\star$.

Observe that a morphism of pairs $\Psi : (X, X') \to (Y, Y')$ induces, naturally a morphism of topological groups:

$$\Psi_\star : \tilde{\mathcal{C}}_p(X, X') \to \tilde{\mathcal{C}}_p(Y, Y').$$

To see this, consider initially the natural morphism $\Psi_\star : \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(Y)$ induced by Ψ. Since $\Psi(X') \subset Y'$, then $\Psi_\star(\tilde{\mathcal{C}}_p(X')) \subset \tilde{\mathcal{C}}_p(Y')$. Hence, we have the following commutative diagram of (abstract abelian) groups and homomorphisms:

$$\begin{array}{ccc} \tilde{\mathcal{C}}_p(X) & \xrightarrow{\Psi_\star} & \tilde{\mathcal{C}}_p(Y) \\ {\scriptstyle \pi_X}\downarrow & & {\scriptstyle \pi_Y}\downarrow \\ \tilde{\mathcal{C}}_p(X)/\tilde{\mathcal{C}}_p(X') & \xrightarrow[\Psi_\star]{} & \tilde{\mathcal{C}}_p(Y)/\tilde{\mathcal{C}}_p(Y'), \end{array}$$

by the universal property of quotients in the category of groups. The continuity of $\Psi_\star$ follows from the corresponding universal property for proclusions, in the category of topological spaces.

Let us assume, from now on, that Ψ is a relative isomorphism. In Proposition 3.1.5 we saw that $\Psi_\#$ takes $\Upsilon_p(X, X')$ into $\Upsilon_p(Y, Y')$, and hence $\Psi_\star : \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(Y)$ restricts to a group homomorphism $\Psi_\star : \tilde{\Upsilon}_p(X, X') \to \tilde{\Upsilon}_p(Y, Y')$, since $\tilde{\Upsilon}_p(X, X')$, (respectively $\tilde{\Upsilon}_p(Y, Y')$), is the group completion of $\Upsilon_p(X, X')$, (respectively $\Upsilon_p(Y, Y')$).

We claim that $\Psi_\star : \tilde{\Upsilon}_p(X, X') \to \tilde{\Upsilon}_p(Y, Y')$ is a group isomorphism.

Proof of claim

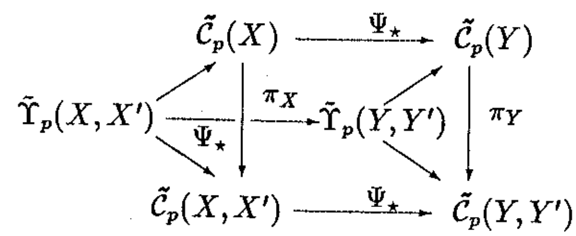
Surjectivity: Let V be irreducible subvariety in $\tilde{\Upsilon}_p(Y, Y')$. We see that $V \setminus Y'$ is an irreducible, p-dimensional quasiprojective subvariety of $Y \setminus Y'$. Define $W = f^{-1}(V \setminus Y')$, it is easily shown that $\Psi(W) = V$ and that $\deg(W/\psi(W)) = 1$, since $\Psi : W \setminus X' \to V \setminus Y'$ is an isomorphism of quasiprojective varieties. Therefore $\Psi_\star(W) = \Psi(W) = V$. For a generic cycle $\sigma = \sum_\lambda n_\lambda V_\lambda$, define W_λ as before, so that $\Psi(W_\lambda) = V_\lambda$, and hence

$$\Psi_\star(\sum_\lambda n_\lambda W_\lambda) = \sum_\lambda n_\lambda \Psi_\star(W_\lambda) = \sum_\lambda n_\lambda V_\lambda = \sigma,$$

i.e., $\Psi_\star$ is surjective.

Injectivity: Suppose $\Psi_\star(\sigma) = 0$. Write $\sigma = \sigma_1 - \sigma_2$, with both σ_1 and σ_2 effective cycles, that is, $\sigma_1 = \sum_{\alpha=1}^r m_\alpha W_\alpha$ and $\sigma_2 = \sum_{\beta=1}^s m_\beta W_\beta$, with $m_\alpha, m_\beta > 0$ and $\{W_\alpha\}$, $\{W_\beta\}$ the distinct irreducible components of σ_1, σ_2 respectively. Since $\Psi_\star(\sigma) = 0$ we then have $\Psi_\star(\sigma_1) = \Psi_\star(\sigma_2)$, which implies that $r = s$ and there is an index permutation so that $\sigma_1 = \sum_{\alpha=1}^r m_\alpha W_\alpha$ and $\sigma_2 = \sum_{\alpha=1}^r m_\alpha W_\alpha^\wedge$, and $\Psi_\star(W_\alpha) = \Psi_\star(W_\alpha^\wedge)$. Now both $\Psi^{-1}(\Psi(W_\alpha \setminus Y'))$ and $\Psi^{-1}(\Psi(W_\alpha^\wedge \setminus Y'))$ are Zariski open subsets of W_α and $W_\alpha^\wedge$ respectively. Furthermore, they are equal, since $\Psi(W_\alpha) = \Psi_\star(W_\alpha) = \Psi_\star(W_\alpha^\wedge) = \Psi(W_\alpha^\wedge)$. By irreducibility we obtain that $W_\alpha = W_\alpha^\wedge$, for all $\alpha = 1, \ldots, r$, and hence $\sigma_1 = \sigma_2$, which shows that $\sigma = 0$ and $\Psi_\star$ is injective.

Summarizing we have the following commutative diagram:



The claim above shows that $\Psi_{\star}|_{\tilde{\Upsilon}_p(X,X')}$ is an isomorphism and the Lemma 3.1.4(b) shows that both $\pi_X|_{\tilde{\Upsilon}_p(X,X')}$ and $\pi_Y|_{\tilde{\Upsilon}_p(Y,Y')}$ are isomorphisms. By the commutativity of the diagram we obtain that $\Psi_\star$ is an (continuous) isomorphism.

In Lemma 3.1.4 we have shown further that $\pi_X(\tilde{\Upsilon}_{p,\leq d}(X,X')) = Q_d(X,X')$ and $\pi_Y(\tilde{\Upsilon}_{p,\leq d}(Y,Y')) = Q_d(Y,Y')$. By Proposition 3.1.5 we know that for every $d > 0$ there exists $D > 0$ such that

$$\Psi_{\#}^{-1}(\Upsilon_{p,\leq d}(Y,Y')) \subseteq \Upsilon_{p,\leq D}(X,X'),$$

and hence

$$\Psi_{\star}^{-1}(\tilde{\Upsilon}_{p,\leq d}(Y,Y')) \subseteq \tilde{\Upsilon}_{p,\leq D}(X,X').$$

Consequently

$$\Psi_\star^{-1}(Q_d(Y,Y')) = (\pi_Y \circ \Psi_\star \circ \pi^{-1}_{X|_{\tilde{\Upsilon}_p(X,X')}})^{-1}(Q_d(Y,Y'))$$

$$= \pi_X(\Psi_\star^{-1}(\pi_Y^{-1}(Q_d(Y,Y')))) = \pi_X(\Psi_\star^{-1}(\tilde{\Upsilon}_{p\leq d}(Y,Y')))$$

$$\subseteq \pi_X(\tilde{\Upsilon}_{p\leq D}(X,X')) = Q_D(X,X').$$

Now let F be a closed subset of $\tilde{\mathcal{C}}_p(X,X')$, and, given $d>0$ choose $D>0$ as above so that

$$\Psi_\star(F)\cap Q_d(Y,Y') = \Psi_\star(F\cap \psi_\star{}^{-1}(Q_d(Y,Y')))\cap Q_d(Y,Y')$$

$$= \Psi_\star(F\cap Q_D(X,X'))\cap Q_d(Y,Y').$$

Since $Q_D(X,X')$ is compact and F is closed in $\tilde{\mathcal{C}}_p(X,X')$, then $F\cap Q_D(X,X')$ is compact and hence $\Psi_\star(F\cap Q_D(X,X'))$ is compact. From this we conclude that $\Psi_\star(F)\cap Q_d(Y,Y')$ is closed, for all d. Therefore $\Psi_\star(F)$ is closed, by Lemma 3.1.4(a). This last conclusion shows that $\Psi_\star$ is a closed map, and hence a homeomorphism, that is, an isomorphism of topological groups. □

Corollary 3.1.8 *If (X,X') and (Y,Y') are relatively isomorphic pairs then they have isomorphic Lawson homology, that is:*

$$(X,X')\overset{\cong}{\leftrightarrow}(Y,Y') \Rightarrow L_pH_{i+2p}(X,X')\cong L_pH_{i+2p}(Y,Y'),$$

for all p and i.

Corollary 3.1.9 *Let (X,X') and (Y,Y') be relatively isomorphic pairs.*

If the relative cycle map

$$s^p_{X,X'} : L_pH_{i+2p}(X,X') \to H_{i+2p}(X,X')$$

is an isomorphism, for fixed $p \geq 1$, *and* i, *then so is* $s^p_{Y,Y'}$ *(in the same bidegree), and vice-versa.*

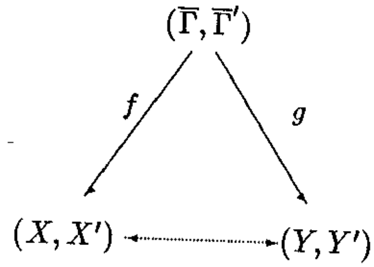
Proof

Suppose there is a morphism of pairs $f : (X,X') \to (Y,Y')$ realizing the relative isomorphism. From 2.2.8 we know that there is a commutative diagram:

$$\begin{array}{ccc} L_pH_{i+2p}(X,X') & \xrightarrow{f_*} & L_pH_{i+2p}(Y,Y') \\ {\scriptstyle s^p_{X,X'}}\downarrow & & \downarrow{\scriptstyle s^p_{Y,Y'}} \\ H_{i+2p}(X,X') & \xrightarrow[f_*]{} & H_{i+2p}(Y,Y'). \end{array}$$

It follows from the above theorem that the top row is an isomorphism. Now, since X' and Y' are algebraic subsets of X and Y respectively, the pairs (X,X') and (Y,Y') are NDR-pairs (because we can give a CW-complex structure to X and Y so that X' and Y' are closed subcomplexes, see [33]). Therefore $H_*(X,X')$ (respectively $H_*(X,X')$) is isomorphic to the reduced homology $H^\#_*(X/X')$ (respectively $H^\#_*(Y/Y')$). Since $f : (X,X') \to (Y,Y')$ is a relative isomorphism, it induces a homeomorphism $\tilde{f} : X/X' \to Y/Y'$ and the result follows from the commutative diagram above.

Now, suppose (X, X') and (Y, Y') are arbitrary relatively isomorphic pairs. Consider the correspondence



described in Remark 3.1.2, and look at the commutative diagram:

$$\begin{array}{ccc} L_pH_{i+2p}(Y, Y') & \xrightarrow{s_{Y,Y'}} & H_{i+2p}(Y, Y') \\ {\scriptstyle g_*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle g_*} \\ L_pH_{i+2p}(\overline{\Gamma})\overline{\Gamma}' & \xrightarrow{s_{\overline{\Gamma},\overline{\Gamma}'}} & H_{i+2p}(\overline{\Gamma}, \overline{\Gamma}') \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle f_*}\downarrow \\ L_pH_{i+2p}(X, X') & \xrightarrow{s_{X,X'}} & H_{i+2p}(X, X') \end{array}$$

Although $s_{\overline{\Gamma},\overline{\Gamma}'}$ is not necessarily an isomorphism, chasing the diagram correctly shows that $s_{Y,Y'}$ is an isomorphism. □

3.2 Some examples

Here we use the previous results to compute the Lawson homology of products of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$ and of hyperquadrics Q_n.

Example 1: Products of projective spaces.

We will show, by induction, that the cycle map $s_{\mathbb{P}^n\times\mathbb{P}^m} : L_pH_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m) \to H_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m)$ is an isomorphism for all n, m.

The induction is on the sum $N = n + m$. For N = 1 we have that $\mathbb{P}^n \times \mathbb{P}^m \equiv \mathbb{P}^1$. In this case the result follows from the Dold-Thom theorem, since $\mathcal{C}_0(\mathbb{P}^1) \cong SP^\infty(\mathbb{P}^1)$ and $\mathcal{C}_1(\mathbb{P}^1) \cong \mathbb{Z}$. See [23] for details.

Suppose the result is true for any product $\mathbb{P}^r \times \mathbb{P}^s$ with $r + s \leq N - 1$, and take $\mathbb{P}^n \times \mathbb{P}^m$ such that $n + m = N$, $N \geq 2$.

Embed $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{nm+m+n}$ via the Segre embedding $j : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+m+n}$. This is the embedding provided by the complete linear system associated to the divisor $D = H_1 + H_2$, where $H_1 = \mathbb{P}^{n-1} \times \mathbb{P}^m$ and $H_2 = \mathbb{P}^n \times \mathbb{P}^{m-1}$ are two effective generators of $\mathrm{Div}(\mathbb{P}^n \times \mathbb{P}^m)$, and $\mathbb{P}^{n-1} = \{pt\}$ when $n \leq 1$. Observe that $H_1 \cap H_2 = \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ and that D is the divisor obtained by a hyperplane section of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+m+n}$.

Remark 3.2.1 Since $D = H_1 + H_2$ has no weights on its irreducible components, we use D to denote both the divisor and the algebraic set $D = H_1 \cup H_2$, indistinctly. Also we assume $n \leq m$, $p \geq 1$.

First of all, let us compute the cycle spaces $\tilde{\mathcal{C}}_p(D)$ associated to the algebraic set D. Notice that the inclusion $i : (H_1, H_1 \cap H_2) \hookrightarrow (D, H_2)$ induces a relative isomorphism.

Here we point out that $H_1 = \mathbb{P}^{n-1} \times \mathbb{P}^m$, $H_2 = \mathbb{P}^n \times \mathbb{P}^{m-1}$ and $H_1 \cap H_2 = \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ satisfy the induction hypothesis, and hence their cycle maps establish an isomorphism between Lawson and singular homologies. From Proposition 2.2.2 we obtain a map of exact sequences:

$$\begin{array}{ccccccc}
\ldots L_pH_{i+2p}(H_1) & \longrightarrow & L_pH_{i+2p}(H_1, H_1 \cap H_2) & \longrightarrow & L_pH_{i-1+2p} & \longrightarrow \\
\cong \big\downarrow s^p_{H_1} & & \big\downarrow s^p_{H_1, H_1 \cap H_2} & & \cong \big\downarrow s^p_{H_1 \cap H_2} & \\
\ldots H_{i+2p}(H_1) & \longrightarrow & H_{i+2p}(H_1, H_1 \cap H_2) & \longrightarrow & H_{i-1+2p}(H_1 \cap H_2) & \longrightarrow
\end{array}$$

which implies that $s^p_{H_1, H_1 \cap H_2}$ is an isomorphism, by the five lemma. Since $i : (H_1, H_1 \cap H_2) \hookrightarrow (D, H_2)$ is a relative isomorphism, this shows that

$$s^p_{D,H_2} : L_pH_{i+2p}(D, H_2) \to H_{i+2p}(D, H_2) \qquad (3.1)$$

is also an isomorphism, according to Corollary 3.1.9 .

Now, the cycle map between the long exact sequences for Lawson and singular homologies of the pair (D, H_2), together with 3.1 and the five lemma shows that

$$s^p_D : L_pH_{i+2p}(D) \to H_{i+2p}(D)$$

is an isomorphism, for all i and p.

Let us go back to $\mathbb{P}^n \times \mathbb{P}^m$. We know that

$$\mathbb{P}^n \times \mathbb{P}^m \setminus D \cong \mathbb{C}^n \times \mathbb{C}^m \cong \mathbb{C}^{n+m} \cong \mathbb{P}^{n+m} \setminus \mathbb{P}^{n+m-1}.$$

Again we have an isomorphism

$$s^p_{\mathbb{P}^n \times \mathbb{P}^m, D} : L_pH_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m, D) \xrightarrow{\cong} H_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m, D)$$

from Corollary 3.1.9, due to the corresponding property for the pair $(\mathbb{P}^{n+m}, \mathbb{P}^{n+m-1})$.

Now we finally use the cycle map 2.2.2 of exact sequences for the pair $(\mathbb{P}^n \times \mathbb{P}^m, D)$:

$$\begin{array}{ccccccc}
\ldots L_pH_{i+2p}(D) & \longrightarrow & L_pH_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m) & \longrightarrow & L_pH_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m, D) & \longrightarrow \\
\cong \downarrow s_D & & \downarrow s^p_{\mathbb{P}^n \times \mathbb{P}^m} & & \cong \downarrow s^p_{\mathbb{P}^n \times \mathbb{P}^m, D} & \\
\ldots H_{i+2p}(D) & \longrightarrow & H_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m) & \longrightarrow & H_{i+2p}(\mathbb{P}^n \times \mathbb{P}^m, D) & \longrightarrow
\end{array}$$

and the five lemma again completes the proof.

Example 2: Hyperquadrics $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$.

Let $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric, that is, a quadric of rank n+1 (and dimension n), and let H be a hyperplane in $\mathbb{P}^{n+1}$ which is tangent to $\mathcal{Q}_n$ at some point $p_0 \in \mathcal{Q}_n$. Recall the following facts:

1. The intersection $H \cap \mathcal{Q}_n$ is a singular quadric of rank n-1 in H, and hence it is isomorphic to the complex suspension $\Sigma\mathcal{Q}_{n-2} = p_0 \# \mathcal{Q}_{n-2}$, where $\mathcal{Q}_{n-2}$ is the intersection $H \cap \mathcal{Q}_n \cap H'$, with H' being any hyperplane in $\mathbb{P}^{n+1}$ not containing p_0.

2. With p_0, H and H' chosen as before, consider the projection $\pi : \mathbb{P}^{n+1} \setminus p_0 \to H'$, away from p_0. Let $p : \mathcal{Q}_n \setminus (\mathcal{Q}_n \cap H) \to H'$ be the restriction of π to $\mathcal{Q}_n \setminus (\mathcal{Q}_n \cap H)$. A standard argument shows that p is, actually, an isomorphism onto $H' \setminus (H \cap H') \equiv \mathbb{P}^n \setminus \mathbb{P}^{n-1}$. From now on, denote $\mathcal{Q}_n \cap H$ by $\Sigma\mathcal{Q}_n$.

From Theorem 3.1.7 we have

$$\tilde{\mathcal{C}}_p(\mathcal{Q}_n, \Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}) \cong \tilde{\mathcal{C}}_p(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong K(\mathbb{Z}, 2(n-p)),$$

and by Corollary 3.1.9 we conclude that

$$s^p_{\mathcal{Q}_n, \Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}} : L_pH_{i+2p}(\mathcal{Q}_n, \Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}) \to H_{i+2p}(\mathcal{Q}_n, \Sigma\!\!\!\!/\,\mathcal{Q}_{n-2})$$

is an isomorphism for all $p \geq 1$, $i \geq 0$. Recall, from the proof of Proposition 2.2.3, that we have the following commutative diagram:

$$\begin{array}{ccc}
L_pH_{i+2p}(\mathcal{Q}_{n-2}) & \xrightarrow{s_{\mathcal{Q}_{n-2}}} & H_{i+2p}(\mathcal{Q}_{n-2}) \\
\cong \downarrow \Sigma\!\!\!\!/ & & \cong \downarrow \tau \\
L_{(p+1)}H_{i+2(p+1)}(\Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}) & \longrightarrow & H_{i+2p+2}(\Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}),
\end{array}$$

where τ is the Thom isomorphism.

Now, let us use induction on the dimension of the quadrics. For $n = 1$, $\mathcal{Q}_1 \cong \mathbb{P}^1$ and the result is already known to be true for this case. Also, for $n = 2$, $\mathcal{Q}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which also has already been studied before. Assume the result is true for all quadrics $\mathcal{Q}_r$, $r \leq n-1$, for some fixed $n \geq 3$. In other words, the cycle map

$$s_{\mathcal{Q}_r} : L_pH_{i+2p}(\mathcal{Q}_r) \to H_{i+2r}(\mathcal{Q}_r)$$

is an isomorphism for all $i \geq 0$, $p \geq 1$, $r \leq n-1$, for some fixed $n \geq 3$. From the above diagram and by the induction hypothesis, $s_{\mathcal{Q}_{n-2}}$ is an isomorphism, and hence $s_{\Sigma\!\!\!\!/\,\mathcal{Q}_{n-2}}$ also is an isomorphism. Now, assemble everything together in the long exact sequences (in Lawson and singular homologies) for the pair $(\mathcal{Q}_{n-2}, \Sigma\!\!\!/\mathcal{Q}_{n-2})$. Since $s_{\Sigma\!\!\!/\mathcal{Q}_{n-2}}$ and $s^p_{\mathcal{Q}_n, \Sigma\!\!\!/\mathcal{Q}_{n-2}}$ are both isomorphisms so is $s_{\mathcal{Q}_n}$ by the five lemma.

Remark 3.2.2 We can rephrase the above result by saying that there is a homotopy equivalence

$$\tilde{\mathcal{C}}_p(\mathcal{Q}_n) \cong \tilde{\mathcal{C}}_p(\mathcal{Q}_{n-2}) \times K(\mathbb{Z}, 2(n-p)),$$

for all $p \geq 1$, $n \geq 2$.

Remark 3.2.3 We remind here that a singular quadric $\mathcal{Q}_n^k \subset \mathbb{P}^{n+1}$, of rank k, is isomorphic to the iterated complex suspension $\Sigma\!\!\!/^{n-k}\mathcal{Q}_k$, where $\mathcal{Q}_k$ is a smooth hyperquadric contained in a linear subspace $\mathbb{P}^{k+1} \subset \mathbb{P}^{n+1}$. Therefore, the complex suspension theorem asserts that

$$\tilde{\mathcal{C}}_p(\mathcal{Q}_n^k) \cong \tilde{\mathcal{C}}_{p+k-n}(\mathcal{Q}_k),$$

which, combined with the above results yields

$$\tilde{\mathcal{C}}_p(\mathcal{Q}_n^k) \cong \tilde{\mathcal{C}}_{p+k-n}(\mathcal{Q}_{k-2}) \times K(\mathbb{Z}, 2(n-p)).$$

3.3 On the projective closure of vector bundles

Let $E \xrightarrow{\pi} X$ be a holomorphic (actually algebraic) vector bundle of rank r over a projective variety X. Recall the following definitions and facts: (see, e.g. [16], [17] or [19])

1. The projectivization of E, denoted $\mathbb{P}(E)$, is the projective bundle $\mathbb{P}(E) \xrightarrow{p} X$ over X, whose fiber $\mathbb{P}(E)_x$ is the projective space $\mathbb{P}(E_x)$ of lines in E_x, where E_x is the fiber of $E \xrightarrow{\pi} X$ over $x \in X$.

2. There is a tautological line bundle $\xi_E^* \to \mathbb{P}(E)$ obtained as a subbundle of p^*E,by taking as fiber over $[v] \in \mathbb{P}(E)$ the line $v \subset E_{p([v])}$ that $[v]$ represents. Notice that its dual, ξ_E, restricts to the hyperplane bundle $H \to \mathbb{P}(E_x)$ on each fiber of the projective bundle $\mathbb{P}(E)$. See

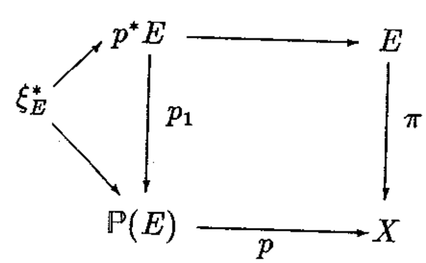


diagram:

3. The projective closure of E is the projective bundle $\mathbb{P}(E \oplus \mathbf{1}_X) \xrightarrow{\bar{p}} X$ over X, where $\mathbf{1}_X$ is the trivial line bundle over X. Recall that, since

$E \oplus \mathbf{1}_X$ carries two canonical subbundles $0 \oplus \mathbf{1}_X$ and $E \oplus 0$, we have a zero section $X \equiv \mathbb{P}(0 \oplus \mathbf{1}_X) \overset{s_0}{\hookrightarrow} \mathbb{P}(E \oplus \mathbf{1}_X)$ and a "section" at infinity $\mathbb{P}(E) \equiv \mathbb{P}(E \oplus 0) \overset{s_\infty}{\hookrightarrow} \mathbb{P}(E \oplus \mathbf{1}_X)$. Furthermore, the composition

$$\mathbb{P}(E) \overset{s_\infty}{\hookrightarrow} \mathbb{P}(E \oplus \mathbf{1}_X) \xrightarrow{p} X.$$

is the projective bundle $\mathbb{P}(E) \xrightarrow{p} X$.

4. The fiber $\mathbb{P}(E \oplus \mathbf{1}_X)_x$ is the projective space $\mathbb{P}(E_x \oplus \mathbb{C})$ and it can be seen as the complex suspension $\not\Sigma \mathbb{P}(E_x)$. This easily shows that the set $\mathbb{P}(E \oplus \mathbf{1}_X) \setminus s_0(X)$ is the total space of the line bundle $\xi_E \to \mathbb{P}(E)$.

5. Analogously, $\mathbb{P}(E \oplus \mathbf{1}_X) \setminus \mathbb{P}(E)$ can be seen to be the total space of the bundle $E \to X$.

6. We define a vector bundle $E \to X$ to be "very ample" if the line bundle $\xi_{E^*} \to \mathbb{P}(E^*)$ provides a projective embedding

$$\mathbb{P}(E^*) \hookrightarrow \mathbb{P}(\Gamma(\mathbb{P}(E^*), \xi_{E^*})^*).$$

7. Our last definition is that of a Grauert-positive vector bundle $E \to X$. It is a vector bundle where we can blow-down the zero-section of its dual $E^* \to X$. In other words, the topological quotient space E^*/X admits a structure of analytic space (or algebraic, depending on the context). It can be seen, see [31], that a very ample (even ample) bundle is also Grauert-positive.

Now we are in a position to state the following result:

Theorem 3.3.1 *Let $E \to X$ be a very ample vector bundle over a projective algebraic variety. Then, the following homotopy equivalence holds:*

$$\tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X)) \cong \tilde{\mathcal{C}}_p(X) \times \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E^*)),$$

for all $p \geq 1$.

Proof

Let $X \overset{s_0}{\hookrightarrow} \mathbb{P}(E^* \oplus \mathbf{1}_X)$ be the zero-section of the projective bundle $\mathbb{P}(E^* \oplus \mathbf{1}_X)$. Denote by $W \overset{p}{\to} \mathbb{P}(E^* \oplus \mathbf{1}_X)$ the blow-up of $\mathbb{P}(E^* \oplus \mathbf{1}_X)$ along X. Recall (from the remarks at the beginning of the section) that $\mathbb{P}(E^* \oplus \mathbf{1}_X) \setminus \mathbb{P}(E^*) \to X$ is the vector bundle E^*, where $\mathbb{P}(E^*)$ is the "section" at infinity; this easily shows that the normal' cone $N_{\mathbb{P}(E^* \oplus \mathbf{1}_X)} X$ of $X \subset \mathbb{P}(E^* \oplus \mathbf{1}_X)$ is E^*, (see [17] or [19]). Therefore, the exceptional divisor in the blow-up W is given by $\mathbb{P}(N_{\mathbb{P}(E^* \oplus \mathbf{1}_X)} X) = \mathbb{P}(E^*)$ and the normal cone to the exceptional $\mathbb{P}(E^*)$ is the tautological line bundle $\xi^*_{E^*}$. Actually, it can be seen that W is the projective closure $\mathbb{P}(\xi^*_{E^*} \oplus \mathbf{1}_X)$ of the tautological bundle $\xi^*_{E^*} \to \mathbb{P}(E^*)$, and that the exceptional divisor is the zero section $\mathbb{P}(E^*) \overset{s_0}{\hookrightarrow} \mathbb{P}(E^* \oplus \mathbf{1}_X)$ with normal bundle $\mathbb{P}(\xi^*_{E^*} \oplus \mathbf{1}_X) \setminus s_\infty \equiv \xi^*_{E^*}$, where s_∞ is the "section" at infinity.

The advantage of this point of view is that it provides us with a flat map $f : W \to \mathbb{P}(E^*)$ and hence we have the following situation

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{f} \\ \mathbb{P}(E^* \oplus \mathbf{1}_X) & & \mathbb{P}(E^*) \end{array}$$

where p is the blow-down map.

Now, observe that the map of pairs $\overline{p} : (W, \mathbb{P}(E^*)) \to (\mathbb{P}(E^* \oplus \mathbf{1}_X), X)$ is a relative isomorphism, with $\mathbb{P}(E^*)$ and X being the zero sections of $\mathbb{P}(\xi^*_{E^*} \oplus \mathbf{1}_X)$ and $\mathbb{P}(E^* \oplus \mathbf{1}_X)$, respectively. It follows from Theorem 3.1.7 that p induces an isomorphism of topological groups

$$p_* : \tilde{\mathcal{C}}_p(W, \mathbb{P}(E^*)) \to \tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X), X). \tag{3.2}$$

Notice also that, since $E \to X$ is very ample, it is Grauert-positive and we can blow-down the zero-section of $E^* \to X$. Therefore we can blowdown the zero section of the projective closure $\mathbb{P}(E^* \oplus \mathbf{1}_X) \to X$ of E^*, obtaining a variety Y and a sequence of blow-downs

$$W = \mathbb{P}(\xi^*_{E^*} \oplus \mathbf{1}_X) \xrightarrow{p} \mathbb{P}(E^* \oplus \mathbf{1}_X) \xrightarrow{d} Y. \tag{3.3}$$

Recall that $\xi_{E^*} \to \mathbb{P}(E^*)$ corresponds to the hyperplane bundle H of an embedding of $\mathbb{P}(E^*)$ into a projective space $\mathbb{P}^N$. Therefore we have that

$$W = \mathbb{P}(\xi_{E^*}^* \oplus \mathbf{1}_X) \equiv \mathbb{P}(\mathbf{1}_{\mathbb{P}(E^*)} \oplus H)$$

and that the composition of blow-downs $W \overset{d \circ p}{\to} Y$ corresponds actually to the blow-down of W to the Thom space Y of the hyperplane bundle $H \equiv \xi_{E^*} \to \mathbb{P}(E^*)$. In other words, Y is the complex suspension $\Sigma\!\!\!/\,\mathbb{P}(E^*)$ of $\mathbb{P}(E^*)$ for the projective embedding of $\mathbb{P}(E^*)$ induced by ξ_{E^*}. It follows that d induces a relative isomorphism

$$\overline{d} : (\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \to (Y, y_0),$$

where y_0 is the image under the blow-down of the zero-section, and therefore, from Theorem 3.1.7 we have an isomorphism of topological groups

$$\overline{d}_* : \tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \to \tilde{\mathcal{C}}_p(Y, y_0).$$

Observe that $\tilde{\mathcal{C}}_p(Y, y_0) \equiv \tilde{\mathcal{C}}_p(Y)$ for $p \geq 1$.

At this point we go back to the projective bundle $W \xrightarrow{f} \mathbb{P}(E^*)$ in the diagram above. As we saw in Remark 1.2.3, f is a flat morphism of relative dimension 1, and hence it induces, by Proposition 1.2.2, a morphism of topological groups

$$f^* : \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E^*)) \to \tilde{\mathcal{C}}_p(W). \tag{3.4}$$

Summarizing, we have the following commutative diagram

$$
\begin{array}{ccccc}
 & & \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E^*)) & & \\
 & \swarrow^{f^*} & & \searrow^{\Sigma\!\!\!\!/}_{\cong} & \\
\tilde{\mathcal{C}}_p(W) & \xrightarrow{p_*} & \tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X)) & \xrightarrow{d_*} & \tilde{\mathcal{C}}_p(\Sigma\!\!\!\!/\,\mathbb{P}(E^*)) \\
\downarrow \pi_1 & & \downarrow \pi_2 & & \cong \downarrow \pi_3 \\
\tilde{\mathcal{C}}_p(W, \Sigma\!\!\!\!/\,\mathbb{P}(E^*)) & \xrightarrow[\overline{p}_*]{\cong} & \tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) & \xrightarrow[\overline{d}_*]{\cong} & \tilde{\mathcal{C}}_p(\Sigma\!\!\!\!/\,\mathbb{P}(E^*), y_0)
\end{array}
$$

where π_1, π_2, π_3 are the quotient maps.

Define a morphism of topological groups by

$$\Psi : \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E^*)) \times \tilde{\mathcal{C}}_p(X) \to \tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X)))$$

$$(\sigma, \tau) \longmapsto p_* \circ f^*(\sigma) + s_{0_*}(\tau),$$

where $s_0 : X \hookrightarrow \mathbb{P}(E^* \oplus \mathbf{1}_X)$ is the zero section. Our aim is to show that Ψ is a homotopy equivalence, or equivalently, that Ψ induces isomorphism in Lawson homology

$$\Psi_* : L_{(p-1)}H_{i+2(p-1)}(\mathbb{P}(E^*)) \oplus L_pH_{i+2p}(X) \xrightarrow{\cong} L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X)).$$

Note that since $\tilde{\mathcal{C}}_p(\mathbb{P}(E^* \oplus \mathbf{1}_X))$ is a topological group, we may take as additive structure for its homotopy groups the one given by pointwise addition of representatives, (see [33]). In particular one sees that the homomorphism Ψ_* is given by

$$\Psi_*(\alpha,\beta) = p_* \circ f^*(\alpha) + s_{0_*}(\beta),$$

where $\alpha \in L_{(p-1)}H_{i+2(p-1)}(\mathbb{P}(E^*)), \beta \in L_pH_{i+2p}(X)$, and p_*, f^* and s_{0_*} are the homomorphisms in Lawson homology induced by p, f and s_0 respectively. See Proposition 1.2.2.

Consider the long exact sequence in Lawson homology for the pair $(\mathbb{P}(E^* \oplus \mathbf{1}_X), X)$:

$$\ldots \to L_pH_{i+2p}(X) \stackrel{s_{0_*}}{\to} L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X)) \stackrel{\pi_{2_*}}{\to} L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \to \ldots$$

and define homomorphisms

$$\varphi : L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \to L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X))$$

for all i, by

$$\varphi \stackrel{def}{=} p_* \circ f^* \circ \not{\Sigma}_*^{-1} \circ \pi_{3_*}^{-1} \circ \overline{d}_*.$$

(See the diagram above). Since

$$\begin{aligned}
\pi_{2_*} \circ \varphi &= (\pi_{2_*} \circ p_*) \circ f^* \circ \not{\Sigma}_*^{-1} \circ \pi_{3_*}^{-1} \circ \overline{d}_* \\
&= \overline{p}_* \circ (\pi_{1_*} \circ f^* \circ \not{\Sigma}_*^{-1} \circ \pi_{3_*}^{-1}) \circ \overline{d}_* \\
&= \overline{p}_* \circ (\overline{p}_*^{-1} \circ \overline{d}_*^{-1}) \circ \overline{d}_* \\
&= id,
\end{aligned}$$

it follows that the long exact sequence breaks into short exact sequences for which φ is a splitting homomorphism. Therefore we obtain an isomorphism

$$T : L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \oplus L_pH_{i+2p}(X) \to L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X))$$

$$(\alpha, \beta) \longmapsto \varphi(\alpha) + s_{0*}(\beta).$$

If we compose T with the isomorphism

$$I : L_{(p-1)}H_{i+2(p-1)}(\mathbb{P}(E^*)) \oplus L_pH_{i+2p}(X) \to L_pH_{i+2p}(\mathbb{P}(E^* \oplus \mathbf{1}_X), X) \oplus L_pH_{i+2p}(X)$$

$$(\sigma, \tau) \longmapsto (\overline{d}_*^{-1} \circ \pi_{3*} \circ \Sigma\!\!\!/_*(\sigma), \tau)$$

we get

$$\begin{aligned} T \circ I(\sigma, \tau) &= \varphi \circ \overline{d}_*^{-1} \circ \pi_{3*} \circ \Sigma\!\!\!/_*(\sigma) + s_{0*}(\tau) \\ &= (p_* \circ f^* \circ \Sigma\!\!\!/_*^{-1} \circ \pi_{3*}^{-1} \circ \overline{d}_*) \circ \overline{d}_*^{-1} \circ \pi_{3*} \circ \Sigma\!\!\!/_*(\sigma) + s_{0*}(\tau) \\ &= p_* \circ f^*(\sigma) + s_{0*}(\tau) \\ &\stackrel{def}{=} \Psi_*(\sigma, \tau), \end{aligned}$$

and the theorem follows. □

Corollary 3.3.2 *For any algebraic vector bundle over a projective variety X there exists an integer $m_0 \geq 0$ such that for $m \geq m_0$ we have:*

$$\tilde{\mathcal{C}}_p(\mathbb{P}(E \oplus H^{\otimes m})) \cong \tilde{\mathcal{C}}_p(X) \times \tilde{\mathcal{C}}_{p-1}(\mathbb{P}(E))$$

for all p, where H is the hyperplane bundle over X.

Proof

Let E^* be the dual of E. Since H is a positive line bundle, it follows (see [31]) that $E^* \otimes H^{\otimes m}$ is very ample for $m \geq m_0$, for some m_0. Apply the theorem to the vector bundle $E^* \otimes H^{\otimes m}$. It says that

$$\tilde{\mathcal{C}}_p(\mathbb{P}((E^* \otimes H^{\otimes m})^* \oplus \mathbf{1}_X)) \cong \tilde{\mathcal{C}}_p(X) \times \tilde{\mathcal{C}}_{p-1}(\mathbb{P}((E^* \otimes H^{\otimes m})^*)).$$

However, observing that

$$\mathbb{P}((E^* \otimes H^{\otimes m})^* \oplus \mathbf{1}_X) = \mathbb{P}((E \otimes H^{*\otimes m}) \oplus \mathbf{1}_X) = \mathbb{P}(E \oplus H^{\otimes m})$$

and that

$$\mathbb{P}((E^* \otimes H^{\otimes m})^*) = \mathbb{P}(E \otimes H^{*\otimes m}) = \mathbb{P}(E),$$

we obtain the corollary. □

Chapter 4

A very special class

4.1 A model

As we have seen in the previous chapter, the "cycle map" $s^p : L_pH_{i+2p}(X) \to H_{i+2p}(X)$ is an isomorphism, for all $p, \ i \geq 0$, for *some* spaces X such as $X = \mathbb{P}^n \times \mathbb{P}^m$ or Q_n. This motivates the following definition:

Definition 4.1.1 *We say that an algebraic variety X lies in the class $\mathcal{L}$ if the cycle map*

$$s^p : L_pH_{i+2p}(X) \to H_{i+2p}(X)$$

is an isomorphism for all $p, \ i \geq 0$.

Before we make a further investigation of the class $\mathcal{L}$, let us first introduce some canonical spaces which will be useful in our future computations.

Definition 4.1.2 *Given a sequence of integers* $1 \leq n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_k$*, choose a flag of affine spaces* $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k$*, where each* V_i *is isomorphic to* $\mathbb{C}^{n_i}$*. This yields a "projective flag"* $\mathbb{P}(V_0) \subseteq \mathbb{P}(V_1) \subseteq \ldots \subseteq \mathbb{P}(V_k)$*. Now, embed* $\mathbb{P}(V_k)$ *linearly in a projective space* $\mathbb{P}(V)$ *of dimension* $n_k + 1$ *and choose* $k+1$ *distinct points* $p_0, \ldots, p_k$ *in* $\mathbb{P}(V) \setminus \mathbb{P}(V_k)$*. Define the space* $P(n_0, \ldots, n_k)$ *to be the union*

$$P(n_0, \ldots, n_k) \stackrel{def}{=} \bigcup_{i=0}^{k} p_i \# \mathbb{P}(V_i).$$

These spaces will be our canonical models.

Observe that $P(n_0, \ldots, n_k)$ can also be described as follows: In the projective space $\mathbb{P}^{n_k+1}$ choose a linear subspace L of dimension $n_k - 1$, and let $\mathcal{P}(L)$ be the pencil of hyperplanes generated by L. Choose $k+1$ distinct hyperplanes $H_0, \ldots, H_k$ in $\mathcal{P}(L)$. In each H_i choose, inductively, a linear subspace $\mathbb{P}^{n_i}$ such that its intersection with the base locus L of $\mathcal{P}(L)$ has dimension $n_i - 1$ and $\mathbb{P}^{n_i} \cap L = \mathbb{P}^{n_i - 1} \cap L$. Then $P(n_0, \ldots, n_k)$ is the union of those $\mathbb{P}^{n_i}$'s.

Proposition 4.1.3 *Given* $1 \leq n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_k$*, let* $P(n_0, \ldots, n_k) = \bigcup_{i=0}^{k} \mathbb{P}^{n_i}$*, be a "standard model" as above. Then:*

(a) *$P(n_0,\ldots,n_k)\setminus L$ is isomorphic to $\coprod_i \mathbb{C}^{n_i}$.*

(b) *The relative cycle map:*

$$s^p : L_pH_{i+2p}(P(n_0,\ldots,n_k),L) \to H_{i+2p}(P(n_0,\ldots,n_k),L)$$

is an isomorphism for all $p \geq 1$, $i \geq 0$.

Proof

(a) This is clear from the construction.

(b) We use induction on the sum $n_k + k$.

For $n_k+k = 1$ we have that $k = 0$ and $n_0 = 1$. Therefore $P(n_0,\ldots,n_k) = \mathbb{P}^1$ and $L = \mathbb{P}^0$. The result is already known to be true in this case.

Assume the result is true for $n_k + k \leq N$ and choose a sequence $1 \leq n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_k$ so that $n_k + k = N + 1$. There is a canonical $P(n_0,\ldots,n_{k-1})$ embedded in $P(n_0,\ldots,n_{k-1},n_k)$ and the mere definition of the objects involved gives an isomorphism

$$P(n_0,\ldots,n_{k-1},n_k)\setminus P(n_0,\ldots,n_{k-1}) \cong \mathbb{P}^{n_k}\setminus L \cong \mathbb{C}^{n_k}.$$

As shown in Corollary 2.1, this implies that the relative cycle map

$$s^p : L_pH_{i+2p}(P(n_0,\ldots,n_k),P(n_0,\ldots,n_{k-1})) \to H_{i+2p}(P(n_0,\ldots,n_k),P(n_0,\ldots,n_{k-1}))$$

is an isomorphism, for all $p \geq 1$, $i \geq 0$. The morphism of long exact sequences induced by the cycle maps together with the induction hypothesis and the corresponding result for projective spaces complete the proof. □

Let us establish a notation here:

Definition 4.1.4 *Given an algebraic pair (X,Y), we say that X is an "algebraic cellular extension" of Y of type $(n_0,\dots,n_k)$, with $1 \leq n_0 \dots \leq n_k$, and denote it by $X = P(Y;n_0,\dots,n_k)$, if*

$$X \setminus Y \overset{iso}{\cong} \mathbb{C}^{n_0} \coprod \dots \coprod \mathbb{C}^{n_k}.$$

In other words, the pairs (X,Y) and $(P(n_0,\dots,n_k),L)$ are relatively isomorphic.

Corollary 4.1.5 *If $X = P(Y;n_0,\dots,n_k)$ for a pair (X,Y) then the cycle map $s_X^p : L_pH_{i+2p}(X) \to H_{i+2p}(X)$ is an isomorphism if and only if so is the map $s_Y^p : L_pH_{i+2p}(Y) \to H_{i+2p}(Y)$.*

Proof

Obvious from Corollary 3.1.9 and the above Proposition, plus standard exact sequence arguments. □

4.2 The class $\mathcal{L}$

As an immediate outcome of the observations in the previous section, we show that the class $\mathcal{L}$ contains a considerable amount of elements.

We take the following definition essentially from [15], Example 1.9.1:

Definition 4.2.1 *Let (X, Y) be an algebraic pair. We say that X is an algebraic cellular extension of Y if X has a filtration $X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = Y$ by closed (algebraic) subsets, with each $X_i \setminus X_{i-1}$ a disjoint union of quasiprojective varieties U_{ij} isomorphic to affine spaces $\mathbb{C}^{n_{ij}}$, for $i \geq 0$. In case $Y = \emptyset$, we recover Fulton's definition of an algebraic set with a cellular decomposition. After ordering the n_{ij}'s for every $i \geq 0$ so that $n_{i1} \leq n_{i2} \leq \ldots n_{ik_i}$, we can re-state the definition by saying that for every pair (X_i, X_{i-1}) in the filtration we have that $X_i = P(X_{i-1}; n_{i1}, \ldots, n_{ik_i})$, for all $0 \leq i \leq n$, according to Definition 4.1.4 .*

Although it is a simple consequence of Corollary 4.1.5, we state the next result as a theorem for the broad class of examples arising from it.

Theorem 4.2.2 *If X is an algebraic cellular extension of Y, then X lies in the class $\mathcal{L}$ if and only if so does Y. In particular one sees that the class $\mathcal{L}$ is closed under algebraic cellular extensions.*

Proof

Once again we use induction. This time on the height n of the filtration. For $n = 0$, $X = X_0 \supset X_{-1} = Y$, we see that $X = P(Y; n_{01}, \ldots, n_{0k_0})$, i.e., X is an extension of Y of type $(n_{01}, \ldots, n_{0k_0})$, and the result is simply Corollary 4.1.5 .

Assume the result is true for filtrations of height $n-1$ and let (X, Y) have a filtration of height n, $n \geq 1$. From the definition we know that $X = X_n = P(X_{n-1}; n_{n1}, \ldots, n_{nk_n})$, therefore X lies in the class $\mathcal{L}$ if and only if so does X_{n-1}, again by 4.1. Now, by the induction hypothesis, X_{n-1} is in the class $\mathcal{L}$ if and only if Y does. This proves the theorem. □

Corollary 4.2.3 *If X is an algebraic set with a cellular decomposition in the sense of Fulton [15], (i.e., X is an algebraic cellular extension of $\emptyset$), then it lies in the class $\mathcal{L}$.*

Remark 4.2.4 Let X and X' both have cellular decompositions $X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$ and $X' = X'_m \supset X'_{m-1} \supset \ldots \supset X'_0 \supset X'_{-1} = \emptyset$, with $X_i \setminus X_{i-1} \overset{iso}{\cong} \coprod_r \mathbb{C}^{n_{ir}}$ and $X'_j \setminus X'_{j-1} \overset{iso}{\cong} \coprod_s \mathbb{C}^{m_{js}}$. It is easy to see that the filtration

$$X \times Y = (X \times Y)_{m+n} \supset (X \times Y)_{m+n-1} \supset \ldots \supset (X \times Y)_0 \supset (X \times Y)_{-1} = \emptyset$$

provides a cellular decomposition for $X \times Y$, with $(X \times Y)_t \setminus (X \times Y)_{t-1} \cong \coprod_{i+j=t,r,s} (\mathbb{C}^{n_{ir}} \times \mathbb{C}^{n_{js}}) \equiv \coprod_{i+j=t,r,s} \mathbb{C}^{n_{ir}+n_{js}}$. In particular we obtain that the product $X \times Y$ also lies in the class $\mathcal{L}$.

4.3 Last example: Generalized Flag Varieties G/P

We start by recalling some standard facts on the theory of linear algebraic groups. Let us introduce the usual (and lengthy) list of notations we use throughout this section:

- G : semisimple linear algebraic group, defined over $\mathbb{C}$;
- B : fixed Borel subgroup of G;
- B^u : unipotent radical of B;
- T : fixed maximal torus of G, $T \subset B$;
- $\mathbf{g}$: Lie algebra of G;

- $\mathbf{h}$: Lie algebra of T;
- $\mathbf{h}^*$: space dual to $\mathbf{h}$;
- $\triangle \subset \mathbf{h}^*$: root system of $\mathbf{h}$ in $\mathbf{g}$;
- $\triangle_+$: set of positive roots;
- $\triangle_- = -\triangle_+$;
- $\Sigma \subset \triangle_+$: system of simple roots;
- $\mathcal{W}$: Weyl group of G. We may face $\mathcal{W}$ either as the quotient $N(T)/T$, where $N(T)$ is the normalizer of T in G, or as the (finite) group generated by the set $\{\sigma_\gamma : \mathbf{h}^* \to \mathbf{h}^* : \gamma \in \triangle\}$, where σ_γ is the reflection across the hyperplane perpendicular to γ (with respect to the Killing form, for example).
- $l(w)$: length of an element $w \in \mathcal{W}$ relative to the set of generators $\{\sigma_\alpha,\ \alpha \in \Sigma\}$ of $\mathcal{W}$, i.e., the least number of factors in the decomposition

  $$w = \sigma_{\alpha_1} \cdot \ldots \cdot \sigma_{\alpha_r}, \quad \alpha_i \in \Sigma.$$

  One such decomposition with $r = l(w)$ is called reduced.
- $w_0 \in \mathcal{W}$: unique element of maximal length.

We use the same letter w to denote either the element $w \in N(T) \subset G$. This makes sense of the following definition:

- $\tilde{B}^u \stackrel{def}{=} w_0 B^u w_0^{-1}$: is the subgroup of G "opposite" to B^u. Recall that $\tilde{B}^u \cap B^u = T$.

Finally, for any $w \in \mathcal{W}$ we define

- $H(w) \stackrel{def}{=} B^u \cap w\tilde{B}^u w^{-1}$.

Parabolic subgroups of G

Here we say that a subgroup P of G is parabolic if it contains a Borel subgroup $B \subset G$. Let us recall some facts about the structure of parabolic subgroups $P \subset G$, see [6]:

Let S be any subset of Σ, and let $\triangle_S$ be the subset of $\triangle_+$ consisting of linear combinations of elements of S. Denote by G_S the subgroups

$$N_\gamma = \{exp\, tE_{-\gamma} : t \in \mathbb{C}\},\ \gamma \in \triangle_S \cup -\triangle_S,$$

where $E_{-\gamma}$ is the standard element of $\mathbf{g}$ in the $(-\gamma)$-root-space. Let N_S be the subgroup of B generated by the N_γ's with $\gamma \in \triangle_+ \setminus \triangle_S$. Then G_S is a reductive group normalizing N_S and $P_S \stackrel{def}{=} G_S N_S$ is a parabolic subgroups of G containing B.

It is a standard fact that every parabolic subgroup $P \subset G$ is conjugate to one of the subgroups P_S. So, we may assume, in what follows, that $P = P_S$ for some fixed subset $S \subset \Sigma$. Let $\mathcal{W}_S$ be the Weyl subgroup of G_S, i.e., the subgroup of $\mathcal{W}$ generated by the reflections σ_α with $\alpha \in S$.

Recall that the space G/P of orbits in G for the action of P on the right is a non-singular projective variety and that $G \to G/P$ is a smooth principal fibration under the group P.

Now, we finally have the following decomposition of G/P under the action of B: (cf. [6] or [5])

(a) $G/P = \cup_{w \in \mathcal{W}} Bw\underline{o}$, where $\underline{o} \in G/P$ is the image of P in G/P;

(b) Two orbits $Bw_1\underline{o}$ and $Bw_2\underline{o}$ are identical if $w_1 W_2^{-1} \in \mathcal{W}_S$, and otherwise are disjoint;

(c) Define $\mathcal{W}_S^1 = \{w \in \mathcal{W} : wS \subset \triangle_+$. Then each coset of $\mathcal{W}/\mathcal{W}_S$ has a unique element of $\mathcal{W}_S^1$. Furthermore, each $w \in \mathcal{W}_S^1$ is characterized by the fact that its length is less than that of any other element in the coset $w\mathcal{W}_S$;

(d) If $w \in \mathcal{W}_S^1$, then the mapping

$$H(w) \to G/P$$

$$n \mapsto nw\underline{o}$$

(see list at beginning of this section for definition of $H(w)$) is an isomorphism of $H(w)$ with a subvariety $X_P(w) \subset G$;

(e) $H(w)$ is a unipotent group isomorphic as variety to an affine space of dimension $l(w)$;

(f) Define the "Schubert varieties" $\overline{X}_P(w)$ as the closure of $X_P(w)$ in G/P. Then, using the partial ordering in $\mathcal{W}_S^1$ induced by that in $\mathcal{W}$, cf. [6], we have that

$$\overline{X}_P(w) = \coprod_{w' \leq w,\, w', w \in \mathcal{W}_S^1} X_P(w').$$

As a consequence of all this we have the following

Proposition 4.3.1 (Example) *Let G be a semisimple algebraic group and let P be a parabolic subgroup of G. Then G/P admits a cellular decomposition in the sense of Definition 4.2.1, and therefore G/P belongs to the class $\mathcal{L}$. In other words, the cycle map*

$$s^p : L_pH_{i+2p}(G/P) \to H_{i+2p}(G/P)$$

is an isomorphism for all $p, i \geq 0$.

Proof

Assume w.l.o.g. that $P = P_S$ for some $S \subset \Sigma$, and let $\mathcal{W}_S^1$ be the set of minimal representatives for the coset space $\mathcal{W}/\mathcal{W}_S$. For every $0 \leq j \leq \dim G/P$, define the algebraic subsets X_j of G/P by

$$X_j \stackrel{def}{=} \cup_{w \in \mathcal{W}_S^1, l(w) \leq j} \overline{X}_P(w).$$

Define $X_0 = \cup_{w \in \mathcal{W}_S^1, l(w)=1} (\overline{X}_P(w) \setminus X_P(w))$, and observe that X_0 consists of at most a finite number of points.

Claim: The filtration $G/P = X_d \supset X_{d-1} \supset \ldots \supset X_0 \supset X_{-1} = \phi$, with $d = \dim G/P$, gives a cellular decomposition for G/P.

This claim (as well as the proposition) is a trivial consequence of the structure of the Schubert varieties $\overline{X}_P(w)$. Observe that

$$X_j \setminus X_{j-1} \subset \bigcup_{w \in \mathcal{W}_S^1, l(w)=j} \overline{X}_P(w),$$

Since $\overline{X}_P(w) = \coprod_{w' \in \mathcal{W}_S^1, w' \leq w} X_P(w')$ and $w' \leq w \Rightarrow l(w') \leq l(w)$ we have that

$$X_j \setminus X_{j-1} = \coprod_{w \in \mathcal{W}_S^1, l(w)=j} X_P(w) \stackrel{iso}{\cong} \underbrace{\mathbb{C}^j \coprod \ldots \coprod \mathbb{C}^j}_{\#\{w \in \mathcal{W}_S^1 : l(w)=j\}}.$$

Therefore G/P admits an algebraic cellular decomposition and hence belongs to the class $\mathcal{L}$ by Corollary 4.2.3 . □

Corollary 4.3.2 *Any hermitian compact symmetric space lies in* *the class $\mathcal{L}$.*

Proof

Any hermitian compact symmetric space can be written as a product of G/P's, with G compact and P parabolic (in fact, maximal parabolic). The corollary now follows from the above Proposition and Remark 4.2.4 . □

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