On twistor spaces of anti-self-dual hermitian surfaces

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Abstract of the Dissertation

On twistor spaces of anti-self-dual hermitian surfaces

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We consider hermitian 4-dimensional manifolds (M, J, g) with Weyl tensor W satisfying the conformally invariant condition $*W = -W$, where $*$ is the Hodge-star operator. An equivalent way to state all this is to say that the twistor space Z of M is a complex 3-dimensional manifold with a certain distinguished divisor X.

In this work we study the interplay between the holomorphic properties of Z and the conformal structure of M.

Following C. Boyer, M belongs to one of two classes, which we call Kähler type and non-Kähler type. The first class consists

exactly of the complex surfaces equipped with a metric of zero scalar curvature. We show that this distinction has a profound influence on the complex structure of its twistor space Z. The main results are:

If K_Z denotes the canonical bundle of Z, then the divisor line bundle $[X]$ is isomorphic to $K_Z^{-\frac{1}{2}}$ if and only if M is of Kähler type. Using this result we are able to simplify the proof of a theorem of Poon which says that the algebraic dimension of the twistor space of a surface of Kähler type is at most 1. In contrast to this result we give the first example of a twistor space of algebraic dimension 2. It is the twistor space of a Hopf surface.

When M is of Kähler type, we also describe the close relation between holomorphic vector fields on M and Z; we also show that this reflects the subdivision between Ricci-flat and non Ricci-flat surfaces of Kähler type.

Then, using new techniques of Donaldson and Friedman for constructing twistor spaces, we prove that the connected sums of any number of copies of the Hopf surface H and the complex projective plane $\mathbb{CP}_2$ admit self-dual-metrics.

We conclude by giving a detailed description of the construction of the twistor space of $H\#\overline{\mathbb{CP}_2}$.

A mio padre

Contents

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Chapter 1

Introduction

The aim of this chapter is to give the basic definitions and results which will be needed later.

1.1 Half-conformally flat manifolds

Let (M, g) be an oriented riemannian manifold of even dimension $2n$, and consider the Hodge-star operator acting on forms of the middle dimension: $* : \bigwedge^n(M) \to \bigwedge^n(M)$ with the property that $*^2 = (-1)^n$.

Now, since in general the Riemann curvature tensor can be interpreted as an operator $\mathcal{R} : \bigwedge^2(M) \to \bigwedge^2(M)$, the case $n = 2$ is of particular interest. In this case in fact, $*^2 = id.$ and we can write $\bigwedge^2(M) = \bigwedge^2_+(M) \oplus \bigwedge^2_-(M)$ where $\bigwedge^2_\pm(M) = \{\eta \in \bigwedge^2(M) | *\eta = \pm\eta\}$. With respect to this decomposition one writes $\mathcal{R} \in \bigodot^2(\bigwedge^2_+(M) \oplus \bigwedge^2_-(M))$, where by $\bigodot$ we denote the symmetric tensor product; as a matrix $\mathcal{R} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$.

Following [23], this gives a complete decomposition of $\mathcal{R}$ into irreducible components: $\mathcal{R} = (trA, B, A - \frac{1}{3}trA, C - \frac{1}{3}trC) \in \mathbb{R} \bigoplus (\mathbb{S}^2_+ \otimes \mathbb{S}^2_-) \bigoplus \mathbb{S}^4_+ \bigoplus \mathbb{S}^4_-$ where $\mathbb{S}^m_+$ and $\mathbb{S}^m_-$ denote the m-th symmetric power of the $+\frac{1}{2}$ and $-\frac{1}{2}$ spin bundle of M respectively. Recall that the irreducible representations of $SO(4)$ are of the form $\mathbb{S}^p_+ \otimes \mathbb{S}^q_-$. Also $\mathbb{S}^{2k}_\pm$ always exist whether M is spin or not.

From now on M will denote a four dimensional manifold.

The above decomposition has important geometric significance because $trA = trC = \frac{1}{4}R$, where R denotes the scalar curvature; furthermore, B is the traceless Ricci tensor ($B = Ric - R/4 \cdot g$) and the last two components, usually denoted by W_+ and W_-, are the self-dual and anti-self-dual parts of the Weyl curvature tensor W.

In general, W is exactly the conformal invariant part of $\mathcal{R}$ and is irreducible in dimension $\neq 4$. That is, a riemannian manifold of dimension ≥ 4 is conformally flat if and only if $W = 0$; but it is only in dimension four that $W = W_+ + W_-$. ($W = 0$ for any three dimensional manifold, but not every such manifold is conformally flat).

An oriented manifold (M, g) of dimension four is then called *self-dual* if $W_- = 0$ and *anti-self-dual* if $W_+ = 0$. Reversing the orientation of M takes W_+ to W_- (since it takes Λ^2_+ to Λ^2_-), so that one often uses the term *half-conformally flat* to indicate a four dimensional manifold where either $W_+ = 0$, or $W_- = 0$.

When M is compact, this symmetry can also be described by the following variational principle. In the theory of conformal gravitational Instantons, one looks for metrics on M minimizing the conformally invariant functional

$$A = \int_M |W_+|^2 + |W_-|^2$$

but, by Chern-Weil theory, the signature τ of M can be expressed by:

$$\tau = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2$$

therefore $A \geq 12\pi^2|\tau|$ with equality if and only if M is half conformally flat.

This gives then an analogy with the theory of Yang-Mills Instantons and in fact, in this context, a riemannian metric is Einstein if and only if the bundle of self-dual 2-forms has a self-dual curvature.

1.2 The Penrose construction

Another important way to characterize half-conformally flat four-manifolds is by means of (the riemannian version of) the twistor theory of Roger Penrose. This was introduced in [2].

Let (M,g) be an oriented four dimensional riemannian manifold. Its twistor space Z, a six dimensional manifold, is a fiber-bundle over M, with fiber S^2. It can be defined in the following equivalent ways:

1.2.1 *1. Z is the bundle of self-dual 2-forms of unit length:* $Z = S(\Lambda^2_+)$

2. Z is the bundle of orthogonal complex structures inducing the positive orientation: $Z = \{J \in O(TM) | J^2 = -1, J > 0)\}$

3. Z is the projectivized positive spin bundle: $Z = \mathbb{P}(\mathbb{S}_+)$

To see this, recall first that an almost complex structure J on a vector space always induces a natural orientation by considering bases of the form $\{e_1, Je_1, e_2, Je_2\}$; when this orientation agrees with the given one on M, we say that $J > 0$. Also, J is *orthogonal*, or equivalently g is *hermitian*, when $g(JX, JY) = g(X, Y)$. The equivalence of the first two definitions is then given by associating to J its fundamental 2-form $\omega(X, JY) := g(X, Y)$, [17].

Concerning the third definition, we have to show how a projective spinor defines an almost complex structure. At any point $p \in M$ let $\phi \in \mathbb{S}_+$ be fixed; Clifford multiplication gives a real isomorphism $\Lambda_1 \to \mathbb{S}_-$ by taking $\alpha \mapsto \alpha\phi$. This defines a complex structure on Λ_1 by identifying it with the complex vector space $\mathbb{S}_-$. Of course, multiplying ϕ by a scalar does not change this complex structure, and all complex structures can be defined in this way, [2, p429].

Now we want to describe a natural almost complex structure on the twistor space Z by using the descriptions above.

Let $t : Z \to M$ be the twistor map. At each point $z \in Z$, we split the tangent space T_zZ into the vertical space $V = ker t_*$, and the horizontal space E given by the Levi-Civita connection: namely $Z = \mathbb{P}(\mathbb{S}_+)$. As we can write $\mathbb{C}TM = \mathbb{S}_+ \bigotimes \mathbb{S}_-$, the covariant derivative on $\mathbb{S}_+$ and $\mathbb{S}_-$ is given by $\nabla_X(\phi \otimes \psi) = (\nabla_X \phi) \otimes \psi + \phi \otimes (\nabla_X \psi)$.

Now that at each point $z \in Z$ we have $TZ = V \bigoplus E$, we define a complex structure $J = J_1 \bigoplus J_2$ by letting J_1 be the natural complex structure of the metric 2-sphere $t^{-1}(t(z))$ to which V is tangent; while E, being isomorphic to $T_{t(z)}M$, is given the tautological almost complex structure J_2, defined by z itself.

The first thing to notice is that the six dimensional almost complex manifold (Z, J) is constructed from the four dimensional manifold (M, g), but its isomorphism class only depends on the conformal class of the metric g.

1.2.2 *[2] If g' is locally conformal to g with twistor space (Z', J'), then Z and Z' are isomorphic almost complex manifolds.*

It is then natural to ask when is J integrable, and to expect the answer to be in terms of the conformal class of the metric g on M. In fact the cornerstone of the subject is the following

Theorem 1.2.3 (Atiyah-Hitchin-Singer)

$$J \text{ is integrable if and only if } W_+ = 0$$

Remark 1.2.4 If $W_- = 0$ instead, then an analogous almost complex structure on $\mathbb{P}(\mathbb{S}_-)$ is integrable. In other words a half-conformally flat four-manifold has an integrable twistor space.

Remark 1.2.5 Since $J = J_1 \oplus J_2$, an easy consequence is that the fibers of the twistor projection are all complex submanifolds of Z, isomorphic to $\mathbb{CP}_1$.

1.3 The twistor space as a complex manifold

From now on, we suppose that our oriented four-manifold M has a half-conformally-flat metric (i.e. either $W_- = 0$ or $W_+ = 0$).

Then, if M' denotes the manifold M with opposite orientation, the twistor space Z of either M or M' is a three-dimensional complex manifold. As already noted, the fibers of the twistor map are holomorphic rational curves in Z, which are called the "real" twistor lines. As an immediate consequence:

1.3.1

1. *Z admits an antiholomorphic bijection $\sigma : Z \to Z$, called the real structure, and defined to be the antipodal map on each S^2-fiber, so that $\sigma^2 = id$. (σ is antiholomorphic because $\sigma J_1 = -J_1$ and $\sigma J_2 = -J_2$).*

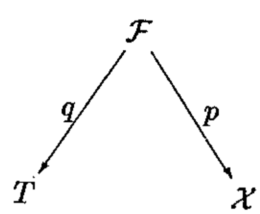
2. For any twistor line $l \subset Z$; $l \cong \mathbb{CP}_1$, and its normal bundle $\nu \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, [2]. Where by $\mathcal{O}(n)$ we denote the (sheaf of sections of the) holomorphic line bundle of Chern class $+n$ over $\mathbb{CP}_k$.

The importance of these two properties lies in the following reverse construction:

1.3.2 *If T is a complex manifold with a "real structure σ" as above, and foliated by holomorphically embedded $\mathbb{CP}_1$'s which are invariant under σ, and with normal bundle $\nu_{\mathbb{CP}_{1/T}} = \mathcal{O}(1) \oplus \mathcal{O}(1)$, then T is a twistor space.*

The reason is that by a theorem of Kodaira [13], $H^1(\mathbb{CP}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 0$ implies that these holomorphic rational curves are part of a complex analytic family $\mathcal{F}$ of submanifolds of T. $\mathcal{F}$ is parametrized by a smooth complex manifold $\mathcal{X}$ whose tangent space $T_x\mathcal{X}$ is naturally isomorphic to $H^0(l_x, \nu)$, where $l_x \cong \mathbb{CP}_1$ is the corresponding curve in Z.

Now, $h^0(l_x, \nu) = dim_{\mathbb{C}} H^0(\mathbb{CP}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 4$ simply says that $\mathcal{X}$ has complex dimension equal to four. So that one has a double fibration



where for each $x \in X$ $q(p^{-1}(x))$ is a twistor line in T. Furthermore, since $H^0(\mathbb{CP}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong H^0(\mathbb{CP}_1, \mathcal{O} \oplus \mathcal{O}) \otimes H^0(\mathbb{CP}_1, \mathcal{O}(1))$, there is a naturally defined complex conformal structure on $H^0(\mathbb{CP}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$, given by defining the null cone to be the set of simple tensor products. This gives $\mathcal{X}$ a natural complex conformal structure. Then the map σ acts on $\mathcal{X}$ as a conjugation, and its fixed-point set $\mathcal{X}^\sigma = X$ is a real four-manifold with conformal structure, which can be shown to be half-conformally flat, [16].

Examples

1.3.3 *The standard metric on S^4 is conformally flat so that its twistor space Z is integrable; in fact $Z = \mathbb{CP}_3$. The real structure σ is given by*

$$[Z_0, Z_1, Z_2, Z_3] \mapsto [-\overline{Z_1}, \overline{Z_0}, -\overline{Z_3}, \overline{Z_2}]$$

while the twistor map t is

$$[Z_0, Z_1, Z_2, Z_3] \mapsto [Z_0 + Z_1 j, Z_2 + Z_3 j].$$

Here we have used homogeneous coordinates and the identification of S^4 with $\mathbb{HP}_1$, [1]. A fiber of t is then a linearly embedded $\mathbb{CP}_1$ which of course has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

1.3.4 *The twistor space of $\mathbb{CP}_2$ is also integrable because $W_- = 0$, and it is in fact the flag manifold $F_{1,2}$, whose points are pairs $(p, l) \in \mathbb{CP}_2 \times \mathbb{CP}_2{}^*$ such that p is a point of $\mathbb{CP}_2$ contained in the line $l \subset \mathbb{CP}_2$. The twistor map is then, $t : (p, l) \mapsto \overline{p}$, where $\overline{p}$ denotes the unique complex line through the origin in $\mathbb{C}^3$, perpendicular to p and contained in l.*

A theorem of Hitchin [12] then states that these are the only two compact twistor spaces which are Kähler (in fact projective algebraic). It is then interesting to investigate "how far is a twistor space from being algebraic", for example, by looking at its algebraic dimension, [20,21, next ch.].

1.4 The Penrose correspondence

We would now like to briefly describe some aspects of the correspondence between holomorphic objects on the twistor space Z and conformal properties of (M, g). This general philosophy, which is fundamental in twistor theory, goes under the name of Penrose correspondence.

To start, we notice that by the conformal invariance of the twistor construction, each biholomorphism f of Z corresponds to a conformal isometry of M. It is a one to one correspondence, because if f induces the identity on M, f has to fix each real twistor line; but from the definition of J we then see that f is holomorphic only if it is the identity.

Similarly,

1.4.1 *Holomorphic vector fields on Z, exactly correspond to conformal killing vector fields on M.*

Recall that a vector field is said to be conformal Killing if its flow consists of conformal isometries.

Next, we want to show how complex deformations of Z give rise to half-conformally flat deformations of (M, g). Let $d : \mathcal{G} \to \Delta$ be a complex deformation of Z, over a small neighborhood of the origin in $\mathbb{C}$, with $Z = d_{-1}(0)$. Let $Z_t = d_{-1}(t)$. As $H^1(\mathbb{CP}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = 0$, every twistor line in Z is "stable under deformations" [14], and so Z_t contains a family of $\mathbb{CP}_1$'s, which also have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

In fact, $H^1(\mathbb{CP}_1, End(\mathcal{O}(1) \oplus \mathcal{O}(1))) = 0$ says that $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is also stable. Finally there is a conjugation τ on Δ, with fixed point set D of real dimension 1, such that Z_t has a "real structure" for each $t \in D$, [8, Lemma 2.11]. Therefore, for each $t \in D$, Z_t is a twistor space, corresponding to a half-conformally flat deformation of g. To summarize:

1.4.2 *Any half-conformally flat small deformation of (M, g) is the "real part" of a small complex deformation of Z.*

When M is compact, the powerful theory of Kodaira and Spencer can then be used to study half-conformally flat deformations.

Consider now a real vector bundle E over an (anti-)self-dual manifold M. We can construct a complex vector bundle F on Z, by simply complexifying E and then pulling it back to Z. A connection on E is said to be (anti-)self-dual if its curvature is an (anti-)self-dual 2-form on M. This sets up an important correspondence:

Theorem 1.4.3 *[2] The above gives a bijection between bundles E on M with (anti-)self-dual connection, and holomorphic bundles F on Z which are trivial along the fiber.*

When one wants to consider the spin bundles of M, it is important to fix an orientation; in what follows we will then assume M to be anti-self-dual.

Suppose for a moment that M is spin, so that $Z = \mathbb{P}(\mathbb{S}_+)$. Then as $\mathbb{S}_+$ is a rank-two complex vector bundle, the complex manifold $\mathbb{P}(\mathbb{S}_+)$ admits a "tautological" holomorphic line bundle $\tau = \{(z, v) \in Z \times \mathbb{S}_+ | v \in z\}$. The restriction of τ to each real twistor line is of course $\cong \mathcal{O}_{\mathbb{CP}_1}(-1)$ and τ is sometimes denoted by H^{-1}, or $\mathcal{O}_Z(-1)$ or simply $\mathcal{O}(-1)$. The holomorphic line bundles $\mathcal{O}_Z(m)$ are always globally defined if m is even; the topological obstruction to the global existence of $\mathcal{O}_Z(-1)$ is $w_2(M) \in H^2(M, \mathbb{Z}_2)$, so that M has to be spin in this case. An important point is that

1.4.4 *[2] The bundles $\mathcal{O}_Z(m)$ are completely determined by the complex structure of Z: in fact $\mathcal{O}_Z(-2) = K^{\frac{1}{2}}$.*

In particular, the canonical line bundle of Z always admits a preferred holomorphic square root, and Z is always spin. Also, a fourth root of K exists if and only if M is spin.

Then, as an important instance of the Penrose correspondence, we mention the following relation, due to Hitchin [11], between the holomorphic cohomology of $\mathcal{O}_Z(m)$ and solutions to certain partial differential equations on M.

To state this, recall that $\mathbb{S}_+^m$ and $\mathbb{S}_-^m$ indicate the m-th symmetric powers of the spin bundles on M. Of course, m is even when M is not spin. One then considers covariant differentiation

$$\nabla : \Gamma(\mathbb{S}_+^m) \to \Gamma(\mathbb{S}_+^m \otimes \mathbb{C}T^*M) = \Gamma(\mathbb{S}_+^m \otimes \mathbb{S}_+ \otimes \mathbb{S}_-)$$

which, together with the orthogonal decomposition [2]

$$\mathbb{S}_+^m \otimes \mathbb{S}_+ \otimes \mathbb{S}_- = (\mathbb{S}_+^{m-1} \otimes \mathbb{S}_-) \otimes (\mathbb{S}_+^{m+1} \otimes \mathbb{S}_-)$$

gives, by projection, the Dirac operator

$$D_m : \Gamma(\mathbb{S}_+^m) \to \Gamma(\mathbb{S}_+^{m-1} \otimes \mathbb{S}_-)$$

and the twistor operator

$$\overline{D}_m : \Gamma(\mathbb{S}_+^m) \to \Gamma(\mathbb{S}_+^{m+1} \otimes \mathbb{S}_-)$$

Is important to notice [2], that

Remark 1.4.5 The operator D_2 above is just the restriction of exterior differentiation d to the self-dual 2-forms.

Theorem 1.4.6 (Hitchin) *[11] If M is compact, for any $m \geq 0$*

$$Ker\overline{D}_m \cong H^0(Z, \mathcal{O}(m)) \quad and \quad Ker D_m \cong H^1(Z, \mathcal{O}(-m-2)$$

Of course the same holds when M is self-dual, after interchanging the roles of the two spin bundles.

Now it is important to notice that on a half-conformally flat manifold, the Dirac and twistor operators both have a Weitzenböck decomposition which only involves the scalar curvature R. In fact there are universal positive constants a and b such that, [11]

$$D_m^* D_m = \nabla^* \nabla + aR \quad \textit{and} \quad \overline{D}_m^* \overline{D}_m = \nabla^* \nabla - bR$$

Corollary 1.4.7 *Let M be compact, with twistor space Z, then*

1. $H^0(Z, \mathcal{O}(m)) = 0$ *for all* $m \geq 0$, *if* $R < O$
2. $H^1(Z, \mathcal{O}(m)) = 0$ *for all* $m \leq -2$, *if* $R > 0$
3. $H^0(Z, \mathcal{O}(m)) \cong H^1(Z, \mathcal{O}(-m-2)) \cong$ *space of parallel sections of* $\mathbb{S}_-^m$ *for all* $m \geq 0$, *if* $R = 0$

This corollary is then particularly useful because [22] one can always choose a metric of constant scalar curvature in the given conformal class.

To conclude we prove a useful equality relating the Betti numbers of M with the Hodge numbers of Z.

Notation For any compact manifold X, we will denote by $b^i(X)$ its i-th Betti number. If X is complex, for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{S}$ on X, we set $h^i(X, \mathcal{S})$ to be the complex dimension of $H^i(X, \mathcal{S})$. By Θ_X or simply Θ we will denote the sheaf of holomorphic vector fields on X. While if $Y \subset X$ is a complex subvariety, $\Theta_{X,Y}$ will denote the subsheaf of vectors which are tangent to Y, along Y.

Proposition 1.4.8 *Let M be half-conformally-flat and compact. Then,*

$$h^1(Z,\mathcal{O}) = b_1(M) \quad \text{and} \quad h^2(Z,\mathcal{O}) = \begin{cases} b^2_+(M) & \text{if } M \text{ is anti-self-dual} \\ b^2_-(M) & \text{if } M \text{ is self-dual} \end{cases}$$

Proof: Let us start by proving the second equality. By Serre duality $h^2(Z,\mathcal{O}) = h^1(Z,K)$, but when M is anti-self-dual, by 1.4.6, $h^1(Z,K) = dim\ Ker(D_2) = b^2_-(M)$, 1.4.

To prove the first part of the proposition, notice that 1.4.3 gives a bijection between line bundles E on M with flat connection, and holomorphic line bundles F on Z, which are trivial along the fibers and horizontally flat; i.e. $c_1(F) = 0$, 2.1. Now, for any compact complex manifold Z the following sequence is exact

$$0 \longrightarrow H^1(Z,\mathbb{Z}) \xrightarrow{a} H^1(Z,\mathcal{O}) \xrightarrow{b} H^1(Z,\mathcal{O}^*) \xrightarrow{c_1} H^2(Z,\mathbb{Z})$$

It follows that $Ker\ c_1 = Im\ b \cong H^1(Z,\mathcal{O})\ /\ H^1(Z,\mathbb{Z})$. Therefore,
$h^1(Z,\mathcal{O}) = dim_{\mathbb{C}}\ (H^1(Z,\mathcal{O})\ /\ H^1(Z,\mathbb{Z})) = dim_{\mathbb{C}}\ Ker\ c_1$
$= dim_{\mathbb{R}}$ {flat line bundles on M} $= dim_{\mathbb{R}}\ Hom(\pi_1(M),\mathbb{R}) = b^1(M)$ □

Chapter 2

Anti-self-dual hermitian surfaces

In what follows M will denote a compact real four-dimensional manifold, with a hermitian metric h satisfying the anti-self-dual equation, $W_+ = 0$, for its conformally invariant Weyl tensor W. According to the work of Boyer [B2], such a complex surface belongs to one of the following two types:

1. *Kähler type*: $b_1(M)$ is even, and in the same conformal class of h there is a Kähler metric of zero scalar curvature.

2. *non-Kähler type*: $b_1(M)$ is odd, and h is locally conformally Kähler ; h is also conformal to a hermitian metric of non-negative scalar curvature which is strictly positive almost everywhere.

All known examples of surfaces of Kähler type are the following:

- Flat tori and $K3$ surfaces with a Yau metric. These are the hyperkähler surfaces and are the universal coverings of:

- The other Ricci-flat Kähler surfaces, i.e. the hyperelliptic and the Enriques surfaces.

- $S_g \times \mathbb{CP}_1$, where S_g is a compact Riemann surface of genus $g \geq 2$ with a metric of constant scalar curvature -1, and $\mathbb{CP}_1$ is the Riemann sphere with constant curvature $+1$. Or, more generally, any flat S^2-bundle over S_g, $g \geq 2$.

The reason why these are hermitian anti-self-dual manifolds is that they have a Kähler metric of zero scalar curvature.

For surfaces of non-Kähler type there is just one known example:

- The Hopf surfaces with their standard conformally flat metric. As a complex manifold a Hopf surface M is defined to be any quotient $(\mathbb{C}^2 \setminus 0)/\Gamma$, where $\Gamma \subset GL(2, \mathbb{C})$ is a discrete subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$. When $\Gamma \subset U(2) \times \mathbb{R}^+$, M has a conformally flat metric and therefore is an anti-self-dual complex surface.

  Since $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}_n$, $b_1(M) = 1$.

Notice that the complex projective plane $\mathbb{CP}_2$ with its standard orientation and metric is self-dual and Kähler , while the same manifold with orientation reversed, $\overline{\mathbb{CP}}_2$, does not even admit an almost complex structure;

otherwise c_1^2 would be equal to $2\chi + 3\tau = 3$, which implies that the first Chern class c_1 cannot be represented by an integral 2-form.

The tecniques used by Boyer are those of differential and algebraic geometry, applied to M. We will instead also consider the twistor space Z of M. Since (M, h) is anti-self-dual, Z is a complex three-dimensional manifold whose complex structure only depends on the conformal class of h. In what follows we will discuss some holomorphic properties of Z and show some relevant differences between the Kähler and non-Kähler types.

2.1 The twistor space

Let $t : Z \to M$ denote the twistor fibration and suppose M is hermitian and anti-self-dual. Two things are clear from the definition 1.2.1 of the almost complex structure of Z: first, t is never a holomorphic map; second, the complex structure J of M defines a cross section $J : M \to Z$, whose image, denoted by Σ, is a complex hypersurface of Z biholomorphic to M. Similarly $-J : M \to Z$ defines a hypersurface $\overline{\Sigma}$. The "real structure" σ of Z switches the two hypersurfaces identifying one with the other in an antiholomorphic fashion. If X denotes the divisor $\Sigma + \overline{\Sigma}$ in Z, we can consider the holomorphic line bundle $[X]$; since $\sigma(X) = \sigma(\Sigma + \overline{\Sigma}) = \overline{\Sigma} + \Sigma = X$, $[X]$ is called a "real" bundle. In what follows $c_1(E)$ will denote the *first* Chern class of the complex vector bundle E and K_N the canonical bundle of the complex manifold N.

For any compact hermitian anti-self-dual surface, the following holds.

Remark 2.1.1 [P3]

$$c_1([X]) = c_1(K_Z^{-\frac{1}{2}})$$

Proof: By the Leray-Hirsch theorem [H2] we can write $c_1([X]) = a \cdot l + b \cdot k$ where l is the cohomology class of a twistor line, $k \in t^*(H^2(M, \mathbb{R}))$ and $a, b \in \mathbb{R}$. Then [H2], $c_1(K_Z) = -4l$ and we only have to show that $c_1([X]) = 2l$. Now the real structure $\sigma : Z \to Z$ induces an automorphism σ^* of $H^*(Z, \mathbb{R})$ such that $\sigma^*(c_1([X])) = a \cdot l - b \cdot k$; but $[X]$ "real" implies $\sigma^*(c_1([X])) = c_1([X])$ and therefore $b = 0$. Counting intersections with a twistor line, we get $a = 2$. □

Now if $H^1(Z, \mathcal{O}) = 0$, the Chern class map $c_1 : H^1(Z, \mathcal{O}^*) \to H^2(Z, \mathbb{Z})$ is injective and the above implies $[X] \cong K_Z^{-\frac{1}{2}}$. Since for every half-conformally flat manifold M, $h^1(Z, \mathcal{O}) = b_1(M)$, M has to be of Kähler type in this case.

But in fact this holds for any surface of Kähler type:

Theorem 2.1.2 *If M is of Kähler type, compact or not, then*

$$[X] \cong K_Z^{-\frac{1}{2}}$$

Proof: The proof is in two steps. We first define a holomorphic section

$\check{\omega} \in H^0(Z, K^{-\frac{1}{2}})$ by using the Kähler form ω of M; then we show that $X = \{\check{\omega} = 0\}$.

In the course of the proof we will often use the following: $Z = \mathbb{P}(\mathbb{S}_+)$; the symplectic form defines a linear isomorphism $\epsilon : \mathbb{S}_+ \to \mathbb{S}_+^*$ and the hermitian form an antilinear isomorphism $h : \mathbb{S}_+ \to \overline{\mathbb{S}}_+$ so that if $\eta \in \mathbb{S}_+$, $\overline{\eta}$ will denote its image and we will write $\eta \otimes \overline{\eta} \in \mathbb{S}_+^2$.

Step 1 Recall that $\Lambda_+^2(M) = \mathbb{S}_+^2$, then the Kähler form ω of M is a section of $\mathbb{S}_+^2$. Now according to [11, sec. 2] any section $\psi \in \mathbb{S}_+^2$ tautologically defines a complex valued function on $\mathbb{S}_+ \setminus 0$ which is a homogeneous polynomial of degree 2 on each fiber; this in turns gives a section $\check{\psi} \in \Gamma(Z, \mathcal{O}(2)) = \Gamma(Z, K^{-\frac{1}{2}})$. And furthermore $\check{\psi}$ is a holomorphic section, i.e. $\check{\psi} \in H^0(Z, \mathcal{O}(K^{-\frac{1}{2}}))$, if and only if ψ satisfies the twistor equation $\overline{D}_2 \psi = 0$. It is clear from the definition of the operators D_m and $\overline{D}_m$ that every parallel section of $\mathbb{S}_+^m$ is a solution to both the Dirac and twistor equations (in fact, by the Weitzenböck formulas, these are the only solutions when M is compact and $R = 0$). Therefore since ω is parallel, $\check{\omega} \in H^0(Z, K^{-\frac{1}{2}})$ is holomorphic.

Step 2 Since M is hermitian we have two sections φ and $\overline{\varphi} : M \to Z$ representing the almost complex structures J and $\overline{J}$. Let $\omega \in \Lambda_+^2(M) = \mathbb{S}_+^2$ be the Kähler form. According to [AHS sec.1], at each point $p \in M$, $\omega = \phi \otimes \overline{\phi}$ where $\phi \in \mathbb{S}_+$ and $\overline{\phi} \in \mathbb{S}_+$ represents φ and $\overline{\varphi}$ respectively. Now let $\alpha \in Z = \mathbb{P}(\mathbb{S}_+)$ be a twistor at p. By using the isomorphism $\epsilon : \mathbb{S}_+ \to \mathbb{S}_+^*$ it makes sense to solve the equation $\phi(\alpha) = 0$. Since ϵ is given by the symplectic form and $\mathbb{S}_+$

has complex dimension 2, the only solution is $\alpha = \varphi$. Similarly for $\overline{\phi}$ and we have shown that $\check{\omega}(\alpha) = 0$ if and only if $\alpha = \varphi$ or $\alpha = \overline{\varphi}$ that is $X = \{\check{\omega} = 0\}$.

□

Using the theorem we can then recover some results of Poon in a more straightforward manner. In what follows we also refer to [P3].

Corollary 2.1.3 *When M is of Kähler type and compact, the normal bundle of X in Z is isomorphic to the anticanonical bundle:*

$$\nu_{X/Z} \cong K_X^{-1}\ , \quad \textit{similarly} \quad \nu_{\Sigma/Z} \cong K_\Sigma^{-1} \quad \textit{and} \quad \nu_{\overline{\Sigma}/Z} \cong K_{\overline{\Sigma}}^{-1}.$$

<u>Proof</u>: The adjunction formulas [GH] state that $\nu_{X/Z} \cong [X]_{|X}$ and $K_X \cong (K_Z[X])_{|X}$ therefore $\nu_{X/Z} \cong K_{Z|X}^{-\frac{1}{2}}$ and $K_X \cong (K_Z \otimes K_Z^{-\frac{1}{2}})_{|X} \cong K_{Z|X}^{\frac{1}{2}}$ as wanted. The rest clearly follows from $X = \Sigma \amalg \overline{\Sigma}$ □

The above theorem says that the line bundle $K^{-\frac{1}{2}}$ has global holomorphic sections and this easily implies that $K^{-\frac{m}{2}}$ has global holomorphic sections for each $m \geq 0$. In fact we next show that these are the only line bundles, with Chern class l, to have global holomorphic sections:

Corollary 2.1.4 *Let M be compact and of Kähler type. If $L \to Z$ is any holomorphic line bundle such that $c_1(L) = c_1(K^{-\frac{m}{2}})$, for some $m \geq 0$, then*

$$H^0(Z, L) \neq 0 \Longleftrightarrow L \cong K^{-\frac{m}{2}}$$

Proof: [P3] we can write $L = FK^{-\frac{m}{2}}$ where $c_1(F) = 0$. We consider three different cases.

Case 1 M is not Ricci flat: then $c_1(F_{|X}K_X^{-m}) = c_1(K_X^{-m}) \neq 0$, therefore by an important theorem of Yau [25], [1] , $H^0(X, F_{|X}K_X^{-m}) = 0$ for any m. By the long exact sequence induced by

$$(1.m) \qquad 0 \to FK^{-\frac{m-1}{2}} \to FK^{-\frac{m}{2}} \to F_{|X}K_X^{-m} \to 0$$

we have $H^0(Z, FK^{-\frac{m-1}{2}}) \cong H^0(Z, FK^{-\frac{m}{2}})$ for any $m \neq 0$ and so is enough to show that $H^0(Z, FK^{-\frac{1}{2}}) = 0$; in fact

$$(1.1) \qquad 0 \to F \to FK^{-\frac{1}{2}} \to F_{|X}K_X^{-1} \to 0$$

shows that $H^0(Z, FK^{-\frac{1}{2}}) \cong H^0(Z, F) = 0$ by [P1, lemma 2.1].

Case 2 M is hyperkähler , suppose $0 \neq H^0(X, F_{|X}K_X^{-m}) = H^0(X, F_{|X})$ then by Yau's theorem $L_{|X}$ is trivial, as M is hyperkähler its twistor space fibers holomorphically over $\mathbb{CP}_1$; the fibers being isometric to M with different complex structures. Therefore L has to be trivial along each fiber and so $L = \pi^*\mathcal{O}_{\mathbb{CP}_1}(k)$, but $c_1(L) = 0 \Rightarrow k = 0$ and $L = \mathcal{O}_Z$. Also in this case $K^{-\frac{m}{2}} = \pi^*\mathcal{O}_{\mathbb{CP}_1}(2m)$, so that $h^0(Z, FK^{-\frac{m}{2}}) = 2m + 1$.

Case 3 the only other possibility is that M is finitely covered by a hyperkähler surface. Let $p' : \tilde{M} \to M$ be this covering, and $p : Z' \to Z$ be the corresponding covering of their twistor spaces. Let $d \neq 1$ be the degree of the coverings. Suppose now that $FK^{-\frac{m}{2}}$ has a global holomorphic section

[1] Warning: proposition 4 in [25] is false, counterexample: $\mathbb{CP}_1 \times S_g$. See also 3.1.14

with zero divisor $Y \subset Z$. Then $p^*(FK^{-\frac{m}{2}})$ also has a section and let $Y' \subset Z'$ be its zero divisor. By the previous case $p^*F = \mathcal{O}_{Z'}$ so that $L^{\otimes d} = \mathcal{O}_Z$ and Y' is the real manifold $\tilde{M}$ with some complex structure compatible with the metric . Since $p_{|Y'} : Y' \to Y$ is a covering map, Y is just the manifold M with a possibly different complex structure compatible with the metric and orientation, this gives M a Kähler structure, but since $b_2^+(M) = 1$, this must be the original Kähler structure and Y is equal to X, so that $L = K^{-\frac{1}{2}}$. □

Now for any holomorphic line bundle F over a compact complex manifold N one can set [24]

Definition 2.1.5 The $F-$ *dimension* $k(N,F)$ of N is $-\infty$ if $h^0(N,F^m) = 0$ for any $m > 0$, or the non-negative integer satisfying $am^{k(N,F)} \leq h^0(N,F^m) \leq bm^{k(N,F)}$ for m sufficiently large, and some constants a, b.

So that $k(N,F)$ gives the rate of grow of $h^0(N,F^m)$ as $m \to +\infty$.

In particular, $k(N,K)$ is called the Kodaira dimension of N.

Then one can also show [U] that for any holomorphic line bundle F, $k(N,F) \leq a(N) \leq dim(N)$ where $a(N)$ denotes the algebraic dimension of N. This is defined to be the degree of transcendency of the field of meromorphic functions on N, denoted by $\mathcal{M}(N)$, over $\mathbb{C}$. Clearly $a(N) = dim(N)$ when N is algebraic. Furthermore [U], there exist a projective manifold V

and a holomorphic map $g : N \to V$ inducing an isomorphism between $\mathcal{M}(N)$ and $\mathcal{M}(V)$. This also implies that $a(N) = k(N, F)$, where $F = g^*(H)$, H being a very ample line bundle on V.

So that, in some sense, $a(N)$ measures how far is N from being algebraic.

When one considers a twistor space $t : Z \to M$ as a complex manifold, it is interesting to investigate its algebraic dimension $a(Z)$ then, because of the following, [12]:

Theorem 2.1.6 (Hitchin) *If Z is a compact, Kähler (in fact algebraic) twistor space then M is either S^4 or $\mathbb{CP}_2$, with its standard conformal structure.*

To this respect Poon has also found some very interesting relations between $a(Z)$ and the geometry of M, [P2, P3]. Now let Z be a twistor space and let $a(Z) = k(Z, S)$, we can construct a real bundle $S\overline{S}$ and easily show that $k(Z, S) = k(Z, S\overline{S})$. As a consequence

Proposition 2.1.7 *[P2,3] [DF] For any compact twistor space Z, $a(Z) = k(Z, F)$ for some* "real" *holomorphic line bundle F.*

Theorem 2.1.8 (Poon) *If M is of Kähler type but not Ricci-flat, then $a(Z) = 0$. If M is Ricci-flat, $a(Z) = 1$.*

<u>Proof</u>: By the last corollary, $a(Z) = k(Z, K^{-\frac{1}{2}})$ in this case. If M is not Ricci-flat, the proof of the same corollary shows that $h^0(Z, K^{-\frac{m}{2}})$ is constant in m, so that $a(Z) = 0$.

If M is Ricci-flat instead, its universal covering $\tilde{M}$ is hyperkähler and

$h^0(Z, K^{-\frac{m}{2}}) = h^0(\tilde{Z}, K^{-\frac{md}{2}}) = h^0(\mathbb{CP}_1, \mathcal{O}(md)) = md + 1$

so that $k(Z, K^{-\frac{1}{2}}) = k(\tilde{Z}, K^{-\frac{1}{2}}) = 1$ □

The situation for surfaces of non Kähler type is different and in the following chapter we will give the first example of a twistor space with algebraic dimension equal to two. It is the twistor space of a Hopf surface. To explain why this can happen we have the following general

Theorem 2.1.9 *If M is a compact anti-self-dual complex surface of non Kähler type, then*

$$[X] \cong K_Z^{-\frac{1}{2}} \otimes F$$

where F is a non trivial holomorphic line bundle of zero Chern class.

Proof: The proof is by contradiction. By 2.1, suppose $[X] \cong K_Z^{-\frac{1}{2}}$ and consider the exact sequence of sheaves on Z, given by restriction

$$0 \to \mathcal{O}_Z(K^{\frac{1}{2}}) \to \mathcal{O}_Z \to \mathcal{O}_X \to 0$$

Since we can assume that M has positive scalar curvature almost everywhere, $H^1(Z, K^{\frac{1}{2}}) = 0$, by a vanishing theorem of Hitchin [11]. On the other hand $H^0(Z, K^{\frac{1}{2}}) = 0$ for any twistor space, by a different vanishing theorem [H2]. The resulting long exact sequence would then imply that $H^0(Z, \mathcal{O}) = H^0(X, \mathcal{O})$ which is impossible because X has two connected components. □

2.2 Holomorphic vector fields

In this section we assume that M is compact and of Kähler type and analyze the close relation between the Lie algebras of holomorphic vector fields of M and Z, which we denote by $H^0(M,\Theta)$ and $H^0(Z,\Theta)$.

We will prove:

Theorem 2.2.1 *If M is Ricci-flat*

$$H^0(Z,\Theta) \cong H^0(M,\Theta)^{\oplus 2}$$

which is also isomorphic to the complexification of the Lie algebra of real *parallel vector fields on M; so that*

$$h^0(Z,\Theta) = b_1(M) = 2h^0(M,\Theta)$$

Theorem 2.2.2 *If M is not Ricci-flat*

$$H^0(Z,\Theta) \cong H^0(M,\Theta)$$

To explain this, recall that in the general case, by the Ward correspondence, $H^0(Z,\Theta)$ is the complexification of the Lie algebra of conformal Killing vector fields on M. This in turn is closely related to $H^0(M,\Theta)$ when M is Kähler .

To prove the above theorems we will use the following : [B]

Theorem 2.2.3 (Bochner) *On a compact riemannian manifold (N, g) with $Ric \leq 0$, every Killing vector field is parallel.*

Similarly if g is Kähler , then every holomorphic vector field is parallel.

Theorem 2.2.4 (Lichnerowicz) *On a compact Kähler manifold of constant scalar curvature*

$$H^0(M, \Theta) \cong \mathbf{a} \oplus \mathbf{h}$$

where $\mathbf{a}$ *is the abelian Lie algebra of all parallel holomorphic vector fields and* $\mathbf{h}$ *is the complexification of a Lie algebra consisting of Killing vector fields.*

Recall that on a pseudo-riemannian manifold a (conformal) Killing vector field is one which generates a flow of (conformal) isometries.

If $\mathcal{L}_V$ denotes the Lie derivative with respect to V, then $\mathcal{L}_V g = fg$ for some function f, if and only if V is conformal Killing.

While $\mathcal{L}_V g = 0$ if and only if V is Killing. So that every parallel vector field is also Killing.

Finally [B], V is real holomorphic if and only if $\mathcal{L}_V J = 0$, i.e. if it is the real part of a (complex) holomorphic vector field.

By ∇ we will denote the covariant derivative of the Levi-Civita connection.

Lemma 2.2.5 *If M is a compact Kähler surface every conformal vector field is real holomorphic and in fact Killing.*

<u>Proof</u>: Suppose $\mathcal{L}_V g = fg$ for some function f; we start by showing that $\mathcal{L}_V \omega = 0$ where ω denotes the Kähler form. In fact let φ_t be the flow of V. For each t, φ_t is a conformal isometry homotopic to the identity. Since ω is a self-dual closed 2-form, it is also harmonic, and it is easy to check that the Hodge-star operator $* : \bigwedge^n \to \bigwedge^n$, on a manifold of real dimension $2n$, is invariant under a conformal rescaling of the metric; so that $\varphi_t^* \omega$ is again harmonic. But $[\varphi_t^* \omega] = [\omega] \in H^2_{dR}(M)$ and so by Hodge theory, $\varphi_t^* \omega = \omega$, i.e. $\mathcal{L}_V \omega = 0$.

Now the complex structure $J = g^{-1} \circ \omega$ as an endomorphism of the tangent bundle, therefore

$$\mathcal{L}_V J = (\mathcal{L}_V g^{-1}) \circ \omega + g^{-1} \circ (\mathcal{L}_V \omega) = f g^{-1} \circ \omega = fJ$$

on the other hand $J^2 = -id$ implies that

$$0 = \mathcal{L}_V(-id) = \mathcal{L}_V J^2 = J(\mathcal{L}_V J) + (\mathcal{L}_V J)J = -2f$$

i.e. $f = 0$, $\mathcal{L}_V g = 0$ and $\mathcal{L}_V J = 0$ □

It has come to my attention that a more general theorem of Lichnerowicz appears in [B].

Lemma 2.2.6 *[B] If M is a compact Kähler manifold and V and JV are both Killing vector fields, they must be parallel.*

<u>Proof:</u> It is straightforward to check that, on any riemannian manifold, a vector field W is Killing if and only if

$$0 = (\mathcal{L}_W g)(A, B) = g(\nabla_A W, B) + g(A, \nabla_B W)$$

And on Kähler manifold, W is holomorphic if and only if

$$0 = (\mathcal{L}_W J)A = J\nabla_A W - \nabla_{JA} W$$

Therefore if V and JV are Killing, and g is Kähler , using the above statements, we have

$$\begin{aligned} g(\nabla_A JV, B) = \quad & g(J\nabla_A V, B) = \quad -g(\nabla_A V, JB) = \\ g(\nabla_A JV, B) = \quad & g(J\nabla_A V, B) = \quad -g(\nabla_A V, JB) = \\ g(A, \nabla_{JB} V) = \quad & g(A, \nabla_B JV) = \quad -g(\nabla_A JV, B). \end{aligned}$$

therefore $\nabla_A JV = 0$ and JV is parallel. But $J\nabla V = \nabla JV = 0$ so that $\nabla V = 0$ also, because J is an isomorphism. □

On a riemannian manifold, the metric defines an isomorphism between vector fields and 1-forms, if V is any vector field and α a 1-form, we will use the following notation:

$$V^\flat = g(V, \cdot) \qquad\qquad \alpha^\sharp = g^{-1}(\alpha, \cdot)$$

Lemma 2.2.7 *On any riemannian manifold, V is parallel if and only if $V^\flat$ is a parallel 1-form.*

Proof: For any vector fields A, B

$$(\nabla_A g(V,\cdot))B = Ag(V,B) - g(V,\nabla_A B) = g(\nabla_A V, B)$$

□

Proof of 2.2.1: by Bochner theorem and 2.2.5 we have that $H^0(Z,\Theta)$ is the complexification of the Lie algebra of parallel vector fields. Now recall the Weitzenböch decomposition of 1-forms:

$$\Delta = dd^* + d^*d = \nabla^*\nabla + Ric$$

it says that on a Ricci-flat riemannian manifold a 1-form is harmonic if and only if is parallel. Using 2.2.7 we then have:

$$h^0(Z,\Theta) = dim_{\mathbb{R}}(\text{Lie algebra of parallel vector fields}) = b_1(M)$$

and we are left to prove that $2h^0(M,\Theta) = b_1(M)$. By Bochner theorem every holomorphic vector field is parallel, so the dual $(0,1)-form$ is parallel; since M is Kähler , $\Delta = 2\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and a $(0,1)-form$ is parallel if and only if is harmonic; we conclude that

$$h^0(M,\Theta) = h^0(M,\Omega^1) = \frac{1}{2}b_1(M)$$

□

Proof of 2.2.2: Suppose M has no parallel vector fields, then by Lichnerowicz theorem and 2.2.5, $H^0(M,\Theta)$ is the complexification of the Lie algebra of all conformal Killing vector fields on $M \cong H^0(Z,\Theta)$, and we have proved the result. Therefore is enough to show that M admits no parallel holomorphic vector fields.

To show this is true, we first reduce to the case of a minimal model: suppose M is not minimal (i.e. it contains a holomorphically embedded, irreducible rational curve C with self-intersection = -1). Then if $\Theta_{M,C}$ denotes the sheaf of holomorphic vector fields on M which are tangent to C, along C, we have an exact sequence $0 \to \Theta_{M,C} \to \Theta_M \to \nu_{C/M} \to 0$. As $H^0(C,\nu_{C/M}) \cong H^0(\mathbb{CP}_1, \mathcal{O}(-1)) = 0$, it follows that every holomorphic tangent vector on M is tangent to C, along C. Since $C \cong \mathbb{CP}_1$, every holomorphic vector field vanishes somewhere. (In fact a direct image argument shows that it has to vanish identically, along C).

If M is minimal, however, and the total scalar curvature is non-negative, Yau [Y1] has shown that $M \cong \mathbb{CP}_2$ or else is a $\mathbb{CP}_1$- bundle over a Riemann surface S_g. This says that $\chi(M) \neq 0$, and therefore has no parallel vector fields, unless M is a $\mathbb{CP}_1$-bundle over a torus; in this case however $c_1^2(M) = 2\chi + 3\tau = 0$. On the other hand, by Chern-Weil theory [B1], $c_1^2 = 2\chi + 3\tau = \int_M -|B|^2$, when M is anti-self-dual with zero scalar curvature; so that $c_1^2(M)$ cannot be zero unless M is Ricci-flat. □

Notice that the result of 2.2.2 holds for any half-conformally flat compact Kähler surface with no parallel holomorphic vector field, e.g. $\mathbb{CP}_2$, or trivially, for any such surface of negative Ricci curvature.

Chapter 3

Two interesting examples

3.1 Kähler type

The following is a class of examples of Kähler compact surfaces M_g, with zero scalar curvature. If S_g denotes a compact Riemann surface of genus $g \geq 2$ then $M_g = S_g \times \mathbb{CP}_1$; the two Riemann surfaces are given their standard metrics of constant curvature and opposite values; it is then clear that M_g is Kähler of zero scalar curvature, and therefore anti-self-dual. In fact since the signature is zero, M_g is conformally flat.

We explicitly describe the twistor space Z_g of M_g and directly show the following:

1. Z_g contains two disjoint complex hypersurfaces Σ_g and $\overline{\Sigma}_g$.

2. If K denotes the canonical line bundle of Z_g with canonical divisor

$|K|$ then, for any $p \geq 0$, the divisor $-\frac{p}{2}|K|$ is linearly equivalent to $p(\Sigma_g + \overline{\Sigma}_g)$.

3. Furthermore $H^0(Z_g, \mathcal{O}(K^{-\frac{p}{2}})) = 1$, $p \geq 0$. And the Lie algebra $H^0(Z_g, \Theta)$ of holomorphic vector fields on the twistor space is isomorphic to the Lie algebra of holomorphic vector fields on M_g.

These three properties are meant to illustrate some of the general facts described in chapter 2: recall from there that when M is a compact anti-self-dual 4-manifold then:

1. holds if and only if M is hermitian
2. holds if and only if M is of Kähler type
3. holds if and only if M is of Kähler type but not Ricci flat.

We start by considering the universal cover M of M_g . This is the riemannian product $M = \mathcal{H} \times \mathbb{CP}_1$, where $\mathcal{H}=\{(t,\rho) \in \mathbb{R}^2 | \rho > 0\}$ is the upper half-plane with the metric of curvature -1 : $g_{\mathcal{H}}=\frac{1}{\rho^2}(dt^2 + d\rho^2)$, and $\mathbb{CP}_1 \cong S^2$ is given the metric of curvature $+1$: $g_{S^2} = \sin^2\varphi d\theta^2 + d\varphi^2$.

We now consider the natural inclusion

$$\imath \colon \mathcal{H} \times \mathbb{CP}_1 \longrightarrow \mathbb{R}^4$$

$$(t,\rho,\varphi,\theta) \longmapsto (t, \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

then

1. $\imath$ is a conformal isometry: the metric on M is

   $\frac{1}{\rho^2}(dt^2 + d\rho^2)+g_{S^2}=\frac{1}{\rho^2}(dt^2 + d\rho^2 + \rho^2 g_{S^2})=\frac{1}{\rho^2}(\text{flat metric on } \mathbb{R}^4).$

2. the image of $\imath$ is $\mathbb{R}^4 \setminus l$ where l is the line (t,0,0,0) in $\mathbb{R}^4$.

In fact it is useful to think of $\imath$ as a map into $S^4 = \mathbb{R}^4 \cup \infty$ and then of S^4 as the quaternionic projective line $\mathbb{HP}_1 = (\mathbb{H}^2 \setminus \{0\}) /\sim$ where $\mathbb{H}$ denotes the non-commutative field of quaternions and $(q_0, q_1) \sim (\tilde{q}_0, \tilde{q}_1)$ if and only if $q_0 = q\tilde{q}_0$ and $q_1 = q\tilde{q}_1$ for some $q \in \mathbb{H}$, notice that we have used left multiplication here to define $\mathbb{HP}_1$.

Since M_g is the quotient of M by a discrete subgroup $\Gamma \subset PSL(2,\mathbb{R})$ acting on $\mathcal{H}$ by fractional linear transformations, we will consider next the action φ of $PSL(2,\mathbb{R})$ on $M \subset \mathbb{HP}_1$ by conformal isometries and its induced holomorphic action Φ on $Z \subset \mathbb{CP}_3$. All of these actions are considered as right actions. For example when $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,\mathbb{H})$ acts on $\mathbb{HP}_1$ by a fractional linear transformation, the map is given by: $[q_0, q_1] \mapsto [q_0 a + q_1 b, q_0 c + q_1 d]$, and this action realizes $PU(2,\mathbb{H}) := U(2,\mathbb{H})/\pm Id.$ as the group of conformal isometries of S^4. This is the quaternionic analogous of $PSL(2,\mathbb{C})$ being the group of conformal transformations of $S^2 = \mathbb{CP}_1$; and the group $U(2,\mathbb{H})$ is defined to be the subgroup of $GL(2,\mathbb{H})$ whose elements have determinant of (quaternionic) norm equal to one, [1].

In fact a direct computation shows that the action φ above is the restriction to $SL(2,\mathbb{R}) \subset SL(2,\mathbb{H})$ of the action by fractional linear transformations:

Proposition 3.1.1 *For any* $\gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,\mathbb{R})$ *the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{H} \times \mathbb{CP}_1 & \xrightarrow{\phi} & \mathcal{H} \times \mathbb{CP}_1 \\ \downarrow t & & \downarrow t \\ \mathbb{HP}_1 & \xrightarrow[\varphi]{} & \mathbb{HP}_1 \end{array}$$

where $\phi : (z,s) \mapsto (\frac{az+b}{cz+d}, s)$.

Now as the inclusion $M \subset \mathbb{HP}_1 = S^4$ is a conformal isometry, the twistor space Z of M is an open set in $\mathbb{CP}_3$ (the twistor space of S^4); and the twistor space Z_g of $M_g = M/\Gamma$ is the quotient Z/Γ. We will then need to look at the Holomorphic action Φ on $\mathbb{HP}_1$, via the twistor projection $t : \mathbb{CP}_3 \to \mathbb{HP}_1$. As explained in [A] this map can be nicely written in homogeneous coordinates as $t : [Z_o, Z_1, Z_2, Z_3] \mapsto [Z_0 + Z_1 j, Z_2 + Z_3 j]$. By the non-commutativity of the quaternions it is interesting to consider both the left and right action of $GL(2,\mathbb{H})$ on $\mathbb{HP}_1$ by fractional linear transformations. The left one lifts to a non holomorphic action on $\mathbb{CP}_3$, but it still has a very important element, namely left multiplication by j acts trivially on $\mathbb{HP}_1$but it lifts to the "real" structure of $\mathbb{CP}_3$,

$$\sigma : [Z_0, Z_1, Z_2, Z_3] \mapsto [-\overline{Z}_1, \overline{Z}_0, -\overline{Z}_3, \overline{Z}_2]$$

The right action instead lifts to an holomorphic action:

Proposition 3.1.2

The conformal isometry of $\mathbb{HP}_1$, $q \mapsto (qc+d)^{-1}(qa+b)$, *given by*

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in PU(2,\mathbb{H}), \quad a=\alpha+\beta j \;,\; b=\gamma+\delta j \;,\; c=\epsilon+\xi j \;, d=\eta+\theta j$$

lifts to the biholomorphism of $\mathbb{CP}_3$ *given by*

$$\begin{pmatrix} \alpha & \beta & \epsilon & \xi \\ -\overline{\beta} & \overline{\alpha} & -\overline{\xi} & \overline{\epsilon} \\ \gamma & \delta & \eta & \theta \\ -\overline{\delta} & \overline{\gamma} & -\overline{\theta} & \overline{\eta} \end{pmatrix} \in GL(4,\mathbb{C})$$

Both matrices are acting on the right.

Proof: In homogeneous coordinates on $\mathbb{HP}_1$, the map $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is given by: $[q_0, q_1] \mapsto [q_0a+q_1b, q_0c+q_1d]$ and one gets the result by identifying $\mathbb{H}$ with the subalgebra of $M(2,\mathbb{C})$ generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = k, \text{ over } \mathbb{R}. \qquad \square$$

Remark 3.1.3 To describe the twistor space Z_g and some of its properties we will need the obvious equivalence between objects on Z_g, such as holomophic sections of vector bundles, and the corresponding Γ-invariant objects on Z. Now the key to the study of holomorphic sections of bundles on Z is that, in some cases, they extend to global sections on all of $\mathbb{CP}_3$. To this respect we start by observing the following: let U be any (compact or not) half-conformally flat manifold with twistor space W. If U is spin, let H denote the holomorphic line bundle on W such that its restriction to each twistor line has Chern class $+1$; then $H^{-4} = K$ is the canonical line bundle of W and H depends only on the complex structure of W. If U is not spin we have to consider H^2 instead. Going back to our situation, notice that when W is an open set of $\mathbb{CP}_3$, $H = \mathcal{O}_{\mathbb{CP}_3}(1)_{|W}$. We now look at holomorphic sections of its powers.

Proposition 3.1.4 *If W is an open neighborhood of a line in $\mathbb{CP}_3$, then any holomorphic section of H^n on W, extends to all of $\mathbb{CP}_3$, i.e.*

$$H^0(W, H^n) \cong H^0(\mathbb{CP}_3, \mathcal{O}(n)) \cong \begin{cases} 0 & \text{if } n < 0 \\ \bigodot^n \mathbb{C}^4 & \text{if } n \geq 0 \end{cases}$$

Proof: When $n = 0$ one can either appeal to a theorem in [H2] which says that twistor spaces only have constant holomorphic functions, whether they are compact or not; or argue as follows: through every point $p \in W$, the set of tangent vectors to projective lines contained in W and passing through p, spans the tangent space T_pW. As any holomorphic function f on W is constant along these compact lines, the differential $df = 0$. Now let $n \neq 0$. Since $H^n = \mathcal{O}_{\mathbb{CP}_3}(n)_{|W}$, a holomorphic section is represented by a homogeneous holomorphic function of degree n, defined on an open subset of $\mathbb{C}^4$, as any first partial derivative of f is holomorphic and homogeneous of degree $n-1$, when $n > 0$, all $n-th$ partial derivatives of f are homogeneous of degree 0 and therefore represent holomorphic functions on W which are constant. It follows that f is a homogeneous polynomial of degree n and can therefore be extended to all of $\mathbb{C}^4$. When $n < 0$, $Z_0^n \cdot f(Z)$ and $Z_1^n \cdot f(Z)$ are both homogeneous of degree 0, and therefore constant $\Rightarrow f = 0$ □

Corollary 3.1.5 *$H^0(Z_g, H^n) = H^0_\Gamma(\mathbb{CP}_3, \mathcal{O}(n))$, where H^0_Γ denotes the Γ-invariant holomorphic sections.*

In order to explicitly find these invariant sections we will need the following basic facts about $\Gamma \subset SL(2,\mathbb{R})$, when $\mathcal{H}/\Gamma = S_g$ is a compact Riemann surface of genus $g \geq 2$. [M].

1. As an abstract group $\Gamma \cong$

   $\pi_1(S_g) = \{A_1, B_1, \ldots, A_g, B_g | A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = 1\}$.

2. The limit set of Γ, defined as $\Lambda(\Gamma) = \{\lim_{n\to\infty} \gamma_n(z) | z \in \mathcal{H}, \gamma_n \in \Gamma\}$ is the real axis $\mathbb{R} \subset \mathbb{C}$, union ∞.

3. Every element $\gamma \in \Gamma$ is hyperbolic, i.e. conjugate to a dilation of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a > 1$. In particular γ has two distinct fixed points, on $\mathbb{R} \cup \infty$, and since no two elements can have a common fixed point, the set $Fix(\Gamma)$ is infinite numerable and in fact dense in $\Lambda(\Gamma)$.

4. For any limit point $r \in \Lambda(\Gamma)$, the orbit $\theta_r = \{\gamma(r) | \gamma \in \Gamma\}$ is dense in $\Lambda(\Gamma)$.

As before we will consider three different actions of the group Γ; the one on the upper half plane $\mathcal{H}$, whose limit set is $\Lambda(\Gamma) = \mathbb{R} \cup \infty = \partial(\mathcal{H})$; the action φ on S^4 whose limit set we denote by $C := \mathbb{HP}_1 \setminus \mathrm{Im}\, \imath = \{[rq, q] : r \in \mathbb{R}\} \cup \{[1,0]\} \cong l \cup \{\infty\} \cong S^1$; and the action Φ on $\mathbb{CP}_3$ whose limit set is a priori only contained in $t^{-1}(C) := D$. Notice that D being the union of all the twistor lines above C is a compact subset of $\mathbb{CP}_3$, homeomorphic to $S^1 \times S^2$.

Lemma 3.1.6 *If $\mathcal{V}$ is a Γ-invariant hypersurface in $\mathbb{CP}_3$, $\mathcal{V}$ contains the set D, which is also the limit set of the action Φ.*

Proof: Let us denote by $L_s \subset D$, the twistor line $t^{-1}(s)$ for $s \in C$. First we notice that to prove the lemma is enough to show that $\mathcal{V}$ contains just one of the twistor lines L_r in D, for some element $r \in F := Fix_{S^4}(\Gamma)$. In fact suppose there is a line $L_r \subset \mathcal{V}$, then $\gamma(L_r) \subset \mathcal{V}$ for any $\gamma \in \Gamma$, but since γ acts by biholomorphisms $\gamma(L_r) = L_{\gamma(r)}$ so that $\mathrm{V} \supset \bigcup_{\gamma \in \Gamma} L_{\gamma(r)}$. By property 4 above , $\mathcal{V}$ then contains a dense set of lines in D , and since $\mathcal{V}$ is closed, it contains D.

To complete the proof it remains to show that V contains an entire twistor line in D. This is more clear if we assume for a moment that Γ contains an elment $\delta = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$ with $h > 1$, i.e. an elment which has $0 \in \mathbb{R}$

as its unstable fixed point and ∞ as its stable fixed point. In this case we will show $\mathcal{V} \supset L_\infty$ or $\mathcal{V} \supset L_0$. Let $P = [W_0, W_1, 0, 0]$ be any point in L_∞, then $\lim_{n\to\infty}[Z_0, Z_1, Z_2, Z_3]\delta^n = \lim_{n\to\infty}[h^n Z_0, h^n Z_1, h^{-n} Z_2, h^{-n} Z_3] = \lim_{n\to\infty}[Z_0, Z_1, h^{-2n} Z_2, h^{-2n} Z_3] = [Z_0, Z_1, 0, 0]$ this shows that for a point $Q \in \mathbb{CP}_3$, $\lim_{n\to\infty} \delta^n(Q) = P$ if and only if Q belongs to $\lambda_P := \{W_1 Z_0 - W_0 Z_1 = 0 : Z_0 \neq 0\} \cong \mathbb{C}^2$, that is λ_P is the projective plane $H_P = \{W_1 Z_0 - W_0 Z_1\}$ minus the projective line $L_0 = \{Z_0 = Z_1 = 0\} = t^{-1}(0)$. Since $\mathcal{V}$ is an hypersurface in $\mathbb{CP}_3$, $\mathcal{V} \cap H_P \neq \emptyset$, for any P so that $\mathcal{V} \cap \gamma_P = \emptyset$ for some $P \in L_\infty$ implies $L_0 \subset \mathcal{V}$. This completes the proof if $\delta = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \in \Gamma$. But the same proof will work with the two fixed points of any elment of Γ, playing the role of 0 amd ∞. □

Remark 3.1.7 One easily checks that the quadric $Q := \{Z_0 Z_3 - Z_1 Z_2 = 0\}$ in $\mathbb{CP}_3$ contains D, is $SL(2, \mathbb{R})$-invariant and "real", i.e. $\sigma(Q) = Q$.

Corollary 3.1.8 *The quadric Q is the only Γ-invariant irreducible hypersurface of* $\mathbb{CP}_3$.

<u>Proof:</u> If $\mathcal{V}$ is any Γ-invariant irreducible hypersurface, $\mathcal{V} \cap Q \subset D$; since D has real dimension 3 and $\mathcal{V} \cap Q$ must be a complex hypersurface, $\mathcal{V} = Q$ (as sets). □

Corollary 3.1.9

$$H^0(Z_g, K^{\frac{-p}{4}}) \cong \begin{cases} \mathbb{C} & \textit{if } p \geq 0 \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

<u>Proof</u>: We know that a section τ is in fact a Γ-invariant global section on $\mathbb{CP}_3$: $\tau \in H^0(\mathbb{CP}_3, \mathcal{O}(p))$ and of course $\tau = 0$ if $p < 0$. When $p \geq 0$, let A denote the zero divisor of τ, A is a Γ-invariant hypersurface of degree p, therefore

$$A \cong \begin{cases} 0 & \text{if } p = 0 \\ \mathbb{CP}_3 & \text{if } p \text{ is odd} \\ \frac{p}{2}Q & \text{if } p \text{ is even} \end{cases}$$

Now if p is even and $\tau' \neq \tau$ is another section, then since the zero divisor of τ and τ' coincide, the meromorphic function τ/τ' is actually holomorphic and therefore constant. □

We can now visualize the surfaces Σ_g and $\overline{\Sigma_g}$ in Z_g given by the complex structure of M_g: consider the Γ-invariant quadric $Q = \{Z_0Z_3 - Z_1Z_2 = 0\}$ and its intersection with the twistor space $Z \subset \mathbb{CP}_3$ of M. We will check that $Z \cap Q$ is the disjoint union of two surfaces Π and $\overline{\Pi}$ in Z, with the property that $\Pi = \sigma(\overline{\Pi})$ and that each twistor line above M meets $Z \cap Q$ in two antipodal points, one of which lies in Π and the other in $\overline{\Pi}$. So that each of Π and $\overline{\Pi}$ is exactly a copy of M inside Z. To see this, we consider the Segre' embedding

$$s : \mathbb{CP}_1 \times \mathbb{CP}_1 \to \mathbb{CP}_3$$

given by

$$([X_0, X_1], [Y_0, Y_1]) \mapsto [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1]$$

then $t^{-1}(D) = \{([1,0],[Y_0,Y_1])\} \cup \{([r,1],[Y_0,Y_1])|r \in \mathbb{R}\}$ which shows that $Q \setminus D$ is the image of two disjoint hemispheres, $\mathcal{H}^+$ and $\mathcal{H}^-$, cross with $\mathbb{CP}_1$; so that $\Pi = \mathcal{H}^+ \times \mathbb{CP}_1$ and $\overline{\Pi} = \mathcal{H} \times \mathbb{CP}_1$. Π is biholomorphic to M while $\overline{\Pi}$ is antibiholomorphic to M. Now the action by $\gamma \in \Gamma$ on Q, pulls back via the Segre' imbedding to an action of the same elment γ on the first $\mathbb{CP}_1$ factor, by a fractional linear trasformation. This action of Γ is then seen to be properly discontinuous on each hemisphere $\mathcal{H}^+$ and $\mathcal{H}^-$; passing to the quotient we get the two hypersurfaces $\Sigma_g = \Pi/\Gamma = \mathcal{H}^+/\Gamma \times \mathbb{CP}_1 \cong S^g \times \mathbb{CP}_1$ and $\overline{\Sigma}_g = \overline{\Pi}/\Gamma \cong \overline{S}_g \times \mathbb{CP}_1$ in Z_g.

Corollary 3.1.10 *For any $q \geq 0$, $K_{Z_g}^{-q/2} \cong q \cdot \Sigma\overline{\Sigma}$.*

<u>Proof</u>: By construction a section of $K_{Z_g}^{-q/2}$ vanishes on $\Sigma \coprod \overline{\Sigma}$ to order q. □

3.1.1 Holomorphic vector fields

Proposition 3.1.11 *Let W_1 and W_2 be two open sets of $\mathbb{CP}_3$, each containing a projective line; then if $h : W_1 \to W_2$ is a biholomorphism h extends to a biholomorphism of all of $\mathbb{CP}_3$; that is $h \in PGL(4, \mathbb{C})$.*

Proof: W_1 and W_2 are twistor spaces of two open sets in S^4, say U_1 and U_2 respectively. Consider the holomorphic line bundles H_{W_1} and H_{W_2}, by remark 3.1, $h^\star H_{W_2} = H_{W_1}$ that is $h^\star \mathcal{O}_{\mathbb{CP}_3}(1)_{|W_2} = O_{\mathbb{CP}_3}(1)_{|W_1}$ and $H_{W_i} = \mathcal{O}_{\mathbb{CP}_3}(1)_{|W_i}, i = 1, 2$; this implies that if Z_i is a homogeneous coordinate on $W_2 \subset \mathbb{CP}_3$, then $h^{-1}(Z_i) = \sum_{j=0}^{3} a_{ij} Z_j$ where Z_j are homogeneous coordinates on$W_1 \subset \mathbb{CP}_3$ and a_{ij} are constants. Therefore $h^{-1} \in PGL(4, \mathbb{C})$.
□

Proposition 3.1.12 *Let W be a neighborhood of a projective line in $\mathbb{CP}_3$. Then any holomorphic vector fields on W extends to all of $\mathbb{CP}_3$.*
That is $H^O(W, \Theta) \cong M(4, \mathbb{C})/\{\lambda I : \lambda \in \mathbb{C}\}$.

Proof: On any open set U of $\mathbb{CP}_3$, the Euler sequence for the sheaf Θ of holomorphic tangent vectors : $0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 4} \to \Theta \to 0$, is an exact sequence of $\mathcal{O}_U$-modules. By taking global homogeneous coordinates it is a split-exact sequence, over U, of complex vector spaces. In particular $H^O(W, \Theta) \cong H^O(W, \mathcal{O}(1)^{\oplus 4})/H^O(W, \mathcal{O})$. And, by proposition 3.1.4 we have done. □

As a corollary we recover:

Theorem 3.1.13 *(Liouville's theorem in dimension 4). Any conformal isometry between two open subsets of S^4, endowed with a conformally flat metric, extends to a conformal isometry of S^4. Also any conformal Killing vector* *field on an open subset, extends to a conformal Killing vector field on all of* S^4.

Proposition 3.1.14 $H^0(Z_g, \Theta) \cong H^0(M_g, \Theta) \cong H^0(\mathbb{CP}_1, \Theta) \cong \mathbb{C}^3$.

Proof: By the previous proposition a holomorphic vector field on Z_g is exactly a Γ-invariant vector field V on $\mathbb{CP}_3$. We will show $H^0_\gamma(\mathbb{CP}_3, \Theta) \cong H^0(\mathbb{CP}_1, \Theta)$. To find all such V's, notice that since the quadric Q is Γ-invariant, V has to be tangent to Q, along Q. That is if $F = Z_0Z_3 - Z_1Z_2$ is the defining function of Q, $dF(V) = \mu F$, $\mu \in \mathbb{C}$. A vector field on $\mathbb{CP}_3$ is written as $V = \sum_{i,j=0}^3 a_{ij} Z_i(\partial/\partial Z_j)$ and is identified with the element $(a_{ij}) \in M(4,\mathbb{C})/\lambda I$ so that $dF(V) = Z_3(\sum_{i=0}^3 a_{i0}Z_i) - Z_2(\sum_{i=0}^3 a_{i1}Z_i) - Z_1(\sum_{i=0}^3 a_{i2}Z_i) + Z_0(\sum_{i=0}^3 a_{i3}Z_i) = \mu(Z_0Z_3 - Z_1Z_2)$, $\mu \in \mathbb{C}$; that is

$$V \leftrightarrow \begin{pmatrix} a_{00} & a_{01} & a_{02} & 0 \\ a_{10} & a_{11} & 0 & a_{02} \\ a_{20} & 0 & \mu - a_{11} & a_{01} \\ 0 & a_{20} & a_{10} & \mu - a_{00} \end{pmatrix} \in M(4,\mathbb{C})/\lambda I$$

Now we impose the condition that V commutes with every element $\gamma \in \Gamma$

$$\gamma = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}$$

using the fact that Γ has at least three free generators, is easy to see that this forces $a_{02} = a_{20} = 0$, $\mu = a_{00} = a_{11}$ so that V is of the form

$$\begin{pmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ 0 & 0 & a_{00} & a_{01} \\ 0 & 0 & a_{10} & a_{11} \end{pmatrix} \in M(4, \mathbb{C})/\lambda I$$

Which is isomorphic to the Lie algebra $H^0(\mathbb{CP}_1, \Theta)$. □

Corollary 3.1.15 *The group of biholomorphisms of Z_g is isomorphic to $PGL(2, \mathbb{C})$.*

Proof: A biholomorphism h of Z_g is an element of $PGL(4, \mathbb{C})$ which commutes with every element of $\Gamma \Rightarrow$

$$h = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ 0 & 0 & a_{00} & a_{01} \\ 0 & 0 & a_{10} & a_{11} \end{pmatrix}$$

□

3.1.2 A word on deformations

We now want to briefly look at small deformations of the product metric of the preceding example; we will be interested of course in conformally flat metrics. One way to get new conformally flat metrics is by considering a representation $\rho : \pi_1(S_g \times \mathbb{CP}_1) = \pi_1(S_g) \to PU(2, \mathbb{HP}_1)$ of the fundamental group of $S_g \times \mathbb{CP}_1$ into the group of conformal isometries of an open set $N \cong \mathcal{H} \times \mathbb{CP}_1$ in S^4. If we denote by Δ the image of ρ then the manifold $N_g = N/\Delta$ is conformally flat. If we consider instead a representation ρ' : $\pi_1(S_g) \to SO(3)$, into the group of isometries of $\mathbb{CP}_1$, the quotient $N'_g = N/\Delta'$ is Kähler of zero scalar curvature. In fact all such metrics are given in this way [BdB]. Next we will show that for generic deformations, the twistor spaces of these manifolds have no holomorphic vector fields.

Lemma 3.1.16 *Let $\gamma, \eta, V \in M(n, \mathbb{C})$; suppose that V commutes with both γ and η, then γ, η have to satisfy the following condition: $\gamma(\Lambda) \subseteq \Lambda$ and $\eta(\Lambda) \subseteq \Lambda$. Where Λ is the eigenspace of any eigenvalue λ of A.*

Proof: For any $v \in \Lambda$; $A(\gamma v) = \gamma(Av) = \lambda(\gamma v)$ and $A(\eta v) = \sigma(Av) = \lambda(\eta v)$ □

Remark 3.1.17 The above condition is non trivial if $A \neq \lambda I$. Therefore two generic elements of $M(n, \mathbb{C})$ both commute with λI only.

Reasoning as in the last section we then have:

Proposition 3.1.18 *If W is the twistor space of a generic conformally flat deformation N, then $H^0(W, \theta) = 0$.*

Corollary 3.1.19 *If N' is a generic deformation with a Kähler metric of zero scalar curvature, then $H^0(N', \Theta) = 0$. See also [BdB].*

We conclude by mentioning a topic for future research:

It is clear from the above discussion that not all conformally flat deformations of the product metric $S_g \times \mathbb{CP}_1$ are Kähler of zero scalar curvature. This could be compared with the following hypothetical situation: suppose for a moment that (as it has been conjectured) (Y_k, g) is the manifold given by blowing up k points on $\mathbb{CP}_2$, with a Kähler metric g of zero scalar curvature (notice that by the condition $c_1^2 < 0$, k has to be at least 10) and consider small deformations of $g_0 = g$ through half-conformally flat metrics g_t. Then we have the following: g_t must be a Kähler metric of zero scalar curvature, if and only if $k = 10$.

3.2 A twistor space of algebraic dimension two

Consider the complement of the origin in the complex plane, $\mathbb{C}^2_* := (\mathbb{C}^2 \setminus \{0\})$, with coordinates $z = (z_1, z_2)$ and hermitian metric $\tilde{h} = ||z||^{-2}(dz_1 \otimes d\overline{z}_1 + dz_2 \otimes d\overline{z}_2)$. Fixed a real number $|\lambda| \neq 0, 1$, the infinite cyclic group of holomorphic isometries $\Lambda := \{z \mapsto \lambda^n : n \in \mathbb{Z}\} \subset GL(2, \mathbb{C})$ acts properly discontinuosly and without fixed points on $\mathbb{C}^2_*$. We can then consider the quotient manifold $M := \mathbb{C}^2_*/\Lambda$ with the induced metric h. M is a complex surface homeomorphic to $S^1 \times S^3$, called a Hopf surface. As $(\mathbb{C}^2_*, \tilde{h})$ and (M, h) are both conformally flat, let W and Z be their respective twistor spaces.

To describe W, we think of $\mathbb{C}^2_*$ as $(S^4 \setminus \{0, \infty\})$. Since $\tilde{h}$ is also conformally flat, their twistor spaces coincide, and W is the open set $(\mathbb{CP}_3 \setminus \{L_0, `\cup L_\infty\})$; where L_0 and L_∞ are the twistor lines above 0 and ∞, in S^4.

As Λ acts on $\mathbb{HP}_1 = S^4$ by conformal isometries: $q \mapsto \lambda^{\frac{n}{2}} q \lambda^{\frac{n}{2}} = \lambda^n q$ when λ is real, it also acts on $\mathbb{CP}_3$ by biholomorphisms :$[Z_0, Z_1, Z_2, Z_3] \mapsto [\lambda^{\frac{n}{2}} Z_0, \lambda^{\frac{n}{2}} Z_1, \lambda^{-\frac{n}{2}} Z_2, \lambda^{-\frac{n}{2}} Z_3] = [\lambda^n Z_0, \lambda^n Z_1, Z_2, Z_3]$.

Λ acts freely on W, and $Z = W/\Lambda$ is the twistor space of M.

Proposition 3.2.1 *Z admits a holomorphic fibration $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$*

<u>Proof</u>: Consider the holomorphic map $\tilde{p} : (\mathbb{CP}_3 \setminus \{L_0 \cup L_\infty\}) \to \mathbb{CP}_1 \times \mathbb{CP}_1$ given by $[Z_0, Z_1, Z_2, Z_3] \mapsto ([Z_0, Z_1], [Z_2, Z_3])$; is clear that $\tilde{p}$ commutes with

the action of Λ, and so descends to $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$.

Notice then that $\tilde{p}$ is a regular map, with fiber

$\tilde{p}^{-1}([X_0, X_1], [Y_0, Z_1]) = [aX_0, aX_1, bY_0, bZ_1] = [\frac{a}{b}X_0, \frac{a}{b}X_1, Y_0, Z_1] \cong (\mathbb{C} \setminus 0)$, also denoted by $\mathbb{C}_*$, where $a, b \in \mathbb{C}_*$. Passing to the quotient by Λ, we get the compact holomorphic fiber bundle $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$ with fiber the compact Riemann surface of genus 1, given by $\mathbb{C}_*/\ (z \sim \lambda z)$ □

We now want to analyze the fibration p more closely: let us denote by F_1 and F_2 the two factors in the base space $\mathbb{CP}_1 \times \mathbb{CP}_1$, and start by considering the fibration $\tilde{p} : W \to F_1 \times F_2$. Let then $\pi_i : F_1 \times F_2 \to F_i$ denote the canonical projections, $i = 1, 2$; and $\tilde{p}_i$ the compositions $\pi_i \circ \tilde{p}$. As $u \in F_1$ varies, $\tilde{p}_1(u)$ describes the set of all hyperplanes in $\mathbb{CP}_3$ passing through the line L_∞, and we notice that each of these hyperplanes meets the line L_0 in exactly one point. It follows that $\tilde{p}_1^{-1}(u) \cong \mathbb{C}_*^2$ for each $u \in F_1$. Similarly $\tilde{p}_2^{-1}(v) \cong \mathbb{C}_*^2$. It is also useful to notice that, as every twistor line L meets any hyperplane through L_0 or L_∞ in exactly one point, we have that $\tilde{p}_1^{-1}(u) \cap L = \{1pt.\}$ and $\tilde{p}_2^{-1}(v) \cap L = \{1pt.\}$, for each twistor line L , $u \in F_1$, $v \in F_2$.

Passing to the quotient by the action of Λ we consider now the fibration $p : Z \to F_1 \times F_2$. Let $p_i = \pi_i \circ p$, $i = 1, 2$; we look at the fibers: if $u \in F_1$ we set $H_u := p_1^{-1}(u) \cong \mathbb{C}_*^2/\ (z \sim \lambda z)$ and if $v \in F_2$, $H^v := p_2^{-1}(v) \cong \mathbb{C}_*^2/\ (z \sim \lambda^{-1} z)$. From the previous discussion we then get the following:

Proposition 3.2.2 *The twistor space Z is foliated by two families of Hopf* *surfaces* $\{H_u\}_{u\in\mathbb{CP}_1}$ *and* $\{H^v\}_{v\in\mathbb{CP}_1}$, *each leaf being biholomorphic to* M. *Furthermore every two leaves* H_u *and* H^v *intersect in the elliptic curve* $p^{-1}(u,v)$, *and each twistor line* L *intersects any of* H_u *and* H^v *in exactly one point.*

We denote the elliptic curves $p^{-1}(u,v)$ by E_u^v. Also notice that when $\lambda\in\mathbb{R}$, the real structure $\sigma:[Z_0,Z_1,Z_2,Z_3]\mapsto[\overline{Z}_0,-\overline{Z}_1,\overline{Z}_2,-\overline{Z}_3]$ of $\mathbb{CP}_3$, commutes with the action of Λ, and we get an induced real structure on Z.

3.2.1 The holomorphic tangent bundle

Let $F:\mathbb{C}^2_*\to\mathbb{C}^2_*$ be a biholomorphism, $F(z)=(f_1(z),f_2(z))$. Since f_1 and f_2 are holomorphic functions, they extend to all of $\mathbb{C}^2$, by Hartog's theorem. Now F descends to a biholomorphism of $M=\mathbb{C}^2_*/\Lambda$ if and only if $F(\lambda z)=\lambda F(z)$. For $i=1,2$, this forces $f_i(\lambda z)=\lambda f_i(z)$, so that each derivative $\frac{\partial}{\partial z_j}f_i$ satisfies $\frac{\partial}{\partial z_j}f_i(\lambda z)=\lambda\frac{\partial}{\partial z_j}f_i(z)$. That is, $\frac{\partial}{\partial z_j}f_i$ is a well defined holomorphic function on M, and therefore constant. It follows that $f_i(z)$ is a linear function of z, and in fact since $F(0)=\lim_{z\to 0}F(\lambda z)=\lambda\lim_{z\to 0}F(z)=\lambda F(0)$, we get $F(0)=0$; therefore $F\in GL(2,\mathbb{C})$.

In an analogous way, every holomorphic vector field on M can be written uniquely as $a_{00}z_0\frac{\partial}{\partial z_0}+a_{01}z_0\frac{\partial}{\partial z_1}+a_{10}z_1\frac{\partial}{\partial z_0}+a_{11}z_1\frac{\partial}{\partial z_1}$. We have proved:

Proposition 3.2.3

$$Aut(M)\cong GL(2,\mathbb{C})\quad and\quad H^0(M,\Theta)\cong\mathbf{gl}(2,\mathbb{C})\cong\mathbb{C}^4$$

We will be interested in computing the cohomology of the tangent bundle of Z. Since the techniques are similar, we start with computing $H^i(M,\Theta)$. This is done by introducing a short exact sequence of sheaves. First notice that M has a "tautological" meromorphic function $f : M \to \mathbb{CP}_1$, given by $[(z_1, z_2)] \mapsto [z_1, z_2]$, and induced by the Hopf fibration on $\mathbb{CP}_1$.

Proposition 3.2.4 $0 \to \mathcal{O}_M \to \Theta_M \to f^*(T\mathbb{CP}_1) \to 0$ *is an exact sequence of vector bundles on* M.

Proof: Since f is a fiber bundle map, with elliptic curves as fibers, $\Theta_M \to f^*(T\mathbb{CP}_1) \to 0$ is exact. But $ker f_* =$ span of $(z_0\frac{\partial}{\partial z_0} + z_1\frac{\partial}{\partial z_1}) \cong \mathcal{O}_M$. □

Now if $g : X \to Y$ is a continuos map of topological spaces, and $\mathcal{S}$ is a sheaf on X, by $g_{*i}\mathcal{S}$ we will denote the $i - th$ direct image sheaf on Y, $i = 0, 1, 2, ...$

Proposition 3.2.5 $f_{*i}\mathcal{O}_M \cong \begin{cases} \mathcal{O}_{\mathbb{CP}_1} & \text{if } i = 0,1 \\ 0 & \text{otherwise} \end{cases}$

Proof: As every fiber is a compact elliptic curve E, the statement is clear for $i \neq 1$.

For each open set $U \subset \mathbb{CP}_1$, when $i = 1$:

$$(f_{*1}\mathcal{O}_M)(U) := H^1(f^{-1}(U), \mathcal{O}_M) = H^1(U \times E, \mathcal{O}) \cong$$

by Kunneth formula,

$$(H^0(U,\mathcal{O}) \otimes H^1(E,\mathcal{O})) \oplus (H^1(U,\mathcal{O}) \otimes H^0(E,\mathcal{O}) =$$

by choosing U to be Stein,

$$H^0(U,\mathcal{O}) \otimes H^1(E,\mathcal{O}) \cong$$

by Serre duality,

$$\mathcal{O}_U \otimes (H^0(E,\Omega^1))^* = (\mathcal{O}_U \otimes (H^0(E,\mathcal{O}))^* := \mathcal{O}^*_{\mathbb{CP}_1}(U) \cong \mathcal{O}_{\mathbb{CP}_1}(U).$$

□

Corollary 3.2.6

$$H^i(M,\mathcal{O}) \cong \begin{cases} \mathbb{C} & \textit{if } i = 0,1 \\ 0 & i{=}2 \end{cases}$$

<u>Proof</u>: By a theorem of Leray, there is a spectral sequence

$$E_\infty^{p,q} \Rightarrow H^{p+q}(M,\mathcal{O}) \text{ , with}$$

$$E_2^{p,q} = H^p(\mathbb{CP}_1, f_{*q}\mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } p = 0 \text{ and } q = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore $E_2^{p,q} = E_\infty^{p,q}$, and the result follows. □

Corollary 3.2.7

$$H^i(M, f^*T\mathbb{CP}_1) \cong \begin{cases} \mathbb{C}^3 & \textit{if } i = 0,1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof: First we notice that, by the projection formula,

$$f_{*q}f^*T\mathbb{CP}_1 = T\mathbb{CP}_1 \otimes f_{*q}\mathcal{O} = \begin{cases} T\mathbb{CP}_1 & \text{if } i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

And again, by the Leray's spectral sequence, we have the result. □

As a consequence of all this, we then have:

Proposition 3.2.8

$$H^i(M, \Theta_M) \cong \begin{cases} \mathbb{C}^4 & i = 0,1 \\ 0 & \textit{otherwise} \end{cases}$$

We now pass to consider the tangent bundle of Z.

Proposition 3.2.9

$$H^i(Z, \Theta_Z) \cong \begin{cases} \mathbb{C}^7 & i = 0,1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof: The proof goes exactly as in the case of M. First, the fiber bundle map $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$, gives an exact sequence

$$0 \to \mathcal{O} \to \Theta_Z \to p^*T(\mathbb{CP}_1 \times \mathbb{CP}_1) \to 0 \qquad (3.1)$$

because $Ker\ p^* = \{\begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix} : I = id \in M(2, \mathbb{C}), a \in \mathbb{C}\} \cong \mathcal{O}_Z$.

Again, Serre duality on he fibers gives:

$$p_{*i}\mathcal{O}_Z \cong \begin{cases} \mathcal{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1} & \text{when } i = 0,1 \\ 0 & i \geq 2 \end{cases} \qquad (3.2)$$

so that the Leray's spectral sequence implies:

$$H^i(Z,\mathcal{O}) \cong \begin{cases} \mathbb{C} & \text{if i=0,1} \\ 0 & i \geq 2 \end{cases} \quad \textit{and} \quad H^i(Z, p^*T(\mathbb{CP}_1 \times \mathbb{CP}_1) \cong \begin{cases} \mathbb{C}^6 & \text{if i=0,1} \\ 0 & i \geq 2 \end{cases} \tag{3.3}$$

Finally, the global holomorphic vector fields on Z are:

$$H^0(Z,\Theta) = \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in M(2,\mathbb{C})\}/kI \cong \mathbb{C}^7$$

The rest then easily follows from 3.1 □

As a corollary of the proof we then have:

Proposition 3.2.10 *The algebraic dimension of Z is equal to 2.*

Proof: The holomorphic fibration $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$, shows that the algebraic dimension $a(Z)$ of Z, is at least 2. The only other possibility then, is $a(z) = 3$, see section 2.1. But in this case Z would be a Moischézon space. All such spaces though, can be blown up to algebraic manifolds [24], in particular they have Hodge symmetry: $h^p(Z,\Omega^q) = h^q(Z,\Omega^p)$. This is then a contradiction, because by 3.3 above, $h^1(Z,\mathcal{O}) = 1$; while $h^0(Z,\Omega^1) = 0$ for any twistor space [12]. □

3.3 Structural differences

We then notice that the twistor space of the Hopf manifold above, is the first example of a twistor space with algebraic dimension 2, [20,21,8]. Furthermore it is the twistor space of a hermitian anti-self-dual surface. As it was shown by Poon [21], $a(z) \leq 1$ when such a surface is of Kähler type. This result is then another instance of the difference in character, between anti-self-dual surfaces of Kähler and non-Kähler type.

Indeed, in marked contrast with the general results on the twistor space of a manifold of Kähler type, see sections 2.1, 2.2, if we let $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$, and denote the holomorphic line bundle $p^*(\mathcal{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1}(m,n))$, by $\mathcal{O}_Z(m,n)$, then:

3.3.1 $[X] \ncong K_Z^{-\frac{1}{2}}$

Proof: From equation 3.1, we have that $K_Z^{-\frac{1}{2}} = \mathcal{O}_Z(1,1)$, on the other hand is clear that $[X] = \mathcal{O}_Z(2,0)$. Therefore, $[X] = K_Z^{-\frac{1}{2}} \otimes \mathcal{O}_Z(1,-1)$ and in fact one can also check that $c_1(\mathcal{O}_Z(1,-1)) = 0$, by intersecting with any twistor line. □

Of course, by 2.1.9, the above holds for any surface of non-Kähler type.

3.3.2 *The normal bundle* $\nu_\Sigma \ncong K_\Sigma^{-1}$

Proof: Since Σ is a fiber, ν_Σ is trivial; but $K_\Sigma^{-1} \cong -2[E]$, where E is the irreducible divisor of Σ. □

Recall now, that $k(Z, L)$ denotes the Kodaira dimension of the holomorphic line bundle L over Z; then

3.3.3 $a(Z) = k(Z, K_Z^{-\frac{1}{2}}) \neq k(Z, [X])$

Proof: By 3.2, $h^0(Z, K_Z^{-\frac{m}{2}}) = h^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(m, m)) = (m+1)^2$ while $h^0(Z, [X]^m) = h^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(2m, 0)) = 2m + 1$ □

The projection $p_1 : Z \to \mathbb{CP}_1$ on the first factor, comes from the hyperhermitian structure of the Hopf surface, [6]. The same is true for the twistor space of a hyperkähler surface, i.e. a torus or a $K3$ surface; in the hyperkähler case, the canonical bundle K_Z is the the pull back of $\mathcal{O}_{\mathbb{CP}_1}(-4)$; we show this is not valid in the hyperhermitian case:

3.3.4 $K_Z = \mathcal{O}_Z(-2, -2) \ncong \mathcal{O}_Z(-4, 0) = p_1^*(\mathcal{O}_{\mathbb{CP}_1}(-4))$

Proof: A direct consequence of equation 3.1. □

finally,

3.3.5 $h^0(Z, \Theta_Z) = 7$, *and therefore is different from both* $h^0(M, \Theta_M) = 4$ *and* $2h^0(M, \Theta_M)$.

Chapter 4

Construction of a twistor space

We will denote by $M \cong S^1 \times S^3$ the Hopf surface described before: $M = \mathbb{C}^2_*/\left(z \sim \lambda z\right)$, $\lambda \in \mathbb{R}$, $\lambda \neq 0, \pm 1$

An interesting consequence of $H^2(Z, \Theta_Z) = 0$ in 3.2.9 is, as we will explain later, that the topological manifold $M\#\overline{\mathbb{CP}_2}$, admits an anti-self-dual metric. (Here we have denoted by $\#$, the connected sum; and by $\overline{\mathbb{CP}_2}$ the manifold $\mathbb{CP}_2$ with orientation reversed). This follows from the general theory of Donaldson and Friedman, which we are going to describe next, together with the twistor space of $M\#\overline{\mathbb{CP}_2}$.

4.1 Singular twistor space

Following the general construction in [8], we want to investigate the twistor space of $M\#\overline{\mathbb{CP}_2}$. This is constructed by first forming a 3-dimensional, singular complex space Z in the following way.

Let $\tilde{Z}_1$ be the twistor space Z_1 of $M(\cong S^1 \times S^3)$, with a twistor line blown up. Notice here, that since $S^1 \times SO(4)$ acts transitively on M by isometries, $Aut(Z_1)$ acts transitively on the family of all twistor lines in it, and therefore the holomorphic structure of $\tilde{Z}_1$ is independent of the choice of a particular line. Now, the twistor space Z_2 of $\overline{\mathbb{CP}_2}$, thought of as an anti-self-dual manifold, is the flag manifold $F_{1,2}$. We then let $\tilde{Z}_2$ be the blow up along a twistor line, and since $\mathbb{CP}_2$ is homogeneous, it doesn't matter again, which line we choose.

Observe then, that if T is any twistor space and L any twistor line in it, its normal bundle is $\nu_{L/Z} = \mathcal{O}(1) \bigoplus \mathcal{O}(1)$. When T is blown up along L, we get a manifold $\tilde{T}$ with exceptional divisor

$$E = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}) \cong \mathbb{CP}_1 \times \mathbb{CP}_1$$

i.e. a 2-quadric. The normal bundle of E in $\tilde{T}$ is, [9]

$$\nu_{E/\tilde{T}} \cong \mathcal{O}_{\mathbb{CP}_1 \times \mathbb{CP}_1}(-1, 1)$$

To summarize, each of our manifolds, $\tilde{Z}_1$ and $\tilde{Z}_2$, contains a copy of a 2-quadric, Q_1 and Q_2 respectively, with normal bundle $\nu_{Q_i} \cong \mathcal{O}(-1, 1)$, $i = 1, 2$.

Finally, Z is defined to be the union of $\tilde{Z}_1$ and $\tilde{Z}_2$, under a biholomorphic map $f : Q_1 \to Q_2$ which identifies the two quadrics, by also reversing the two $\mathbb{CP}_1$-factors. $Z = \tilde{Z}_1 \bigcup_Q \tilde{Z}_2$, where $Q \cong Q_1 \cong Q_2$ under f.

Z is then a singular complex space with a normal crossing singularity, along Q, satisfying Friedman's d-semistable condition: $\nu_Q := \nu_{Q_1} \otimes \nu_{Q_2} \cong \mathcal{O}_Q$, of having trivial normal bundle, [8].

A twistor space of the connected sum $M\#\overline{\mathbb{CP}_2}$ is then constructed by "smoothing out" Z.

4.1.1 Smooth deformations

To describe how this is done, we briefly outline the general theory developed in [8].

Let $V = V_1 \bigcup_W V_2$ be a complex space, as the one constructed before, with only normal crossing singularities, along the hypersurface W; and satisfying the d-semistable condition $\nu_1 \otimes \nu_2 \cong \mathcal{O}_W$. Then,

Definition 4.1.1 A *smooth deformation* of V, is a complex analytic family $d : \mathcal{F} \to \Delta$, with smooth total space $\mathcal{F}$, over an open set Δ in $\mathbb{C}^n$, containing the origin, such that $d^{-1}(0) = V$, and $d^{-1}(t) = V_t$ is a smooth complex manifold, for each t outside of a complex hypersurface $\Delta' \subset \Delta$.

To explain this, let us consider the situation locally. A normal crossing singularity is of the form $z_1 z_2 = 0$ in $\mathbb{C}^{k+1}$; and its deformations are given by $z_1 z_2 = t(z_3, ..., z_{k+1})$. The map d will then look like $d(z_1, ..., z_{k+1}, t_2, ..., t_n) = (z_1 z_2, t_2, ..., t_n)$, with smooth fibers, when $t_1 \neq 0$. i.e. $\Delta' = \{t_1 = 0\}$.

Going back to the global situation, the relevant exact cohomology sequence, for studying smoothings of V, is, [8]:

$$0 \to H^1(V, \tau_V^0) \to T_V^1 \to H^0(W, \nu_1 \otimes \nu_2) \to H^2(V, \tau_V^0) \to T_V^2 \to H^1(W, \nu_1 \otimes \nu_2) \tag{4.1}$$

Let us explain the notation: τ_V^0 is the sheaf of derivations of the structure sheaf, $\mathcal{O}_V$. While the groups $T_V^i := Ext^i(\omega_V^1, \mathcal{O}_V)$, represent infinitesimal complex deformations of V, for $i = 1$; with obstructions lying in T_V^2. That is, there is a Kodaira-Spencer map $\phi : T_V^1 \to T_V^2$, defined on a neighborhood of the origin in T_V^1, such that $\phi^{-1}(0)$ is the base of a versal deformation of V. If the two conditions $T_V^1 \neq 0$ and $T_V^2 = 0$ are satisfied, V has a smooth deformation, as above, with $dim\ \Delta = dim\ T_V^1$. Furthermore the term $H^1(V, \tau_V^0)$ in 4.1, represents locally trivial deformations; while the map $T_V^1 \to H^0(W, \nu_1 \otimes \nu_2)$ measures the change in the singularity, [8].

Remark 4.1.2 In the cases we are interested in, $H^0(W, \nu_1 \otimes \nu_2) \cong \mathbb{C}$ (i.e. W is connected), and $H^1(W, \nu_1 \otimes \nu_2) \cong \mathbb{C} = 0$. So that one way to assure that $T_V^1 \neq 0$ and $T_V^2 = 0$, and therefore that smoothings exist, is to show that $H^2(V, \tau_V^0) = 0$.

4.1.2 Some exact sequences

We now proceed, following [8], to give some general propositions which are useful to study the cohomology of τ_V^0.

Let us start by considering the sheaf $\mathcal{O}_V$ of holomorphic functions on $V = V_1 \cup_W V_2$. We denote by $V' = V_1 \amalg V_2$ the normalization of V. The map $q : V' \to V$ is then, simply the identification of W_1 and W_2.

Proposition 4.1.3 *[8]*

$$0 \to \mathcal{O}_V \to q_* \mathcal{O}_{V'} \to \mathcal{O}_W \to 0$$

is an exact sequence of sheaves on V.

Proof: First notice that this is a local statement that only needs to be proved around the singularity W. Locally a normal crossing singularity looks like $z_1 z_2 = 0$ in $\mathbb{C}^{n+1}$. So, for some polycilinder D in $\mathbb{C}^{n+1}$, let $D \cap V = \{z_1 z_2 = 0\}$, $D \cap V_i = \{z_i = 0\}$, $i = 1,2$, $D \cap W = \{z_1 = z_2 = 0\}$. Then $\mathcal{O}_V$ is defined to be $\mathcal{O}_D / < z_1 z_2 >$; where by $< z_1 z_2 >$ we denote the ideal generated by $z_1 z_2$. We now define the ring $R = \{(f_1, f_2) \ : \ f_i \in \mathcal{O}_{V_i}\ ,\ i = 1,2 \textit{ and } f_{1|W} = f_{2|W}\}$; and to complete the proof we have to show $\mathcal{O}_V = R$. For this purpose define the ring homomorphism $\psi : \mathcal{O}_D \to R$ by $f(z_1, z_2, ..., z_{n+1}) \mapsto (f(0, z_2, ..., z_{n+1}), f(z_1, 0, ..., z_{n+1}))$. The first thing to show is that ψ is onto; but for any $(f_1, f_2) \in R$, is easy to see that $f(z_1, ..., z_{n+1}) := f_1(z_2, ..., z_{n+1}) + f_2(z_1, z_3, ..., z_{n+1}) - f_1(0, z_3, ..., z_{n+1})$, satisfies $\psi(f) = (f_1, f_2)$. What is left to show, is that $Ker\ \psi = < z_1 z_2 >$. To do this we use the following weak form of the Nullenstellensatz, [9]: if $h \in \mathcal{O}_{\mathbb{C}^{n+1}}$ is irreducible and $f \in \mathcal{O}_{\mathbb{C}^{n+1}}$ is zero where h is zero; then, h divides f. So let $f \in Ker\ \psi$, i.e. $f = 0$ when $z_1 = 0$ or $z_2 = 0$; since z_1 and z_2 are irreducible, $f = z_1 z_2 k$; i.e. $f \in < z_1 z_2 >$ and therefore $Ker\ \psi \subset < z_1 z_2 >$. The other inclusion being trivial, the proof is now complete. □

As a consequence, consider the sheaf of derivations of $\mathcal{O}_V$, denoted by τ_V^0; it is the sheaf of germs of pairs (Y_1, Y_2) such that: Y_j is a holomorphic vector field on V_j, which is tangent to W_j, along W_j, $j = 1, 2$. Furthermore $Y_1 = Y_2$ under the map $q : V' \to V$ which identifies W_1 and W_2. This shows:

Proposition 4.1.4 *There exists a "Mayer-Vietoris" exact sequence, [8]*

$$0 \to \tau_V^0 \to q_* \Theta_{V', W_1 \amalg W_2} \to i_* \theta_W \to 0$$

Proposition 4.1.5 *[8] Let $\mathcal{S}$ be any coherent sheaf on V', then for any i,*

$$H^i(V, q_* \mathcal{S}) \cong H^i(V_1, \mathcal{S}) \oplus H^i(V_2, \mathcal{S})$$

Proof: By the Leray spectral sequence, is enough to prove that the sheaves $q_{*j} = 0$ on V, for $j \geq 1$. This is clear because if U is a small open set in V, with $U \cap W = \emptyset$, then $q^{-1}(U) \cong U$, which we can assume to be like a small ball in $\mathbb{C}^n$; therefore $(q_{*j} \mathcal{S}) U := H^j(U, \mathcal{S}) = 0$ when $j \geq 1$, by Cartan's theorem B. The other possibility si that $U \cap W \neq \emptyset$, in which case $q^{-1}(U) = U_1 \amalg U_2$ which we can assume to be Stein again, and the same argument applies. □

4.1.3 Cohomology of a blow up manifold

In the general case of twistor spaces, and also in the other case we will need to consider, the manifolds V_1 and V_2 are both blow ups. To study their cohomology, we consider the following general situation.

Let $b : \tilde{M} \to M$ denote the blow up of a complex manifold M along a complex submanifold N, with exceptional divisor the hypersurface $E = b^{-1}(N)$. We will then use the following notations: $m = dim\ \tilde{M} = dim\ M$, $n = dim\ N$, so that $m - n \geq 2$. Now $b_{|E} : E \to N$ is a fiber bundle with fiber $F \cong \mathbb{CP}_r$, $r = m - n - 1$. By $\mathcal{O}(l)$ we will then denote $\mathcal{O}_{\mathbb{CP}_r}(l) = \mathcal{O}_F(l)$. Finally ν will be the normal bundle $\nu_{F/\tilde{M}}$, of any fiber. We start by proving:

Lemma 4.1.6 $\nu \cong \mathcal{O}_{\mathbb{CP}_r}(-1) \bigoplus (\mathcal{O}_{\mathbb{CP}_r})^{\oplus n}$

Proof: ν fits into the exact sequence

$$0 \to \nu_{F/E} \to \nu \to (\nu_{E/\tilde{M}})_{|F} \to 0$$

But, from the (local) definition of blowing up, it is easy to see that $(\nu_{E/\tilde{M}})_{|F} \cong \mathcal{O}_{\mathbb{CP}_r}(-1)$. On the other hand, $\nu_{F/E}$ is trivial of rank n, because F is the fiber of a holomorphic bundle. Therefore

$$0 \to \mathcal{O}^{\oplus n} \to \nu \to \mathcal{O}(-1) \to 0$$

is an exact sequence of vector bundles on $\mathbb{CP}_r$.

But since $H^1(\mathbb{CP}_r, \mathcal{O}(1) \otimes \mathcal{O}^{\oplus n}) = 0$, the sequence must split and we have the result. □

For any $k \geq 0$ we will denote by $S^k(\nu^{-1})$, the k-th symmetric tensor power of ν^{-1}.

Lemma 4.1.7 *$H^i(F, S^k(\nu^{-1})) = 0$ for all $i \geq 1$ and $k \geq 0$.*

Proof: $S^k(\nu^{-1}) = S^k(\mathcal{O}(1) \bigoplus \mathcal{O}^{\oplus n}) =$

$\mathcal{O}(k) \bigoplus [\mathcal{O}(k-1) \otimes \mathcal{O}^{\oplus n}] \bigoplus \cdots \bigoplus [S^{k-1}(\mathcal{O}^{\oplus n}) \otimes \mathcal{O}(1)] \bigoplus S^k(\mathcal{O}^{\oplus n}) =$

$\mathcal{O}(k) \bigoplus \mathcal{O}(k-1)^{\oplus a_{k-1}} \bigoplus \cdots \bigoplus \mathcal{O}(1)^{\oplus a_1} \bigoplus \mathcal{O}^{\oplus a_0}$

for some $1 < a_{k-1} \leq \cdots \leq a_1 \leq a_0$. The result is now clear. □

We can now look at the direct images of the sheaf $\mathcal{O}_{\tilde{M}}$.

Proposition 4.1.8 *$b_{*0}\mathcal{O}_{\tilde{M}} = \mathcal{O}_M$ and $b_{*i}\mathcal{O}_{\tilde{M}} = 0$, for all $i \geq 1$*

Proof: Let $U \subset M$ be a small Stein ball. If $U \cap N = \emptyset$, we can just apply Cartan's theorem B, to prove the assersion, because $b^{-1}(U) \cong U$. Otherwise, $(b_{*0}\mathcal{O}_{\tilde{M}})U = \mathcal{O}_{U \setminus N} = \mathcal{O}_U$ by Riemann's extension theorem, since *codim* $N \geq 2$.

What is left to prove is that for any $p \in N$, the stalk $\mathcal{S}_i := (b_{*i}\mathcal{O}_{\tilde{M}})_p = 0$. To see this we fix p, and let $F = b^{-1}(p)$. Then if $\mathcal{I}$ denotes its ideal sheaf, we can consider $\mathcal{O}_{\tilde{M}}/\mathcal{I}^{k+1}$; this is a sheaf supported on F, and called the *k-th* *order formal neighborhood* of F in $\tilde{M}$. Then one has the exact sequence:

$$0 \to \mathcal{O}(S^k\nu^{-1}) \to \mathcal{O}_{\tilde{M}}/\mathcal{I}^{k+1} \to \mathcal{O}_{\tilde{M}}/\mathcal{I}^k \to 0$$

By 4.1.7, $H^i(\mathcal{O}_{\tilde{M}}/\mathcal{I}^{k+1}) = H^i(\mathcal{O}_{\tilde{M}}/\mathcal{I}^k)$ for all $i \geq 1$, $k \geq 0$. Since $\mathcal{O}_{\tilde{M}}/\mathcal{I}^1 = \mathcal{O}_F \cong \mathcal{O}_{\mathbb{CP}_r}$, this implies that $H^i(\mathcal{O}_{\tilde{M}}/\mathcal{I}^k) = 0$ for all $i \geq 1$, $k \geq 0$. Now, we apply Grothendieck's theorem on formal functions, [10]: it says that if $\hat{\mathcal{S}}_i$ is the completion of the stalk $\mathcal{S}_i$, then $\hat{\mathcal{S}}_i \cong lim_{\leftarrow} H^i(\mathcal{O}_{\tilde{M}}/\mathcal{I}^k)$; therefore $\hat{\mathcal{S}}_i = 0$ by the above. But now, by a theorem of Grauert, $b_{*i}\mathcal{O}_{\tilde{M}}$ is coherent, because b is proper, therefore $\mathcal{S}_i = \hat{\mathcal{S}}_i = 0$ □

We then look at the tangent bundle of $\tilde{M}$.

Proposition 4.1.9 $b_{*0}(\Theta_{\tilde{M}}) = b_{*0}(\Theta_{\tilde{M},E}) = \Theta_{M,N}$

Proof: It is enough to show that for any open set $U \subset M$, which meets N, $H^0(b^{-1}(U), \Theta_{\tilde{M}}) = H^0(b^{-1}(U), \Theta_{\tilde{M},E}) \cong H^0(U, \Theta_{M,N}$. To prove the first equality we notice that from the definition of the normal bundle ν_E, there is an exact sequence of sheaves on $\tilde{M}$:

$$0 \to \Theta_{\tilde{M},E} \to \Theta_{\tilde{M}} \to \nu_E \to 0$$

and we only have to show that $H^0(b^{-1}(U), \nu_E) = H^0(b^{-1}(U \cap N), \nu_E) = 0$. This is clear, because $b^{-1}(U \cap N)$ is foliated by the fibers of b, and $\nu_{E|_F} \cong \mathcal{O}_{\mathbb{CP}_r}(-1)$. It follows that every section of ν_E has to vanish on each fiber, and therefore identically.

Now the second equality easily follows, because any holomorphic vector field on $b^{-1}(U)$ obviously defines a holomorphic vector field on $U \setminus N$, which is tangent to N, and can again be extended to all of N, because *codim* $N \geq 2$. □

Proposition 4.1.10 *The direct image sheaves $b_{*i}(\Theta_{\tilde{M}})$ and $b_{*i}(\Theta_{\tilde{M},E})$ all vanish, when $i \geq 1$.*

Proof: Followig the proof of 4.1.8, we only have to show that $\lim_{\leftarrow} H^i(\Theta_{\tilde{M}}/\, \mathcal{I}^k\Theta_{\tilde{M}}) = 0$ for all $i \geq 1$. In fact, we start by considering the exact sequence

$$0 \to S^k(\nu^{-1}) \otimes \Theta_{\tilde{M}|_F} \to \Theta_{\tilde{M}}/\, \mathcal{I}^{k+1}\Theta_{\tilde{M}} \to \Theta_{\tilde{M}}/\, \mathcal{I}^k\Theta_{\tilde{M}} \to 0$$

and we want to show that,

$$for\ each\ k, i \geq 1, \qquad H^i(F, S^k(\nu^{-1}) \otimes \Theta_{\tilde{M}|_F}) = 0 \tag{4.2}$$

We can indeed identify $\Theta_{\tilde{M}|_F}$, by looking at the following exact sequence of vector bundles on $F \cong \mathbb{CP}_r$:

$$0 \to T\mathbb{CP}_r \to \Theta_{\tilde{M}|_F} \to \nu \to 0$$

is then easy to check, using 4.1.6 and the Euler sequence for $T\mathbb{CP}_r$ [9], that $H^1(F, \nu^{-1} \otimes \Theta_{\tilde{M}|_F}) = 0$ so that the sequence splits and $\Theta_{\tilde{M}|_F} = T\mathbb{CP}_r \bigoplus \nu$.

Now, from 4.1.7 and the Euler sequence again, is clear that equation 4.2 holds. This implies that for any $i \geq 1$, $H^i(\Theta_{\tilde{M}}/\, \mathcal{I}^{k+1}\Theta_{\tilde{M}}) = H^i(\Theta_{\tilde{M}}/\, \mathcal{I}^k\Theta_{\tilde{M}})$

all k. But $H^i(\Theta_{\tilde{M}}/\ \mathcal{I}^1\Theta_{\tilde{M}}) = H^i(F, \Theta_{\tilde{M}|_F}) = 0$ for all $i \geq 1$, and the the first part of the proposition follows. The proof of the remaining part is analogous, but easier. □

As a direct consequence of the Leray spectral sequence and all of the above, we have proved:

Theorem 4.1.11 *[8] If $b : \tilde{M} \to M$ denotes the blow up of a complex manifold M along a complex submanifold N, with exceptional divisor $E = b^{-1}(N)$, then for any $i \geq 0$*

$$H^i(\tilde{M}, \mathcal{O}) = H^i(M, \mathcal{O}) \ \ \textit{and} \ \ H^i(\tilde{M}, \Theta_{\tilde{M}}) = H^i(\tilde{M}, \Theta_{\tilde{M},E}) = H^i(M, \Theta_{M,N})$$

4.2 Application to twistor spaces

In the particular case when $V = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ is a singular twistor space, with Z_i the twistor space of a compact self-dual manifold X_i ; $i = 1, 2$, a "smooth deformation", $d : Z \to \Delta$ with $d^{-1}(0) = Z$, can be given a "real structure", (extending the natural one on Z) and is then called a *standard deformation* in [8]. If $D \subset \Delta$ is the fixed locus of the real structure on Δ, then the main result of [8] is:

Theorem 4.2.1 (Donaldson-Friedman) *If $d : Z \to \Delta$ is a standard deformation with $d^{-1}(0) = Z$ as above, then for each small t, in the real submanifold D of Δ and not contained in the complex hypersurface $\Delta' = \{t_1 =$*

0}, *the complex 3-manifold* $Z_t = d^{-1}(t)$ *is the twistor space of a self-dual metric on* $X_1 \# X_2$.

We now return to our particular twistor space $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$, to apply some of the general theory we illustrated. Recall then that Z_1 is the twistor space of the Hopf surface with its standard conformally flat metric, $M = \mathbb{C}^2_*/(z \sim \lambda z)$ and Z_2 is the twistor space of $\mathbb{CP}_2$. By definition, Z_2 is also the twistor space of the anti-self-dual manifold $\overline{\mathbb{CP}_2}$. We now consider smoothings of Z: in the general case of twistor spaces, remark 4.1.1, together with the results on sections 4.1.2, 4.1.3 give the following: (we will go over the details of a very similar calculations later).

Theorem 4.2.2 *[8] If* $H^2(\Theta_{Z_i}) = 0$, *for* $i = 1, 2$ *then* $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ *admits a "standard deformation" and therefore* $X_1 \# X_2$ *has a self-dual metric.*

As a corollary we have:

Proposition 4.2.3 *The compact four-dimensional manifold* $M \# \mathbb{CP}_2$ *admits self-dual metrics, where* M *is a Hopf surface. While the manifold* $M \# \overline{\mathbb{CP}_2}$ *admits anti-self-dual metrics.*

Proof: If Z_1 is the twistor space of M, we have shown that $H^2(\Theta_{Z_1}) = 0$, 3.2.9. While it is well known that the twistor space of $\mathbb{CP}_2$ is the flag manifold $F_{1,2}$ and $H^2(\Theta_{F_{1,2}}) = 0$ also. This shows that $M \# \mathbb{CP}_2$ is self-dual. But now M is conformally flat, so it is also anti-self-dual as $\overline{\mathbb{CP}_2}$ is. $\square$

Proposition 4.2.4 *For any natural numbers p and q the manifold*

$$(\#_{i=1}^{p} M)(\#_{j=1}^{q} \mathbb{CP}_2)$$

admits self-dual metrics, while

$$(\#_{i=1}^{p} M)(\#_{j=1}^{q} \overline{\mathbb{CP}_2})$$

admits anti-self-dual metrics.

Proof: [8] As in the last theorem, let $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ be a singular twistor space with $H^2(\Theta_{\tilde{Z}_i}) = 0,\ i = 1,2$. Then one proves (see 4.4.3) that $H^2(\tau_2^0) = 0$, under these hypothesis; and by remark 4.1.1 there is a "standard deformation" $d : \mathcal{Z} \to \Delta$ with $d^{-1}(0) = Z$. If $\Theta_{\mathcal{Z}}$ denotes the sheaf of sections of the tangent bundle of $\mathcal{Z}$, we indicate by $(\Theta_{\mathcal{Z}})_t$ the tangent sheaf of the fiber Z_t. In particular when $t = 0, (\Theta_{\mathcal{Z}})_0 = \tau_2^0$. By Grauert's semicontinuity theorem then, $0 = h^2(\tau_2^0) \geq h^2((\Theta_Z)_t)$; it follows that if Z_t is a twistor space of $X_1 \# X_2$,then $h^2(\Theta_{Z_t}) = h^2(\Theta_{Z_1}) = h^2(\Theta_{Z_2}) = 0$. And the theorem can be applied again. □

4.3 The hermitian condition

Now that we have shown that $M\#\overline{\mathbb{CP}_2}$ admits an anti-self-dual metric, we can ask the following question: does the complex surface $N = M\#\overline{\mathbb{CP}_2}$ (i.e. a Hopf surface with a point blown up) admit a hermitian anti-self-dual metric?

To answer this question we look for a holomorphic smooth hypersurface (or effective divisor) S in the twistor space Z_t, with the property that the restriction to S of the twistor map $Z_t \to S$, gives a biholomorphism $S \cong N$.

For this purpose let $b : \tilde{Z}_1 \to Z_1$ be the blowing down map to the twistor space Z_1 of M. Then from the proof of 3.2.2, the map
$p \circ b : \tilde{Z}_1 \to \mathbb{CP}_1 \times \mathbb{CP}_1$ is regular, and since each twistor line in Z_1 meets every element of the families $\{H_u\}_{u \in \mathbb{CP}_1}$ and $\{H^v\}_{v \in \mathbb{CP}_1}$ in exactly one point, the complex 3-manifold $\tilde{Z}_1$ is foliated by two families of Hopf surfaces with a point blown up: $\{\tilde{H}_u\}_{u \in \mathbb{CP}_1}$ and $\{\tilde{H}^v\}_{v \in \mathbb{CP}_1}$ where $\tilde{H}_u = (p_2 \circ b)^{-1}(u)$ and $\tilde{H}^v = (p_2 \circ b)^{-1}(v)$. Furthermore, for any u and v, $\tilde{H}_u$ and $\tilde{H}^v$ are biholomorphic, because H_u and H^v are, and $Aut(M)$ acts transitively on H_u and H^v.

Because of this, let now $\tilde{\Sigma}_1$ denote any element of $\{\tilde{H}_u\}$ or $\{\tilde{H}^v\}$; we notice that since $\tilde{\Sigma}_1 = (p_i \circ b)^{-1}(u)$, $i = 1$ or 2, we have that
$\nu_{\tilde{\Sigma}_1/\tilde{Z}_1} = [\tilde{\Sigma}_1]|_{\tilde{\Sigma}_1} = \mathcal{O}_{\tilde{\Sigma}_1}$.

Proposition 4.3.1

$$H^i(\tilde{\Sigma}_1, \nu_{\tilde{\Sigma}_1/\tilde{Z}_1}) \cong \begin{cases} \mathbb{C} & i = 0, 1 \\ 0 & i \geq 2 \end{cases}$$

Proof: First apply 4.1.11 and then recall 3.2.6. □

We then proceed by looking at $\tilde{Z}_2$. As we said before the twistor space Z_2 of $\mathbb{CP}_2$ is the flag manifold $F_{1,2} = \{([z],[w]) \in \mathbb{CP}_2 \times \mathbb{CP}_2{}^* : \sum_{j=0}^{2} z_j w_j = 0\}$

whose points correspond to pairs (l,π) with l a complex line in $\mathbb{C}^3$ contained in the complex hyperplane π. We start by noticing a hypersurface $\Sigma_2 \subset Z_2$ given by the equation $\{z_0 = 0\}$.

Proposition 4.3.2 *Σ_2 is biholomorphic to $\mathbb{CP}_2$ with a point blown up.*

Proof: Σ_2 is the smooth complex hypersurface of codimension two, contained in $\mathbb{CP}_2 \times \mathbb{CP}_2$ given by $\{z_0 = z_1w_1 + z_2w_2 = 0\}$. But $z_0 = 0$ is just the inclusion $\mathbb{CP}_1 \times \mathbb{CP}_2 \hookrightarrow \mathbb{CP}_2 \times \mathbb{CP}_2$, so that Σ_2 is biholomorphic to the hypersurface $\{z_1w_1 + z_2w_2 = 0\} \subset \mathbb{CP}_1 \times \mathbb{CP}_2$ which is by definition the blow up of $\mathbb{CP}_2$ at the point $[1,0,0]$. □

Now the twistor map $t : F_{1,2} \to \mathbb{CP}_2$ is given by [2], $t : (l,\pi) \mapsto l^\perp$ where $l^\perp$ is the unique complex line in π, perpendicular to l. Notice that t is not holomorphic.

Proposition 4.3.3 *Σ_2 contains a unique twistor line, $L_2 := t^{-1}([1,0,0])$, and it meets any other twistor line in exactly one point. Furthermore if $E \subset \Sigma_2$ denotes the exceptional divisor, $L_2 \cap E = \emptyset$ and in fact $\nu_{L_2/\Sigma_2} \cong \mathcal{O}_{\mathbb{CP}_1}(1)$.*

Proof: A point $l = [v_0, v_1, v_2]$ in $\mathbb{CP}_2$ is a complex line in $\mathbb{C}^3$ and its preimage is the set of pairs $t^{-1}(l)$={(all complex lines $k \subset \mathbb{C}^3$ which are perpendicular to l, hyperplane spanned by k and l)} ≅{all lines contained in the hyperplane perpendicular to l} ≅ $\mathbb{CP}_1$. Now the statements of the proposition can be easily checked by writing the corresponding equations in homogeneous coordinates $[z]$ and $[w]$. □

Remark 4.3.4 A consequence of this is that $\overline{\mathbb{CP}_2}$ admits a complex structure in the complement of a point, $([1,0,0])$.

Remark 4.3.5 The real structure σ of $F_{1,2} = Z_2$ is $\sigma : (l, \pi) \to (\bar{\pi}, \bar{l})$ where for example, $\bar{\pi}$ is the complex line perpendicular to π. In coordinates $\sigma : ([z],[w]) \mapsto ([\bar{w}],[\bar{z}])$. So that σ takes Σ_2 to $\bar{\Sigma}_2 = \{w_0 = 0\}$. The hypersurface $\{w_0 z_0 = 0\}$ is "real" and it either meets a twistor line in two antipodal points or it contains it: $L_2 = \{z_0 = w_0 = 0\}$.

Now let $\tilde{Z}_2$ be the blow up of Z_2 along L_2. $\tilde{Z}_2$ contains a hypersurface $\tilde{\Sigma}_2$ which is the proper transform of Σ_2, that is $\tilde{\Sigma}_2 = b^{-1}(\Sigma_2)$ where $b : \tilde{Z}_2 \to Z_2$ is the blowing down map. However $L_2 \subset \Sigma_2$ has codimension 1, therefore $b|_{\tilde{\Sigma}_2} : \tilde{\Sigma}_2 \to \Sigma_2$ is a biholomorphism. We denote by Q_2 the quadric $b^{-1}(L_2)$. Then $Q_2 \cap \tilde{\Sigma}_2 = b^{-1}|_{\tilde{\Sigma}_2}(L_2) \cong L_2$. Also $\nu_{L_2/\tilde{\sigma}_2} \cong \nu_{L_2/\Sigma_2} \cong \mathcal{O}_{\mathbf{P}_1}(1)$. While $\nu_{L_2/Q_2} \cong \mathcal{O}_{\mathbb{CP}_1}(2)$ of course.

We now construct the singular twistor space $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$.

Let $Q_i \subset \tilde{Z}_i$ be the exceptional quadric, $i = 1,2$ and let $f : Q_2 \to Q_1$ be a biholomorphism as described before; then the twistor line $L_2 \subset Q_2$ is sent to a line $L_1 \subset Q_1$ which is the exceptional divisor of $\tilde{\Sigma}_1 \subset \tilde{Z}_1$, because f interchanges the two factors of the quadric. So that the singular twistor space Z contains a singular hypersurface $\Sigma = \tilde{\Sigma}_1 \cup_L \tilde{\Sigma}_2$ which is smooth everywhere except along the line $L := L_1 \cong L_2$ where it has a normal crossing.

4.4 Smoothings of the hypersurface Σ

In this section we show that Σ can be smoothed out and that any of its smooth deformations is biholomorphic to a Hopf surface with a point blown up, which is the kind of surface we want. This is proved by means of the general theory outlined in section 4.1.1, and some theorems of Kodaira on compact complex surfaces.

First notice that the normal bundle of L in Σ is trivial:

$$\nu_L = \nu_{L_1/\tilde{\Sigma}_1} \otimes \nu_{L_2/\tilde{\Sigma}_2} \cong \mathcal{O}_{\mathbb{CP}_1}(-1) \otimes \mathcal{O}_{\mathbb{CP}_1}(1) = \mathcal{O}_{\mathbb{CP}_1},$$

so that Σ satisfies the d-semistable condition and we can consider the exact cohomology sequence, cfr. sequence 4.1:

$$0 \to H^1(\tau^0_\Sigma) \to T^1_\Sigma \to H^0(L, \mathcal{O}) \to H^2(\tau^0_\Sigma) \to T^2_\Sigma \to H^1(L, \mathcal{O}) = 0$$

to analyze the terms of this sequence, recall that

$$0 \to \tau^0_\Sigma \to q_* \Theta_{\Sigma', L'} \to i^* \Theta_L \to 0$$

is an exact sequence for the sheaf of derivations of $\mathcal{O}_\Sigma$, where we denoted by Σ' the normalization $\tilde{\Sigma}_1 \coprod \tilde{\Sigma}_2$ of Σ, and $L' = L_1 \coprod L_2$. By 4.1.5 we then have to look at the cohomology of $\Theta_{\tilde{\Sigma}_i,L_i}$ $i = 1,2$. And by 4.1.11: $H^j(\Theta_{\tilde{\Sigma}_i,L_i}) \cong H^j(b_{0*}\Theta_{\tilde{\Sigma}_i,L_i})$ for any j; $i = 1,2$.

Proposition 4.4.1

$$H^j(\tilde{\Sigma}_1, \Theta_{\tilde{\Sigma}_1,L_1}) \cong \begin{cases} \mathbb{C}^2 & j = 0 \\ \mathbb{C}^4 & j = 1 \\ 0 & j \geq 2 \end{cases}$$

Proof: By the last proposition we have to compute the cohomology of $\Theta_{\Sigma_1,p}$: the sheaf of holomorphic vector fields on $\Sigma_1 \cong M$, which vanish at a point p. This fits into the exact sequence

$$0 \to \Theta_{\Sigma_1,p} \to \Theta_{\Sigma_1} \to T_p\Sigma_1 \to 0$$

where $T_p\Sigma_1$ is the "skyscraper" sheaf given by the tangent space at p, so that $H^0(T_p\Sigma_1) \cong \mathbb{C}^2$ and $H^j(T_p\Sigma_1) = 0$ for $i \geq 1$. Now $H^0(\Theta_{\Sigma_1,p})$ is the Lie algebra of the group of automorphisms of Σ_1 fixing p, this is given by matrices of the form $\begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \in GL(2,\mathbb{C})$, so that $H^0(\Theta_{\Sigma_1,p}) \cong \mathbb{C}^2$. The proposition then follows from 3.2.8, and the exact sequence above. □

Proposition 4.4.2

$$H^j(\tilde{\Sigma}_2, \Theta_{\tilde{\Sigma}_2,L_2}) \cong \begin{cases} \mathbb{C}^4 & j = 0 \\ 0 & j \geq 0 \end{cases}$$

Proof: First recall that $\tilde{\Sigma}_2$ is the blow up of $\mathbb{CP}_2$ at a point p not in the line L_2, and that $\nu_{L/\mathbb{CP}_2} \cong \mathcal{O}_{\mathbb{CP}_1}$. Therefore we have

$$H^j(\tilde{\Sigma}_2, \Theta_{\tilde{\Sigma}_2,L_2}) \cong H^j(\mathbb{CP}_2, \Theta_{\mathbb{CP}_2,L_2,p})$$

and two exact sequences:

$$0 \to \Theta_{\mathbb{CP}_2,L_2} \to \Theta_{\mathbb{CP}_2} \to \mathcal{O}_{\mathbb{CP}_1} \to 0$$

$$0 \to \Theta_{\hat{\Sigma}_2,L_2,p} \to \Theta_{\hat{\Sigma}_2,L_2} \to T_p\mathbb{CP}_2 \to 0$$

Now, the automorphisms of $\mathbb{CP}_2$ fixing a line are

$$\{\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \in GL(3,\mathbb{C})\} \,/\, \lambda I$$

so that $H^0(\Theta_{\hat{\Sigma}_2,L_2}) \cong \mathbb{C}^6$ and similarly $H^0(\Theta_{\hat{\Sigma}_2,L_2,p}) \cong \mathbb{C}^4$. The result then follows by recalling $H^i(\Theta_{\mathbb{CP}_2}) \cong \begin{cases} \mathbb{C}^8 & i = 0 \\ 0 & i \neq 0 \end{cases}$ □

Corollary 4.4.3

$$H^j(\tau^0_\Sigma) \cong \begin{cases} \mathbb{C}^3 & j = 0 \\ \mathbb{C}^4 & j = 1 \\ 0 & j \geq 2. \end{cases}$$

Proof: From 4.1.4 we get an exact sequence

$$0 \to H^0(\tau^0_\Sigma) \to H^0(\Theta_{\hat{\Sigma}_1,L_1}) \oplus H^0(\Theta_{\hat{\Sigma}_2,L_2}) \to$$

$$\to H^0(\Theta_L) \to H^1(\tau^0_\Sigma) \to H^1(\Theta_{\hat{\Sigma}_1,L_1}) \to 0$$

now, it is easy to see that the map $H^0(\Theta_{\hat{\Sigma}_2,L_2}) \to H^0(\Theta_L)$ is onto, by looking at a similar map $H^0(\Theta_{\mathbb{CP}_2,L}) \to H^0(\Theta_L)$ where $L \hookrightarrow \mathbb{CP}_2$ is a linear embedded $\mathbb{CP}_1$. Then recall that $H^0(\Theta_L) \cong \mathbb{C}^3$ and apply the last two propositions to get the thesis. □

Corollary 4.4.4 *For the Ext groups of* Σ *we have:*

$$T^i_\Sigma \cong \begin{cases} \mathbb{C}^5 & i = 1 \\ 0 & i = 2 \end{cases}$$

Proof: A direct consequence of the the exact sequence 4.1 □

As a consequence of this, see remark 4.1.1, we have a "smooth deformation" $d : \mathcal{E} \to \Delta$ of Σ, with smooth total space $\mathcal{E}$ over a neighborhood of the origin in $\mathbb{C}^5$; furthermore there is a complex hypersurface $\Delta' \subset \Delta$ such that $d^{-1}(t)$ is smooth for $t \notin \Delta'$. By restricting this deformation to a smooth complex curve $\Gamma \subset \Delta$ with the properties that $0 \in \Gamma$ and Γ is transverse to Δ', we get a smooth deformation $c : \mathcal{D} \to \Gamma$ such that $c^{-1}(0) = \Sigma$ and $c^{-1}(t) = \Sigma_t$ is a smooth compact complex surface for any t.

With the next two propositions we will then prove, using two theorems of Kodaira, that such a Σ_t is always biholomorphic equivalent to a Hopf surface with a point blow up.

Proposition 4.4.5 *For any smooth deformation* $c : \mathcal{D} \to \Gamma$ *as above and for any* $t \neq 0$, *the surface* Σ_t *is homeomorphic to* $(S^1 \times S^3)\#\overline{\mathbb{CP}_2}$.

Proof: This is a consequence of a construction of Kodaira. In [15], he shows that in our situation, which he calls a singular surface with an ordinary double curve of the first kind, every smoothing Σ_t of $\Sigma = \tilde{\Sigma}_1 \cup_L \tilde{\Sigma}_2$ is topologically obtained as follows. Consider first $L \subset \tilde{\Sigma}_1$ and let U be the manifold with boundary, gotten by removing from $\tilde{\Sigma}_1$, a tubular neighborhood of L (namely the bundle of normal vectors of *length* $\leq \epsilon$). Since $\nu_{L/\tilde{\Sigma}_1} \cong \mathcal{O}_{\mathbb{CP}_{(1)}}(-1)$, the boundary ∂U is homeomorphic to S^3 in our case, and U is homeomorphic to Σ_1, with a ball B_1 removed. So that $\partial B_1 = \partial U \cong S^3$; the reason for this is that topologically we can think of $\tilde{\Sigma}_1$ as being constructed by removing B_1 from Σ_1 and replace it by a neighborhood of the zero section of the line bundle $\mathcal{O}_{\mathbb{CP}_1}(-1)$.

Applying the same transformation to $\tilde{\Sigma}_2$, yields a manifold V with boundary $\partial V \cong S^3$, where V is homeomorphic to $\overline{\mathbb{CP}_2}$ with a ball B_2 removed, so that $\partial B_2 = \partial V$. The smooth surface Σ_t is then given by identifying U and V by means of an orientation reversing diffeomorphism $\partial U \leftrightarrow \partial V$. Therefore $\Sigma_t \cong (S^1 \times S^3)\#\overline{\mathbb{CP}_2}$, topologically. □

Proposition 4.4.6 *With the same hypothesis as before, Σ_t is biholomorphic to a Hopf surface with a point blown up.*

Proof: We start by recalling a different theorem of Kodaira, [14]. Let N be a compact complex submanifold of a (non necessarily compact) complex manifold M, such that $H^1(\nu_{N/M}) = 0$. Then N is stable in M, i.e. N survives in any deformation of M.

We will use this theorem to show that Σ_t contains a $(-1) -$ *curve*, i.e. a copy of $\mathbb{CP}_1$ with self-intersection (-1). With this in mind, let $M \subset \Sigma$ be

$U_{1\,|L} = U_{2\,|L}$, and we need to show that

$$(\nu_{\tilde{\Sigma}_1/\tilde{Z}_1})_{|L} = (\nu_{\tilde{\Sigma}_2/\tilde{Z}_2})_{|L} = \mathcal{O}_L$$

First 4.3.1, $\nu_{\tilde{\Sigma}_1/\tilde{Z}_1} = \mathcal{O}_{\tilde{\Sigma}_1}$ so that $(\nu_{\tilde{\Sigma}_1/\tilde{Z}_1})_{|L} = \mathcal{O}_L \cong \mathcal{O}_{\mathbb{CP}_1}$. To show the corresponding statement for $\tilde{\Sigma}_2$, recall that, as for any proper transform, $[\tilde{\Sigma}_2] = b^*[\Sigma_2] \otimes [Q]^{-1}$; therefore $\nu_{\tilde{\Sigma}_2/\tilde{Z}_2})_{|L} = [\tilde{\Sigma}_2]_{|L} = b^*[\Sigma_2]_{|L} \otimes [Q]^{-1}_{|L}$ by the adjunction formula; since of course $[Q]^{-1}_{|L} \cong \mathcal{O}_{\mathbb{CP}_1}(-1)$ we only need to show that $(b^*[\Sigma_2])_{|L} \cong \mathcal{O}_{\mathbb{CP}_1}(1)$; but $b : \tilde{\Sigma}_2 \to \Sigma_2$ is a biholomorphism, so that $(b^*[\Sigma_2])_{|L} \cong [\Sigma_2]_{|L}$. Now, by embedding everything in $\mathbb{CP}_2 \times \mathbb{CP}_2$, is easy to see that $[\Sigma_2]_{|L} \cong \mathcal{O}_{\Sigma_2}(1,0)_{|L} \cong \mathcal{O}_L(1)$ as wanted. □

Corollary 4.5.2

$$H^i(\Sigma, \nu_{\Sigma/Z}) \cong \begin{cases} 0 & \textit{for } i = 0,2 \\ \mathbb{C} & i = 1 \end{cases}$$

Proof: We will use the long exact sequence given by the last proposition. From proposition 4.1.5 $H^i(q_*\nu_{\Sigma'/Z'}) = H^i(\nu_{\tilde{\Sigma}_1/\tilde{Z}_1}) \oplus H^i(\nu_{\tilde{\Sigma}_2/\tilde{Z}_2})$. Recall, 4.3.1, that

$$H^i(\nu_{\tilde{\Sigma}_1/\tilde{Z}_1}) \cong H^i(\mathcal{O}_{\tilde{\Sigma}_1}) \cong \begin{cases} \mathbb{C} & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

To compute the cohomology of $\nu_{\tilde{\Sigma}_2/\tilde{Z}_2}$ instead, we use the exact sequence

$$0 \to \nu_{\tilde{\Sigma}_2} \to b^*\nu_{\Sigma_2} \to (b^*\nu_{\Sigma_2})_{|L} \to 0 \tag{4.3}$$

gotten by restricting to $\tilde{\Sigma}_2$ the isomorphism $[\tilde{\Sigma}_2] \cong b^*[\Sigma_2] \otimes [Q]^{-1}$. Now $H^i(b^*\nu_{\Sigma_2}) \cong H^i(\nu_{\Sigma_2})$ by 4.1.11; but from the last proof, $\nu_{\Sigma_2} = [\Sigma_2]_{|\Sigma_2} =$

$\mathcal{O}_{\Sigma_2}(1,0)$, which cohomology can easily be computed by using the embedding of Σ_2 into $\mathbb{CP}_1 \times \mathbb{CP}_2$. This indeed yields

$$H^i(b^*\nu_{\Sigma_2}) \cong \begin{cases} \mathbb{C}^2 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Since we already know that $(b^*\nu_{\Sigma_2})_{|L} \cong \mathcal{O}_{\mathbb{CP}_1}(1)$, the last ingredient we need is the observation that the restriction map

$$H^0(\Sigma_2, \mathcal{O}_{\Sigma_2}(1,0)) \to H^0(L, \mathcal{O}_{\Sigma_2}(1,0))$$

is an isomorphism; this can be easily checked by embedding everything in $\mathbb{CP}_1 \times \mathbb{CP}_2$. As a consequence, the exact sequence from 4.3 :

$$0 \to H^0(\nu_{\tilde{\Sigma}_2}) \to H^0(b^*\nu_{\Sigma_2}) \to H^0((b^*\nu_{\Sigma_2})_{|L}) \to H^1(\nu_{\tilde{\Sigma}_2}) \to 0$$

implies that $H^i(\nu_{\tilde{\Sigma}_2}) = 0$ for any i.

We are now in a position to use the long exact sequence from the last proposition:

$$0 \to H^0(\nu_\Sigma) \to H^0(\nu_{\tilde{\Sigma}_1}) \to H^0(\mathcal{O}_L) \to H^1(\nu_\Sigma) \to H^1(\nu_{\tilde{\Sigma}_1}) \to 0$$

and the proof is now complete because the restriction map $H^0(\nu_{\tilde{\Sigma}_1}) \to H^0(\mathcal{O}_L)$ is obviously an isomorphism of $\mathbb{C}$. □

The fact that $H^1(\Sigma, \nu_{\Sigma/Z}) \neq 0$, says that there might be obstructions to the "survival" of Σ when Z is deformed to a smooth twistor space Z_t, [14]. Because of this, we try a different approach, which, by the way, will not work either.

We consider $\Sigma \subset Z$ to be a divisor, and we indicate by F the corresponding line bundle $[\Sigma]$ over Z. Using the last corollary we can easily compute its cohomology; but we first need:

Proposition 4.5.3

$$H^i(Z,\mathcal{O}) \cong \begin{cases} \mathbb{C} & \textit{when } i = 0,1 \\ 0 & \textit{for } i \geq 2 \end{cases}$$

Proof: From proposition 4.1.3, the sequence

$$0 \to \mathcal{O}_Z \to q_{*0}\mathcal{O}_{Z'} \to i_{*0}\mathcal{O}_Q \to 0 \tag{4.4}$$

is exact, while $H^j(i_{*0}\mathcal{O}_Q) = 0$ unless $j = 0$, and $H^i(\mathcal{O}_{Z'}) = H^i(\mathcal{O}_{\tilde{Z}_1}) \oplus H^i(\mathcal{O}_{\tilde{Z}_2}) \cong H^i(\mathcal{O}_{Z_1}) \oplus H^i(\mathcal{O}_{Z_2})$, by 4.1.11. Finally, by the twistor correspondence, $h^1(\mathcal{O}_{Z_2}) = b_1(\mathbb{CP}_2) = 0$, $h^2(\mathcal{O}_{Z_2}) = b_2^+(\mathbb{CP}_2) = 0$, and the proposition follows from 4.4 □

Proposition 4.5.4

$$H^i(Z,F) \cong \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C}^2 & i = 1 \\ 0 & i \geq 2 \end{cases}$$

in particular $\chi(Z,F) = -1$

Proof: Since $F = [\Sigma]$, $F_{|\Sigma} = \nu_\Sigma$ and we have an exact sequence of sheaves

$$0 \to \mathcal{O}_Z \to F \to \nu_{\Sigma/Z} \to 0$$

whose induced long exact sequence easily implies the result. □

One can also define a "real" line bundle $F\overline{F}$ on Z, by letting $\overline{F}$ be the line bundle associated to the divisor $\overline{\Sigma} := \sigma(\Sigma)$, where σ is the real structure of Z. First $H^i(Z,\overline{F}) \cong H^i(Z,F)$, because $\sigma^*F = \overline{F}$; then we use the exact sequence

$$0 \to \overline{F} \to F\overline{F} \to (F\overline{F})_{|\Sigma} \to 0$$

where, since $\Sigma \cap \overline{\Sigma} = \emptyset$, $(F\overline{F})_{|\Sigma} = F_{|\Sigma} = \nu_{\Sigma/Z}$, we have $H^0((F\overline{F})_{|\Sigma} = 0$; the resulting exact sequence easily implies

Proposition 4.5.5

$$H^i(Z,F\overline{F}) \cong \begin{cases} \mathbb{C} & i=0 \\ \mathbb{C}^3 & i=1 \\ 0 & \textit{otherwise} \end{cases}$$

in particular $\chi(Z,F\overline{F}) = -2$

We now pass to the problem of extending a holomorphic line bundle on Z, to the total space of a deformation. By corollary 4.4.4, using a reasoning analougus to that of section 4.1.1, we can consider a "smooth deformation" of Z, that is a family $d : \mathcal{Z} \to \Delta$ with the following properties:

1. Δ is a smooth, simply connected neighborhood of the origin in $\mathbb{C}$
2. $\mathcal{Z}$ is a smooth complex four dimensional manifold
3. $d^{-1}(0) = Z$ and for every $t \in \Delta$, $d^{-t}(0) = Z_t$ is a smooth complex three dimensional manifold

4. Z_t is a twistor space of $M\#\overline{\mathbb{CP}_2}$, for any t in a "real " submanifold of Δ

The following proposition is useful for extending a line bundle on Z, to $\mathcal{Z}$.

Proposition 4.5.6 *For any family $d : \mathcal{Z} \to \Delta$ as above,*

$$H^i(\mathcal{Z}\ ,\mathcal{O}) \cong \begin{cases} H^0(\mathcal{O}_\Delta) & \text{for } i = 0,1 \\ 0 & i \geq 2 \end{cases}$$

Proof: By the Leray spectral sequence is enough to prove that

$$d_{*q}\mathcal{O}_{\mathcal{Z}} \cong \begin{cases} \mathcal{O}_\Delta & \text{when } q = 0,1 \\ 0 & \text{when } q \geq 2 \end{cases}$$

To show this we use a theorem of Grauert [10, p288] which says that $d_{*q}\mathcal{O}_{\mathcal{Z}}$ is locally free when $h^q(Z_t, \mathcal{O})$ is constant in t. We shoved in 4.5.3 that

$$h^q(Z_0, \mathcal{O}) = \begin{cases} 1 & q = 0,1 \\ 0 & q \geq 2 \end{cases} \quad \textit{so that} \ \ \chi(Z_0, \mathcal{O}) = 0$$

By Grauert's semi-continuity theorem then, $h^q(Z_t, \mathcal{O}) = 0$ for any $q \geq 2$ and $\chi(Z_t, \mathcal{O}) = 0$, for any t. Since Z_t is compact, $h^0(Z_t, \mathcal{O}) = 1$, forcing $h^1(Z_t, \mathcal{O}) = 1$ for all T. We conclude that the Hodge numbers $h^q(Z_t, \mathcal{O})$ are constant in t, while $d_{*q}\mathcal{O}_{\mathcal{Z}}$ is locally free fo rank 1 when $q = 0,1$ and it is 0 for $q \geq 2$. Since Δ is simply connected, we have the assertion. □

Corollary 4.5.7 *Every holomorphic line bundle on Z extends to a holomorphic line bundle on $\mathcal{Z}$. But not in a unique way.*

Proof: Since Z is defined by the global equation $d = 0$, the ideal sheaf of Z is trivial: $\mathcal{I}_Z \cong \mathcal{O}_{\mathcal{Z}}$. Then, it follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \xrightarrow{exp} \mathcal{O}^*_{\mathcal{Z}} \xrightarrow{r} \mathcal{O}^*_Z \longrightarrow 0$$

that the following sequence is also exact

$$0 \to H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to H^1(\mathcal{Z}, \mathcal{O}^*_{\mathcal{Z}}) \to H^1(Z, \mathcal{O}^*_Z) \to H^2(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$$

and by the previous proposition the restriction : $Pic(\mathcal{Z}) \to Pic(Z)$ is onto but not injective. □

This result then assures that the divisor line bundle $F = [\Sigma]$ on Z, extends to a non-unique line bundle $\mathcal{F}$ on $\mathcal{Z}$. The problem of finding a hermitian anti-self-dual metric on the blow up of M, is then reduced to finding a holomorphic section of $\mathcal{F}$, because this would imply that $h^0(Z_t, \mathcal{F}_{|Z_t}) \neq 0$, for any twistor space Z_t and any $0 < t < \epsilon$.

Unfortunately this last statement can not be proved by using just Grauert's semicontinuity theorem, because $\chi(Z, F) = -1$.

Another application of the last corollary is the following: we can extend the real line bundle $F\overline{F}$ to a line bundle $\mathcal{F}\overline{\mathcal{F}}$ on $\mathcal{Z}$. If we then restrict to a twistor space Z_t, the holomorphic line bundle $\mathcal{F}\overline{\mathcal{F}}_{|Z_t}$ is "real" and has Chern class equal to $\frac{1}{2}c_1(Z_t)$. The Riemann-Roch formula then gives: $\chi(Z_t, \mathcal{F}\overline{\mathcal{F}}_{|Z_t}) = -2$, which agrees with 4.5.5.

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