Vanishing Theorems for Quaternionic Kähler Manifolds

A Dissertation Presented

by

Mark Thornber

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1989

State University of New York
at Stony Brook

The Graduate School

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Graduate School

Abstract of the Dissertation

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In this thesis we give a corrected version of a Weitzenböck formula originally due to Griffiths. We use this formula to establish vanishing of certain cohomology groups on the twistor space of a compact quaternionic Kähler manifold with negative scalar curvature. Specifically: if L is the line bundle associated to the complex contact structure on such a twistor space, $\mathcal{Z}$, we show that $H^1(\mathcal{Z}, \mathcal{O}(L^{\otimes n})) = 0$ for $n \geq 1$. Furthermore if L has a square root bundle, $L^{\frac{1}{2}}$, we show that $H^1(\mathcal{Z}, \mathcal{O}(L^{\frac{1}{2}})) = 0$. Finally we use these vanishing theorems to show that quaternionic Kähler metrics with negative scalar curvature on a compact man-

ifold have no one-parameter deformations through quaternionic Kähler metrics.

To my parents

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Acknowledgements

Thanks are due to many people.

First to my family for their unwavering support.

Next, I owe a deep debt of gratitude to Claude LeBrun for his time and encouragement, without his guidance this work could not have been completed. Most of all I thank him for believing in me, even when I didn't believe in myself.

I'd like to thank Massimiliano Pontecorvo, our conversations helped to form my ideas in many ways.

Brian Steer introduced me to much of mathematics and it was at his suggestion that I came to Stony Brook. The wisdom of his advice becomes more apparent with each passing year.

My friendship with Shivaji Sondhi made living conditions bearable for more than four years. Our conversations are always a delight.

Fortunately my friends here at Stony Brook are too numerous for individual mention, I thank you all. Special mention is, however, owed to Jim Anderson, Ara and Sue Basmajian, Deborah DeWitt, David Groisser, Gerry Palisi, Efton and Molly Park, Larry Polin, and Ralf Spatzier.

Chapter 1

Introduction

In this chapter we summarize some of the definitions and results concerning quaternionic-Kähler manifolds and self-dual four-manifolds which will be useful in what is to come. In some respects these theories parallel one another so that in subsequent chapters theorems we prove for quaternionic-Kähler manifolds will be true for self-dual Einstein four-manifolds with only minor modifications.

1.1 Self-Dual Einstein Four-Manifolds

Using the decomposition $SO(4) \cong SU(2) \times_{\mathbb{Z}_2} SU(2)$ we may construct, at least locally, on any Riemannian four-manifold, M, two $\mathbb{C}^2$-bundles, $\mathbb{S}_+$ and $\mathbb{S}_-$, in the following way. Let P be the frame bundle of M, this is a principal

$SO(4)$ bundle which we may locally lift to a principal $SU(2)\times SU(2)$ bundle, $\tilde{P}$. Now let $\mathbb{S}_+$ and $\mathbb{S}_-$ be the bundles associated to $\tilde{P}$ by the standard representations of the first and second factors of the fibre $SU(2)\times SU(2)$ on $\mathbb{C}^2$. Note that these "spin-bundles" are quaternionic in the sense that they possess antilinear involutions whose squares are -1, and their determinant bundles, $\bigwedge^2\mathbb{S}_+$ and $\bigwedge^2\mathbb{S}_-$, have canonical trivializations ε and $\bar{\varepsilon}$ respectively. From this we see that $\mathbb{C}T^*M = \mathbb{S}_+\bigotimes\mathbb{S}_-$, and we may identify the metric on M as

$$g=\varepsilon\otimes\bar{\varepsilon}\in\Gamma(\bigwedge^2\mathbb{S}_+\bigotimes\bigwedge^2\mathbb{S}_-)\subset\Gamma(\bigodot^2T^*M). \tag{1.1}$$

So far this is standard for any riemannian four-manifold. The case of self-dual or anti-self-dual conformal curvature is interesting in that there is then a natural complex three-manifold, $\mathcal{Z}$, associated to M, constructed from spaces of almost complex structures on TM. Let us therefore consider some of the ways of producing these spaces. For details the reader is referred to [3], [4], [16], [2], or any of the many other treatments available. We concentrate on the role of $\mathbb{S}_+$, and give three essentially equivalent constructions. All of these take place at a point $x\in M$.

1. Notice that $\bigodot^2\mathbb{S}_{+x}$ acts on $\mathbb{C}T_xM$ in the following way. Let lowercase indices denote TM and use primed and unprimed uppercase indices for $\mathbb{S}_-$ and $\mathbb{S}_+$ respectively, then, if $\beta\in\bigodot^2\mathbb{S}_{+x}$, define $\beta(v)^{AC'}=\beta^A{}_Bv^{BC'}$,

where indices are raised and lowered with ε. If we restrict to the real slices and only consider elements of $\bigodot^2 \mathbb{S}_{+x}$ of length $\sqrt{2}$ we see that these are almost complex structures compatible with the metric and orientation. Thus we get a two-sphere of such almost complex structures, $S_{\sqrt{2}}(\Re(\bigodot^2 \mathbb{S}_{+x}))$.

2. Given $\alpha \in \mathbb{S}_{+x}$ we get a map $\mathbb{C}T_xM = \mathbb{S}^*_{+x} \otimes \mathbb{S}^*_{-x} \to \mathbb{S}^*_{-x}$, given by contraction in the first slot with α. Restricting to the real slice, T_xM, gives an isomorphism $T_xM \cong \mathbb{S}^*_{-x}$ if $\alpha \neq 0$. The almost complex structure produced in this way depends only on the projective class of α, so that the space $\mathbb{P}(\mathbb{S}_{+x})$ again gives a two sphere of almost complex structures at x, each of which is compatible with the metric and orientation on M.

3. This is identical to method 1 but we may hide the role of spinors. Notice that $\bigwedge^2 \mathbb{C}T^*M = (\bigodot^2 \mathbb{S}_+ \otimes \bigwedge^2 \mathbb{S}_-) \oplus (\bigwedge^2 \mathbb{S}_+ \otimes \bigodot^2 \mathbb{S}_-)$. Using ε and $\bar{\varepsilon}$ we may think of this as $\bigwedge^2 \mathbb{C}T^*M = \bigodot^2 \mathbb{S}_+ \oplus \bigodot^2 \mathbb{S}_-$. On the level of real slices this is exactly the decomposition of $\bigwedge^2 T^*M$ into self-dual and anti-self-dual two-forms. That is $\bigwedge^2 T^*M = \bigwedge^2_+ \oplus \bigwedge^2_-$ where the two spaces on the right of this equation are the $+1$ and -1 eigenspaces of the Hodge star operator, $* : \bigwedge^2 T^*M \to \bigwedge^2 T^*M$. Having said all this we now note that, given a self-dual two-form ω_{ab}, of length $\sqrt{2}$,

we get an almost complex structure J_a^b by raising an index of ω with the metric g. Again we get a two-sphere of almost complex structures compatible with the metric and orientation, this time it is $S_{\sqrt{2}}(\Lambda^2_+)$.

In all three cases above we get S^2-bundles over M which we choose to call $\mathcal{Z}$, since they are all equivalent. In the case of a four-manifold we see from general principles that $\mathcal{Z}$ parametrizes all of the almost complex structures compatible with the metric and orientation. The manifold $\mathcal{Z}$ carries a natural almost complex structure described in the following way. At a point $z \in \mathcal{Z}$ use the connection on M to split the tangent space into a vertical space tangent to the fibre and a horizontal space isomorphic to $T_{\pi(z)}M$, where π is the projection, $\pi : \mathcal{Z} \to M$. The vertical space has a natural almost complex structure, in one case it is a copy of $\mathbb{CP}_1$ and in the other two it is a metric two-sphere. On the horizontal space we put the almost complex structure which is z.

In considering the construction of the space $\mathcal{Z}$ it should not be too surprising that the relationship between curvature and the decomposition $\Lambda^2 T^*M = \Lambda^2_+ \oplus \Lambda^2_-$ plays an important role in the integrability of the almost complex structure defined above. To summarize: the Riemann curvature tensor can be interpreted as an operator $\mathcal{R} : \Lambda^2 TM \to \Lambda^2 TM$ so that, with respect to the above decomposition, we may write $\mathcal{R} \in \odot^2(\Lambda^2_+ \oplus \Lambda^2_-)$.

As a matrix we have $\mathcal{R} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. This gives a complete decomposition of $\mathcal{R}$ into irreducible components [18], $\mathcal{R} = (trA, B, A - \frac{1}{3}trA, C - \frac{1}{3}trC)$. We may identify these components as; $trA = trC = \frac{1}{4}R$ where R is the scalar curvature, B is the trace-free Ricci tensor, and the last two components are the self-dual and anti-self-dual parts of the Weyl curvature, usually denoted by W_+ and W_-, respectively. In general the Weyl curvature W is exactly that part of $\mathcal{R}$ which is invariant under conformal rescalings of the metric, W is irreducible in dimensions bigger than four.

Definition 1.1.1 An oriented riemannian four-manifold is called self dual if $W_- = 0$ and anti-self-dual if $W_+ = 0$.

It should be noted that the isomorphism class of the almost complex manifold $\mathcal{Z}$ depends only on the conformal class of the riemannian manifold (M, g), so that it is natural to expect that integrability of $\mathcal{Z}$ is related to the conformal class of g. In fact this is so and the important result is the following [3]:

Theorem 1.1.2 (Atiyah-Hitchin-Singer) *The almost complex manifold $\mathcal{Z}$ described above is integrable if and only if $W_+ = 0$.*

Remark 1.1.3 If $W_- = 0$ instead, then since reversing the orientation of M has the effect of reversing the roles of $\mathbb{S}_+$ and $\mathbb{S}_-$ and of W_+ and W_-, we see that an analogous almost complex structure on $\mathbb{P}(\mathbb{S}_-)$ is integrable instead. Thus any half conformally flat four manifold, M, has an associated complex manifold $\mathcal{Z}$, usually called the *twistor space* of M.

Since the twistor space will play an important part in our later results we summarize some of the salient structures here. First note that the fibres of the projection $\pi : \mathcal{Z} \to M$ are holomorphically embedded $\mathbb{P}_1$'s, and that each of these "twistor lines" has normal bundle $\nu \cong \mathcal{O}(1) \bigoplus \mathcal{O}(1)$. Furthermore $\mathcal{Z}$ admits a real structure, that is: an antiholomorphic map, $\sigma : \mathcal{Z} \to \mathcal{Z}$, such that $\sigma^2 = id$, in this case σ is given by the antipodal map on the fibres. In fact these two properties allow us to reconstruct the conformal class of M via the following construction.

Theorem 1.1.4 *Let $\mathcal{Z}$ be a complex manifold foliated by holomorphically embedded $\mathbb{P}_1$'s with normal bundles $\mathcal{O}(1) \bigoplus \mathcal{O}(1)$ and suppose $\mathcal{Z}$ has a real structure σ, with no fixed points, so that the leaves of this foliation are invariant under σ. Then we may construct a half conformally flat, conformal riemannian manifold M, so that $\mathcal{Z}$ is the twistor space of M. If we begin* *with the twistor space of N this reconstructs N.*

The details of this construction will be important to us later, however since the construction in the case of a quaternionic-Kähler-manifold is very similar and will be described in the next section we will be content with a very brief sketch here. The main point is the deformation theory of Kodaira [12] which asserts that, if $\mathcal{Z}$ admits a holomorphically embedded $\mathbb{P}_1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ then in fact this is one of a complex four-dimensional family of such curves. If $\mathcal{M}$ denotes this family we may construct local bundles, $\mathbb{S}_+$ and $\mathbb{S}_-$, over $\mathcal{M}$ and use these to define a complex conformal riemannian metric r on $\mathcal{M}$ via equation 1.1. If we let M denote those curves in the family $\mathcal{M}$ invariant under σ, then since these foliate $\mathcal{Z}$ we see that the restriction of r to M gives a riemannian conformal structure. We may then check that $\mathcal{Z}$ is the twistor space of M.

It is part of the general philosophy of, and motivation for twistor theory that geometric objects on M are related to holomorphic objects on $\mathcal{Z}$. An example of this is the Ward correspondence [3]

Theorem 1.1.5 *There is a bijection between bundles on M with self-dual connection and holomorphic bundles on $\mathcal{Z}$ which are trivial along the fibres.*

More generally there is a program, going under the name of the Penrose transform, which relates cohomology of vector bundles on $\mathcal{Z}$ to solutions of geometric partial differential operators on M, see for example [9], [6].

The case of most interest to us is the structure on $\mathcal{Z}$ corresponding to an Einstein metric in the conformal class of g on M. It turns out that this is a complex contact structure described in the following way: Given an explicit metric in the conformal class of g let D be the corresponding horizontal distribution, $D \subset T\mathcal{Z}$. D is a distribution of complex hyperplanes transverse to the vertical space $V \subset T\mathcal{Z}$, so that $L = T\mathcal{Z}/D$ restricts to $\mathcal{O}(2)$ on each fibre. Let θ be the canonical projection $\theta : T\mathcal{Z} \to L$. The obstruction to integrability of D is then $\theta \wedge \partial\theta$. It turns out that if M is Einstein with non-zero scalar curvature then $\theta \wedge \partial\theta \in H^0(\mathcal{Z}, \mathcal{O}(\kappa \otimes L^{\otimes 2}))$ is never zero so that D is a complex contact structure on $\mathcal{Z}$. If M is Einstein with scalar curvature zero then D is integrable and the leaves give a two-sphere of complex structures on M so that M is in fact hyperkähler.

Accompanying this contact structure is a Kähler metric on $\mathcal{Z}$ constructed by using the standard metric on the fibres and lifting the metric from the base to D. In the case of positive scalar curvature this gives a positive definite Kähler metric on $\mathcal{Z}$, if R is negative this metric is no longer Kähler however multiplying by -1 in the fibre direction gives a metric of indefinite signature which is Kähler. For details of these constructions and the proofs the reader is referred to the excellent treatment in Besse [4].

In the case of positive scalar curvature Hitchin showed [10], by classifying those twistor spaces which admit a Kähler metric, that the only simply connected, self-dual, Einstein, four-manifolds with positive scalar curvature are the round four-sphere, S^4, and complex projective space, $\mathbb{CP}_2$, with the Fubini-Study metric.

In the case of negative scalar curvature less is known [13]. Our results are concerned with the rigidity of such manifolds under deformations through self-dual Einstein metrics. Since the techniques are also applicable to quaternionic-Kähler manifolds with negative scalar curvature and the results are, perhaps, a little more interesting in this case we now give a slightly more detailed account of the theory of these spaces.

1.2 Quaternionic-Kähler manifolds

Our summary here is essentially that of Salamon [17], for the construction of twistor spaces, and LeBrun [15] for the inverse construction.

Definition 1.2.1 A quaternionic-Kähler manifold, M, is an oriented $4n$-manifold, $n > 1$, whose holonomy group is contained in the subgroup $Sp(n)Sp(1) \subset SO(4n)$, where $Sp(n)Sp(1) = Sp(n) \times_{\mathbb{Z}_2} Sp(1)$.

To begin with we summarize the "E-H" formalism of Salamon. This produces locally defined bundles, E and H, over M, which play the role of the spin bundles in the four-dimensional case already discussed. More precisely H will be a quaternionic line bundle and will play the role of $\mathbb{S}_+$, in that the twistor space will be $\mathbb{P}(H)$. To this end let P denote the frame bundle of M, this is a principal $Sp(n)Sp(1)$-bundle. Using P we may, given a representation of $Sp(n) \times Sp(1)$, construct, at least locally, an associated vector bundle V. To do this we choose a local lifting of P to a $Sp(n) \times Sp(1)$ bundle, $\tilde{P}$, and use the usual associated bundle construction. Via this construction we may consider the bundles E, and H associated to the standard representations of $Sp(n)$ and $Sp(1)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ respectively. These bundles are quaternionic in the sense that they possess anti-linear structure maps $v \mapsto \tilde{v}$ satisfying $\tilde{\tilde{v}} = -v$, and there are canonical sections $\omega_H \in \bigwedge^2 H$ and $\omega_E \in \bigwedge^2 E$ with $\omega_H(\tilde{h}, \tilde{h}) = \omega_H(h,h)$, $\omega_H(h, \tilde{h}) > 0$ if $h \neq 0$, and similarly for E.

From the construction of E and H, we may identify $\mathbb{C}T^*M = E \otimes H$. Note that the quaternionic structures on E and H induce a real structure on any subspace of $(\bigotimes^p E) \otimes (\bigotimes^q H)$ when $p+q$ is even, so that we may think of this as an identification of the underlying real spaces. Furtheremore we may identify the Riemannian metric as

$$g = \omega_E \otimes \omega_H \in \Gamma(\Lambda^2 E \otimes \Lambda^2 H) \subset \Gamma(\odot^2 T^* M). \tag{1.2}$$

If, by abuse of notation, we let $\odot^2 H$ denote the real vectors in the object of the same name, then there is an action of $\odot^2 H$ on TM in the following way. Use lowercase roman indices for TM and primed and unprimed uppercase indices for E and H respectively, so that e.g. $v^a = v^{AA'}$, then we may write this action as

$$w_{AB} : v^{CC'} \mapsto w^A{}_C v^{CC'}, \tag{1.3}$$

where $w_{AB} \in \odot^2 H$ and indices are raised and lowered with ω_H. It follows from this that if $I, J \in \odot^2 H$, then as edomorphisms of TM we have

$$JK + KJ = -\langle J, K \rangle 1. \tag{1.4}$$

Where $\langle\ ,\ \rangle$ is the inner product on $\odot^2 H$ induced by ω_H. Using this we may construct local bases of $\odot^2 H$, $\{I, J, K\}$ satisfying the identities

$$I^2 = J^2 = -1, \quad IJ = -JI = K. \tag{1.5}$$

From equation 1.4 we just choose I, J, K orthogonal of length $\sqrt{2}$. Thus we may view the real three-dimensional bundle $\odot^2 H \subset End(TM)$ as a coefficient bundle of imaginary quaternions acting on the tangent space at each point as if by right multiplication. Now we may set $\mathcal{Z} = S_{\sqrt{2}}(\Re(\odot^2 H))$, the set of real vectors in $\bigodot^2 H$ of length $\sqrt{2}$, in exact analogy with the first method for constructing $\mathcal{Z}$ outlined in the previous section.

Similar analogies exist for the other methods of constructing $\mathcal{Z}$, we mention only the second since this will prove useful. In this method, given a non-zero vector $h \in H$, we get an isomorphism $TM \cong \Re(H^* \otimes E^*) \to E^*$, given by contraction in the first slot with h. This gives an almost complex structure on TM which again depends only on the projective class of h, so that $\mathcal{Z} = \mathbb{P}(H)$ is a two-sphere bundle of almost complex structures over M.

There is a natural almost complex structure on $\mathcal{Z}$ defined in the same way as before, namely at $z \in \mathcal{Z}$ we split $T_z\mathcal{Z}$ into horizontal and vertical spaces using the connection from M, on the vertical space we put the almost complex structure of the fibre S^2, and on the horizontal space which is isomorphic to $T_{\pi(z)}M$ where π is the projection, $\pi : \mathcal{Z} \to M$, we put the almost complex structure which the point z is.

Again the integrability of this almost complex structure is tied up with the decomposition of the curvature tensor of M under the decomposition $\mathbb{C}T^*M = E \otimes H$, however in this case the quaternionic structure forces the curvature to be of the right type, we have:

Theorem 1.2.2 *Any quaternionic-Kähler manifold is Einstein, and it's Rie-*

mannian curvature has the form

$$\mathcal{R} = t\mathcal{R}_0 + \mathcal{R}_1 \tag{1.6}$$

where t is the scalar curvature, $\mathcal{R}_0$ is the curvature tensor of quaternionic projective space, and $\mathcal{R}_1$ is a section of $\bigodot^2(\bigodot^2 H)$.

Careful analysis of the curvature of M then yields

Theorem 1.2.3 (Salamon) *Let M be a quaternionic-Kähler manifold, then the associated almost complex manifold $\mathcal{Z}$ is integrable.*

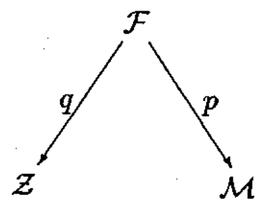
Since we have seen that quaternionic-Kähler manifolds are Einstein it is natural to expect, by analogy with the preceding section, that when $t \neq 0$, $\mathcal{Z}$ should possess a complex contact structure. Indeed let $D \subset T\mathcal{Z}$ be the horizontal distribution induced by the connection on M, this is transverse to the vertical space L so that $T\mathcal{Z}/D \cong L$, as before. Let θ be the projection map $\theta : T\mathcal{Z} \to L$ then we have

Theorem 1.2.4 (Salamon) *Let M be a quaternionic-Kähler manifold of dimension $4n$, with scalar curvature $t \neq 0$, then the distribution of hyperplanes D defined above is a complex contact structure on M, that is $\theta \wedge \partial\theta^{\wedge n} \in H^0(\mathcal{Z}, \mathcal{O}(\kappa \otimes L^{\otimes n+1}))$ is never zero.*

As in the four-dimensional case there is an inverse construction for these manifolds too. Namely, given a complex contact manifold, $\mathcal{Z}$, of dimension

$2n+1$, with a family of holomorphically embedded $\mathbb{P}_1$'s whose normal bundles are isomorphic to $\mathcal{O}(1)^{\oplus 2n}$, it is possible to produce a complex riemannian $4n$-manifold $(\mathcal{M}, \Gamma)$ with holonomy a subgroup of $Sp(n, \mathbb{C})Sp(1, \mathbb{C})$ so that $\mathcal{Z}$ is the twistor space of $\mathcal{M}$. In addition, if $\mathcal{Z}$ posseses a real structure with no fixed points which preserves the contact structure we may single out a real slice of $\mathcal{M}$ and a metric of signature $(n-l, l)$ on this slice with holonomy in $Sp(n-l, l)Sp(1)$ which has $\mathcal{Z}$ as its twistor space. This construction was first carried out in detail by LeBrun [15], in the case of a hyperkähler manifold the details were given earlier in [11]. Since we will need some properties of this construction, in particular the extent to which we may recover Γ from the 1-jets of the contact structure and real structures, the rest of this chapter is devoted to a more detailed exposition of this inverse construction. Of course this is based extensively on [15] and the reader is referred to this paper for clarification and the proofs which we omit.

To begin, let $\mathcal{Z}$ be a complex contact manifold of dimension $2n+1$, suppose that $\mathcal{Z}$ has a holomorphically embedded $\mathbb{P}_1$ with normal bundle $\mathcal{O}(1)^{\oplus 2n}$ which is transverse to the contact distribution on $\mathcal{Z}$. The deformation theory of Kodaira then guarantees the existence of a complex $4n$-parameter family, $\mathcal{M}$, of such curves. An open subset of $\mathcal{M}$ will consist of curves transverse to the contact distribution, redefine $\mathcal{M}$ to consist of just these curves. There is a double-fibration as follows,



In this fibration the fibre over each $x \in \mathcal{M}$ is a $\mathbb{P}_1$ and $q(p^{-1}(x))$ is a twistor line in $\mathcal{Z}$ which we will refer to as $\mathbb{P}_x$. The bundles E and H on $\mathcal{M}$ wiil be constructed by a local method, for general $\mathcal{M}$ they need not exist globally. We will need the 0th-direct image construction defined as follows: for any sheaf $\mathcal{S}$ on $\mathcal{F}$ we produce a new sheaf $p_*^0\mathcal{S}$ on $\mathcal{M}$ by the requirement that the sections of $p_*^0\mathcal{S}$ over a set $U \subset \mathcal{M}$ are to be the sections of $\mathcal{S}$ over $p^{-1}(U)$. In the cases we are interested in, if F is a vector bundle over $\mathcal{F}$ then $p_*^0\mathcal{O}(F)$ is a sheaf of sections of a vector bundle and the fibre of this bundle at a point $m \in \mathcal{M}$ is the vector space $H^0(\mathbb{P}_x, \mathcal{O}(F))$ which is finite dimensional since $\mathbb{P}_x$ is compact. With this tool in hand, let $L^{\frac{1}{2}}$ be a local square root of the contact line bundle L on $\mathcal{Z}$, and let D be the contact distribution.

Definition 1.2.5

1. Define H by $\mathcal{O}(H) = p_*^0(q^*\mathcal{O}(L^{\frac{1}{2}}))$.

2. Define E by $\mathcal{O}(E) = p_*^0(q^*\mathcal{O}(D \otimes L^{-\frac{1}{2}}))$.

Remark 1.2.6 These bundles are actually the duals of the bundles E and H discussed earlier.

Lemma 1.2.7 $T\mathcal{M} = E \otimes H$.

Proof

By assumption $D|_{\mathbb{P}_x}$ is the normal bundle of $\mathbb{P}_x$ hence $H^0(\mathbb{P}_x, \mathcal{O}(D)) = T\mathcal{M}$. But $H^0(\mathbb{P}_x, \mathcal{O}(D)) = E_x \otimes H_x$ so we are done. □

Definition 1.2.8 $d\theta$ is a well defined section of $\bigwedge^2(D^*) \otimes L = \bigwedge^2(D \otimes L^{\frac{1}{2}})^*$ so we set $\omega_E = d\theta \in \bigwedge^2 E^*$.

Definition 1.2.9 We produce a connection ∇_E on E as follows. Let Ω^1_q be the sections of $(ker\ q_*) \to \mathcal{F}$, and let $d_q : \mathcal{O}(q^*(D \otimes L^{\frac{1}{2}})) \to \Omega^1_q(q^*(D \otimes L^{\frac{1}{2}}))$ be differentiation up the fibres of q. On $\mathcal{F}$ we have the exact sequence

$$0 \to q^*\mathcal{O}(D) \to p^*\Omega^1 \to \Omega^1_q \to 0$$

so that $p^0_*\Omega^1_q = \Omega^1$. Thus the 0th direct image of d_q under p gives a connection, $\nabla_E : \mathcal{O}(E) \to \Omega^1(E)$.

Lemma 1.2.10 $\nabla_E \omega_E = 0$ *so the holonomy of* ∇_E *is a subgroup of* $Sp(n, \mathbb{C})$.

Proof

This is clear since $d\theta$ is closed. □

This constructs the bundle E with connection, the construction of H is more elaborate.

Definition 1.2.11 For brevity set $\hat{L} = ker\, p_*$ and $\hat{D} = q_*^{-1}(D)$, so that $\hat{L} = T\mathcal{F}/\hat{D}$. Let $d_p : \mathcal{O} \to \mathcal{O}(\hat{L}^{-1})$ be differentiation up the fibres of p, then the Wronskian

$$W : \mathcal{O}(\hat{L}^{\frac{1}{2}}) \times \mathcal{O}(\hat{L}^{\frac{1}{2}}) \;\to\; \mathcal{O} \tag{1.7}$$

$$(s_1, s_2) \;\mapsto\; s_1 \otimes d_p s_2 - s_2 \otimes d_p s_1 \tag{1.8}$$

is independent of choice of local trivialization used to compute the derivatives. Let $\omega_H \in \Gamma(\mathcal{M}, \mathcal{O}(\bigwedge^2 H^*))$ be the direct image of W.

Definition 1.2.12 To define a connection, ∇_H, on H let $\hat{\theta} = q^*\theta : T\mathcal{F} \to \hat{L}$ be the canonical projection, and consider the operator

$$\mathcal{L} : \mathcal{O}(\hat{D}) \times \mathcal{O}(\hat{L}) \;\to\; \mathcal{O}(\hat{L}) \tag{1.9}$$

$$(u, v) \;\mapsto\; \hat{\theta}([u, v]) \tag{1.10}$$

which is a derivation in the second slot. Produce a new operator

$$\hat{\mathcal{L}} : \mathcal{O}(\hat{D}) \times \mathcal{O}(\hat{L}^{\frac{1}{2}}) \to \mathcal{O}(\hat{L}), \tag{1.11}$$

by requiring $\hat{\mathcal{L}}$ to be a derivation in the second slot and setting $w\hat{\mathcal{L}}(u,w) = \frac{1}{2}\mathcal{L}(u,w^2)$. Finally, if s is a section of H identified with $\hat{s} \in H^0(\mathcal{O}(\hat{L}^{\frac{1}{2}}))$, and if t is a section of $T\mathcal{M}$ identified with $p_*^{-1}t \in H^0(\mathcal{O}(\hat{D}))$ via the rule $p_*(p_*^{-1}t) = t$ we may at last define $\nabla_H : \mathcal{O}(H) \to \Omega^1(H)$ via the rule $(\nabla_H s)(t) = \hat{\mathcal{L}}(p_*^{-1}t, \hat{s})$.

Lemma 1.2.13 $\nabla_H \omega_H = 0$ *so that the holonomy of* ∇_H *is in* $Sp(1,\mathbb{C})$.

Proof

(Sketch) The point here is that $\mathcal{F} = \mathbb{P}(H)$ and the distribution $\hat{D}$ on $\mathcal{F}$ is just the projectivization of the horizontal distribution of ∇_H. Thus if we identify the fibres of p over the ends of a curve, c, in $\mathcal{M}$ by lifting that curve tangent to $\hat{D}$ and use this to identify $H_{c(0)} = H^0(p^{-1}(c(0)), \mathcal{O}(\hat{L}^{\frac{1}{2}}))$ with $H_{c(1)}$, this gives us parallel transport with respect to ∇_H . Now invariance of the Wronskian under coordinate changes guarantees invariance of ω_H under parallel transport. □

Remark 1.2.14 Since $d\hat{\theta} = q^*\theta$ is non-zero we also see that the curvature of ∇_H is everywhere non-zero.

Definition 1.2.15 Now define a complex riemannian metric Γ on $\mathcal{M}$ by the rule $\Gamma = \omega_E \otimes \omega_H$ and a connection, ∇, by $\nabla(e \otimes h) = \nabla_E e \otimes h + e \otimes \nabla_H h$.

Clearly $\nabla\Gamma = 0$ and the holonomy of ∇ is a subgroup of $Sp(n, \mathbb{C})Sp(1, \mathbb{C})$.

Proposition 1.2.16 ∇ *is torsion-free and so is the Levi-Civita connection of* Γ *.*

Proof

The proof is in two parts, in the first we show that it suffices to do the calculation at a point in the model space $\mathbb{CP}_{2n+1}$, in the second we carry this out. For the first part we need a lemma:

Lemma 1.2.17 *Let* $\mathbb{P}_x$ *be any curve of the family* $\mathcal{M}$. *There is an isomorphism between the second infinitesimal neighbourhood of* $\mathbb{P}_x \subset \mathcal{Z}$ *and the second infinitesimal neighbourhood of a line* $\mathbb{CP}_1 \subset \mathbb{CP}_{2n+1}$. *Moreover this can be chosen to send the* 1*-jet of the contact distribution to the* 1*-jet of the unique contact distribution on* $\mathbb{CP}_{2n+1}$.

Proof

(of lemma) This requires some careful analysis. The first neighbourhoods agree because we may use the contact distribution to write the tangent bundle of $\mathcal{Z}$ restricted to a line as $T\mathcal{Z}|_{\mathbb{P}_m} = \mathcal{O}(2) \oplus 2n\mathcal{O}(1)$. For the second neighbourhoods analysis, given in [15], reveals that the obstruction to extending the isomorphism lies in $H^1(\mathbb{CP}_1, \mathcal{O}(2) \otimes \odot^2(2n\mathcal{O}(-1)))$ but this is zero. □

The point of this lemma is that it now suffices to check in the case $\mathcal{Z} = \mathbb{CP}_{2n+1}$ since by the lemma we may identify $\mathcal{M}$ with the corresponding space for $\mathcal{Z} = \mathbb{CP}_{2n+1}$ to second order at a point in such a manner that the connections agree to first order.

Now when $\mathcal{Z} = \mathbb{CP}_{2n+1}$ we may let $\omega \in \Lambda^2(\mathbb{C}^{2n+2})$ be any non-degenerate two form, and $\theta \in \Gamma(\mathbb{CP}_{2n+1}, \mathcal{O}(2))$ be the corresponding contact structure, defined by $\theta_{[V]}(W) = \omega(V, W)$. The space of lines to consider is $G_2^t(\mathbb{C}^{2n+2})$, the space of planes in $\mathbb{C}^{2n+2}$ on which the restriction of ω is non-degenerate. The construction is invariant under $Sp(n+1, \mathbb{C})$ so the torsion is invariant under the isotropy subgroup, $Sp(n, \mathbb{C})Sp(1, \mathbb{C})$, and in particular under $Sp(1, \mathbb{C})$. If $T = T_p G_2^t(\mathbb{C}^{2n+2})$, then as an $Sp(1, \mathbb{C})$-module $T = 2n\mathbf{H}$ where $\mathbf{H}$ is the fundamental representation of $Sp(1, \mathbb{C})$. Now torsion lives in $T \otimes \Lambda^2 T$, since $T \otimes \Lambda^2 T = 8n^3\mathbf{H} \oplus 2n^2(n-1) \odot^3 \mathbf{H}$ and this has no 1-dimensional component the torsion must vanish □

To recap: starting with $\mathcal{Z}_{2n+1}$ we've produced a complex riemannian manifold $\mathcal{M}_{4n}$ with holonomy in $Sp(n,\mathbb{C})Sp(1,\mathbb{C})$. Since H has non-zero curvature the holonomy is not in $Sp(n,\mathbb{C})$ and for $n \geq 2$ $\mathcal{M}$ is complex Einstein with non-zero scalar curvature. For $n = 1$ one may also show that $\mathcal{M}$ is self-dual.

Finally we consider the role of real structures. Suppose σ is an antiholomorphic involution of $\mathcal{Z}$ without fixed points and suppose further that σ preserves the underlying real distribution of the contact structure . Thus $\sigma_*\theta = \bar{\theta}$. If $\mathbb{P}_x$ is a twistor line then so is $\sigma(\mathbb{P}_x)$ so σ gives rise to an antiholomorphic involution $\hat{\sigma} : \mathcal{M} \to \mathcal{M}$.

Definition 1.2.18 M_σ will denote the fixed point set of $\hat{\sigma}$, a real-analytic 4n-manifold.

Now given $x \in M_\sigma$ we have $\sigma : \mathbb{P}_x \to \mathbb{P}_x$ an antiholomorphic involution. But $H_x = H^0(\mathbb{P}_x, \mathcal{O}(L^{\frac{1}{2}}))$ and $E_x = H^0(\mathbb{P}_x, \mathcal{O}(D \otimes L^{-\frac{1}{2}}))$, where $L|_{\mathbb{P}_x} \cong T\mathbb{P}_x$ in a canonical way induced by θ. This gives rise to antilinear maps $\sigma_{*H} : H_x \to H_x$ and $\sigma_{*E} : E_x \to E_x$ with $\sigma_{*H}^2 = \pm 1$ and $\sigma_{*E}^2 = \pm 1$. Since $\sigma_{*E \bigotimes H}$ is just $\hat{\sigma}_*$ these signs must be the same. Further, $\mathbb{P}(H_x) = \mathbb{P}_x$ in a natural way which implies that the projectivization of σ_{*H} is just $\sigma : \mathbb{P}_x \to \mathbb{P}_x$. This has no fixed points so $\sigma_{*H}^2 = -1$ and therefore $\sigma_{*E}^2 = -1$ also. This makes E and H into quaternionic bundles. We get an $Sp(1)$ structure on H via the inner product $\langle h_1, h_2\rangle_H = \omega_H(h_1, \sigma_{*H} h_2)$, and an $Sp(l, n-l)$ structure on E for some l, depending on the signature of the corresponding object on E. Now

$$\langle h_1, h_2\rangle_H \langle e_1, e_2\rangle_E = \Gamma\ (h_1 \otimes e_1, \hat{\sigma_*}(h_2 \otimes e_2)) = \Gamma\ (h_1 \otimes e_1, \overline{h_2 \otimes e_2}) \qquad (1.12)$$

Thus σ induces a $Sp(l, n-l)Sp(1)$ structure on M_σ compatible with the Levi-Civita connection.

To conclude we remark that if we began with $\mathcal{Z}$ the twistor space of M then this construction rebuilds all of the information from M and, at least locally, reconstructs M.

Chapter 2

Preliminaries

2.1 Notation and Conventions

For tensor calculations we use the abstract index convention of Penrose. An excellent reference for this is [16]. In this formalism indices are used as placemarkers only and do not indicate components. Thus V^a is a vector (with components$V^a_{\mathbf{1}}, \ldots V^a_{\mathbf{n}}$) α_a is a 1-form, and $\alpha_b V^b$ denotes $\alpha(V)$ etc.

We will be working with a complex manifold $\mathcal{Z}$, so a little more notation is appropriate. The complex structure, J, defines a splitting, $\mathbb{C}T\mathcal{Z} = T^{1,0} \oplus T^{0,1}$. Thus, Greek indices will denote $\mathbb{C}T\mathcal{Z}$ and barred or unbarred Roman indices will denote $T^{0,1}$ and $T^{1,0}$ respectively, ie. $T^\alpha = T^a \oplus T^{\bar{a}}$.

It is a standard convention in the abstract index formalism that the ordering of indices of different types is irrelevant (since eg. $E^\alpha \otimes F^B \cong F^B \otimes E^\alpha$). Unfortunately in the case of $T^{1,0}$ and $T^{0,1}$ this leads to confusion, since both are subbundles of $\mathbb{C}T\mathcal{Z}$ and the types of symmetries we study (in the case of hermitian metrics and kähler forms) use this fact. For example $g_{a\bar{b}}$ and $g_{\bar{b}a}$ usually denote the same tensor g in $T_a \otimes T_{\bar{b}}$, but in our case $g_{a\bar{b}}$ usually means that part of $g_{\alpha\beta}$ in $T_{a\bar{b}}$, which is not the same as $g_{\bar{b}a}$. Thus we will distinguish carefully between $T_{a\bar{b}}$ and $T_{\bar{a}b}$.

2.2 Hermitian Differential Geometry

In addition to [16] the presentation in this section is based on [7]. Let h be an hermitian riemannian metric on T$\mathcal{Z}$, ie. $h(V,W) = h(JV,JW)$. This gives rise to an hermitian inner product, g, on $\mathbb{C}T\mathcal{Z}$ via $g(X,Y) = \mathbb{C}h(X,\overline{Y})$, where $\mathbb{C}h$ denotes the extension of h to $\mathbb{C}T\mathcal{Z}$ via complex linearity. If V and W are real vectors then $V - iJV$ and $W + iJW$ are typical elements of $T^{1,0}$ and $T^{0,1}$ respectively. It's easy to see that $g(V - iJV, W + iJW) = 0$ and thus $T^{1,0} = (T^{0,1})^\perp$. Hence, if $g(X,Y) = g_{\alpha\beta}X^\alpha\overline{Y}^\beta$, then $g_{ab} = 0 = g_{\bar{a}\bar{b}}$ and, under the decomposition $T_{\alpha\beta} = T_{ab} \oplus T_{a\bar{b}} \oplus T_{\bar{a}b} \oplus T_{\bar{a}\bar{b}}$, we have

$$g_{\alpha\beta} = g_{a\bar{b}} \oplus g_{\bar{a}b}. \tag{2.1}$$

Now

$$g(X,Y) = g_{\alpha\beta}X^{\alpha}\overline{Y}^{\beta} \quad (2.2)$$

$$= g_{a\bar{b}} \oplus g_{\bar{a}b}(X^a \oplus X^{\bar{a}})\overline{(Y^b \oplus Y^{\bar{b}})} \quad (2.3)$$

$$= g_{a\bar{b}}X^a\overline{Y}^{\bar{b}} + g_{\bar{a}b}X^{\bar{a}}\overline{Y}^b. \quad (2.4)$$

Since $g(X,Y) = \overline{g(Y,X)}$ we see that

$$g_{a\bar{b}} = \overline{g_{b\bar{a}}} = \bar{g}_{\bar{b}a}. \quad (2.5)$$

In our case g is the complexification of a real tensor and so $\bar{g} = g$, ie.

$$g_{a\bar{b}} = g_{\bar{b}a} \quad (2.6)$$

The Kähler form, or associated two form, ω, is the complexification of the real form $h(JV, W)$, i.e. $\omega(X,Y) = g(JX,\overline{Y})$. Thus,

$$\omega(X^a \oplus X^{\bar{a}}, Y^b \oplus Y^{\bar{b}}) = g(JX^a \oplus JX^{\bar{a}}, \overline{Y}^{\bar{b}} \oplus \overline{Y}^b) \quad (2.7)$$

$$= g(iX^a \oplus -iX^{\bar{a}}, \overline{Y}^{\bar{b}} \oplus \overline{Y}^b) \quad (2.8)$$

$$= ig_{a\bar{b}}X^aY^{\bar{b}} - ig_{\bar{a}b}X^{\bar{a}}Y^b \quad (2.9)$$

From this, $\omega_{\alpha\beta} = \omega_{a\bar{b}} \oplus \omega_{\bar{a}b} = i\{g_{a\bar{b}} \oplus -g_{\bar{a}b}\}$, i.e. $\omega_{a\bar{b}} = g_{a\bar{b}}$ and $\omega_{\bar{a}b} = -g_{\bar{a}b}$

A connection on $\mathcal{Z}$ is an operator $\nabla : T\mathcal{Z} \rightarrow T^*\mathcal{Z} \otimes T\mathcal{Z}$ satisfying $\nabla(hV) = dh \otimes V + h\nabla V$ for all vector fields V and all smooth functions h. In abstract indices this is an operator $\nabla_\alpha : T^\beta \rightarrow T^\beta_\alpha$. Given a coordinate system $\{x_1, \ldots, x_n\}$ on some open set $U \subset \mathcal{Z}$, we get a local tensor field on U, $\Gamma_{\alpha\beta}{}^{\gamma}$, defined by

$$\Gamma_{\alpha\beta}{}^{\gamma} V^{\beta} = \nabla_{\alpha} V^{\gamma} - D_{\alpha} V^{\gamma} \tag{2.10}$$

where D_α is the connection on U induced by the coordinate diffeomorphism with $\mathbb{R}^n$. It is important to note that, with this definition, Γ is a *tensor* on U. Confusion may arise from the similarity of the formula $\nabla_\alpha V^\gamma = D_\alpha V^\gamma + \Gamma_{\alpha\beta}{}^{\gamma} V^\beta$ to those found in classical texts. In our case $D_\alpha V^\gamma$ does not mean "differentiate the coordinates" for in writing V^γ we have not yet chosen a frame for $\mathbb{C}T\mathcal{Z}$ with which to obtain coordinates. There is a natural frame for $\mathbb{C}T\mathcal{Z}$ since we have coordinates, however if this is not used then the expression $D_\alpha V^\gamma$ means the following: first transform to the frame $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n}\}$, then differentiate the components, finally transform back to the original frame.

Definition 2.2.1 We denote by ∇ the unique connection compatible with the metric and complex structure on M, i.e. $\nabla_\alpha g_{\beta\gamma} = 0$, $\Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} = \overline{\Gamma_{ab}{}^{c}}$, and all other components of Γ are zero.

Definition 2.2.2 Torsion is defined by $T_{ij}{}^{k} = \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} = 2\Gamma_{[ij]}{}^{k}$.

Remark 2.2.3 This definition of torsion gives the usual formula:

$$T_{\alpha\beta}{}^{\gamma}X^{\alpha}Y^{\beta} = T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.11}$$

We conclude this section with some definitions and observations which will be relevant in the later computations.

Using the condition, $\Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} = \overline{\Gamma_{ab}{}^{c}}$, and all other components of Γ are zero, we compute

$$\nabla_i g_{j\bar{k}} = D_i g_{j\bar{k}} - \Gamma_{ij}{}^{l} g_{l\bar{k}}. \tag{2.12}$$

Then $\nabla_\alpha g_{\beta\gamma} = 0$ implies that $D_i g_{j\bar{k}} = \Gamma_{ij}{}^{l} g_{l\bar{k}}$ and thus

$$\partial\omega = D_{[i}\omega_{j]\bar{k}} = D_{[i}g_{j]\bar{k}} = \Gamma_{[ij]}{}^{l} g_{l\bar{k}} = \frac{1}{2}T_{ij}{}^{l} g_{l\bar{k}}. \tag{2.13}$$

Definition 2.2.4 Curvature is defined by:

$$\nabla_\alpha\nabla_\beta V^\gamma - \nabla_\beta\nabla_\alpha V^\gamma = \mathcal{R}_{\alpha\beta\delta}{}^{\gamma}V^\delta + T_{\alpha\beta}{}^{\delta}\nabla_\delta V^\gamma \tag{2.14}$$

For a general connection, with non-vanishing torsion, many of the symmetries of the Riemann tensor are lost or retained in a weaker form. The following are two symmetries which we will have cause to use.

Lemma 2.2.5 *1.* $\mathcal{R}_{\alpha\beta\gamma\delta} = \mathcal{R}_{[\alpha\beta]\gamma\delta} = \mathcal{R}_{\alpha\beta[\gamma\delta]}$. *Where square brackets indicate the operation of skew symmetrization; i.e. if A is any tensor then $A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$.*

2. $\mathcal{R}_{\bar{a}b\bar{c}}{}^{d} - \mathcal{R}_{\bar{a}\bar{c}b}{}^{d} = D_b T_{\bar{a}\bar{c}}{}^{d}$.

Proof

1. The first of these is clear from the definition. The second follows from the observation that $\nabla_\alpha \nabla_\beta g_{\gamma\delta} = 0$.

2. This is an application of the Jacobi identity, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, to vector fields of the appropriate type. For details we refer the reader to [7] or [16].

□

With our conventions the Ricci form of $\mathcal{Z}$, $\Phi_{\mathcal{Z}}$, which is the curvature of the canonical bundle, takes the form $(\Phi_{\mathcal{Z}})_{\bar{a}b} = \mathcal{R}_{\bar{a}b\bar{c}}{}^{\bar{c}}$. We may compute $\mathcal{R}_{\bar{a}b\bar{c}}{}^{d} = D_b \Gamma_{\bar{a}\bar{c}}{}^{d}$, from which it follows that $\Phi_{\mathcal{Z}} = \partial\bar{\partial} \log \det(g)$ as usual. One final curvature identity is appropriate at this point.

Lemma 2.2.6 *If $T_{\bar{a}\bar{b}}{}^{\bar{b}} = 0$ then $\mathcal{R}_{\bar{a}b\bar{c}}{}^{\bar{c}} = \mathcal{R}^{\bar{c}}{}_{\bar{a}\bar{c}b}$.*

Proof

Since $T_{\bar{a}\bar{b}}{}^{\bar{b}} = 0$, it follows from lemma 2.2.5 part 2 that $\mathcal{R}_{\bar{a}b\bar{c}}{}^{\bar{c}} = \mathcal{R}_{\bar{a}\bar{c}b}{}^{\bar{c}}$. Now two applications of part 1 of the same lemma yield the result. □

If $\phi_{\bar{\imath}_1\bar{\imath}_2\ldots\bar{\imath}_n A}$ is a (0,n)-form with values in a holomorphic bundle E, we see that

$$(\bar{\partial}\phi)_{\bar{\imath}\bar{\imath}_1\ldots\bar{\imath}_n A} = \nabla_{[\bar{\imath}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n]A} - n\Gamma_{[\bar{\imath}\bar{\imath}_1}{}^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} \quad (2.15)$$

$$= \nabla_{[\bar{\imath}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n]A} - \frac{n}{2}T_{[\bar{\imath}\bar{\imath}_1}{}^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} \quad (2.16)$$

This leads us to the following definition.

Definition 2.2.7 Define $T : \Lambda^{0,n}(E) \longrightarrow \Lambda^{0,n+1}(E)$ by,

$$(T(\phi))_{\bar{\imath}\bar{\imath}_1\ldots\bar{\imath}_n A} = \frac{n}{2}T_{[\bar{\imath}\bar{\imath}_1}{}^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A}, \quad (2.17)$$

and $\bar{\nabla} : \Omega^{0,n}(E) \longrightarrow \Omega^{0,n+1}(E)$ by,

$$(\bar{\nabla}\phi)_{\bar{\imath}\bar{\imath}_1\ldots\bar{\imath}_n A} = \nabla_{[\bar{\imath}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n]A}. \quad (2.18)$$

With this we have $\bar{\partial}\phi = \bar{\nabla}\phi - T(\phi)$. If $\mathcal{Z}$ is compact and E is hermitian we let $\bar{\delta}$ denote the adjoint of $\bar{\partial}$ with respect to the inner product on $\Omega^{0,n}(E)$ given by $(\phi, \psi) = \int_{\mathcal{Z}} n!\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A}\overline{\psi}^{\bar{\imath}_1\ldots\bar{\imath}_n A}$. We may compute $\bar{\delta}\phi = \bar{\nabla}^\dagger\phi - T^\dagger\phi$, where

$$(\bar{\nabla}^\dagger\phi)_{\bar{\imath}_1\ldots\bar{\imath}_n A} = -n\nabla^{\bar{\imath}}\phi_{\bar{\imath}\bar{\imath}_2\ldots\bar{\imath}_n A} \quad (2.19)$$

$$(T^\dagger(\phi))_{\bar{\imath}_1\ldots\bar{\imath}_n A} = \frac{n(n-1)}{2}T^{\bar{\imath}\bar{\jmath}}{}_{[\bar{\imath}_2}\phi_{|\bar{\imath}\bar{\jmath}|\bar{\imath}_3\ldots\bar{\imath}_n]A} \quad (2.20)$$

Then an easy computation yields

$$\bar{\delta}\bar{\partial} + \bar{\partial}\bar{\delta} = (\bar{\nabla}, \bar{\nabla}^{\dagger}) - (T^{\dagger}, T) - (T^{\dagger}, \bar{\partial}) - (T, \bar{\delta}), \qquad (2.21)$$

where for operators A and B, (A,B) = AB + BA.

Chapter 3

Technical Lemmas

3.1 A Weitzenböck Formula

In this section we develop a Weitzenböck formula and a Bochner style vanishing theorem. It should be noted that both of these are very similar to those in [8], indeed it was in an attempt to use the results of [8] that these formulas were computed. Due to the appearance of an extra term in our formulas, and because of the need to ensure the accuracy of all the relevant constants, we reproduce them in full. First, with the conventions of chapter 2, we have:

Lemma 3.1.1 *Let $\mathcal{Z}$ be a complex hermitian manifold satisfying $T_{\bar{a}\bar{b}}{}^{\bar{b}} = 0$, and let L be a holomorphic line bundle over $\mathcal{Z}$ with an hermitian metric.*

Then if $\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A}$ is a (0,n)-form on $\mathcal{Z}$ with values in L we have

$$(\bar{\delta}\bar{\partial}+\bar{\partial}\bar{\delta})\phi \;=\; \Delta\phi + n\Phi(\phi) - (T^{\dagger},T)\phi - (T^{\dagger},\bar{\partial})\phi - (T,\bar{\delta})\phi \quad (3.1)$$

$$-\frac{n(n-1)}{2}(\partial T)(\phi), \quad (3.2)$$

where $\triangle$ is the Laplacian on (0,n)-forms, Φ is the Ricci form of $\mathcal{Z}$ plus the curvature of L and ∂T is an operator which will be identified in the proof.

Proof

Recall, from (2.21), we have:

$$\bar{\delta}\bar{\partial}+\bar{\partial}\bar{\delta} = (\bar{\nabla},\bar{\nabla}^{\#}) - (T^{\#},T) - (T^{\#},\bar{\partial}) - (T,\bar{\delta}). \quad (3.3)$$

We compute:

$$(\bar{\nabla},\bar{\nabla}^{\dagger})\phi_{\bar{\imath}_1\ldots\bar{\imath}_n} = -n\nabla_{[\bar{\imath}_1}\nabla^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} - (n+1)\nabla^{\bar{q}}\nabla_{[\bar{q}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n]A} \quad (3.4)$$

$$= -n\nabla_{[\bar{\imath}_1}\nabla^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} + n\nabla^{\bar{q}}\nabla_{[\bar{\imath}_1}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$$

$$-\nabla^{\bar{q}}\nabla_{\bar{q}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A} \quad (3.5)$$

$$= -n(\nabla_{[\bar{\imath}_1}\nabla^{\bar{q}}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} - \nabla^{\bar{q}}\nabla_{[\bar{\imath}_1}\phi_{|\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A})$$

$$-\nabla^{\bar{q}}\nabla_{\bar{q}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A} \quad (3.6)$$

$$= -\nabla^{\bar{q}}\nabla_{\bar{q}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A} - nT_{[\bar{\imath}_1}{}^{\bar{q}\bar{e}}\nabla_{|\bar{e}}\phi_{\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A} - n\mathcal{R}^{\bar{q}}{}_{[\bar{\imath}_1|\bar{q}}{}^{\bar{e}}\phi_{\bar{e}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$$

$$-n(n-1)\mathcal{R}^{\bar{q}}{}_{[\bar{\imath}_1\bar{\imath}_2}{}^{\bar{e}}\phi_{|\bar{q}\bar{e}|\bar{\imath}_3\ldots\bar{\imath}_n]A} - n(\Phi_L)_{[\bar{\imath}_1}{}^{\bar{e}}\phi_{|\bar{e}|\bar{\imath}_2\ldots\bar{\imath}_n]A}, \quad (3.7)$$

where Φ_L denotes the curvature of L. Let us examine the terms in this last expression one by one

1. $-\nabla^{\bar{q}}\nabla_{\bar{q}}\phi_{\bar{\imath}_1\ldots\bar{\imath}_n A}$. This is the Laplacian on (0,n)-forms, which we denote by $\triangle\phi$.

2. $\mathcal{R}^{\bar{q}}{}_{[\bar{\imath}_1|\bar{q}}{}^{\bar{e}}\phi_{\bar{e}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$. From the observations in chapter 2 this is the Ricci form of $\mathcal{Z}$ contracted with ϕ, $-(\Phi_{\mathcal{Z}})_{[\bar{\imath}_1}{}^{\bar{e}}\phi_{|\bar{e}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$.

3. $(\Phi_L)_{[\bar{\imath}_1}{}^{\bar{e}}\phi_{|\bar{e}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$. This is the curvature form of L contracted with ϕ. We group this with the previous term and denote the sum of these curvature forms contracted with ϕ by $\Phi(\phi)$

4. $\mathcal{R}^{\bar{q}}{}_{[\bar{\imath}_1\bar{\imath}_2}{}^{\bar{e}}\phi_{|\bar{q}\bar{e}|\bar{\imath}_3\ldots\bar{\imath}_n]A}$. Using the identity $\mathcal{R}_{a[\bar{b}\bar{c}]}{}^{\bar{d}} = \frac{1}{2}D_a T_{[\bar{b}\bar{c}]}{}^{\bar{d}}$ and the fact that ϕ is skew we may rewrite this as $\frac{1}{2}D^{[\bar{q}}T_{[\bar{\imath}_1\bar{\imath}_2}{}^{\bar{e}]}\phi_{|\bar{q}\bar{e}|\bar{\imath}_3\ldots\bar{\imath}_n]A}$. We denote this by $\frac{1}{2}(\partial T)(\phi)$.

5. $T_{[\bar{\imath}_1}{}^{\bar{q}\bar{e}}\nabla_{|\bar{e}}\phi_{\bar{q}|\bar{\imath}_2\ldots\bar{\imath}_n]A}$. Since $T_{\bar{a}}{}^{\bar{b}\bar{c}} = g^{\bar{b}d}T_{\bar{a}d}{}^{\bar{c}}$ and $T_{\bar{a}d}{}^{\bar{c}} \equiv 0$, this term vanishes.

Adding these terms together we get the desired result. □

Remark 3.1.2 This formula is very similar to the one found by Griffiths, [8] with the exception of the additional term $(\partial T)(\phi)$, which does not appear in his version.

The standard vanishing argument will now give us the following theorem:

Theorem 3.1.3 *Let $\mathcal{Z}$ be a compact complex hermitian manifold satisfying $T_{\bar{a}\bar{b}}{}^{\bar{b}} = 0$, and let L be a holomorphic line bundle over $\mathcal{Z}$ with an hermitian* *metric. Then, if*

$$n\langle\Phi(\phi),\phi\rangle - \|T(\phi)\|^2 - \|T^\dagger(\phi)\|^2 - \frac{n(n-1)}{2}\langle(\partial T)(\phi),\phi\rangle > 0 \qquad (3.8)$$

for all (0,n)-forms on $\mathcal{Z}$ with values in L, we have $H^n(\mathcal{Z},\mathcal{O}(L)) = 0$.

Proof

First note that by Hodge theory every element of $H^n(\mathcal{Z},\mathcal{O}(L))$ has a unique harmonic representative so it suffices to show that any harmonic $(0,n)$-form with values in L is zero. Suppose ϕ is such a form, by the previous lemma we have that

$$\begin{aligned} 0 &= \triangle\phi + n\Phi(\phi) - (T^\dagger,T)\phi - (T^\dagger,\bar\partial)\phi - (T,\bar\delta)\phi \\ &\quad -\frac{n(n-1)}{2}(\partial T)(\phi). \end{aligned} \qquad (3.9)$$

Now taking the inner product with ϕ and integrating over $\mathcal{Z}$ we get

$$0 = \|\nabla\phi\|^2 + n(\Phi(\phi),\phi) - \|T(\phi)\|^2 - \|T^\dagger(\phi)\|^2 - \frac{n(n-1)}{2}((\partial T)(\phi),\phi) \quad (3.10)$$

If the conditions of the theorem are satisfied then the right hand side of this expression is positive unless ϕ is zero, thus ϕ must be zero as required. □

3.2 Some Calculations

In this section we list a number of exterior algebra calculations which will be useful in what is to come. Throughout we will have an hermitian metric on a complex $(2n+1)$-manifold, $\mathcal{Z}$, whose associated two form is given, at a point, by

$$\omega_\varepsilon = \frac{i}{2}\left(2\varepsilon^2 dz_1 \wedge d\bar{z}_1 + 2\sum_{i=2}^{2n+1} dz_i \wedge d\bar{z}_i\right). \tag{3.11}$$

Thus an orthonormal frame for the cotangent bundle at this point is given by $\omega_1 = \sqrt{2}\varepsilon dz_1$ and $\omega_i = \sqrt{2}dz_i, i > 1$.

Definition 3.2.1 For convenience of notation let $d\bar{z}^{\{i_1 \dots i_k\}}$ denote the $(2n+1-k)$-form $d\bar{z}_1 \wedge \dots \wedge \widehat{d\bar{z}_{i_1}} \wedge \dots \wedge \widehat{d\bar{z}_{i_k}} \wedge \dots \wedge d\bar{z}_{2n+1}$, where carats indicate that a term is to be omitted.

Lemma 3.2.2

1. $(dz_i)^{\bar{a}}(d\bar{z}_i)_{a} = \begin{cases} \frac{1}{2\varepsilon^2} & \textit{if } i = 1 \\ \frac{1}{2} & \textit{if } i > 1 \end{cases}$

2. $(dz_i \wedge dz_j)^{\bar{a}\bar{b}}(d\bar{z}_i \wedge d\bar{z}_j)_{\bar{a}\bar{b}} = \begin{cases} \frac{1}{8\varepsilon^2} & \textit{if } i = 1 \textit{ or } j = 1 \\ \frac{1}{8} & \textit{otherwise} \end{cases}$

3. $(dz_i \wedge d\bar{z}_i)_{[\bar{a}}{}^{\bar{b}}(d\bar{z}^{\{k\}})_{|\bar{b}|\bar{c}\ldots\bar{d}]} = \begin{cases} \frac{1}{4n\epsilon^2} d\bar{z}^{\{k\}} & \text{if } i = 1 \text{ and } k \neq 1 \\ \frac{1}{4n} d\bar{z}^{\{k\}} & \text{if } i \neq 1, k \end{cases}$

4. $(dz_k \wedge dz_{k+1})^{\bar{a}\bar{b}}(d\bar{z}^{\{i\}})_{\bar{a}\bar{b}[\bar{c}\ldots\bar{d}]} =$

$$= \begin{cases} \frac{1}{8n(2n-1)\epsilon^2} d\bar{z}^{\{i,k,k+1\}} & \text{if } k > 1 \text{ and } i \neq 1, k, k+1 \\ \frac{1}{8n(2n-1)} d\bar{z}^{\{i,k,k+1\}} & \text{if } k > 1 \text{ and } i = 1 \end{cases}$$

Proof

1. Since $(dz_i)^{\bar{a}}(d\bar{z}_i)_{\bar{a}} = \|d\bar{z}_i\|^2$, this is clear from the form of the metric.

2. If we expand

$$(dz_i \wedge dz_j)^{\bar{a}\bar{b}}(d\bar{z}_i \wedge d\bar{z}_j)_{\bar{a}\bar{b}} =$$

$$\frac{1}{4}\left(d\bar{z}_i \otimes d\bar{z}_j - d\bar{z}_j \otimes d\bar{z}_i\right)^{\bar{a}\bar{b}}\left(d\bar{z}_i \otimes d\bar{z}_j - d\bar{z}_j \otimes d\bar{z}_i\right)_{\bar{a}\bar{b}}, \quad (3.12)$$

then use part one, we get the desired result.

3. First note that if we expand the wedge product in the first slot we have;

$$d\bar{z}^{\{k\}} = \frac{1}{2n}(d\bar{z}_1 \otimes d\bar{z}^{\{1,k\}} - d\bar{z}_2 \otimes d\bar{z}^{\{2,k\}} + \cdots$$
$$+ \cdots + (-1)^p d\bar{z}_i \otimes d\bar{z}^{\{i,k\}} \ldots), \quad (3.13)$$

where $p = i + 1$ if $i < k$ and $p = i$ otherwise. The only term in this expression which contributes is $(-1)^p d\bar{z}_i \otimes d\bar{z}^{\{i,k\}}$ so

$$(dz_i \wedge d\bar{z}_i)_{[\bar{a}}{}^{\bar{b}}(d\bar{z}^{\{k\}})_{|\bar{b}|\bar{c}\ldots\bar{d}]} = \frac{(-1)^p}{2n}\|d\bar{z}_i\|^2 d\bar{z}_i \wedge d\bar{z}^{\{i,k\}} \quad (3.14)$$

$$= \frac{1}{2n}\|d\bar{z}_i\|^2 d\bar{z}^{\{k\}}. \qquad (3.15)$$

Using part one now gives the result.

4. As in part three we expand;

$$\begin{aligned} d\bar{z}^{\{i\}} &= \frac{2!(2n-2)!}{(2n)!}((d\bar{z}_1 \wedge d\bar{z}_2) \otimes d\bar{z}^{\{1,2,i\}} \\ &+ (d\bar{z}_2 \wedge d\bar{z}_3) \otimes d\bar{z}^{\{2,3,i\}} + \cdots). \end{aligned} \qquad (3.16)$$

This time the only term contributing is $(d\bar{z}_k \wedge d\bar{z}_{k+1}) \otimes d\bar{z}^{\{i,k,k+1\}}$, so we have;

$$(dz_i \wedge dz_j)^{\bar{a}\bar{b}}(d\bar{z}_i \wedge d\bar{z}_j)_{\bar{a}\bar{b}} = \frac{1}{n(2n-1)}\|dz_k \wedge dz_{k+1}\|^2 d\bar{z}^{\{i,k,k+1\}}. \quad (3.17)$$

Now using part two will yield the desired conclusion.

□

Chapter 4

Vanishing Theorems

4.1 Introduction

In this chapter we will calculate the operator

$$A(\phi) = 2n\langle\Phi(\phi), \phi\rangle - \|T(\phi)\|^2 - \|T^\dagger(\phi)\|^2 - n(2n-1)\langle(\partial T)(\phi), \phi\rangle, \quad (4.1)$$

used in theorem 3.1.3, for $(0, 2n)$-forms with values in the contact line bundle on the twistor space of $\mathbb{HP}^n_+ = Sp(n-1,1)/Sp(n-1)Sp(1)$, the non-compact dual of $\mathbb{HP}^n$, using a metric we will introduce. Following this we will observe that the same calculation will give $A(\phi)$ on the twistor space of any quaternionic-Kähler manifold with negative scalar curvature and use these calculations to obtain some vanishing theorems to be used in the rigidity results to come.

First note that we may think of $\mathbb{HP}^n_+$ as a certain upper half-space in $\mathbb{HP}^n$. More precisely, using quaternionic homogeneous coordinates,

$$\mathbb{HP}^n_+ = \{\,[q] \in \mathbb{HP}^n \mid q_0\bar{q}_0 - q_1\bar{q}_1 - \cdots - q_n\bar{q}_n > 0\,\}. \tag{4.2}$$

Note that the action of $Sp(n,1)$ leaves invariant the quadratic form used in this definition. With this definition the twistor space of $\mathbb{HP}^n_+$ is seen to be $\mathbb{CP}^{2n+1}_+$, defined by

$$\mathbb{CP}^{2n+1}_+ = \{[z] \in \mathbb{CP}^{2n+1} \mid z_0\bar{z}_0 + z_1\bar{z}_1 - z_3\bar{z}_2 - \cdots z_{2n+1}\bar{z}_{2n+1} > 0\,\}. \tag{4.3}$$

The twistor fibration $\pi : \mathbb{CP}^{2n+1}_+ \to \mathbb{HP}^n_+$ is then given by

$$\pi : [z_0, \ldots, z_{2n+1}] \mapsto [z_0 + z_1 j, \ldots, z_{2n} + z_{2n+1} j]. \tag{4.4}$$

The contact line bundle is $\mathcal{O}(2)$, and, as discussed in chapter 1, we get an indefinite Kähler-Einstein metric on the twistor space by using $|s|^2$ as a Kähler potential, where s is a section of $\mathcal{O}(1)$ and the norm is as dicussed in chapter 1. In our case it is easier to find a the norm on $\mathcal{O}(-1)$, this must be invariant under the action of $Sp(n,1)$ and therefore a multiple of

$$\|z\|^2 = z_0^2 + z_1^2 - \sum_{k=2}^{2n+1} z_k\bar{z}_k. \tag{4.5}$$

Thus $\omega = i\partial\bar{\partial}\log\|z\|^2$ is indefinite Kähler-Einstein. It's restriction to the fibres of π is positive definite and, when restricted to the horizontal spaces, it is the negative of the lift of the metric from the base. Because of the symmetry of the situation we will only need to compute things at the point [1,0, ...,0]. Here ω has the form

$$\omega = i(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 - \ldots - dz_{2n+1} \wedge d\bar{z}_{2n+1}). \tag{4.6}$$

Note that ω is normalized so that if $\mathcal{S} = \{[z_0, z_1, 0, \ldots, 0]\}$ then

$$\int_{\mathcal{S}} \omega = \int_{\mathbb{C}} \frac{idz \wedge d\bar{z}}{\|z\|^4} = \int_{\mathbb{C}} \frac{2idxdy}{(1+r^2)^2} = 4\pi i. \tag{4.7}$$

To find the contact form we note that this too must be $Sp(n,1)$-invariant, so that if $\langle\, , \rangle$ denotes the inner product on $\mathbb{C}^{2n+2}$ obtained by polarizing the norm in equation 4.5, then we may define $\Theta_V(W) = \langle V, JW\rangle$, where J is the quaternionic structure on $\mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$. This is a one form on $\mathbb{C}^{2n+2} - \{0\}$, it is clearly homogeneous of degree two so gives rise to a one form on $\mathbb{CP}^{2n+1}_+$, ϑ, via $\vartheta_{[V]}(W) = \Theta_V(W)$, which is invariant under $Sp(n-1,1)$ and has the right horizontal space at [1,0, ...,0], and so must be the right contact structure. We compute

$$\vartheta = -z_0 dz_1 + z_1 dz_0 + z_2 dz_3 - z_3 dz_2 + \ldots + z_{2n} dz_{2n+1} - z_{2n+1} dz_{2n}. \tag{4.8}$$

For our vanishing theorems we will need a positive definite metric. To this end use the metric on L whose curvature is ω, (as already noted $c_1(L) = [\omega]$). Then make the following definition

Definition 4.1.1 If ϑ is a $(1,0)$-form with values in L, we may write $\vartheta = \vartheta_{aE}$,where E is a bundle index. We define $(\vartheta \wedge \bar{\vartheta})_{a\bar{b}} = \vartheta_{[a|E}\bar{\vartheta}_{\bar{b}]}{}^{E}$, so that $\vartheta \wedge \bar{\vartheta}$ is an ordinary $(1,1)$-form.

We will work with the metric whose associated two form is

$$\omega_\varepsilon = -\omega + i(1+\varepsilon^2)\vartheta \wedge \bar{\vartheta}. \tag{4.9}$$

Notice the introduction of a parameter, ε, this will be adjusted later to give positivity of the operator A. In local coordinates, if σ is a local section of $\mathcal{O}(1)$ near [1,0, ...,0], then ϑ is represented by the 1-form θ, where $\theta_{[V]}(W) = \sigma^2_{[V]}(\vartheta_{[V]}(W)) = \sigma_{[V]}(\Theta_{\sigma_{[V]}}(W))$ and we have:

$$\vartheta \wedge \bar{\vartheta} = \frac{1}{|\sigma|^4}\theta \wedge \bar{\theta}. \tag{4.10}$$

In particular if we choose $\sigma = (1, z_1, \ldots, z_{2n+1})$ then, since $|\sigma|^4 = 1$, and $\partial(|\sigma|^4) = 0$ at [1,0, ...,0], we can see $\theta = -dz_1$, and $\partial\theta = 2\sum_{k=1}^{n} dz_{2k} \wedge dz_{2k+1}$ at [1,0, ...,0].

If $\{\omega_i\}$ is an orthonormal basis for the cotangent bundle then $\omega_\varepsilon = \frac{i}{2}\sum \omega_i \wedge \bar{\omega}_i$. In our case, at [1,0, ...,0] this has the form

$$\omega_\varepsilon = \frac{i}{2}\left(2\varepsilon^2 dz_1 \wedge d\bar{z}_1 + 2\sum_{i=2}^{2n+1} dz_i \wedge d\bar{z}_i\right), \tag{4.11}$$

thus an orthonormal basis at [1,0, ...,0] is given by $\omega_1 = \sqrt{2}\varepsilon dz_1$ and $\omega_i = \sqrt{2}dz_i, i > 1$, as noted in chapter 3.

If L denotes the contact bundle $\mathcal{O}(2)$, we want a vanishing theorem for $H^1(\mathcal{Z}, \mathcal{O}(L))$, where $\mathcal{Z}$ is a compact quotient of $\mathbb{CP}_+^{2n+1}$. Since the extra effort is minimal we will consider sections of $L^{\otimes m}$. By Serre duality $H^1(\mathcal{Z}, \mathcal{O}(L^{\otimes m})) = H^{2n}(\mathcal{Z}, \mathcal{O}(\kappa \otimes (L^*)^{\otimes m}))$, where κ is the canonical bundle of $\mathcal{Z}$, so we will work with $(0, 2n)$-forms with values in $\kappa \otimes (L^*)^{\otimes m}$.

4.2 Torsion

First we will compute T and then the quantities $\|T(\phi)\|^2$ and $\|T^{\#}(\phi)\|^2$. To begin with

$$
\begin{aligned}
\frac{1}{2} T_{ab}{}^{c} g_{c\bar{d}} &= \partial \omega_{\varepsilon} && (4.12)\\
&= \partial \omega + i(1+\varepsilon^2)\partial \left(\frac{1}{|\sigma|^4} \theta \wedge \overline{\theta} \right) && (4.13)\\
&= i(1+\varepsilon^2) \left(\partial \left(\frac{1}{|\sigma|^4} \right) \wedge \theta \wedge \bar{\theta} + \frac{1}{|\sigma|^4} \partial\theta \wedge \bar{\theta} \right). && (4.14)
\end{aligned}
$$

Thus at [1,0, ...,0] we have

$$
T_{ab}{}^{c} = -4i(1+\varepsilon^2) \sum_{k=1}^{n} dz_{2k} \wedge dz_{2k+1} \otimes (d\bar{z}_1^{\#}), \qquad (4.15)
$$

where # denotes the operation of raising an index with the (positive-definite) metric. First an important lemma:

Lemma 4.2.1 *For this example we have* $T_{\bar{a}\bar{b}}{}^{\bar{b}} = 0$.

Proof

This is clear from the form of T above and the fact that the forms $\{d\bar{z}_k\}_{k=1}^{2n+1}$ are orthogonal at [1,0, ... ,0]. □

Remark 4.2.2 Note that without this fact theorem 3.1.3 would not apply.

Thus

$$T(\phi) = -4in(1+\varepsilon^2)\sum_{k=1}^{n}\sum_{l=1}^{2n+1}(\phi_l)_A(d\bar{z}_{2k}\wedge d\bar{z}_{2k+1}\wedge dz_1)_{[\bar{\imath}_1\bar{\imath}_2}{}^{\bar{a}}(dz^{\{l\}})_{|a|\bar{\imath}_3\ldots\bar{\imath}_{2n+1}]}, \tag{4.16}$$

but this is zero since every term in the sum contains $d\bar{z}_{2k}\wedge d\bar{z}_{2k+1}\wedge dz^{\{1,l\}}$, and all of these vanish since at least one of $d\bar{z}_{2k}$ or $d\bar{z}_{2k+1}$ appears twice. We have proved:

Lemma 4.2.3 *With the conventions above* $\|T(\phi)\|^2 = 0$.

The computation of $T^\dagger(\phi)$ is a little more tricky! To begin with, writing $C = -4in(2n-1)(1+\varepsilon^2)$ for brevity,

$$T^{\#}(\phi) = C\sum_{k=1}^{2n+1}(\phi_k)_A\sum_{l=1}^{n}(dz_{2l}\wedge dz_{2l+1}\wedge d\bar{z}_1)^{\bar{a}\bar{b}}{}_{[\bar{\imath}_1}(dz^{\{k\}})_{|\bar{a}\bar{b}|\bar{\imath}_2\ldots\bar{\imath}_{2n-1}]} \tag{4.17}$$

so we must compute $(dz_{2l}\wedge dz_{2l+1}\wedge d\bar{z}_1)^{\bar{a}\bar{b}}{}_{[\bar{\imath}_1}(dz^{\{k\}})_{|\bar{a}\bar{b}|\bar{\imath}_2\ldots\bar{\imath}_{2n-1}]}$. Observe that we may rewrite this as $(d\bar{z}_1)\wedge\left\{(dz_{2l}\wedge dz_{2l+1})^{\bar{a}\bar{b}}(dz^{\{k\}})_{\bar{a}\bar{b}\bar{\imath}_2\ldots\bar{\imath}_{2n-1}}\right\}$, but the term in braces was computed in lemma 3.2.2, it is zero if $k = 2l$ or $k = 2l + 1$ and otherwise we get a multiple of $dz^{\{k,2l,2l+1\}}$. Since we then wedge with $d\bar{z}_1$, the only non-zero terms arise when k or $2l$ or $2l+1$ is 1, that is, $k = 1$. We conclude

$$
\begin{aligned}
T^{\#}(\phi) &= C\phi_1 \sum_{l=1}^{n} (d\bar{z}_1) \wedge \left\{ (dz_{2l} \wedge dz_{2l+1})^{\bar{a}\bar{b}} (dz^{\{1\}})_{\bar{a}\bar{b}\overline{i_2}\ldots\overline{i_{2n-1}}} \right\} &(4.18)\\
&= C\phi_1 \sum_{l=1}^{n} (d\bar{z}_1) \wedge \left\{ \frac{1}{8n(2n-1)} dz^{\{1,2l,2l+1\}} \right\} &(4.19)\\
&= \frac{-i(1+\varepsilon^2)}{2} \phi_1 \sum_{l=1}^{n} dz^{\{2l,2l+1\}}. &(4.20)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|T^{\#}(\phi)\|^2 &= \frac{(1+\varepsilon^2)^2}{4} |\phi_1|^2 \sum_{l=1}^{n} \|dz^{\{2l,2l+1\}}\|^2 &(4.21)\\
&= \frac{n(1+\varepsilon^2)^2}{2^{2n+1}\varepsilon^2} |\phi_1|^2. &(4.22)
\end{aligned}
$$

To conclude:

Lemma 4.2.4 *With the conventions above*

$$
\|T^{\#}(\phi)\|^2 = \frac{n(1+\varepsilon^2)^2}{2^{2n+1}\varepsilon^2} |\phi_1|^2. \qquad (4.23)
$$

4.3 The Operator ∂T

We will also need to compute $(\partial T)(\phi)$ at the point $[1,0,\ldots,0]$. We have

$$
D_{[a} T_{|[\bar{b}\bar{c}]|d]} = 2\partial\bar{\partial}\omega_e \qquad (4.24)
$$

$$
\begin{aligned}
&= 2i(1+\varepsilon^2)\partial\left\{\bar{\partial}\left(\frac{1}{|\sigma|^4}\right)\wedge\theta\wedge\bar{\theta}+\frac{1}{|\sigma|^4}\theta\wedge\overline{\partial\theta}\right\} && (4.25)\\
&= 2i(1+\varepsilon^2)\left\{\partial\left[\bar{\partial}\left(\frac{1}{|\sigma|^4}\right)\wedge\theta\wedge\bar{\theta}\right]+\partial\left(\frac{1}{|\sigma|^4}\theta\wedge\overline{\partial\theta}\right)\right\} && (4.26)\\
&= 2i(1+\varepsilon^2)\left\{\partial\bar{\partial}\left(\frac{1}{|\sigma|^4}\right)\wedge\theta\wedge\bar{\theta}+\frac{1}{|\sigma|^4}\partial\theta\wedge\overline{\partial\theta}\right\} && (4.27)\\
&= 2i(1+\varepsilon^2)\left\{-2\omega\wedge\theta\wedge\bar{\theta}+\partial\theta\wedge\overline{\partial\theta}\right\}. && (4.28)
\end{aligned}
$$

Thus

$$
\begin{aligned}
D^{[\bar{a}}T_{\bar{b}\bar{c}}{}^{\bar{d}]} &= 4i(1+\varepsilon^2)\left\{-\sum_{k=2}^{2n+1}(dz_1\wedge dz_k)^{\bar{a}\bar{d}}\otimes(d\bar{z}_1\wedge d\bar{z}_k)_{\bar{b}\bar{c}}\right.\\
&\quad\left.+2\sum_{k=1}^{n}(dz_{2k}\wedge dz_{2k+1})^{\bar{a}\bar{d}}\otimes(d\bar{z}_{2k}\wedge d\bar{z}_{2k+1})_{\bar{b}\bar{c}}\right\}. \qquad (4.29)
\end{aligned}
$$

Therefore, if we set $C = 4i(1+\varepsilon^2)$,

$$
\begin{aligned}
&D^{\bar{a}}T_{[\bar{\imath}_1\bar{\imath}_2}{}^{\bar{b}}\phi_{|\bar{a}\bar{b}|\bar{\imath}_3\ldots\bar{\imath}_{2n}]A} =\\
&\quad -C\sum_{k=2}^{2n+1}\sum_{l=1}^{2n+1}(\phi_l)_A(d\bar{z}_1\wedge d\bar{z}_k)_{[\bar{\imath}_1\bar{\imath}_2}\left\{(dz_1\wedge dz_k)^{\bar{a}\bar{b}}(dz^{\{l\}})_{|\bar{a}\bar{b}|\bar{\imath}_3\ldots\bar{\imath}_{2n}]}\right\}\\
&\quad +2C\sum_{k=1}^{n}\sum_{l=1}^{2n+1}(\phi_l)_A(d\bar{z}_{2k}\wedge d\bar{z}_{2k+1})_{[\bar{\imath}_1\bar{\imath}_2}\left\{(dz_{2k}\wedge dz_{2k+1})^{\bar{a}\bar{b}}(dz^{\{l\}})_{|\bar{a}\bar{b}|\bar{\imath}_3\ldots\bar{\imath}_{2n}]}\right\}.
\end{aligned}
\qquad (4.30)
$$

For the first of these sums note that, by lemma 3.2.2, we have

$$
(d\bar{z}_1\wedge d\bar{z}_k)_{[\bar{\imath}_1\bar{\imath}_2}\left\{(dz_1\wedge dz_k)^{\bar{a}\bar{b}}(dz^{\{l\}})_{|\bar{a}\bar{b}|\bar{\imath}_3\ldots\bar{\imath}_{2n}]}\right\} = \frac{1}{8n(2n-1)\varepsilon^2}dz^{\{l\}}, \qquad (4.31)
$$

unless $l = 1$ or k when it is zero. Thus $l = 1$ does not occur and, for every other value of l, we get $2n-1$ terms all the same. The first sum is therefore

$$
\frac{-4i(1+\varepsilon^2)}{8n\varepsilon^2}\sum_{l=2}^{2n+1}\phi_l dz^{\{l\}}. \qquad (4.32)
$$

For the second sum we first consider the term when $l = 1$. Again from lemma 3.2.2 we have

$$(d\bar{z}_{2k} \wedge d\bar{z}_{2k+1})_{[\bar{i}_1\bar{i}_2} \left\{(dz_{2k} \wedge dz_{2k+1})^{\bar{a}\bar{b}}(dz^{\{1\}})_{|\bar{a}\bar{b}|\bar{i}_3\ldots\bar{i}_{2n}]}\right\} = \frac{1}{8n(2n-1}dz^{\{1\}}. \tag{4.33}$$

So we have n terms when $l = 1$, each of which is the same. When $l \neq 1$ we get one less term since, for each value of l there is a k so that $d\bar{z}_{2k}$ or $d\bar{z}_{2k+1}$ is missing from $dz^{\{l\}}$. The second sum therefore becomes

$$\frac{8i(1+\varepsilon^2)}{8(2n-1)}\phi_1 dz^{\{1\}} + \frac{8i(1+\varepsilon^2)(n-1)}{8n(2n-1)}\sum_{l=2}^{2n+1}\phi_l dz^{\{l\}}. \tag{4.34}$$

Putting this together we get

$$\partial T(\phi) = \frac{i(1+\varepsilon^2)}{(2n-1)}\phi_1 dz^{\{1\}} + \left(\frac{i(1+\varepsilon^2)(n-1)}{n(2n-1)} - \frac{i(1+\varepsilon^2)}{2n\varepsilon^2}\right)\sum_{l=2}^{2n+1}\phi_l dz^{\{l\}} \tag{4.35}$$

The object we want is $\langle n(2n-1)\partial T(\phi), \phi\rangle$. From the above we get

$$\langle n(2n-1)\partial T(\phi), \phi\rangle =$$

$$n(1+\varepsilon^2)|\phi_1|^2\|dz^{\{1\}}\|^2 + (1+\varepsilon^2)\left((n-1) - \frac{(2n-1)}{2\varepsilon^2}\right)\sum_{l=2}^{2n+1}|\phi_l|^2\|dz^{\{1\}}\|^2 \tag{4.36}$$

$$= \frac{n(1+\varepsilon^2)}{2^{2n}}|\phi_1|^2 + \frac{(1+\varepsilon^2)}{2^{2n}\varepsilon^2}\left((n-1) - \frac{(2n-1)}{2\varepsilon^2}\right)\sum_{l=2}^{2n+1}|\phi_l|^2. \tag{4.37}$$

This gives us

Lemma 4.3.1 *With the conventions above*

$$\langle n(2n-1)\partial T(\phi),\phi\rangle =$$
$$\frac{n(1+\varepsilon^2)}{2^{2n}}|\phi_1|^2+\frac{(1+\varepsilon^2)}{2^{2n+1}\varepsilon^4}\left(2\varepsilon^2(n-1)-(2n-1)\right)\sum_{l=2}^{2n+1}|\phi_l|^2. \quad (4.38)$$

4.4 Curvature

We want to compute $\langle 2n\Phi(\phi),\phi\rangle$, where Φ is the Ricci form of ω_ε plus the curvature of κ plus the curvature of $(L^*)^{\otimes m}$. Now ω is Einstein so its Ricci form is proportional to ω, from the observations in the previous section it must be $-(n+1)\omega$. Since ω_ε and ω have the same volume form up to constant multiples they must have the same Ricci form. In the previous section we saw that ω was chosen so that $[\frac{1}{2\pi}\omega]=c_1(L)$, so we choose a metric on L with curvature ω. From this we see $\Phi=-(2n+2+m)\omega$. Thus, at [1,0, …,0],

$$\Phi=-(2n+2+m)(dz_1\wedge d\bar z_1-dz_2\wedge d\bar z_2-\cdots-dz_{2n+1}\wedge d\bar z_{2n+1}) \quad (4.39)$$

Now if $\phi_{\bar\imath_1\ldots\bar\imath_{2n}A}=\sum_{k=1}^{2n+1}\phi_k dz^{\{k\}}$ in the notation of chapter 3, then

$$\begin{aligned}\Phi(\phi) &= \sum_{k=1}^{2n+1}(\phi_k)_A\Phi_{[\bar\imath_1}{}^{\bar a}(dz^{\{k\}})_{|\bar a|\bar\imath_2\ldots\bar\imath_{2n}]} && (4.40)\\ &= (2n+2+m)\sum_{k=1}^{2n+1}(\phi_k)_A\Big\{-(dz_1\wedge d\bar z_1)_{[\bar\imath_1}{}^{\bar a}(dz^{\{k\}})_{|\bar a|\bar\imath_2\ldots\bar\imath_{2n}]}\\ &\quad+\sum_{l=2}^{2n+1}(dz_l\wedge d\bar z_l)_{[\bar\imath_1}{}^{\bar a}(dz^{\{k\}})_{|\bar a|\bar\imath_2\ldots\bar\imath_{2n}]}\Big\} && (4.41)\end{aligned}$$

$$= \quad (2n+2+m)\left\{\sum_{k=2}^{2n+1}\frac{-\phi_k}{4n\varepsilon^2}dz^{\{k\}}+\sum_{k=1}^{2n+1}\phi_k\sum_{\substack{l=2\\ l\neq k}}^{2n+1}\frac{1}{4n}dz^{\{k\}}\right\} \quad (4.42)$$

$$= \quad (2n+2+m)\left\{\frac{1}{2}\phi_1 dz^{\{1\}}+\sum_{k=2}^{2n+1}\frac{(2n-1)\varepsilon^2-1}{4n\varepsilon^2}\phi_k dz^{\{k\}}\right\} \quad (4.43)$$

Here we have made use of lemma 3.2.2 to get line 4.42. From this

$$\langle 2n\Phi(\phi),\phi\rangle = 2n(2n+2+m)\left(\frac{|\phi_1|^2}{2}\|dz^{\{1\}}\|^2 + \frac{2n((2n-1)\varepsilon^2-1)}{4n\varepsilon^2}\sum_{k=2}^{2n+1}\|dz^{\{k\}}\|^2\right). \quad (4.44)$$

Thus we have proved:

Lemma 4.4.1 *With the conventions above*

$$\langle 2n\Phi(\phi),\phi\rangle = \frac{2n+2+m}{2^{2n}}\left(n|\phi_1|^2+\frac{(2n-1)\varepsilon^2-1}{2\varepsilon^4}\sum_{k=2}^{2n+1}|\phi_k|^2\right). \quad (4.45)$$

4.5 The Operator A

At last we put together all these calculations to compute $A(\phi)$ as in equation 4.1. From our earlier results we see that $A(\phi) = C_1|\phi_1|^2 + C_2\sum_{l=2}^{2n+1}|\phi_l|^2$ where

$$C_1 = \frac{(2n+2+m)n}{2^{2n}} - \frac{n(1+\varepsilon^2)}{2^{2n}} - \frac{n(1+\varepsilon^2)^2}{2^{2n+1}\varepsilon^2} \quad (4.46)$$

$$= \frac{n}{2^{2n+1}\varepsilon^2}\{(4n+1+2m)\varepsilon^2-1-2\varepsilon^4\}, \quad (4.47)$$

and

$$C_2 = \frac{2n+2+m}{2^{2n+1}\varepsilon^4}\{(2n-1)\varepsilon^2-1\} - \frac{(1+\varepsilon^2)}{2^{2n+1}\varepsilon^4}\{2\varepsilon^2(n-1)-(2n-1)\}$$

(4.48)

$$= \frac{1}{2^{2n+1}\varepsilon^4}\{(4n^2+2nm+2n-m-1)\varepsilon^2 - 3 - m + (2-2n)\varepsilon^4\}. \quad (4.49)$$

To guarantee positivity of A it will suffice to find a value of ε which makes $\{(4n+1+2m)\varepsilon^2-1-2\varepsilon^4\}$ and $\{(4n^2+2nm+2n-m-1)\varepsilon^2-3-m+(2-2n)\varepsilon^4\}$ simultaneously positive for all values of m and n. Having come this far we note simply that, when $\varepsilon = \frac{3}{2}$, these expressions become $9n + \frac{9m}{2} - \frac{71}{8}$ and $9n^2 - \frac{45n}{8} + \frac{9nm}{2} - \frac{13m}{4} + \frac{39}{8}$ respectively, and that both of these are positive for $n \geq 1$ and $m \geq \frac{1}{2}$. Thus we have proved the result we wanted, namely

Proposition 4.5.1 *If ϕ is any $(0,2n)$-form on $\mathbb{CP}_+^{2n+1}$ with values in the bundle $L^{\otimes m}$,$m \geq \frac{1}{2}$, and $\mathbb{CP}_+^{2n+1}$ is given the metric from equation 4.6 with $\varepsilon = \frac{3}{2}$. Then $A(\phi) \geq 0$ and $A(\phi) = 0$ if and only if $\phi = 0$.*

With this we get immediately the vanishing theorem:

Theorem 4.5.2 *Suppose $\mathcal{Z}$ is a compact manifold constructed as a quotient of $\mathbb{CP}_+^{2n+1}$ by a discrete subgroup of $Sp(n,1)$, and suppose L is the contact line bundle on $\mathcal{Z}$, then $H^1(\mathcal{Z}, \mathcal{O}(L^{\otimes m})) = 0$ for all $m > 1$. Furthermore, if L has a square root bundle $L^{\frac{1}{2}}$ on $\mathcal{Z}$, then $H^1(\mathcal{Z}, \mathcal{O}(L^{\frac{1}{2}})) = 0$ also.*

Proof

Just note that, since $A(\phi) > 0$, by proposition 4.5.1, we may appeal to theorem 3.1.3 for our result. □

4.6 The General Case of Negative Scalar Curvature

In this section we will extend the estimates and vanishing theorems of the previous sections to the most general case of the twistor space, $\mathcal{Z}$, of a quaternionic-Kähler manifold, M, with negative scalar curvature. The method will be to use an observation, due to LeBrun, [15], that we can osculate to second order the contact structure of such a manifold at a point, by that of the standard model space $\mathbb{CP}_{+}^{2n+1}$ we will also show that we can simultaneously osculate the real structure to second order. This will be sufficient to give the 1-jet of the metric at a point in M and hence the 1-jet of the metric and contact forms on $\mathcal{Z}$. Since our calculations only involve knowledge of the first derivatives of these objects, (the metric is Einstein so the Ricci form is a multiple of the associated two-form), we may choose coordinates at a point so that the requisite formulae are exactly those of the preceding chapter.

Remark 4.6.1 Of course the use of exponential charts allows us to make any two metrics equivalent to second order at a point, and Darboux's theorem says any two contact structures are locally equivalent. The point is that, in the case of a twistor space, we may accomplish both simultaneously since both are encoded in the complex structure of the manifold.

To begin with

Lemma 4.6.2 (LeBrun) *Let $\mathcal{L}$ be any twistor line in $\mathcal{Z}$. There is an isomorphism between the second infinitesimal neighbourhood of $\mathcal{L} \subset \mathcal{Z}$ and the second infinitesimal neighbourhood of a twistor line $\mathbb{P}_1 \subset \mathbb{CP}_+^{2n+1}$. Moreover, this isomorphism sends the 1-jet of the contact distribution to the 1-jet of the distribution on $\mathbb{CP}_+^{2n+1}$.*

Proof

This is given in [15]. The idea is that, by well known calculations, the second neighbourhood of a line in a twistor space is standard. That is, isomorphic to the second neighbourhood of the zero-section in the normal bundle, $2n\mathcal{O}(1)$. Now since the contact structure allows us to identify this bundle as the contact distribution, D, we get a contact isomorphism on the level of first neighbourhoods. LeBrun computes that the contact automorphisms of the second neighbourhood which leave fixed the first neighbourhood are given by the sheaf $\mathcal{O}(2) \otimes \odot^2(2n\mathcal{O}(-1))$, (i.e. $\mathcal{O}(L) \otimes \odot^2 \mathcal{O}(D^*)$). Since $H^1(\mathbb{P}_1, \mathcal{O}(2) \otimes \odot^2(2n\mathcal{O}(-1))) = 0$ we see that this isomorphism may be extended to the second neighbourhood. □

We also want to show that we can extend this isomorphism to include the real structures. To this end let σ_1 be the real structure on the second neighbourhood of a line in $\mathbb{CP}_+^{2n+1}$ inducing the standard metric of negative scalar curvature on $\mathbb{HP}_+^n$, and let σ_2 be the real structure induced by that on $\mathcal{Z}$. Note that σ_2 induces another metric of scalar curvature -1 by the construction outlined in chapter 1 so that the forms ω_E are both definite and the metrics $\langle\ ,\ \rangle_H \otimes \langle\ ,\ \rangle_E$ are either both positive definite or both negative definite. Furthermore the fact that σ_i preserves the contact structure translates, in the case of the second neighbourhood of the zero section in $2n\mathcal{O}(1)$ to the assertion that $\sigma_{i*}(fibre) = fibre$.

Consider $\hat{\sigma}_i = \sigma_i|_{\mathbb{P}_1}$, these are two real structures on $\mathbb{P}_1$ without fixed points. These arise exactly as the projectivization of quaternionic structures on $\mathbb{C}^2$. Now any two such are conjugate by a linear map which descends to a biholomorphism of $\mathbb{P}_1$ conjugating $\hat{\sigma}_1$ and $\hat{\sigma}_2$. By extending this to any contact isomorphism on the second neighbourhood we may assume that σ_1 and σ_2 are the same on $\mathbb{P}_1$ and induce the same 1-jet of a quaternionic structure on $H^0(\mathbb{P}_1, \mathcal{O}(1))$.

Note that $\sigma_1\sigma_2$ is a biholomorphism of second neighbourhoods fixing $\mathbb{P}_1$

and sending fibres to fibres. On the first neighbourhood this is an element of $Aut(2n\mathcal{O}(1))$ but these are determined by their action on a single fibre. Now σ_1 and σ_2 both induce quaternionic structures on $H^0(\mathbb{P}_1, \mathcal{O}(D \otimes L^{-\frac{1}{2}}))$ inducing definite metrics of the same sign via ω_E. It is easy to see that two quaternionic structures on a vector space which induce a metric of the same signature with respect to a non-degenerate two form are conjugate via a linear map, (just choose one which sends a pseudo-orthonormal basis with respect to the first metric to one which is pseudo-orthonormal with respect to the second.) By tensoring with the identity in $Aut(\mathcal{O}(-1))$ we get an element of $Aut(2n\mathcal{O}(1))$ which conjugates σ_{1*} and σ_{2*}. Thus we may assume that σ_1 and σ_2 agree to order two on $\mathbb{P}_1$ and their one jets agree on the fibres. Thus $\sigma_1\sigma_2$ is a contact isomorphism of the second neighbourhood fixing the first neighbourhood. As computed by LeBrun these are exactly $\mathcal{O}(2) \otimes \odot^2(2n\mathcal{O}(-1))$. Now the "standard " structure σ_1 is actually induced from one on $\mathbb{CP}_{2n+1}$, namely $[z_0, \ldots, z_{2n+1}] \mapsto [-\bar{z_1}, \bar{z_0}, \ldots, -\bar{z_{2n+1}}, \bar{z_{2n}}]$ and this has second derivative equal to zero. Thus the second derivative of σ_2 which is obtained from that of σ_1 by an element of $\mathcal{O}(2) \otimes \odot^2(2n\mathcal{O}(-1))$ must also be zero. We have proved:

Lemma 4.6.3 *Let $\mathcal{L}$ be any twistor line in $\mathcal{Z}$. There is an isomorphism between the second infinitesimal neighbourhood of $\mathcal{L} \subset \mathcal{Z}$ and the second* *infinitesimal neighbourhood of a twistor line* $\mathbb{P}_1 \subset \mathbb{CP}_+^{2n+1}$. *Moreover, this isomorphism can be chosen to send the 1-jet of the contact distribution to the 1-jet of the distribution on* $\mathbb{CP}_+^{2n+1}$ *and the 2-jet of the real structure to the 2-jet of the real structure on* $\mathbb{CP}_+^{2n+1}$.

We note that the construction from [15] outlined in chapter 1 allows us to construct the 1-jet of the indefinite Kähler metric at a point in $\mathcal{Z}$ from the second infinitesimal neighbourhood of a twistor line through that point. If we then choose coordinates near $p \in \mathcal{Z}$ so that the 1-forms $\{dz_k\}_{k=1}^{2n+1}$ at p agree with the same forms at [1,0, ...,0] in $\mathbb{CP}_+^{2n+1}$, then the indefinite kähler metric, ω, will agree with that of (4.9), for $\mathbb{CP}_+^{2n+1}$, and then if we choose a local section, σ, of $L^{\frac{1}{2}}$ so that $|\sigma|^4 = 1$, and $d(|\sigma|^4) = 0$ at p, then the corresponding local representative, θ, for the contact form will satisfy $\theta = -dz_1$, and $\partial\theta = \sum_{k=1}^{n} dz_{2k} \wedge dz_{2k+1}$ at p, in agreement with the formulas in section 4.1 for the standard form at [1,0, ...,0].

Bearing this in mind we define a new metric on $\mathcal{Z}$ via

$$\omega_\varepsilon = -\omega + \frac{1}{|\sigma|^4}\theta \wedge \bar{\theta}, \tag{4.50}$$

and note that this has the same torsion and ∂T operators at p as the metric on $\mathbb{CP}_+^{2n+1}$ has at [1,0, ...,0]. Similarly, the curvature of L^* is $-\omega$, and the Ricci form, which is also the curvature of κ, is $-(n+1)\omega$ since the volume form for ω_ε is a constant multiple of that for ω. Therefore the operator $A(\phi)$

as in 4.1, is the same, in these coordinates as that for $\mathbb{CP}_+^{2n+1}$ computed in the previous section. Since we already noted that when $\varepsilon = \frac{3}{2}$ this is positive for $n \geq 1$ and $m \geq \frac{1}{2}$, we have:

Theorem 4.6.4 *Let M be a compact quaternionic-Kähler manifold of dimension $4n$ with negative scalar curvature. In the case $n = 1$ we take this to mean a self-dual, Einstein 4-manifold with negative scalar curvature. Let $\mathcal{Z}$ be the twistor space of M, and let L denote the contact line bundle of $\mathcal{Z}$. Then $H^1(\mathcal{Z}, \mathcal{O}(L^{\otimes m})) = 0$ for all $m \geq 1$. Furthermore, if L has a square root bundle, $L^{\frac{1}{2}}$, then $H^1(\mathcal{Z}, \mathcal{O}(L^{\frac{1}{2}})) = 0$ also.*

Proof

As noted in section 4.5, since $A(\phi)$ is positive we may immediately appeal to theorem 3.1.3 for the result. □

Chapter 5

A Rigidity Theorem

5.1 Introduction

In this chapter we will use the vanishing theorem for $H^1(\mathcal{Z}, \mathcal{O}(L))$ given in the previous chapter to prove that quaternionic Kähler metrics with negative scalar curvature on a compact manifold have no non-trivial deformations through quaternionic-Kähler metrics. This was first proved by LeBrun [14] in the case of positive scalar curvature. The idea is that such deformations of a quaternionic-Kähler manifold will correspond to infinitesimal deformations of the twistor space preserving the contact structure and that these objects are exactly $H^1(\mathcal{Z}, \mathcal{O}(L))$. More precisely:

Proposition 5.1.1 (LeBrun) *Let $\mathcal{Z}$ be a complex contact manifold with contact distribution $D \subset T\mathcal{Z}$; let $L = T\mathcal{Z}/D$ be the contact line bundle of $\mathcal{Z}$, and suppose that $H^1(\mathcal{Z}, \mathcal{O}(L)) = 0$. Then any small complex contact deformation of $\mathcal{Z}$ is trivial in the following sense: if $\pi : \hat{\mathcal{Z}} \to \mathbb{R}$ is a smooth proper map whose fibres $\mathcal{Z}_t = \pi^{-1}(t)$ are complex contact manifolds with complex contact structure depending smoothly on t, and if $\mathcal{Z} = \mathcal{Z}_0$, there is a neighbourhood I of $0 \in \mathbb{R}$ such that $\pi^{-1}(I) \cong \mathcal{Z} \times I$ in a fibre-wise complex contact manner. Moreover, if $H^1(\mathcal{Z}_t, \mathcal{O}(L_t)) = 0$ for all $t \in \mathbb{R}$, where $L_t \to \mathcal{Z}_t$ is the contact line bundle analogous to $L \to \mathcal{Z}$, then $\hat{\mathcal{Z}} \cong \mathcal{Z} \times \mathbb{R}$ in a fibre-wise complex contact manner.*

Proof

(Sketch) First note that by Darboux's theorem [1] $\mathcal{Z}$ has a holomorphic atlas in which the contact distribution is the orthogonal space of

$$\theta = dz_{2n+1} + \sum_{k=1}^{n} z_k dz_{n+k} \tag{5.1}$$

Notice that $\mathcal{O}(L)$ is isomorphic to the sheaf of holomorphic vector fields on $\mathcal{Z}$ preserving the contact structure, $\mathcal{C}$, consisting of those vector fields V for which $\mathcal{L}_V\mathcal{O}(D) \subset \mathcal{O}(D)$. If $\theta_{\mathcal{Z}}$ is the contact form this is given by $V \mapsto V \lrcorner \theta_{\mathcal{Z}}$, the inverse is given in local coordinates by solving

$$V \lrcorner \theta = f, \quad V \lrcorner d\theta \equiv -df \bmod \theta, \tag{5.2}$$

where $f\theta$ is any given local section of $\mathcal{O}(L)$. Denote this inverse by μ. Now cover $\hat{\mathcal{Z}}$ with charts (U_J, ϕ_J), which take open sets in $\mathcal{Z}_t$ to open sets in $\mathbb{C}^{2n+1} \times \{t\}$ and take D_t to the orthogonal space of θ. Now the idea is to proceed as in Kodaira-Spencer theory [12], we produce a cocyle, $V_{JK}(t) \in \Gamma(U_J \cap U_K, \mathcal{O}(L_t))$ by differentiating the transition functions,

$$V_{JK}(t) = (\phi_J^{-1}(t))_* \frac{d}{dt} \phi_{JK}(t) \tag{5.3}$$

where $\phi_J(t)(z) = \phi_J(z,t)$. Now set $\theta_{JK}(t) = \mu_t^{-1}(V_{JK}(t))$.

Now since $H^1(\mathcal{Z}, \mathcal{O}(L)) = 0$ we have $H^1(\mathcal{Z}_t, \mathcal{O}(L_t)) = 0$ in a neighbourhood I of 0 by upper semi-continuity of dimension. Thus we find sections $\theta_J(t)$ with $\theta_{JK} = \theta_J - \theta_K$. Set $V_J(t) = \mu_t \theta_J(t)$ and then on $\theta_J(U_J \cap U_K)$ solve the equation

$$d\hat{Z}/dt = -\phi_{J*} V_J, \quad \hat{Z}(Z,0) = Z \tag{5.4}$$

The solutions give new coordinates $(\hat{Z}, t)$ and by refining the cover we get new transition functions which are independent of t.

Now notice that this gives rise to isomorphisms of $\pi^{-1}(I_\alpha)$ with $\mathcal{Z}_{t_\alpha} \times I_\alpha$ for intervals I_α covering I. Finally if $H^1(\mathcal{Z}_t, \mathcal{O}(L_t)) = 0$ for all $t \in \mathbb{R}$ we may take $I = \mathbb{R}$. □

5.2 The Theorem

Now we may prove the promised rigidity theorem, the proof is modelled on that in [14] but we save ourselves a step because of the negative scalar curvature.

Theorem 5.2.1 *Let M be a compact $4n$-manifold and let $\{g_t\}$ be a family of quaternionic-Kähler metrics on M of fixed volume, depending smoothly on $t \in \mathbb{R}$. (In the case $n = 1$ we take quaternionic-Kähler to mean self-dual-Einstein). If g_0 has negative scalar curvature then there is a family of diffeomorphisms $\{\psi_t : M \to M\}$ depending smoothly on t such that $\psi_t^* g_t = g_0$.*

Proof

First note that we may assume all g_t have negative scalar curvature. For, if not, let $R(t)$ be the (constant) scalar curvature of g_t, and let I be the largest open interval containing 0 on which $R(t)$ is negative. The theorem applied to I yields that R is constant on I, and so by maximality $I = \mathbb{R}$.

Now, under this assumption, let $\pi : \hat{\mathcal{Z}} \to \mathbb{R}$ be the family whose fibres are the twistor spaces, $\mathcal{Z}_t$ of the manifolds (M, g_t). This is a family of complex contact manifolds satisfying $H^1(\mathcal{Z}_t, \mathcal{O}(L_t)) = 0$ for all $t \in \mathbb{R}$, so that by the previous proposition there is a diffeomorphism $\hat{\psi} : \mathcal{Z}_0 \times \mathbb{R} \to \hat{\mathcal{Z}}$ which sends $\mathcal{Z}_0 \times \{t\}$ biholomorphically onto $\mathcal{Z}_t$ in a manner preserving the contact structure. Set $\hat{\psi}_t(z) = \hat{\psi}(z,t)$.

Let σ_t be the real structure of $\mathcal{Z}_t$, pull this back to $\mathcal{Z}_0$ to get $\rho_t = \hat{\psi}_t^{-1}\sigma_t\hat{\psi}_t$. Now ρ_t is a family of real structures on $\mathcal{Z}_0$ preserving the contact structure so $\phi_t = \rho_t\rho_0$ is a family of contact transformations of $\mathcal{Z}_0$. The derivative of such a family is a holomorphic vector field on $\mathcal{Z}_0$ but, by the Penrose transform, these are in one-to-one correspondence with conformal-Killing fields on M. By a theorem of Bochner [5], if M has negative definite Ricci curvature then there are no conformal-Killing fields on M. Thus ϕ_t is constant and since $\phi_0 = id$ we have that $\rho_t = \rho_0$ for all t.

The fibres of $\pi_t : \mathcal{Z}_t \to M$ are precisely the σ_t-invariant elements of a complex analytic family of curves so that $\hat{\psi}_t$ sends the fibres of π_0 to the fibres of π_t. Therefore there are diffeomorphisms $\psi_t : M \to M$, making the diagram

$$\begin{array}{ccc} \mathcal{Z}_0 & \xrightarrow{\hat{\psi}_t} & \mathcal{Z}_t \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_t} \\ M & \xrightarrow{\psi_t} & M \end{array}$$

commute. Since $\hat{\psi}_t$ preserves contact and real structures, and since these determine the metric up to an overall constant we have that $\psi_t^* g_t = c_t g_0$ for some $c_t > 0$. But since g_t and g_0 have the same volume $c_t = 1$ and the result is proved. □

Remark 5.2.2 As noted by LeBrun in the same paper, it should be possible to give a proof which avoids mention of the twistor space. The space $H^1(\mathcal{Z}, \mathcal{O}(L))$ corresponds via the Penrose transform to a space of solutions of a linear differential equation which should be interpreted as "linearized quaternionic metrics modulo diffeomorphisms", and the vanishing theorem of chapter 4 should correspond to a Bochner style vanishing theorem on M. Unfortunately neither of these interpretations is very clear at this time.

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