PSEUDODIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS WITH DILATIONS

A Dissertation presented

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Abstract of the Dissertation

Pseudodifferential Operators on Nilpotent Lie Groups with Dilations

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In this thesis we will be concerned with polynomial group laws on $\mathbb{R}^n$ which respect weighted dilations. The kernel of a pseudodifferential operator is a translate of what is called a core. Our main result is that if compactly supported cores $g_{1u}(x)$ and $g_{2u}(x)$ have asymptotic expansions in quasihomogeneous distributions of increasing order, then the composition of their associated pseudodifferential operators has an asymptotic expansion whose terms may be written as a certain adopted convolution of the terms in the original expansions.

Dedicated to my parents,

Dr. Raymond Polin

and

Constance Faye Polin.

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Notations and Abbreviations

$v \cdot u$: group addition (it will also be denoted by vu).

AB: $\{a \cdot b / a \in A,\ b \in B\}$

A^{-1}: $\{a^{-1} / a \in A\}$

$\mathbb{R}^n \setminus 0$: $\mathbb{R}^n \setminus \{0\} = \{x \in \mathbb{R}^n / x \neq 0\}$

iff: if and only if

w.r.t.: with respect to

ETS: Enough to show

WLOG: Without lost of generality

$A \subsetneq\subsetneq B$: A has compact closure which is a subset of the interior of B.

$f \in L^1_\infty$: $\int_{|x|>\varepsilon} |f|\,dx < \infty, \quad \forall \varepsilon > 0.$

X_j, Y_j, U_j etc.: As will be explained later, left invariant vector fields w.r.t. x,y,u respectively.

$\int dx$: Not only for use of Liebesgue integral, but also as an abuse of notation when working with distributions.

Preface

Consider the usual Euclidean addition on $\mathbb{R}^n$. Given pseudodifferential operators P,Q with properly supported symbols s_P, s_Q, the Kohn-Nirenberg formula gives an asymptotic expansion for the symbol s_{QP} of the composition QP as

$$\sum_{\alpha} \frac{i^{\|\alpha\|}}{\alpha!} D_{\xi}^{\alpha} s_{Q,u}(\xi) D_{u}^{\alpha} s_{P,u}(\xi). \tag{0.1}$$

Taking the inverse fourier transform, we formally expect a type of asymptotic expansion $\sum_{\alpha} \frac{i^{\|\alpha\|}}{\alpha!}(\#^{\alpha}\check{s}_{Q,u}) * D_{u}^{\alpha}\check{s}_{P,u}$, where $\#^{\alpha}\check{s}(x) = (-x)^{\alpha}\check{s}(x)$.

In this thesis we produce a comparable result in consideration of more general addition laws, $\check{s}_u$'s with a type of asymptotic expansion in quasihomogeneous distributions, and by use of a type of generalized form of convolution.

It would seem appropriate to state some of the history behind this dissertation.

Although (0.1) is called the Kohn-Nirenberg formula [K], the original idea of ψDO's (i.e. pseudodifferential operators) was in essence due to Mikhlin [Mi] and Calderon and Zymund [Ca]. This bit of history is important since Mikhlin, Calderon and Zymund used

$$(\mathcal{K}f)(x) = (K_u * f)(x) \tag{0.2}$$

which is a convolution of a kernel and a function.

Kohn and Nirenberg had the idea of introducing symbols and the fourier transform into the definition of ψDO's instead of (0.2). This has the appealing effect of converting convolution into multiplication; but this ceses to be given when working with nilpotent groups (i.e. when convolutoin $f_* g$ is defined $\int f(xy^{-1})g(y)dy$).

One should note that a crucial paper in the origin of ψDO's was written by Hormander [H].

The idea of using operators of the form (0.2) on groups, when doing analysis, is due to Folland and Stein [FS]. Dynin [D] then sketched a DO calculus for H^n (i.e. the Heisenberg group), and other groups, using symbols of various kinds. Melin [Me] also had a sort of calculus on groups.

Taylor [T] adopted (0.2) as his definition of a DO on $\mathbf{H}^n$, then quickly switched gears and developed a "symbolic calculus." Beals and Greiner [Be], last year, published a book detailing a calculus of DO on $\mathbb{H}^n$, again using symbols. Cummins [Cu] discussed 3-step nilpotent Lie groups, using (0.2).

Analytic ψDO's on R^n were developed by Boutet de Monvel and Kree [Bo]. Using (0.2), Geller [G] discovered the product rule on $\mathbb{H}^n$, and developed an analytic calculus on $\mathbb{H}^n$.

This year, Nagel, Rosay, Stein, and Wainger [NR] (and unpublished work) have invented "Non-Isotropic Smoothing Operators" which extended the ideas implicit in (0.2) beyond groups. Again, the idea is to construct a calculus using just kernels. But in their situation, one can't hope for exact formulas.

For readers familiar with the general literature (e.g. Nagel and Stein [R]), skimming Chapters 1,2,4 is encouraged. For those comfortable with Geller [G], the same may be done with Chapter 3.

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1. Homogeneous Functions and Rhom_k Distributions.

Purpose of this chapter:

We "review" basic facts concerning weighted homogeneous distributions.

The notation and definitions which precede (1.16) are taken from Geller [G].

Consider the weight $a = (a_1,\ldots,a_n)$ where the a_i's are positive rationals.

(1.1) Define Q as $\sum a_i$.

(1.2) Define dilation $D_r x$ as $(r^{a_1}x_1,\ldots,r^{a_n}x_n)$, $\forall x \in \mathbb{R}^n$, $\forall r > 0$.

(1.3) Define $D^r f(x)$ as $f(D_r x)$, $\forall$ functions $f : \mathbb{R}^n \to \mathbb{C}$.

(1.4) Define $(f,g) = \int f(x),\ g(x)dx$, $\forall$ functions f,g.

(1.5) Define (F,g) as functional F acting on g.

Notice $(f(D_r(x)),g(x)) = (f(x),r^{-Q}g(D_{1/r}(x)))$ which suggests...

(1.6) Define $D_r f(x)$ as $r^{-Q}f(D_{1/r}(x))$ meaning $r^{-Q}D^{1/r}f(x)$.

Hence $(f(D_r x),g(x)) = (D^r f,g) = (f,D_r g)$.

Considering distribution F,

(1.7) Define $(D^r F,g)$ as $(F,D_r g)$.

(1.8) Define $(D_r F, g)$ as $(F, D^r g)$.

Consider complex number k.

(1.9) Define "f homogeneous function of degree k" iff $D^r f = r^k f$, $\forall r > 0$.

(1.10) Define "F homogeneous distribution of degree k" iff $D^r F = r^k F$, $\forall r > 0$.

Note: Since we can multiply a_i's (and k) by the least common multiple of $\{a_i\}$, we have WLOG a_i's being positive integers.

(1.11) Define $|x|$ as "homogeneous norm function" meaning a homogeneous function of degree 1, smooth away from 0, such that $|x| \geqq 0$ $\forall x$ with $|x| = 0$ iff $x = 0$.

(1.12) Example: If $A = \prod_{i=1}^{n} a_i$ and $a_i' = \frac{A}{a_i}$ then we could define $|x| = (\sum x_i^{2a_i'})^{1/2A}$.

Note: $|D_r x| = r|x|$ but $|cx|$ is usually not $|c||x|$.

(1.13) Define $\|x\|$ as $(\sum_{j=1}^{n} (x_j)^2)^{1/2}$ meaning the usual Euclidean norm, meaning example (1.12) with $a = (1,1,\ldots,1)$.

Note: $\|x\| = 0(|x|)$ as $x \to 0$ while $|x| = 0(\|x\|)$ as $\|x\| \to \infty$.

Consider the multiindex $\beta \in (\mathbb{Z}^+)^n$ where $\mathbb{Z}^+ = \{0,1,2,3,\dots\}$.

(1.14) Define $|\beta| = a\cdot\beta$.

(1.15) Define $\|\beta\| = \sum\beta_i$.

Having stated our first round of definitions, we state the following trivial propositions:

(1.16) Proposition: If f is C^∞ near $D_r x$ then $\partial_x^\beta(f(D_r x)) = r^{|\beta|}(\partial^\beta f)(D_r x)$.

(1.17) Proposition: If f is homogeneous distribution degree k and C^∞ on $\mathbb{R}^n\backslash 0$ then $\partial^\beta f$ is homogeneous degree $k - |\beta|$ while $x^\beta f$ is homogeneous degree $k + |\beta|$.

Consider the map $\mathbb{R}^n \to M' : u \mapsto g_u$ where M' is the continuous linear functionals on $M = C^\infty, C_c^\infty$, or S.

(1.18) Define "g_u is a C^∞ distribution" iff (g_u, φ) is C^∞ $\forall\varphi \in M$ meaning iff $\partial_u^\alpha(g_u,\varphi)$ exists $\forall\alpha \in (\mathbb{Z}^+)^n$ $\forall u \in \mathbb{R}^n$ $\forall\varphi \in M$.

(1.19) Proposition: If g_u is a C^∞ distribution; $\mathbb{R}^n \to D^1$ then $\partial_u^\ell g_u$ is also, and $(\partial_u^\ell g_u,\varphi) = \partial_n^\ell(g_n,\varphi)$ $\forall\varphi \in C_c^\infty$.

Proof: ∂u_j; g_u is itself in D^1 (w.r.t. x) by Banach-Steinhaus Theorem, see Peterson [P].

(1.20) <u>Define</u> $F(g)(\xi)$ and $\hat{g}(\xi)$ as both meaning the fourier transform of g, meaning $(2\pi)^{-n/2}\int g(x)e^{-ix.\xi}dx$.

(1.21) <u>Define</u> $F^{-1}(g)(x)$ and $\check{g}(x)$ as both meaning the inverse fourier transform of g, meaning $(2\pi)^{-n/2}\int g(\xi)e^{+ix.\xi}d\xi$.

Definitions (1.20) and (1.21) are from Taylor [T] as well as the following principle: "the author will be found guilty of lapses in the text regarding factors of powers of 2π, which may be omitted from many formulas."

(1.22) <u>Define</u> D_j as $\frac{1}{i}\partial_j$ where $i^2 = -1$

<u>Note</u>: $\widehat{D^{\alpha}f} = \xi^{\alpha}\hat{f}$.

(1.23) <u>Proposition</u>: If g_u is a distribution: $\mathbb{R}^n \to S'$ and C^∞ in u then $\hat{g_u}$ is also.

(1.24) "$K_u \varepsilon \text{Rhom}_k$" as "$K_u$ is a regular homogeneous C^∞ distribution of degree k" iff $u \mapsto K_u$ is a map $\mathbb{R}^n \to S'$, such that $K_u(x)$ is C^∞ in $\mathbb{R}^n \times (\mathbb{R}^n\backslash 0)$, and $D^r\partial_u^{\ell}K_u = r^k\partial_u^{\ell}K_u$, $\forall r > 0$.

<u>Note</u>: We have the implicit condition that K_u is C^∞ distribution.

It is trivial to show...

(1.25) <u>Proposition</u>: If $K_u \in \text{Rhom}_k$ then $\partial_x^\beta K_u \in \text{Rhom}_{k-|\beta|}$, $x^\beta K_u \in \text{Rhom}_{k+|\beta|}$, and $\partial_u^\ell K_u \in \text{Rhom}_k$.

Consider for a moment $a = (1,1,\ldots,1)$. While we know that $\Delta^n|\xi|^1$ must be in Rhom_{1-2n}, it cannot simply be $c|\xi|^{1-2n}$ which is not in L^1_{loc}. Hence the questions of "what do elements of Rhom_k look like?" and "what are their basic properties?"

Answering the former is done in the next chapter on the Λ-transform. Answering the latter question begins immediately.

(1.26) <u>Proposition</u>: If $K_u \in \text{Rhom}_k$, and $G_u(x)$ is $= K_u(x)$ when $x \neq 0$ while $= 0$ when $x = 0$ then $\partial_u^\ell G_u$ is C^∞ in $\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$ and homogeneous degree k.

The remaining propositions of this chapter are, in effect, results from Nagel and Stein [N]. "In effect" meaning we include our parameter u in K_u.

(1.27) <u>Proposition</u>: $|x|^k \in L^1_{loc}$ iff Re $k > -Q$ while $|x|^k \in L^1_\infty$ iff Re $k < -Q$. In other words, Q is the "critical index."

(1.28) <u>Proposition</u>: $\exists$ measure σ on the unit sphere $\{x \in \mathbb{R}^n / |x| = 1\}$; $\forall$ functions $f(y)$ homogeneous degree k and define on $0 \leq a < |y| < b \leq \infty$

we have $\int_{0<|y|<b} f(y)\,dy = \left(\int_{|x|=1} f(x)\,d\sigma(x)\right)\left(\int_a^b r^{k+Q-1}dr\right).$

where $\int_n^b r^{k+Q^{-1}}\,dr = \begin{cases} \frac{1}{K+Q} r^{k+Q}\Big|_a^b, & k \neq -Q \\ \log(^b/a), & k = -Q. \end{cases}$

(1.29) <u>Proposition</u>: Given $0 < \int_{|x|=1} |f|\,d\sigma < \infty$,

f homogeneous degree K,

we have $f \varepsilon L^1_{loc}$ iff Re $k > -Q$

while $f \varepsilon L^1_{\infty}$ iff Re $k < -Q$.

As a result of these we have:

(1.30) <u>Proposition</u>: Consider $f_n \not\equiv 0$ a.e. w.r.t. x.

$f_u(x)$ is a L^1_{loc} function (w.r.t. x), is C^∞ in $\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$,

and $D_r \partial^{\ell}_n f_n(x) = r^k \partial^{\ell}_n (x)$, $\forall x \neq 0$, $\forall r > 0$

iff $f \varepsilon$ $Rhom_k$ where Re $k > -Q$.

Realizing that the dot product of $D_r x$ and y = the dot product of x and $D_r y$, we have...

(1.31) <u>Proposition</u>: $\widehat{D^r F} = D_r \hat{F}$ and $\widehat{D_r F} = D^r \hat{F}$, $\forall F \varepsilon S^1$.

(1.32) <u>Proposition</u>: If f homogeneous degree k on $\mathbb{R}^n \backslash 0$ and $f \varepsilon L^1$ then $\hat{f}$ is homogeneous degree $-Q-k$.

(1.33) <u>Proposition</u>: If F homogeneous distribution of degree k then $\hat{F}$ is homogeneous distribution of degree $-Q-k$.

(1.34) Example: The delta function is homogeneous of degree $-0-k = -k$ while the constant function is of degree 0.

(1.35) Lemma: If N is a bounded neighborhood of 0, $K_n \in Rhom_k$, and $\varphi \in C_0^\infty$ with $\varphi \equiv 1$ on N then $[(1-\varphi)K_u]^\wedge$ is C^∞ on $\mathbb{R}^n \times (R^n \setminus 0)$.

(1.36) Proposition: If $K_u \in Rhom_k$ then $\widehat{K_u} \in Rhom_{-Q-k}$. See Nagel and Stein [N] p.9.

2. Λ-transform and Quasihomogeneous Distributions K^κ.

Background: Consider $L(D)u = f$, or in other words, $(L(\xi)\hat{u}(\xi))^\vee = f$. Formally, the solution u is $(\frac{1}{L(\xi)}\hat{f})^\vee = (\frac{1}{L(\xi)})^\vee * f$. If $(\frac{1}{L(\xi)})^\vee$ can be made to make sense, then it would be the "fundamental solution of $L(D)$." The function $L(\xi)$ is said to be the "symbol" for $L(D)$. Consider $L(\xi)$ to be homogeneous.

For example, the symbol of the Laplacian Δ is $\|\xi\|^2$. For $n > 2$, $(\frac{1}{\|\xi\|^2}) \in \text{Rhom}_{-2}$ when $a = (1,\ldots,1)$, while $(\frac{1}{\|\xi\|^2})^\vee$ is a rotation-invariant element of Rhom_{2-n}, namely $\frac{c}{\|\xi\|^{n-2}}$ which is the well-known fundamental solution of the Laplacian.

Notice that the symbol of Δ^p is $\|\xi\|^{2p}$ whose reciprocal is not L^1_{loc} (nor a distribution) when $p \geq n/2$. Hence we must take a fourier transform that is in some sense more general than that used on distributions. Of course, this new fourier transform should agree with the old on $\text{Rhom}_{k>-Q}$.

One approach is to somehow transform the homogeneous function $\frac{1}{L}$ into a distribution and then apply the distributional fourier transform. The easiest way to remove the non-L^1_{loc}-ness of $\frac{1}{L(\xi)}$ would be to use a principal-value-style deletion of Taylor series terms at 0.

Purpose of this chapter:

We will use such a deletion of terms of $\frac{1}{L(\xi)}$ to present a basic study of Rhom_k. The material in this chapter is standard; see e.g. Geller [G]

(2.1) Define $M(f) = \int_{|x|=1} f(x)\,d\sigma(x)$

(2.2) Define δ^{α} as $\partial^{\alpha}\delta$ where δ is "the delta function."

Consider $G_u(x) : \mathbb{R}^n\mathbb{R}^n \times (\mathbb{R}^n\setminus 0) \to \mathbb{C}$ where $G_u(x)$ is C^∞ in $\mathbb{R}^n \times (\mathbb{R}^n\setminus 0)$ and where $D^r\partial_n^{\ell}G_u(x) = r^k\partial_n^{\ell}G_n(x)$, $\forall x \in R^n\setminus 0$, $\forall r > 0$.

(2.3) Define Λ_G as the "Λ transform of G" meaning

$$\int_{|\xi|\leq 1} G_u(\xi)\,[\varphi(\xi) - \sum_{|\alpha|\leq N} \partial^{\alpha}\varphi(0)\frac{\xi^{\alpha}}{\alpha!}]\,d\xi$$

$$+ \int_{|\xi|>1} G_u(\xi)\varphi(\xi)\,d\xi + \sum_{\substack{|\alpha|\leq N \\ |\alpha|\neq -Q-k}} \frac{M(\frac{\xi^{\alpha}}{\alpha!}G(\xi))}{|\alpha| + k+Q}\,\delta^{\alpha}\varphi.$$

(2.4) Define remainder R_N as $[\varphi(\xi) - \sum_{|\alpha|\leq N} \partial^{\alpha}\varphi(0)\,\frac{\xi^{\alpha}}{\alpha!}]$.

We will now develop the framework to understand R_N.

Consider multiindices α,β and $M \in \mathbb{Z}^+$.

(2.5) Define "$\alpha \leq \beta$" as $\alpha_i \leq \beta_i\ \forall i = 1$ to n.

(2.6) Define "$\alpha < \beta$" as $\alpha \leq \beta$ where $\exists i \ni \alpha_i < \beta_i$.

(2.7) Define 1_j as $(0,\ldots,0,1,0,\ldots,0)$ where the 1 appears as the jth component.

(2.8) <u>Define</u> I(M) as $\{\alpha \in (\mathbb{Z}^+)^n / |\alpha| \geq M$ and $j \ni |\alpha - 1_j| < M\}$.

(2.9) <u>Lemma</u>: $\forall |\beta| \geq M \; \exists \gamma \in I(M) \ni \gamma \leq \beta$.

<u>Proof</u>: WLOG. Assume $\beta_1, \beta_2 = 0$.

Consider as we let $b = 1, \ldots, \beta_1$ that $|\beta - b1_1| =$ $|\beta| - ba_1$ strictly decreases to $|\beta - \beta_1 1_1| = |\beta| - \beta_1 a_1$.

Consider as we let $b = 1, \ldots, \beta_2$, that $|\beta - \beta_1 1_1 - b1_2|$ (which $< |\beta - \beta_1 1_1|$) also strictly decreases.

Finally, note that $|\beta - \sum \beta_j 1_j| = |0| = 0 \leq M$. Hence there exists some first point at which such a process has $|\beta - \sum_{i<j} \beta_i 1_i - b1_j$ being $\leq M$.

Let $\beta^1 = \beta - \sum_{i<j} \beta_i 1_i - b1_j$.

Let $\gamma = \beta^1$ if $|\beta^1| = M$

and let $\gamma = \beta^1 + 1_j$ if $|\beta^1| < M$.

γ will be in I(M) as desired and clearly $\gamma \leq \beta$. ∎

(2.10) <u>Proposition</u>: If f is C^∞ on V which is any open convex neighborhood of D then

$$f(v) = \sum_{|\alpha|<M} \frac{v^\alpha \partial^\alpha f(0)}{\alpha!} + \sum_{\alpha \in I(M)} v^\alpha g_\alpha(v)$$

where the g_α's are C^∞ on V, $M \in \mathbb{Z}^+$.

Proof: By the usual Taylor's theorem,

$$f(v) = \sum_{\|\alpha\| \leq QM} \frac{v^{\alpha}\partial^{\alpha}f(0)}{\alpha!} + \sum_{\|\beta\| = QM+1} v^{\beta} f_{\beta}(v)$$

where the f_{β}'s are C^{∞} on V.

If $|\alpha| < M$ then $\|\alpha\| \leq QM$ while

if $\|\beta\| = QM + 1$ then $|\beta| > M$.

Hence,

$$f(v) = \Big(\sum_{|\alpha| < M} \frac{v^{\alpha}\partial^{\alpha}f(0)}{\alpha!}\Big) + \Big(\sum_{\substack{\|\alpha\| \leq QM \\ |\alpha| \geq M}} \frac{v^{\alpha}\partial^{\alpha}f(0)}{\alpha!}\Big) + \Big(\sum_{\|\beta\| = QM+1} v^{\beta} f_{\beta}(v)\Big).$$

Of the three parts of the sum, the last two is by (2.09) sums of objects of the form $v^{\alpha}g_{\alpha}$ as desired. ■

Of course we may replace "$|\alpha| < M$" and "$\alpha \in I(M)$" with "$|\alpha| \leq N$" and "$\alpha \in I(N+1)$." We could then derive...

(2.11) Corollary: If f is C^{∞} on neighborhood of $|\xi| \leq c$, then $R_N(\xi) = [f(\xi) - \sum_{|\alpha| \leq N} \frac{\xi^{\alpha}\partial^{\alpha}f(0)}{\alpha!}]$ dies $O(|v|^{N+1})$ at 0. In fact, $R_N(\xi) = g(\xi)|\xi|^{N+1}$ with

$$\sup_{|\xi| \leq c} |g(\xi)| \leq \sum_{\substack{|\alpha| \geq N+1 \\ \|\alpha\| \leq Q(N+2)}} \frac{1}{\alpha!} \sup_{|\xi| \leq c} |\partial^{\alpha}f(\xi)|.$$

Proof: $f_{\beta}(\xi)$ of (2.10) is $\frac{\partial^{\beta}f(p_{\beta}(\xi))}{\beta!}$ where $p_{\beta}(\xi)$ is on line $\{t\xi / t \in [0,1]\} \subseteq \{v / |v| \leq c\}$, hence giving the desired bound on $|g|$. ■

(2.12) <u>Define</u> $A : \mathbb{R} \to \mathbb{Z} : p \mapsto A(p)$ the greatest integer less than or equal to p.

(2.13) <u>Proposition:</u> If $N \geq A(-Q-\text{Re } k)$ then Λ is a functional on S. Except for that condition, N is arbitrary.

<u>Proof:</u> Recall k is the degree of G and recall that R_N dies $O(|\xi|^{N+1})$. Hence for $R_N(\xi)G(\xi)$ to be in L^1_{loc}, we need $(\text{Re } k) + (N+1) + Q > 0$. Thus for Λ_G to be a functional on S, N should be greater than $-Q-\text{Re } k-1$. Hence if we assume N is an integer, N should be greater than or equal to $A(-Q-\text{Re } k)$.

Now consider any $|\alpha| \ni |\alpha| > -Q-\text{Re } k$. Then $G(\xi)\frac{\xi^\alpha}{\alpha!} \in L^1_{loc}$ and

$$\int_{|\xi|\leq 1} G(\xi)\,\partial^\alpha\varphi(0)\frac{\xi^\alpha}{\alpha!}d\xi) - \frac{1}{|\alpha|+k+Q} M(\frac{\xi^\alpha}{\alpha!}G(\xi))\partial^\alpha\varphi(0) = 0.$$

Thus, WLOG, we can think of $N = A(-Q-\text{Re } k)$. ■

(2.14) <u>Proposition:</u> If $G_u(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $G_u(x)$ is C^∞ on $\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$, and

$$D^r \partial_u^\ell G_u(x) = r^k \partial_u^\ell G_u(x), \ \forall n \in \mathbb{R}^n, \ \forall x \in \mathbb{R}^n \backslash 0, \ \forall r > 0$$

then Λ_G is C^∞ on $\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$, $\partial_u^\ell \Lambda_G = \Lambda_{\partial_u^\ell G}$,

and $\Lambda_G \in S^1 \quad \forall u \in \mathbb{R}^n$.

Proof: Consider the first third of the Λ_G sum. Consider φ_i converging to 0 in S. In light of (2.11), consider the corresponding g_i.

Then
$$\sup_{|\xi|\leq 1}|g_i(\xi)| \leq \sum_{\|\alpha\|\leq Q(N+2)} \frac{1}{\alpha!} \sup_{|\xi|\leq 1}|\partial^\alpha \varphi_i(\xi)|$$
which are going to 0 as $i \to \infty$. Of course, $\sup_{|\xi|\leq 1}|g_i(\xi)|$ is less than 1 for i large.

Hence
$$|G(\xi)|\xi|^{N+1}g_i(\xi)| \leq |G(\xi)|\xi|^{N+1}1|$$
is in L^1_{loc} and $(G(\xi)|\xi|^{N+1}g_i(\xi)) \to 0$, $\xi \neq 0$.

Thus by Lebesgue Dominated Convergence Theorem, the first part of the sum $\Lambda_G\varphi_i$ converges to 0 in $\mathbb{R}$.

By simpler arguments, we see all of $\Lambda_G\varphi_i$ converges to 0 in $\mathbb{R}$ whenever φ_i converges to 0 in S. The other results are also trivial. ∎

In the trivial case of $Q + \text{Re } k > 0$ (i.e., N can be taken as negative), Λ_G is simply G_n. A reasonable quesiton then is when else is Λ_G homogeneous? First,

(2.15) Define "dilation difference of f" as $r^k f - D^r f$.

(2.16) Lemma: $\langle r^k\Lambda_G - D^r\Lambda_G, \varphi\rangle$

$$= -r^k \sum_{|\alpha|=-Q-k} M\left(\frac{\xi^\alpha}{\alpha!}G(\xi)\right)\log r\,\partial^\alpha\varphi(0).$$

Now pick $\varphi \in S \ni \varphi \equiv 0$ near 0.

Then by the definition of Λ_G, $\langle\Lambda_G,\varphi\rangle = \langle G,\varphi\rangle$. Hence if Λ_G is a homogeneous distribution and G is not identically 0 on $\mathbb{R}^n\setminus 0$ then they must be of same degree, namely k. Hence $r^k\Lambda_G - D^r\Lambda_G$ must equal 0, in the sense of distributions. Hence, ...

(2.17) Proposition: Λ_G homogeneous degree k iff $M(\xi^\alpha G_n(\xi)) = 0$, $\forall\alpha \ni |\alpha| = -Q-k$, $\forall n \in \mathbb{R}^n$.

Note: This would include the trivial result: if $Q + \mathrm{Re}\, k > 0$ then $\Lambda_G \in \mathrm{Rhom}_k$.

(2.18) Define "M = 0" as meaning $M(\xi^\alpha \partial_n^\ell G_j(\xi)) = 0$, $\forall\ell \in (\mathbb{Z}^+)^n$, $\forall u \in \mathbb{R}^n$, $\forall\alpha \ni |\alpha| = -Q-k$.

Combining the last two propositions, we have...

(2.19) Proposition: Consider $G_u(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $G_u(x)$ is C^∞ on $\mathbb{R}^n \times (\mathbb{R}^n\setminus 0)$ and $D^r\partial_u^\ell G_u(x) = r^k\partial_u^\ell G_u(x)$, $\forall u \in \mathbb{R}^n$, $\forall x \in \mathbb{R}^n\setminus D$, $\forall r > 0$. Then $\Lambda_G \in \mathrm{Rhom}_k$ if M = 0.

(2.20) Define P,R to be homogeneous polynomials with coefficients C in u.

In keeping with our comments on L(D) at the start of the chapter, "$P(\partial)$" is P with derivatives ∂_j in place of x_j.

(2.21) Proposition: $\mathrm{Rhom}_k = \{\Lambda_G/M = 0\} + \{P(\partial)\delta/P \text{ degree } -Q-k\}$.

<u>Proof</u>: Rhom_k contains $\{\Lambda_G/M = 0\}$ by (2.19) and contains $P(\partial)\delta$ by (1.25).

Pick $T \in \mathrm{Rhom}_k$. Let $G_u = T$, $x \neq 0$. Note that $T - \Lambda_G$ is supported at the origin and hence is a finite sum of $c_\alpha \delta^\alpha$.

By (1.26), the dilation difference of $T = 0$. The dilation difference of $\sum c_\alpha \delta^\alpha$ grows polynomially. Thus, the sum of (2.16) is $=0$, $\forall\varphi$.

Hence $M = 0$, as desired. ∎

(2.22) <u>Corollary</u>: If $-Q-k \notin \mathbb{Z}^+$ then $\mathrm{Rhom}_k = \{\Lambda_G/M = 0\}$.

For the following results, see Geller [G].

(2.23) <u>Proposition</u>: If T is in $\{\Lambda_G\} + \{R(\partial)\delta/R \text{ degree } -Q-k\}$. Then $\hat{T}$ is in $\mathrm{Rhom}_{-Q-k} + \{P(\xi)\log|\xi|/P \text{ degree } -Q-k\}$. If $M = 0$ for G_u, then $\hat{T}$ is simply in Rhom_{-Q-k}.

To complete our description of such objects...

(2.24) <u>Define</u> K^k as Rhom_k for $k \in \mathbb{C}\setminus \mathbb{Z}^+$ and as $\mathrm{Rhom}_k + \{P(x)\log|\psi|/P(x) \text{ degree } k\}$ for $k \in \mathbb{Z}^+$.

(2.25) <u>Define</u> J^j as Rhom_j for $-j-Q \in \mathbb{C}\setminus\mathbb{Z}^+$ and as $\{\Lambda_G/G \text{ degree } j\} + \{R_u(\partial)\delta/R(x) \text{ degree } -j-Q\}$.

(2.26) <u>Proposition</u>: If $k + j = -Q$ then $\widehat{K^k} = J^j$.

(2.27) <u>Proposition</u>: If $K_u \in K^k$

then $\partial_t^\alpha K_u \in K^{k-|\beta|}$, $x^\beta K_u \in K^{k+|\beta|}$

and $\partial_u^\ell K_u \in K^k$.

3. Poincaré Lemma for K^k

Purpose of this chapter:

To explain the method of assembling a K_u from what could be thought of as derivatives of K_u.

First we should explain what happens when $D_j^m f = 0$, $\forall_j \ldots$.

(3.1) Proposition: If $f \in \mathcal{S}'$, $m \in \mathbb{Z}^+$, $D_j^m f \equiv 0$, $\forall j = 1$ to n, then $f(x) = \sum_{\alpha_j < m, \forall j} c_\alpha x^\alpha$.

In addition, if f_n is a C^∞ distribution then the c_α's are C^∞ functions of u.

Proof: Use the fourier transform with the following:

(3.2) Proposition: If $g \in \mathcal{S}'$, $m \in \mathbb{Z}^+$, and $x_j^m \equiv 0$, $\forall j = 1$ to n, then $g(x) = \sum_{\alpha_j < m, \forall j} c_\alpha \delta^\alpha$.

In addition, if g_n is a C^∞ distribution then the c_α's are C^∞ functions of u.

A useful result of (3.1) is:

(3.3) Proposition: Consider homogeneous $K \in \mathcal{S}'$. K is in Rhom_k and has C^∞ extension to origin iff K is polynomial away from 0.

(3.4) Lemma: Consider a fixed $i \in \{1,\ldots,n\}$, $k \in \mathbb{C}$, $m \in \mathbb{Z}^+$, and $a \in (\mathbb{Z}^+)^n$. If $\mu \cdot a \leq \text{Re } k$ and $\text{Re } k < ma_i$ then $m/i \nleq \mu$ (where $(m/j)_i = m\delta_{ij}$).

Proof: Assume $\mu \cdot a \leq \text{Re } k$ and $m/i \leq \mu$.
Then since all $a_j > 0$, $(m/i)\cdot a \leq \mu \cdot a \leq \text{Re } k$.
Then $ma_i \leq \text{Re } k$ contradicting $\text{Re } k < ma_i$. ■

(3.5) Poincaré Lemma: Consider $k \in \mathbb{C}$ and $m \in \mathbb{Z}^+$, such that $\text{Re } k - ma_j < 0$, $\forall j = 1$ to n. If $K_{ju} \in K^{k-ma_j}$ (which is Rhom_{k-ma_j}), $\forall j = 1$ to n and $D_j^m K_u^i = D_i^m K_u^j$, $\forall i,j = 1$ to n then there exists a $K_u \in K^k$ such that $D^m K_u = K_u^j$, $\forall j = 1$ to n.

Proof: $\hat{K}^j$ is in $\text{Rhom}_{-Q-k+ma_j}$.
Now since $-Q-(\text{Re } k-ma_j) > -Q$, $\widehat{K^j}$ may be considered as a function on $\mathbb{R}^n$ and $= 0$ at 0. Notice x_j^m is homogeneous degree ma_j, so loosely speaking $\frac{\widehat{K^j}}{x_j^m}$ is homogeneous (on $\mathbb{R}^n \setminus 0$) of degree $-Q-k$. There is the problem that x_j could be 0 while $-Q-\text{Re } k$ could be negative. To remove the problem, define f_u as $\frac{\widehat{K^j}}{x_j^m}$ when $x_j \neq 0$ and as 0 when $x = 0$. The function f is well defined since if x_i and $x_j \neq 0$ then $x_i^m \widehat{K^j}$ is equal to $x_j^m \widehat{K^i}$, giving us $\frac{\widehat{K^j}}{x_j^m} = \frac{\widehat{K^i}}{x_i^m}$.

We see that f_n is C^∞ on $\mathbf{R}^n \times \mathbf{R}^n \setminus 0$, homogeneous degree $-Q-k$ and is equal to 0 at the origin.

Define $K_u = (\Lambda_f)^v \in K^k$

where $$(\Lambda_f, \varphi) = \int_{|x|\leq 1} f_u(x)\,[\varphi(x) - \sum_{\mu\cdot a \leq \mathrm{Re}\, k} \frac{x^\mu \partial^\mu \varphi(0)}{\mu!}]$$

$$+ \int_{|x|>1} f_u(x)\varphi(x)\,dx + \sum_{\mu\cdot a<\mathrm{Re}\, k} c_\mu \delta^\mu \varphi.$$

The $\mu\cdot a \leq \mathrm{Re}\, k$ is, of course, equivalent to the expected $-Q-\mathrm{Re}\, k + \mu\cdot a \leq -Q$.

Since $D_i^m K_u = D_i^m(\Lambda)^v = (x_i^m \Lambda_f)^v$, it will be enough to prove $x_i^m \Lambda_f = \hat{K}_u^i$ to show $D_i^m K_u = K_u^i$.

We have $(x_i^m \Lambda_f, \varphi) = (\Lambda_f, x_i^m \varphi)$

$$= \int_{|x|\leq 1} f(x)\,[x_i^m \varphi - \sum_{\mu\cdot a \leq \mathrm{Re}\, k} \partial^\mu|_0 (x_i^m \varphi) \frac{x^\mu}{\mu!}]$$

$$+ \int_{|x|>1} f(x) x_i^m \varphi(x)\,dx + \sum_{\mu\cdot a<\mathrm{Re}\, k} c_\mu \delta^\mu (x_i^m \varphi).$$

But notice that

$\partial^\alpha|_0 (x_i^m \varphi)$ is 0 if $ml_i \not\leq \alpha$.

But by the lemma, we know $ml_i \not\leq \mu$.

Thus $$(x_i^m \Lambda_f, \mu) = \int_{|x|\leq 1} f(x)\,[x_i^m \varphi - 0] + \int_{|x|>1} f x_i^m \mu + \sum 0$$

$$= (x_i^m f, \mu).$$

So as distributions $x_i^m \Lambda_f = x_i^m f$.

Hence we must show $x_i^m f = \widehat{K^i}$ as distributions. Since both are 0 at the origin, it is certainly enough to show $x_i^m f = \widehat{K^i}$ for $x \neq 0$.

Assuming $x \neq 0$, there is a j such that $x_j \neq 0$. Then $x_i^m f$ is $\frac{x_i^m \widehat{K^j}}{x_j^m}$ which, from before is $\frac{x_j^m \widehat{K^i}}{x_j^m} = \widehat{K^i}$. ∎

4. Polynomial Group Laws and Vector Fields.

Purpose of this chapter:

We will consider groups $(\mathbb{R}^n, \cdot)$ where any result of group addition $z = x \cdot y$ has each component z_i being a polynomial in $x_1, \ldots, x_n, y_1, \ldots, y_n$ and where D_r is an automorphism, meaning $D_r(x,y) = D_r x \cdot D_r y$, $\forall r > 0$, $x, y \in \mathbb{R}^n$.

Note: By our definition of D_r, we have $D_r D_s x = D_{(rs)} x$ but usually not $= D_{r+s} x$.

(4.1) Proposition: Identity e must be $(0, \ldots, 0)$.

(4.2) Proposition: Reorder components (and addition) of each element of the group so that $a_K \leq a_{K+1}$ for $K = 1$ to $n - 1$.

If $w = x \cdot y$ then the jth component of w is

$$w_j = x_j + y_j + \sum_{\substack{|\alpha| + |\beta| = a_j \\ |\alpha|, |\beta| < a_j}} c_{j\alpha\beta} x^\alpha y^\beta$$

where $c_{j\alpha\beta}$'s are real numbers.

(4.3) Corollary: Pick a fixed y in $\mathbb{R}^n$

$$\det\left[\frac{\partial(x \cdot y)}{\partial x}\right] = \det\left[\frac{\partial(y \cdot x)}{\partial x}\right] = 1.$$

(4.4) Proposition: $D_r(y^{-1}) = (D_r y)^{-1}$

(4.5) Corollary: If $|y^{-1}| = C|y|$ then $|D_r y^{-1}| = C|D_r y|$.

(4.6) <u>Proposition</u>: For $j = 1$ to n, there exists a homogeneous degree a_j polynomial, p_j, such that $w^{-1} = (p_1(w), \ldots, p_n(w))$.

In fact, $p_j(w) = -w_j + \sum_{\substack{|r|=a_j \\ r_i' < a_j}} d_r w^r$ where d_r is a universal polynomial in $c_{j\alpha\beta}$ (of 4.2) only.

(4.7) <u>Corollary</u>. $\det[\frac{w^{-1}}{w}] = (-1)^n$.

So fortunately, whenever we replace the variable of integration w with w^{-1}, we have the absolute value of the determinant (of the Jacobian being simply 1.

(4.8) <u>Triangle Inequality</u>: $\exists C \in (0,\infty)$, $\forall u, v \in \mathbb{R}^n$,

$$|u \cdot v| \leq C(|u|+|v|).$$

<u>Proof</u>: Let $C = \sup_{|z|,|w| \leq 1} |z \cdot w|$.

Let $B = |u| + |v|$.

Then
$$\begin{aligned} |u \circ v| &= |D_\beta D_{\frac{1}{\beta}}(u \circ v)| \\ &= B|D_{\frac{1}{\beta}}(u) \circ D_{\frac{1}{\beta}}(v)| \\ &\leq BC = C(|u|+|v|). \end{aligned}$$ ■

(4.9) <u>Corollary</u>: $\forall r > 0 \ \exists \varphi \in C_c^\infty \ni \varphi \equiv 1$ near 0 and $\operatorname{supp}_y \varphi(x \cdot y^{-1}) \cap \operatorname{supp}_y \varphi(y) = \emptyset$, $\forall x \ni |x| \geq r$.

Unfortunately $\partial_y^\mu \varphi(x \cdot y) \neq (\partial^\mu \varphi)(x \cdot y)$, so we will need...

(4.10) <u>Proposition</u>: Consider $f(x,y) =$

$$\sum_{\substack{\|r\| \leq M \\ |\alpha|+|\beta| \leq mQ}} b_{r\alpha\beta} x^{\alpha} y^{\beta} (\partial^{r}\varphi)(x \cdot y)$$

where m and M are in $\mathbb{Z}^{+}$.

$\frac{\partial}{\partial y_j} f(x,y)$ will be of a similar form, meaning

$$\frac{\partial}{\partial y_j} f(x,y) = \sum_{\substack{\|r\| \leq M+1 \\ |\alpha|+|\beta| \leq (m+1)Q}} A_{r\alpha\beta} x^{\alpha} y^{\beta} (\partial^{r}\varphi)(x,y).$$

(4.11) <u>Corollary</u>: Let $S = \|\mu\|$.

$$\partial_y^{\mu}\varphi(x,y) = \sum_{\substack{\|r\| \leq S \\ |\alpha|+|\beta| \leq SQ}} A_{r\alpha\beta} x \; y^{\beta} (\partial^{r}\varphi)(x,y).$$

(4.12) <u>Define</u>: $f_w(\_\_)$ as the "left translation of f by w" meaning $f(w \circ \_\_)$.

(4.13) <u>Define</u>: $f_{Rw}(\_\_)$ as the "right translation of f by w" meaning $f(\_\_ \circ w)$.

Hence $(f_u)_v(\_\_) = f_v(u \circ \_\_) = f(u \circ v \circ \_\_)$

$= f_{u \circ v}(\_\_)$ while

$(f_{Ru})_{Rv}(\_\_) = f_{Rv}(\_\_ \circ u) = f(\_\_ \circ v \circ u)$

$= f_{Rv \circ u}(\_\_)$.

(4.14) <u>Define</u> X as a "left (translation) invariant vector field" iff $Xf(w) = Xf_w(0)$, $\forall w$, $\forall f$.

(4.15) <u>Define</u> Y as a "right (translation) invariant vector field" iff $Yf(w) = Yf_{Rw}(0)$, $\forall w$, $\forall f$.

Hence $Xf(u \circ v) = Xf_{u \circ v}(0) = Xf_u(v \circ D) = Xf_u(v)$

while $Yf(u \cdot v) = Yf_{Ru \circ v}(0) = Yf_{Rv}(D \cdot u) = Yf_{Rv}(u)$.

(4.16) Proposition: Uniqueness of invariant vector field agreeing with $\frac{\partial}{\partial x_j}$ at 0. Consider $X_j|_w = \sum_{m=1}^{n} c_m(w) \frac{\partial}{\partial x_m}\Big|_{x=w}$ with c_m in C^∞. X_j is a left or right invariant vector field with $(X_j f)(0) = (\frac{\partial}{\partial x_j} f)(0)$ iff $c_m(w) = \frac{\partial (w \cdot x)_m}{\partial x_j}\Big|_{x=0}$, $\forall m = 1$ to n or $\frac{\partial (x \cdot w)_m}{\partial x_j}\Big|_{x=0}$, $\forall m = 1$ to n, respectively.

Note: $\frac{\partial (w \cdot x)_m}{\partial x_j}$ and $\frac{\partial (x \cdot w)_m}{\partial x_j}$ are both =0 when $a_j > a_m$ and =1 when j = m. Actually X_j would $= \frac{\partial}{\partial x_j} +$ $+ \sum_{a_m > a_j} c_m(w) \frac{\partial}{\partial x_j}$.

Consider $X = \sum b_j(w) \frac{\partial}{\partial x_j}\Big|_{x=w}$.

(4.17) Define X as a "homogeneous order k vector field" iff $X(D^r f)\big|_x = r^k (Xf)\big|_{D_r x}$ $\forall f$ C^∞ near x and $D_r x$.

A few completely trivial results of this definition are:

(4.18) <u>Proposition</u>: If X and Y are homogeneous vector fields of order k_1 and k_2 respectively, then XY is a homogeneous vector field of order $k_1 + k_2$.

(4.19) <u>Proposition</u>: ∂^β is an order $|\beta|$ vector field.

(4.20) <u>Proposition</u>: If X homogeneous order k vector field and f homogeneous degree k^1 and C^∞ on $\mathbb{R}^n \setminus 0$ then Xf is homogeneous degree $k^1 - k$.

(4.21) <u>Corollary</u>: $X = \sum b_j(x) \frac{\partial}{\partial x_j}\Big|_{x=w}$ is a homogeneous order k vector field iff $b_j(w)$ are homogeneous degree $a_j - k$.

<u>Define</u> U_j as the left and U_j^R as the right invariant vector field agreeing with $\frac{\partial}{\partial u_j}$ at 0.

(4.22) <u>Proposition</u> U_j and U_j^R are order a_j vector fields.

(4.23) <u>Proposition</u>: $U_m = \frac{\partial}{\partial v_m} + \sum_{a_j > a_m} a_{m,j} \frac{\partial}{\partial v_j}$

and $\frac{\partial}{\partial v_j} = U_j + \sum_{a_m > a_j} p_{j,m} U_m$ where $a_{m,j}$ and $p_{j,m}$ are explicitly computable homogeneous polynomials of degree $a_j - a_m$ and $a_m - a_j$ respectively. The same result holds for U_m^R's.

(4.24) <u>Proposition</u>: The product rule holds.

Consider a differentiable function f and a distribution g. Then $U_m(fg) = (U_m f)g + f(U_m g)$. The same result holds for U_n^R's

Consider polynomial p which is <u>not</u> necessarily homogeneous.

(4.25) <u>Define</u> "m is the degree of polynomial p" as meaning m is the least non-negative integer such that $p(D_r x) = 0(r^m)$ as $r \to \infty \; \forall x$.

(4.25) <u>Proposition</u>: For i = 1 and 2, consider $X_i = b_i \frac{\partial}{\partial v_i}$ where b_i is a polynomial of degree $\leq a_i - d_i$ where d_i is a positive integer. Then $[X_1, X_2] = c_1 \frac{\partial}{\partial v_1} - c_2 \frac{\partial}{\partial v_2}$ where c_m are polynomials of degree $\leq a_m - d_1 - d_2$.

Such reasoning gives us a rough estimate of order of nilpotency...

(4.27) <u>Proposition</u>: The Lie Algebra (formed by linear combinations of U_j) is nilpotent of order $\leq Q$.

5. Convolutions and Pseudodifferential Operators ψDO

Purpose of this chapter:

Pseudodifferential operators can be viewed as convolution operators whose kernels are translates of "cores." In this chapter we form the core from the composition of two such operators.

Consider $G \in E^1$, $\varphi \in C^\infty$, and $u \in \mathbb{R}^n$. Now abusing notation...

(5.1) Define $(G*\varphi)(u) = (G(uw^{-1}),\varphi(w))$

$$= (G(w^{-1}),\varphi(wu)) = (G(w),\varphi(w^{-1}u)).$$

Consider $\alpha \in (\mathbb{Z}^+)^n$.

(5.2) Define U^α as meaning some ordered composition of left invariant vectors where U_1 appears α_1 times, U_2 appears α_2 times, etc.

(5.3) Define $U^{R\alpha}$ in the same way with respect to $U_1^R,\ldots,U_n^R$,

(5.4) Proposition: $(G*\varphi)(u)$ is C^∞ and in fact

$$U^\alpha[(G*\varphi)(u)] = [G*(U^\alpha\varphi)](u)$$

while $$U^{R\alpha}[(G*\varphi)(u)] = [(U^{R\alpha}G)*\varphi](u).$$

Likewise $(G(w),\varphi(uw^{-1}))$ is C^∞ and

$$U^\alpha(G(w),\varphi(uw^{-1})) = ((U^{R\alpha}G)(w),\varphi(uw^{-1}))$$

while $$U^{R\alpha}(G(w),\varphi(uw^{-1})) = (G(w),(U^{R\alpha}\varphi)(uw^{-1})).$$

The same results are obtained if C^∞, E^1 are replaced by $\mathcal{S},\mathcal{S}^1$ or $C_c^\infty,\mathcal{S}^1$.

Note: Of course, since we do not require that the group be abelian, $(G*\varphi)(u) = (G(w),\varphi(w^{-1}u))$ is usually not $(G(w),\varphi(uw^{-1}))$.

(5.5) Define $\sim : h(x) \mapsto \overline{h(x^{-1})}$

(5.6) Proposition: Consider $K \in S^1$.

$\_\_*K$ has an adjoint $\_\_*\tilde{K}$.

Consider $K_1, K_2 \in E^1$. We have as a well defined distribution...

(5.7) Define $K_2 * K_1 : \varphi \mapsto (K_2, \varphi * \tilde{K}_1)$.

By duality we can now say...

(5.8) Proposition: $U^{\alpha R}(K_2 * K_1) = (U^{\alpha R}K_2) * K_1$

and $U^{\alpha}(K_2 * K_1) = K_2 * (U^{\alpha}K_1)$.

(5.9) Define F_w as the fourier transform with w as the dual variable (e.g. $e^{-iw\cdot y}$ is in the integral and y is the variable of integration).

Consider open $X \subseteq \mathbb{R}^n$ and $0 \leq \delta \leq \rho \leq 1$. From Folland [F], we consider symbol class $S^m_{\rho,\delta}(X)$

$= \{a \in C^\infty(X \times \mathbb{R}^n) / \forall \alpha, \beta,\ \forall \Lambda \subseteq\subseteq X,$

$\exists c \ni \sup_{x \in \Lambda} \left| D^\beta_x D^\alpha_\xi a(x,\xi) \right| \leq c(1+\|\xi\|)^{m-\rho\|\alpha\|+\delta\|\beta\|}\}$.

The distributional kernel of $a(x,D)$ is $K(x,y) = c(x,x-y) = F^{-1}_{(x-y)}a(x,x-y)$ where $x - y$ means regular Euclidean subtraction. Notice that $K(x,y)$ is C^∞ in x and y away from $(x-y) = 0$.

(5.10) Define "Euclidean core of a" as $c(u,w) = F_w^{-1}a(u,w)$.

Consider that $a(u,\xi)$ is S^1 in ξ. Or in other words, $a : U \mapsto S^1$. Then since $F(S^1) = S^1$ we have $c : U \to S^1$.

More completely...

(5.11) Proposition: Consider Euclidean core $c(u,w)$. $c : U \mapsto S^1$, $c(u,w)$ is C^∞ as a function in $u \in U$ and $w \in \mathbb{R}^n \setminus 0$, and $(c(u,x),\varphi(x))$ is C^∞ in u, $\forall\varphi \in S$.

We will now expand our concept of cores. Consider an open set $\tilde{U} \subseteq \mathbb{R}^n$.

(5.12) Define "$g_u(x)$ is a core" as meaning $g_u : \tilde{U} \to S^1$ such that $g_n(x)$ is smooth on $\tilde{U} \times (\mathbb{R}^n \setminus 0)$ and such that (g_u,φ) is C^∞ in u, $\forall\varphi \in S$.

Consider $h \in S^1$. Consider the largest open set V such that $(h,\varphi) = 0$ for all $\varphi \in S \ni \varphi \equiv 0$ on $\mathbb{R}^n \setminus V$.

(5.13) Define "supp h" as the "support for h" meaning the closure of V.

Consider core $g_u(x)$ with compact support contained in open set X. Since (g_u,φ) is C^∞ in u, $\forall\varphi \in S$, it is thus C^∞ in u, $\forall\varphi \in C_c^\infty$. By a result of the Banach Steinhaus Theorem (see e.g. Petersen [P]), the limit (of difference quotients) $\frac{\partial}{\partial u_j}(g\ ,\_\_)$ is itself a map from $\tilde{U}$ to E^1. Hence $\partial_u^\beta g_u : \varphi \mapsto \partial_u^\beta (g_u, \varphi)$ is itself a core.

Consider compact set U. Consider an open neighborhood X of 0.

(5.14) <u>Define</u> "$G \in \psi DO(U,X)$" as meaning "G is a pseudodifferential operator with respect to a polynomial group law," meaning ∃ bounded open $\tilde{U} \supsetneq U$, ∃ core $g_u \ni$ supp $g_u \leq X$, $\forall u \in U$ and $G(\varphi) = \int g_u(uv^{-1})\varphi(v)dv$.

<u>Note</u>: (φ) is in $C^\infty(\tilde{U})$.

Since compact U is contained within open $\tilde{U}$, there exists another open bounded set U^1 and its closure $\overline{U^1}$ such that

$$U \leq U^1 \leq \overline{U^1} \leq \tilde{U}.$$

Sometimes the U in generic proposition A will be thought of (but not stated as) $\overline{U^1}$ in the proof of Proposition B in which Proposition A is being applied. Likewise, something will often be shown about U^1 rather than $\tilde{U}$; a distinction which won't matter since both contain U. Having explicitly all this, we will tend not to trouble the reader with it in the future.

In order to have the option of certain manipulations of integrals containing cores g_u, we now create a counterpart to the fact that any object in E^1 is actually a sum of derivatives of a continuous function. This will be our most useful lemma.

First consider the following form of the uniform boundedness principle for Frechet spaces (e.g., Reed and Simon, p. 132 [Re]).

(5.15) Proposition: Consider a Frechet space S with directed semi-norms $\{d_j\}_{j=1}^{\infty}$. Consider a family M of continuous linear maps from S to $\mathbb{R}$. If $\sup_{g \in M} |g(\varphi)|$ is finite for all $\varphi \in S$, then $\exists j \in \mathbb{Z}^+ \; \exists C \in \mathbb{R}^+ \ni |g(\varphi)| \leq Cd_j(\varphi) \; \forall \varphi \in S \; \forall g \in M$.

Of course, we are more interested in cores so....

(5.16) Corollary: Consider $g_u : U \to E^1$ with support in bounded open $X \leq \mathbb{R}^n$ and with U compact. If $\sup_{u \in U} |(g_u, \varphi)|$ is finite for all $\varphi \in C_c^\infty$ then $\exists M \in \mathbb{Z}^+ \; \forall u \in U$, g_u is of order $\leq M$ and $(\bigcup_{u \in u} \text{supp } g_u) \subseteq\subseteq X$.

Consider $h \in E^1$, hence supp h is compact.

(5.17) Define "order of h" as the least $N \in \mathbb{Z}^+$,

$$\exists C \in \mathbb{R}^+, \forall \varphi \in C^\infty, \; |h(\varphi)| \leq C \sup_{\substack{\|\alpha\| \leq N \\ x \in \text{supp } h}} |\varphi(x)|.$$

(5.18) Lemma: Consider $m \in \mathbb{Z}^+$. Consider core $g_u(x)$ with support in bounded open $X \subseteq \mathbb{R}^n$, $\forall u \in$ compact U. There exists $M \in \mathbb{Z}^+$ and $f_u \in C(U \times \mathbb{R}^n)$ such that $\partial_u^\beta f \in C(U \times \mathbb{R}^n)$ $\forall \beta \ni \|\beta\| \leq m$, $f_u(x)$ vanishes as $x \to \infty$ for all $u \in U$, and $g_u = (1+\Delta^M)f_u$ for all $u \in U$. If the order of $\partial_u^\beta g_u$ is less than some N^1 for all $\beta \in (\mathbb{Z}^+)^n$, $\forall u \in U$, then f_u may be chosen to give the same results with "C^m" replaced by "C^∞."

Proof: Let $\varphi_\xi(x) = \psi(x)e^{-ix\cdot\xi}$ where $\psi \in C_c^\infty \ni \psi \equiv 1$ on X. Since $\partial_u^\beta g_u$ is itself a core with support in X, by (5.16) we have a finite sum

$$(C_\beta \sum \sup_{x \in X} |\partial^\alpha \varphi_\xi(x)|) \geq |(\partial_u^\beta g_u, \varphi_\xi)|, \ \forall \xi \in \mathbb{R}^n, \forall u \in U.$$

Then $\exists C \in \mathbb{R}^+$, $\exists j \in \mathbb{Z}^+$,

$$\ni |(\partial_u^\beta g_u, \varphi_\xi)| \leq (\sum_{\|\alpha\| \leq 2j} \sup_{x \in X} |\partial_x^\alpha \varphi_\xi|,$$

$$\forall \beta \in (Z^+)^n \ni \|\beta\| \leq m, \forall \xi \in \mathbb{R}^n, \forall u \in U.$$

Thus $|\widehat{\partial_u^\beta g_u}(\xi)| \leq C \sum_{\|\alpha\| \leq 2j} |\xi^\alpha|$, $\forall \xi \in \mathbb{R}^n$, $\forall n \in U$.

Hence $\exists C^1 > 0 \ni |\widehat{\partial_u^\beta g_u}(\xi)| \leq C^1(1+\|\xi\|^{2j})$, $\forall \xi \in \mathbb{R}^n$, $\forall u \in U$.

Define $h_u(\xi) = \dfrac{\hat{g}(\xi)}{1+\|\xi\|^{2(j+n)}}$, $\forall u \in U$.

Since $\hat{g}_u = (g_u, \varphi_\xi)$ is C^∞ in $u \in U$ and C^∞ (actually, real analytic) in $\xi \in \mathbb{R}^n$, we have h_u is also $C^\infty(U\ \xi)$. Consider that when $\|\beta\| \leq m$, that

$$[1 + \|\xi\|^{2(j+n)}]\,\partial_u^\beta h_u(\xi) = \partial_u^\beta \hat{g}_u$$

$$= \partial_u^\beta (g_u, \varphi_\xi) = (\partial_u^\beta g_u, \varphi) = \widehat{\partial_u^\beta g_u}.$$

Hence $|\partial_u^\beta h_u(\xi)| = O(\|\xi\|^{-2n})$ as $\xi \to \infty$ and is thus L^1, $\forall u \in U$. Hence $(\partial_u^\beta h_u)^\vee(x)$ is continuous in x and vanishes as $x \to \infty$. Hence $\partial^\beta h_u^\vee(x)$ is continuous in u and x.

Let $M = j+n$.

Then $$(1+\Delta^M)\,h_u^\vee(x) = [(1+\|\xi\|^{2M})\,h_u]^\vee(x)$$

$$= (\hat{g}_u)^\vee(x) = g_u \text{ as desired.}$$

Thus let $f_n = h_u^\vee$.

Regardless of whether or not U is bounded, if we know a bound on the order of $\partial_u^\beta g_u$'s then we can do without (5.16) and still derive $|\widehat{\partial^\beta g_n}(\xi)| \leq C^1(1+\|\xi\|^{2j})$. Obviously if this holds all $\beta \in (\mathbb{Z}^+)^n$ then the preceding argument gives f_u being C^∞ in $u \in U$. ∎

Consider cores with compact support $g_{2u}h_{1u}$. In keeping with earlier in the chapter, $(g_{2u}*h_{1u}, \varphi) = (g_{2u}, \varphi*h_{1u})$ meaning, by an abuse of notation, $\int g_{2u}(v)[\int h_{1u}(v^{-1}w)\varphi(w)dw]dv$.

Now, we replace u by $v^{-1}u$ in the inner integral.

(5.19) Define $g_{2n}*h_{1\#u}$ as the "core of the composition" meaning $\varphi \mapsto (g_{2u}*h_{1\#u},\varphi)$

$= \int g_{2u}(v)[\int h_{1v^{-1}u}(v^{-1}w)\varphi(w)dw]dv.$

This is justified by the following...

(5.20) Theorem: Consider $G_1 \varepsilon \psi DO(U_1,X_1)$ with X_1 bounded and core h_{1u}. Consider $G_2 \varepsilon \psi DO(U_2,X_2)$ with X_2 bounded and core g_{2u}. We'll require the order of both $\partial_u^\alpha g_{2u}$ and $\partial_u^\alpha h_{1u}$ to be bounded as α varies over $(\mathbb{Z}^+)^n$.

Define $G : \varphi \mapsto G_2(G_1(\varphi))$ when $U = U_2 \setminus [X_2(\mathbb{R}^n \setminus U_1)]$ is nonempty. Then $G \varepsilon \psi DO(U,X_2X_1)$ and $X_2X_1 = \{x\cdot y/x \varepsilon X_2, y \varepsilon X_1, \cdot \text{denoting group addition}\}$. Also, G has the core $g_u(w) = (g_{2u}*h_{1\#u})(w)$.

Note: As will be seen in the next chapter, we automatically have a bound on the order of $\partial_u^\alpha g_{2u}$, $\partial_u^\alpha h_{1u}$ when we work with asymptotic expansions.

Proof:

Finding U: Define $(g,\varphi) = G(\varphi)$, $\forall\varphi \varepsilon S$.

Consider that $(g(w),\varphi(w)) = (g_u(uw^{-1})\varphi(w))\forall\varphi \varepsilon S$ iff $(g(w),\varphi(uw^{-1})) = (g_u(w),\varphi(w))$, $\forall\varphi \varepsilon S$.

Hence, define $(g_u,\varphi) = (g(w),\varphi(uw^{-1}))$

$$= \int g_{2u}(uv^{-1})[\int h_{1v}(vw^{-1})\varphi(uw^{-1})dw]dv$$

$$= \int g_{2u}(uv^{-1})[\int h_{1v}(vw)\varphi(uw)dw]dv.$$

To be well defined, it is sufficient that u be in $\tilde{U}_2$ and either v be in $\tilde{U}_1$ or uv^{-1} not in X_2. Hence it is sufficient if u is in $\tilde{U}_2$ and u is not in $X_2(\mathbb{R}^n\setminus\tilde{U}_1)$. In turn, this means

$$u \in \tilde{U}_2\setminus[X_2(\mathbb{R}^n\setminus\tilde{U}_1)] \text{ which contains } U_2\setminus[X_2(\mathbb{R}^n\setminus U_1)]$$

which we now consider to be U.

It can be seen that if $X_2^{-1}U_2 \subset U_1$, then U can simply be defined as U_2. Does this mean that without knowing such a condition, we can simply define U as $U_2 \cap X_2U_1$? No. Unfortunately,

$$X_2(\mathbb{R}^n\setminus U_1) \not\subseteq X_2\mathbb{R}^n\setminus X_2U_1 = \mathbb{R}^n\setminus(X_2U_1).$$

Hence $U_2\setminus[X_2(\mathbb{R}^n\setminus U_1)]$ will have to do unless we place restrictions on (or related to) the nature of X_i and/or U_i (e.g. φ has support outside of X_2X_1).

<u>Finding the support of g_u:</u>

Let X_3 = supp φ. Then $\varphi(uw)$ has support $u^{-1}X_3$. Since $h_{1v}(vw)$ has support $v^{-1}X_1$, the product $h_{1v}(vw)\varphi(uw)$ is supported in $w \in U^{-1}X_3 \cap v^{-1}X_1$.

Now notice $g_{2u}(uv^{-1})$ has support $uv^{-1} \varepsilon X_2$. Hence WLOG, assume the "integration" w.r.t. dv is restricted so $v^{-1} \varepsilon u^{-1}X_2$. Thus the product $(g_2(uv^{-1})h_{1v}(vw)\varphi(uw) \equiv 0$ for w not in $u^{-1}X_3 \cap (u^{-1}X_2)X_1$.

Hence $(g_u, \varphi) = 0$ when $X_3 \cap X_2X_1 = \phi$.

Hence $g_u \varepsilon E^1(X_2X_1), \forall u \varepsilon \tilde{U}$.

(g_u, φ) is C^∞ in u:

Consider $(g_u,\varphi) = \int g_{2x}(yv^{-1})[\int h_{1v}(vw)\varphi(zw)dw]dv$ where $x = y = z = u$.

By the definition of core, (g_u,φ) is C^∞ in x. Since $[\int h\, dw] \varepsilon C^\infty$, (g_u,φ) is C^∞ in y. By construction, and limit of difference of quotient arguments, $(g_u\varphi)$ is continuous in y

and $$Z_j^R\int g[\int h\varphi dw]dv = \int gZ_j^R[\int h\varphi dw]dv = \int g[\int h(Z_j^R\varphi)dw]dv.$$

Hence (g_u,φ) is C^∞ in z.

Hence (g_u,φ) is C^∞ in $u = x = y = z$.

g_u is C^∞ on $\tilde{U} \times (\mathbb{R}^n\backslash 0)$:

$$(g_u,\varphi) = \int g_{2u}(v^{-1})[\int h_{1vu}(vw)\varphi(w)dw]dv.$$

For this part of the theorem, WLOG, we replace $g_{2u}(v^{-1})$ with $g_{2u}(v)$.

Assume $\varphi(w)$ has compact support in $\mathbb{R}^n\setminus(NN)$ where N is a symmetric neighborhood of 0 (i.e. $N = N^{-1}$). Pick $\psi \in C_c^\infty(N) \ni \psi(\mathbb{R}^n) \subseteq [0,1]$ and $\psi(v) \equiv 1$ near $v = 0$. Consider that if $v \in \text{supp}\,\psi \subseteq N$ and $vw \in \text{supp}\,\psi \subseteq N$ then $w = v^{-1}(vw) \in N^{-1}N = NN$ and hence $w \notin \text{supp}\,\varphi$. Hence $\psi(v)\psi(vw)\varphi(w) = 0$ $\forall v\, \forall w$. Hence the parametrized distribution given by $\varphi \mapsto \int(\psi g_{2u})(v)[\int(\psi h_{1vu})(vw)\varphi(w)dw]dv$ is the zero distribution and hence C^∞ in $u \in \tilde{U}$ and in $w \in \mathbb{R}^n \supseteq \mathbb{R}^n\setminus NN$.

Consider that $(1-\psi)g_{2u}$ and $(1-\psi)h_{1vu}$ are both C_c^∞ functions.

Hence $\int(1-\psi)g_{2u}(v)[\int(1-\psi)h_{1vu}(vw)\varphi(w)dw]dv$

$$= \int[\int(1-\psi)g_{2u}(v)(1-\psi)h_{1vu}(vw)dv]\varphi(w)dw$$

where $[\int dv]$ is an absolutely convergent integral, uniformly bounded w.r.t. w and C^∞ in $u \in \overline{U^1}$ and in $w \in R^n\setminus NN$.

Consider $\int(1-\psi)g_{2u}(v)[\int\psi h_{1vu}(vw)\varphi(w)dw]dv$. Notice that ψh_{1vu} is itself a core of compact support. Hence by Lemma (5.18), we have

$$\int(1-\psi)g_{2u}(v)[\int(1+\Delta_w^M)f_{vu}(w)\varphi(v^{-1}w)dw]dv$$

where $f_{vu}(w)$ is C^∞ in vu and continuous in w. This integral is a finite sum of objects of the form

$$\int(1-\psi)g_{2u}(v)[\int f_{vu}(w)\partial_w^\alpha\varphi(v^{-1}w)dw]dv.$$

Notice $\partial_w^{\alpha}\varphi(v^{-1}w)$ is a finite sum of objects of the form

$$v^{\beta}w^{\gamma}(\partial^{\omega}\varphi)(v^{-1}w).$$

Throwing v^{β} onto $g_{2u}(v)$ and w^{γ} onto $f_{vu}(w)$, we can WLOG consider

$$\int(1-\psi)g_{2u}(v)[\int f_{vu}(w)(\partial^{\omega}\varphi)(v^{-1}w)dw]dv$$

$$= \int(1-\psi)g_{2u}(v)[\int f_{vu}(vw)(\partial^{\omega}\varphi)(w)dw]dv.$$

Since $\operatorname{supp}(1-\varphi)g_{2u} \leq$ bounded X_2, we have by Fubini's theorem

$$\int[\int(1-\varphi)g_{2u}(v)f_{vu}(vw)dv](\partial^{\omega}\varphi)(w)dw.$$

The $[\int dv]$ is C^{∞} in $u \in \overline{U^1}$ and (by a change of variables $v \to vw^{-1}$ in the inner integral) is C^{∞} in $w \in \mathbf{R}^n \supseteq \mathbf{R}^n \setminus NN$. Hence, so is $-1^{\|\omega\|}\partial_{\omega}[\int dv]$.

Now consider,

$$\int(\psi g_{2u})(v)[\int(1-\psi)h_{1vu}(vw)\varphi(w)dw]dv.$$

Again by Lemma (5.18), we have

$$\int(1+\Delta_v^M)f_u(v)[\int(1-\psi)h_{1vu}(vw)\varphi(w)dw]dv$$

$$= \int f_u(v)(1+\Delta_v^M)[\int(1-\psi)h_{1vu}(vw)\varphi(w)dw]dv$$

which is a finite sum of terms of the form

$$\int f_u(v)\int[u^{\alpha}w^{\beta}v^{\gamma}\partial^{\mu}_{(vu)_i}\partial^{\omega}_{(vw)_j}(1-\psi)h_{1vu}(vw)]\varphi_{(w)}dwdv.$$

Again by Fubini's theorem we have

$$\iint f_u(v)[\ ]dv\varphi(w)dw,$$

where $\int f_u(v)[\ ]dv$ is C^∞ in $u \in \overline{U^1}$ and (by throwing derivatives onto $(1-\psi)h_{1vu}$) is C^∞ in $w \in \mathbb{R}\setminus NN$.

In light of $h_{1u} = \psi h_{1u} + (1-\psi)h_{1u}$ and $g_{2u} = \psi g_{2u} + (1-\psi)g_{2u}$, we have shown that $g_u(w)$ is C^∞ in u and C^∞ in $w \in \mathbb{R}^n\setminus NN$ for N arbitrarily small, hence

$$C^\infty \text{ in } w \in \mathbb{R}^n\setminus 0.$$

Notice that by virtue of the final integral in each of the four cases and by g_u's compact support, we may assume $\varphi \in S$.

<u>Finally, what is g_u?</u>

For all $\varphi \in S$,

$$\begin{aligned}
(g_u,\varphi) &= \int g_{2u}(uv^{-1})[\int h_{1v}(vw^{-1})\varphi(uw^{-1})dw]dv \\
&= \int g_{2u}(uv^{-1})[\int h_{1v}(vu^{-1}w^{-1})\varphi(w^{-1})dw]dv \\
&= \int g_{2u}(v^{-1})[\int h_{1vu}(vw^{-1})\varphi(w^{-1})dw]dv \\
&= \int g_{2u}(v)[\int h_{1v^{-1}u}(v^{-1}w^{-1})\varphi(w^{-1})dw]dv \\
&= (g_{2u}*h_{1\#u},\varphi) \text{ by definition and as desired.}
\end{aligned}$$

We will note that this is also

$$\int g_{2u}(v)[\int \check{h}_{1v^{-1}u}(wv)\varphi(w^{-1})dw]dv$$

which is in a sense $(g_{2u},\varphi * \tilde{h}_{1v^{-1}u})$ as might be expected. ■

(5.21) <u>Define</u> $g_{2u}*(\#)^{\beta}h_{1u}$ by

$$\varphi \mapsto (g_{2u}*(\#)^{\beta}h_{1u},\varphi) = \int g_{2u}(v)\,[\int (v^{-1})^{\beta}h_{1u}(v^{-1}w)\varphi(w)\,dw]\,dv.$$

(5.22) <u>Corollary</u>: Consider h_{1u} and g_{2u} as in Theorem (5.2).

<u>Define</u> $G^1 : \varphi \to \int g_{2u}(uv^{-1})\,[\int (vu^{-1})^{\beta}h_{1u}(vw^{-1})\varphi(w)\,dw]\,dv.$

Then $G^1 \in \psi DO(U, X_2X_1)$ with U and X_2X_1 as in Theorem (5.20). Also, G^1 has the core $(g_{2u}*(\#)^{\beta}h_{1u},\varphi)(w)$.

<u>Proof</u>: Replace $g_{2u}(v)$ by $(v^{-1})^{\beta}g_{2u}(v)$. ■

6. Error Class B^k and Asymptotics.

Purpose of this chapter:

We will develop an asymptotic expansion of cores and show that the core of a composition may be considered to have only a finite number of terms.

Consider $\tilde{U} \subseteq \mathbb{R}^n$, $k \in \{1,2,\ldots\}$.

(6.1) Define $g_u \in B^k$ as "g_u is in the kth error class" meaning $g_u(x)$ is a core and a function $\forall \ell \in (\mathbb{Z}^+)^n$ $\forall \alpha \in (\mathbb{Z}^+)^n \ni |\alpha| < k$, $\partial_u^\ell \partial_x^\alpha g_u$ has a continuous extension over $\tilde{U} \times \mathbb{R}^n$.

(6.2) Proposition: Consider $m \in \mathbb{C}$ and positive integers $k \leq \operatorname{Re} m$. Then $K^m \subseteq B^k$.

Actually if $k < \operatorname{Re} m$ then $K^{m-1} \subseteq B^k$, but for simplicity we will use the above (less sharp) proposition.

Recall $A(p)$ is the greatest integer less than or equal to p.

Consider core $g_u \ni \operatorname{supp} g_u \subseteq$ bounded open X, $\forall u \in \tilde{U}$.

(6.3) Define $g_u \sim \sum K_u^j$ as "g_u has an asymptotic series in K," meaning $\exists k \in \mathbb{C}$, $\exists \varphi \in C_c^\infty \ni \varphi \equiv 1$ near 0, $\exists K_u^j \in K^{k+j}$, $\forall j \in \mathbb{Z}^+$,

$g_u - \varphi \sum_{j=0}^{n} K_u^j \in B^{A(\operatorname{Re} k+M+1)}$, $\forall$ nonnegative integers $M \geq -\operatorname{Re} k$.

Note: WLOG "$\exists\varphi$" can be replaced by "$\forall\varphi$".

(6.4) Proposition: Consider $K_u \in \mathcal{K}^k$ and $\varphi \in C_c^\infty \ni \varphi(\mathbb{R}^n) = [0,1]$ and $\varphi \equiv 1$ near 0. Then $N \in \mathbb{Z}^+ \ni$ the order of $\partial_u^\alpha(\varphi K_u)$ is less than N, $\forall\alpha \in (\mathbb{Z}^+)^n$.

Proof: Express $K = \Lambda_G + \sum c_\alpha \delta^\alpha$ and use Corollary 2.11. ■

(6.5) Corollary: If $g_u \sim \sum K_u^j$ then $\exists N \in \mathbb{Z}^+$ the order of $\partial_u^\alpha g_u$ is less than N, $\forall\alpha \in (\mathbb{Z}^+)^n$.

Proof: Pick nonnegative $M \geq -\operatorname{Re} k$. By the definition of B^k and $g_u \sim \sum K_u^j$, $\partial_u^\alpha(g_u - \varphi \sum_{j=0}^{M} K_u^j)$ is to be considered continuous and compact support. Hence order 0.

By the preceding proposition, $\exists N \in \mathbb{Z}^+ \ni$ the order of $\partial_u^\alpha(\varphi \sum_{j=0}^{M} K_u^j)$ is less than N, $\forall\alpha \in (\mathbb{Z}^+)^n$. Hence the same is true of $\partial_u^\alpha g_u$. ■

(6.6) Proposition: Consider cores h_{1u} and g_{2u} which have compact support in x and with $h_{1u} \sim \sum K_{1u}^j$. Consider $M \in \mathbb{Z}^+$. If $M^1 \in \mathbb{Z}^+$ is large enough then $g_{2u} * (h_{1\#u} - \varphi \sum_{j=0}^{M'} K_{1\#u}^j) \in B^M$.

Proof: By Theorem (5.20) and the definition of $\sim\sum K^j$, the above "condition" is a core of compact support.

By the last corollary and Lemma (5.18), $\exists m' \in \mathbb{Z}^+ \ni g_{2u} = (1+\Delta^{m'})f_{2u}$ where $f_{2u} \in C(U^1 \times \mathbb{R}^n)$ and C^∞ in u.

Hence $(g_{2u} * (h_{1\#u} - \varphi \sum^{M'} K^j_{1\#u}), \psi)$

$$= \int g_{2u}(v)\,[\int (h_{1v^{-1}u} - \varphi \sum^{M'} K^j_{iv^{-1}u})(v^{-1}w)\psi(w)\,dw]\,dv$$

$$= \int f_{2u}(v)\,(1+\Delta_v^{m'})\,[\int (h_{1v^{-1}u} - \varphi \sum^{M'} K^j_{1v^{-1}u})(v^{-1}w)\psi(w)\,dw]\,dv.$$

Due to the support of $(1+\Delta^{m'})(h-\varphi\sum K^j)$ being compact and Fubini's Theorem, for M^1 large enough we have

$$\int [\int f_{2u}(v)\,(1+\Delta_v^{m'})\,(h_{1v^{-1}u} - \varphi \sum^{M'} K^j_{1v^{-1}u})(v^{-1}w)\,dv]\,\psi(w)\,dw.$$

By definition of $\sim\sum K^j$ and despite the presence of $\Delta_v^{m'}$,

$$\partial_u^\ell \partial_w^\alpha [\int f_{2u}(v)\,(1+\Delta_v^{m'})\,(h_{1v^{-1}u} - \varphi \sum^{M'} K^j_{1v^{-1}u})(v^{-1}w)\,dv]$$

is continuous over $U^1 \times \mathbb{R}^n$ $\forall \ell \in (\mathbb{Z}^+)^n$, $\forall \alpha \in (\mathbb{Z}^+)^n \ni |\alpha| < M$, when M' is large enough. ■

(6.7) <u>Proposition</u>: Consider cores h_{1u} and g_{2u} with $h_{1u} \sim \sum K^j_{1u}$ and $g_{2u} \sim \sum K^i_{2u}$. Consider $M \in \mathbb{Z}^+$. If $M'' \in \mathbb{Z}^+$ is large enough then

$$(g_{2u} - \varphi \sum_{i=0}^{M''} K^i_{2u}) * (\varphi \sum_{j=0}^{M'} K^j_{1\#u}) \in B^M$$

independent of whatever $M' \in \mathbb{Z}^+$ is.

As in the last proof, we have a core of compact support created by convolution. Let $M'\ E\{0,1,2,\ldots, A(-\mathrm{Re}\ k2 + 1)\}$. By Proposition (6.4) and Lemma (5.18) again, $\exists m'' \in \mathbb{Z}^+ \ni \varphi \sum_{j=0}^{M'} K^j_{1u} = (1+\Delta^{m''}) f_{1u}$ where $f_{1u} \in C(U \times \mathbb{R}^n)$ and C^∞ in u.

Hence $((g_{2u} - \varphi \sum^{M''} K^i_{2u}) * (\varphi \sum^{M'} K^j_{1\#u}), \psi)$

$$= \int (g_{2u} - \varphi \sum^{M''} K^i_{2u})(v) [\int \varphi \sum^{M'} K^j_{1v^{-1}u}(w) \psi(vw) dw] dv$$

$$= \int (g_{2u} - \varphi \sum^{M''} K^i_{2u})(v) [\int (1+\Delta^{m''}_w) f_{1v^{-1}u}(w) \psi(vw) dw] dv$$

$$= \int (g_{2u} - \varphi \sum^{M''} K^i_{2u})(v) [\int f_{1v^{-1}u}(w) (1+\Delta^{m''}_w) \psi(vw) dw] dv$$

WLOG we consider objects of the form

$$\int (g_{2u} - \varphi \sum^{M''} K^i_{2u})(v) [\int f_{1v^{-1}u}(w) v^\alpha w^\beta (\partial^\tau_j \psi)(vw) dw] dv.$$

In turn WLOG consider objects of the form

$$\int (g_{2u} - \varphi \sum^{M''} K^i_{2u}(v) [\int f_{1v^{-1}u}(v^{-1}w) v^\gamma w^\omega (\partial^\tau_j \psi)(w) dw] dv.$$

Since $(g_{2u} - \varphi \sum^{N''} K^i_{2u})(v)$ and $\varphi \sum^{M'} K^j_{1v^{-1}u}(v^{-1}w)$ have compact support.

Only those ψ with support in a fixed compact set need be considered, so assume the region of integration w.r.t. dw to be compact. Hence by Fubini's theorem we have WLOG

$$\int W^\omega \partial^\tau_{W_j} [\int v^\gamma (g_{2u} - \varphi \sum^{M''} K^i_{2u})(v) f_{1v^{-1}u}(v^{-1}w) dv] \psi(w) dw.$$

By definition of $\sim \sum K^i$ and because f_{1z} is C^∞ in z, we have

$$\partial^\ell_u \partial^\alpha_w W^\omega \partial^2_{W_j} [\int (wv)^\gamma (g_{2u} - \varphi \sum^{M''} K^i_{2u})(wv) f_{1v^{-1}w^{-1}u}(v^{-1}) dv]$$

is continuous over $U^1 \times \mathbb{R}^n$

$$\forall \ell \in (\mathbb{Z}^+)^n, \ \forall \alpha \in (\mathbb{Z}^+)^n \ni |\alpha| < M$$

when M" is large enough.

But recall from the beginning of the proof that we choose a nonnegative integer $M' \leqq A(-\mathrm{Re}\ k_2+1)$. Since this is a finite # of choices, $\exists M''$ that will work for all such M'.

What of $M' > A(-\mathrm{Re}\ k_2+1)$? Any extra K^j_{1u}'s will be continuous and thus we can concentrate on only the differentiability of $(g_{2u}-\varphi\sum^{M''} K^i_{2u})$. Hence the only additional requirement on M" is merely that it be $\geqq \max(A(1+M-\mathrm{Re}\ k_2),0)$. ■

(6.8) <u>Theorem</u>: Consider cores h_{1u} and g_{2u} with $h_{1u} \sim \sum K^j_{1u}$ and $g_{2u} \sim \sum K^i_{2u}$. Consider $M \in \mathbb{Z}^+$. If N is large enough then

$$g_{2u}*h_{1\#u} = (\varphi\sum_{i=0}^{N} K^i_{2u})*(\varphi\sum_{j=0}^{N} K^j_{1\#u})$$

modulo B^M.

<u>Proof</u>: By Proposition (6.6), if M' large enough then

$$g_{2u}*h_{1\#u} = g_{2u}*(\varphi\sum_{j=0}^{M'} K^j_{1\#n}) \bmod B^M.$$

By Proposition (6.7), if M" large enough then

$$g_{2u}*(\varphi\sum_{j=0}^{M'} K^j_{1\#u}) = (\varphi\sum_{i=0}^{M''} K^i_{2u})*(\varphi\sum_{j=0}^{M'} K^j_{1\#u})$$

mod B^M, regardless of the size of M'.

Hence, we can simply let N = max(M',M").

7. Taylor Expansions in Distribution's Parameter.

Purpose of this chapter:

To further reduce the core of composition to a finite number of $\varphi K^i_{2u} * \varphi K^j_{1u}$. We also discuss the following operator.

(7.1) Define ρ_α as the left invariant operator $= \partial^\alpha$ at 0. In other words, $\rho_\alpha \varphi(u) = \partial^\alpha_w \psi(0)$ where $\psi(w) = \varphi(u \cdot w)$, $\forall w$.

(7.2) Define ρ^R_α as the right invariant operator $= \partial^\alpha$ at 0. In other words,
$\rho^R_\alpha \varphi(n) = \partial^\alpha_n \psi(0)$ where $\psi(w) = \varphi(w \cdot u)$, $\forall w$.

Recall $I(M) = \{\alpha \varepsilon (Z^+)^n / |\alpha| \geq M$ and $\exists_j \exists |\alpha - 1_j| < M\}$ and

(7.3) Proposition: If h is C^∞ on V which is any open convex neighborhood of u then

$$h(vu) = \sum_{|\alpha|<M} \frac{v^\alpha \rho^R_\alpha h(u)}{\alpha!} + \sum_{\alpha \varepsilon I(M)} v^\alpha g_{\alpha,u}(v)$$

where $g_{\alpha,u} \varepsilon C^\infty(v)$.

Proof: Consider that $\rho^R_\alpha h(u) = \partial^\alpha \Big|_{x=0} h(x \cdot u) = \partial_x f_u(0)$

where $f(x) = h(x \cdot u)$, $\forall x$.

By Proposition (2.10), $f_u(r) = \sum \frac{v^\alpha \partial^\alpha f_u(0)}{\alpha!} + \sum v^\alpha g_{\alpha,u}(v)$.

Hence $\quad h(vu) = \sum \frac{v^\alpha \rho^R_\alpha h(u)}{\alpha!} + \sum v^\alpha g_{\alpha,u}(v)$ as desired. ∎

(7.4) Proposition: Consider cores h_{1u} and g_{2u} with $h_{1u} \sim \sum K^j_{1u}$ and $g_{2u} \sim \sum K^i_{2u}$. Consider $M \in \mathbb{Z}^+$.

If N' is large enouth then

$$(\varphi \sum_{i=0}^{N} K^i_{2u}) * \varphi \sum_{j=0}^{N} (K^j_{1\#u} - \sum_{|\alpha|<N'} \frac{(\#)^\alpha}{\alpha!} \rho^R_\alpha K^j_{1u}) \in B^M$$

regardless of the size of $N \in \mathbb{Z}^+$.

Note: Since $D \in X$, $\tilde{U}_2 \setminus [X(\mathbb{R}^n \setminus U_1]$ is a subset of $\tilde{U}_2 \setminus [0(\mathbb{R}^n \setminus \tilde{U}_1)] = \tilde{U}_2 \setminus [\mathbb{R}^n \setminus \tilde{U}_1] = \tilde{U}_2 \cap \tilde{U}_1$, as would be desired by the presence of K^j_{1u}.

Proof: By Theorems (5.20) and (5.22), we are dealing with a core. We must show, as in Propositions (6.6) and (6.7), $\partial^\ell_u \partial^\alpha_w$ of it is continuous.

Notice that the order of φK^j_u is less than or equal to $\max(0,A(-\text{Re } k))$, $\forall j \in \mathbb{Z}^+$. Hence, by Lemma (5.18), $\exists m \in \mathbb{Z}^+$ $\forall j \in \mathbb{Z}^+$ $\varphi K_{1vu}(w) = (1+\Delta^m_w) f_{1vu}(w)$.

Hence, $\varphi^R_{\rho\alpha} K^j_{1vu}(w) = (1+\Delta^m_w) \rho^R_\alpha f^j_{1u}(w)$

with $\varphi^R_{\rho\alpha} K^j_{1u}(w) = 1+\Delta^m_w) \rho^R_\alpha f^j_{1u}(w)$.

Hence, $\varphi(K_{1v^{-1}u}(w) - \sum_{|\alpha|<N'} \frac{(v^{-1})^\alpha}{\alpha!} \rho^R_\alpha K^j_{1u}(w))$

$$= (1+\Delta^m_w)(f^j_{1v^{-1}u}(w) - \sum_{|\alpha|<N'} \frac{(v^{-1})^\alpha}{\alpha!} \rho^R_\alpha f^j_{1u}(w)).$$

By our definitions involving #,

$$((\varphi K^i_{2u}) * \varphi (K^j_{1\#u} - \sum_{|\alpha|<N'} \frac{(\#)^\alpha}{\alpha!} \rho^R_\alpha K^j_{1u}), \psi)$$

$$= \int \varphi K^i_{2u}(v) [\int (\varphi K^j_{1v^{-1}u} - \sum_{|\alpha|<N'} \frac{(v^{-1})^\alpha}{\alpha!} \rho^R_\alpha K^j_{1u})(w) \psi(vw) dw] dv$$

$$= \int |v|^{N'} \varphi K^i_{2u}(v) [\int \frac{1}{|v|^{N'}} (f^j_{1v^{-1}u} - \sum_{|\alpha|<N'} \frac{(v^{-1})^\alpha}{\alpha!} \rho^R_\alpha f_{1u})$$

$$(v^{-1}w) v^\gamma w^\omega (\partial^\beta \psi)(w) dw] dv$$

as in the proof of Proposition (6.7) and with $\|\beta\| \leqq 2m$.

Notice that $\frac{1}{|v|^{N'}}(f_{1v^{-1}u} - \sum_{|\alpha|<N'} \frac{(v^{-1})^\alpha}{\alpha!} \rho^R_\alpha f^j_{1u})(w)$ is C^∞ in u, continuous in w, and by Corollary (2.11) is continuous in v. Due to φK_{2u} and φK_{1u} being of compact support, ψ may be considered as such, and in turn f_{1u} also.

Likewise, $|v|^{N'} \varphi K^i_{2u}(v)$ is in a desired B^k when N' is large enough and is also of compact support. Continuing as in the proof of Proposition (6.7), we have continuity of the core over $U^1 \times \mathbb{R}^n$ after $\partial^\ell_u \partial^\alpha_w$, $\forall \ell \in (\mathbb{Z}^+)^n$ $\forall \alpha \in (\mathbb{Z}^+)^n \ni |\alpha| < M$ when N' is large enough. ∎

(7.5) <u>Theorem</u>: Consider $M \in \mathbb{Z}^+$. If N is large enough then

$$g_{2u} * h_{1\#u} = \sum \frac{(\#)^{\alpha}}{\alpha !} \varphi K^{i}_{2u} * \rho^{R}_{\alpha} \varphi K^{j}_{1u} \text{ modulo } B^{M}$$

$$0 \leqq i \leqq N$$

$$0 \leqq j \leqq N$$

$$|\alpha| < N$$

with $(\#)^{\alpha}\varphi K^{i}_{2u}(s)$ meaning $(s^{-1})^{\alpha}\ K^{i}_{2u}(s)$ as expected.

Proof: Theorem (6.8) and Proposition (7.4). ■

Hence, in the next chapter, we analyze objects of the form $\varphi K_{2u,i} * \varphi K_{1u,j}$.

The remainder of this chapter is a discussion of ρ_{α} versus the "symmetrization" δ_{α} (as seen in Berkoff-Poincaré-Witt Theorem).

Let us first review basic facts about σ_{α}. Consider an operator of composition $x \mapsto T_1(T_2(\ldots(T_m(x))))$ associated with the sequence of operators $T_1,\ldots,T_m$ on algebraic A.

(7.6) Define S as the sum (of $\alpha!$ terms) of the compositions corresponding to each ordering of the sequence. If the sequence is α_1 copies of L_1, α_2 copies of $L_2,\ldots$, α_n copies of L_n where $\|\alpha\| = m$, then each of the terms (when the sequence is reordered as $T_1,\ldots,T_m$) are created $\alpha!$ times.

(7.7) <u>Define</u> S_α as the sum of operators of composition of <u>distinct</u> orderings of $L_1,\ldots,L_1, L_2,\ldots,L_n$.

By the above remarks, $S = \alpha! S_\alpha$. Hence a type of average composition of $T_1,\ldots,T_m$ is given by $\frac{1}{m!}S$.

Consider $L_1,\ldots,L_1, L_2,\ldots,L_n$.

(7.8) <u>Define</u> $\sigma_\alpha(L)$ as the α symmetrization of L meaning $\frac{1}{m!}S$, meaning $\frac{\alpha!}{\|\alpha\|!}S_\alpha$.

Consider $u \in A^n = A\times\ldots\times A$. Then $(u\cdot L)^m = (u_1L_1+\ldots+u_nL_n)^m = (\sum_{\|\alpha\|=m} u^\alpha S_\alpha)$ since the L_i's do not act on the u_j's.

Hence.....

(7.9) <u>Proposition</u>: σ_α (L) is the coefficient of u^α in the expansion of $\frac{\alpha!}{\|\alpha\|!}(u\underline{.}L)^{\|\alpha\|}$ where $\underline{.}$ means dot product.

(7.10) <u>Example</u>: Consider $D_r(x) = (rx_1, r^2x_2)$, $\forall r > 0\ \forall x \in \mathbb{R}^2$.

Consider addition: $\mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}^2$: $(x,y) \mapsto (x_1+y_1, x_2+y_2+x_1y_1)$. Then $(\mathbb{R}^2,\cdot)$ is an abelian Lie group such that $D_rxD_ry = D_r(xy)$, <u>but</u> $\sigma_{(2,0)}(X) \neq \rho_{(2,0)}$. More specifically $\sigma_{(2,0)}(X)f(0) = X^{(2,0)}f(0) = \frac{\partial f}{\partial t_2}(0) + \frac{\partial^2 f}{\partial w_1^2}(0)$ while $\rho_{(2,0)} = \frac{\partial^2 f}{\partial w_1^2}(0)$ by definition.

However, inspired by Geller [G]....

(7.11) Proposition: Consider group addition $\cdot$ on $\mathbb{R}^n$ such that $\forall h \in \mathbb{R}\ \forall u \in \mathbb{R}^n$, $hu \cdot u = hu + u + h^2 v$ where v is a polynomial in h and u.

Define $X = (X_1, \ldots, X_n)$ as the listing of our left invariant vector fields $X_j = \frac{\partial}{\partial x_j}$ at 0.

Then $[(x \cdot \partial_x)^m F](sx) = \frac{\partial}{\partial_s m}[F(sx)] = [(x \cdot X)^m F](sx)$ where $s \in \mathbb{R}$.

Proof: ETS for m = 1. Consider that

$$\frac{\partial}{\partial s}[F(sx)] = \sum \frac{\partial sx_j}{\partial s}\left[\frac{\partial}{\partial sx_j} F(sx)\right]$$

$$= x_j(\partial_j\big|_{sx} F) = [x \cdot \partial_x) F](sx).$$

Let $G(v) = F(v \cdot sx)$. Then

$$\frac{\partial}{\partial s}[F(sx)] = \lim_{h \to 0} \frac{F(hx+sx) - F(sx)}{h}$$

$$= \lim_{h \to 0} \frac{F(hx+sx) - F(hx+sx+h^2 v)}{h} + \lim_{h \to 0} \frac{F(hx \cdot sx) - F(sx)}{h}$$

$$= 0 + \left[\frac{\partial G(hx)}{h}\right]\Bigg|_{h=0} = [(x \cdot X) G](0) = [(x \cdot X) F](sx). \quad \blacksquare$$

(7.12) Corollary: If $hu \cdot u = hu + u + h^2 v$ then $\sigma_\alpha(X) = \rho_\alpha$.

Proof: $[(x \cdot \partial_x)^m F](0) = [(x \cdot X)^n F](0)$.

Hence $\sigma_\alpha(X)$ will equal ∂_x^α at 0, as desired. $\blacksquare$

Note: By choosing canonical coördinates, one can always arrange straight lines thru the origin to be parameter subgroup. Hence, $hu \cdot u = hu + u$.

8. Generalized Convolution $\underset{-}{*}$ and the Asymptotics of Composition.

Purpose of this chapter:

To express the asymptotic expansion of the core of composition in terms of $K^i_{2n}\underset{-}{*}K^j_{1u}$'s rather than $\varphi K^i_{2u}*\varphi K^j_{1u}$'s

Recall...

(5.7) Define $K_2 * K_1 : \varphi \mapsto (K_2, \varphi * \tilde{K}_1)$ for $K_1, K_2 \in E^1$.
Consider $K_2, K_1 \in \text{Rhom} \leqq S^1$.
Even though $K_2, K_1 \in E^1$, if they don't grow too quickly...

(8.1) Proposition: Consider $K_{2u} \in \text{Rhom}_{k_2}$, $K_{1u} \in \text{Rhom}_{k_1}$, and $\text{Re}(k_2+k_1) < -Q$.
Then $K_{2u} * K_{1u} \in \text{Rhom}_{k_2+k_1+Q}$.

Although for k_1, k_2 real and K_1, K_2 not parametrized, see Christ and Geller [C] Lemma 9.5, page 592-3 for a method of proof.

Recall that $\frac{\partial}{\partial x_j} = X_j + \sum_{a_m > a_j} p_{j,m}(x) X_m$ where X_i is a left invariant vector field agreeing with $\frac{\partial}{\partial x_i}$ at 0 and $p_{j,m}(x)$ are homogeneous degree $a_m - a_j$ polynomials.

(8.2) Proposition: Consider $M \in \mathbb{Z}^+$. Then $(\frac{\partial}{\partial x_j})^M$ is a finite sum of terms of the form

$$p(x)X_{b_1}\ldots X_{b_9}$$

such that (the weighted degree of p) - $(\sum_{i=1}^{9} a_{(b_i)})$ $= -Ma_j$.

<u>Proof</u>: Notice that $\frac{\partial}{\partial x_j}$ is of that form.

Now consider

$$(\frac{\partial}{\partial x_j})(pX_{b_1}\ldots X_{b_9})$$

$$= (X_j + \sum_{a_m > a_j} p_{j,m}X_m)(pX_{b_1}\ldots X_{b_9}).$$

Enough to consider

$$(p_{j,m}X_m)(pX_{b_1}\ldots X_{b_9})$$

which by product rule (4.24) is

$$p_{j,m}pX_mX_{b_1}\ldots X_9 + p_{j,m}(X_mp)X_{b_1}\ldots X_{b_9}.$$

Both parts of the sum have (weighted degree polynomial) - (sum of weighted order of X_i's) = $(-M-1)a_j$ as desired. ■

Straightforward calculations give

(8.3) <u>Lemma</u>: Consider $K_{1u} \in Rhom_{k_1}$ and p_u homogeneous polynomial degree k_2.

Then $\exists m \in \mathbb{Z}\ \forall j = 1$ to n,

$D_j^m(p_u \log_* K_{1u}) \in Rhom_{k_1 - ma_j + k_2 + \bar{Q}}$.

(8.4) Proposition: Consider $K_{2u} \varepsilon K^{k_2}$, $K_{1u} \varepsilon K^{k_1}$, and $Re(k_2+k_1) < -Q$.
Then $K_{2u} * K_{1u} \varepsilon \text{Rhom}_{k_1+k_2+Q} = K^{k_1+k_2+Q}$.

Proof: In view of Proposition (8.1) WLOG, it is enough to consider the case of $K_u = p_u(x) \log|x| * K_{1u}(x)$.

Since $Re(k_1+k_2) < -Q$, K_u is a well defined core by the same reasoning as Proposition (8.1).

We can claim that we are finished by use of Lemma (8.3), Poincaré Lemma (3.5), and Proposition (3.1).

Assume $Re(k_1+k_2) \geq -Q$. Then $(1-\varphi)K_{2u} * (1-\varphi)K_{1u}$ would be (in general) undefined. However, if m_j is large enough, then we can claim that "$D_j^{m_j}(K_{2u} * K_{1u})$" exists in the sense that we could expand D_j in terms of left invariant vector fields and move them onto K_{1u}.

In other words, "$D_j^{m_j}(K_{2u} * K_{1u})$" equals a sum of terms of the form

$$p(x)(K_{2u} * X^{\alpha} K_{1u})(x)$$

where $$Re(k_1+k_2-|\alpha|) < -Q,$$

p is the appropriate polynomial, and X^{α} is some (ordered) sequencing of

$$X_1, \ldots, X_1 (\alpha_1 \text{ times}), X_2 \ldots X_2 (\alpha_2 \text{ times}), \ldots,$$

$$X_n \ldots X_n (\alpha_n \text{ times}).$$

Recall that $D_j = c\frac{\partial}{\partial x_j} \ni \widehat{D_j f} = \xi_j \hat{f}$. By Proposition (8.2), there exist a finite number of polynomials c_α, d_β such that $D_i^m = \sum c_\alpha X^\alpha$, $D_j^m = \sum d_\beta X$.

Pick $\psi \in C_c^\infty \ni \psi : \mathbb{R}^n \to [0,1]$ and $\psi \equiv 1$ near 0.

(8.5) <u>Define</u> $K_{1u}^0 = \psi K_{1u}$, $K_{2u}^0 = \psi K_{2u}$, $K_{1u}^\infty = (1-\psi)K_{1u}$, and $K_{2u}^\infty = (1-\psi)K_{2u}$.

(8.6) <u>Lemma</u>: Consider $m \in Z^+ \ni \mathrm{Re}(k_1+k_2) - ma_j < -Q, \forall j$. Then $D_i^m \sum d_\beta [K_{2u}^\infty * X^\beta K_{1u}^\infty] = D_j^m \sum c_\alpha [K_{2u} * X^\alpha K_{1u}^\infty]$.

<u>Proof</u>: $\psi \in C_c^\infty \ni \psi : \mathbb{R}^n \to [0,1]$ and $\psi \equiv 1$ near 0. There exists $C > 0$, $\forall r \in (0,1)$,

$$|(D^r\psi)K_{2u}(xy^{-1})| \leq |K_{2u}^\infty(xy^{-1})| \leq C|xy^{-1}|$$

and $\quad |(X^r K_1^\infty)(y)| \leq C|y|$ for all relevant $\gamma \in (\mathbb{Z}^+)^n$.

Use these inequalities as was done in Proposition (8.1) and then the Lebesgue Dominated Convergence Theorem, we have

$$D_i^m \sum d_\beta [(D^r\psi)K_{2u}^\infty * X^\beta K_{1u}](x) \to D_i^m \sum d_\beta [K_{2u}^\infty * X^\beta K_{1u}^\infty](x)$$

and $D_j^m \sum c_\alpha [(D^r\psi)K_{2u}^\infty * X^\alpha K_{1u}^\infty](x) \to D_j^m \sum c_\alpha [K_{2u}^\infty * X^\beta K_{1u}^\infty](x)$

as r goes to 0, for all x in $\mathbb{R}^n$.

These limits are equal since

$$D_i^m \sum d_\beta[(D^r\psi)K_{2u}^\infty * X^\beta K_{1u}^\infty]$$

$$= D_i^m D_j^m[(D^r\psi)K_{2u}^\infty * K_{1u}^\infty] = D_j^m D_i^m[(D^r\psi)K_{2u}^\infty * K_{1u}^\infty]$$

$$= D_j^m \sum c_\alpha[(D^r\psi)K_{2u}^\infty * X^\alpha K_{1u}^\infty], \ \forall r \in (0,1).$$ ∎

(8.7) <u>Proposition</u>: Consider $m \in \mathbb{Z}^+ \ni \mathrm{Re}(k_1+k_2) - ma_j < -Q, \ \forall j$.

Then $D_i^m "D_j^m(K_{2u}*K_{1u})" = D_j^m "D_i^m(K_{2u}*K_{1u})"$.

<u>Proof</u>: By definition,

$$D_i^m "D_j^m(K_{2u}*K_{1u})"$$

$$= D_i^m \sum d_\beta(K_{2u}*X^\beta K_{1u}) = D_i^m(\sum d_\beta[(K_{2u}^0 * X^\beta K_{1u}^0)$$

$$+ (K_{2u}^0 * X^\beta K_{1u}^\infty) + (K_{2u}^\infty * X^\beta K_{1u}^0)] + d_\beta[K_{2u}^\infty * X^\beta K_{1u}^\infty]).$$

Since K_{iu}^0 is in E' and K_{iu}^∞ is in C^∞, we have

$$D_i^m D_j^m[K_{2u}^0 * K_{1u}^0) + (K_{2u}^0 * K_{1u}^\infty) + (K_{2u}^\infty * K_{1u}^0)] + D_i^m[\sum d_\beta(K_{2u}^\infty * X^\beta K_{1u}^\infty)].$$

Using our Lemma (8.6), we produce

$$D_j^m D_i^m[(K_{2u}^0 * K_{1u}^0) + (K_{2u}^0 * K_{1u}^\infty) + (K_{2u}^\infty * K_{1u}^0)]$$

$$+ D_j^m[\sum c_\alpha(K_{2u}^\infty * X^\alpha K_{1u}^\infty)].$$

Now by reversing the steps used, we derive $D_j^m "D_i^m(K_{2u}*K_{1u})"$.

Consider $m \in \mathbb{Z}^+ \quad \mathrm{Re}(k_2+k_1+Q) - ma_j < 0, \ \forall j$.

(8.8) <u>Define</u> $K_{2u} \underline{*} K_{1u}$ as the "generalized convolution of K_{2u} and K_{1u}" by which we mean the K_u constructed by Poincaré Lemma (3.5) from using "$D_j^m(K_{2u} * K_{1u})$"'s.

Is $K_{2u} \underline{*} K_{1u}$ well defined? Yes, since m is large enough for "$D_j^m(K_{2u} * K_{1u})$"'s to be well defined homogeneous distributions and due to Proposition (8.7).

Is $K_{2u} \underline{*} K_{1u}$ uniquely defined? Depends on $k_1 + k_2$. Pick $m' \in \mathbb{Z}^+ \ni m' > m$. Then $D_j^{m'}$ of the two derivations of $K_{2u} \underline{*} K_{1u}$ are equal.

Hence, by Proposition (3.1), $K_{2u} \underline{*} K_{1u}$ is unique modulo, a polynomial of degree $k_2 + k_1 + Q$.

Hence, if $k_2 + k_1 + Q \notin \mathbb{Z}^+$, then $K_{2u} \underline{*} K_{1u}$ is simply unique.

(8.9) <u>Proposition</u>: Consider $K_{2u} \in \mathcal{K}^{k_2}$ and $K_{1u} \in \mathcal{K}^{k_1}$. Pick $\varphi \in C_c^\infty \ni \varphi \equiv 1$ near 0.

Then $K_{2u} \underline{*} K_{1u} = \varphi K_{2u} * \varphi K_{1u} + \psi_u$

with ψ being a C^∞ function in u and x.

<u>Proof</u>: Pick $m \in \mathbb{Z}^+ \ni \mathrm{Re}(k_1+k_2) + ma_j < -Q, \forall j$.

$$D_j^m(K_{2u} \underline{*} K_{1u}) = \sum d_\beta (K_{2u} * X^\beta K_{1u})$$

plus possibly a polynomial.

In turn, $(K_{2u} * X^\beta K_{1u})$

$= ((1-\varphi)K_{2u} * X^\beta K_{1u}) + X^\beta(\varphi K_{2u} * (1-\varphi)K_{1u}) + X^\beta(\varphi K_{2u} * \varphi K_{1u})$.

The first two of the three parts of that sum are C^∞ in x.

Hence, $\quad X^\beta(K_{2u}\underline{*}K_{1u}) - X^\beta(\varphi K_{2u}*\varphi K_{1u})$ is C^∞ in x.

Hence, $\quad \sum_{j=1}^{n} D_j^{2m}(K_{2u}\underline{*}K_{1u}-\varphi K_{2u}*\varphi K_{1u}]$ is C^∞ in x.

But $\quad \sum D_j^{2m}$ is elliptic,

hence, $\quad (K_{2u}\underline{*}K_{1u}-\varphi K_{2u}*\varphi K_{1u})$ is C^∞ in x.

The same holds for

$$\partial_u^\alpha(K_{2u}\underline{*}K_{1u}) - \partial_u^\alpha(\varphi K_{2u}*\varphi K_{1u}).$$

Hence, $\quad (K_{2u}\underline{*}K_{1u}-\varphi K_{2u}*\varphi K_{1u})$ is C^∞ in x and u. ■

Recall

(7.5) Theorem: Consider $M \in \mathbb{Z}^+$.

If N is large enough then

$$g_{2u}*h_{1\#u} = \sum_{\substack{0\leq i\leq N\\ 0\leq j\leq N\\ |\alpha|<N}} \frac{(\#)^\alpha}{\alpha!}\,\varphi K_{2u}^i*\rho_\alpha^R\varphi K_{1u}^j \text{ modulo } B^M$$

with $(\#)^\alpha\varphi K_{2u}^i(s)$ meaning $(s^{-1})^\alpha\varphi K_{2u}^i(s)$.

By Proposition (8.9), $(\#)^\alpha\varphi K_{2u}^i*\varphi_{\rho\alpha}^R K_{1u}^j$

$-(\#)^\alpha K_{2u}^i\underline{*}\,\rho_\alpha^R K_{1u}^j$ is in $C^\infty \subsetneq B^\infty$.

Hence, $$g_{2u}*h_{1\#u} = \sum_{\substack{0\leq i\leq N\\ 0\leq j\leq N\\ |\alpha|<N}} \frac{(\#)^{\alpha}}{\alpha!} K^{i}_{2u}\underline{*}_{\rho} R_{\alpha} K^{j}_{1u}$$

modulo B^M when N is large enough.

Notice that $(\#)^{\alpha}K^{i}_{2u}\underline{*}_{\rho}R_{\alpha}K^{j}_{1u} \in K^{m+k_1+k_2+Q}$ where $m = i + j + |\alpha|$. Hence we finally have...

(8.10) <u>Theorem</u>: Consider $G_1 \in \psi DO(U_1,X_1)$ with core $h_{1u} \sim \sum K^{j}_{1u}$ and $G_2 \in \psi DO(U_2,X_2)$ with core $g_{2u} \sim \sum K^{i}_{2u}$.

<u>Define</u> $G : S \to C^{\infty}(\tilde{U}) : \varphi \mapsto G_2(G_1(\varphi))$.

Then $G \in \psi DO(U,X_2X_1)$ with $U = U_2\setminus[X_2(\mathbb{R}^n\setminus U_1)]$.

The core $g_u \sim \sum K^{m}_{u}$ where

$$K^{m}_{u} = \sum_{i+j+|\alpha|=m} \frac{1}{\alpha!}(\#)^{\alpha}K^{i}_{2u}\underline{*}_{\rho}R_{\alpha}K^{j}_{1u}.$$

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