Smooth Tame Fréchet Algebras and Lie Groups of Pseudodifferential Operators

by

Kevin Ray Payne

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

August, 1989

State University of New York

at Stony Brook

The Graduate School

Kevin Ray Payne

We, the dissertation committee

for the above candidate for the doctoral degree,

hereby recommend acceptance

of this dissertation.

Michael Taylor, Professor of Mathematics, University of North Carolina
Dissertation Director

Bernard Maskit, Professor of Mathematics
Dissertation Co-Director

Anthony Phillips, Professor of Mathematics
Defense Chairman

Daryl Geller, Professor of Mathematics

Nandor Balazs, Professor of Physics
Outside Member

This dissertation is accepted by

the Graduate School

Abstract of the Dissertation

Smooth Tame Fréchet Algebras and Lie Groups of Pseudodifferential Operators

by

Kevin Ray Payne

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

1989

For certain pseudodifferential operator algebras with globally defined 0-order symbols, belonging to either the Hörmander class $S^0_{0,0}(\mathbf{R}^{2n})$ or certain prescribed subspaces, it is shown that these algebras possess important *smooth tame structures* that make them compatible with Nash-Moser implicit function theory. In particular, the algebras investigated possess a graded Fréchet topological structure and a family of smoothing operators that make them into *tame Fréchet spaces*. With respect to this structure, the product map on these algebras is shown to be smooth tame, making the algebras into *smooth tame Fréchet algebras*. These algebras are also invariant under the L^2-adjoint, which is shown to be a smooth tame map, and the algebras are closed under the inversion in their parent C*-algebra, which is the bounded linear operators on $L^2(\mathbf{R}^n)$, and thus form *smooth tame* ψ**-subalgebras* of the parent C*-algebra. Finally, the inversion map on the set of elements in the algebra nearby the identity is shown to be smooth tame, which gives rise to the

structure of a *smooth tame Lie group* to the group of invertible elements in these operator algebras. The work exploits the characterization of Cordes for such algebras as special subsets of the bounded linear operators on $L^2(\mathbf{R}^n)$, where the defining condition is a smoothness criterion involving the conjugation of an operator in the algebra by a special strongly continuous unitary representation of a finite dimensional Lie group. Such a characterization allows one to identify a pseudodifferential operator with a smooth map form a finite dimensional Lie group into a Banach space, Banach algebra, or C*-algebra. In this setting, the desired tameness estimates are then essentially the result of the Leibniz formula acting in concert with the appropriate interpolation estimates coming from the tame Fréchet space structure, which is analyzed in detail.

Table of Contents

List of Symbols

Acknowledgments

I am deeply indebted to my parents for their unwavering personal support and their constant interest in my work. Their awareness of the uncertainty of the venture encouraged the development of the work for the sake of academic integrity alone. I also wish to extend my gratitude to the Departments of Mathematics at SUNY-Stony Brook, Caltech, and UNC-Chapel Hill for their providing a network of academic support and an environment of personal concern during the development of this dissertation.

For additional technical assistance, I wish to thank the members of my dissertation committee for their critique of my work. I give hearty thanks to Professor Mark Williams at UNC for his providing a generous ear throughout the development of the results. Also, to Greg Peterson at UNC, I give my thanks for valuable discussions on the theory of Lie groups and Lie algebras.

For additional personal assistance, I am thankful for the many friends who helped along the way. In particular, I am indebted to Brian Kasper for our shared development in the early years at Stony Brook and to Corinne Silva for affection and encouragement during the trying period of pulling everything together.

Finally, I am proud to acknowledge the assistance of my dissertation director, Michael Taylor. I am grateful for both his confidence in me by encouragement to follow him and his direction of me by exposure to an open and fertile research area. Most importantly, I thank him for being a constant source of inspiration through the ferocity of his mathematical pursuits.

0. Introduction.

It is our intent in this work to begin a systematic treatment of the analysis involved to justify the use of pseudodifferential and Fourier integral operators in Nash-Moser schemes for the production of smooth solutions to nonlinear partial differential equations. In particular, for a smooth function f, if:

$$P(u) = f$$

is a nonlinear partial differential equation, and P is a smooth function of its arguments, then to conclude the that smooth solutions locally exist by applying the Nash-Moser inverse function theorem one needs, for u near to some appropriate u_0, a family of inverses E(u) to the linearized operator P'(u) that satisfy the so-called *smooth tame estimates* with respect to u. We are motivated by the recent work of Goodman and Yang [**G/Y**], in which a family of inverses is constructed for general nonlinear partial differential equation of *real principal type*. Their construction is built as a composition of various operators whose u dependence is given in terms of elliptic Fourier integral operators of order 0, their inverses and adjoints. For this reason, one would like to know whether the products, inverses and adjoints of pseudodifferential and Fourier integral operators are compatible with the Nash-Moser machinery.

For the present, we deal only with the following pseudodifferential question. Does there exist a large algebra of 0-order pseudodifferential operators $\mathcal{A}$ whose subgroup $\mathcal{A}_*$ of invertible elements forms a *smooth tame Lie group* in the sense of Hamilton **[H]** ? In order to address this question, we pose the following questions:

1) is there a large class $\mathcal{A}$ for which the composition map:

$$C : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \quad \text{defined by} \quad C(A, B) = A \circ B$$

is smooth tame with respect to a suitable topology on $\mathcal{A}$ which yields $\mathcal{A}$ as a tame Fréchet space?

2) If one restricts attention to the invertible elements $\mathcal{A}_*$, is the inversion map:

$$V : \mathcal{A}_* \to \mathcal{A}_* \text{ defined by } \quad V(A) = A^{-1}$$

smooth tame?

The answer to these questions is yes, and we will demonstrate this for $\mathcal{A} = OPS^{0,0}_{gs}$, the pseudodifferential operator algebra of Cordes **[C2]**, which consists of global 0-order operators whose symbols are smooth functions on $\mathbf{R}^{2n}$ which remain bounded after an arbitrary finite application of phase space differential operators of a prescribed type.

The key ingredient in our treatment is the exploitation of the Cordes characterization of such algebras as subsets of $\mathcal{L}(\mathcal{H})$, the bounded linear operators on $L^2(\mathbf{R}^n)$ defined by the following smoothness condition. Let U(t) be a strongly continuous unitary representation of a Lie group $G \ni t$ onto $\mathcal{L}(\mathcal{H})$. We will say that $A \in \mathcal{L}(\mathcal{H})$ *satisfies the Cordes criterion with respect to U* if the map $A(t) = U^{-1}(t)\, A\, U(t)$ is a smooth map from G into $\mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ carries the operator norm topology. If a class of pseudodifferential operators can be characterized by their satisfying the Cordes criterion with respect to some unitary representation, then there is a natural topology induced on such algebras as as a closed subspace of a certain space of smooth maps. As a result, the verification of tame estimates for products and inverses reduces to such questions on spaces of smooth maps from a Lie group into a Banach algebra, where in this setting, the estimates are essentially the result of the Leibniz formula.

Before presenting the organization of this paper, we would like to make a few remarks. There are a few advantages of using the Cordes characterization presented here for the purpose of obtaining tame estimates and generating a Lie group structure on an algebra of 0-order pseudodifferential operators. First, experience indicates that, when trying to obtain tame estimates for pseudodifferential products, natural attempts at the symbolic level using stationary phase methods fail to produce sharp enough estimates unless the operators have large negative order. In particular, while these integration by parts arguments do give continuity estimates, they result in bounds that involve too many derivatives of the symbol factors to be tame estimates. Thus, the desire to work directly with a suitable operator topology. Second, the Lie group structure of these operator algebras, which is an *infinite* dimensional notion, is being captured by a *finite* dimensional underlying group, with the aid of the *ψ*-subalgebra property* that they in turn posses. This allows one to avoid the delicacies of the infinite dimensional theory. It is then reasonable to ask for which algebras of pseudodifferential operators does one have a Cordes type characterization with a finite dimensional underlying group ? To this end, we include at the end of the first section an additional feature of the Cordes characterization in the present case of operators on the noncompact manifold $\mathbf{R}^n$.

We hope that these observations may serve as another model for where to look for such algebras. We mention the related work of R. Beals [**B2**], A. Connes [**Co**], J. Dunau [**Du**], R. Seeley [**Se**], E. Schrohe [**Sr2**], and M. Taylor [**T3**].

The paper is organized as follows. In section 1, we explore the Cordes characterization of certain algebras of global 0-order pseudodifferential operators on $\mathbf{R}^n$ and provide a self-contained proof that the algebra $\mathcal{A} = \mathrm{OPS}^{0,0}_{gs}$ can be characterized by its satisfying the Cordes criterion with respect to an explicit representation of a finite dimensional group. In so doing, the foundation of the Fréchet topological structure of $\mathcal{A}$ is displayed, and in section 2, this structure is treated carefully with an eye on the Nash-Moser categories, where we show that the natural symbol topology and operator topology supply *tamely equivalent* gradings on $\mathcal{A}$. In section 3, we provide the necessary results on the smooth tame structures of spaces of smooth maps defined on a compact neighborhood of the identity in a real Lie group taking values in a Banach space, Banach algebra, or C*-algebra. Finally, in section 4, we translate the abstract results of section 3 by way of the characterization of section 1 to demonstrate the smooth tame structures of $\mathrm{OPS}^{0,0}_{gs}$ and $[\mathrm{OPS}^{0,0}_{gs}]_*$.

1. The Cordes Characterization of Pseudodifferential Operators

In this section, we wish to review the machinery developed by Cordes in **[C1]** - **[C3]** for characterizing certain algebras of global pseudodifferential operators as special sub-algebras of $\mathcal{L}(\mathcal{H})$, the C*- algebra of bounded linear operators on the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^n)$. In particular, we will reintroduce the pseudodifferential operator algebras $OPS^{0,0}_{\mathcal{A}}$ after having described the Cordes operator algebras ΨGX which characterize them by a suitable smoothness criterion. The main goal is to present a self-contained proof of the bijective correspondence between the algebras $OPS^{0,0}_{\mathcal{g}s}$ and ΨGS which was stated in [**C2**], but whose proof has appeared only in the as yet unpublished form [**C3**]. Then, we proceed to analyzing the group structure of the underlying Lie groups in the smoothness criterion as a means of establishing a further connection in the bijective correspondence between these Lie groups and the Lie algebras which define the symbol spaces of the pseudodifferential realization. We begin by describing where the underlying Lie groups come from, in the framework of [**C2**].

Consider the "parameter space" $\{(s,\lambda)\}$, where $s = s(x)$ will be a coordinate transformation on $\mathbf{R}^n$, belonging to a subgroup of the diffeomorphism group, and $\lambda = \lambda(x)$ will be a real valued function. Then, the map:

(1.1) $\quad (s, \lambda) \to T_{s,\lambda} \qquad \text{where} \qquad T_{s,\lambda}u(x) = e^{i\lambda(x)}u(s(x))\sqrt{\mathrm{Jac}[s(x)]},$

where Jac[s(x)] is the Jacobian determinant of s(x), may be viewed as a special unitary representation of $\{(s, \lambda)\}$ onto $\mathcal{L}(\mathcal{H})$ for $\mathcal{H} = L^2(\mathbf{R}^n)$, after noting that $\{(s,\lambda)\}$ forms a Lie group with respect to the operations (cf. [**C2**]):

$$(1.2) \qquad (s,\lambda)\,\Delta\,(s',\lambda') = (s'\circ s,\ \lambda + \lambda'\circ s) \quad \text{and} \quad (s,\lambda)^{-1} = (s^{-1}, -\lambda\circ s^{-1}).$$

The image of this strongly continuous representation, that is, $GC = \{T_{s,\lambda}\}$, forms a subgroup of $\mathcal{U}(\mathcal{H})$ = the group of unitary operators on $\mathcal{H}$ and is thought to be built up from a coordinate transformation $u(x) \to u(s(x))$, and a gauge transformation, multiplication by $e^{i\lambda(x)}$, hence the label *GC*. Cordes has analyzed three special cases of finite dimensional groups, namely, those given by the following choices:

(1.3) $\{s(x) = gx + z \text{ and } \lambda(x) = \zeta\cdot x + \varphi : \ g\in GL(n,\mathbf{R}),\ z, \zeta\in\mathbf{R}^n, \text{ and } \varphi\in\mathbf{R}\}$

(1.4) $\{s(x) = \sigma o x + z \text{ and } \lambda(x) = \zeta\cdot x + \varphi:\ \sigma\in\mathbf{R}^+, o\in O(n), z,\zeta\in\mathbf{R}^n, \text{ and } \varphi\in\mathbf{R}\}$

(1.5) $\{s(x) = x + z \text{ and } \lambda(x) = \zeta\cdot x + \varphi:\ z,\zeta\in\mathbf{R}^n \text{ and } \varphi\in\mathbf{R}\}$.

One then defines the following subgroups of $\mathcal{U}(\mathcal{H})$:

(1.6) $\qquad GL = \{T_{g,z,\zeta,\varphi}:\ g\in GL(n,\mathbf{R}),\ z, \zeta\in\mathbf{R}^n,\ \varphi\in\mathbf{R}\}$

(1.7) $\qquad GS = \{T_{\sigma o,z,\zeta,\varphi} : \sigma\in\mathbf{R}^+, o\in O(n), z,\zeta\in\mathbf{R}^n, \varphi\in\mathbf{R}\}$

(1.8) $\qquad GT = \{E_{z,\zeta,\varphi}:\ z,\zeta\in\mathbf{R}^n, \varphi\in\mathbf{R}\}$,

where $E_{z,\zeta,\varphi} = T_{-z,\zeta,\varphi}$, and the notation is to suggest the Gauge-Linear, Gauge-Similarity and Gauge-Translation subgroups of $\mathcal{U}(\mathcal{H})$. The underlying Lie groups, whose algebraic structure descends from the parameter space $\{(s, \lambda)\}$ by the group laws (1.2), are denoted *gl, gs,* and *gt* respectively.

One uses the above representations to conjugate elements in $\mathcal{L}(\mathcal{H})$ by defining:

(1.9) $$A_{g,z,\zeta} = A_{g,z,\zeta,\varphi} = (T_{g,z,\zeta,\varphi})^{-1} A T_{g,z,\zeta,\varphi}$$

(1.10) $$A_{\sigma o,z,\zeta} = A_{\sigma o,z,\zeta,\varphi} = (T_{\sigma o,z,\zeta,\varphi})^{-1} A T_{\sigma o,z,\zeta,\varphi}$$

(1.11) $$A_{z,\zeta} = A_{z,\zeta,\varphi} = (E_{z,\zeta,\varphi})^{-1} A E_{z,\zeta,\varphi} ,$$

where the notation is intended to suggest the independence of the conjugation on the parameter φ , since multiplication by the scalar $e^{i\varphi}$ commutes with everything in $\mathcal{L}(\mathcal{H})$. In particular, one denotes by GL', GS', and GT' the subgroups formed by modding out by the normal subgroup $\{e^{i\varphi} : \varphi \in \mathbf{R}\}$ and by gl', gs', and gt' the corresponding "reduced" Lie groups. One is now prepared to define the operator algebras:

(1.12) $$\Psi GL \overset{d}{=} \{ A \in \mathcal{L}(\mathcal{H}) : \quad A_{g,z,\zeta} \in C^\infty(gl', \mathcal{L}(\mathcal{H}))\},$$

(1.13) $$\Psi GS \overset{d}{=} \{ A \in \mathcal{L}(\mathcal{H}) : \quad A_{\sigma o,z,\zeta} \in C^\infty(gs', \mathcal{L}(\mathcal{H}))\}, \text{ and}$$

(1.14) $$\Psi GT \overset{d}{=} \{ A \in \mathcal{L}(\mathcal{H}) : \quad A_{z,\zeta} \in C^\infty(gt', \mathcal{L}(\mathcal{H}))\},$$

where $\mathcal{L}(\mathcal{H})$ carries the operator norm topology. That is, one requires that the derivatives exist in the sense that the relevant difference quotients converge uniformly on compact subsets with respect to the L^2 -operator norm.

We pause here momentarily to recall some of the basic facts about these operator spaces. The claim that the above ΨGX form algebras results from the Leibniz formula and will be addressed carefully for the case $X = S$ when we discuss the tameness of products. In addition, these algebras are invariant under the L^2 -adjoint and may be supplied with Fréchet space topologies for which the inversion of them as $L(\mathcal{H})$ elements guarantees that the inverses remain in ΨGX; which is to say that they are *ψ*-subalgebras of* $L(\mathcal{H})$ in the sense of Gramsch [**G**], and as such, have nice perturbation properties.

Finally, and most importantly for now, these algebras ΨGX, for $X = T,S,L$, are in fact algebras of 0-order pseudodifferential operators, whose symbols belong to the classes $S^{0,0}_{gt}$, $S^{0,0}_{gs}$, and $S^{0,0}_{g\ell}$ defined as examples of the following symbol classes.

Definition 1.1. Let $\mathcal{A}$ be a finite dimensional Lie subalgebra of $X(R^{2n})$, the smooth vector fields on R^{2n}, with generators $\{ X_k : 1 \le k \le M\}$. Then, the symbol class $S^{0,0}_{\mathcal{A}}$ is defined to be:

$$\{a \in C^\infty(\mathbf{R}^{2n}) : \ \forall N = 0,1,2,\ldots, \ \left| \prod_{j=1}^{N} X_{k_j} a \right| = O(1) \text{ on } \mathbf{R}^{2n} \},$$

where by $\prod_{j=1}^{N} X_{k_j} a$ one means an arbitrary N-fold product of the generators X_k applied to a, with the conventions $\prod_{j=1}^{0} X_{k_j} a = a$ and $\prod_{j=1}^{N} X_{k_j} = X_{k_N} \circ \ldots \circ X_{k_1}$.

Then define:

(1.15) $\quad gt = \text{Real linear span of } \{\partial_{x_p}, \partial_{\xi_q} : 1 \le p,q \le n \}$,

(1.16) $\quad gs = \text{Real linear span of } \{\eta_{00}, \eta_{p0}, \eta_{0q}, \eta_{pq}\}$,

where:

$$\eta_{00} = \sum_{j=1}^{n} [\xi_j \partial_{\xi_j} - x_j \partial_{x_j}]$$

$$\eta_{p0} = \partial_{x_p},\ 1 \le p \le n; \qquad \eta_{0q} = \partial_{\xi_q},\ 1 \le q \le n;$$

$$\eta_{pq} = (\xi_p \partial_{\xi_q} - \xi_q \partial_{\xi_p}) + (x_p \partial_{x_q} - x_q \partial_{x_p}),\ 1 \le p < q \le n,$$

and:

(1.17) $\quad \mathcal{gl} = $ Real linear span of $\{\varepsilon_{p0}, \varepsilon_{0q}, \varepsilon_{pq}\}$

where:

$$\varepsilon_{p0} = \partial_{x_p},\ 1 \le p \le n\ ; \qquad \varepsilon_{0q} = \partial_{\xi_q},\ 1 \le q \le n$$

$$\varepsilon_{pq} = \xi_p\partial_{\xi_q} - x_q\partial_{x_p},\ 1 \le p,q \le n\ ,$$

We remark that these are precisely the symbol classes $\psi t_0, \psi s_0$, and ψl_0 of Cordes, where two redundant aspects of his definition have been removed. Namely, it is not necessary to explicitly assume that $S^{0,0}_{\mathcal{gs}}$ and $S^{0,0}_{\mathcal{gl}}$ are subsets of $S^{0,0}_{\mathcal{gt}}$, nor is it necessary to explicitly insist that the application of products of the operators $\eta_{p_jq_j}$ and $\varepsilon_{p_jq_j}$ to symbols a belong to $S^{0,0}_{\mathcal{gt}}$. Both conditions follow automatically because the η's and the ε's include those differentiations defining $S^{0,0}_{\mathcal{gt}}$.

We also note that the space $S^{0,0}_{\mathcal{gt}}$ is just the space $S^0_{0,0}$ of uniform Hörmander symbols of order 0 and type 0,0. It is well known that such a symbol space, while producing bounded pseudodifferential operators on $\mathbf{R}^n$, does not come equipped with the convenient symbol calculus. On the other hand, it is not difficult to show that the symbol spaces $S^{0,0}_{\mathcal{gs}}$ and $S^{0,0}_{\mathcal{gl}}$ consist of symbols which are locally of Hörmander type $S^0_{1,0}$. That is, on any relatively compact subset Ω of $\mathbf{R}^n$, the restrictions of $S^{0,0}_{\mathcal{gs}}$ and $S^{0,0}_{\mathcal{gl}}$ to $\Omega \times \mathbf{R}^n$ obey the estimates:

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1 + |\xi|)^{-|\alpha|} \quad \forall\ (x,\xi) \in \Omega \times \mathbf{R}^n.$$

The new notation is to suggest a few of their salient features.The superscript pair (0,0) is to suggest that their is a differentiation order as well as a multiplication order, both of which are zero in this case. The differentiation order is the standard notion of the order of growth at infinity in the fiber variable ξ and the multiplication order is the analogous notion of the allowable growth at infinity in the spatial variable x, where the symbol is viewed as a function on the cotangent bundle of $\mathbf{R}^n$. This multiplication order is one systematic way of imposing growth restictions at infinity in the spacial variable to allow for a global pseudodifferential theory on the noncompact manifold $\mathbf{R}^n$, and the explicit reference to the pair of indeces is to suggest the existence of a theory of general orders, which has been carried out by Cordes. The subscripts gt, gs, and $g\ell$ are related to this global problem by providing a basis for a suitable family of seminorms for their Fréchet space topology, and will be examined further, with an eye on the symbol topology as well as a means of understanding the Cordes criterion for the above operator algebras.

The corresponding spaces of pseudodifferential operators are denoted by $OPS^{0,0}_{gt}$, $OPS^{0,0}_{gs}$ and $OPS^{0,0}_{g\ell}$ respectively, where the action of a pseudodifferential operator a(x,D) with symbol $a(x,\xi)$ on a suitable function is given by its Fourier integral representation:

(1.18) $$a(x,D) = \int e^{ix\cdot\xi}\, a(x,\xi)\hat{u}(\xi)đ\xi, \qquad \text{where} \quad đ\xi = (2\pi)^{-n/2}d\xi,$$

and:

(1.19) $$\hat{u}(\xi) = \int e^{-ix\cdot\xi}\, u(x)đx\ , \qquad \text{where} \quad đx = (2\pi)^{-n/2}dx$$

and for x and ξ in $\mathbf{R}^n$: $x\cdot\xi = x_1\xi_1 + ... + x_n\xi_n$.

With this as background, we state,(cf. **[C2]**, Theorems 1.1 and 3.2) :

Theorem 1.2: For $X = T, S, L$ and $\mathcal{A} = gt, gs, g\ell$ one has the bijective correspondence

(1.20) $$\mathrm{OPS}^{0,0}_{\mathcal{A}} = \Psi GX\ .$$

Remark 1.3: (Sketch of the $X = T$ case)

One first shows that : given A = a(x,D) with $a \in S^{0,0}_{gt}$, the conjugation by GT', given in (1.11), yields $A_{z,\zeta} = a(x+z,D+\zeta)$. Then, using a fundamental solution to $(\partial_t + 1)^m$, the expression for $A_{z,\zeta}$, a trace formula, the Fourier -Parseval-Weyl formula, and facts about Hilbert-Schmidt and trace class ideals, one shows that :

1) any A = a(x,D) with $a \in S^{0,0}_{gt}$, belongs to $\mathcal{L}(\mathcal{H})$,

and

2) given any $A \in \Psi GT$, one may produce a symbol for A according to the formula :

(1.21) $$\sigma(A) = (2\pi)^n \mathrm{tr}[Q^* P(\partial_z,\partial_\zeta)\, A_{z,\zeta}]\ ,$$

where P is a differential operator built up from operators like $(\partial_t + 1)^m$ and Q is a suitable fundamental solution. This symbol can be shown to lie in $S^{0,0}_{gt}$ and the symbol map :

$$A \;\rightarrow\; a = \sigma(A)$$

can be shown to be injective, hence

$$\Psi GT \;\subseteq\; \mathrm{OPS}^{0,0}_{gt}$$

is an injection.

On the other hand, proceeding from the expression for $A_{z,\zeta}$, one shows that derivatives with respect to z and ζ exist as L^2-operator norm limits of the relevant difference quotients by exploiting a refinement of the Calderon-Vaillancourt theorem, which gives $\Psi GT \supset OPS^{0,0}_{gt}$. Finally, one argues that the operator map and the symbol map are inverses.

Remark 1.4: (Reduction of the $X = S$ and L cases)

In light of how one defines the symbol classes $S^{0,0}_{\mathcal{A}}$ and the Lie groups gx', one has the following diagram of inclusions:

$$\begin{array}{ccccc} \Psi GL & \subset & \Psi GS & \subset & \Psi GT \\ & & & & \| \\ OPS^{0,0}_{g\ell} & \subset & OPS^{0,0}_{gs} & \subset & OPS^{0,0}_{gt} . \end{array} \tag{1.22}$$

As a result, one can restrict the symbol map $\sigma\colon \Psi GT \to S^{0,0}_{gt}$ to a map defined on ΨGS or ΨGL to produce symbols which are apriori in $S^{0,0}_{gt}$. Also, since $OPS^{0,0}_{g\ell}$ and $OPS^{0,0}_{gs}$ are contained in $OPS^{0,0}_{gt}$, one knows that any such operator is bounded on $L^2(\mathbf{R}^n)$, and, in fact, belongs to ΨGT. As a result, to prove the theorem for the cases $X = S,L$, one only needs to verify that:

(1) If $A = a(x,D)$, with $a \in S^{0,0}_{\mathcal{A}}$, then the relevant derivatives of the conjugation by $T_{s,\lambda}$ exist in a suitable fashion, and

(2) If $A \in \Psi GX$ and $A = a(x,D)$ with $a \in S^{0,0}_{gt}$, then a is even better, namely, that $a \in S^{0,0}_{\mathcal{A}}$.

Showing (1) is just a computation and an application of the aforementioned version of the Calderon-Vaillancourt theorem, and will be done for $X = S$ in Lemmas 1.6 - 1.9. Showing (2) is aided by establishing some "connecting identities" which was done for $X = L$ in [C1] and will be done for $X = S$ in Proposition 1.10. We remark that Cordes has followed this reduction through for the $X = L$ case in **[C1]**, so we will concentrate on the larger class $X = S$ and imitate his $X = L$ proof, which has already been done to a large extent in **[C3]**.

Remark 1.5: (Convenient parameterization of the group gs', **[C3]**)

Recalling that $gs' = \{ (g,z,\zeta) : g = \sigma o \text{ for } \sigma \in \mathbf{R}^+, o \in O(n), \text{ and } z,\zeta \in \mathbf{R}^n \}$, one wishes to parameterize the O(n) component by skew symmetric matrices in the standard way; namely, for $o \in O(n)$, one can realize it as:

$$o = e^h \text{ for } h = (h_{ij})_{n\times n} \text{ with } h_{ij} = -h_{ji}.$$

More precisely, what one is doing is choosing a convenient coordinate system near the identity $(g,z,\zeta) = (I,0,0)$, in which a local chart :

$$Q = \{ \sigma o : 0<\sigma<\infty, o \in O(n). \text{ and } ||o - I|| < 1 \},$$

$||\cdot||$ being the matrix norm, is linked with :

$$P = \{ h \text{ skew symmetric} : ||h|| < \pi/4 \},$$

by the maps:

$$o = e^h = \sum_{k=0}^{\infty} h^k/k! \qquad \text{and} \qquad h = \log(o) = -\sum_{k=1}^{\infty} (I - o)^k/k.$$

This having been done, one denotes the conjugated operators $A_{\sigma o,z,\zeta}$ by $A_{\sigma,h,z,\zeta}$, and examines the derivatives with respect to σ,h,z, and ζ, thinking of it as a map defined on a neighborhood of $e = (1,0,0,0)$. One denotes by $\partial_{h_{km}}$ the directional derivative in the direction of γ_{pq} where γ_{pq} is the elementary skew-symmetric matrix whose entry in row μ and column ν is $(\gamma_{pq})_{\mu\nu} = (\delta_{k\mu}\delta_{p\nu} - \delta_{k\nu}\delta_{p\mu})$.

Proof of Theorem 1.2 ($X = S$ case) :

Step 1: (Existence of gs' derivatives)

For A = a(x,D) with $a \in S^{0,0}_{gs}$, one verifies (Lemma 1.6) that :

$$A_{\sigma,h,z,\zeta} = a(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}D+z). \tag{1.23}$$

Armed with this formula, one shows (Lemma 1.7) that the first derivatives ∂_σ, $\partial_{h_{pq}}$, ∂_{z_p}, and ∂_{ζ_q} of $A_{\sigma,h,z,\zeta}$ exist at $e = (1,0,0,0)$ as L^2-operator norm convergent limits of the relevant difference quotients. Furthermore, these first order derivatives are $(\eta_{pq}a)(x,D)$ for some η_{pq} ,and hence the transition formulas (Lemma 1.8) give $\partial_{\sigma,h,z,\zeta}(A_{\sigma,h,z,\zeta})$ in terms of first order derivatives at e. Hence, the first order derivatives exist everywhere on gs', being uniform L^2-operator norm convergent limits on compact subsets of gs'. Moreover, since the derivatives at the identity are equal to some $(\eta_{pq}a)(x,D)$, one invokes the obvious invariance of $S^{0,0}_{gs}$ under the operators η_{pq} (Lemma 1.9) to repeat the process with $\eta_{pq}a \in S^{0,0}_{gs}$. One concludes that the second derivatives exist on gs' and so on. Finally, since the derivatives are uniform limits of continuous maps : $gs' \rightarrow \mathcal{L}(\mathcal{H})$, one concludes that :

$$A_{\sigma,h,z,\zeta} \in C^\infty(gs', \mathcal{L}(\mathcal{H})),$$

and hence that :

$$OPS^{0,0}_{gs} \subseteq \Psi GS. \tag{1.24}$$

Step 2 : (For $A \in \Psi GS$, the symbol of A defined by (1.21) , which is apriori in $S^{0,0}_{gt}$, is in fact in $S^{0,0}_{gs}$.

Take any $A \in \Psi GS$ where $A = a(x,D)$ with $a \in S^{0,0}_{gt}$. One needs to show that arbitrary finite products of the η_{pq}'s applied to a remain bounded on $\mathbf{R}^{2n}$. Pick any η_{pq}. By the Proposition 1.10 below, one can realize $\eta_{pq}a$ by way of :

$$[(\eta_{pq}a)(x,D)]_{\sigma,h,z,\zeta} = \tilde{\eta}_{pq}(A_{\sigma,h,z,\zeta}) , \tag{1.25}$$

where $\tilde{\eta}_{pq}$ is a prescribed gs' vector field corresponding to η_{pq},and the right hand side in (1.25) is well defined since $A \in \Psi GS$. Moreover, this $\tilde{\eta}_{pq}(A_{\sigma,h,z,\zeta})$ will still be a smooth map from gs' into $\mathcal{L}(\mathcal{H})$ and is the GS' conjugation of a pseudodifferential operator with symbol $\eta_{pq}a$. Thus, $\eta_{pq}a(x,D) \in \Psi GS \subset \Psi GT$, and since $\Psi GT = OPS^{0,0}_{gt}$, one concludes that $\eta_{pq}a$ is a bounded function on $\mathbf{R}^{2n}$ for any η_{pq}. This argument can be iterated to conclude that arbitrary finite products of the η_{pq}'s remain bounded functions on $\mathbf{R}^{2n}$, and hence :

$$\Psi GS \subseteq OPS^{0,0}_{gs} . \tag{1.26}$$

This completes the proof of Theorem 1.2, once one has verified the technical claims made above, and this will be done in what follows. In particular, what Step 1 shows is that the restriction of the canonical operator map to $S^{0,0}_{gs}$, which is well defined as a map from $S^{0,0}_{gt}$ to ΨGT, actually has its image in ΨGS. Similarly, Step 2 says that the symbol map from ΨGT to $S^{0,0}_{gt}$ restricts to ΨGS, producing $S^{0,0}_{gs}$ symbols, and hence the bijective correspondence of $\Psi GT = OPS^{0,0}_{gt}$ restricts to one of $\Psi GS = OPS^{0,0}_{gs}$. Q.E.D.

Lemma 1.6 :(Conjugation formula)

Let $a(x,\xi) \in C^\infty(\mathbf{R}^{2n})$ have bounded derivatives of all orders and let its corresponding pseudodifferential operator be $A = a(x,D)$, then:

a) $A_{\sigma,h,z,\zeta} = (T_{\sigma,h,z,\zeta})^{-1} A T_{\sigma,h,z,\zeta}$ is an element of $\mathcal{L}(\mathcal{H})$,

and,

b) for σ,h,z,ζ in a neighborhood of the identity $e = (1,0,0,0)$ in gs' :

$$A_{\sigma,h,z,\zeta} = a(\sigma^{-1}e^{-h}(x-z), \sigma e^{-h}D+\zeta)\,, \tag{1.27}$$

where $T_{\sigma,h,z,\zeta}u(x) = \sigma^{1/2}e^{i\zeta\cdot x}u(\sigma e^{h}x+z)$ for $\sigma \in \mathbf{R}^+$, h a skew symmetric $n \times n$ matrix, and $z,\zeta \in \mathbf{R}^n$.

Proof: First notice that part **a)** is just an application of a version of the Calderon-Vaillancourt theorem (cf. **[C1]**, Theorem 2.1), and if one verifies (1.27), then $A_{\sigma,h,z,\zeta}$ will be a family of bounded linear operators on $L^2(\mathbf{R}^n)$, parameterized by σ,h,z,ζ. The verification of (1.27) is just an exercise in the change of variables theorem, which is aided by the density of $\mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$, where $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz space. This allows one to regard the elements $T_{\sigma,h,z,\zeta} \in GS'$ as operators on $\mathcal{S}(\mathbf{R}^n)$ and to exploit the Fourier representation of the action of pseudodifferential operators. To indicate a more general formula, we will verify (1.27) as a special case involving $A_{g,z,\zeta}$, where in this case, for $g \in GL(n,\mathbf{R})$:

$$T_{g,z,\zeta}u(x) = |\det g|^{1/2}\, e^{ix\cdot\zeta}u(gx+z) \text{ for } g \in GL(n,\mathbf{R}).$$

Notice that :

$$T_{g,z,\zeta} = e^{i\zeta\cdot M}(T_g)e^{iz\cdot D},$$

where $e^{i\zeta\cdot M}u(x) = e^{i\zeta\cdot x}u(x)$, $T_g u(x) = |\det g|^{1/2} u(gx)$, and $e^{iz\cdot D}u(x) = u(x+z)$.

Then :

$$(T_{g,z,\zeta})^{-1} = e^{-iz\cdot D}(T_{g^{-1}})e^{-i\zeta\cdot M},$$

and so, for $u \in \mathcal{S}(\mathbf{R}^n)$, one has the integral representation:

$$(1.28)\quad A_{g,z,\zeta}u(x) = e^{-iz\cdot D}(T_{g^{-1}})e^{-i\zeta\cdot M}\{\int e^{ix\cdot\xi}a(x,\xi)\,[e^{i\zeta\cdot M}T_g e^{iz\cdot D}u]^\wedge(\xi)đ\xi\},$$

where: $$[e^{i\zeta\cdot M}T_g e^{iz\cdot D}u]^\wedge(\xi) = \int e^{-iy\cdot\xi}e^{i\zeta\cdot y}u(gy+z)\,|\det g|^{1/2}\,đy,$$

and placing $y' = gy + z$, so that $đy' = |\det g|đy$ and $y = g^{-1}(y'-z)$, yields :

$$(1.29)\quad [e^{i\zeta\cdot M}T_g e^{iz\cdot D}u]^\wedge(\xi) = \int e^{-ig^{-1}(y'-z)\cdot(\xi-\zeta)}u(y')\,|\det g|^{-1/2}đy'$$

$$= \int e^{-i(y'-z)\cdot g^{-t}(\xi-\zeta)}u(y')\,|\det g|^{-1/2}đy'$$

$$= e^{iz\cdot g^{-t}(\xi-\zeta)}\int e^{-iy'\cdot g^{-t}(\xi-\zeta)}u(y')\,|\det g|^{-1/2}đy'.$$

So, substituting (1.29) into (1.28) yields :

(1.30) $A_{g,z,\zeta}u(x) =$

$$e^{-iz\cdot D}(T_{g^{-1}})e^{-i\zeta\cdot M}\int e^{ix\cdot\xi}\,e^{iz\cdot g^{-t}(\xi-\zeta)}\,a(x,\xi)\,\hat{u}(g^{-t}(\xi-\zeta))\,|\det g|^{-1/2}đ\xi$$

$$= \int e^{-i\zeta\cdot(g^{-1}(x-z))}\,e^{ig^{-1}(x-z)\cdot\xi}\,e^{iz\cdot g^{-t}(\xi-\zeta)}\,a(g^{-1}(x-z),\xi)\,\hat{u}(g^{-t}(\xi-\zeta))\,|\det g|^{-1}đ\xi$$

$$= \int e^{ig^{-1}(x-z)\cdot(\xi-\zeta)}\, e^{ig^{-1}z\cdot(\xi-\zeta)}\, a(g^{-1}(x-z),\xi)\, \hat{u}(g^{-t}(\xi-\zeta)\,|\,\det g\,|^{-1} d\!\!{}^-\xi$$

$$= \int e^{ig^{-1}x\cdot(\xi-\zeta)}\, a(g^{-1}(x-z),\xi)\, \hat{u}(g^{-t}(\xi-\zeta))\,|\,\det g\,|^{-1} d\!\!{}^-\xi\ ,$$

and placing $\xi' = g^{-t}(\xi-\zeta)$, so that $d\!\!{}^-\xi' = |\det g|^{-1} d\!\!{}^-\xi$ and $\xi = g^t\xi' + \zeta$, yields :

(1.31) $$A_{g,z,\zeta}u(x) = \int e^{ix\cdot\xi'}\, a(g^{-1}(x-z),g^t\xi'+\zeta)\, \hat{u}(\xi')d\!\!{}^-\xi' \qquad \text{for } u \in \mathcal{S}(\mathbf{R}^n),$$

and so:

(1.32) $$A_{g,z,\zeta} = a(g^{-1}(x-z),g^tD+\zeta).$$

Formula (1.32) is present in [**C3**, Proposition 1.2] and validates (1.27) after choosing $g = \sigma e^h$, where $g^{-1} = \sigma^{-1}e^{-h}$ and $g^t = \sigma e^{-h}$. Q.E.D.

Armed with this formula, one can address the existence of *gs'* derivatives of $A_{\sigma,h,z,\zeta}$ for $A = a(x,D)$ with $a \in S^{0,0}_{gs}$.

Lemma 1.7 :(Existence of derivatives at the identity)

If $a(x,\xi) \in S^{0,0}_{gs}$, then $A_{\sigma,h,z,\zeta}$ has first partial derivatives $\partial_\sigma, \partial_{z_p}, \partial_{z_q}$ for $1 \le p,q \le n$, and $\partial_{h_{pq}}$ for $1 \le p < q$ at the identity $e = (1,0,0,0)$, and moreover :

(1.33) $$B_{00} \stackrel{d}{=} \partial_\sigma(A_{\sigma,h,z,\zeta})\,|_{\,e} = (\eta_{00}a)(x,D)$$

(1.34) $$B_p \stackrel{d}{=} \partial_{z_p}(A_{\sigma,h,z,\zeta})\,|_{\,e} = -(\eta_{p0}a)(x,D)$$

(1.35) $$B^q \stackrel{d}{=} \partial_{\zeta_q}(A_{\sigma,h,z,\zeta})\,|_{\,e} = (\eta_{0q}a)(x,D)$$

(1.36) $$B_{pq} \stackrel{d}{=} \partial_{h_{pq}}(A_{\sigma,h,z,\zeta})\,|_{\,e} = (\eta_{pq}a)(x,D)\ .$$

Proof : The idea is to show that each difference quotient is a pseudodifferential operator whose symbol converges to some $\eta_{pq}a$ at the identity. One is guided by a refinement of the Calderon-Vaillancourt theorem, due to Cordes [**C4**], which says; for a 0-order pseudodifferential operator with symbol $a \in C^\infty(\mathbf{R}^{2n})$ with bounded derivatives:

$$(1.37) \qquad \| a(x,D) \|_{L^2\text{-op}} \leq C \sum \| a^{(\alpha)}_{(\beta)}(x,\xi) \|_{L^\infty(\mathbf{R}^{2n})},$$

where the sum runs over $|\, a \,|$ and $|\, b \,| \leq (n+2)/2$, and $a^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\beta_x \partial^\alpha_\xi a(x,\xi)$.

Hence, one needs to show that for the symbol of each difference quotient $(\nabla_\varepsilon a(x,\xi)/\varepsilon)$ satisfies, for every $|\,\alpha\,|, |\,\beta\,| \leq (n+2)/2$:

$$(1.38) \quad \lim_{\varepsilon\to 0} \| (\nabla_\varepsilon a(x,\xi)/\varepsilon)^{(\alpha)}_{(\beta)} - (\eta_{pq} a(x,\xi)) \|_{L^\infty(\mathbf{R}^{2n})} = 0 .$$

$\partial_{\mathbf{h}_{\mathbf{pq}}}(\mathbf{A}_{\sigma,\mathbf{h},\mathbf{z},\zeta})\,|\, e$ **:**

Let γ_{pq} be the elementary skew-symmetric $n \times n$ matrix previously defined.

Then, for $1 \leq p < q$:

$$(1.39) \qquad \partial_{h_{pq}}(A_{\sigma,h,z,\zeta})\,|\, e = \lim_{\varepsilon\to 0} (\nabla_\varepsilon A)/\varepsilon,$$

where:

$$(1.40) \qquad \nabla_\varepsilon A = A_{1,\varepsilon\gamma_{pq},0,0} - A_{1,0,0,0}.$$

By virtue of Lemma 1.6, $\nabla_\varepsilon A = a(e^{-\varepsilon\gamma_{pq}} x, e^{-\varepsilon\gamma_{pq}} D) - a(x,D)$, or:

(1.41) $$\nabla_\varepsilon A = \nabla_\varepsilon a(x,D) \quad \text{with} \quad \nabla_\varepsilon a(x,\xi) = a(e^{-\varepsilon\gamma_{pq}}x, e^{-\varepsilon\gamma_{pq}}\xi) - a(x,\xi)\,.$$

Now notice that, where in what follows $a_{|\mu} = (\partial_{x_\mu} a)(e^{-t\varepsilon\gamma_{pq}}x, e^{-t\varepsilon\gamma_{pq}}\xi)$, and $a^{|\mu}$ is the analogous expression with ∂_{ξ_μ} in place of ∂_{x_μ} :

$$\partial_t[a(e^{-t\varepsilon\gamma_{pq}}x, e^{-t\varepsilon\gamma_{pq}}\xi)] = \sum_{\mu=1}^{n} a_{|\mu}\,\partial_t(e^{-t\varepsilon\gamma_{pq}}x)_\mu + a^{|\mu}\,\partial_t(e^{-t\varepsilon\gamma_{pq}}\xi)_\mu$$

$$= \sum_{\mu=1}^{n} a_{|\mu}\,(-\varepsilon\gamma_{pq}e^{-t\varepsilon\gamma_{pq}}x)_\mu + a^{|\mu}(-\varepsilon\gamma_{pq}e^{-t\varepsilon\gamma_{pq}}\xi)_\mu$$

$$= (-\varepsilon)\sum_{\mu=1}^{n}\Big[a_{|\mu}\sum_{\nu=1}^{n}(\gamma_{pq})_{\mu\nu}(e^{-t\varepsilon\gamma_{pq}}x)_\nu + a^{|\mu}\sum_{\nu=1}^{n}(\gamma_{pq})_{\mu\nu}(e^{-t\varepsilon\gamma_{pq}}\xi)_\nu\Big],$$

where $(\gamma_{pq})_{\mu\nu} = \delta_{p\mu}\delta_{q\nu} - \delta_{p\nu}\delta_{q\nu}$, so the sums over ν and μ collapse to yield:

$$= (-\varepsilon)[\, a_{|p}(e^{-t\varepsilon\gamma_{pq}}x)_q - a_{|q}(e^{-t\varepsilon\gamma_{pq}}x)_p + a^{|p}(e^{-t\varepsilon\gamma_{pq}}\xi)_q - a^{|q}(e^{-t\varepsilon\gamma_{pq}}\xi)_p\,]$$

$$= \varepsilon(\eta_{pq}a)(e^{-t\varepsilon\gamma_{pq}}x, e^{-t\varepsilon\gamma_{pq}}\xi).$$

Hence:

(1.42) $$\partial_\tau[\, a(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi)\,] = (\varepsilon t)(\eta_{pq}a)(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi)$$

and:

(1.43) $$\partial_t\partial_\tau[\, a(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi)\,] = \varepsilon(\eta_{pq}a)(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi) +$$

$$(\varepsilon t)\partial_\tau[(\eta_{pq}a)(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi)].$$

Now, integrate (1.43), multiplied by $1/\varepsilon$, over the region $(t,\tau) \in I^2 = [0,1] \times [0,1]$ to find:

$$(1.44) \qquad 1/\varepsilon[\, a(e^{-\varepsilon\gamma_{pq}}x, e^{-\varepsilon\gamma_{pq}}\xi)\,] = \int_{I^2} [\eta_{pq} a(\bullet,\bullet) + (\varepsilon\tau t)\eta_{pq}^2 a(\bullet,\bullet)]dt d\tau,$$

where we have denoted the arguments $(e^{-t\tau\varepsilon\gamma_{pq}}x, e^{-t\tau\varepsilon\gamma_{pq}}\xi)$ by the symbol $(\bullet,\bullet)$.

This gives:

$$\nabla_\varepsilon a(x,\xi)/e - \eta_{pq} a(x,\xi) = -\eta_{pq} a(x,\xi) + \int_{I^2} [\eta_{pq} a(\bullet,\bullet) + (\varepsilon\tau t)\eta_{pq}^2 a(\bullet,\bullet)]dt d\tau\,,$$

and hence:

$$\| \nabla_\varepsilon a(x,\xi)/\varepsilon - \eta_{pq} a(x,\xi) \|_{L^\infty(R^{2n})} \le \| \int_{I^2} \eta_{pq} a(\bullet,\bullet)dt d\tau - \eta_{pq} a(x,\xi) \|_{L^\infty(\mathbf{R}^{2n})}$$

$$+ \varepsilon \| \int_{I^2} t\tau(\eta_{jl}^2 a)(\bullet,\bullet)dt d\tau \|_{L^\infty(\mathbf{R}^{2n})},$$

where the terms on the left are uniformly bounded for all ε, since $a \in S^{0,0}_{gs}$ and I^2 is compact, and tend to 0 as $\varepsilon \to 0$. Hence:

$$(1.45) \qquad \lim_{\varepsilon \to 0} \| (\nabla_\varepsilon a(x,\xi))/\varepsilon - \eta_{pq}(x,\xi) \|_{L^\infty(\mathbf{R}^{2n})}$$

as will its derivatives of order $|\, a \,|, |\, b \,| \le (n+2)/2$, and therefore:

$$(\nabla_\varepsilon a(x,D))/\varepsilon \to \eta_{pq} a(x,D) \qquad \text{as } \varepsilon \to 0.$$

$\partial_\sigma(\mathbf{A}_{\sigma,\mathbf{h},\mathbf{z},\zeta})\,|_{\,e}$:

One begins by noting that :

(1.46) $\quad \partial_\sigma(A_{\sigma,h,z,\zeta})\,|_{\,e} = \lim_{\varepsilon\to 0} (\nabla_\varepsilon A)/\varepsilon \qquad$ with $\quad \nabla_\varepsilon A = A_{1+\varepsilon,0,0,0} - A_{1,0,0,0}.$

By Lemma 1.6, $\nabla_\varepsilon A = a(\frac{x}{1+\varepsilon},(1+\varepsilon)D) - a(x,D)$, or :

(1.47) $\qquad \nabla_\varepsilon A = \nabla_\varepsilon a(x,D) \qquad$ with $\nabla_\varepsilon a(x,\xi) = a(\frac{x}{1+\varepsilon},(1+\varepsilon)\xi)$.

Proceeding as above, notice that :

$$\partial_t[\, a(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi)\,] = \sum_{\mu=1}^{n} [a_{|\mu}\, \partial_t[(\tfrac{x}{1+t\varepsilon})_\mu] + a^{|\mu}\, \partial_t[((1+t\varepsilon)\xi)_\mu]$$

$$= \sum_{\mu=1}^{n} \varepsilon\, a_{|\mu} \frac{-x_\mu}{(1+t\varepsilon)^2} + \varepsilon\, a^{|\mu}\, \xi_p$$

$$= \frac{\varepsilon}{1+t\varepsilon} \sum_{\mu=1}^{n} [(1+t\varepsilon)\xi_\mu \partial_{\xi_\mu} a(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi) - (\tfrac{x}{1+t\varepsilon})_\mu \partial_{x_\mu} a(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi)]$$

$$= \frac{\varepsilon}{1+t\varepsilon} (\eta_{00} a)(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi).$$

Then :

$$(1.48)\qquad 1/\varepsilon[\, a(\tfrac{x}{1+\varepsilon},(1+\varepsilon)\xi) - a(x,\xi)\,] = \int_0^1 \tfrac{1}{1+t\varepsilon}(\eta_{00}a)(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi)dt,$$

and so :

$$(1.49)\qquad ||\, (\nabla_\varepsilon a(x,\xi))/\varepsilon - \eta_{00}a(x,\xi)\, ||_{L^\infty(\mathbf{R}^{2n})} =$$

$$||\int_0^1 \tfrac{1}{1+t\varepsilon}(\eta_{00}a)(\tfrac{x}{1+t\varepsilon},(1+t\varepsilon)\xi)dt \;-\; \eta_{00}a(x,\xi)\,||_{L^\infty(\mathbf{R}^{2n})}.$$

From this it is clear that:

$$\lim_{\varepsilon\to 0} ||\, (\nabla_\varepsilon a(x,\xi))/\varepsilon - \eta_{00}(x,\xi)\, ||_{L^\infty(\mathbf{R}^{2n})},$$

as do the necessary derivatives, thus $(\nabla_\varepsilon a(x,D))/\varepsilon \to \eta_{00}(x,D)$ in $\mathcal{L}(\mathcal{H})$,as $\varepsilon \to 0$.

$\partial_{z_p}[A_{\sigma,h,z,\zeta}]\,|_{e}$:

One begins by noting that :

$$(1.50)\qquad \partial_{z_p}[A_{\sigma,h,z,\zeta}]|_{e} = \lim_{\varepsilon\to 0}(\nabla_\varepsilon A)/\varepsilon, \quad \text{where } \nabla_\varepsilon A = A_{1,0,\varepsilon e_p,0} - A_{1,0,0,0},$$

and so, by Lemma 1.6, $\nabla_\varepsilon A = a(x-\varepsilon e_p,D) - a(x,D)$, or :

$$(1.51)\qquad \nabla_\varepsilon A = \nabla_\varepsilon a(x,D) \quad \text{with} \quad \nabla_\varepsilon a(x,\xi) = a(x-\varepsilon e_p,\xi) - a(x,\xi).$$

Then, one finds that $\partial_t[\, a(x\text{-}t\varepsilon e_p,\xi)\,] = -\varepsilon\partial_{x_p} a(x\text{-}t\varepsilon e_p,\xi)$, and so :

(1.52) $$[\, a(x-\varepsilon e_p,\xi) - a(x,\xi)\,]/\varepsilon = -\int_0^1 \partial_{x_p} a(x-t\varepsilon e_j,\xi)dt\ ,$$

or:

(1.53) $$\|\ (\nabla_\varepsilon a(x,\xi))/\varepsilon - (-\partial_{x_p} a(x,\xi))\,\|_{L^\infty(\mathbf{R}^{2n})}$$

$$= \|\ -\int_0^1 \partial_{x_p} a(x-t\varepsilon e_p,\xi)dt + \partial_{x_p} a(x,\xi)\,\|_{L^\infty(\mathbf{R}^{2n})},$$

and so: $(\nabla_\varepsilon a(x,D)/\varepsilon) \to -\eta_{p0} a(x,D)$ in $\mathcal{L}(\mathcal{H})$, as $\varepsilon \to 0$.

$\partial_{\zeta_q}[\mathbf{A}_{\sigma,\mathbf{h},\mathbf{z},\zeta}]|_{e}$:

Showing (1.35) is just a repetition of the above argument for (1.34), where now:

$$\nabla_\varepsilon A = A_{1,0,0,\varepsilon e_q} - A_{1,0,0,0},\ \text{and}\ \ \nabla_\varepsilon A = \nabla_\varepsilon a(x,D)$$

with $a(x,\xi) = a(x,\xi+\varepsilon e_q) - a(x,\xi)$. This yields:

$$(\nabla_\varepsilon a(x,D))/\varepsilon \to \eta_{0q} a(x,D) = (\partial_{x_q} a)(x,D)\ \text{in}\ \mathcal{L}(\mathcal{H})\ \text{as}\ \varepsilon \to 0.$$ Q.E.D.

Using these expressions for the derivatives at the identity, one may produce transition formulas for the derivatives away from the identity (cf. **[C3]**, Proposition 3.6.).

Lemma 1.8: (Transition formulas for *gs′* derivatives)

For $a \in S^{0,0}_{gs}$, let B_{00}, B_p, B^q, and B_{pq} be the first derivatives at the identity of $A = a(x,D)$ as given in Lemma 1.7. Then one has the following formulas :

(1.54) $$\partial_\sigma(A_{\sigma,h,z,\zeta}) = \sigma^{-1}[\,(B_{00})_{\sigma,h,z,\zeta} - \sum_{p=1}^{n} \zeta_p (B^p)_{\sigma,h,z,\zeta}\,]$$

(1.55) $$\partial_{z_j}(A_{\sigma,h,z,\zeta}) = \sigma^{-1} \sum_{p=1}^{n} (e^{-h})_{pj} (B_p)_{\sigma,h,z,\zeta}$$

(1.56) $$\partial_{z_k}(A_{\sigma,h,z,\zeta}) = (B^k)_{\sigma,h,z,\zeta}$$

(1.57) $$\partial_{h_{jk}}(A_{\sigma,h,z,\zeta}) = \sum_{1 \le p < q} \varphi^{jk}_{pq} \{\,(B_{pq})_{\sigma,h,z,\zeta} - \zeta_p (B^q)_{\sigma,h,z,\zeta} + \zeta_q (B^p)_{\sigma,h,z,\zeta}\,\},$$

where $\varphi^{jk}_{pq} = (e^{-h}\partial_{h_{jk}} e^{h})_{pq}$.

Proof: The idea is to reduce the Lemma to computations on the symbolic level in the following way. The operator $A_{\sigma,h,z,\zeta}$ is, by Lemma 1.6, a family of pseudodifferential operators with symbol $a(\sigma^{-1}e^{-h}(x-z), \sigma e^{-h}\xi+\zeta)$, where the σ,h,z, and ζ are regarded as parameters. The symbol, being a smooth function, may be differentiated freely with respect to the group parameters via the chain rule, and one checks that this differentiated symbol agrees with the symbol of the corresponding operator on the right hand side of (1.54)-(1.57). Then, on any compact subset of *gs′*, since the functions weighting the expressions B_{jk} are smooth, the difference between the difference quotient and its conjectured limit will be uniformly bounded in L^2- operator norm, again making use of (1.37). Hence, the derivative exists in the appropriate sense and will have the form as claimed. We proceed to the symbolic calculations.

(1.54') $$\partial_\sigma[\, a(\sigma^{-1}(x-z),\sigma e^{-h}\xi+\zeta)\,] = \sum_{p=1}^{n} [-\sigma^{-2}(e^{-h}(x-z))_p\, a_{|p} + (e^{-h}\xi)_p\, a^{|p}]$$

$$= \sigma^{-1} \sum_{p=1}^{n} [\, -(\sigma^{-1}e^{-h}(x-z))_p a_{|p} + (\sigma e^{-h}\xi)_p a^{|p}\,]$$

$$= \sigma^{-1} \sum_{p=1}^{n}\Big[(\sigma e^{-h}\xi+\zeta)_p a^{|p} - (\sigma^{-1}e^{-h}(x-z))_p a_{|p}\Big] - \sigma^{-1} \sum_{p=1}^{n} \zeta_p a^{|p}$$

$$= \sigma^{-1}\Big[(\eta_{00}a)(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta) - \sum_{p=1}^{n} \zeta_p(\partial_{\xi_p} a)(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta)\Big],$$

which is nothing other than the symbol of $\sigma^{-1}[(B_{00})_{\sigma,h,z,\zeta} - \sum_{p=1}^{n} \zeta_p(B^p)_{\sigma,h,z,\zeta}]$, in light of (1.27) and (1.33). The identity (1.55) follows from :

(1.55') $$\partial_{z_j}[\, a(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta)\,] = \sum_{p=1}^{n} a_{|p}\, \partial_{z_j}(\sigma^{-1}e^{-h}(x-z))_p$$

$$= \sum_{p=1}^{n} \Big[a_{|p}\, \sigma^{-1}\partial_{z_j}\{\sum_{\nu=1}^{n}(e^{-h})_{p\nu}(x-z)_\nu\}\Big]$$

$$= \sigma^{-1} \sum_{p=1}^{n} \Big[a_{|p} \sum_{\nu=1}^{n}(e^{-h})_{p\nu}(-\delta_{j\nu})\Big] = -\sigma^{-1}\sum_{p=1}^{n} a_{|p}(e^{-h})_{pj}$$

$$= \sigma^{-1}\sum_{p=1}^{n}(e^{-h})_{pj}(-\partial_{x_j}a)(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta),$$

and this is the desired symbol in light of (1.27) and (1.34). Similarly:

(1.56') $$\partial_{\zeta_k}[\, a(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta)\,] = \sum_{p=1}^{n} a^{|p}\,\partial_{\zeta_k}(\sigma e^{-h}\xi+\zeta)_p$$

$$= \sum_{p=1}^{n} a^{|p}\,\delta_{kp} = (\partial_{\xi_k}a)(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta),$$

and this is the desired symbol in light of (1.27) and (1.35).

Justifying (1.57) on the symbolic level is a little more involved and makes use of the chain rule with $g = \sigma e^{h}$ and a derivative formula of Cordes **[C3]** for the *gl'* case. In what follows, $C_{rq} = (\varepsilon_{rq}a)(x,D)$, in analogy with $B_{rq} = (\eta_{rq}a)(x,D)$, and so $B_{rq} = C_{rq} - C_{qr}$, since $\eta_{rq} = \varepsilon_{rq} - \varepsilon_{qr}$. Also denote by b_{rq} and c_{rq} the symbols of B_{rq} and C_{rq}. Now, by making use of the conjugation formula (1.32) for $A_{g,z,\zeta}$, one can show that :

(1.58) $$\partial_{g_{pq}}[\, a(g^{-1}(x-z),g^{-t}\xi+\zeta)\,] = \sum_{r=1}^{n}\left[\bar{g}_{rp}(c_{rq})_{g,z,\zeta} - \bar{g}_{rp}\zeta_r(b^{q})_{g,z,\zeta}\right],$$

where $\bar{g}_{rp} = (g^{-1})_{rp}$, and $(b^{q})_{g,z,\zeta}$ is the symbol of $(B^{q})_{g,z,\zeta}$. Now, the chain rule with $g = \sigma e^{h}$ says that :

(1.57') $$\partial_{h_{jk}}[\, a(\sigma^{-1}e^{-h}(x-z),\sigma e^{-h}\xi+\zeta)\,]$$

$$= \sum_{p,q}\left[\partial_{g_{pq}}\{a(g^{-1}(x-z),g^{-t}\xi+\zeta)\}|_{g=\sigma e^{-h}}\right]\left[\partial_{h_{jk}}(\sigma e^{h})_{pq}\right]$$

$$= \sum_{p,q} \Big[\sum_{r} \{ g^{-}_{rp} (c_{rq})_{g,z,\zeta} - (g^{-}_{rp}\zeta_r)(b^q)_{g,z,\zeta} \}\big|_{g=\sigma e^h} \Big] \sigma(\partial_{h_{jk}} e^h)_{pq},$$

where for $g = \sigma e^h$:

$$g^{-}_{rp} = (\sigma e^h)^{-}_{rp} = (\sigma^{-1} e^{-h})_{rp} \text{ and } (b^q)_{g,z,\zeta}\big|_{g=\sigma e^h} = (b^q)_{\sigma,h,z,\zeta},$$

as both are equal to $(\partial_{\xi_q} a)_{\sigma,h,z,\zeta}$. Hence (1.57') is equal to:

$$= \sum_{p,q,r} (\sigma^{-1} e^{-h})_{rp}\, \sigma(\partial_{h_{jk}} e^h)_{pq} \left\{ (c_{rq})_{g,z,\zeta}\big|_{g=\sigma e^h} - \zeta_r (b^q)_{\sigma,h,z,\zeta} \right\}$$

$$= \sum_{q,r} (e^{-h} \partial_{h_{jk}} e^h)_{rq} \left\{ (c_{rq})_{g,z,\zeta}\big|_{g=\sigma e^h} - \zeta_r (b^q)_{\sigma,h,z,\zeta} \right\}$$

since the sum over p represents a matrix multiplication. Now, $\varphi = e^{-h} \partial_{h_{jk}} e^h$ is a skew symmetric matrix. Indeed:

(1.59) $$\varphi^t = (\partial_{h_{jk}} e^h)^t (e^{-h})^t = \partial_{h_{jk}} (e^h)^t e^{(-h)^t} = (\partial_{h_{jk}} e^{-h})(e^h)$$

but

(1.60) $$0 = \partial_{h_{jk}} (\mathrm{Id}) = \partial_{h_{jk}} (e^{-h} e^h) = e^{-h} (\partial_{h_{jk}} e^h) + (\partial_{h_{jk}} e^{-h})(e^h) = \varphi + \varphi^t.$$

Hence the terms in the sum for q = r are zero, so summing over $1 \leq r < q$ gives (1.57') as:

$$\sum_{1 \leq r < q} \varphi^{jk}_{rq} \left\{ (c_{rq} - c_{qr})_{\sigma,h,z,\zeta} - \zeta_r (b^q)_{\sigma,h,z,\zeta} + \zeta_q (b^r)_{\sigma,h,z,\zeta} \right\}$$

$$= \sum_{1 \leq r < q} \varphi^{jk}_{rq} \left\{ (b_{rq})_{\sigma,h,z,\zeta} - \zeta_r (b^q)_{\sigma,h,z,\zeta} + \zeta_q (b^r)_{\sigma,h,z,\zeta} \right\},$$

which is the desired symbol for $\partial_{h_{jk}} [A_{\sigma,h,z,\zeta}]$. Q.E.D.

Next we state the needed invariance lemma, which is the last claim in the argument of Step 1 in the proof of the theorem, (cf. Cordes [**C3**]).

Lemma 1.9: The symbol space $S^{0,0}_{gs}$ is invariant under the phase space differential differential operators η_{pq} defined in (1.16), that is, if $a \in S^{0,0}_{gs}$, then $\eta_{pq}a \in S^{0,0}_{gs}$, for any such η_{pq}.

Proof: In order for $\eta_{pq}a$ to belong to $S^{0,0}_{gs}$, one needs first that $\eta_{pq}a \in C^{\infty}(\mathbf{R}^{2n})$, but this follows from a being smooth and η_{pq} being a differential operator with smooth coefficients. Then, one needs to know that for any N = 0,1,2,...: $\prod_{j=1}^{N}\eta_{p_jq_j}(\eta_{pq}a)$ is bounded on $\mathbf{R}^{2n}$, but this is just $\prod_{j=1}^{N+1}\eta'_{p_jq_j}a$, which must be bounded on $\mathbf{R}^{2n}$ since $a \in S^{0,0}_{gs}$.

Q.E.D.

Finally, we state the key proposition in the argument of Step 2 of the proof of the theorem, where we recall for convenience that:

$$\eta_{00} = \sum_{k=1}^{n}[\xi_k\partial_{\xi_k} - x_k\partial_{x_k}],$$

$$\eta_{j0} = \partial_{x_j} \quad \text{for } 1 \leq j \leq n, \qquad \eta_{0k} = \partial_{\xi_k} \quad \text{for } 1 \leq k \leq n,$$

and:

$$\eta_{jk} = (\xi_j\partial_{\xi_k} - \xi_k\partial_{\xi_j}) + (x_j\partial_{x_k} - x_k\partial_{x_j}) \quad \text{for } 1 \leq j < k \leq n.$$

Proposition 1.10: (Connecting identities) Let:

(1.61) $$\tilde{\eta}_{00} = \sigma\partial_\sigma + \sum_{k=1}^{n} \zeta_k \partial_{\zeta_k}$$

(1.62) $$\tilde{\eta}_{j0} = -\sigma\sum_{k=1}^{n} (e^h)_{kj}\partial_{z_k} \qquad \text{for } j = 1,\dots,n$$

(1.63) $$\tilde{\eta}_{0k} = \partial_{\zeta_k} \quad \text{for } k = 1,\dots,n$$

(1.64) $$\tilde{\eta}_{jk} = \sum_{1\le\mu<\nu} \Phi^{jk}_{\mu\nu}\partial_{h_{\mu\nu}} + \zeta_j\partial_{\zeta_k} - \zeta_k\partial_{\zeta_j} \qquad \text{for } 1\le j<k,$$

where:

(1.65) $$\varphi = \begin{pmatrix} \varphi^{12}_{12} & \cdots & \varphi^{12}_{pq} & \cdots & \varphi^{12}_{n-1,n} \\ \vdots & & \vdots & & \vdots \\ \varphi^{jk}_{12} & \cdots & \varphi^{jk}_{pq} & \cdots & \varphi^{jk}_{n-1,n} \\ \vdots & & \vdots & & \vdots \\ \varphi^{n-1,n}_{12} & \cdots & \varphi^{n-1,n}_{pq} & \cdots & \varphi^{n-1,n}_{n-1,n} \end{pmatrix} \quad \text{is an } N\times N \text{ matrix,}$$

with $N = (n^2-n)/2$,

and: $\varphi^{jk}_{pq} = (e^{-h}\partial_{h_{jk}} e^h)_{pq}$; i.e., $1\le j<k$ indexes the rows of φ and $1\le p<q$ indexes the columns, and:

(1.66) $\Phi^{jk}_{\mu\nu} = (\varphi^{-1})^{jk}_{\mu\nu}$ = the entry of φ^{-1} in the row indexed by $1\le j<k$ and the column indexed by $1\le\mu<\nu$. In particular, $\Phi^{jk}_{\mu\nu}$ is well defined in a neighborhood of the identity ($h = 0$) and may be written as $(\det\varphi)^{-1}\times$ (determinant of an appropriate cofactor matrix of φ).

Then, for $a \in S^{0,0}_{gs}$ and $j=k=0$, $j=0$, $k=0$, or $1\le j<k$:

(1.67) $$\tilde{\eta}_{jk}[\, a(x,D)_{\sigma,h,z,\zeta}\,] = [\,(\eta_{jk}a)(x,D)\,]_{\sigma,h,z,\zeta}.$$

Proof: One just makes use of the formula (1.27) for $(a(x,D))_{\sigma,h,z,\zeta}$ and then applies the given $\tilde{\eta}_{jk}$ to this expression, using the transition formulas of Lemma 1.8, to verify the formulas (1.67).

For j=k=0: One has by Lemma 1.8, formula (1.54):

$$\partial_\sigma[A_{\sigma,h,z,\zeta}] = \sigma^{-1}[(B_{00})_{\sigma,h,z,\zeta} - \sum_{k=1}^{n} \zeta_k (B^k)_{\sigma,h,z,\zeta}]$$

and one wants to isolate $(B_{00})_{\sigma,h,z,\zeta} = [(\eta_{00}a)(x,D)]_{\sigma,h,z,\zeta}$ as the right hand side of (1.67). Well:

$$(B_{00})_{\sigma,h,z,\zeta} = \sigma\partial_\sigma[A_{\sigma,h,z,\zeta}] + \sum_{k=1}^{n} \zeta_k (B^k)_{\sigma,h,z,\zeta}$$

$$= \sigma\partial_\sigma[A_{\sigma,h,z,\zeta}] + \sum_{k=1}^{n} \zeta_k \partial_{\zeta_k}[A_{\sigma,h,z,\zeta}],$$

where we have applied formula (1.56) of Lemma 1.8 to the second term on the right hand side, and hence $[\eta_{00}a(x,D)]_{\sigma,h,z,\zeta} = \tilde{\eta}_{00}[A_{\sigma,h,z,\zeta}]$ as desired.

For k=0, j=1,...,n: One has by Lemma 1.8, formula (1.55):

$$\partial_{z_k}(A_{\sigma,h,z,\zeta}) = \sigma^{-1} \sum_{\mu=1}^{n} \left[(e^{-h})_{\mu k} (B_\mu)_{\sigma,h,z,\zeta}\right],$$

and one wants to isolate $-(B_j)_{\sigma,h,z,\zeta} = [(\partial_{x_j} a)(x,D)]_{\sigma,h,z,\zeta}$, which is achieved by taking the correct linear combination. One finds:

$$-\sigma \sum_{k=1}^{n} \left[(e^h)_{kj} \partial_{z_k}(A_{\sigma,h,z,\zeta})\right] = -\sigma \sum_{k=1}^{n} (e^h)_{kj} \left[\sum_{\mu=1}^{n} \sigma^{-1} (e^{-h})_{\mu k} (B_\mu)_{\sigma,h,z,\zeta}\right]$$

$$= - \sum_{\mu=1}^{n} \Big[\sum_{k=1}^{n} (e^{-h})_{\mu k} (e^{h})_{kj} \Big] (B_\mu)_{\sigma,h,z,\zeta}$$

$$= - \sum_{\mu=1}^{n} \delta_{j\mu} (B_\mu)_{\sigma,h,z,\zeta}.$$

Therefore, $\tilde{\eta}_{j0}(A_{\sigma,h,z,\zeta}) = (-B_j)_{\sigma,h,z,\zeta} = [\eta_{j0} a(x,D)]_{\sigma,h,z,\zeta}$, as desired.

For j=0, k=1,...,n: One has by Lemma 1.8, formula (1.56):

$$\partial_{\zeta_k}[A_{\sigma,h,z,\zeta}] = (B^k)_{\sigma,h,z,\zeta} = [(\partial_{\xi_k} a)(x,D)]_{\sigma,h,z,\zeta},$$

and hence: $\tilde{\eta}_{0k}(A_{\sigma,h,z,\zeta}) = [(\eta_{0k} a)(x,D)]_{\sigma,h,z,\zeta}$, as desired.

For $1 \le j < k$: The situation here is a little more delicate since the coefficients:

$\varphi^{jk}_{pq} = (e^{-h} \partial_{h_{jk}} e^{h})$ appearing in the transition formulas for $\partial_{h_{jk}}[A_{\sigma,h,z,\zeta}]$ are not easily written down explicitly. As a result, we content ourselves with showing the existence of the $\tilde{\eta}_{jk}$ in a neighborhood of the identity, and in so doing, we will verify the claim in (1.66) about their form. By Lemma 1.8, (1.57), one has the $N = (n^2-n)/2$ dimensional linear system of equations:

$$\text{(1.68)} \quad \partial_{h_{\mu\nu}}(A_{\sigma,h,z,\zeta}) = \sum_{1 \le p < q} \varphi^{\mu\nu}_{pq} \{ (B_{pq})_{\sigma,h,z,\zeta} - \zeta_p (B^q)_{\sigma,h,z,\zeta} + \zeta_q (B^p)_{\sigma,h,z,\zeta} \},$$

where one wants to isolate $(B_{jk})_{\sigma,h,z,\zeta}$ for each $1 \le j < k$. To do this, one wants to invert the matrix φ whose row indexed by $\mu\nu$ appears in (1.68) by forming the correct linear combination.

First, reindex as follows:

let $K = (\mu,\nu)$ and $J = (p,q)$, so K and J run over $1,\ldots,N = (n^2-n)/2$,

so, for example:

$K=1 \Leftrightarrow \mu=1$ and $\nu=2$ and $K=2 \Leftrightarrow \mu=1$ and $\nu=3$,

which is to say that K is a linear ordering of $1 \le \mu < \nu$, and J is the same linear ordering of $1 \le p < q$. Then place:

$$\begin{aligned} A_K &= \partial_{h_{\mu\nu}} A_{\sigma,h,z,\zeta} \quad \text{for the appropriate } K \Leftrightarrow (\mu,\nu) \\ B_J &= (B_{pq})_{\sigma,h,z,\zeta} \quad \text{for the appropriate } J \Leftrightarrow (p,q) \\ Z_J &= -\zeta_p (B^q)_{\sigma,h,z,\zeta} + \zeta_q (B^p)_{\sigma,h,z,\zeta} \\ \varphi_J^K &= \varphi_{pq}^{\mu\nu}, \end{aligned}$$

which yields the system (1.68) as:

$$A_K = \sum_{J=1}^{N} \varphi_J^K (B_J + Z_J) \qquad K, J = 1,2,\ldots,N, \tag{1.69}$$

or :

$$\begin{pmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_N^1 \\ \varphi_1^2 & \varphi_2^2 & \cdots & \varphi_N^2 \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_1^N & \varphi_2^N & \cdots & \varphi_N^N \end{pmatrix} \bullet \begin{pmatrix} B_1 + Z_1 \\ B_2 + Z_2 \\ \vdots \\ B_N + Z_N \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix},$$

and, hence $B_J = (\varphi^{-1} A)_J - Z_J$, where φ is the matrix φ_J^K above.

Claim: φ is invertible in a neighborhood of the identity $e = (1,0,0,0) = (\sigma,h,z,\zeta)$.

Indeed, notice that the general element of φ is $\varphi^{jk}_{pq} = (e^{-h}\partial_{h_{jk}}e^{h})_{pq}$, and so at the identity, since h = 0, one has by standard considerations:

(1.70) $$(e^{-h}\partial_{h_{jk}}e^{h})_{pq}|_{h=0} = \Big(\sum_{m=0}^{\infty}(-1)^m/(m+1)!\bullet(\mathrm{ad}h)^m[\gamma_{jk}]\Big)_{pq}|_{h=0} = (\gamma_{jk})_{pq},$$

where $\mathrm{ad}h[A] = hA - Ah$ and γ_{jk} is the elementary skew symmetric matrix. Therefore, (1.70) reduces, for $1 \le j < k$ and $1 \le p < q$ to being: 1 if j=p and k=q and 0 otherwise. Hence:

(1.71) $$\varphi|_{h=0} = \mathrm{Id}_{N\times N}$$

and, so φ is invertible at the identity. Furthermore, by the continuity of the determinant map, $\det\varphi \neq 0$ in a neighborhood of the identity, and hence the claim.

Consequently, if Φ^J_M represents the entry in the Jth row and Mth column of φ^{-1}, one has:

(1.72) $$B_J = \sum_{M=1}^{N}\Phi^J_M A_M - Z_J,$$

which, for J = (j,k) and M = (μ,ν), yields:

(1.73) $$(B_{jk})_{\sigma,h,z,\zeta} = \sum_{1\le\mu<\nu}\Phi^{jk}_{\mu\nu}\partial_{h_{\mu\nu}}A_{\sigma,h,z,\zeta} - [-\zeta_j(B^k)_{\sigma,h,z,\zeta} + \zeta_k(B^j)_{\sigma,h,z,\zeta}],$$

where the right hand side of (1.73) is $\tilde{\eta}_{jk}[A_{\sigma,h,z,\zeta}]$, since $(B^p)_{\sigma,h,z,\zeta}$ is equal to $\partial_{\zeta_p}[A_{\sigma,h,z,\zeta}]$. This completes the proof of Proposition 1.10, and hence the Theorem 1.2 as well. Q.E.D.

We end this section by exploring an additional feature of the Cordes criterion for the operator algebras considered above. In particular, the bijective correspondence:

$$\Psi GX \;=\; \mathrm{OPS}^{0,0}_{gx} \tag{1.74}$$

can be interpreted as equivalent realizations of a family of subalgebras of $\mathcal{L}(\mathcal{H})$. On one side, membership is determined by a Lie group gx' which appears in the Cordes criterion, and on the other side, membership is determined by a Lie algebra gx which characterizes the symbol class of the resulting pseudodifferential operator realization. A natural question then arises. Is the Lie algebra gx isomorphic to the Lie algebra of the Lie group gx'? The answer is yes, and we begin by noting that the group gt' is isomorphic to $\mathbf{R}^{2n}$ and the groups gs' and gl' have semidirect product structures.

We recall that the groups gs' and gl' described set theoretically by:

$$\mathbf{R}^{+} \times \mathrm{O}(n) \times \mathbf{R}^{2n} \qquad \text{and} \qquad \mathrm{GL}(n,\mathbf{R}) \times \mathbf{R}^{2n},$$

where their group laws were induced by (1.2). Explicitly these laws are:

$$(\sigma,o,z,\zeta)\ \Delta\ (\sigma,o',z',\zeta') \;=\; (\sigma'\sigma,\ o'o,\ \sigma'o'z+z',\ \zeta+o^t\zeta') \tag{1.75}$$

and:

$$(g,z,\zeta)\ \Delta\ (g',z',\zeta') \;=\; (g'g,\ g'z+z',\ \zeta+g^t\zeta'). \tag{1.76}$$

Next we recall the notion of the semidirect product of Lie groups as given in Varadarajan [**V**]. See also Taylor [**T2**]. Let G and H be Lie groups where G acts on H by automorphisms; i.e.,there exists a group homomorphism:

$$\alpha\colon G \to \mathrm{Aut}(H) = \text{automorphism group of H}. \tag{1.77}$$

Then, the *semidirect product* $H \times_\alpha G$ is the set $H \times G$ with the group law:

$$(h,\ g) \times_\alpha (h',\ g') = (h\alpha(g)h',\ gg'). \tag{1.78}$$

Now, $\mathbf{R}^+ \times O(n)$ and $GL(n,\mathbf{R})$ can be made to act by automorphisms on $\mathbf{R}^{2n}$ by defining:

(1.79) $$\alpha : \mathbf{R}^+ \times O(n) \to \mathbf{R}^{2n} \qquad \text{with} \quad \alpha(\sigma,o)(\zeta,z) = (\tfrac{1}{\sigma} o\zeta,\ \sigma o z)$$

and:

(1.80) $$\alpha : GL(n,\mathbf{R}) \to \mathbf{R}^{2n} \qquad \text{with} \quad \alpha(g)(\zeta,z) = (g^{-t}\zeta,\ gz).$$

These choices of α give the group law on $\mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$ as:

(1.81) $$(\zeta,z,\sigma,o) \cdot (\zeta',z',\sigma',o') = (\zeta + \tfrac{1}{\sigma} o\zeta',\ z + \sigma o z',\ \sigma\sigma',\ oo'),$$

and on $\mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R})$:

(1.82) $$(\zeta,z,g) \cdot (\zeta',z',g') = (\zeta + g^{-t}\zeta',\ z + gz',\ gg').$$

Now define the maps:

(1.83) $$\varphi : gs' \to \mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$$
$$(\sigma,o,z,\zeta) \to (\zeta,\ \tfrac{1}{\sigma} o^{-1} z,\ \tfrac{1}{\sigma},\ o^{-1})$$

and:

(1.84) $$\varphi : gl' \to \mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R})$$
$$(g,z,\zeta) \to (\zeta,\ g^{-1}z,\ g^{-1}),$$

which are clearly well defined and bijective. Moreover:

Proposition 1.11: The maps φ defined by (1.83) and (1.84) are group homomorphisms, and hence one has the group isomorphisms:

(1.85) $$gs' \cong \mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$$

and:

(1.86) $$gl' \cong \mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R}).$$

Proof: One merely computes:

$$\varphi[\,(\sigma,o,z,\zeta)\;\Delta\;(\sigma,o',z',\zeta')\,] = \varphi[(\sigma'\sigma,\; o'o,\; \sigma'o'z+z',\; \zeta+o^t\zeta')] \tag{1.87}$$

$$= (\zeta+\sigma o^t\zeta', \frac{1}{\sigma\sigma'}(o'o)^{-1}(\sigma'o'z+z'),\; \frac{1}{\sigma\sigma'}, (o'o)^{-1}$$

$$= (\zeta+\sigma o^t\zeta', \frac{1}{\sigma}o^{-1}z + \frac{1}{\sigma\sigma'}o^{-1}(o')^{-1}z',\; \frac{1}{\sigma\sigma'}, o^{-1}(o')^{-1}),$$

but:

$$\varphi(\sigma,o,z,\zeta) \bullet \varphi(\zeta',z',\sigma',o') \;=\; (\zeta, \frac{1}{\sigma}o^{-1}z, \frac{1}{\sigma}o^{-1}) \bullet (\zeta', \frac{1}{\sigma'}(o')^{-1}z', \frac{1}{\sigma'}, (o')^{-1}) \tag{1.88}$$

$$= (\zeta+\sigma o^{-1}\zeta', \frac{1}{\sigma}o^{-1}z + \frac{1}{\sigma}o^{-1}\frac{1}{\sigma'}(o')^{-1}z',\; \frac{1}{\sigma\sigma'}, o^{-1}(o')^{-1}),$$

and, hence φ defined by (1.83) is a homomorphism, where in the ζ component one notes that o being orthogonal means that $o^t = o^{-1}$. Similarly:

$$\varphi[(g,z,\zeta)\;\Delta\;(g',z',\zeta')] \;=\; \varphi[(g'g,\; g'z+z',\; \zeta+g^t\zeta')] \tag{1.89}$$

$$= (\zeta+g^t\zeta',\; (g'g)^{-1}(g'z+z'),\; (g'g)^{-1})$$

$$= (\zeta+g^t\zeta',\; g^{-1}z+(g)^{-1}(g')^{-1}z',\; (g)^{-1}(g')^{-1}),$$

but:

$$\varphi(\zeta,z,g) \bullet \varphi(\zeta',z',g') \;=\; (\zeta,\; g^{-1}z,\; g^{-1}) \bullet (\zeta',\; (g')^{-1}z',\; (g')^{-1}) \tag{1.90}$$

$$= (\zeta+g^t\zeta',\; g^{-1}z+(g)^{-1}(g')^{-1}z',\; (g)^{-1}(g')^{-1}),$$

and so φ defined by (1.84) is also a homomorphism. Q.E.D.

Remark 1.12: In light of the isomorphisms in the above proposition, one could have formulated the Cordes criterion for the operator algebras ΨGS and ΨGL in terms of these semidirect structures on the groups *gs'* and *gl'*. More precisely, one might choose the following unitary subgroups of $\mathcal{L}(\mathcal{H})$, which have the aforementioned semidirect product group laws:

$$\left\{ U(\zeta,z,\sigma,o) = T_{(g)^{-1}}\, e^{i\zeta\cdot M}\, e^{iz\cdot D} \right\} \tag{1.91}$$

and:

$$\left\{ U(\zeta,z,g) = T_{(\sigma o)^{-1}}\, e^{i\zeta\cdot M}\, e^{iz\cdot D} \right\}. \tag{1.92}$$

These subgroups are, therefore, isomorphic to the subgroups *GS'* and *GT'* of $\mathcal{L}(\mathcal{H})$ in the previous formulation. Consequently, an operator $A \in \mathcal{L}(\mathcal{H})$ will be in ΨGS (respectively ΨGL) if and only if conjugation by the unitary representation (1.91) (respectively (1.92)) yields a smooth map from $\mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$ (respectively $\mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R})$) into $\mathcal{L}(\mathcal{H})$. This equivalent formulation has the advantage of decoupling some of the transition formulas and connecting identities of Lemma 1.8 and Proposition 1.10.

Finally, we would like to relate the collections of phase space differential operators defining the symbol classes $S^{0,0}_{gx}$ to the Lie groups *gx'*.

Proposition 1.13: The vector subspaces of the Lie algebra $X(\mathbf{R}^{2n})$ of smooth vector fields on $\mathbf{R}^{2n}$ defined by:

(1.93) $\mathfrak{gt}$ = Real linear span of $\{\partial_{x_p}, \partial_{\xi_q} : 1 \leq p,q \leq n\}$,

(1.94) $\mathfrak{gs}$ = Real linear span of $\{\eta_{00}, \eta_{p0}, \eta_{0q}, \eta_{pq}\}$,

where:

$$\eta_{00} = \sum_{j=1}^{n} [\xi_j \partial_{\xi_j} - x_j \partial_{x_j}]$$

$$\eta_{p0} = \partial_{x_p},\ 1 \leq p \leq n; \qquad \eta_{0q} = \partial_{\xi_q},\ 1 \leq q \leq n;$$

$$\eta_{pq} = (\xi_p \partial_{\xi_q} - \xi_q \partial_{\xi_p}) + (x_p \partial_{x_q} - x_q \partial_{x_p}),\ 1 \leq p < q \leq n,$$

and:

(1.95) $\mathfrak{gl}$ = Real linear span of $\{\varepsilon_{p0}, \varepsilon_{0q}, \varepsilon_{pq}\}$

where:

$$\varepsilon_{p0} = \partial_{x_p},\ 1 \leq p \leq n; \qquad \varepsilon_{0q} = \partial_{\xi_q},\ 1 \leq q \leq n$$

$$\varepsilon_{pq} = \xi_p \partial_{\xi_q} - x_q \partial_{x_p},\ 1 \leq p,q \leq n,$$

a) form finite dimensional Lie subalgebras of $X(\mathbf{R}^{2n})$ with respect to the standard commutator bracket, and

b) are isomorphic to the Lie algebras of the Lie groups:

(1.93') $\mathbf{R}^{2n}$

(1.94') $\mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$

(1.95') $\mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R}))$.

Proof: $\mathfrak{gt}$ is clearly a 2n-dimensional abelian Lie algebra, and hence isomorphic to the Lie algebra of $\mathbf{R}^{2n}$. The vector subspaces $\mathfrak{gs}$ and $\mathfrak{gl}$ will be Lie subalgebras of $X(\mathbf{R}^{2n})$ if they are closed under the commutator bracket, and so, it suffices to derive the *structure constants with respect to the bases* $\{\eta_{pq}\}$ *and* $\{\varepsilon_{pq}\}$. For $\mathfrak{gs}$, one finds that the only nontrivial commutators of the basis elements are:

(1.96) $$[\eta_{j0}, \eta_{00}] = -\eta_{j0} \qquad [\eta_{0k}, \eta_{00}] = \eta_{0k}$$

$$[\eta_{j0}, \eta_{pq}] = \delta_{jp}\eta_{q0} - \delta_{jq}\eta_{p0} \qquad [\eta_{0k}, \eta_{pq}] = \delta_{kp}\eta_{0q} - \delta_{kq}\eta_{0p}$$

and: $$[\eta_{jk}, \eta_{pq}] = \delta_{jp}\eta_{qk} + \delta_{jq}\eta_{kp} + \delta_{kp}\eta_{jq} + \delta_{kq}\eta_{pj}.$$

While, for $\mathfrak{gl}$ one has the nontrivial commutators:

(1.97) $$[\varepsilon_{j0}, \varepsilon_{pq}] = -\delta_{jq}\varepsilon_{p0} \qquad [\varepsilon_{0k}, \varepsilon_{pq}] = \delta_{kp}\varepsilon_{0q}$$

and: $$[\varepsilon_{jk}, \varepsilon_{pq}] = \delta_{kp}\varepsilon_{jq} - \delta_{jq}\varepsilon_{pk}.$$

Since the commutators of the basis elements are linear combinations of the basis elements, the bracket is closed on these subspaces, which completes part **a**. We remark that the commutator identities for $[\eta_{jk}, \eta_{pq}]$ and $[\varepsilon_{jk}, \varepsilon_{pq}]$ are precisely those of skew-symmetric matrices and general matrices respectively, with respect to the standard bases γ_{pq} and e_{pq} for those matrix Lie algebras, where e_{pq} is the $\mathfrak{gl}(n,\mathbf{R})$ matrix with a one in its pth row and qth column, and γ_{pq} is the skew symmetric matrix equal to $e_{pq} - e_{qp}$.

Next, we want to derive the structure constants for the Lie algebras of the semidirect product groups $\mathbf{R}^{2n} \times_\alpha (\mathbf{R}^+ \times O(n))$ and $\mathbf{R}^{2n} \times_\alpha GL(n,\mathbf{R}))$ with respect to the bases inherited from the standard bases of the Lie algebras of the factors.

We recall (cf. [V]) that if $K = H \times_\alpha G$ is the semidirect product of H and G relative to α, and if $\mathfrak{h}$ and $\mathfrak{g}$ are the Lie algebras of H and G, then the Lie algebra $\mathfrak{k}$ of K is isomorphic to *the semidirect product of the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ with respect to β*, where β is derived from α in a prescribed way.

The standard construction is as follows. From the given homomorphism:

(1.98) $$\alpha : G \to \mathrm{Aut}(H),$$

one defines the automorphism $\alpha_g \in \mathrm{Aut}(H)$ as the image of g under α:

(1.99) $$\alpha_g : H \to H,$$

whose differential as an automorphism on $\mathfrak{h}$ is denoted:

(1.100) $$d\alpha_g : \mathfrak{h} \to \mathfrak{h},$$

which in turn defines the homomorphism:

(1.101) $$\tau : G \to \mathrm{Aut}(\mathfrak{h}) \qquad \text{with } \tau(g) = d\alpha_g.$$

The differential of τ yields the desired β as:

(1.102) $$\beta : \mathfrak{g} \to \mathrm{End}(\mathfrak{h}) = \text{Lie algebra of endomorphisms of } \mathfrak{h}.$$

In fact, because τ is a homomorphism, the image of β actually lies in $\mathrm{Der}(\mathfrak{h})$ which is the Lie algebra of derivations on $\mathfrak{h}$. Given such a map β from one Lie algebra into the derivations of another, one defines the *semidirect product of $\mathfrak{h}$ and $\mathfrak{g}$ relative to β*, which is denoted by $\mathfrak{h} \times_\beta \mathfrak{g}$, to be the set $\mathfrak{h} \times \mathfrak{g}$ with the Lie algebra bracket:

(1.103) $$[(X,Y), (X',Y')] = ([X,X'] + \beta(Y)X' - \beta(Y')X, [Y,Y']),$$

for all $X, X' \in \mathfrak{h}$ and $Y, Y' \in \mathfrak{g}$, where the brackets on the right are the brackets in $\mathfrak{h}$ and $\mathfrak{g}$ respectively. Finally, the Lie algebra isomorphism between $\mathfrak{h} \times_\beta \mathfrak{g}$ and $\mathfrak{k}$ is provided by the map:

(1.104) $$\varphi : \mathfrak{h} \times_\beta \mathfrak{g} \to \mathfrak{k},$$

where: $$\varphi(X, Y) = X + Y \quad \text{for all } X \in \mathfrak{h}, Y \in \mathfrak{g}.$$

Now, let:

(1.105) $\{ X_{j0}, X_{k0} : 1 \leq j,k \leq n \}$ be the standard basis of the Lie algebra $\mathfrak{h}$ of left invariant vector fields on $H = \mathbf{R}^{2n}$; where, by choosing coordinates (ζ,z) on $\mathbf{R}^{2n}$, the canonical isomorphisms $\mathfrak{h} \cong T_{e_H}(\mathbf{R}^{2n}) \cong \mathbf{R}^{2n}$, allow us to identify:

$$X_{0k} \leftrightarrow \partial_{\zeta_k}|_{e_H} \leftrightarrow e_k = (k)^{\text{th}} \text{ basis vector in } \mathbf{R}^{2n}$$

$$X_{j0} \leftrightarrow \partial_{z_j}|_{e_H} \leftrightarrow e_{n+j} = (n+j)^{\text{th}} \text{ basis vector in } \mathbf{R}^{2n},$$

(1.106) $\{ Y_{pq} : 1 \leq p , q \leq n \}$ be the standard basis for the Lie algebra $\mathfrak{g}$ of $G = GL(n,\mathbf{R})$); where, by thinking of G as an open subset of $\mathbf{R}^{n^2}$ with coordinates g_{jk} near the identity I, the canonical isomorphisms $\mathfrak{g} \cong T_I G \cong \mathfrak{gl}(n,\mathbf{R})$, allow us to identify:

$$Y_{pq} \leftrightarrow \partial_{g_{pq}}|_I \leftrightarrow e_{pq} = pq^{\text{th}} \text{ standard basis vector in } \mathfrak{gl}(n,\mathbf{R}),$$

and:

(1.107) $\{ Z_{00}; Z_{pq} : 1 \leq p < q \leq n \}$ be the standard basis for the Lie algebra $\mathfrak{g}'$ of $G' = \mathbf{R}^+ \times O(n)$; where, by choosing coordinates (σ,h) near the identity $e = (\sigma,h) = (1,0)$, the canonical isomorphisms $\mathfrak{g} \cong T_e G' \cong \mathbf{R} \times \mathfrak{o}(n)$ allow us to identify:

$$Z_{00} \leftrightarrow \partial_\sigma|_e \leftrightarrow (1,0) \qquad Z_{pq} \leftrightarrow \partial_{h_{pq}}|_e \leftrightarrow (0,\gamma_{pq}),$$

with γ_{pq} the standard basis of $\mathfrak{o}(n)$ = skew-symmetric matrices.

By computing the structure constants of the semidirect product Lie algebras with respect to these bases, we find that part **b** of the proposition follows from the following lemma.

Lemma 1.14: Relative to the bases of $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{g}'$ given by (1.105) - (1.107), the map β defined by (1.102) satisfies:

(1.108) $\beta(Y_{pq})X_{0k} = -\delta_{kp}X_{0q}$ for $1 \le p,q \le n$ and $1 \le k \le n$,

(1.109) $\beta(Y_{pq})X_{j0} = \delta_{jq}X_{p0}$ for $1 \le p,q \le n$ and $1 \le j \le n$,

when G = GL(n,**R**), and when G' = $\mathbf{R}^+ \times O(n)$:

(1.110) $\beta(Z_{00})X_{0k} = -X_{0k}$ for $1 \le k \le n$,

(1.111) $\beta(Z_{00})X_{j0} = X_{j0}$ for $1 \le j \le n$,

(1.112) $\beta(Z_{pq})X_{0k} = \delta_{kq}X_{0p} - \delta_{kp}X_{0q}$ for $1 \le p,q \le n$, $1 \le k \le n$,

(1.113) $\beta(Z_{pq})X_{j0} = \delta_{jq}X_{p0} - \delta_{jp}X_{q0}$ for $1 \le p,q \le n$, $1 \le j \le n$.

The proposition follows from the lemma by defining the maps:

(1.114) $\Phi : \mathfrak{gl} \rightarrow \mathfrak{k} \cong \mathfrak{h} \times_\beta \mathfrak{g}$

(1.115) $\Phi : \mathfrak{gs} \rightarrow \mathfrak{k}' \cong \mathfrak{h} \times_\beta \mathfrak{g}'$

by linearly extending the identifications of the bases:

(1.116) $\Phi(\varepsilon_{j0}) = X_{j0}$, $\Phi(\varepsilon_{0k}) = X_{0k}$, and $\Phi(\varepsilon_{pq}) = Y_{pq}$

(1.117) $\Phi(\eta_{j0}) = X_{j0}$, $\Phi(\eta_{0k}) = X_{0k}$, $\Phi(\eta_{00}) = Z_{00}$, and $\Phi(\eta_{pq}) = Z_{pq}$,

where these vector space isomorphisms are also bracket homomorphisms, and hence Lie algebra isomorphisms. Indeed, the structure constants of $\mathfrak{k}$ and $\mathfrak{k}'$ are easily computed in terms of β.

G = GL(n,R) case: Choosing bases for $\mathfrak{h}$ and $\mathfrak{g}$ as in (1.105) and (1.106) also provides the choice of basis for $\mathfrak{k} \cong \mathfrak{h} \times_\beta \mathfrak{g}$ in light of the canonical isomorphism φ of (1.104). Now, since $\mathfrak{h} \cong \mathbf{R}^{2n}$, this factor is abelian and yields the trivial commutators:

(1.118) $$[X,X'] = \varphi[\ (X,0),\ (X',0)\ ] \ = \ \varphi(\ [X,X'] + \beta(0)0 - \beta(0)X,\ [0,0]\ ) = \varphi(0,0) = 0,$$

which correspond to the abelian subalgebra, span$\{\varepsilon_{j0}, \varepsilon_{0k}\}$, of $\mathfrak{gl}$. The nontrivial commutators are:

(1.119) $$\begin{aligned}[X_{j0}, Y_{pq}] &= \varphi[\ (X_{j0},0),\ (0,Y_{pq})\ ] = \\ &= \varphi(\ [X_{j0},0] + \beta(0)0 - \beta(Y_{pq})X_{j0},\ [0,Y_{pq}]\ ) \\ &= \varphi(-\beta(Y_{pq})X_{j0},\ 0) \\ &= -\beta(Y_{pq})X_{j0},\end{aligned}$$

(1.120) $$\begin{aligned}[X_{0k}, Y_{pq}] &= \varphi[\ (X_{0k},0),\ (0,Y_{pq})\ ] = \\ &= -\beta(Y_{pq})X_{0k},\end{aligned}$$

and:

(1.121) $$\begin{aligned}[Y_{jk}, Y_{pq}] &= \varphi([\ (0,Y_{jk}),\ (0,Y_{pq})\ ]) = \varphi(\ 0,\ [Y_{jk}, Y_{pq}]\ ) \\ &= [Y_{jk}, Y_{pq}]\ ,\end{aligned}$$

where (1.119) - (1.121) yield the same structure constants as those in the nontrivial commutators of $\mathfrak{gl}$ given in (1.97); the first two depend on the lemma, and the last being already noted in the remark following (1.97).

G' = R$^+$ × O(n) case: One again has trivial commutators within the $\mathfrak{h}$ factor, corresponding to the abelian subalgebra, span$\{\eta_{j0}, \eta_{0k}\}$, of $\mathfrak{gs}$. In addition, there are the trivial commutators:

(1.122) $$[Z_{00}, Z_{00}] = [Z_{00}, Z_{pq}] = 0,$$

coming from the one dimensional subalgebra, span$\{Z_{00}\}$, as well as the presence of the abelian multiplicative subgroup $\mathbf{R}^+$ within G'. The nontrivial commutators then correspond exactly to those in $\mathfrak{gs}$, provided that β acts as claimed in the lemma. Q.E.D.

Proof of Lemma 1.14:

G = GL(n,R) case: The map α_g of (1.99) is given in this case by:

(1.123) $$\alpha_g(\zeta,z) = (g^{-t}\zeta, gz),$$

and so the map:

(1.124) $$\tau : G \rightarrow \mathrm{Aut}(\mathfrak{h})$$

may be identified with:

(1.125) $$g \rightarrow \begin{pmatrix} g^{-t} & 0 \\ 0 & g \end{pmatrix} \in \mathrm{Aut}(\mathfrak{h}),$$

where this matrix represents the automorphism on $\mathfrak{h}$ with respect to the basis $\{X_{0k}, X_{j0}\}$ previously defined for $\mathfrak{h}$. To compute $\beta(Y_{pq})X_{0k}$ and $\beta(Y_{pq})X_{j0}$, we need to compute:

(1.126) $$\beta = d\tau : \mathfrak{g} \rightarrow \mathrm{Der}(\mathfrak{h}) \subset \mathrm{End}(\mathfrak{h}),$$

where we will identify $\mathfrak{g}$ with the tangent space to the identity I in GL(n,**R**) as noted in (1.106). We will also regard the group $\Gamma = \mathrm{Aut}(\mathfrak{h})$ as an open subset of the vector space $V = \mathrm{End}(\mathfrak{h})$, where we choose the basis of V:

$$\{ V^{\alpha\beta}_{\mu\nu} : 1 \le \alpha,\beta \le n,\ 1 \le \mu,\nu \le n \},$$

relative to the basis on h, i.e.:

(1.127) $\quad V^{\alpha\beta}_{\mu\nu}(X_{0k}) = \delta_{\beta k}X_{0\alpha} + \delta_{\nu k}X_{\mu 0}$ and $V^{\alpha\beta}_{\mu\nu}(X_{j0}) = \delta_{\beta j}X_{0\alpha} + \delta_{\nu j}X_{\mu 0}$.

This basis for V gives:

(1.128) $$\tau(g) = \sum_{1\le\alpha,\beta\le n} (g^{-t})_{\alpha\beta} V^{\alpha\beta}_{00} + \sum_{1\le\mu,\nu\le n} (g)_{\mu\nu} V^{00}_{\mu\nu},$$

where $(g^{-t})_{\alpha\beta} = g^{-t}_{\alpha\beta}$ is the entry in row α and column β of g^{-t}.

Next, the tangent space to Γ at the identity automorphism in Γ is identified with the tangent space to the vector space V at the identity endomorphism, which is, in turn, identified with the vector space V itself. With these identifications, one finds that:

(1.129) $$\begin{aligned} d\tau(Y_{pq}) &= d\tau_I(\partial_{g_{pq}}|_I) \\ &= \sum_{1\le\alpha,\beta\le n} [\partial_{g_{pq}}|_I (g^{-t}_{\alpha\beta})] V^{\alpha\beta}_{00} + \sum_{1\le\mu,\nu\le n} [\partial_{g_{pq}}|_I (g_{\mu\nu})] V^{00}_{\mu\nu} \\ &= \sum_{1\le\alpha,\beta\le n} [-\delta_{\alpha q}\delta_{\beta p}] V^{\alpha\beta}_{00} + \sum_{1\le\mu,\nu\le n} [\delta_{p\mu}\delta_{q\nu}] V^{00}_{\mu\nu} \\ &= -V^{qp}_{00} + V^{00}_{pq}, \end{aligned}$$

and, thus:

$$\beta(Y_{pq})(X_{0k}) = -V^{qp}_{00}(X_{0k}) + V^{00}_{pq}(X_{0k}) = -\delta_{kp}X_{0q}$$

and:

$$\beta(Y_{pq})(X_{j0}) = -V^{qp}_{00}(X_{j0}) + V^{00}_{pq}(X_{j0}) = \delta_{jq}X_{p0}\text{, as claimed.}$$

$G' = R^+ \times O(n)$ case:

Here the map τ, whose differential we need to compute, is given by:

$$(1.130) \qquad \tau(\sigma,h) = \begin{pmatrix} \frac{1}{\sigma}(e^h) & 0 \\ 0 & \sigma e^h \end{pmatrix} \in \mathrm{Aut}(\mathfrak{h}),$$

where we have chosen coordinates (σ,h) near the identity $e = (\sigma,o) = (1,0)$, with h a skew-symmetric matrix. By the same reasoning as above, one has:

$$(1.131) \qquad \tau(\sigma,h) = \sum_{1\le\alpha,\beta\le n} (\tfrac{1}{\sigma} e^h)_{\alpha\beta} V_{00}^{\alpha\beta} + \sum_{1\le\mu,\nu\le n} (\sigma e^h)_{\mu\nu} V_{\mu\nu}^{00} ,$$

and hence:

$$(1.132) \qquad d\tau(Z_{00}) = d\tau_e(\partial_{\sigma}|_e)$$

$$= \sum_{1\le\alpha,\beta\le n} [\partial_{\sigma}|_e (\tfrac{1}{\sigma} e^h)_{\alpha\beta}] V_{00}^{\alpha\beta} + \sum_{1\le\mu,\nu\le n} [\partial_{\sigma}|_e (\sigma e^h)_{\mu\nu}] V_{\mu\nu}^{00}$$

$$= \sum_{1\le\alpha,\beta\le n} [-\delta_{\alpha\beta}] V_{00}^{\alpha\beta} + \sum_{1\le\mu,\nu\le n} [\delta_{\mu\nu}] V_{\mu\nu}^{00} ,$$

and:

$$(1.133) \qquad d\tau(Z_{pq}) = d\tau_e(\partial_{h_{pq}}|_e)$$

$$= \sum_{1\le\alpha,\beta\le n} [\partial_{h_{pq}}|_e (\tfrac{1}{\sigma} e^h)_{\alpha\beta}] V_{00}^{\alpha\beta} + \sum_{1\le\mu,\nu\le n} [\partial_{h_{pq}}|_e (\sigma e^h)_{\mu\nu}] V_{\mu\nu}^{00}$$

$$= \sum_{1\le\alpha,\beta\le n} [\delta_{p\alpha}\delta_{q\beta} - \delta_{p\beta}\delta_{q\alpha}] V_{00}^{\alpha\beta} + \sum_{1\le\mu,\nu\le n} [\delta_{p\mu}\delta_{q\nu} - \delta_{p\nu}\delta_{q\mu}] V_{\mu\nu}^{00}$$

$$= (V_{00}^{pq} - V_{00}^{qp}) + (V_{pq}^{00} - V_{qp}^{00}) .$$

Thus:

(1.134) $$\beta(Z_{00})(X_{0k}) = \sum_{1\le\alpha,\beta\le n}[-\delta_{\alpha\beta}]V_{00}^{\alpha\beta}(X_{0k})$$

$$= \sum_{1\le\alpha,\beta\le n}[-\delta_{\alpha\beta}]\delta_{k\beta}X_{0\alpha} = -X_{0k},$$

(1.135) $$\beta(Z_{00})(X_{j0}) = \sum_{1\le\mu,\nu\le n}[\delta_{\mu\nu}]V_{\mu\nu}^{00}(X_{j0})$$

$$= \sum_{1\le\mu,\nu\le n}[\delta_{\mu\nu}]\delta_{j\nu}X_{\mu 0} = X_{j0},$$

(1.136) $$\beta(Y_{pq})(X_{0k}) = (V_{00}^{pq} - V_{00}^{qp})(X_{0k}) = \delta_{kq}X_{0p} - \delta_{kp}X_{0q},$$

and:

(1.137) $$\beta(Y_{pq})(X_{j0}) = (V_{pq}^{00} - V_{qp}^{00})(X_{j0}) = \delta_{jq}X_{p0} - \delta_{jp}X_{q0},$$

as desired. Q.E.D.

2. Fréchet Space Topologies and the Nash-Moser Category.

Having recalled Cordes characterization of the pseudodifferential operator algebras $OPS^{0,0}_{\mathcal{A}}$, we want to proceed towards establishing this correspondence as one between Fréchet spaces in the Nash-Moser category. In this section, we define the two natural Fréchet space topologies on $OPS^{0,0}_{gs} = \Psi GS$, a *symbol topology* which comes from the pseudodifferential realization $OPS^{0,0}_{gs}$, and an *operator topology* which comes from its characterization as ΨGS by the Cordes criterion. Then we show, (as in **[C3]**, Proposition 4.5), that these topologies make the identification maps of Theorem 1.2 continuous. In fact, more is true; these maps are *tame maps* between *graded Fréchet spaces* (Proposition 2.10), which combined with the claim that ΨGS is a *tame Fréchet space* (Proposition 4.1), yields this topological isomorphism as one between tame Fréchet spaces. We remark that while we concentrate on the particular algebra $OPS^{0,0}_{gs}$, the largest algebra which retains locally classical symbols, all of the results stated here will also hold for any pseudodifferential operator algebra which has such a characterization. We begin by recalling some of the basic definitions and properties of objects in the Nash-Moser category of tame Fréchet spaces and smooth tame maps (cf. Hamilton **[H]** , Goodman/Yang **[G/Y]**, and Sergeraert **[Sr1]**-**[Sr3]**).

Definition 2.1: A *tame Fréchet space* is a graded Fréchet space $(\mathcal{F}, |\bullet|_k)$, i.e., a Fréchet space $\mathcal{F}$ with an increasing sequence of seminorms, $|\bullet|_0 \le |\bullet|_1 \le |\bullet|_2 \le \dots$, which supports a family of *smoothing operators* S_θ, satisfying, for $\theta \ge 1$:

(2.1) $$S_\theta : \mathcal{F} \to \mathcal{F}$$

(2.2) $$|S_\theta u|_k \le C_k \theta^{k-j} |u|_j \qquad \forall\, j \le k, \forall\, u \in \mathcal{F}$$

(2.3) $$|u - S_\theta|_j \le C_k \theta^{j-k} |u|_k \qquad \forall\, j \le k, \forall\, u \in \mathcal{F}$$

(2.4) $$|u - S_\theta|_k \to 0 \quad \text{as } \theta \to \infty \qquad \forall\, k \ge 0, \forall\, u \in \mathcal{F}.$$

We remark that this definition, emphasizing the existence of the smoothing operators, is in the spirit of **[G/Y]** as opposed to the treatment in **[H]**, where the tameness is stated in terms of a tame linear isomorphism from $\mathcal{F}$ to $\sum(\mathcal{B})$ = the space of exponentially decreasing sequences of some Banach space $\mathcal{B}$. These notions are equivalent for spaces of smooth maps, and for our purposes the smoothing operator characterization is useful. In what follows, we will adopt the terminology of Schwartz **[Sw]** in calling (2.2) - (2.4) "efficiency estimates". Also, we note that there is no need to have ones sequence of seminorms begin with index 0, in fact, if $(\mathcal{F}, |\bullet|_k : k\ge 0)$ is a tame Fréchet space, then so is $(\mathcal{F}, |\bullet|_{k+r} : k\ge 0)$.

Definition 2.2: (Hamilton [**H**]) Let $\Phi: U \subseteq \mathcal{F} \to \mathcal{G}$ be a map defined on an open subset U of a graded Fréchet space $(\mathcal{F}, |\bullet|_k)$ with values in the graded Fréchet space $(\mathcal{G}, ||\bullet||_k)$, then one says that *$\Phi$ satisfies a tame estimate of degree r and base b on U* if $\forall\, k \geq b\ \exists\, C_k$ and r such that:

$$(2.5) \qquad ||\Phi(f)||_k \leq C_k(1 + |f|_{k+r}) \qquad \text{for all } f \in U.$$

One then says that *Φ is a tame map* if it is defined on an open subset of $\mathcal{F}$ and is continuous and satisfies a tame estimate on a neighborhood of each point.

Remark 2.3: In general, one allows the degree and the base to vary from neighborhood to neighborhood, but often they may be chosen uniformly. The constants C_k, however, almost always depend on fixing a neighborhood, which is often described by bounding f in a lower order seminorm.

Remark 2.4: If Φ is linear, or more generally, if $\Phi(0) = 0$, then the estimate (2.5) may be replaced by:

$$(2.6) \qquad ||\Phi(f)||_k \leq C_k||f||_{k+r}.$$

In addition, if Φ is linear, then the tameness estimates of course imply continuity, but for nonlinear Φ, continuity does not automatically follow from tameness estimates.

Definition 2.5: A map $\Phi : U \subseteq \mathcal{F} \to \mathcal{G}$ between graded Fréchet spaces is said to be *smooth tame* if Φ and all of its tangent maps satisfy tame estimates on the relevant Cartesian product $(U \subseteq \mathcal{F}) \times \mathcal{F} \times \ldots \times \mathcal{F} \to \mathcal{G}$, where the derivatives are taken in the sense of Gateaux (cf. [**H**], Definition. I.3.1.1).

Remark 2.6: In practice, the verification of Φ being smooth tame is often reduced just to the tameness of Φ by making use of the general result that the composition of tame maps are tame (cf. [H], Theorem. II.2.1.6) and by examining the form of the derivatives on the space $(U \subseteq \mathcal{F}) \times \mathcal{F} \times \ldots \times \mathcal{F}$, where one knows that the Cartesian product of tame Fréchet spaces is a tame Fréchet space with respect to the standard topology built on products. So, for example, a bilinear map $\Phi : \mathcal{F} \times \mathcal{F} \to \mathcal{G}$ *satisfies a tame estimate of degree r and base b in both factors if:*

$$(2.7) \qquad || \Phi(u,v) ||_k \le C_k(| u |_{k+r} + | v |_{k+r}) \quad \forall\, k \ge b,$$

where C_k depends only on k and seminorms of order < k+r.

We are now ready to define Fréchet space topologies on $OPS^{0,0}_{gs} = \Psi GS$.

Definition 2.7: (Symbol topology)

On $S^{0,0}_{gs} = \{ a(x,\xi) \in C^\infty(\mathbf{R}^{2n}) : \prod_{j=1}^{N} \eta_{p_j q_j} a \in L^\infty(\mathbf{R}^{2n}) \quad \forall N=0,1,2,.. \}$,

one defines the seminorms:

$$(2.8) \qquad \rho_k[a] = \sup_{N \le k} || \prod_{j=1}^{N} \eta_{p_j q_j} a ||_{L^\infty(\mathbf{R}^{2n})},$$

where we recall that the symbol $\prod_{j=1}^{N} \eta_{p_j q_j}$ stands for an arbitrary N-fold product of the phase space differential operators η_{pq} of (1.16), with the adopted conventions $\prod_{j=1}^{0} \eta_{p_j q_j} a = a$ to contain the boundedness of a, and $\prod_{j=1}^{N} \eta_{p_j q_j} = \eta_{p_N q_N} \cdots \eta_{p_1 q_1}$ to address the lack of commutivity.

Then, in light of the characterization $\mathrm{OPS}^{0,0}_{gs} = \Psi GS$ as a subset of $C^\infty(gs', \mathcal{L}(\mathcal{H}))$, one may also endow these operator spaces with a sensible topology.

Definition 2.8: (Operator topology)

For $A \in \Psi GS = \mathrm{OPS}^{0,0}_{gs}$, define the seminorms:

$$(2.9) \qquad N_k[A] = \max_{|\alpha| \le k} \| \partial^\alpha (A_{\sigma,h,z,\zeta})|_e \|_{L^2\text{-}op},$$

where by ∂^α one means the derivative with respect to σ,h,z,ζ of length α.

Remark 2.9: One should notice that the operator topology defined by (2.9) is generated by information about the conjugated operator near the identity in gs'. This makes sense because the transition formulas of Lemma 1.8 relate the derivatives at an arbitrary point in gs' to those at the identity, and hence, on any compact neighborhood K of e:

$$(2.10) \qquad \sup_K \max_{|\alpha| \le k} \| \partial^\alpha (A_{\sigma,h,z,\zeta})|_e \|_{L^2\text{-}op} \le C_k N_k(A).$$

This says that the seminorms in (2.9) give the topology of uniform L^2-operator norm convergence on compact subsets for $A_{\sigma,h,z,\zeta}$ and its derivatives. Both families of seminorms induce Fréchet space topologies on $\mathrm{OPS}^{0,0}_{gs} = \Psi GS$, in fact, the topologies are tamely equivalent, which is the content of the following proposition.

Proposition 2.10: Endow $S^{0,0}_{gs}$ with its symbol topology (2.8) and $OPS^{0,0}_{gs}$ with its operator topology (2.9), then the maps :

(2.11) $\sigma : OPS^{0,0}_{gs} \rightarrow S^{0,0}_{gs}$ given by $\sigma(a(x,D)) = a(x,\xi)$

and

(2.12) $Op : S^{0,0}_{gs} \rightarrow OPS^{0,0}_{gs}$ given by $Op(a(x,\xi)) = a(x,D)$

are tame.

Remark 2.11: The maps, being linear and tame, will be continuous in accordance with Remark 2.4, and so $OPS^{0,0}_{gs} \cong \Psi GS$ are isomorphic as Fréchet spaces. In addition, equipping $OPS^{0,0}_{gs} \cong \Psi GS$ with either the symbol topology or the operator topology yields tamely equivalent gradings. Finally, since these maps are linear, their tangent maps will also be tame, provided that the product map is a tame bilinear map on $OPS^{0,0}_{gs} \times OPS^{0,0}_{gs}$, which will be shown in section 4. Hence, the maps will be smooth tame.

Proof: (of Proposition 2.10)

σ is tame: One has:

(2.13) $$\rho_k[\sigma(A)] = \sup_{N \leq k} \left|\left| \prod_{j=1}^{N} \eta_{p_j q_j} a(x,\xi) \right|\right|_{L^\infty(\mathbf{R}^{2n})}$$

Now put $b(x,\xi) = \prod_{j=1}^{N} \eta_{p_j q_j} a(x,\xi)$ and make use of the following estimate, which results from the trace formula (1.21).

Lemma 2.12: Let $b(x,\xi) \in CB^{\infty}(\mathbf{R}^{2n})$ = {smooth functions on R^{2n} with bounded derivatives of all orders}. Then:

(2.14) $$||\, b \,||_{L^{\infty}(\mathbf{R}^{2n})} \;\leq\; C \sum_{|\beta|,|\gamma|\leq 2n} ||\, b^{(\beta)}_{(\gamma)}(x,D) \,||_{L^2\text{- op}}.$$

Proof: (of Lemma 2.12)

For $B = b(x,D)$, one recovers the symbol $b(x,\xi)$ by way of the trace formula (1.21)

(2.15) $$b(z,\zeta) \;=\; (2\pi)^{-n/2}\mathrm{tr}[Q^{*}_{-}P(\partial_z,\partial_\zeta)B_{z,\zeta}]$$

where:

(2.16) $$B_{z,\zeta} = e^{iz\cdot D}e^{-i\zeta\cdot M}Be^{i\zeta\cdot M}e^{-iz\cdot D}$$

and:

(2.17) $$P(\partial_z,\partial_\zeta) = \prod_{j=1}^{n} (\partial_{z_j} + 1)^2(\partial_{\zeta_j} + 1)^2$$

and:

(2.18) Q^{*}_{-} = the reflected adjoint of Q, a fundamental solution for $P(\partial_z,\partial_\zeta)$.

Then, one estimates:

(2.19) $$|\, b(z,\zeta) \,| = |\, (2\pi)^{-n/2}\mathrm{tr}[Q^{*}_{-}P(\partial_z,\partial_\zeta)B_{z,\zeta}] \,|$$

$$\leq (2\pi)^{-n/2}\, ||\, Q^{*}_{-} \,||_{tr}\, ||\, P(\partial_z,\partial_\zeta)B_{z,\zeta} \,||_{L^2\text{ -op}}$$

where $||\bullet||_{tr}$ is the trace class norm, Q^{*}_{-} is of trace class, and $P(\partial_z,\partial_\zeta)B_{z,\zeta}$ belongs to $\mathcal{L}(\mathcal{H})$, as per Cordes **[C1]**. Now:

$$(2.20) \qquad P(\partial_z,\partial_\zeta)B_{z,\zeta} = \sum_{|\beta|,|\gamma|\leq 2n} C_{\beta\gamma}(B^{(\beta)}_{(\gamma)})_{z,\zeta},$$

follows by a simple calculation based on:

$$\partial_{z_j}[B_{z,\zeta}] = [(\partial_{x_j}b)(x,D)]_{z,\zeta} \quad \text{and} \quad \partial_{\zeta_j}[B_{z,\zeta}] = [(\partial_{\xi_j}b)(x,D)]_{z,\zeta}.$$

Hence:

$$(2.21) \quad |b(z,\zeta)| \leq (2\pi)^{-n/2}||Q^*_-||_{tr} \bullet$$

$$\sum_{|\beta|,|\gamma|\leq 2n} ||C_{\beta\gamma}e^{iz\cdot D}e^{-i\zeta\cdot M}b^{(\beta)}_{(\gamma)}(x,D)\,e^{i\zeta\cdot M}e^{-iz\cdot D}||_{L^2\text{-op}},$$

but $e^{\pm iz\cdot D}$ and $e^{\pm i\zeta\cdot M}$ are unitary, so:

$$(2.22) \qquad |b(z,\zeta)| \leq (2\pi)^{-n/2}||Q^*_-||_{tr} \sum_{|\beta|,|\gamma|\leq 2n} C_{\beta\gamma}||b^{(\beta)}_{(\gamma)}(x,D)||_{L^2\text{-op}},$$

and hence the bound on $||b||_{L^\infty}$, completing the lemma. Q.E.D.

Applying (2.14) to (2.13), with $b = \prod_{j=1}^{N}\eta_{p_jq_j}a$, where $b \in CB^\infty(\mathbf{R}^{2n})$ because $a \in OPS^{0,0}_{gs}$, one finds:

$$(2.23) \qquad \rho_k[\sigma(A)] \leq \sup_{N\leq k} C_N \sum_{|\beta|,|\gamma|\leq 2n} ||(b)^{(\beta)}_{(\gamma)}(x,D)||_{L^2\text{-op}}$$

$$= \sup_{N\leq k} C_N \sum_{|\beta|,|\gamma|\leq 2n} ||(\eta_{10}^{\gamma_1}\cdots\eta_{n0}^{\gamma_n}\eta_{01}^{\beta_1}\cdots\eta_{0n}^{\beta_n}\prod_{j=1}^{N}\eta_{p_jq_j}a)(x,D)||_{L^2\text{-op}},$$

and now, by Lemma 1.8, since each $(\eta_{pq}a)(x,D)$ is some first derivative of $A_{\sigma,h,z,\zeta}$ evaluated at the identity, (2.23) becomes:

$$(2.24) \qquad \rho_k[\sigma(A)] \leq \sup_{N\leq k} C_N \sum || \partial^{\alpha+\beta+\gamma}[A_{\sigma,h,z,\zeta}]|_e ||_{L^2\text{-op}},$$

where the sum runs over $|\alpha| = N$ and $|\beta|, |\gamma| \leq 2n$, and ∂ represents differentiation with respect to σ,h,z, and ζ. Therefore, for $A \in \Psi GS$:

$$(2.25) \qquad \rho_k[\sigma(A)] \leq C_k N_{k+4n}[A],$$

which is to say that the symbol map is tame of degree $\leq 4n$ and base 0.

Before proceeding to the tameness of the operator map, we want to remark that the degree of the symbol map in (2.25) is not claimed to be sharp. In fact, as Cordes remarks in **[C1]**, since the formula (1.21) involves a choice of the differential operator P, which has degree 2n in both variables z and ζ, a different choice of a lower degree operator may produce an equivalent trace formula that uses fewer derivatives. At any rate, the symbol map is tame.

Op is tame: For $a(x,\xi) \in S^{0,0}_{gs}$ and $A = a(x,D)$:

$$(2.26) \qquad N_k[Op(a)] = \max_{|\alpha|\leq k} || \partial^{\alpha}(A_{\sigma,h,z,\zeta})|_e ||_{L^2\text{-op}}$$

$$= \max_{|\alpha|\leq k} || \prod_{j=1}^{|\alpha|} (\eta_{p_jq_j}a)(x,D) ||_{L^2\text{-op}},$$

in accordance with Lemma 1.8. Now, one invokes the form of the Calderon-Vaillancourt theorem given in (1.37), which bounds the L^2- operator norm of a pseudodifferential operator in $Op(CB^\infty(\mathbf{R}^n))$ by L^∞ norms of up to (n+2)/2 derivatives of its symbol, to give:

$$(2.27) \qquad N_k[Op(a)] \;\le\; \max_{|\alpha|\le k} C_\alpha \sum_{|\beta|,|\gamma|} \left\| \left(\prod_{j=1}^{|\alpha|} \eta_{p_j q_j} a\right)^{(\beta)}_{(\gamma)}(x,\xi) \right\|_{L^\infty(\mathbf{R}^{2n})},$$

where the sum runs over $|\beta|$ and $|\gamma| \le (n+2)/2$. Now, each derivative ∂_x^γ and ∂_ξ^β is itself a product of η_{pq}'s , and so, if one puts $\mu = \alpha + \beta + \gamma$:

$$(2.28) \qquad N_k[Op(a)] \;\le\; C_k \max_{|\mu|\le k+n+2} \left\| \prod_{j=1}^{|\mu|} (\eta'_{p_j q_j} a)(x,\xi) \right\|_{L^\infty(R^{2n})}$$

$$\le\; C_k \rho_{k+n+2}[a].$$

Hence, the map Op is tame of degree $\le n + 2$ and base 0, and the proposition is finished.

Q.E.D.

3. General Smooth Tame Structures: Spaces of Smooth Maps.

In this section, we wish to provide general results which will form the basis of our demonstration that certain global classes of 0-order pseudodifferential operators have smooth tame structures. The important point here is that the characterization of the operator topology for algebras like $OPS^{0,0}_{gs}$ in terms of Fréchet spaces of smooth maps from a Lie group into a C^*-algebra yields the tameness of products and inverses, when the latter exist, as an application of the Leibniz formula, provided one can "interpolate" in the operator seminorms:

$$N_k[A] \;=\; \max_{|\alpha|\leq k} \| \, \partial^\alpha(A_{\sigma,h,z,\zeta})|_e \, \|_{L^2\text{-op}}. \tag{3.1}$$

By interpolation, we mean the type of estimates interrelating the seminorms on a tame Fréchet space. Consequently, one wants to investigate the tame Fréchet space structure of spaces like $C^\infty(gs', \mathcal{L}(L^2(\mathbf{R}^n)))$. In particular, since the operator topology on $OPS^{0,0}_{gs}$ is defined using only a neighborhood of the identity in gs', one only needs to consider spaces of smooth maps whose domains are fixed *coordinate balls* containing the identity e of a real Lie group G, where this neighborhood is any convenient preimage of a true ball in $\mathbf{R}^n$ under a local diffeomorphism about $e \in$ G. This fact greatly simplifies the needed work.

In this context, we seek to justify the interpolation properties of the $OPS^{0,0}_{gs}$ seminorms by showing that the space $C^\infty(M, \mathcal{B})$ of smooth maps from M, a compact coordinate ball containing the identity *e* of a real Lie group G into a Banach space $\mathcal{B}$ is a tame Fréchet space with respect to a grading by C^k norms, in the sense of Definition 2.1. The needed smoothing operators will be first constructed on $\mathbf{R}^n$ as maps on the spaces $C^\infty(\bar{\Omega}, \mathcal{B})$, where Ω is an open region with compact closure and smooth boundary. For the special case of Ω being a ball, these smoothing operators are then easily transferred to M by a fixed coordinate chart valid in a neighborhood of M. This would be sufficient for providing the tameness of products and inverses, but we include for future use a more general treatment; namely, if $\mathcal{B}$ is a Banach algebra, then $C^\infty(M, \mathcal{B})$ becomes a *smooth tame Fréchet algebra* , which is to say that the pointwise product map:

$$C : C^\infty(M, \mathcal{B}) \times C^\infty(M, \mathcal{B})$$

is smooth tame (Proposition 3.13). Also, if $\mathcal{B}$ is a C^*-algebra, then $C^\infty(M, \mathcal{B})$ forms a *smooth tame Fréchet ∗-algebra*, which is to say that the involution ∗ on $\mathcal{B}$ gives rise to a smooth tame map from $C^\infty(M, \mathcal{B})$ into itself (Proposition 3.14). Finally, if $[C^\infty(M, \mathcal{B})]_*$ denotes the subgroup of invertible elements in the smooth tame Fréchet algebra $C^\infty(M, \mathcal{B})$, then $[C^\infty(M, \mathcal{B})]_*$ forms a *smooth tame Lie group*, which is to say that, in addition, the inversion map is smooth tame (Proposition 3.17).

We begin by working in Ω, an open subset of $\mathbf{R}^n$ with compact closure, where additional conditions on Ω will be imposed later.

Definition 3.1: For $(\mathcal{B}, ||\bullet||_{\mathcal{B}})$ a Banach space, let :

(3.2) $$C_0^\infty(\Omega, \mathcal{B}) = \{\, u : \Omega \to \mathcal{B} \mid u \text{ possesses derivatives of all orders, and } \operatorname{supp}(u) = K \subset \Omega, \text{ for some compact } K \}.$$

To say that $\partial_{x_j} u(x)$ exists is to say:

(3.3) $$\lim_{\varepsilon \to 0} || \varepsilon^{-1} \nabla_{j,\varepsilon} u(x) - \partial_{x_j} u(x) ||_{\mathcal{B}} = 0,$$

where $\nabla_{j,\varepsilon} u$ is the relevant difference quotient. Also define:

(3.4) $$C^\infty(\bar{\Omega}, \mathcal{B}) = \{\, u : \bar{\Omega} \to \mathcal{B} \mid u \text{ is smooth in a neighborhood of } \Omega, \text{ with continuous derivatives on } \bar{\Omega} \,\}.$$

We consider the following seminorms on these spaces:

(3.5) $$||u||_{\Omega;k} = \max_{|\alpha| \le k} \sup_{x \in \bar{\Omega}} || \partial_x^\alpha u(x) ||_{\mathcal{B}}.$$

These seminorms generate a Fréchet space topology on $C^\infty(\bar{\Omega}, \mathcal{B})$ and are bounded linear functionals on $C_0^\infty(\Omega, \mathcal{B})$, which is not itself a Fréchet space.

Now, for smooth a function $u \in C_0^\infty(\Omega, \mathcal{B})$, one convolutes u with a suitably chosen mollifier. Following Nash **[N1]**, (also cf. Goodman/Yang **[G/Y]** and Schwartz **[Sw]**), for any $\delta > 0$, one can find a $\hat{\phi}(\xi) \in C_0^\infty(\mathbf{R}^n)$ such that:

(3.6) $$\hat{\phi}(\xi) = (2\pi)^{-n/2} \qquad \forall\, |\xi| \le \delta,$$

(3.7) $$\hat{\phi}(\xi) = 0 \qquad \forall\, |\xi| \ge 2\delta, \quad \text{and}$$

(3.8) $$\hat{\phi}(\xi) \text{ is monotone decreasing on } \delta \le |\xi| \le 2\delta.$$

Now define:

(3.9) $$\varphi(x) = \int e^{ix\cdot\xi}\, \hat{\phi}(\xi)\, đ\xi \qquad \text{for } đ\xi = (2\pi)^{-n/2} d\xi,$$

and it is straightforward to check:

Lemma 3.2: $\varphi(x)$ defined by (3.9) is such that:

i) $\varphi(x) \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz space of rapidly decreasing functions,

ii) $\int \varphi(x)dx = 1$

iii) $\int x^\alpha \varphi(x)dx = 0 \qquad \forall\, |\alpha| > 0.$

Proof: Since $\hat{\phi} \in C_0^\infty(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$, and the Fourier transform is an isomorphism on $\mathcal{S}(\mathbf{R}^n)$, i) follows, which is to say:

(3.10) $$\forall\, \alpha, \beta\ \exists\, C_{\alpha\beta} \text{ such that } \quad |x^\beta \partial_x^\alpha \varphi(x)| \le C_{\alpha\beta}\, ,$$

or:

(3.11) $$\forall\, N, \alpha\, \exists\, C_{\alpha,N} \text{ such that } \quad |\partial_x^\alpha \varphi(x)| \le C_{\alpha,N}\, (1+|x|)^{-N}.$$

Since:

(3.12) $$\int \varphi(x)dx = (2\pi)^{n/2} \int \varphi(x)đx = (2\pi)^{n/2} \hat{\phi}(o),$$

the condition (3.6) yields ii). Similarly:

(3.13) $$\int x^\alpha \varphi(x)dx = (2\pi)^{n/2} \int x^\alpha \varphi(x)đx = (2\pi)^{n/2}\, (x^\alpha \varphi)^\wedge(0)$$

$$= (2\pi)^{n/2}\, (D_\xi^\alpha \hat{\phi})(0),$$

which is zero by (3.5), unless $|\alpha| = 0$, and hence iii). Q.E.D.

Now, one defines:

(3.14) $$\varphi_\theta(x) = \theta^n \varphi(\theta x), \quad \text{for } \theta \geq 1,$$

and by a change of variables $z = \theta x$, one sees that φ_θ is also a Schwartz function with integral one. Then:

Definition 3.3: For $u \in C_o^\infty(\Omega, \mathcal{B})$, the operator S_θ determined by φ_θ is given by:

(3.15) $$S_\theta u(x) = (\varphi_\theta * u)(x) = \int \varphi_\theta(x-y)u(y)dy,$$

where one can think of the integral as being taken over $\mathbf{R}^n$ after extending u to be zero outside of Ω.

One needs to verify that these operators are well defined linear maps from $C_o^\infty(\Omega, \mathcal{B})$ into $C^\infty(\bar{\Omega}, \mathcal{B})$ and possess the necessary "efficiency estimates" introduced in (2.2)-(2.4), with respect to the seminorms (3.5). These S_θ are not quite smoothing operators in the sense we need since they are to act on a given Fréchet space, but they posses the key properties which allow them to form the basis of true smoothing operators. We begin with an important lemma.

Lemma 3.4: For each $\theta \geq 1$, the operator S_θ, defined by (3.15), is well defined as a linear map: $C_o^\infty(\Omega, \mathcal{B}) \to C^\infty(\bar{\Omega}, \mathcal{B})$, and satisfies for all multi-indices α, β:

(3.16) $$\partial_x^\alpha(S_\theta u) = \partial_x^\alpha \varphi_\theta * u = \varphi_\theta * \partial_x^\alpha u = \partial_x^{\alpha-\beta} \varphi_\theta * \partial_x^\beta u.$$

Proof: For each $x \in \mathbf{R}^n$, the integral in (3.15) makes sense and produces an element $S_\theta u(x) \in \mathcal{B}$, because after extending u to be zero outside of Ω, the integral agrees with the integral over $K = \mathrm{supp}(u)$, and, as such, is the integral of a continuous map from a compact measure space into a Banach space. Therefore, the integral is well defined, and, in fact, one has (cf. Rudin **[R]**, Theorem 3.29):

(3.17) $$\| S_\theta u(x) \|_{\mathcal{B}} \leq \int \| \varphi_\theta(x-y)u(y) \|_{\mathcal{B}} \, dy ,$$

but this is:

$$\leq \sup_{(x-y)\in K} | \varphi_\theta(x-y) | \sup_{y\in K} \| u(y) \|_{\mathcal{B}} \, \mathrm{vol}(K) < \infty ,$$

for each $x \in \bar{\Omega}$, where the bound is uniform in x. Therefore, $S_\theta u(x)$ is a function on $\bar{\Omega}$ with uniformly bounded values in $\mathcal{B}$, and it is clear that S_θ is a linear operator by the linearity of the integral. The question that remains is whether $S_\theta u(x)$ is a smooth function for x in a neighborhood of Ω with derivatives continuous on $\bar{\Omega}$.

Claim 1: $S_\theta u(x)$ is a continuous map from $\bar{\Omega}$ into $\mathcal{B}$.

Proof of claim 1: Pick any $x_0 \in \bar{\Omega}$, and given $\varepsilon > 0$, one needs a $\delta > 0$ such that:

(3.18) $$| x - x_0 | < \delta \quad \Rightarrow \quad \| S_\theta u(x) - S_\theta u(x_0) \|_{\mathcal{B}} < \varepsilon .$$

One has,

(3.19) $$\| S_\theta u(x) - S_\theta(x_0) \|_{\mathcal{B}} = \| \int_K (\varphi_\theta(x-y) - \varphi_\theta(x_0-y))u(y)dy \|_{\mathcal{B}}$$

$$\leq \mathrm{vol}(K) \sup_{y\in K} \| u(y) \|_{\mathcal{B}} \sup_{y\in K} | (\varphi_\theta(x-y) - \varphi_\theta(x_0-y)) |$$

$$\leq \text{vol}(K)\,||\,u\,||_{\Omega;0}\,\omega(\varphi_\theta,\delta),$$

where $\omega(f,\delta) = \sup_{|z-w|<\delta} |f(z) - f(w)|$ is the modulus of continuity, and $|x - x_0| < \delta$. Since φ_θ is uniformly continuous on K, $\omega(\varphi_\theta,\delta)$ can be made less than $\varepsilon/(\text{vol}(K)\,||\,u\,||_{\Omega;0}$ for δ small enough; whence the claim. Next, one examines first derivatives.

Claim 2: $S_\theta u(x)$ is a differentiable function and:

$$\partial_{x_j}[S_\theta u(x)] = (\partial_{x_j}(\varphi_\theta)*u)(x) = (\varphi_\theta*\partial_{x_j}u)(x). \tag{3.20}$$

Proof of claim 2: By examining the limit of the difference quotient when forced onto φ_θ, one finds:

$$\partial_{x_j}(S_\theta u(x)) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\{\varphi_\theta(x+\varepsilon e_j-y) - \varphi_\theta(x-y)\}u(y)dy \tag{3.21}$$

$$= \int\partial_{x_j}[\varphi_\theta(x-y)]u(y)dy\ ,$$

where the last identity is in the appropriate sense; namely, that the difference in the last two expressions converges to 0 in the $\mathcal{B}$ norm. Indeed, as in (3.19):

$$\lim_{\varepsilon\to 0}||\frac{1}{\varepsilon}\int\{\varphi_\theta(x+\varepsilon e_j-y) - \varphi_\theta(x-y)\}u(y)dy - \int\partial_{x_j}[\varphi_\theta(x-y)]u(y)dy\,||_{\mathcal{B}} \tag{3.22}$$

$$\leq \text{vol}(K)\,||\,u\,||_{\Omega;\,0}\lim_{\varepsilon\to 0}\sup_{y\in K}|\{\frac{1}{\varepsilon}\varphi_\theta(x+\varepsilon e_j-y) + \varphi_\theta(x-y)\} - \partial_{x_j}[\varphi_\theta(x-y)]\,|,$$

but this limit is zero as the convergence of the difference quotient for the smooth function φ_θ is uniform on the compact set K for each $x \in \bar{\Omega}$. Also notice that:

$$(3.23) \qquad \partial_{x_j}[\varphi_\theta(x)] = \partial_{x_j}[\theta^n \varphi(\theta x)] = \theta^n \theta(\partial_{x_j}\varphi)(\theta x) = \theta(\partial_{x_j}\varphi)_\theta(x).$$

As a result, one knows that $S_\theta u(x)$ is differentiable, and by examining the limit:

$$(3.24) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \| \int \varphi_\theta(y)\{u(x+\varepsilon e_j-y) - u(x-y)\}dy - \int \varphi_\theta(y)\partial_{x_j}[u(x-y)]dy \|_{\mathcal{B}}$$

$$\leq \| \varphi \|_{L^1} \lim_{\varepsilon \to 0} \int \frac{1}{\varepsilon} \| u(x+\varepsilon e_j-y) - u(x-y) - \partial_{x_j}[u(x-y)] \|_{\mathcal{B}},$$

where $\varphi \in \mathcal{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$, the above limit is zero since u is a smooth function with compact support, and, as such, the difference quotient converges uniformly on the support of u.

Now, the limit in (3.24) is an expression for $\partial_{x_j}[S_\theta u(x)] - (\varphi_\theta * \partial_{x_j} u)(x)$ and hence the identity (3.20), which completes the claim. To finish the lemma, one just iterates the above arguments, by pushing the derivatives onto u or φ as needed, where we remark that the content of (3.16) is:

$$(3.25) \qquad [\partial_x^\alpha, S_\theta] = 0. \qquad \text{Q.E.D.}$$

Next, we address the necessary "efficiency estimates" that characterize the operators' smoothing properties.

Lemma 3.5: If $u \in C_0^\infty(\Omega, \mathcal{B})$, then, $\forall$ pairs (k,j) with $j \leq k$, $\exists$ constants C_k, independent of u, such that:

(3.26) $$\| S_\theta u \|_{\Omega;k} \leq C_k \theta^{k-j} \| u \|_{\Omega;j}$$

Proof: First, one may notice that the estimates for j=0 imply those for j≠0 in light of (3.25). However, it is not difficult to produce the estimate in one stroke by exploiting the last identity in (3.16). Indeed:

(3.27) $$\| S_\theta u \|_{\Omega;k} = \max_{|\alpha| \leq k} \sup_{x \in \bar{\Omega}} \| \partial_x^{\alpha-\beta}[\varphi_\theta] * \partial_x^\beta u \|_{\mathcal{B}},$$

where we have split off some derivative ∂_x^β of order $|\beta| = j$. By (3.23), this is:

$$= \max_{|\alpha| \leq k} \sup_{x \in \bar{\Omega}} \| \int \theta^{|\alpha-\beta|} (\partial_x^{\alpha-\beta}\varphi)_\theta(x-y) \partial_x^\beta u(y) dy \|_{\mathcal{B}}$$

$$\leq \theta^{k-j} \max_{|\alpha| \leq k} \sup_{x \in \bar{\Omega}} \{ \mathrm{vol}(\Omega) \sup_{y \in \bar{\Omega}} | (\partial_x^{\alpha-\beta}\varphi)_\theta(x-y) | \| \partial_x^\beta u(y) \|_{\mathcal{B}} \},$$

where we have used that $\theta \geq 1$ and u has compact support within Ω. Now, since $|\beta| = j$, $\bar{\Omega}$ is compact, and φ is smooth, one obtains (3.27) with:

(3.28) $$C_k = \mathrm{vol}(\Omega) \max_{|\alpha| \leq k} \sup_{x,y \in \bar{\Omega}} | (\partial_x^{\alpha-\beta}\varphi)_\theta(x-y) | < \infty .$$ Q.E.D.

Lemma 3.6: If $u \in C_0^\infty(\Omega, \mathcal{B})$, then, $\forall$ pairs (k,j) with $j \leq k$, $\exists$ constants C_k, independent of u, such that:

(3.29) $$\| u - S_\theta u \|_{\Omega;j} \leq C_k \theta^{j-k} \| u \|_{\Omega;k} .$$

Proof: One may break the argument into three cases. First, one notices that the j=k=0 case follows from Lemma 3.5 by the triangle inequality. Then one estimates the case j=0 and $k \geq 1$ by exploiting the identities ii) and iii) from Lemma 3.2 by way of an appropriate finite order Taylor expansion with $u - S_\theta u$ appearing as its remainder term. This remainder is then estimated. Finally, by making use of the property (3.25), one generates the estimates for $0 < j \leq k$ from the j=0 estimates.

j=k=0 case: $\| u - S_\theta u \|_{\Omega;0} \leq \| u \|_{\Omega;0} + \| S_\theta u \|_{\Omega;0}$

$$\leq \| u \|_{\Omega;0} + C_0 \| u \|_{\Omega;0} \leq C_0' \| u \|_{\Omega;0} ,$$

where C_0 comes from Lemma 3.5.

j=0, k ≥ 1 case: Here, one makes use of the finite Taylor expansion with integral remainder of a smooth function $f : [0, 1] \to \mathcal{B}$, which states:

(3.30) $$f(1) = \sum_{j=0}^{N} \frac{f^{(j)}(0)}{j!} + \frac{1}{N!} \int_0^1 (1-s)^N f^{(N+1)}(s) ds ,$$

where $f^{(j)}(0) = [(\frac{d}{dt})^j f](0)$. To apply this to the present situation, one defines:

(3.31) $$f(t) = u(x - t\theta^{-1} z) \qquad \text{for } t \in [0, 1],$$

which is smooth in t as a map from [0, 1] into $\mathcal{B}$, given that $u \in C_0^\infty(\Omega, \mathcal{B})$. Then, one notices that, under a change of variables $z = \theta(x-y)$, the expression for $S_\theta u(x)$ becomes:

(3.32) $$S_\theta u(x) = \int \varphi_\theta(x-y)u(y)dy = \int \varphi(\theta(x-y))u(y)\theta^n dy$$

$$= \int \varphi(z)u(x-\theta^{-1}z)dz.$$

Hence, for f(t) defined by (3.31), one has:

(3.33) $$S_\theta u(x) = \int \varphi(z)f(1)dz.$$

Now, applying the expansion (3.30) with $N = k-1 \geq 0$, one finds, by making use of the chain rule, that:

(3.34) $$f^{(j)}(0) = \sum_{|\alpha|=j} (\partial_x^\alpha u)(x)(-\theta^{-1}z)^\alpha = (-\theta)^{-j} \sum_{|\alpha|=j} z^\alpha(\partial_x^\alpha u)(x) ,$$

and:

(3.35) $$f^{(k)}(s) = (-\theta)^{-k} \sum_{|\alpha|=k} z^\alpha(\partial_x^\alpha u)(x-s\theta^{-1}z),$$

and hence:

(3.36) $$S_\theta u(x) = \int \varphi(z) \Big[\sum_{j=0}^{k-1} \frac{(-\theta)^{-j}}{j!} \sum_{|\alpha|=j} z^\alpha(\partial_x^\alpha u)(x) +$$

$$\frac{(-\theta)^{-k}}{(k-1)!} \sum_{|\alpha|=k} \int_0^1 (1-s)^{k-1} z^\alpha(\partial_x^\alpha u)(x-s\theta^{-1}z)ds \Big]dz.$$

Now, making use of the fact that $\int z^{\alpha}\varphi(z)dz$ is 0 if $|\alpha| > 0$ and is 1 if $|\alpha| = 0$, one sees that the difference $S_{\theta}u(x) - u(x)$ is the remainder term in the expansion (3.36). One wants to estimate the 0- norm of this difference, which is the supremum over $x \in \bar{\Omega}$, in terms of the k- norm of u. One finds:

(3.37) $$|| u - S_{\theta}u ||_{\Omega;0} \leq$$

$$\frac{\theta^{-k}}{(k-1)!} \sum_{|\alpha|=k} \sup_{x\in\Omega} \int \left[| z^{\alpha}\varphi(z) | \int_0^1 | 1-s |^{k-1} || (\partial_x^{\alpha}u)(x-s\theta^{-1}z) ||_{\mathcal{B}} ds\right]dz .$$

Then, examination of the inner integral reveals:

(3.38) $$\int_0^1 | 1-s |^{k-1} || (\partial_x^{\alpha}u)(x-s\theta^{-1}z) ||_{\mathcal{B}} ds \leq C_k \max_{s\in[0,1]} || (\partial_x^{\alpha}u)(x-s\theta^{-1}z) ||_{\mathcal{B}},$$

where $C_k = \int_0^1 | 1-s |^{k-1} ds < \infty$ as $k \geq 1$. Furthermore, $z^{\alpha}\varphi(z) \in \mathcal{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$ for all α, and so:

(3.39) $$|| u - S_{\theta}u ||_{\bar{\Omega};0} \leq C'_k \theta^{-k} \sum_{|\alpha|=k} C_{\alpha} \sup_{x\in\bar{\Omega}} \sup_{z\in\mathbf{R}^n} \max_{s\in[0,1]} || (\partial_x^{\alpha}u)(x-s\theta^{-1}z) ||_{\mathcal{B}},$$

where $C'_k = C_k \frac{1}{(k-1)!}$ and $C_{\alpha} = || z^{\alpha}\varphi(z) ||_{L^1}$. Finally, since u is smooth and has compact support, the expression within the sum of (3.39) is bounded by $C_{\alpha} || \partial_x^{\alpha}u ||_{\Omega;0}$, and hence, one has:

(3.40) $$\|u - S_\theta u\|_{\bar{\Omega};0} \leq C_k'' \theta^{-k} \|u\|_{\Omega;k},$$

where $C_k'' = C_k' \sum_{|\alpha|=k} C_\alpha$ is independent of $u \in C_0^\infty(\Omega, \mathcal{B})$, which completes this case.

j ≥ 1, k ≥ 1 case: Here, one just translates the previous estimates via:

(3.41) $$\|u - S_\theta u\|_{\bar{\Omega};j} = \max_{|\alpha|\leq j} \sup_{x\in\bar{\Omega}} \|\partial_x^\alpha(u - S_\theta u)\|_{\mathcal{B}},$$

which, by (3.25), is

$$= \max_{|\alpha|\leq j} \sup_{x\in\bar{\Omega}} \|\partial_x^\alpha u - S_\theta(\partial_x^\alpha u)\|_{\mathcal{B}} = \max_{|\alpha|\leq j} \|\partial_x^\alpha u - S_\theta(\partial_x^\alpha u)\|_{\bar{\Omega};0}$$

$$\leq \max_{|\alpha|\leq j} C_{k,\alpha}\, \theta^{j-k} \|\partial_x^\alpha u\|_{\Omega;k-j},$$

by using the j=0 estimates on $\partial_x^\alpha u \in C_0^\infty(\Omega, \mathcal{B})$. Then, the continuity of ∂_x^α on the C^k spaces provides:

$$\leq \max_{|\alpha|\leq j} C_{k,\alpha}'\, \theta^{k-j} \|u\|_{\Omega;k} \leq C_k\, \theta^{k-j} \|u\|_{\Omega;k},$$

where C_k is independent of u. This completes the Lemma. Q.E.D.

Lemma 3.7: If $u \in C_0^\infty(\Omega, \mathcal{B})$, then, $\forall\, k \geq 0$, one has:

(3.42) $$\lim_{\theta\to\infty} \|u - S_\theta u\|_{\Omega;k} = 0.$$

Proof: Again, it suffices to know the result for k = 0, since by (3.25),

(3.43)
$$\lim_{\theta\to\infty} || u - S_\theta u ||_{\Omega;k} = \lim_{\theta\to\infty} \max_{|\alpha|\leq k} || \partial_x^\alpha u - S_\theta u ||_{\Omega;\,0} ,$$

and, for $u \in C_0^\infty(\Omega, \mathcal{B})$, $\partial_x^\alpha u$ is also. For the k = 0 case:

(3.44)
$$|| u - S_\theta u ||_{\Omega;0} = \sup_{x\in\Omega} || u(x) - \int \varphi_\theta(y)u(x-y)dy ||_{\mathcal{B}} ,$$

but φ has integral one, so:

$$= \sup_{x\in\Omega} || \int \varphi_\theta(y)\{u(x) - u(x-y)\}dy ||_{\mathcal{B}} ,$$

and, by making a change of variables $z = \theta y$, one finds:

(3.45)
$$\lim_{\theta\to\infty} || u - S_\theta u ||_{\Omega;0} = \lim_{\theta\to\infty} \sup_{x\in\Omega} || \int \varphi(z)\{u(x) - u(x-\theta^{-1}z)\}dz ||_{\mathcal{B}} .$$

Now, $u \in C_0^\infty(\Omega, \mathcal{B})$, so its first order derivatives are uniformly bounded over its support, and by the Mean Value Theorem:

(3.45)
$$|| u(x) - u(x - \frac{z}{\theta}) ||_{\mathcal{B}} \leq || u ||_{\Omega;1} \, | \frac{z}{\theta} | .$$

Hence:

$$\lim_{\theta\to\infty} || u - S_\theta u ||_{\Omega;0} \leq || u ||_{\Omega;1} \lim_{\theta\to\infty} \frac{1}{\theta} \int | z\varphi(z) | dz ;$$

but, $z\varphi(z) \in L^1(\mathbf{R}^n)$, and so the above limit is zero. Q.E.D.

We are now ready to construct the desired smoothing operators on the Fréchet space $C^\infty(\bar{\Omega}, \mathcal{B})$. Consider two open regions Ω and Ω' with compact closure and smooth boundaries in $\mathbf{R}^n$ with $\Omega \subset \bar{\Omega} \subset \Omega' \subset \bar{\Omega}'$. Under these assumptions on Ω and Ω', we can find a continuous linear extension operator:

(3.47) $$E : C^\infty(\bar{\Omega}, \mathcal{B}) \to C_0^\infty(\Omega', \mathcal{B}),$$

which respects the smoothing procedure in the sense that:

(3.48) $$\| Eu \|_{\Omega';k} \leq C_k \| u \|_{\Omega;k} .$$

One such extension is described in Hamilton [**H**], Corollary II.1.3.7, where, for Ω compact without boundary, a tubular neighborhood is used to extend functions to be constant along the fibres, and a smooth bump function is employed to cut off the support. Then a "doubling of Ω" is exploited to treat the case when Ω has boundary. However, since we will only need this extension for the special case when Ω and Ω' are balls, we will make use of a related, but simpler procedure. The method is taken from Nash's fundamental paper [**N1**],where he treated compact manifolds which were analytically embedded in $\mathbf{R}^n$, and is carried out via a regularized distance function. The only modification here is that the maps are Banach space valued.

Let Ω and Ω' be concentric balls in $\mathbf{R}^n$, and for $u \in C^\infty(\bar{\Omega}, \mathcal{B})$, define:

(3.49) $$Eu(x) = \begin{cases} s(x)u(r(x)) & x \in \Omega' \\ 0 & x \in \mathbf{R}^n \setminus \Omega' , \end{cases}$$

where:

(3.50) $$r(x) = \text{unique point in } \Omega \text{ nearest to } \Omega' ,$$

(3.51) $$s(x) = \beta\left(\frac{d(x,r(x))}{\varepsilon}\right),$$

where

(3.52) $\beta(t) \in C^\infty(\mathbf{R})$ with $\beta(t) = 1$ for $t \leq 1$

$\beta(t)$ monotonically decreases for $1 \leq t \leq 2$

$\beta(t) = 0$ for $t \geq 2$,

and:

(3.53) d is the Euclidian distance function.

The constant ε is chosen suitably so that the resulting cutoff function $s(x)$ will be a smooth function with compact support in Ω', see Nash [**N2**]. This operator E is clearly linear and extends u, since if $x \in \bar{\Omega}$, $r(x) = x$ and $s(x) = 1$. The importance of this particular extension is that it preserves the "efficiency" of the smoothing done above, which is why Nash was considering it. This preservation claim is the content of the following lemma.

Lemma 3.8: The extension operator $E : C^\infty(\bar{\Omega}, \mathcal{B}) \to C_0^\infty(\Omega', \mathcal{B})$,defined by (3.49) - (3.53) satisfies the following continuity estimates with respect to the seminorms (3.5). For each $k \geq 0$ $\exists$ constants C_k, independent of u, such that:

(3.54) $$\| Eu \|_{\Omega';k} \leq C_k \| u \|_{\Omega;k} .$$

Proof: The result follows from the Leibniz formula and the chain rule because $s(x) \in C_0^\infty(\Omega')$ and $r(x)$ is a real analytic, and hence smooth, function for $x \in \Omega'$, and E just composes u with r and multiplies by s. Q.E.D.

In addition, the natural restriction map, $R : C^\infty(\bar{\Omega}, \mathcal{B}) \to C_0^\infty(\Omega', \mathcal{B})$, will also be continuous, with:

(3.55)
$$||Ru||_{\Omega;k} \leq ||u||_{\Omega';k}.$$

As a result, one has the smoothing operators $S_\theta^\# : C^\infty(\bar{\Omega}, \mathcal{B}) \to C^\infty(\bar{\Omega}, \mathcal{B})$, by forming the composition:

(3.56)
$$S_\theta^\# = R \circ S_\theta \circ E.$$

Proposition 3.9: Let Ω be a ball in R^n. Then, the Fréchet space $C^\infty(\bar{\Omega}, \mathcal{B})$ supplied with its C^k grading $||\bullet||_{\Omega;k}$ is a tame Fréchet space. In particular, the operators $S_\theta^\#$, defined by (3.56), are smoothing operators $C^\infty(\bar{\Omega}, \mathcal{B})$ in the sense that they satisfy the efficiency estimates (2.2) - (2.4) with respect to the C^k grading $||\bullet||_{\Omega;k}$ on $C^\infty(\bar{\Omega}, \mathcal{B})$.

Proof: The proof is just a combination of Lemmas 3.5 - 3.7 on S_θ efficiency estimates, together with the continuity estimates (3.54) and (3.55) for the extension E and the restriction R. For example:

(3.57)
$$||S_\theta^\# u||_{\Omega;k} = ||(R \circ S_\theta \circ E)u||_{\Omega;k} \leq ||S_\theta(Eu)||_{\Omega';k},$$

by the continuity of the restriction map, and by Lemma 3.5:

$$\leq C_k \, \theta^{k-j} \, || \, Eu \, ||_{\Omega';j} ,$$

and by (3.54)

$$\leq C'_k \, \theta^{k-j} \, || \, u \, ||_{\Omega;j} ,$$

which is just the estimate (2.2). The same considerations yield (2.3) and (2.4).

Q.E.D.

We are now ready to transfer the smoothing operators $S^{\#}_{\theta}$ on $C^{\infty}(\bar{\Omega}, \mathcal{B})$ to a neighborhood of the identity of a real Lie group. We fix a coordinate neighborhood of the identity e in the following way. Let (V,ψ) be any convenient coordinate neighborhood of e where V is an open set with compact closure and ψ is a diffeomorphism:

(3.58) $$\psi : V \to \Omega'' \subset\subset R^n.$$

Inside Ω'' one can pick concentric balls Ω and Ω' with center $\psi(e)$ such that:

(3.59) $$\Omega \subset \bar{\Omega} \subset \Omega' \subset \bar{\Omega}' \subset \Omega''.$$

Then define the *compact coordinate ball M about the identity e* as the preimage:

(3.60) $$M = \psi^{-1}(\bar{\Omega}).$$

With respect to these fixed coordinate neighborhoods, we define the spaces of smooth maps below.

Definition 3.10: For $\mathcal{B}$ a Banach space and M a compact coordinate ball about the identity e in a Lie group G, place:

(3.61) $$C^\infty(M, \mathcal{B}) = \{ f : M \to \mathcal{B} \mid f \text{ is the restriction to M of a smooth map from a neighborhood of M into } \mathcal{B} \},$$

where since we have selected M to lie entirely within a single coordinate patch (V,ψ), the smoothness may be checked by the single condition that the map:

$$(\psi^{-1})^*f \in C^\infty(\psi(V), \mathcal{B}),$$

where:

$$(\psi^{-1})^*f\,(x) = f(\psi^{-1}(x)),$$

and the smoothness of this map on $\psi(V) \subset \mathbf{R}^n$ is as in definition 3.1.

This space $C^\infty(M, \mathcal{B})$ may then be identified by way of the fixed diffeomorphism ψ with the space $C^\infty(\bar{\Omega}, \mathcal{B})$ for $\bar{\Omega} = \psi(M)$, and becomes a Fréchet space with respect to the following seminorms:

(3.62) $$N_k[f] = \max_{|\alpha| \le k} \sup_{x \in \psi(M)} \| \partial_x^\alpha[(\psi^{-1})^*f] \|_{\mathcal{B}} .$$

Notice that:

(3.63) $$N_k[f] = \| (\psi^{-1})^*f \|_{\bar{\Omega};k} \qquad \text{for } \bar{\Omega} = \psi(M),$$

and, hence the identification by ψ is a Fréchet space isomorphism. Consequently, the transferral of the smoothing operators to $S_\theta^\#$ on $C^\infty(\bar{\Omega}, \mathcal{B})$ to ones on $C^\infty(M, \mathcal{B})$ is a triviality.

Define:

(3.64) $$S_\theta : C^\infty(M, \mathcal{B}) \to C^\infty(M, \mathcal{B})$$

by:

(3.65) $$S_\theta = (\psi)^* \circ S^\#_\theta \circ (\psi^{-1})^*.$$

Proposition 3.11: For $\mathcal{B}$ a Banach space and M a compact coordinate ball about the identity in a real Lie group, the space $C^\infty(M, \mathcal{B})$ forms a tame Fréchet space. In particular, the operators S_θ defined in (3.65) satisfy the efficiency estimates with respect to the C^k grading by the seminorms of (3.62) on $C^\infty(M, \mathcal{B})$.

Proof: This follows directly from smoothing properties in $\mathbf{R}^n$ in light of the identity (3.63). For example, by definition:

(3.66) $$N_k[S_\theta f] = \max_{|\alpha|\leq k} \sup_{x\in\psi(M)} \| \partial^\alpha_x[(\psi^{-1})^* \circ (\psi)^* \circ S^\#_\theta \circ (\psi^{-1})^* f] \|_{\mathcal{B}}$$

$$= \max_{|\alpha|\leq k} \sup_{x\in\psi(M)} \| \partial^\alpha_x[S^\#_\theta [(\psi^{-1})^* f]] \|_{\mathcal{B}},$$

which by (3.63):

$$= \| S^\#_\theta [(\psi^{-1})^* f] \|_{\bar{\Omega};k} \qquad \text{for } \bar{\Omega} = \psi(M),$$

and Proposition 3.9:

$$\leq C_k\, \theta^{k-j} \| (\psi^{-1})^* f \|_{\bar{\Omega};j},$$

and (3.63) again:

$$= C_k\, \theta^{k-j} N_j[f],$$

which is the first efficiency estimate, and similarly for the other estimates. Q.E.D.

Remark 3.12: Knowing that $C^\infty(M, \mathcal{B})$ is a tame Fréchet space allows one to interpolate products of seminorms in a way which yields tame estimates for products and inverses of $C^\infty(M, \mathcal{B})$ elements whenever these products and inverse are defined. In particular, if $\{ \mathcal{F}, |\bullet|_k \}$ is a Fréchet space admitting smoothing operators in the above sense, then one has the *interpolation estimates* (cf. Goodman/Yang [G/Y]):

$$(3.67) \qquad |u|_j \le C_k (|u|_i)^{k-j/k-i}(|u|_k)^{j-i/k-i} \qquad \text{if } u \in \mathcal{F} \text{ and } i < j < k,$$

and:

$$(3.68) \qquad |u|_b |v|_c \le C_d (|u|_a |v|_d + |u|_d |v|_a) \qquad \text{if } u,v \in \mathcal{F},$$

with $a < b$, $c < d$, and $a+d = b+c$. Armed with these estimates, it is straightforward to show:

Proposition 3.13: Let M be a compact coordinate ball about the identity in a real Lie group and $\mathcal{B}$ a Banach algebra. Then $C^\infty(M, \mathcal{B})$, as in Proposition 3.11, forms a *smooth tame Fréchet algebra* with respect to the pointwise product. In particular, one has, for $f,g \in C^\infty(M, \mathcal{B})$:

$$(3.69) \qquad N_k[fg] \le C_k(N_0[f]\ N_k[g] + N_k[f]\ N_0[g]) .$$

Proof: The basic claim is that the product map:

$$C : C^\infty(M, \mathcal{B}) \times C^\infty(M, \mathcal{B}) \to C^\infty(M, \mathcal{B})$$
$$(f, g) \to (fg)(p) = f(p)g(p)$$

is a smooth tame map. It is smooth being bilinear, and the tameness of the tangent maps will follow from the tameness of C because the tangent maps are compositions of derivatives and products, which are tame maps. Hence, C will be smooth tame if it is tame. To see that C is tame, one just applies the Leibniz formula and then interpolates as per (3.68). Indeed:

$$(3.70) \qquad N_k[fg] \stackrel{d}{=} \max_{|\alpha|\leq k} \sup_{x\in\psi(M)} \| \partial_x^{\alpha}[(\psi^{-1})^*[fg]] \|_{\mathcal{B}},$$

and by applying the Leibniz formula:

$$= \max_{|\alpha|\leq k} \sup_{x\in\psi(M)} \| \sum_{\beta\leq\alpha} C_{\alpha\beta}\partial_x^{\alpha-\beta}[f(\psi^{-1}(x))]\partial_x^{\beta}[g(\psi^{-1}(x))] \|_{\mathcal{B}}$$

$$\leq \max_{|\alpha|\leq k} \sum_{\beta\leq\alpha} C_{\alpha\beta} \sup_{x\in\psi(M)} \| \partial_x^{\alpha-\beta}[f(\psi^{-1}(x))] \|_{\mathcal{B}} \| \partial_x^{\beta}[g(\psi^{-1}(x))] \|_{\mathcal{B}}$$

$$\leq \max_{|\alpha|\leq k} \sum_{\beta\leq\alpha} C_{\alpha\beta}\, N_{|\alpha-\beta|}[f]\, N_{|\beta|}[g]$$

$$\leq \sum_{\beta\leq k} C_{k\beta}\, N_{|k-\beta|}[f]\, N_{|\beta|}[g]\,,$$

where we have used that the seminorms form an increasing sequence and chosen $C_{k\beta} = \max_{|\alpha|\leq k} C_{\alpha\beta}$. Then we interpolate by $|\beta|$ units in each term of the sum to get the desired estimate (3.69). Q.E.D.

Proposition 3.14: If $\mathcal{B}$ is a C*-algebra with involution $*$, and M is a compact coordinate ball containing the identity in a real Lie group, then $C^\infty(M, \mathcal{B})$ forms a *smooth tame Fréchet *-algebra*. That is, $C^\infty(M, \mathcal{B})$ is $*$-invariant and the adjoint map defined by :

$$\Phi : C^\infty(M, \mathcal{B}) \to C^\infty(M, \mathcal{B}) \qquad (3.71)$$
$$f(p) \to (f(p))^*$$

is smooth tame. In particular, the adjoint map is seminorm preserving, i.e., $\forall\, k \geq 0, \forall\, f \in C^\infty(M, B)$:

$$N_k[\Phi(f)] \;=\; N_k[f]\,. \qquad (3.72)$$

Proof: That $C^\infty(M, \mathcal{B})$ is $*$-invariant is the result of the fact that in any local chart, the operations of taking derivatives and taking adjoints commute, hence the derivatives of adjoints are adjoints of derivatives of a smooth map, and hence the adjoint of a $C^\infty(M, \mathcal{B})$ map is also smooth. Also, since Φ is linear ($\mathcal{B}$ being a C*-algebra means that $*$ is linear), Φ will be smooth tame if it is tame because its tangent maps will be compositions of products, adjoints, and derivatives, which are all tame maps. Consequently, it remains to verify (3.72). Indeed:

$$N_k[\Phi(f)] \overset{\mathrm{d}}{=} \max_{|\alpha|\leq k}\ \sup_{x\in\psi(M)} \|\ \partial_x^\alpha[(\psi^{-1})^*[\Phi(f)]]\ \|_{\mathcal{B}} \qquad (3.73)$$

$$= \max_{|\alpha|\leq k}\ \sup_{x\in\psi(M)} \|\ \partial_x^\alpha[f(\psi^{-1}(x))^*]\ \|_{\mathcal{B}}\,.$$

For notational convenience, place $\tilde{f} = f(\psi^{-1}(x))$, and notice that:

$$\text{(3.74)} \qquad \partial_x^\alpha[(\tilde{f}(x))^*] = [\partial_x^\alpha(\tilde{f}(x))]^*.$$

Indeed:

$$\partial_{x_k}[(\tilde{f}(x))^*] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\ [\tilde{f}(x+\varepsilon e_k)]^* - [\tilde{f}(x)]^*\ \}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\ [\tilde{f}(x+\varepsilon e_k) - \tilde{f}(x)]^*\},$$

and, by the continuity of $*$ on $\mathcal{B}$, the above limit is $(\partial_{x_k}\tilde{f}(x))^*$. One may iterate this argument, making use of the smoothness of f and the continuity of $*$, to produce:

$$\text{(3.75)} \qquad N_k[\Phi(f)] = \max_{|\alpha|\leq k}\ \sup_{x\in\psi(M)} ||\ \{\partial_x^\alpha[f(\psi^{-1}(x))]\ \}^*\ ||_{\mathcal{B}}$$

$$= \max_{|\alpha|\leq k}\ \sup_{x\in\psi(M)} ||\ \partial_x^\alpha[f(\psi^{-1}(x))]\ ||_{\mathcal{B}},$$

which is just $N_k[f]$.

Q.E.D.

Remark 3.15: In what follows, we will find it convenient to notice that the zero order seminorm on $C^\infty(M, \mathcal{B})$, defined as in (3.62), that is:

$$\text{(3.76)} \qquad N_0[f] = \sup_{x\in\psi(M)} ||\ f(\psi^{-1}(x))\ ||_{\mathcal{B}},$$

is nothing other than the expression $\sup_{p\in M} ||\ f(p)\ ||_{\mathcal{B}}$.

Finally, in preparation for the final result of this section, we recall a definition of Hamilton:

Definition 3.16: A *smooth tame Lie group G* is a smooth tame Fréchet manifold, (i.e. a Hausdorff topological space with local charts valued in a tame Fréchet space such that the transition maps are smooth tame), supplied with an algebraic structure such that the product map:

(3.77) $$C: G \times G \to G, \text{ given by } C(g_1, g_2) = g_1 g_2,$$

and the inversion map:

(3.78) $$V: G \to G, \text{ given by } V(g) = g^{-1},$$

are smooth tame maps.

With this in mind, denote by $[C^{\infty}(M, \mathcal{B})]_*$ the subgroup of the smooth tame Fréchet algebra $C^{\infty}(M, \mathcal{B})$ consisting of the elements that are pointwise invertible, which is to say:

(3.79) $f \in [C^{\infty}(M, \mathcal{B})]_*$ if there exists $f^{-1} \in C^{\infty}(M, \mathcal{B})$ such that:

$$ff^{-1}(p) = f^{-1}f(p) = \mathrm{Id}(p) = I \quad \forall\, p \in M, \text{ I the identity in } \mathcal{B}.$$

Now, we have seen that $C^{\infty}(M, \mathcal{B})$ is a tame Fréchet space, and by standard Banach algebra considerations, (cf. Douglas **[D]**, Proposition 2.7), the set of invertible elements $[C^{\infty}(M, \mathcal{B})]_*$ is an open subset of $C^{\infty}(M, \mathcal{B})$. Thus, one may say that $[C^\infty(M, \mathcal{B})]_*$ is a smooth tame Fréchet manifold. Also, since the product map on $C^\infty(M, \mathcal{B}) \times C^\infty(M, \mathcal{B})$ is smooth tame by Proposition 3.13, $[C^\infty(M, \mathcal{B})]_*$ will form a smooth tame Lie group if the inversion map:

$$\text{(3.80)} \qquad V: [C^\infty(M, \mathcal{B})]_* \to [C^\infty(M, \mathcal{B})]_* \quad \text{defined by} \quad V(f) = f^{-1}$$

is smooth tame.

Proposition 3.17: For M a compact coordinate ball about the identity in a real Lie group and $\mathcal{B}$ a Banach algebra, the space $[C^\infty(M, \mathcal{B})]_*$, defined by (3.79), forms a *smooth tame Lie group in the sense of Hamilton.* In particular, for f near enough to the identity map: Id: $M \to \mathcal{B}$ with respect to $N_0[\bullet]$, the inversion map defined by (3.80) is smooth, and:

$$\text{(3.81)} \qquad N_k[f^{-1}] \le C_k (1 + N_k[f]) \quad \forall\, k \ge 0 \quad \text{and } f \in [C^\infty(M,\mathcal{B})]_*,$$

where C_k depends only on k and $N_0[f]$.

Proof: One can reduce the question of the tameness of V to its tameness near the identity map, as in Hamilton [H] for the case of the diffeomorphism group of a compact manifold, in the following manner.

If $f_o \in [C^\infty(M, \mathcal{B})]_*$ is fixed, then for f near f_o, $g = f_o^{-1}f$ will be near the identity map, and since $f = f_o g$, one has:

(3.82) $$f^{-1} = g^{-1}f_o^{-1} = (f_o^{-1}f)^{-1} ,$$

and:

(3.83) $$V(f) = C[\, V(C(\, V(f_o), f\,)),\, V(f_o)\,] ,$$

which is to say that V near f is built up from products in $[C^\infty(M, \mathcal{B})]_*$, which are smooth tame, and inverses of g near the identity. Then, one can generate the $k = 0$ estimate in the following way:

Claim 1: If f is near the identity in $N_0[\bullet]$ norm, then f^{-1} will also be near to the identity, and:

(3.84) $$N_0[f^{-1}] \le C_0\, N_0[f] .$$

Proof of Claim 1: First notice that since f is invertible by assumption, $f(p) \neq 0$ for all p in M, where 0 is the additive identity in the Banach algebra $\mathcal{B}$. Hence, $\| f(p) \|_{\mathcal{B}} \ge \delta_p > 0$ for all p in M since $\mathcal{B}$ is a Banach space. Moreover, since the norm on $\mathcal{B}$ is a continuous function and M is compact, the δ_p may be chosen uniformly, so that, in light of Remark 3.15:

(3.85) $$N_0[f] = \sup_{p \in M} \| f(p) \|_{\mathcal{B}} \ge \delta > 0.$$

Now, if one assumes that $N_0[Id - f] < \varepsilon < 1$, then:

$$(3.86) \qquad \| (Id - f)(p) \|_{\mathcal{B}} < \varepsilon < 1,$$

holds uniformly on M, and by standard considerations (cf. **[D]**, Proposition 2.5), one has:

$$(3.87) \qquad \| f^{-1}(p) \|_{\mathcal{B}} \leq \frac{1}{1 - \| (Id - f)(p) \|_{\mathcal{B}}} \qquad \forall\, p \in M,$$

or $N_0[f^{-1}] < \dfrac{1}{1 - \varepsilon}$, and since $N_0[f] \geq \delta > 0$, one has that:

$$(3.88) \qquad N_0[f^{-1}] \leq \frac{1}{(1 - \varepsilon)N_0[f]} N_0[f],$$

which is (3.84) with $C_0 = \dfrac{1}{(1 - \varepsilon)N_0[f]}$. Also, one finds that:

$$(3.89) \qquad N_0[f^{-1} - Id] = N_0[f^{-1}(Id - f)] < N_0[f^{-1}]\varepsilon < \frac{\varepsilon}{1 - \varepsilon},$$

whence f^{-1} is near the identity if f is; completing the claim.

This claim gives the 0-order tame estimates of the inversion map and specifies the dependence of the constant C_0 on the 0-norm of f. Using this estimate, one can produce the higher order estimates by induction; exploiting the Leibniz formula, the interpolation properties of the $C^\infty(M, \mathcal{B})$ norms, and the identity $f^{-1}f = ff^{-1} = Id$.

Claim 2: $N_k[f^{-1}] \leq C_k (1 + N_k[f])$, where C_k depends on k and $N_0[Id - f]$.

Proof of Claim 2: Given that :

$$N_k[f^{-1}] = \max_{|\alpha|\le k} \sup_{x\in\psi(M)} \| \partial_x^\alpha[f^{-1}(\psi^{-1}(x))] \|_{\mathcal{B}},$$

we will prove the estimate by induction on $|\alpha|$, where the $|\alpha| = 0$ case is Claim 1. We assume that, for $|\alpha| < k$:

$$N_{|\alpha|}[f^{-1}] \le C_{|\alpha|}(1 + N_{|\alpha|}[f]), \tag{3.90}$$

and we seek to verify (3.90) for $|\alpha| = k$. First, since $f^{-1}f = \mathrm{Id}$, one notices that for every $x \in \bar{\Omega} = \psi(M)$:

$$\partial_x^\alpha[(f^{-1}f)(\psi^{-1}(x))] = 0, \tag{3.91}$$

and, hence, by the Leibniz formula on the product of $f^{-1}(\psi^{-1}(x))$ and $f(\psi^{-1}(x))$:

$$0 = \partial_x^\alpha[(f^{-1}f)(\psi^{-1}(x))] = \sum_{\beta\le\alpha} C_{\alpha\beta}\partial_x^\beta[f^{-1}(\psi^{-1}(x))]\,\partial_x^{\alpha-\beta}[f(\psi^{-1}(x))], \tag{3.92}$$

and by splitting off the top derivative of f^{-1}:

$$\partial_x^\alpha[f^{-1}(\psi^{-1}(x))]\,f(\psi^{-1}(x)) = -\sum_{\beta<\alpha} C_{\alpha\beta}\partial_x^\beta[f^{-1}(\psi^{-1}(x))]\,\partial_x^{\alpha-\beta}[f(\psi^{-1}(x))]. \tag{3.93}$$

Then, by multiplying both sides by $f^{-1}(\psi^{-1}(x))$ and taking $\mathcal{B}$ norms:

$$\| \partial_x^\alpha[f^{-1}(\psi^{-1}(x))] \|_{\mathcal{B}} \le \tag{3.94}$$

$$\| f^{-1}(\psi^{-1}(x)) \|_{\mathcal{B}} \Big\{ \sum_{\beta<\alpha} C_{\alpha\beta} \| \partial_x^\beta[f^{-1}(\psi^{-1}(x))] \|_{\mathcal{B}} \| \partial_x^{\alpha-\beta}[f(\psi^{-1}(x))] \|_{\mathcal{B}}.$$

Hence:

(3.95)
$$N_k[f^{-1}] = \max_{|\alpha|\le k} \sup_{x\in\psi(M)} \| \partial_x^\alpha [f^{-1}(\psi^{-1}(x))] \|_{\mathcal{B}}$$

$$\le N_0[f^{-1}] \left\{ \max_{|\alpha|\le k} \sum_{\beta<\alpha} C_{\alpha\beta} N_{|\beta|}[f^{-1}] N_{|\alpha-\beta|}[f] \right\},$$

where we have used the definition of the seminorms $N_k[\cdot]$ to say that the supremum over $\psi(M)$ of any derivative of length $|\beta|$ is bounded by the $|\beta|$ seminorm. Then, by applying the induction hypothesis since $|\beta|< k$ and the estimate on the 0 order norm from claim 1, one finds:

(3.96)
$$N_k[f^{-1}] \le C_0 N_0[f] \left\{ \max_{|\alpha|\le k} \sum_{\beta<\alpha} C_{\alpha\beta} C_{|\beta|} (1 + N_{|\beta|}[f]) N_{|\alpha-\beta|}[f] \right\}$$

$$\le C_0 N_0[f] \left\{ \sum_{|\beta|<k} C_{k\beta} (1 + N_{|\beta|}[f]) N_{k-|\beta|}[f] \right\}.$$

where we have the fact that the seminorms form an increasing sequence when taking the maximum over the length of α. The quantity on the right of (3.96) is then rearranged to set up the interpolation:

$$= C_0 N_0[f] \left\{ \sum_{|\beta|<k} C_{k\beta} (N_{k-|\beta|}[f] + N_{|\beta|}[f] N_{k-|\beta|}[f]) \right\}$$

$$\le C_0 N_0[f] \left\{ C_k N_k[f] + \sum_{|\beta|<k} C_{k\beta} N_{|\beta|}[f] N_{k-|\beta|}[f]) \right\},$$

where we have chosen the constant C_k as the maximum of the constants $C_{k\beta}$, and used the increasing sequence property again. Finally, we interpolate the products in the sum by $|\beta|$ units to find:

$$(3.97) \qquad N_k[f^{-1}] \leq C_0 N_0[f] \left\{ C_k N_k[f] + \sum_{|\beta|<k} C'_{k\beta} N_0[f]\, N_k[f] \right\}$$

$$\leq C'_k (1 + N_k[f]) ,$$

with C'_k depending only on k and $N_0[f]$, as claimed. Q.E.D.

4. Smooth Tame Structures on Pseudodifferential Operators

We are now ready to exploit the characterization of the pseudodifferential algebra $OPS^{0,0}_{gs}$ as a ψ^*-subalgebra of $\mathcal{L}(L^2(\mathbf{R}^n))$ with the tameness properties of the spaces $C^\infty(M, \mathcal{B})$ to conclude that $OPS^{0,0}_{gs}$ has the structure of a smooth tame ψ^*-subalgebra of $\mathcal{L}(L^2(\mathbf{R}^n))$ and that $[OPS^{0,0}_{gs}]_*$, defined as the subset of $OPS^{0,0}_{gs}$ consisting of operators whose inverses exist in $\mathcal{L}(L^2(\mathbf{R}^n))$, has the structure of a smooth tame Lie group. To accomplish this, we state a sequence of results which formalizes much of what has already been done. In particular, we show that $OPS^{0,0}_{gs}$ is a tame Fréchet space in Proposition 4.1 and that it is a smooth tame Fréchet algebra in Proposition 4.2. Then we investigate the smooth tameness of the inversion map on various subspaces of $OPS^{0,0}_{gs}$, by exploiting the ψ^*-subalgebra property, which is proven in Proposition 4.3. We conclude with the smooth tame Lie group structure of $[OPS^{0,0}_{gs}]_*$ in Theorem 4.9.

Proposition 4.1: $OPS^{0,0}_{gs}$, together with its operator topology (2.9), forms a tame Fréchet space in the sense of Definition 2.1; in particular, one may interpolate in the operator norms as per (3.68).

Proof: The idea is to show that there is a tame isomorphism between $OPS^{0,0}_{gs}$ and a closed subspace of the tame Fréchet space $C^\infty(M, \mathcal{B})$ for M a compact coordinate ball about the identity e in the Lie group gs' and $\mathcal{B}$ the Banach space $\mathcal{L}(L^2(R^n))$ with its usual norm.

Indeed, choose M to be a compact coordinate ball about the identity e in the Lie group gs'. In particular, choose M so that it lies entirely within an open ball U on which the local coordinate system $\psi(\sigma,o,z,\zeta) = (\sigma,h,z,\zeta)$ is valid, as addressed in Remark 1.5, shrinking U if necessary so that the transition formulas of Lemma 1.8 are also valid. Using this coordinate system, a family of seminorms on $C^\infty(M, \mathcal{B})$ is given by:

(4.1) $$N_k'[A] = \max_{|\alpha|\leq k} \sup_{\psi(M)} \| \partial^\alpha[A(\sigma,h,z,\zeta)] \|_{L^2\text{-op}},$$

where ∂^α is a coordinate derivative of length α.

Now, $OPS^{0,0}_{gs}$ is characterized by:

(4.2) $$OPS^{0,0}_{gs} = \{ A \in \mathcal{B} : A_{\sigma,h,z,\zeta} \in C^\infty(gs', \mathcal{B}) \},$$

where $A_{\sigma,h,z,\zeta} = U^{-1}(\sigma,h,z,\zeta)AU(\sigma,h,z,\zeta)$ and U is the strongly continuous unitary representation of gs' on $\mathcal{B}$ used in the Cordes criterion. The operator topology on $OPS^{0,0}_{gs}$ was given by:

(4.3) $$N_k[A] = \max_{|\alpha|\leq k} \| \partial^\alpha[A_{\sigma,h,z,\zeta}]\,|_e \|_{L^2\text{-op}}.$$

Let $\mathcal{F}$ denote the tame Fréchet space $\{C^\infty(M, \mathcal{B}), N_k'[\cdot]\}$ and define a subspace $\mathcal{G}$ of $\mathcal{F}$ as the image of $OPS^{0,0}_{gs}$ under the conjugation by $U(\sigma,h,z,\zeta)$. That is, set:

(4.4) $$\mathcal{G} = \{A(\sigma,h,z,\zeta) : A(\sigma,h,z,\zeta) = U^{-1}(\sigma,h,z,\zeta)\, A\, U(\sigma,h,z,\zeta) \text{ for some } A \in OPS^{0,0}_{gs}\},$$

where we define the map:

(4.5) $$\Phi \; : \; OPS^{0,0}_{gs} \rightarrow \mathcal{G} \, ,$$
$$A \quad \rightarrow \quad A(\sigma,h,z,\zeta) = A_{\sigma,h,z,\zeta} \, ,$$

where we will temporarily preserve the distinction between an arbitrary function $A(\sigma,h,z,\zeta)$ in $\mathcal{F}$ and those $A_{\sigma,h,z,\zeta}$ coming from $OPS^{0,0}_{gs}$.

Claim 1: $\mathcal{G}$ is a closed subspace of $\mathcal{F}$.

Proof of Claim 1: By Theorem 1.2, the conjugation by U of an operator in $OPS^{0,0}_{gs}$ is a smooth map in a neighborhood of M with values in $\mathcal{B}$, hence Φ is well defined, and $\mathcal{G}$ is a subspace. To show that $\mathcal{G}$ is closed, one demonstrates that $\mathcal{G}$ contains its limit points.

Let $A_j(\sigma,h,z,\zeta)$ be a sequence in $\mathcal{G}$ that converges with respect to the $\mathcal{F}$ topology. Since $\mathcal{F}$ is a Fréchet space, the sequence has a limit, call it $A_\infty(\sigma,h,z,\zeta)$, that belongs to $\mathcal{F}$. As such, this function A_∞ is smooth in a neighborhood of M taking values in $\mathcal{B}$. The question remains as to whether this function looks like the conjugation of an element in $\mathcal{B}$. However, the function $A_\infty(\sigma,h,z,\zeta)$ is the uniform limit over M as $j \rightarrow \infty$ of :

(4.6) $$U^{-1}(\sigma,h,z,\zeta) \, A_j \, U(\sigma,h,z,\zeta),$$

which, by the strong continuity of the representation $U(\sigma,h,z,\zeta)$ is nothing other than:

(4.7) $$U^{-1}(\sigma,h,z,\zeta) \, A_\infty \, U(\sigma,h,z,\zeta),$$

with $A_\infty = \lim_{j\to\infty} [A_j]$, the limit being in the $\mathcal{B}$ norm, and so the operator A_∞ belongs to $\mathcal{B}$. Hence, the function $A_\infty(\sigma,h,z,\zeta)$ is a smooth map in a neighborhood of M that is the conjugation by $U(\sigma,h,z,\zeta)$ of an element in $\mathcal{B}$, and thus belongs to $\mathcal{G}$. Therefore, $\mathcal{G}$ is closed.

Claim 2: Φ is a tame isomorphism between $\{OPS^{0,0}_{gs}, N_k[\bullet]\}$ and $\{\mathcal{G}, N'_k[\bullet]\}$.

Proof of Claim 2: Φ maps $OPS^{0,0}_{gs}$ onto $\mathcal{G}$ by definition as $\mathcal{G}$ is the image of $OPS^{0,0}_{gs}$ under Φ. Φ is also one-to-one since:

(4.8) $$U^{-1}(\sigma,h,z,\zeta)\ A\ U(\sigma,h,z,\zeta) \ = \ U^{-1}(\sigma,h,z,\zeta)\ B\ U(\sigma,h,z,\zeta)$$

if and only if:

(4.9) $$A \ = \ B.$$

In fact, one has the inverse for Φ defined by:

(4.10) $$\Psi : G \to OPS^{0,0}_{gs}$$
$$A_{\sigma,h,z,\zeta} \to A_{\sigma,h,z,\zeta}|_e = A.$$

To complete the claim, we need to show that the maps Φ and Ψ are tame.

Notice that:

(4.11) $$N_k[\ \Psi[A_{\sigma,h,z,\zeta}]] \ = \ \max_{|\alpha|\le k} ||\ \partial^\alpha[A_{\sigma,h,z,\zeta}]\,|_e\ ||_{L^2\text{-op}}$$
$$\le \ \max_{|\alpha|\le k} \sup_{\psi(M)} ||\ \partial^\alpha[A_{\sigma,h,z,\zeta}]\ ||_{L^2\text{-op}}$$
$$= \ N'_k[A],$$

where the inequality comes from the fact that the value of the derivatives at the identity are trivially bounded by those over a neighborhood, and the equality is just the definition of the N'_k norm of the function $A(\sigma,h,z,\zeta) = A_{\sigma,h,z,\zeta}$ in $\mathcal{G}$. Finally, by the transition formulas of Lemma 1.8 which hold over M, one has:

$$(4.12) \qquad N'_k[\Phi(A)] = \max_{|\alpha|\le k} \sup_{\psi(M)} \| \partial^\alpha [A_{\sigma,h,z,\zeta}] \|_{L^2\text{-op}}$$

$$\le \max_{|\alpha|\le k} \sup_{\psi(M)} \| \sum_\beta \varphi_{\alpha\beta}(\sigma,h,z,\zeta)\, \partial^\alpha [A_{\sigma,h,z,\zeta}]\,|_e \|_{L^2\text{-op}} ,$$

where the functions $\varphi_{\alpha\beta}$ are smooth real valued functions in a neighborhood of $e \in gs'$, and hence bounded over the compact set $\psi(M)$, so:

$$\le \max_{|\alpha|\le k} C_k \| \partial^\alpha [A_{\sigma,h,z,\zeta}]\,|_e \|_{L^2\text{-op}} = C_k N_k[A].$$

Q.E.D.

Given that one may now interpolate in the $OPS^{0,0}_{gs}$ norms, one might proceed to verify the following tame estimates directly on $OPS^{0,0}_{gs}$ by mocking the section 3 arguments in this case. Rather than repeating them, we choose to translate the previous results by way of the inequalities (4.11) and (4.12), which we can be stated for further reference as: $\forall\, A \in OPS^{0,0}_{gs}$:

$$(4.13) \qquad N_k[A] \le N'_k[A_{\sigma,h,z,\zeta}]$$

and:

$$(4.14) \qquad N'_k[A_{\sigma,h,z,\zeta}] \le C_k N_k[A] .$$

Proposition 4.2: $OPS^{0,0}_{gs}$, equipped with its operator topology, forms a smooth tame Fréchet algebra. In particular, the product map:

$$C : OPS^{0,0}_{gs} \times OPS^{0,0}_{gs} \to OPS^{0,0}_{gs} \,,$$

given by $C(A, B) = AB = A \circ B$, is *smooth tame*, and for $A, B \in OPS^{0,0}_{gs}$:

$$(4.15) \qquad N_k[AB] \le C_k(N_k[A]\, N_0[B] + N_0[A]\, N_k[B]) \qquad \forall\, k \ge 0.$$

Proof: By Proposition 4.1, one knows that $\{ OPS^{0,0}_{gs}, N_k[\bullet] \}$ forms a tame Fréchet space, and for $A, B \in OPS^{0,0}_{gs}$:

$$(4.16) \qquad (AB)_{\sigma,h,z,\zeta} = (T_{\sigma,h,z,\zeta})^{-1}(AB)T_{\sigma,h,z,\zeta}$$

$$= (T_{\sigma,h,z,\zeta})^{-1}AT_{\sigma,h,z,\zeta}(T_{\sigma,h,z,\zeta})^{-1}BT_{\sigma,h,z,\zeta}$$

$$= A_{\sigma,h,z,\zeta}\, B_{\sigma,h,z,\zeta} \,.$$

Therefore, the composition map makes $OPS^{0,0}_{gs}$ into an algebra since $AB \in \mathcal{L}(\mathcal{H})$ if A and B are, and (4.16) shows that $(AB)_{\sigma,h,z,\zeta} \in C^\infty(gs', \mathcal{L}(\mathcal{H}))$, being the composition of two such maps. As a result, one only needs to show that C is a smooth tame map; however, C being bilinear will be smooth tame if it is tame, as was indicated in the proof of Proposition 3.13. Consequently, it suffices to verify the estimate (4.15).

Now, after choosing M as above, one makes use of Proposition 3.15 for this M and $\mathcal{B} = \mathcal{L}(\mathcal{H})$. Indeed, by (4.13) and (4.16):

$$(4.17) \qquad N_k[AB] \leq N'_k[(AB)_{\sigma,h,z,\zeta}] = N'_k[A_{\sigma,h,z,\zeta}B_{\sigma,h,z,\zeta}] .$$

Now, both $A_{\sigma,h,z,\zeta}$ and $B_{\sigma,h,z,\zeta}$ belong to $C^\infty(M, \mathcal{B})$, so one may apply the product estimate (3.69) to the last quantity in (4.17) to yield:

$$(4.18) \quad N_k[AB] \leq C_k(N'_k[A_{\sigma,h,z,\zeta}]\, N'_0[B_{\sigma,h,z,\zeta}] + N'_0[A_{\sigma,h,z,\zeta}]\, N'_k[B_{\sigma,h,z,\zeta}]) .$$

Finally, one applies (4.14) to conclude the result. Q.E.D.

Remark 4.3: $OPS^{0,0}_{gs}$ is more than just subalgebra of $\mathcal{L}(\mathcal{H})$. Propositions 4.1 and 4.2 justify calling $OPS^{0,0}_{gs}$ a *smooth tame Fréchet subalgebra of* $\mathcal{L}(\mathcal{H})$. In addition, it is easy to show that $OPS^{0,0}_{gs}$ is invariant under the L^2-adjoint, and that this adjoint map is smooth tame; therefore, one can say that $OPS^{0,0}_{gs}$ forms a *smooth tame Fréchet *-subalgebra of the C*-algebra* $\mathcal{L}(\mathcal{H})$. Moreover, as is indicated by Cordes in **[C2]** and [C3], $OPS^{0,0}_{gs}$ possesses the *ψ*–subalgebra property of Gramsch* [G]. This means that, in addition to being a * -invariant subalgebra of $\mathcal{L}(\mathcal{H})$ with a Fréchet space topology, it is closed under inversion in $\mathcal{L}(\mathcal{H})$. It is this last property, the so called *relative inversion* that we wish to exploit in what follows.

Proposition 4.4: $OPS^{0,0}_{gs}$, together with its operator topology, forms a smooth tame ψ^*-subalgebra of the C*-algebra $\mathcal{L}(L^2(\mathbf{R}^n))$.

Proof: First notice that $OPS^{0,0}_{gs}$ is indeed invariant under the L^2- adjoint since:

$$(4.19) \qquad (A_{\sigma,h,z,\zeta})^* = [(T_{\sigma,h,z,\zeta})^{-1}AT_{\sigma,h,z,\zeta}]^* = (T_{\sigma,h,z,\zeta})^{-1}A^*T_{\sigma,h,z,\zeta}$$

$$= (A^*)_{\sigma,h,z,\zeta},$$

because the representation T is unitary. Thus $(A^*)_{\sigma,h,z,\zeta}$ is the adjoint of the smooth map $A_{\sigma,h,z,\zeta}$ from gs' into $\mathcal{L}(\mathcal{H})$. As a result, the GS' conjugation of A* is also a smooth map from gs' into $\mathcal{L}(\mathcal{H})$, or $A^* \in OPS^{0,0}_{gs}$. Furthermore, the smooth tame property of the adjoint map follows from (4.19), Proposition 3.14, and the inequalities (4.13) and (4.14). Indeed, by the seminorm inequality (4.13) and the adjoint identity (4.19), one has:

$$(4.20) \qquad N_k[A^*] \leq N'_k[(A)^*_{\sigma,h,z,\zeta}] = N'_k[(A_{\sigma,h,z,\zeta})^*],$$

and by the adjoint estimate (3.92) applied to the function $A_{\sigma,h,z,\zeta}$ and the seminorm inequality (4.14), this is:

$$= N'_k[A_{\sigma,h,z,\zeta}] \leq C_k N_k[A].$$

It remains to demonstrate the relative inversion property. That is, if $A \in OPS^{0,0}_{gs}$ has an inverse $A^{-1} \in \mathcal{L}(\mathcal{H})$, does it follow that $A^{-1} \in OPS^{0,0}_{gs}$ as well? In particular, since:

$$OPS^{0,0}_{gs} = \{ A \in \mathcal{L}(\mathcal{H}) : A_{\sigma,h,z,\zeta} \in C^{\infty}(gs', \mathcal{L}(\mathcal{H})) \},$$

one only needs to verify that $(A^{-1})_{\sigma,h,z,\zeta} \in C^{\infty}(gs', \mathcal{L}(\mathcal{H}))$ because $A^{-1} \in \mathcal{L}(\mathcal{H})$ by assumption. However:

$$(4.21) \qquad (A^{-1})_{\sigma,h,z,\zeta} = (T_{\sigma,h,z,\zeta})^{-1} A^{-1} T_{\sigma,h,z,\zeta} = [\, (T_{\sigma,h,z,\zeta})^{-1} A T_{\sigma,h,z,\zeta} \,]^{-1}$$

$$= (A_{\sigma,h,z,\zeta})^{-1},$$

so that $(A^{-1})_{\sigma,h,z,\zeta}$, which is well defined for $A^{-1} \in \mathcal{L}(\mathcal{H})$, is just the inverse of the smooth map $A_{\sigma,h,z,\zeta}$,and hence smooth by the Banach space inverse function theorem.

Q.E.D.

Remark 4.5: Of course, one can go on to say that if A^{-1} and B^{-1} exist in $\mathcal{L}(\mathcal{H})$ for A and B in $OPS^{0,0}_{gs}$, then:

$$A^{-1} \text{ and } B^{-1} \text{ belong to } OPS^{0,0}_{gs},$$

and since $(AB)^{-1} = B^{-1}A^{-1} \in \mathcal{L}(\mathcal{H})$, one may invoke the algebra result of Proposition 4.2 to conclude that $(AB)^{-1} \in OPS^{0,0}_{gs}$ because :

$$(4.22) \qquad [(AB)^{-1}]_{\sigma,h,z,\zeta} = (B^{-1})_{\sigma,h,z,\zeta} (A^{-1})_{\sigma,h,z,\zeta} .$$

That is, the GS' conjugation of $(AB)^{-1}$ is the composition of $C^\infty(gs', \mathcal{L}(\mathcal{H}))$ maps. In this way, one can adapt Proposition 3.17 to show that $[OPS^{0,0}_{gs}]_*$ forms a smooth tame Lie group, as will be done in Theorem 4.10. However, in preparation for this, we wish to exploit the ψ*-subalgebra property of $OPS^{0,0}_{gs}$ to distinguish some subcollections of $OPS^{0,0}_{gs}$ for which the inversion map is well defined and smooth tame, without explicitly assuming the invertibility in $\mathcal{L}(\mathcal{H})$. The following three propositions give sufficient conditions for which one has a smooth tame inversion.

Proposition 4.6: Let :

(4.23) $$\mathcal{A}_0 \overset{\mathrm{d}}{=} \{ A \in OPS^{0,0}_{gs} : \ || A - Id ||_{L^2\text{-op}} < 1 \}.$$

Then, the map :

(4.24) $$V : \mathcal{A}_0 \to OPS^{0,0}_{gs} \qquad \text{given by} \qquad V(A) = A^{-1}$$

is well defined and a smooth tame map. In particular, one has: $\forall\ A \in \mathcal{A}_0$:

(4.25) $$N_k[A^{-1}] \le C_k (1 + N_k[A]),$$

with C_k depending on k and $N_0[A] = || A ||_{L^2\text{-op}}$.

Proof: First, one notices that if $|| A - Id ||_{L^2\text{-op}} < 1$, then $A^{-1} \in \mathcal{L}(\mathcal{H})$ exists because $\mathcal{L}(\mathcal{H})$ is a Banach algebra (cf. Proposition 2.5 of [**D**]). Therefore, by the ψ*-subalgebra property, $A^{-1} \in OPS^{0,0}_{gs}$. Hence, V is well defined as a map from $\mathcal{A}_0$ into $OPS^{0,0}_{gs}$.

As for V being smooth tame, notice that if $A, B \in \mathcal{A}_0$, then:

$$\text{(4.26)} \qquad DV(A)(B) \stackrel{d}{=} \lim_{h \to 0} \frac{1}{h}\left(V(A + hB) - V(A) \right) = -A^{-1}A^{-1}B.$$

Indeed, the triangle inequality:

$$\| A + hB - Id \|_{L^2\text{-op}} \leq \| A - Id \|_{L^2\text{-op}} + \| hB \|_{L^2\text{-op}}$$

and $B \in \mathcal{A}_0$:

$$\| B \|_{L^2\text{-op}} = \| (B - Id) + Id \|_{L^2\text{-op}} < 2,$$

give, for h small enough:

$$\| (A + hB) - Id \|_{L^2\text{-op}} < 1,$$

and so, $(A + hB)^{-1}$ exists. Furthermore, the difference quotient in (4.26) may be written as:

$$\text{(4.27)} \qquad \frac{1}{h} A^{-1}(-hB)(A + hB)^{-1} = A^{-1}(-B)(A + hB)^{-1},$$

which has the advertised limit. Given this inversion formula (4.26), one sees that the tangent maps to V are given by products of inverses on $\mathcal{A}_0$. Thus, the tangent maps will be tame if V itself is tame, as taking products in $\mathcal{A}_0$ is a tame operation by Proposition 4.2.

We produce the estimate (4.25) from Proposition 3.19 and the seminorm inequalities (4.13) and (4.14), in the spirit of Proposition 4.2. For $A \in \mathcal{A}_0$:

$$N_k[A^{-1}] \leq N_k'[(A^{-1})_{\sigma,h,z,\zeta}] , \tag{4.28}$$

by the seminorm inequality (4.13). Now, by invoking Proposition 3.17 concerning the tame inversion in $C^\infty(M, \mathcal{L}(\mathcal{H}))$, the estimate (3.81) applied to the last quantity in (4.28) yields :

$$N_k[A^{-1}] \leq C_k(1 + N_k'[A_{\sigma,h,z,\zeta}]), \tag{4.29}$$

where the constant C_k depends only on k and the L^2-operator norm of $A_{\sigma,h,z,\zeta}$. Finally, the desired estimate follows from applying the seminorm inequality (4.14) to bound (4.29) by C^k noms at the identity. Q.E.D.

Remark 4.7: The collection $\mathcal{A}_0$ is not itself a group, or even a subalgebra of $OPS^{0,0}_{gs}$ as one might hope, but it does represent something useful; namely, one knows that inversion in $OPS^{0,0}_{gs}$ is a smooth tame operation provided one is near the identity in L^2-operator norm. One can also build extensions of V to other subcollections by defining $\mathcal{A}_0$ as above, and then iterating by:

$$\mathcal{A}_k = \{ A_k \in OPS^{0,0}_{gs} : || A_k A_{k-1} - Id ||_{L^2\text{-op}} < 1 \text{ for some } A_{k-1} \in \mathcal{A}_{k-1} \}. \tag{4.30}$$

One can show that $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ and that the inverse map V extends to any such $\mathcal{A}_k$. We indicate the first extension in the following proposition.

Proposition 4.8: Let:

(4.31) $\mathcal{A}_1 = \{ B \in OPS^{0,0}_{gs} : || BA^{-1} - Id ||_{L^2} < 1, \text{ for some } A \in \mathcal{A}_0 \}$,

with $\mathcal{A}_0$ as in Proposition 4.6. then the inversion map V extends to a smooth tame (right) inverse on $\mathcal{A}_1$ with the following estimate valid $\forall B \in \mathcal{A}_1$:

(4.32) $$N_k[B^{-1}] \leq C_k(1 + N_k[B]) \qquad \forall k \geq 0.$$

Proof: First notice that $\mathcal{A}_1$ is well defined because if $A \in \mathcal{A}_0$, then A^{-1} exists by Proposition 4.6. Then notice that V is well defined on $\mathcal{A}_1$, which is to say that: if $B \in \mathcal{A}_1$, then B^{-1} exists and lies in $OPS^{0,0}_{gs}$. By the ψ*-subalgebra property, one only needs to show that $B^{-1} \in \mathcal{L}(\mathcal{H})$ exists. However, the condition on B that $|| BA^{-1} - I ||_{L^2\text{-op}} < 1$ implies that $(BA^{-1})^{-1}$ exists in $\mathcal{L}(\mathcal{H})$, and so one may define a (right) inverse by :

(4.33) $$B^{-1} = A^{-1}(BA^{-1})^{-1}.$$

Hence, the inverse V on $\mathcal{A}_0$ extends to a (right) inverse on $\mathcal{A}_1$, in light of the obvious inclusion $\mathcal{A}_0 \subset \mathcal{A}_1$ (i.e., if B is already in $\mathcal{A}_0$, pick A = Id in $\mathcal{A}_0$ for the $\mathcal{A}_1$ condition). The estimate (4.32) is then clear for $B \in \mathcal{A}_1$:

(4.34) $$N_k[B^{-1}] \leq C_k(N_k[A^{-1}]\, N_0[(BA^{-1})^{-1}] + N_0[A^{-1}]\, N_k[(BA^{-1})^{-1}]),$$

by applying the tameness of products in $OPS^{0,0}_{gs}$ to (4.33). Then, since both A and $(A^{-1}B)$ belong to $\mathcal{A}_0$, one can use Proposition 4.6 to estimate the inverses in the right hand side of (4.34).

Indeed:

(4.35) $$N_k[B^{-1}] \le C'_k \{(1+N_k[A])\, N_0[BA^{-1}] + N_0[A]\,(1+N_k[BA^{-1}])\},$$

and by absorbing the norms of A into the constant:

$$\le C''_k (1 + N_0[BA^{-1}] + N_k[BA^{-1}]),$$

and by taking products again:

$$\le C^{(3)}_k (1 + N_0[B]\, N_0[A^{-1}] + N_k[B]\, N_0[A^{-1}] + N_0[B]\, N_k[A^{-1}]).$$

Now, applying (4.25) again to A^{-1} yields:

$$N_k[B^{-1}] \le C^{(4)}_k \{1 + N_0[B]\, N_0[A] + N_k[B]\, N_0[A] + N_0[B]\,(1+N_k[A])\},$$

and by absorbing $N_0[B]$ and the norms of A into the constant, one obtains:

(4.36) $$N_k[B^{-1}] \le C^{(5)}_k (1 + N_k[B])$$ Q.E.D.

Proposition 4.9: Let:

(4.37) $$\mathcal{B}_1 \overset{d}{=} \{ B \in OPS^{0,0}_{gs} : \| B - A \|_{L^2\text{-op}} < 1/\| A^{-1} \|_{L^2\text{-op}} \text{ with } A \in \mathcal{A}_0 \}$$

Then the inversion map V of Proposition 4.6 extends to a smooth tame inverse on $\mathcal{B}_1$ with the estimate (4.32) valid for all $B \in \mathcal{B}_1$.

Proof: If $B \in \mathcal{B}_1$, then :

$$(4.38) \qquad \|BA^{-1} - I\|_{L^2\text{-op}} = \|(B - A)A^{-1}\|_{L^2\text{-op}}$$

$$\leq \|B - A\|_{L^2\text{-op}} \|A^{-1}\|_{L^2\text{-op}} < 1,$$

and :

$$(4.39) \qquad \|A^{-1}B - I\|_{L^2\text{-op}} = \|A^{-1}(B - A)\|_{L^2\text{-op}}$$

$$\leq \|A^{-1}\|_{L^2\text{-op}} \|B - A\|_{L^2\text{-op}} < 1.$$

Therefore, $(BA^{-1})^{-1}$ and $(A^{-1}B)^{-1}$ exist and lie in $\mathcal{A}_0$; so, one may define right and left inverses by:

$$(4.40) \qquad B_R^{-1} = A^{-1}(BA^{-1}) \qquad \text{and} \qquad B_L^{-1} = (A^{-1}B)^{-1}A^{-1},$$

and estimate $N_k[B_R^{-1}]$ and $N_k[B_L^{-1}]$ in the manner of Proposition 4.8. Q.E.D.

Finally, we conclude with the main result, which is merely a recapitulation of the previous considerations.

Theorem 4.10: Let :

$$(4.41) \qquad \mathcal{G} = [OPS^{0,0}_{gs}]_* = \{ A \in OPS^{0,0}_{gs} : A^{-1} \in \mathcal{L}(L^2(\mathbf{R}^n)) \text{ exists } \}.$$

Then $\mathcal{G}$ forms a smooth tame Lie group with respect to its grading by the $C^k(gs', \mathcal{L}(L^2(\mathbf{R}^n)))$ seminorms in the sense of Definition 3.16.

Proof: First of all, $\mathcal{G}$ is a subgroup of the smooth tame Fréchet algebra $OPS^{0,0}_{gs}$ as argued in Remark 4.5. Furthermore, $\mathcal{G}$ has a Fréchet manifold structure because it sits as an open set inside the Fréchet space $OPS^{0,0}_{gs}$. Consequently, $\mathcal{G}$ will be a smooth tame Lie group if the inversion map, which is well defined on all of $\mathcal{G}$ is a smooth tame map, but this is clear in light of Propositions 4.6 and 4.8 as the following argument indicates.

For any $A \in \mathcal{G}$, fix an A_0 with $|| A - A_0 ||_{L^2\text{-op}} < (|| A_0^{-1} ||_{L^2\text{-op}})^{-1}$. Such an A_0 exists since the group of invertible elements of the Banach algebra $\mathcal{L}(L^2(\mathbf{R}^n))$ is open with respect to the L^2-operator norm (cf. Douglas [**D**], Proposition .2.7). Then put $B = A_0^{-1}A$ so that :

(4.42) $$|| B - I ||_{L^2\text{-op}} = || A_0^{-1}A - I ||_{L^2\text{-op}} = || A_0^{-1}(A - A_0^{-1}) ||_{L^2\text{-op}} < 1,$$

and so $B \in \mathcal{A}_0$, with $\mathcal{A}_0$ as in Proposition 4.6. Hence $V(B) = B^{-1}$ defines a smooth tame map, and:

(4.43) $$A^{-1} = B^{-1}A_0^{-1} = (A_0^{-1}A)^{-1}A_0^{-1} ,$$

so that the inversion of $A \in \mathcal{G}$ is just the product of the inversion map on $B \in \mathcal{A}_0$ and the inverse of a fixed element A_0; that is :

(4.44) $$V(A) = C(V(C(V(A_0), A)), V(A_0)).$$

As such, for all A near a fixed $A_0 \in \mathcal{G}$, one produces the estimate:

$$(4.45) \qquad N_k[A^{-1}] \le C_k(1 + N_k[A]) \qquad \text{for all } k \ge 0,$$

where C_k depends only on k and $\| A - A_0 \|_{L^2\text{-op}}$, and, therefore, $V : \mathcal{G} \to \mathcal{G}$ is tame. Finally, by considerations like those in Proposition 4.6, V is smooth tame.

Q.E.D.

References

[B1] R. Beals - *A general calculus of pseudo-differential operators*, Duke Math. J. **42** (1975), pp. 1-42.

[B2] ______ - *Characterization of pseudodifferential operators and applications*, Duke Math. J. **44** (1977), pp. 45-57; correction, Duke Math J. **46** (1979), p.215.

[BGY] R. Bryant, P. Griffiths, D. Yang - *Characteristics and existence of isometric embeddings*, Duke Math J. **50** (1983), pp. 893-994.

[Co] A. Connes - *C* algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris Série A, **290** (1980), pp. 599-604.

[C1] H.O. Cordes, - *On pseudodifferential operators and smoothness of special Lie group representations*, Manuscripta Math. **28** (1979), pp. 51-69.

[C2] __________ - *On some C*-algebras and Fréchet*-algebras of pseudodifferential operators*, Proc. Sympos. Pure Math., Vol. 43, Amer. Math. Soc., Providence, R.I., 1985, pp. 79-104.

[C3] __________ - *Seminar notes*, Univ. of California at Berkeley, 1984.

[C4] __________ - *On the compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975), pp. 115-131.

[D] R. Douglas - *C*-algebra techniques in operator theory*, Academic Press, New York, 1972.

[Du] J. Dunàu - *Fonctions d'un operateur elliptique sur une varietée compacte*, J. Math. Purees Appl. **56** (1977), pp.367-391.

[F] G. Folland - *Harmonic analysis in phase space*, Ann. of Math. Studies, Number 122, Princeton Univ. Press, Princeton, N.J., 1989.

[G/Y] J. Goodman / D. Yang - *Local solvability of nonlinear partial differential equations of real principal type*, preprint.

[G] B. Gramsch - *Relative Inversion in der Störungstheorie von Operatoren und ψ- Algebren*, Math. Ann. **269** (1984), pp. 27-71.

[H] R. S. Hamilton - *The inverse function theorem of Nash and Moser,* Bull. Amer. Math. Soc. **7** (1982), pp. 65-222.

[H/Z] J. Hong / C. Zuilly - *Existence of C^∞ local solutions for the Monge-Ampère equation*, Invent. math. **89** (1987), pp. 645-661.

[Hö] L. Hörmander - The Weyl calculus of pseudo-differential operators, Comm. Pure Appl. Math **32** (1979), pp. 359-443.

[L/Z] S. Lojasiewicz, Jr. / E. Zehnder - *An inverse function theorem in Fréchet-Spaces*, J. Funct. Anal. **33** (1979), pp. 165-174.

[M] J. Moser - *A new technique for the construction of solutions of nonlinear partial differential equations*, Proc. Natl. Acad. Sci. USA **47** (1961), pp. 1824-1831.

[N1] J. Nash - *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), pp.20-63.

[N2] ______ - *Real algebraic manifolds*, Ann. of Math. **56** (1952), pp. 405-421.

[R] W. Rudin - *Functional analysis*, McGraw-Hill, New York, 1973.

[S] R. Seeley - *Topics in pseudodifferential operators*, C.I.M.E.Conf., Rome, 1969, pp. 138-305.

[Sc1] E. Schrohe - *The symbols of an algebra of pseudodifferential operators*, Pac. J. Math. **125** (1986), pp. 211-224.

[Sc2] ________ - *A Ψ^* algebra of pseudodifferential operators on noncompact manifolds*, Arch. Math **51** (1988), pp. 81-86.

[Sr1] F. Sergeraert - *Une generalization du théorème des fonctions implicites de Nash*, C.R. Acad. Sci. Paris Série A, **270** (1970), pp. 861-863.

[Sr2] ________ - *Un théorème des fonctions implicites sur certains espaces de Fréchet et quelques applications*, Ann. scien. École. Norm. Sup., 4e série, **5** (1972), pp. 599-660.

[Sr3] ________ - *Une extension d'un théorèm implicites de Hamilton*, Bull. Soc. Math. France, Mem. 46, Journees sur la Géométrie de la Dimension Infinite, 1976, pp. 163-171.

[Sw] J. Schwartz - *On Nash's implicit functional theorem*, Comm. Pure Appl. Math. **13** (1960), pp. 509-530.

[T1] M. Taylor - *Pseudodifferential operators*, Princeton Univ. Press, Princeton, N.J., 1981.

[T2] ________ - *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs, Number **22**, Amer. Math. Soc., Providence, R.I., 1986.

[T3] ________ - *Cordes type characterizations*, Partial Differential Equations Seminar Notes, Univ. of North Carolina at Chapel Hill, 1988.

[V] V.S. Varadarajan - *Lie groups Lie algebras and their representations*, Springer-Verlag, New York, 1984.

[W] A. Weinstein - *A symbol class for some Schrödinger equations on* $\boldsymbol{R}^n$, Amer. J. Math **107** (1985), pp. 1-21.