The Index Theory of Toeplitz Operators on the Skew Quarter-Plane

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Efton Lilborn Park

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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In this thesis, the index theory of the Toeplitz operators associated to a skew quarter-plane in $\mathbb{Z}^2$ is studied by examining the C^*-algebra generated by these operators. Criteria for operators in this C^*-algebra to be Fredholm is established, and cyclic cohomology is used to construct an explicit index formula for the Fredholm operators. In addition, the K-theory of many of these C^*-algebras is computed, as well as the K-theory of some related algebras.

To the memory of my father.

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Introduction

We begin by reviewing the development of the study of Toeplitz operators.

The first class of Toeplitz operators considered were those associated to the circle. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, equipped with Haar measure, and consider the Hilbert space $L^2(\mathbb{T})$ of square-integrable functions on $\mathbb{T}$. Define $H^2(\mathbb{T})$ to be the subspace of functions in $L^2(\mathbb{T})$ that extend holomorphically to the interior of the unit disk. Note that the functions $\{\chi_n \mid n \in \mathbb{Z}\}$, where $\chi_n(\theta) = e^{in\theta}$, form an orthonormal basis for $L^2(\mathbb{T})$, and $H^2(\mathbb{T})$ is the closed subspace spanned by $\{\chi_n \mid n \geq 0\}$.

Given a continuous function φ on the circle, define the multiplication operator M_φ on $L^2(\mathbb{T})$ by $M_\varphi f = \varphi f$. Let P denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Define the Toeplitz operator T_φ on $H^2(\mathbb{T})$ by $T_\varphi = PM_\varphi$, and define the Toeplitz algebra $\mathcal{T}$ to be the C^*-algebra generated by the T_φ.

The index theory of the Toeplitz algebra is well known and discussed in several places (see [11]). Specifically, there exists a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0,$$

where $\sigma : \mathcal{T} \longrightarrow C(\mathbb{T})$ is defined by requiring that $\sigma(T_\varphi) = \varphi$, and where $\mathcal{K}$ denotes the compact operators on $H^2(\mathbb{T})$. This exact sequence immediately yields the following index result:

Theorem 0.1: *Let $T \in \mathcal{T}$. Then T is a Fredholm operator if and only if $\sigma(T) \in C(\mathbb{T})$ is nonvanishing.*■

There is also the following index formula:

Theorem 0.2: *Let $T \in \mathcal{T}$ be a Fredholm operator. Then the index of T equals the negative of the winding number of $\sigma(T)$.*■

These results can be recast to provide index results for another class of operators. Let $\mathbb{Z}$ denote the integers endowed with counting measure. Then consider the Hilbert space $\ell^2(\mathbb{Z})$ of square-summable functions from $\mathbb{Z}$ to $\mathbb{C}$. The functions $\{e_n \mid n \in \mathbb{Z}\}$, where $e_n(k)$ equals 1 if k equals n and 0 otherwise, form an orthonormal basis for $\ell^2(\mathbb{Z})$. Let $H^2(\mathbb{Z})$ be the closed span of the e_n for $n \geq 0$, and let P be the orthogonal projection from $\ell^2(\mathbb{Z})$ onto $H^2(\mathbb{Z})$. Next, given $n \in \mathbb{Z}$, define the translation operator $M_n : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ by $(M_n f)(k) = f(n+k)$, and define the operator $T_n : H^2(\mathbb{Z}) \longrightarrow H^2(\mathbb{Z})$ by $T_n = PM_n$. The Fourier transform gives an isomorphism between $L^2(T)$ and $\ell^2(\mathbb{Z})$, and under this isomorphism, $H^2(T)$ and $H^2(\mathbb{Z})$ correspond. Moreover, the Fourier transform is a unitary equivalence mapping T_{χ_n} to T_n. The C^*-algebra generated by the T_n is isomorphic to the Toeplitz algebra $\mathcal{T}$ via this unitary equivalence. Therefore, the index results for the Toeplitz algebra on the circle can also be interpreted as index results for the Toeplitz algebra on the integers.

We now discuss more general types of Toeplitz operators. Specifically, we will consider Toeplitz algebras on $\mathbb{Z}^2$. We could also study Toeplitz algebras on the two-torus T^2, since the Fourier transform will implement isomorphisms between the appropriate C^*-algebras, but we prefer to work on $\mathbb{Z}^2$ instead.

Let $\mathbb{Z}^2$ be the collection of pairs of integers, endowed with counting measure, and let $\ell^2(\mathbb{Z}^2)$ be the

Hilbert space of square-summable functions from $\mathbb{Z}^2$ to $\mathbb{C}$. Given a pair of integers (m,n), define the translation operator $M_{m,n}$ on $\ell^2(\mathbb{Z}^2)$ by

$$(M_{m,n}f)(k,\ell) = f(m+k,n+\ell).$$

To define Toeplitz operators, we need a subspace on which to project, and there are several candidates. One possibility is to project onto the subspace of sequences in $\ell^2(\mathbb{Z}^2)$ that are supported on a half space in $\mathbb{Z}^2$.

For each pair of integers (m,n), let $e_{m,n}$ be the function in $\ell^2(\mathbb{Z}^2)$ defined by $e_{m,n}(k,\ell) = 1$ if $(k,\ell) = (m,n)$ and 0 otherwise. Then it is clear that the $e_{m,n}$ form an orthonormal basis for $\ell^2(\mathbb{Z}^2)$. Choose a real number α, and define

$$\mathcal{H}^\alpha = \text{closed span of } \{e_{m,n} \mid -\alpha m + n \geq 0\}.$$

We could, of course, consider the closed span of the $e_{m,n}$ for which $-\alpha m + n \leq 0$, but the results obtained in this case will be essentially the same.

Let P^α be the orthogonal projection of $\ell^2(\mathbb{Z}^2)$ onto $\mathcal{H}^\alpha$, and define the half-plane Toeplitz algebra $\mathcal{T}^\alpha$ to be the C^*-algebra generated by the operators $P^\alpha M_{m,n} P^\alpha$. The half-plane Toeplitz algebras have been studied by several authors, and the index theory of these algebras is outlined in [9]. Specifically, there exists a short exact sequence

$$0 \longrightarrow \mathcal{C}^{\alpha} \longrightarrow \mathcal{T}^{\alpha} \xrightarrow{\sigma^{\alpha}} C(T^2) \longrightarrow 0,$$

where $\mathcal{C}^{\alpha}$ denotes the commutator ideal of $\mathcal{T}^{\alpha}$ and σ^{α} is defined by requiring $\sigma^{\alpha}(P^{\alpha}M_{m,n}P^{\alpha}) = \chi_{m,n}$, where $\chi_{m,n}(\theta_1,\theta_2) = e^{im\theta_1}e^{in\theta_2}$. Unlike the Toeplitz algebra on $\mathbb{Z}$, the commutator ideal here does not coincide with the ideal of compact operators on $\mathcal{H}^{\alpha}$. In fact, there are no nonzero compact operators at all in $\mathcal{T}^{\alpha}$. Therefore, the index result one obtains here is much different from that in the integer case:

Theorem 0.3: *An operator in $\mathcal{T}^{\alpha}$ is a Fredholm operator if and only if it is invertible.*■

As we can see, the Fredholm index theory of the half-plane Toeplitz algebras is not very interesting. It is possible to consider a different kind of index which comes from a type II_{∞} von Neumann algebra, but this is a topic which we will not discuss further.

The results above show that projecting onto a half-plane does not provide the proper generalization of the Toeplitz operators, at least as far as ordinary Fredholm index is concerned. Let us now consider a different setup with better possibilites for such development.

Choose distinct real numbers α and β, and consider the lines with slopes α and β that pass through the origin in $\mathbb{R}^2$. Suppose without loss of generality that $\alpha < \beta$, and define

$$\mathcal{H}^{\alpha} = \text{closed span of } \{e_{m,n} \mid -\alpha m + n \geq 0\}$$

$$\mathcal{H}^{\beta} = \text{closed span of } \{e_{m,n} \mid -\beta m + n \leq 0\}.$$

Let P^{α} and P^{β} be the orthogonal projections of $\ell^2(\mathbb{Z}^2)$ onto $\mathcal{H}^{\alpha}$ and $\mathcal{H}^{\beta}$, respectively. The subspace in which we are interested is $\mathcal{H}^{\alpha,\beta} = \mathcal{H}^{\alpha} \cap \mathcal{H}^{\beta}$, and $P^{\alpha}P^{\beta}$ is the orthogonal projection onto this subspace.

Define $\mathcal{T}^{\alpha,\beta}$ to be the C^*-algebra generated by the operators $P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta}$ for $(m,n) \in \mathbb{Z}^2$. This C^*-algebra will be called a skew quarter-plane Toeplitz algebra; we wish to consider the index theory of this algebra.

The skew quarter-plane Toeplitz algebra was first considered in [13] in the case $\alpha = 0$, $\beta = \infty$ (in this case, $\mathcal{H}^{\infty}$ is defined to be the closed span of the $e_{m,n}$ such that $m \geq 0$). In this case, $\mathfrak{T}^{\alpha,\beta}$ can be identified as the tensor product of the Toeplitz algebra $\mathfrak{T}$ with itself. Moreover, the Toeplitz algebras associated to the upper half plane and to the right half plane are each isomorphic to the tensor product of $\mathfrak{T}$ and $C(T)$. Using these identifications, it is easy to see that there exist surjective algebra homomorphisms γ^{0} and γ^{∞} from $\mathfrak{T}^{0,\infty}$ to $\mathfrak{T}^{0}$ and $\mathfrak{T}^{\infty}$, respectively, and these homomorphisms allow one to obtain the following index result:

Theorem 0.4: An operator T in $\mathfrak{T}^{0,\infty}$ is Fredholm if and only if $\gamma^{0}(T)$ and $\gamma^{\infty}(T)$ are invertible in $\mathfrak{T}^{0}$ and $\mathfrak{T}^{\infty}$, respectively.∎

Moreover, in [6], a method for computing the index of the Fredholm operators in $\mathfrak{T}^{0,\infty}$ was established. The method works in theory, but it is impractical, since it depends upon the construction of operator-valued homotopies. Thus, it is nearly impossible to use this result to determine the index of Fredholm operators in $\mathfrak{T}^{0,\infty}$.

In this dissertation, the Fredholm index result in

[13] is extended to the general skew quarter-plane Toeplitz algebra $\mathfrak{T}^{\alpha,\beta}$. Since it is not always possible to represent $\mathfrak{T}^{\alpha,\beta}$ as a tensor product, the techniques used in [13] do not apply. In the general case, we use a number theoretic result concerning the approximation of real numbers by rational numbers to establish surjective homomorphisms γ^{α} and γ^{β} from $\mathfrak{T}^{\alpha,\beta}$ to $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$. We use these homomorphisms to construct a noncommutative symbol algebra $\mathfrak{S}^{\alpha,\beta}$ for Fredholm operators in the skew quarter-plane Toeplitz algebra, and we use this symbol algebra to prove that an operator in $\mathfrak{T}^{\alpha,\beta}$ is Fredholm if and only if its image is invertible under the aforementioned maps.

We compute the K-theory of the symbol algebra $\mathfrak{S}^{\alpha,\beta}$ as well as some of the skew quarter-plane Toeplitz algebras. We also compute the K-theory of some related C^*-algebras. We use these calculations to show that when at least one of α and β is rational, then index is a complete stable deformation invariant for Fredholm operators in $\mathfrak{T}^{\alpha,\beta}$. Finally, we consider the possibilities for the K-theory of $\mathfrak{T}^{\alpha,\beta}$ when α and β are both irrational.

We use cyclic cohomology to construct an explicit index formula for Fredholm operators in the skew quarter-plane Toeplitz algebra. Unlike the index formula in [13], our index formula involves calculations that are easy to perform for many Fredholm operators. Finally, we use our index formula to calculate the index of some specific operators.

The index theory of the skew quarter-plane Toeplitz algebra is interesting for several reasons. First, the skew quarter plane operators are a natural generalization of the ordinary quarter-plane operators. Second, the half plane Toeplitz algebras are related to Connes's foliation algebras on the torus ([7],[13]) and it is likely that the skew quarter-plane Toeplitz algebras can be used to study the case of two transverse foliations on the torus. Finally, the skew quarter-plane operators are related to Upmeier's work on Toeplitz operators on bounded irreducible symmetric domains ([26],[27]); our work here provides a carefully worked out example of the index theory of a Toeplitz algebra associated to a *reducible* symmetric domain.

This dissertation is organized as follows. In the first chapter, the skew quarter-plane Toeplitz algebra $\mathcal{T}^{\alpha,\beta}$ and the symbol algebra $\mathcal{S}^{\alpha,\beta}$ are defined, and we give necessary and sufficient conditions for operators in $\mathfrak{T}^{\alpha,\beta}$ to be Fredholm. In the second chapter, we compute the K-theory of many of the relevant algebras. We also show that when at least one of α and β is rational, index is a complete stable deformation invariant for Fredholm operators in $\mathfrak{T}^{\alpha,\beta}$. In the third chapter, cyclic cohomology is used to construct an explicit index formula for many Fredholm operators in $\mathfrak{T}^{\alpha,\beta}$. In the fourth chapter, we use our results to calculate the index of some specific Fredholm operators. Finally, in the fifth chapter, we consider some open questions and speculate on what their answers might be.

I. The skew quarter-plane Toeplitz algebra

In this chapter we define the skew quarter-plane Toeplitz algebra $\mathfrak{T}^{\alpha,\beta}$ and examine several relevant ideals in this algebra. We also study the relationship between $\mathfrak{T}^{\alpha,\beta}$ and the half-plane Toeplitz algebras $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$.

Endow $\mathbb{Z}^2$ with counting measure, and let $\ell^2(\mathbb{Z}^2)$ denote the Hilbert space of square-summable functions from $\mathbb{Z}^2$ to $\mathbb{C}$.

Definition 1.1: Let $(m,n) \in \mathbb{Z}^2$. Then $e_{m,n} \in \ell^2(\mathbb{Z}^2)$ is defined by

$$e_{m,n}(k,\ell) = \begin{cases} 1 & \text{if } (k,\ell) = (m,n) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2: Let $(m,n) \in \mathbb{Z}^2$. Then the translation operator $M_{m,n} : \ell^2(\mathbb{Z}^2) \longrightarrow \ell^2(\mathbb{Z}^2)$ is defined by

$$M_{m,n}(e_{k,\ell}) = e_{m+k,n+\ell}.$$

Regard $\mathbb{Z}^2$ as a subset of $\mathbb{R}^2$, and consider distinct lines through the origin with slopes α and β; without loss of generality we may assume that $\alpha < \beta$. We will also allow the possibility that $\beta = \infty$; that is, one of the lines is vertical. We will not usually explicitly mention this possibility in the definitions that follow, however.

Definition 1.3: Let $\alpha < \beta$. The subspaces $\mathcal{H}^\alpha$, $\mathcal{H}^\beta$, and $\mathcal{H}^{\alpha,\beta}$ of $\ell^2(\mathbb{Z}^2)$ are defined to be

$$\mathcal{H}^\alpha = \text{closed span of } \{e_{m,n} \mid -\alpha m + n \geq 0\}$$

$$\mathcal{H}^\beta = \text{closed span of } \{e_{m,n} \mid -\beta m + n \leq 0\}$$

$$\mathcal{H}^{\alpha,\beta} = \mathcal{H}^\alpha \cap \mathcal{H}^\beta.$$

Also, define the subspace $\mathcal{H}^\infty$ of $\ell^2(\mathbb{Z}^2)$ by

$$\mathcal{H}^\infty = \text{closed span of } \{e_{m,n} \mid m \geq 0\}.$$

Definition 1.4: The operators P^α and P^β are defined to be the orthogonal projections

$$P^{\alpha} : \ell^2(\mathbb{Z}^2) \longrightarrow \mathcal{H}^{\alpha}$$

$$P^{\beta} : \ell^2(\mathbb{Z}^2) \longrightarrow \mathcal{H}^{\beta}.$$

Also, P^{∞} is the orthogonal projection

$$P^{\infty} : \ell^2(\mathbb{Z}^2) \longrightarrow \mathcal{H}^{\infty}.$$

Note that $P^{\alpha}P^{\beta}$ is the orthogonal projection onto $\mathcal{H}^{\alpha,\beta}$.

Definition 1.5: The C^*-algebra $\mathcal{T}^{\alpha,\beta}$ is defined to be the C^*-algebra generated by the set of operators

$$\{P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta} \mid (m,n) \in \mathbb{Z}^2\}.$$

This C^*-algebra is called a *skew quarter-plane Toeplitz algebra*.

We are interested in the following questions: which operators in $\mathcal{T}^{\alpha,\beta}$ are Fredholm operators, and if T in $\mathcal{T}^{\alpha,\beta}$ is Fredholm, what is its index?

Before we go further, several comments are in order. First, $\mathcal{T}^{\alpha,\beta}$ is called a skew quarter-plane algebra because the region onto which $P^{\alpha}P^{\beta}$ projects is a skew quarter-plane in $\mathbb{Z}^2$. Secondly, note that the two lines of slopes α and β define four quarter-planes in $\mathbb{Z}^2$, and we could project onto any one of them. However, no generality is lost considering the skew quarter-plane we have chosen. For if φ is an isomorphism of $\mathbb{Z}^2$ that takes one skew quarter-plane to another, then φ gives rise to a unitary operator U_{φ} which implements an isomorphism between the associated Toeplitz algebras.

Definition 1.6: Let $\mathcal{T}^{\alpha,\beta}$ be a skew quarter-plane Toeplitz algebra. Then $\mathcal{C}^{\alpha,\beta}$ is the commutator ideal of $\mathcal{T}^{\alpha,\beta}$.

The following theorem is proved in [4]:

Theorem 1.7: *There exists a short exact sequence*

$$0 \longrightarrow \mathcal{C}^{\alpha,\beta} \longrightarrow \mathcal{T}^{\alpha,\beta} \xrightarrow{\sigma^{\alpha,\beta}} C(\mathbb{T}^2) \longrightarrow 0,$$

where $\sigma^{\alpha,\beta}$ *is defined by*

$$\sigma^{\alpha,\beta}(P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta}) = \chi_{m,n},$$

where

$$\chi_{m,n}(\theta_1,\theta_2) = e^{im\theta_1}e^{in\theta_2}.$$

Furthermore, this exact sequence has a linear splitting

$$\xi^{\alpha,\beta} : C(\mathbb{T}^2) \longrightarrow \mathfrak{T}^{\alpha,\beta}$$

defined by

$$\xi^{\alpha,\beta}(\chi_{m,n}) = P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta}. \blacksquare$$

Unfortunately, Theorem 1.7 provides little index information. The problem is that the commutator ideal $\mathfrak{C}^{\alpha,\beta}$ contains the compact operators $\mathfrak{K}$ on $\mathfrak{H}^{\alpha,\beta}$, but $\mathfrak{C}^{\alpha,\beta}$ is considerably larger than $\mathfrak{K}$.

We begin our study of the index theory of the skew quarter-plane Toeplitz algebra $\mathfrak{T}^{\alpha,\beta}$ by considering a C^*-algebra that contains $\mathfrak{T}^{\alpha,\beta}$, but which is ostensibly larger.

Definition 1.8: Define $\mathfrak{R}^{\alpha,\beta}$ to be the C^*-algebra generated by the collection of operators

$$P^\alpha P^\beta M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j}\right]P^\alpha P^\beta,$$

where each Q_j is either P^α, P^β, or $P^\alpha P^\beta$.

Note that a dense subalgebra of $\mathfrak{R}^{\alpha,\beta}$ consists of operators of the form

$$\left\{\sum_{i=0}^{\ell} c_{ij} P^\alpha P^\beta M_{m_{i0},n_{i0}}\left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij},n_{ij}}\right]P^\alpha P^\beta\right\},$$

where the c_{ij} are constants. Also observe that $\mathfrak{R}^{\alpha,\beta}$ contains $\mathfrak{J}^{\alpha,\beta}$, but $\mathfrak{R}^{\alpha,\beta}$ contains operators that are not obviously contained in $\mathfrak{J}^{\alpha,\beta}$. For example, the operator $P^\alpha P^\beta M_{k,\ell} P^\alpha M_{m,n} P^\alpha P^\beta$ is in $\mathfrak{R}^{\alpha,\beta}$, but this operator does not appear to be in $\mathfrak{J}^{\alpha,\beta}$. We will later show that $\mathfrak{R}^{\alpha,\beta}$ and $\mathfrak{J}^{\alpha,\beta}$ are in fact the same C^*-algebra.

Definition 1.9: Let α be a real number. Then define $\mathfrak{J}^\alpha$ to be the C^*-algebra generated by the collection of operators

$$\{P^{\alpha}M_{m,n}P^{\alpha} \mid (m,n) \in \mathbb{Z}^2\}.$$

Such an algebra is called a *half-plane Toeplitz algebra*.

Definition 1.10: Let α and β be distinct real numbers. Define linear maps

$$\rho^{\alpha} : \mathfrak{T}^{\alpha} \longrightarrow \mathfrak{R}^{\alpha,\beta}$$

$$\rho^{\beta} : \mathfrak{T}^{\beta} \longrightarrow \mathfrak{R}^{\alpha,\beta}$$

by

$$\rho^{\alpha}(X) = P^{\beta}XP^{\beta}$$

$$\rho^{\beta}(Y) = P^{\alpha}YP^{\alpha}.$$

These maps are clearly linear, but they are not multiplicative.

We will construct algebra homomorphisms from $\mathfrak{R}^{\alpha,\beta}$ to $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$, but we first need the following technical lemma.

Lemma 1.11: *Let $\{(m_i, n_i)\}$ be a finite collection of pairs of integers. Then there exists a pair of integers (p,q) such that, for all i,*

(i). $-\alpha(m_i+p) + (n_i+q) \geq 0$ *if and only if* $-\alpha m_i + n_i \geq 0$.

(ii). $-\beta(m_i+p) + (n_i+q) \leq 0$.

Proof: Choose positive numbers ϵ and M so that

$$M > \max_i \{-\beta m_i + n_i\}$$

$$\epsilon < \min_i \{|-\alpha m_i + n_i|\}$$

$$\epsilon < M.$$

Then it suffices to show that there exist integers p and q so that $|-\alpha p + q| < \epsilon$ and $-\beta p + q < -M$.

In [14], it is proved that there exist an infinite number of integers p for which there exists an integer q with $|\alpha - \frac{q}{p}| < \frac{1}{p^2}$. Choose such a p so large that $p > \frac{1}{\epsilon}$ and $p > \frac{2M}{\beta - \alpha}$. Then

$$|-\alpha p + q| < \frac{1}{p} < \epsilon$$

and

$$-\beta p + q = -(\beta - \alpha)p + (-\alpha p + q)$$

$$< -(\beta - \alpha)p + \epsilon$$

$$< -2M + \epsilon$$

$$< -M,$$

as desired.∎

Proposition 1.12: There exist contractive algebra homomorphisms

$$\gamma^{\alpha} : \mathfrak{R}^{\alpha,\beta} \longrightarrow \mathfrak{T}^{\alpha}$$

$$\gamma^{\beta} : \mathfrak{R}^{\alpha,\beta} \longrightarrow \mathfrak{T}^{\beta}$$

such that $\gamma^{\alpha}\rho^{\alpha} = id$, $\gamma^{\beta}\rho^{\beta} = id$.

Proof: Let T be an operator in $\mathfrak{R}^{\alpha,\beta}$ of the form

$$\left\{ \sum_{i=0}^{\ell} c_{ij} P^{\alpha} P^{\beta} M_{m_{i0},n_{i0}} \left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij},n_{ij}} \right] P^{\alpha} P^{\beta} \right\},$$

and define

$$\gamma^{\alpha}(T) = \left\{ \sum_{i=0}^{\ell} c_{ij} P^{\alpha} M_{m_{i0},n_{i0}} \left[\prod_{j=1}^{k_i} Q^{\alpha}_{ij} M_{m_{ij},n_{ij}} \right] P^{\alpha} \right\},$$

where each Q^{α}_{ij} equals P^{α}, I, or P^{α}, depending on whether Q_{ij} equals P^{α}, P^{β}, or $P^{\alpha}P^{\beta}$. To show that γ^{α} is well defined and can be extended to an algebra homomorphism on $\mathfrak{R}^{\alpha,\beta}$, it suffices to show that $\| \gamma^{\alpha}(T) \| \leq \| T \|$.

Fix $\epsilon > 0$, and choose f in $\ell^2(\mathbb{Z}^2)$ so that

1. f has finite support.

2. f has norm one.

3. $\| \gamma^{\alpha}(T) \| \leq \| \gamma^{\alpha}(T) f \| + \epsilon$.

Now, since f has finite support,

$$(*) \qquad \left[\prod_{j=N}^{k_i} M_{m_{ij},n_{ij}} \right] f$$

is also finitely supported for all $0 \leq i \leq \ell$ and $0 \leq N \leq k_i$, whence Tf is finitely supported. Now, consider the set

$$S = \{ (m,n) \in \mathbb{Z}^2 \mid e_{m,n} \in \text{range of } (*) \text{ for some } i, N \}.$$

Apply the previous lemma to the pairs of integers in S to obtain a pair of integers (p,q). A moment's thought yields that

$$M_{p,q}\gamma^{\alpha}(T)f = \gamma^{\alpha}(T)M_{p,q}f.$$

Furthermore, our choice of (p,q) implies that in the expression for $TM_{p,q}f$, the projection P^{β} is unnecessary each place that it appears. Therefore

$$\gamma^{\alpha}(T)M_{p,q}f = TM_{p,q}f,$$

and, since $M_{p,q}$ is a unitary operator,

$$
\begin{aligned}
\| \gamma^{\alpha}(T) \| &\le \| \gamma^{\alpha}(T)f \| + \epsilon \\
&= \| M_{p,q}\gamma^{\alpha}(T)f \| + \epsilon \\
&= \| TM_{p,q}f \| + \epsilon \\
&\le \| TM_{p,q} \| + \epsilon \\
&\le \| T \| \, \| M_{p,q} \| + \epsilon \\
&\le \| T \| + \epsilon .
\end{aligned}
$$

Since ϵ was arbitrary, $\| \gamma^{\alpha}(T) \| \le \| T \|$, and thus γ^{α} extends to a contractive algebra homomorphism from $\mathfrak{R}^{\alpha,\beta}$ to $\mathfrak{T}^{\alpha}$. Similarly, there exists a contractive algebra homomorphism γ^{β} from $\mathfrak{R}^{\alpha,\beta}$ to $\mathfrak{T}^{\beta}$. Finally, it is obvious that ρ^{α} is a splitting of γ^{α} and ρ^{β} is a splitting of γ^{β}.■

Definition 1.13: The C^*-algebras $\mathfrak{C}^{\alpha}$ and $\mathfrak{C}^{\beta}$ are defined to be the commutator ideals of $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$.

The results in [4] yield the following theorem:

Theorem 1.14: *There exist short exact sequences*

$$0 \longrightarrow \mathfrak{C}^{\alpha} \longrightarrow \mathfrak{I}^{\alpha} \xrightarrow{\sigma^{\alpha}} C(\mathbb{T}^2) \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{C}^{\beta} \longrightarrow \mathfrak{I}^{\beta} \xrightarrow{\sigma^{\beta}} C(\mathbb{T}^2) \longrightarrow 0,$$

where σ^{α} *and* σ^{β} *are defined by*

$$\sigma^{\alpha}(P^{\alpha}M_{m,n}P^{\alpha}) = \chi_{m,n} = \sigma^{\beta}(P^{\beta}M_{m,n}P^{\beta}).$$

Furthermore, these sequences have linear splittings

$$\xi^{\alpha} : C(\mathbb{T}^2) \longrightarrow \mathfrak{I}^{\alpha}$$

$$\xi^{\beta} : C(\mathbb{T}^2) \longrightarrow \mathfrak{I}^{\beta}$$

defined by

$$\xi^{\alpha}(\chi_{m,n}) = P^{\alpha}M_{m,n}P^{\alpha}$$

$$\xi^{\beta}(\chi_{m,n}) = P^{\beta}M_{m,n}P^{\beta}. \blacksquare$$

We now record several useful relations between the various maps we have defined.

Lemma 1.15: $\sigma^{\alpha}\gamma^{\alpha} = \sigma^{\beta}\gamma^{\beta}$.

Proof: It suffices to show that $\sigma^{\alpha}\gamma^{\alpha}(T) = \sigma^{\beta}\gamma^{\beta}(T)$ for T an operator of the form

$$\left\{ \sum_{i=0}^{\ell} c_{ij} P^{\alpha} P^{\beta} M_{m_{i0}, n_{i0}} \left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij}, n_{ij}} \right] P^{\alpha} P^{\beta} \right\},$$

and this is a direct computation:

$$\sigma^{\alpha}\gamma^{\alpha}(T)$$

$$= \sigma^{\alpha}\gamma^{\alpha}\left\{ \sum_{i=0}^{\ell} c_{ij} P^{\alpha} P^{\beta} M_{m_{i0}, n_{i0}} \left[\prod_{j=1}^{k_i} Q_{ij} M_{m_{ij}, n_{ij}} \right] P^{\alpha} P^{\beta} \right\}$$

$$= \sigma^{\alpha}\left\{ \sum_{i=0}^{\ell} c_{ij} P^{\alpha} M_{m_{i0}, n_{i0}} \left[\prod_{j=1}^{k_i} Q^{\alpha}_{ij} M_{m_{ij}, n_{ij}} \right] P^{\alpha} \right\}$$

$$= \sum_{i=0}^{\ell} c_{ij} \left[\prod_{j=0}^{k_i} M_{m_{ij}, n_{ij}} \right]$$

$= \sigma^{\beta}\gamma^{\beta}(T)$.■

Note that if T is in $\mathfrak{T}^{\alpha,\beta}$,

$$\sigma^{\alpha,\beta}(T) = \sigma^{\alpha}(\gamma^{\alpha}(T)) = \sigma^{\beta}(\gamma^{\beta}(T)),$$

so the homomorphism $\sigma^{\alpha,\beta}$ from $\mathfrak{T}^{\alpha,\beta}$ to $C(T^2)$ extends to a homomorphism from $\mathfrak{R}^{\alpha,\beta}$ to $C(T^2)$.

Lemma 1.16: *$\gamma^{\beta}\rho^{\alpha} = \xi^{\beta}\sigma^{\alpha}$ and $\gamma^{\alpha}\rho^{\beta} = \xi^{\alpha}\sigma^{\beta}$.*

Proof: Consider an operator in $\mathfrak{T}^{\alpha}$ that has the form

$$T = \sum_{i=0}^{\ell} c_i \Big[\prod_{j=0}^{k_i} P^{\alpha} M_{m_{ij},n_{ij}} P^{\alpha} \Big].$$

It will suffice to prove that $\gamma^{\beta}\rho^{\alpha}(T) = \xi^{\beta}\sigma^{\alpha}(T)$ for operators of the above form, since these operators are dense in $\mathfrak{T}^{\alpha}$. We will only verify that $\gamma^{\beta}\rho^{\alpha} = \xi^{\beta}\sigma^{\alpha}$; showing that $\gamma^{\alpha}\rho^{\beta} = \xi^{\alpha}\sigma^{\beta}$ involves a similar calculation.

$$\gamma^{\beta}\rho^{\alpha}(T) = \gamma^{\beta}\rho^{\alpha}\left\{ \sum_{i=0}^{\ell} c_i \left[\prod_{j=0}^{k_i} P^{\alpha} M_{m_{ij},n_{ij}} P^{\alpha} \right] \right\}$$

$$= \gamma^{\beta}\left\{ \sum_{i=0}^{\ell} c_i P^{\beta} \left[\prod_{j=0}^{k_i} P^{\alpha} M_{m_{ij},n_{ij}} P^{\alpha} \right] P^{\beta} \right\},$$

$$= \sum_{i=0}^{\ell} c_i P^{\beta} \left[\prod_{j=0}^{k_i} M_{m_{ij},n_{ij}} \right] P^{\beta}.$$

On the other hand,

$$\xi^{\beta}\sigma^{\alpha}(T) = \xi^{\beta}\sigma^{\alpha}\left\{ \sum_{i=0}^{\ell} c_i \left[\prod_{j=0}^{k_i} P^{\alpha} M_{m_{ij},n_{ij}} P^{\alpha} \right] \right\}$$

$$= \xi^{\beta}\left\{ \sum_{i=0}^{\ell} c_i \left[\prod_{j=0}^{k_i} M_{m_{ij},n_{ij}} \right] \right\}$$

$$= \sum_{i=0}^{\ell} c_i P^{\beta} \left[\prod_{j=1}^{k_i} M_{m_{ij},n_{ij}} \right] P^{\beta}. \blacksquare$$

Lemma 1.17: $\xi^{\alpha} = \gamma^{\alpha}\xi^{\alpha,\beta}$ *and* $\xi^{\beta} = \gamma^{\beta}\xi^{\alpha,\beta}$.

Proof: It is enough to verify the lemma for the functions $\chi_{m,n}$, since the span of these functions is dense in $C(T^2)$:

$$\gamma^{\alpha}\xi^{\alpha,\beta}(\chi_{m,n}) = \gamma^{\alpha}(P^{\alpha}P^{\beta}M_{m,n}P^{\alpha}P^{\beta})$$

$$= P^{\alpha}M_{m,n}P^{\alpha}$$

$$= \xi^{\alpha}(\chi_{m,n}).$$

Similarly, $\gamma^{\beta}\xi^{\alpha,\beta}(\chi_{m,n}) = \gamma^{\beta}(\chi_{m,n})$. ∎

Definition 1.18: Define the ideals $\mathfrak{J}_{\alpha}^{\beta}$, $\mathfrak{J}_{\beta}^{\alpha}$ and $\mathfrak{J}^{\alpha,\beta}$ by

$$\mathfrak{J}_{\beta}^{\alpha} = \ker \gamma^{\beta}$$

$$\mathfrak{J}_{\alpha}^{\beta} = \ker \gamma^{\alpha}.$$

$$\mathfrak{J}^{\alpha,\beta} = \mathfrak{J}_{\beta}^{\alpha} \cap \mathfrak{J}_{\alpha}^{\beta}.$$

Proposition 1.19*: There exists the following commutative diagram with exact rows and columns:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{J}^{\alpha,\beta} & \longrightarrow & \mathfrak{J}^{\alpha}_{\beta} & \longrightarrow & \mathfrak{C}^{\alpha} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{J}^{\beta}_{\alpha} & \longrightarrow & \mathfrak{R}^{\alpha,\beta} & \xrightarrow{\gamma^{\alpha}} & \mathfrak{T}^{\alpha} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \gamma^{\beta}} & & \downarrow{\scriptstyle \sigma^{\alpha}} & & \\
0 & \longrightarrow & \mathfrak{C}^{\beta} & \longrightarrow & \mathfrak{T}^{\beta} & \xrightarrow{\sigma^{\beta}} & C(\mathrm{T}^2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Proof: We have already shown that $\sigma^{\alpha}\gamma^{\alpha} = \sigma^{\beta}\gamma^{\beta}$. Since all other maps in the diagram are inclusions, we need only check that γ^{α} and γ^{β} map $\mathfrak{J}^{\alpha}_{\beta}$ to $\mathfrak{C}^{\alpha}$ and $\mathfrak{J}^{\beta}_{\alpha}$ to $\mathfrak{C}^{\beta}$, respectively, to check that the diagram commutes.

Take X in $\mathfrak{J}^{\alpha}_{\beta}$. Then $\gamma^{\alpha}(X)$ is in $\mathfrak{T}^{\alpha}$. Now, $\sigma^{\alpha}\gamma^{\alpha}(X) = \sigma^{\beta}\gamma^{\beta}(X)$, and since X is in $\mathfrak{J}^{\alpha}_{\beta} = \ker \gamma^{\beta}$, $\sigma^{\alpha}\gamma^{\alpha}(X) = \sigma^{\beta}\gamma^{\beta}(X) = 0$. This implies that $\gamma^{\alpha}(X)$ is in $\ker \sigma^{\alpha} = \mathfrak{C}^{\alpha}$. Similarly, γ^{β} takes $\mathfrak{J}^{\beta}_{\alpha}$ to $\mathfrak{C}^{\beta}$.

Next, let us verify that the rows and columns of the diagram are exact. The last row and last column are exact by Theorem 1.14, and Definition 1.18 implies that the second row and column are exact.

It remains to show that the first row and column are exact. We will show that the first row is exact; the proof that the first column is exact is essentially identical.

Let Y be in $\mathfrak{C}^{\alpha}$. Then $\rho^{\alpha}(Y)$ is in $\mathfrak{R}^{\alpha,\beta}$. Furthermore, since $\gamma^{\beta}\rho^{\alpha}(Y) = \xi^{\beta}\sigma^{\alpha}(Y) = 0$, we see that $\rho^{\alpha}(Y)$ is in $\mathfrak{J}^{\alpha}_{\beta}$. Thus the sequence is exact at $\mathfrak{C}^{\alpha}$. Next, take X in $\mathfrak{J}^{\alpha}_{\beta}$ to be in the kernel of γ^{α}. Then obviously X will be in $\mathfrak{J}^{\alpha,\beta}$. Conversely, let X be in $\mathfrak{J}^{\alpha,\beta}$. Then X is in $\mathfrak{J}^{\beta}_{\alpha}$, so $\gamma^{\alpha}(X) = 0$, and hence X is in the kernel of γ^{α} restricted to $\mathfrak{J}^{\alpha}_{\beta}$. Therefore the sequence is exact at $\mathfrak{J}^{\alpha}_{\beta}$. Finally, the sequence is exact at $\mathfrak{J}^{\alpha,\beta}$, since the map is inclusion.∎

Definition 1.20: Define the C^*-algebra $\mathcal{J}^{\alpha,\beta}$ by

$$\mathcal{J}^{\alpha,\beta} = \{(T^{\alpha},T^{\beta}) \in \mathfrak{T}^{\alpha} \oplus \mathfrak{T}^{\beta} \mid \sigma^{\alpha}(T^{\alpha}) = \sigma^{\beta}(T^{\beta})\}.$$

Proposition 1.21: *There exists a short exact sequence*

$$0 \longrightarrow \mathfrak{J}^{\alpha,\beta} \longrightarrow \mathfrak{R}^{\alpha,\beta} \xrightarrow{\ \gamma\ } \mathcal{J}^{\alpha,\beta} \longrightarrow 0,$$

where

$$\gamma(T) = (\gamma^{\alpha}(T),\ \gamma^{\beta}(T)).$$

Furthermore, this sequence has a linear splitting

$$\rho\ :\ \wp^{\alpha,\beta} \longrightarrow \mathfrak{R}^{\alpha,\beta}$$

defined by

$$\rho(T^{\alpha},T^{\beta}) = \rho^{\alpha}(T^{\alpha}) + \rho^{\beta}(T^{\beta}) - \xi^{\alpha,\beta}\sigma^{\beta}(T^{\beta})$$

$$= \rho^{\alpha}(T^{\alpha}) + \rho^{\beta}(T^{\beta}) - \xi^{\alpha,\beta}\sigma^{\alpha}(T^{\alpha}).$$

Proof: Let T be in $\mathfrak{R}^{\alpha,\beta}$. Then $\sigma^{\alpha}\gamma^{\alpha}(T) = \sigma^{\beta}\gamma^{\beta}(T)$, so γ maps $\mathfrak{R}^{\alpha,\beta}$ into $\wp^{\alpha,\beta}$. Next, suppose T is in $\mathfrak{J}^{\alpha,\beta}$. Then $\gamma^{\alpha}(T) = 0$ and $\gamma^{\beta}(T) = 0$, so T is in ker γ. Conversely, suppose that T is in ker γ. Then T is in ker γ^{α} and ker γ^{β}, whence T is in $\mathfrak{J}^{\alpha,\beta}$. Therefore the sequence is exact at $\mathfrak{R}^{\alpha,\beta}$. Clearly the sequence is exact at $\mathfrak{J}^{\alpha,\beta}$, since the map is inclusion. To show that the sequence is exact at $\wp^{\alpha,\beta}$, it suffices to show that the map ρ defined above is a splitting.

Choose (T^{α},T^{β}) in $\wp^{\alpha,\beta}$. Then to show that ρ is a splitting, we must show that $\gamma^{\alpha}\rho(T^{\alpha},T^{\beta}) = T^{\alpha}$ and that $\gamma^{\beta}\rho(T^{\alpha},T^{\beta}) = T^{\beta}$.

$$\gamma^{\alpha}\rho(T^{\alpha},T^{\beta}) = \gamma^{\alpha}\Big[\ \rho^{\alpha}(T^{\alpha}) + \rho^{\beta}(T^{\beta}) - \xi^{\alpha,\beta}\sigma^{\beta}(T^{\beta})\ \Big]$$

$$= \gamma^{\alpha}\rho^{\alpha}(T^{\alpha}) + \gamma^{\alpha}\rho^{\beta}(T^{\beta}) - \gamma^{\alpha}\xi^{\alpha,\beta}\sigma^{\beta}(T^{\beta})$$

$$= T^{\alpha} + \gamma^{\alpha}\rho^{\beta}(T^{\beta}) - \gamma^{\alpha}\xi^{\alpha,\beta}\sigma^{\beta}(T^{\beta})$$

$$= T^{\alpha} + \xi^{\alpha}\sigma^{\beta}(T^{\beta}) - \gamma^{\alpha}\xi^{\alpha,\beta}\sigma^{\beta}(T^{\beta})$$

$$= T^{\alpha}.$$

The proof that $\gamma^{\beta}\rho(T^{\alpha},T^{\beta}) = T^{\beta}$ is similar.■

We make use of this short exact sequence by identifying $\mathfrak{J}^{\alpha,\beta}$ as a more familiar object.

Proposition 1.22: $\mathfrak{J}^{\alpha,\beta} = \mathfrak{K}(\mathcal{H}^{\alpha,\beta})$.

Proof: We first show the $\mathfrak{J}^{\alpha,\beta}$ is contained in $\mathfrak{K}(\mathcal{H}^{\alpha,\beta})$. Since ρ is a splitting, every element of $\mathfrak{J}^{\alpha,\beta}$ can be written in the form T - $(\rho\gamma)(T)$ for some T in $\mathfrak{T}^{\alpha,\beta}$. Now, operators consisting of finite sums and products of translations and the projections P^{α} and P^{β} are dense in $\mathfrak{R}^{\alpha,\beta}$, so the operators T - $\rho\gamma(T)$, where T has this form, are dense in $\mathfrak{I}^{\alpha,\beta}$. Therefore, to show that $\mathfrak{I}^{\alpha,\beta}$ is contained in $\mathfrak{K}(\mathfrak{H}^{\alpha,\beta})$, it suffices to show that T - $(\rho\gamma)(T)$ is a finite rank operator for operators T having this form. Furthermore, since $\rho\gamma$ is linear, it suffices to show that T - $\rho\gamma(T)$ has finite rank for T of the form

$$T = P^{\alpha}P^{\beta}M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j}\right]P^{\alpha}P^{\beta}.$$

In this case,

$$T - (\rho\gamma)(T) = P^{\alpha}P^{\beta}M_{m_0,n_0}\left[\prod_{j=1}^{k} Q_j M_{m_j,n_j} - \prod_{j=1}^{k} Q_j^{\alpha} M_{m_j,n_j} - \prod_{j=1}^{k} Q_j^{\beta} M_{m_j,n_j} + \prod_{j=1}^{k} M_{m_j,n_j}\right]P^{\alpha}P^{\beta}.$$

Now, let $\lambda^{\alpha} = -\sum_{j=0}^{k} \min\{-\alpha m_j + n_j, 0\}$, and choose (m,n) so that $-\alpha m + n \geq \lambda^{\alpha}$. Then the definition of λ^{α}

implies that the projection P^{α} makes no contribution in $(T - \rho\gamma(T))(e_{m,n})$. Therefore,

$$\prod_{j=1}^{k} Q_j M_{m_j,n_j}(e_{m,n}) = \prod_{j=1}^{k} Q_j^{\beta} M_{m_j,n_j}(e_{m,n})$$

and

$$\prod_{j=1}^{k} Q_j^{\alpha} M_{m_j,n_j}(e_{m,n}) = \prod_{j=1}^{k} M_{m_j,n_j}(e_{m,n}),$$

so $(T - \rho\gamma(T))(e_{m,n}) = 0$ when $-\alpha m + n \geq -\lambda^{\alpha}$.

Similarly, define $\lambda^{\beta} = \sum_{j=0}^{k} \max\{-\beta m_j + n_j, 0\}$, and choose (m,n) so that $-\beta m + n \leq -\lambda^{\beta}$. Then the projection P^{β} has no effect in $(T - \rho\gamma(T))(e_{m,n})$, so

$$\prod_{j=1}^{k} Q_j M_{m_j,n_j}(e_{m,n}) = \prod_{j=1}^{k} Q_j^{\alpha} M_{m_j,n_j}(e_{m,n})$$

and

$$\prod_{j=1}^{k} Q_j^{\beta} M_{m_j,n_j}(e_{m,n}) = \prod_{j=1}^{k} M_{m_j,n_j}(e_{m,n}),$$

and thus $(T - \rho\gamma(T))(e_{m,n}) = 0$ for $-\beta m + n \leq -\lambda^{\beta}$. Therefore, the range of $T - \rho\gamma(T)$ is contained in the span of the $e_{m,n}$ that satisfy the inequalities

$$0 \leq -\alpha m + n \leq \lambda^{\alpha}$$

$$-\lambda^{\beta} \leq -\beta m + n \leq 0.$$

It is easy to see that since α and β are distinct, there are only finitely many pairs of integers satisfying both inequalities. Therefore, $T - \rho\gamma(T)$ is a finite rank operator when T has the above form, and thus for arbitrary X in $\mathfrak{R}^{\alpha,\beta}$, we see that $X - \rho\gamma(X)$ is compact.

To show that $\mathfrak{J}^{\alpha,\beta}$ contains $\mathfrak{K}(\mathfrak{H}^{\alpha,\beta})$, it is enough to show that $\mathfrak{R}^{\alpha,\beta}$, and hence $\mathfrak{J}^{\alpha,\beta}$, is irreducible. That $\mathfrak{R}^{\alpha,\beta}$ is irreducible follows from the fact that $\mathfrak{T}^{\alpha,\beta}$ is irreducible [5] and that $\mathfrak{T}^{\alpha,\beta} \subset \mathfrak{R}^{\alpha,\beta}$.∎

Proposition 1.23: $\mathfrak{R}^{\alpha,\beta} = \mathfrak{T}^{\alpha,\beta}$.

Proof: Since $\gamma^{\alpha}(P^{\alpha}P^{\beta}MP^{\alpha}P^{\beta}) = P^{\alpha}MP^{\alpha}$ for every translation operator M, we see that γ^{α} maps $\mathfrak{T}^{\alpha,\beta}$ onto $\mathfrak{T}^{\alpha}$. Similarly, γ^{β} maps $\mathfrak{T}^{\alpha,\beta}$ onto $\mathfrak{T}^{\beta}$. Let $\mathfrak{J}$ be the kernel of γ^{α} restricted to $\mathfrak{T}^{\alpha,\beta}$. Then we have the commutative square

$$\begin{array}{ccc} \mathfrak{J} & \longrightarrow & \mathfrak{T}^{\alpha,\beta} \\ \downarrow \gamma^{\beta} & & \downarrow \gamma^{\beta} \\ \mathfrak{C}^{\beta} & \longrightarrow & \mathfrak{T}^{\beta}, \end{array}$$

where the horizontal maps are inclusions. Now, $\mathfrak{J}$ is an ideal in $\mathfrak{T}^{\alpha,\beta}$ and γ^{β} maps $\mathfrak{T}^{\alpha,\beta}$ onto $\mathfrak{T}^{\beta}$, so $\gamma^{\beta}(\mathfrak{J})$ is an ideal in $\mathfrak{T}^{\beta}$. Furthermore, the commutativity of the square above implies that $\gamma^{\beta}(\mathfrak{J})$ is an ideal in $\mathfrak{C}^{\beta}$. But $\mathfrak{C}^{\beta}$ is simple [9], so $\gamma^{\beta}(\mathfrak{J})$ is either zero or all of $\mathfrak{C}^{\beta}$. It is easy to show that $\gamma^{\beta}(\mathfrak{J})$ contains nonzero operators, so γ^{β} maps $\mathfrak{J}$ onto $\mathfrak{C}^{\beta}$. Next, [5] implies that $\mathfrak{T}^{\alpha,\beta}$ is irreducible, so $\mathfrak{J}$ is also irreducible. Therefore $\mathfrak{J}$ contains all the compact operators, and we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \mathfrak{J} & \longrightarrow & \mathfrak{C}^{\beta} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{K} & \longrightarrow & \mathfrak{J}^{\beta}_{\alpha} & \longrightarrow & \mathfrak{C}^{\beta} & \longrightarrow & 0,
\end{array}$$

where the first and third vertical maps are the identity and the middle vertical map is inclusion. Since the first and third maps are isomorphisms, the Five Lemma implies that the middle map is also an isomorphism, so $\mathfrak{J} = \mathfrak{J}^{\beta}_{\alpha}$.

We now have another commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{J}^{\beta}_{\alpha} & \longrightarrow & \mathfrak{T}^{\alpha,\beta} & \longrightarrow & \mathfrak{T}^{\alpha} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{J}^{\beta}_{\alpha} & \longrightarrow & \mathfrak{R}^{\alpha,\beta} & \longrightarrow & \mathfrak{T}^{\alpha} & \longrightarrow & 0,
\end{array}$$

where the first and third maps are again identity maps and the middle map is inclusion. A second application of the Five Lemma yields $\mathfrak{R}^{\alpha,\beta} = \mathfrak{T}^{\alpha,\beta}$.∎

Corollary 1.24: *The following sequence is exact:*

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{T}^{\alpha,\beta} \xrightarrow{\ \gamma\ } \mathfrak{P}^{\alpha,\beta} \longrightarrow 0,$$

and has a linear splitting $\rho : \mathfrak{P}^{\alpha,\beta} \longrightarrow \mathfrak{T}^{\alpha,\beta}$.∎

This short exact sequence gives an index formula for the Fredholm operators in the skew quarter-plane Toeplitz algebra:

Theorem 1.25: *An operator T in $\mathfrak{T}^{\alpha,\beta}$ is Fredholm if and only if $\gamma(T)$ is invertible in $\mathfrak{S}^{\alpha,\beta}$, or equivalently, if $\gamma^{\alpha}(T)$ and $\gamma^{\beta}(T)$ are invertible in $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$, respectively.*∎

It should be noted that the exact sequence above remains exact when tensored by $M_n(\mathbb{C})$, so the above index theorem also applies to the matrix-valued skew quarter-plane Toeplitz algebra.

There is one more exact sequence we will need when we compute the K-theory of the various algebras considered in this chapter:

Proposition 1.26: *Let $\mathfrak{C}^{\alpha,\beta}$ be the commutator ideal of the skew quarter-plane algebra $\mathfrak{T}^{\alpha,\beta}$. Then the following sequence is exact:*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{C}^{\alpha,\beta} \xrightarrow{\ \gamma\ } \mathfrak{C}^{\alpha} \oplus \mathfrak{C}^{\beta} \longrightarrow 0.$$

Furthermore, the splitting $\rho : \mathcal{P}^{\alpha,\beta} \longrightarrow \mathcal{T}^{\alpha,\beta}$ *restricts to a splitting* $\rho : \mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta} \longrightarrow \mathcal{C}^{\alpha,\beta}$.

Proof: Take X in $\mathcal{K}$. Then since $\mathcal{K} = \mathcal{G}^{\alpha}_{\beta} \cap \mathcal{G}^{\beta}_{\alpha}$, $\gamma^{\alpha}(X)$ and $\gamma^{\beta}(X)$ are both zero. Since $\sigma^{\alpha,\beta} = \sigma^{\alpha}\gamma^{\alpha} = \sigma^{\beta}\gamma^{\beta}$, $\sigma^{\alpha,\beta}(X) = 0$, whence X is in $\mathcal{C}^{\alpha,\beta}$. Therefore $\mathcal{K}$ is contained in $\mathcal{C}^{\alpha,\beta}$ and hence the sequence is exact at $\mathcal{K}$. Next, choose Y in $\mathcal{C}^{\alpha,\beta}$. Then $0 = \sigma^{\alpha,\beta}(Y) = \sigma^{\alpha}\gamma^{\alpha}(Y)$, so $\gamma^{\alpha}(Y)$ is in $\mathcal{C}^{\alpha}$. Similarly, $0 = \sigma^{\alpha,\beta}(Y) = \sigma^{\beta}\gamma^{\beta}(Y)$, so $\gamma^{\beta}(Y)$ is in $\mathcal{C}^{\beta}$. Therefore γ maps $\mathcal{C}^{\alpha,\beta}$ into $\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}$.

Let us show that the sequence is exact at $\mathcal{C}^{\alpha,\beta}$. If X is in $\mathcal{K}$, then $\gamma(X) = 0$, so $\mathcal{K} \subset \ker(\gamma|_{\mathcal{C}^{\alpha,\beta}})$. Conversely, let X be in $\mathcal{C}^{\alpha,\beta}$ and $\gamma(X) = 0$. Then since X is also in $\mathcal{T}^{\alpha,\beta}$ and since the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{P}^{\alpha,\beta} \longrightarrow 0$$

is exact, it follows that X is compact. Therefore the sequence is exact at $\mathcal{C}^{\alpha,\beta}$. Finally, we must show that the γ maps $\mathcal{C}^{\alpha,\beta}$ onto $\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}$. This will follow immediately once we have established that the linear map $\rho : \mathcal{P}^{\alpha,\beta} \longrightarrow \mathcal{T}^{\alpha,\beta}$ restricts to give a splitting from $\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}$ to $\mathcal{C}^{\alpha,\beta}$.

Take (T^{α},T^{β}) in $\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}$. To show that ρ is a splitting for the exact sequence in the statement of the theorem, we need only show that $\rho(T^{\alpha},T^{\beta})$ is in $\mathcal{C}^{\alpha,\beta} = \ker \sigma^{\alpha,\beta}$, since it has already been shown that $\gamma\rho$ is the identity map on $\mathcal{S}^{\alpha,\beta}$. But this is easy: since T^{α} is in $\mathcal{C}^{\alpha}$, $\gamma^{\alpha}(T^{\alpha}) = 0$, whence

$$\sigma^{\alpha,\beta}\rho(T^{\alpha},T^{\beta}) = \sigma^{\alpha}\gamma^{\alpha}(T^{\alpha}) = 0.$$

Therefore ρ is a splitting for the sequence, and the sequence is exact at $\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}$.∎

II. K-theory

In this chapter, we will compute the K-theory of the symbol algebra $\mathcal{S}^{\alpha,\beta}$. In the cases where at least one of α and β is rational we will calculate the K-theory of the other algebras as well, and we will show that index is a complete stable deformation invariant for Fredholm operators in $\mathcal{T}^{\alpha,\beta}$. Finally, we will discuss the case when both α and β are irrational, and give some partial results on the K-theory of the various algebras in this case. Throughout this chapter we use [1] as a standard reference.

We begin by discussing the K-theory of the half-plane Toeplitz algebra $\mathcal{T}^{\alpha}$ and its commutator ideal C^{α}. First consider the case when $\alpha = 0$. It was shown in [13] that $\mathcal{T}^{0} \cong \mathcal{T} \otimes C(T)$, where $\mathcal{T}$ denotes the Toeplitz algebra on the circle; the isomorphism is given by sending $PM_kP \otimes \chi_{\ell}$ to the operator $P^0M_{k,\ell}P^0$. Now, to compute the K-theory of $\mathcal{T}^0$, we need to know the K-theory of $\mathcal{T}$:

$$K_0(\mathcal{T}) \cong \mathbb{Z}$$

$$K_1(\mathcal{T}) \cong 0.$$

The first isomorphism is given by sending the integer k to the identity matrix in $M_k(\mathcal{T})$. Henceforth, we will refer to this situation by saying that K_0 consists of trivial projections, since the identity matrices in the matrix algebras are the simplest kind of projections.

We also have

$$K_0(C(T)) \cong \mathbb{Z}$$

$$K_1(C(T)) \cong \mathbb{Z}.$$

Now, $K_0(C(T))$ consists of trivial projections, and the second isomorphism is given by sending an integer k to $[x_k]$, where $x_k(\theta) = e^{ik\theta}$. We may therefore use the Künneth formula [19] to obtain

$$K_0(\mathcal{T}^0) \cong K_0(\mathcal{T} \otimes C(T))$$

$$\cong \left[K_0(\mathcal{T}) \otimes K_0(C(T))\right] \oplus \left[K_1(\mathcal{T}) \otimes K_1(C(T))\right]$$

$$\cong K_0(\mathcal{T}) \otimes K_0(C(T))$$

$$\cong \mathbb{Z}.$$

Just as in the case of $\mathcal{T}$ and $C(T)$, $K_0(\mathcal{T})$ consists of only the trivial projections.

We can also use the Künneth formula to compute $K_1(\mathcal{T}^0)$:

$$K_1(\mathcal{T}^0) \cong K_1(\mathcal{T} \otimes C(T))$$

$$\cong \left[K_0(\mathcal{T}) \otimes K_1(C(T))\right] \oplus \left[K_1(\mathcal{T}) \otimes K_0(C(T))\right]$$

$$\cong K_0(\mathcal{T}) \otimes K_1(C(T))$$

$$\cong \mathbb{Z}.$$

The isomorphism between $\mathbb{Z}$ and $K_1(\mathcal{T} \otimes C(T))$ is given by sending the integer k to $[I \otimes x_k]$. Therefore, in the case of $K_1(\mathcal{T}^0)$, the integer k is sent to $[P^0 M_{(k,0)} P^0]$.

We can also compute the K-theory of $\mathcal{C}^0$. It is also shown in [13] that the isomorphism between $\mathcal{T}^0$ and $\mathcal{T} \otimes C(T)$ restricts to an isomorphism between $\mathcal{C}^0$ and $\mathcal{K} \otimes C(T)$. Therefore,

$$K_0(\mathcal{C}^0) \cong K_0(\mathcal{K} \otimes C(T)) \cong K_0(C(T)) \cong \mathbb{Z}$$

$$K_1(\mathcal{C}^0) \cong K_1(\mathcal{K} \otimes C(T)) \cong K_1(C(T)) \cong \mathbb{Z}.$$

Next, let α be a rational number and consider $\mathcal{T}^\alpha$. Write $\alpha = \frac{p}{q}$, where p and q are relatively prime, and choose integers m,n so that pm + qn = 1. Then the matrix $\begin{bmatrix} q & -m \\ p & n \end{bmatrix}$ gives an automorphism of $\mathbb{Z}^2$, and this automorphism defines a unitary operator on $\ell^2(\mathbb{Z}^2)$ that induces an isomorphism between $\mathcal{T}^\alpha$ and $\mathcal{T}^0$. Therefore

$$K_0(\mathcal{T}^\alpha) \cong K_0(\mathcal{T}^0) \cong \mathbb{Z}$$

$$K_1(\mathcal{T}^\alpha) \cong K_1(\mathcal{T}^0) \cong \mathbb{Z}.$$

In particular, the operator $P^\alpha M_{q,p} P^\alpha$ is a generator for $K_1(\mathcal{T}^\alpha)$. Also,

$$K_0(\mathcal{C}^\alpha) \cong K_0(\mathcal{C}^0) \cong \mathbb{Z}$$

$$K_1(\mathcal{C}^\alpha) \cong K_1(\mathcal{C}^0) \cong \mathbb{Z}.$$

We make the observation that if $\alpha = \pm\infty$, then we can easily find automorphisms of $\mathbb{Z}^2$ that induce isomorphisms between $\mathfrak{T}^0$ and $\mathfrak{T}^{+\infty}$ and between $\mathfrak{T}^0$and $\mathfrak{T}^{-\infty}$, so the K-theory of $\mathfrak{T}^{\pm\infty}$ and $\mathfrak{C}^{\pm\infty}$ is the same as that of $\mathfrak{T}^0$ and $\mathfrak{C}^0$.

Finally, consider the case where α is irrational. Calculating the K-theory of $\mathfrak{C}^\alpha$ and $\mathfrak{T}^\alpha$ is considerably more difficult in this case. It has been shown in [15] and [28] that $K_0(\mathfrak{T}^\alpha)$ consists of the trivial projections and that $K_1(\mathfrak{T}^\alpha)$ is zero. Also, it is shown that $K_0(\mathfrak{C}^\alpha) \cong \mathbb{Z}^2$ and $K_1(\mathfrak{C}^\alpha) \cong \mathbb{Z}$.

We can use the K-theory calculations for the half-plane Toeplitz algebras to obtain K-theory results for the skew quarter-plane Toeplitz algebra and the various ideals we discussed in Chapter 1. For simplicity, we shall assume that $\alpha < 0 < \beta$. We lose no generality in considering this case for two reasons. First, for any skew quarter-plane the calculations will be essentially the same. Second, it is easy to check that given any skew quarter-plane in $\mathbb{Z}^2$, there is always an automorphism of $\mathbb{Z}^2$ that takes the given skew quarter-plane to one of the desired form.

Our calculations will break up into three cases:

CASE I: α and β both rational

Write $\alpha = \frac{p}{q}$, where p and q are relatively prime and $q > 0$, $p < 0$. Similarly, write $\beta = \frac{r}{s}$, where r and s are relatively prime and both positive. We begin by considering the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{J}^{\alpha}_{\beta} \longrightarrow \mathcal{C}^{\alpha} \longrightarrow 0.$$

This gives us the following exact diagram in K-theory:

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{J}^{\alpha}_{\beta}) & \longrightarrow & K_0(\mathcal{C}^{\alpha}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}^{\alpha}) & \longleftarrow & K_1(\mathcal{J}^{\alpha}_{\beta}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

Now, we have $pm + qn = 1$, so m and n are either both positive or both negative. We will suppose that m and n are both positive; the case where m and n are both negative gives similar results.

We define operators

$$A = P^{\alpha} M_{-q,-p} P^{\alpha}$$

$$B = P^{\alpha} M_{m,n} P^{\alpha} M_{-m,-n} P^{\alpha},$$

and define

$$T = I - (I - A)(I - B).$$

Since I - B is in $\mathcal{C}^{\alpha}$, T is in $(\mathcal{C}^{\alpha})^{+}$, where $(\mathcal{C}^{\alpha})^{+}$ is the algebra $\mathcal{C}^{\alpha}$ with a unit adjoined. Furthermore, A is unitary, B is a projection, and A and B commute, so it is easily checked that T is unitary. Next, lift A and B to

$$\tilde{A} = P^{\alpha} P^{\beta} M_{-q,-p} P^{\alpha} P^{\beta}$$

$$\tilde{B} = P^{\alpha} P^{\beta} M_{m,n} P^{\alpha} M_{-m,-n} P^{\alpha} P^{\beta}$$

in $\mathcal{T}^{\alpha,\beta}$. Then T lifts to

$$\tilde{T} = I - (I - \tilde{A})(I - \tilde{B}),$$

and since $(I - \tilde{B})$ is in $\mathcal{J}^{\alpha}_{\beta}$, $\tilde{T}$ is in $(\mathcal{J}^{\alpha}_{\beta})^{+}$. Furthermore, T^{*} lifts to the operator

$$\tilde{T}^* = I - (I - \tilde{A}^*)(I - \tilde{B}).$$

Direct calculation yields

$$\tilde{T}\tilde{T}^* = I$$

$$\tilde{T}^*\tilde{T} = I - (I - \tilde{A}^*\tilde{A})(I - \tilde{B}).$$

Therefore, $\tilde{T}^*$ is injective, whence $\ker \tilde{T}^* = \{0\}$. Also, $\ker \tilde{T}^*\tilde{T} = \ker \tilde{T}$. Now, it is easy to check that $\tilde{T}^*\tilde{T}(e_{x,y}) = 0$ if and only if the integers x and y satisfy the inequalities

$$0 \leq -\alpha x + y < \alpha m + n$$

$$0 \geq -\beta x + y > -\beta q + p,$$

and $\tilde{T}^*\tilde{T}\,(e_{x,y}) = e_{x,y}$ otherwise.

Consider the first inequality. If we write α as $\frac{p}{q}$ and multiply through by q, we obtain

$$0 \leq -px + qy < pm + qn = 1.$$

Therefore, $-px + qy = 0$, or $\alpha x = y$.

Now consider the second inequality. If we substitute αx for y, we obtain

$$0 \geq (\alpha - \beta)x > -\beta q + p,$$

or, since $\alpha - \beta$ is negative,

$$0 \leq x < \frac{-\beta q + p}{\alpha - \beta} = q.$$

Now, since $y = \alpha x = \frac{p}{q} x$, we see that the only pair of integers that satisfy both inequalities is $x = y = 0$. Therefore $\tilde{T}$ is a Fredholm operator, and

$$\text{index } \tilde{T} = \dim \ker \tilde{T} - \dim \ker \tilde{T}^* = 1 - 0 = 1.$$

Since the connecting homomorphism from $K_1(\mathcal{C}^\alpha)$ to $K_0(\mathcal{K})$ coincides with the index map, T is a generator of $K_1(\mathcal{C}^\alpha)$. Moreover, the connecting homomorphism is an isomorphism in this case. Since $K_1(\mathcal{K}) \cong 0$, the exact diagram in K-theory above implies that

$$K_0(\mathcal{J}^\alpha_\beta) \cong \mathbb{Z}$$

$$K_1(\mathcal{J}^\alpha_\beta) \cong 0.$$

We can similarly write down a generator of $K_1(\mathcal{C}^\beta)$ that lifts to a Fredholm operator with index one, so

$$K_0(\mathcal{J}_\alpha^\beta) \cong \mathbb{Z}$$

$$K_1(\mathcal{J}_\alpha^\beta) \cong 0.$$

Next, we calculate the K-theory of $\mathcal{C}^{\alpha,\beta}$. We have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}^{\alpha,\beta} \longrightarrow \mathcal{C}^\alpha \oplus \mathcal{C}^\beta \longrightarrow 0,$$

which yields the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{C}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta) & \longleftarrow & K_1(\mathcal{C}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

Consider the element (T,I) in $(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta)^+$, where T is defined as above. Then (T,I) lifts to $\tilde{T}$ in $(\mathcal{C}^{\alpha,\beta})^+$, and we have already observed that $\tilde{T}$ is a Fredholm operator of index 1. Therefore the connecting homomorphism from $K_1(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta)$ to $K_0(\mathcal{K})$ is onto and has kernel isomorphic to $\mathbb{Z}$. Thus

$$K_0(\mathcal{C}^{\alpha,\beta}) \cong K_0(\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}) \cong \mathbb{Z}^2$$

$$K_1(\mathcal{C}^{\alpha,\beta}) \cong \mathbb{Z}.$$

Before we compute the K-theory of the symbol algebra $\mathcal{S}^{\alpha,\beta}$, we need to know the K-theory of $C(T^2)$:

$$K_0(C(T^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$K_1(C(T^2)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

In $K_0(C(T^2))$, one copy of $\mathbb{Z}$ corresponds to the trivial projections. The other copy of $\mathbb{Z}$ is generated by the projection that corresponds to the complex line bundle over T^2 with chern class one [16]. As for $K_1(C(T))$, the two copies of $\mathbb{Z}$ are generated by $[x_{1,0}]$ and $[x_{0,1}]$.

We can now compute the K-theory of $\mathcal{S}^{\alpha,\beta}$. Since $\mathcal{S}^{\alpha,\beta}$ is the pullback of $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$ along $C(T^2)$, we can apply the Mayer-Vietoris sequence in K-theory [20] to obtain the following exact diagram:

$$\begin{array}{ccccc}
K_0(\mathcal{J}^{\alpha,\beta}) & \longrightarrow & K_0(\mathfrak{J}^{\alpha}) \oplus K_0(\mathfrak{J}^{\beta}) & \longrightarrow & K_0(C(T^2)) \\
\uparrow & & & & \downarrow \\
K_1(C(T^2)) & \longleftarrow & K_1(\mathfrak{J}^{\alpha}) \oplus K_1(\mathfrak{J}^{\beta}) & \longleftarrow & K_1(\mathcal{J}^{\alpha,\beta}).
\end{array}$$

Consider the map from $K_0(\mathfrak{J}^{\alpha}) \oplus K_0(\mathfrak{J}^{\beta})$ to $K_0(C(T^2))$. The elements of $K_0(\mathfrak{J}^{\alpha})$ and $K_0(\mathfrak{J}^{\beta})$ consist of trivial projections, and thus the map from $K_0(\mathfrak{J}^{\alpha}) \oplus K_0(\mathfrak{J}^{\beta})$ to $K_0(C(T^2))$ maps onto the trivial projections. Combining this fact with the calculation of $K_0(C(T^2))$ above, we see that $K_1(\mathcal{J}^{\alpha,\beta})$ contains at least a factor of $\mathbb{Z}$, and there is also at least one factor of $\mathbb{Z}$ in $K_0(\mathcal{J}^{\alpha,\beta})$. Next, consider the map from $K_1(\mathfrak{J}^{\alpha}) \oplus K_1(\mathfrak{J}^{\beta})$ to $K_1(C(T^2))$. The operators $P^{\alpha}M_{q,p}P^{\alpha}$ and $P^{\beta}M_{s,r}P^{\beta}$ are generators of $K_1(\mathfrak{J}^{\alpha})$ and $K_1(\mathfrak{J}^{\beta})$, and these operators map to $\chi_{q,p}$ and $\chi_{s,r}$, respectively. In general, $[\chi_{q,p}]$ and $[\chi_{s,r}]$ generate a proper subgroup H of $K_1(C(T^2))$, so

$$K_0(\mathcal{J}^{\alpha,\beta}) \cong \mathbb{Z} \oplus \mathbb{Z}/_{H}.$$

Moreover, it is easy to check that since α and β are distinct, the map from $K_1(\mathfrak{J}^{\alpha}) \oplus K_1(\mathfrak{J}^{\beta})$ to $K_1(C(T^2))$ is injective, so

$$K_1(\mathcal{J}^{\alpha,\beta}) \cong \mathbb{Z}.$$

Finally, we will calculate the K-theory of the skew quarter-plane Toeplitz algebra $\mathcal{T}^{\alpha,\beta}$. We have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{S}^{\alpha,\beta} \longrightarrow 0$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{T}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{S}^{\alpha,\beta}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{S}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{T}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

Let T and $\tilde{T}$ be defined as above. Then (T,I) is an invertible element in $\mathcal{S}^{\alpha,\beta}$ that lifts to $\tilde{T} \in \mathcal{T}^{\alpha,\beta}$, and we have shown that $\tilde{T}$ is a Fredholm operator with index one. Therefore, the connecting homomorphism between $K_1(\mathcal{S}^{\alpha,\beta})$ and $K_0(\mathcal{K})$ is an isomorphism, whence

$$K_0(\mathcal{T}^{\alpha,\beta}) \cong K_0(\mathcal{S}^{\alpha,\beta}) \cong \mathbb{Z} \oplus \mathbb{Z}/_H$$

$$K_1(\mathcal{T}^{\alpha,\beta}) \cong 0.$$

CASE II. α rational and β irrational

We begin by considering the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{J}_{\beta}^{\alpha} \longrightarrow C^{\alpha} \longrightarrow 0.$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{J}_{\beta}^{\alpha}) & \longrightarrow & K_0(C^{\alpha}) \\ \uparrow & & & & \downarrow \\ K_1(C^{\alpha}) & \longleftarrow & K_1(\mathcal{J}_{\beta}^{\alpha}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

We define T in the same fashion that we did in Case I; again T lifts to a Fredholm operator $\tilde{T}$ with index one. Thus

$$K_0(\mathcal{J}_{\beta}^{\alpha}) \cong K_0(C^{\alpha}) \cong \mathbb{Z}$$

$$K_1(\mathcal{J}_{\beta}^{\alpha}) \cong 0.$$

Next consider the short exact sequence

$$0 \longrightarrow \mathcal{J}_{\beta}^{\alpha} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{T}^{\beta} \longrightarrow 0$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{J}^{\alpha}_{\beta}) & \longrightarrow & K_0(\mathcal{T}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{T}^{\beta}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}^{\beta}) & \longleftarrow & K_1(\mathcal{T}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{J}^{\alpha}_{\beta}). \end{array}$$

Since β is irrational, $K_1(\mathcal{T}^{\beta}) \cong 0$. Furthermore, we showed above that $K_1(\mathcal{J}^{\alpha}_{\beta}) \cong 0$, so both connecting homomorphisms are zero. Therefore

$$K_0(\mathcal{T}^{\alpha,\beta}) \cong K_0(\mathcal{J}^{\alpha}_{\beta}) \oplus K_0(\mathcal{T}^{\beta}) \cong \mathbb{Z}^2$$

$$K_1(\mathcal{T}^{\alpha,\beta}) \cong K_1(\mathcal{J}^{\alpha}_{\beta}) \oplus K_1(\mathcal{T}^{\beta}) \cong 0.$$

Next, consider the short exact sequence

$$0 \longrightarrow \mathcal{J}^{\beta}_{\alpha} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{T}^{\alpha} \longrightarrow 0$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{J}^{\beta}_{\alpha}) & \longrightarrow & K_0(\mathcal{T}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{T}^{\alpha}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}^{\alpha}) & \longleftarrow & K_1(\mathcal{T}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{J}^{\beta}_{\alpha}). \end{array}$$

The group $K_0(\mathfrak{J}^{\alpha})$ consists of trivial projections, and it is easy to see that $K_0(\mathfrak{J}^{\alpha,\beta})$ maps onto $K_0(\mathfrak{J}^{\alpha})$. We combine this observation with the calculation of the K-theory of $\mathfrak{J}^{\alpha,\beta}$ above to obtain

$$K_0(\mathfrak{J}^{\beta}_{\alpha}) \cong \mathbb{Z}^2$$

$$K_1(\mathfrak{J}^{\beta}_{\alpha}) \cong 0.$$

We now compute the K-theory of $\mathcal{C}^{\alpha,\beta}$. We have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}^{\alpha,\beta} \longrightarrow \mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta} \longrightarrow 0$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{C}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta}) & \longleftarrow & K_1(\mathcal{C}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

Again consider the element (T,I) in $(\mathcal{C}^{\alpha} \oplus \mathcal{C}^{\beta})^{+}$. As above, (T,I) lifts to $\tilde{T}$ in $(\mathcal{C}^{\alpha,\beta})^{+}$, and $\tilde{T}$ is a Fredholm operator of index one. Therefore the connecting homomorphism from $K_1(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta)$ to $K_0(\mathcal{K})$ is onto and has kernel isomorphic to $\mathbb{Z}$, and hence

$$K_0(\mathcal{C}^{\alpha,\beta}) \cong K_0(\mathcal{C}^\alpha \oplus \mathcal{C}^\beta) \cong \mathbb{Z}^3$$

$$K_1(\mathcal{C}^{\alpha,\beta}) \cong \mathbb{Z}.$$

Finally, we compute the K-theory of the symbol space $\mathcal{S}^{\alpha,\beta}$. The short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{S}^{\alpha,\beta} \longrightarrow 0$$

yields the exact diagram

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(\mathcal{T}^{\alpha,\beta}) & \longrightarrow & K_0(\mathcal{S}^{\alpha,\beta}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{S}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{T}^{\alpha,\beta}) & \longleftarrow & K_1(\mathcal{K}). \end{array}$$

Let T be defined as above. Then (T,I) is an invertible element of $\mathcal{S}^{\alpha,\beta}$ that lifts to a Fredholm operator of index one. We have shown that $K_1(\mathcal{T}^{\alpha,\beta})$ is zero, so

$$K_0(\mathcal{I}^{\alpha,\beta}) \cong K_0(\mathfrak{I}^{\alpha,\beta}) \cong \mathbb{Z}^2.$$

$$K_1(\mathcal{I}^{\alpha,\beta}) \cong K_0(\mathcal{K}) \cong \mathbb{Z}.$$

Before we consider the case where α and β are both irrational, let us make some definitions and observations. First, we note that the algebra $\mathfrak{I}^{\alpha,\beta}$ can be imbedded in any of its matrix algebras $M_n(\mathfrak{I}^{\alpha,\beta})$ by putting $\mathfrak{I}^{\alpha,\beta}$ in the upper left hand corner of $M_n(\mathfrak{I}^{\alpha,\beta})$. With this identification, we make the following definition:

Definition 3.1: Let S and T be Fredholm operators in $\mathfrak{I}^{\alpha,\beta}$. If there exists a positive integer n and a path of Fredholm operators in $M_n(\mathfrak{I}^{\alpha,\beta})$ connecting S and T, then S and T are said to be *stably connected by Fredholm operators*.

It is clear that a necessary condition for S and T to be stably connected is that they have the same index. We have shown that in Cases I and II above, $K_1(\mathcal{I}^{\alpha,\beta})$ is isomorphic to $\mathbb{Z}$, and that isomorphism is implemented by the index map. Therefore, the index condition is not only necessary, but sufficient as well:

Theorem 3.2: Let α and β be distinct numbers, at least one of which is rational. Then index is a complete stable deformation invariant for Fredholm operators in the skew quarter-plane Toeplitz algebra $\mathcal{T}^{\alpha,\beta}$; i.e., if S and T are Fredholm operators in $\mathcal{T}^{\alpha,\beta}$, they are stably connected by Fredholm operators if and only if S and T have the same index.∎

This theorem holds not only for $\mathcal{T}^{\alpha,\beta}$, but also for the matrix algebras over $\mathcal{T}^{\alpha,\beta}$.

CASE III. α and β both irrational

Finally, let us consider the case where α and β are both irrational. We begin with the K-theory of the symbol space $\mathcal{S}^{\alpha,\beta}$. When α and β are irrational, then $K_0(\mathcal{T}^{\alpha})$ and $K_0(\mathcal{T}^{\beta})$ consist of trivial projections and $K_1(\mathcal{T}^{\alpha})$ and $K_1(\mathcal{T}^{\beta})$ are both zero. It is therefore easy to see from the Mayer-Vietoris sequence that

$$K_0(\mathcal{F}^{\alpha,\beta}) \cong \mathbb{Z}^3$$

$$K_1(\mathcal{F}^{\alpha,\beta}) \cong \mathbb{Z}.$$

Calculating the K-theory of the remaining algebras is considerably more difficult. The difficulty lies in in producing an invertible element in $(\mathfrak{C}^\alpha)^+$ that generates $K_1(\mathfrak{C}^\alpha)$. Now, we have the short exact sequence

$$0 \longrightarrow \mathfrak{C}^\alpha \longrightarrow \mathfrak{T}^\alpha \longrightarrow C(T^2) \longrightarrow 0$$

and the exact diagram

$$\begin{array}{ccccc} K_0(\mathfrak{C}^\alpha) & \longrightarrow & K_0(\mathfrak{T}^\alpha) & \longrightarrow & K_0(C(T^2)) \\ \uparrow & & & & \downarrow \\ K_1(C(T^2)) & \longleftarrow & K_1(\mathfrak{T}^\alpha) & \longleftarrow & K_1(\mathfrak{C}^\alpha). \end{array}$$

The connecting homomorphism from $K_0(C(T^2))$ to $K_1(\mathfrak{C}^\alpha)$ maps the projection corresponding to the complex line bundle with chern class one to a generator of $K_1(\mathfrak{C}^\alpha)$. It is possible to actually write down this projection in $K_0(C(T^2))$, but the expression for it is so complicated that producing a generator of $K_1(\mathfrak{C}^\alpha)$ in this manner is infeasible. The inability to explicitly produce such a generator makes it impossible to analyze the connecting homomorphism between $K_1(\mathcal{C}^{\alpha})$ and $K_0(\mathcal{K})$. We encounter the same difficulties in considering the connecting homomorphism between $K_1(\mathcal{J}^{\alpha,\beta})$ and $K_0(\mathcal{K})$. If one of these connecting homomorphisms is an isomorphism, so is the other, in which case

$$K_0(\mathcal{J}^{\alpha}_{\beta}) \cong \mathbb{Z}^2$$

$$K_0(\mathcal{J}^{\alpha,\beta}) \cong \mathbb{Z}^3$$

$$K_1(\mathcal{J}^{\alpha}_{\beta}) \cong K_1(\mathcal{J}^{\alpha,\beta}) \cong 0.$$

On the other hand, if one of the connecting homomorphisms is the zero map, so is the other, and then

$$K_0(\mathcal{J}^{\alpha}_{\beta}) \cong \mathbb{Z}^3$$

$$K_0(\mathcal{J}^{\alpha,\beta}) \cong \mathbb{Z}^4$$

$$K_1(\mathcal{J}^{\alpha}_{\beta}) \cong K_1(\mathcal{J}^{\alpha,\beta}) \cong \mathbb{Z}.$$

There are, of course, other possibilities for the connecting homomorphisms, but it seems unlikely that they could occur. We will have more to about this in Chapter 5.

III. Cyclic cohomology

In the introduction, we asked the following questions: When is an operator in $\mathfrak{T}^{\alpha,\beta}$ Fredholm, and if T in $\mathfrak{T}^{\alpha,\beta}$ is Fredholm, what is its index? We answered the first question in Chapter 1, where we established criteria for an operator in $\mathfrak{T}^{\alpha,\beta}$ to be Fredholm. In this chapter, we answer the second question. We use Connes's cyclic cohomology to construct an explicit index formula for many operators in $\mathfrak{T}^{\alpha,\beta}$. Throughout this chapter we shall assume for concreteness that $\alpha < 0 < \beta$; as we have pointed out in previous chapters, no generality will be lost in making this assumption.

An index theorem for Fredholm operators in $\mathfrak{T}^{0,\infty}$ was given in [6]; the theorem uses the existence of certain operator-valued homotopies in the symbol algebra $\mathfrak{S}^{0,\infty}$, and these are in practice difficult to produce. Presumably this theorem generalizes to arbitrary skew quarter-plane Toeplitz algebras, but we seek an index formula that is easier to compute. We use Connes's cyclic cohomology to produce the desired index formula, and we begin by outlining those aspects of cyclic cohomology that we need.

Definition 3.1: Let A be a normed algebra with unit. For $n \geq 0$, let $C^n_\lambda(A)$ denote the A-module of $(n+1)$-linear continuous complex functionals φ on A such that

$$\varphi(a^1, a^2, \ldots, a^n, a^0) = (-1)^n \varphi(a^0, a^1, \ldots, a^n),$$

and for $n < 0$, define $C^n_\lambda(A)$ to be zero. Also, define the graded A-module

$$C^*_\lambda(A) = \sum_{n \in \mathbb{Z}} C^n_\lambda(A).$$

Definition 3.2: Let A be a normed algebra with unit. The graded A-module homomorphism b on $C^*_\lambda(A)$ is defined by

$$(b\varphi)(a^0,a^1,\ldots,a^{n+1}) =$$

$$\sum_{j=0}^{n-1} \varphi(a^0,\ldots,a^ja^{j+1},\ldots,a^{n+1}) +$$

$$(-1)^n\varphi(a^{n+1}a^0,a^1,\ldots,a^n)$$

for $n \geq 0$, and b is the zero map for $n < 0$.

One can check that $b^2 = 0$, and we can therefore consider the cohomology of the complex $(C_\lambda^*(A), b)$.

Definition 3.3: $H_\lambda^*(A)$ is the cohomology of the complex $(C_\lambda^*(A), b)$.

These are Connes's cyclic cohomology groups (actually, Connes takes a direct limit to obtain two groups $H^{even}(A)$ and $H^{odd}(A)$, but it is simpler for us to work with the groups $H_\lambda^n(A)$). The study of the groups $H_\lambda^n(A)$ naturally splits into the cases where n is even and where n is odd. It is the odd case that concerns us, since it will be the case that will yield our index formula. Therefore, we shall restrict our discussion to the odd case, specifically the case $n = 1$.

We can construct elements of $H^1_\lambda(A)$ in the following manner. Let $\mathcal{H}$ be a Hilbert space, and let ρ be a continuous (not necessarily unital) linear map from A into $L(\mathcal{H})$ with the property that $\rho(xy) - \rho(x)\rho(y)$ is a trace class operator. Such a map is called an almost multiplicative map. We can associate to ρ a cyclic 1-cocycle τ defined as follows:

$$\tau(a^0,a^1) = \text{Trace}\ (\epsilon_0 - \epsilon_1),$$

where

$$\epsilon_0 = \rho(a^0a^1) - \rho(a^0)\rho(a^1)$$

$$\epsilon_1 = \rho(a^1a^0) - \rho(a^1)\rho(a^0).$$

We would like to produce a cyclic 1-cocycle in this manner for the symbol algebra $\mathcal{S}^{\alpha,\beta}$. Consider the following short exact sequence from Chapter 1:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}^{\alpha,\beta} \longrightarrow \mathcal{S}^{\alpha,\beta} \longrightarrow 0.$$

We showed that this sequence has a linear splitting ρ from $\mathcal{S}^{\alpha,\beta}$ to $\mathcal{T}^{\alpha,\beta} \subset L(\mathcal{H}^{\alpha,\beta})$, and therefore for all X and

Y in $\mathcal{F}^{\alpha,\beta}$, $\rho(XY) - \rho(X)\rho(Y)$ is compact. Unfortunately, in this generality, this is the most we can say; $\rho(XY) - \rho(X)\rho(Y)$ is not always a trace class operator. We will get around this problem by restricting our choices of X and Y to a dense subalgebra $\mathcal{F}_{\infty}^{\alpha,\beta}$ of $\mathcal{F}^{\alpha,\beta}$ which we will define presently.

Definition 3.4: An operator T in $\mathcal{T}^{\alpha,\beta}$ of the form

$$T = P^{\alpha}P^{\beta}M_{m_0,n_0}\left[\prod_{j=1}^{\ell} Q_j M_{m_j,n_j} \right]P^{\alpha}P^{\beta},$$

where each Q_j is either P^{α}, P^{β}, or $P^{\alpha}P^{\beta}$, is called a *finite product*.

We recall some definitions from Chapter 1:

Definition 3.5: Let T be a finite product written as in Definition 3.4. Then define

$$\lambda^{\alpha}(T) = -\sum_{j=0}^{\ell} \min\{-\alpha m_j + n_j, 0\}.$$

$$\lambda^{\beta}(T) = \sum_{j=0}^{\ell} \max\{-\beta m_j + n_j, 0\}.$$

$$\Lambda(T) = \max\{1, \lambda^{\alpha}(T), \lambda^{\beta}(T)\}.$$

Definition 3.6: $\mathcal{T}_{\infty}^{\alpha}$ is the collection of operators T that can be written in the form

$$T = \sum_{k=0}^{\infty} c_k T_k,$$

where each T_k is a finite product, and where the sequence $\{c_k(\Lambda(T_k))^2\}$ is absolutely summable.

Note that in particular the sequence $\{c_k\}$ is absolutely summable, and since each finite product T_k has norm 1, the infinite sum above is well defined.

Proposition 3.7: *$\mathcal{T}_{\infty}^{\alpha,\beta}$ is an algebra.*

Proof: Clearly $\mathcal{T}_{\infty}^{\alpha,\beta}$ is closed under addition and scalar multiplication. The only nonobvious point to check is that $\mathcal{T}_{\infty}^{\alpha,\beta}$ is closed under multiplication.

Let S and T be finite products. Then it is easy to see from the definition of λ^{α} and λ^{β} that

$$\lambda^{\alpha}(ST) = \lambda^{\alpha}(S) + \lambda^{\alpha}(T)$$

$$\lambda^{\beta}(ST) = \lambda^{\beta}(S) + \lambda^{\beta}(T),$$

and therefore

$$\Lambda(ST) \leq \Lambda(S) + \Lambda(T).$$

Now let $S = \sum_{\ell=0}^{\infty} b_{\ell} S_{\ell}$ and $T = \sum_{k=0}^{\infty} c_k T_k$ be in $\mathcal{T}_{\infty}^{\alpha,\beta}$. Then

$$ST = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} b_{\ell} c_k S_{\ell} T_k,$$

and

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |b_{\ell} c_k| (\Lambda(S_{\ell} T_k))^2$$

$$\leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |b_{\ell} c_k| (\Lambda(S_{\ell}) + \Lambda(T_k))^2,$$

$$\leq \sum_{k=0}^{\infty} |c_k| \sum_{\ell=0}^{\infty} |b_\ell| (\Lambda(S_\ell))^2 + \sum_{\ell=0}^{\infty} |b_\ell| \sum_{k=0}^{\infty} |c_k| (\Lambda(T_k))^2$$

$$+ 2 \sum_{\ell=0}^{\infty} |b_\ell| \Lambda(S_\ell) \sum_{k=0}^{\infty} |c_k| \Lambda(T_k),$$

which is finite. Therefore ST is in $\mathcal{T}_{\infty}^{\alpha,\beta}$, as desired.∎

Lemma 3.8: Let T be a finite product in $\mathcal{T}^{\alpha,\beta}$. Then $T - \rho\gamma(T)$ is a partial isometry.

Proof: Since T is a finite product, T can be written in the form

$$T = P^{\alpha}P^{\beta}M_{m_0,n_0}\Big[\prod_{j=1}^{\ell} Q_j M_{m_j,n_j}\Big]P^{\alpha}P^{\beta}.$$

Therefore,

$$T - \rho\gamma(T) = T - \rho^{\alpha}\gamma^{\alpha}(T) - \rho^{\beta}\gamma^{\beta}(T) + \xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)$$

$$= P^{\alpha}P^{\beta}M_{m_0,n_0}\Big[\prod_{j=1}^{\ell} Q_j M_{m_j,n_j}\Big]P^{\alpha}P^{\beta}$$

$$- P^{\alpha}P^{\beta}M_{m_0,n_0}\Big[\prod_{j=1}^{\ell} Q_j^{\alpha}M_{m_j,n_j}\Big]P^{\alpha}P^{\beta}$$

$$- P^{\alpha}P^{\beta}M_{m_0,n_0}\Big[\prod_{j=1}^{\ell} Q_j^{\beta}M_{m_j,n_j}\Big]P^{\alpha}P^{\beta}$$

$$+ P^{\alpha}P^{\beta}M_{m_0,n_0}\Big[\prod_{j=1}^{\ell} M_{m_j,n_j}\Big]P^{\alpha}P^{\beta}.$$

Note that each of these summands must either annihilate a basis element $e_{m,n}$ or map it to the basis element $e_{M,N}$, where

$$M = m + \sum_{j=0}^{\ell} m_j$$

$$N = n + \sum_{j=0}^{\ell} n_j.$$

Moreover, note that each of the four summands above contains the same translations; only the projections that appear in each term differ. Therefore, if two summands do not annihilate a basis element, they must each map that basis element to the same place.

Suppose that (m,n) is a pair of integers for which $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)(e_{m,n}) = 0$. Then it is clear that the other three terms in the sum above also annihilate $e_{m,n}$, so $(T - \rho\gamma(T))(e_{m,n}) = 0$.

Now suppose $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)(e_{m,n})$ is not zero. We consider several cases.

Case I: $\rho^{\alpha}\gamma^{\alpha}(T)(e_{m,n}) = 0$, $\rho^{\beta}\gamma^{\beta}(T)(e_{m,n}) = 0$.

Since $\rho^{\alpha}\gamma^{\alpha}(T)(e_{m,n}) = 0$, it is clear that $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)(e_{m,n}) = 0$ as well. Therefore,

$$(T - \rho\gamma(T))(e_{m,n}) = e_{M,N} - 0 - 0 + 0 = e_{M,N}.$$

Case II: $\rho^{\alpha}\gamma^{\alpha}(T)(e_{m,n}) = 0$, $\rho^{\beta}\gamma^{\beta}(T)(e_{m,n}) \neq 0$.

Just as in case I, $\rho^{\alpha}\gamma^{\alpha}(T)$ annihiliates $e_{m,n}$, so $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)$ must also annihilate $e_{m,n}$, and thus

$$(T - \rho\gamma(T))(e_{m,n}) = e_{M,N} - 0 - e_{M,N} + 0 = 0.$$

Case III: $\rho^{\alpha}\gamma^{\alpha}(T)(e_{m,n}) \neq 0$, $\rho^{\beta}\gamma^{\beta}(T)(e_{m,n}) = 0$.

This case is essentially the same as case II; since $\rho^{\beta}\gamma^{\beta}(T)(e_{m,n}) = 0$, $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T) = 0$ also. Thus,

$$(T - \rho\gamma(T))(e_{m,n}) = e_{M,N} - e_{M,N} - 0 + 0 = 0.$$

Case IV: $\rho^{\alpha}\gamma^{\alpha}(T)(e_{m,n}) \neq 0$, $\rho^{\beta}\gamma^{\beta}(T)(e_{m,n}) \neq 0$.

If neither of these terms annihilates $e_{m,n}$, then the projections that appear in the expressions for these two terms are unnecessary. However, since these projections are the same ones that appear in $\xi^{\alpha,\beta}\sigma^{\alpha,\beta}(T)$, this term will not annihilate $e_{m,n}$ either. Therefore,

$$(T - \rho\gamma(T))(e_{m,n}) = e_{M,N} - e_{M,N} - e_{M,N} + e_{M,N} = 0.$$

Thus in each case a basis element $e_{m,n}$ is either annihilated or mapped to a basis element by $T - \rho\gamma(T)$. Therefore $T - \rho\gamma(T)$ is a partial isometry whose initial space is the space spanned by the $e_{m,n}$ which are not annihilated.∎

Lemma 3.9: Let T be a finite product in $\mathfrak{T}^{\alpha,\beta}$, and let $\| \ \|_1$ denote trace norm. Then

$$\| T - \rho\gamma(T) \|_1 \leq C\, \Lambda(T)^2,$$

where C is a constant depending only on α and β.

Proof: Since T is a finite product, we may write T in the form

$$T = P^{\alpha}P^{\beta}M_{m_0,n_0}\left[\prod_{j=1}^{\ell} Q_j M_{m_j,n_j} \right]P^{\alpha}P^{\beta}.$$

The previous lemma yields that $T - \rho\gamma(T)$ is a partial isometry, so $[T - \rho\gamma(T)][T - \rho\gamma(T)]^*$ is a projection onto the final space of $T - \rho\gamma(T)$. Therefore,

$$\| T - \rho\gamma(T) \|_1 = \| (T - \rho\gamma(T))^* \|_1$$

$$= \text{trace } \{[T - \rho\gamma(T))][T - \rho\gamma(T)]^*\}$$

$$= \dim \text{ran } (T - \rho\gamma(T)).$$

The dimension of the range of $T - \rho\gamma(T)$ is the number of $e_{m,n}$ such that $(T - \rho\gamma(T))(e_{m,n}) \neq 0$, and we noted in Chapter 1 that this number is bounded by the number of pairs of integers (m,n) that satisfy the following two inequalities:

$$0 \leq -\alpha m + n \leq \lambda^{\alpha}(T)$$

$$-\lambda^{\beta}(T) \leq -\beta m + n \leq 0.$$

On one hand, we can combine these two inequalities to obtain

$$0 \leq (\beta - \alpha)m \leq \lambda^{\alpha}(T) + \lambda^{\beta}(T),$$

so the number of different possible values of m that can appear in a solution to the inequalities is bounded by

$$\frac{\lambda^{\alpha}(T) + \lambda^{\beta}(T)}{\beta - \alpha} + 1.$$

On the other hand, we can also combine the two inequalities to obtain

$$\alpha\lambda^{\beta}(T) \leq (\beta - \alpha)n \leq \beta\lambda^{\alpha}(T),$$

so the number of possible values of n that can appear in a solution to the inequalities is bounded by

$$\frac{\beta\lambda^{\alpha}(T) - \alpha\lambda^{\beta}(T)}{\beta - \alpha} + 1.$$

Hence, the total number of possible solutions, and hence the dimension of $T - \rho\gamma(T)$, is bounded by

$$\left[\frac{\lambda^{\alpha}(T) + \lambda^{\beta}(T)}{\beta - \alpha} + 1 \right]\left[\frac{\beta\lambda^{\alpha}(T) - \alpha\lambda^{\beta}(T)}{\beta - \alpha} + 1 \right].$$

We get the bound in the statement of the lemma by recalling the definition on $\Lambda(T)$ as the maximum of 1, $\lambda^{\alpha}(T)$ and $\lambda^{\beta}(T)$.∎

Definition 3.10: $\mathcal{S}_{\infty}^{\alpha,\beta} = \gamma(\mathcal{T}_{\infty}^{\alpha,\beta})$.

Proposition 3.11: Let X and Y be in $\mathcal{S}_{\infty}^{\alpha,\beta}$. *Then* $\rho(XY) - \rho(X)\rho(Y)$ *is trace class.*

Proof: Choose operators S and T in $\mathcal{T}_{\infty}^{\alpha,\beta}$ such that $\gamma(S) = X$ and $\gamma(T) = Y$. We first consider the case where S and T are both finite products. Now,

$$\rho(XY) - \rho(X)\rho(Y) = \rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T))$$

$$= S[T - \rho\gamma(T)] + [S - \rho\gamma(S)]\rho\gamma(T) - [ST - \rho\gamma(ST)].$$

Therefore,

$$\| \rho(XY) - \rho(X)\rho(Y) \|_1 \le \| S \|_\infty \| T - \rho\gamma(T) \|_1$$

$$+ \| S - \rho\gamma(S) \|_1 \| T \|_\infty + \| ST - \rho\gamma(ST) \|_1.$$

Using the fact that $\| S \|_\infty = \| T \|_\infty = 1$, along with the estimate from Lemma 3.9, we obtain

$$\| \rho(XY) - \rho(X)\rho(Y) \|_1 \le C[\Lambda(S)^2 + \Lambda(ST)^2 + \Lambda(T)^2].$$

Thus $\rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T))$ is a trace class operator, and is in fact a finite rank operator.

Now, let $S = \sum_{\ell=0}^{\infty} b_\ell S_\ell$ and $T = \sum_{k=0}^{\infty} c_k T_k$ be operators in $\mathfrak{T}_\infty^{\alpha,\beta}$, where the S_ℓ and T_k are finite products. Since ρ and γ are linear,

$$\rho(\gamma(ST)) - \rho(\gamma(S))\rho(\gamma(T)) =$$

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} b_\ell c_k [\rho(\gamma(S_\ell T_k)) - \rho(\gamma(S_\ell))\rho(\gamma(T_k))],$$

and therefore

$$\| \rho(XY) - \rho(X)\rho(Y) \|_1 \leq \cdot$$

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |b_\ell c_k| \| \rho(\gamma(S_\ell T_k)) - \rho(\gamma(S_\ell))\rho(\gamma(T_k)) \|_1$$

$$\leq C \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |b_\ell c_k| [\Lambda(S_\ell)^2 + \Lambda(S_\ell T_k)^2 + \Lambda(T_k)^2].$$

Now, since $\Lambda(S_\ell T_k) \leq \Lambda(S_\ell) + \Lambda(T_k)$ for all ℓ and k, the above sum is finite, and therefore $\rho(XY) - \rho(X)\rho(Y)$ is a trace class operator.∎

The proposition above shows that ρ yields a cyclic 1-cocycle on $\mathcal{J}_\infty^{\alpha,\beta}$. We now consider how such a cocycle will enable us to compute the index of Fredholm operators in $\mathfrak{T}^{\alpha,\beta}$.

One of the most important features of the group $H^1_\lambda(A)$ is that there is a bilinear pairing $\langle \ , \ \rangle$ of $H^1_\lambda(A)$ with $K_1(A)$. Let u be an invertible element of A, let ρ be an almost multiplicative map, and let τ be the cyclic 1-cocycle associated to ρ. Then

$$\langle [u], [\tau] \rangle = \text{index } \rho(u),$$

where [u] denotes the class of u in $K_1(A)$ and $[\tau]$ denotes the class of τ in $H^1_\lambda(A)$. In the case we are considering, the pairing is particularly simple:

$$\langle [u],[\tau] \rangle = \tau(u - 1, u^{-1} - 1).$$

If addition, ρ is a unital map, as it is in our case, the pairing is

$$\langle [u],[\tau] \rangle = \tau(u,u^{-1}).$$

It is this result that we will use to obtain our index formula.

Let T be an operator in the skew quarter-plane Toeplitz algebra $\mathfrak{T}^{\alpha,\beta}$. Then it is easy to see that T and $\rho\gamma(T)$ differ by a compact operator, so T is Fredholm if and only if $\rho\gamma(T)$ is. Moreover, if T is Fredholm, then T and $\rho\gamma(T)$ have the same index. Therefore, to determine the index of T, it suffices to compute the index of $\rho\gamma(T)$. Combining the pairing above with the definitions of ρ and γ, we obtain the desired index formula:

Theorem 3.12: *Let T in $\mathfrak{T}^{\alpha,\beta}$ be a Fredholm operator such that $\gamma(T)$ and $\gamma(T)^{-1}$ are in $\mathcal{S}_{\infty}^{\alpha,\beta}$. Then the index of T is given by the following formula:*

$$Index\ T = Trace\left[\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T))\right] \blacksquare.$$

We will have more to say about this index formula in Chpater 5, but we also make some remarks here. First, the class of Fredholm operators we consider above is not the largest class for which this formula will work. However, the above formula applies to many cases of interest. Second, this index formula will also give the index of matrix Toeplitz operators, where the trace in the above formula is taken to be the usual matrix trace. Finally, this index formula gives the index of Fredholm operators in $\mathfrak{T}_{\infty}^{\alpha,\beta}$ for all values of α and β. However, when α and β are both irrational, it is not known if there are any operators of nonzero index, so in this case the index theorem may turn out to be uninteresting.

IV. Examples

We now use our index formula to compute the index of some specific Fredholm operators.

EXAMPLE 1

Let α be rational and negative, and let β be any positive number, rational or irrational. Write $\alpha = \frac{p}{q}$, with p and q relatively prime and $p < 0$ and $q > 0$. Also, choose positive integers r and s. Then let

$$T = P^{\alpha}P^{\beta}M_{q,p}P^{\alpha}P^{\beta} +$$

$$(I - P^{\alpha}P^{\beta}M_{q,p}P^{\alpha}P^{\beta})P^{\alpha}P^{\beta}M_{s,r}P^{\alpha}P^{\beta}M_{-s,-r}P^{\alpha}P^{\beta}.$$

Then

$$\gamma^{\alpha}(T) = P^{\alpha}M_{q,p}P^{\alpha} + (I - P^{\alpha}M_{q,p}P^{\alpha})P^{\alpha}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}.$$

It is easy to check that $P^{\alpha}M_{q,p}P^{\alpha}$ is a unitary operator in $\mathcal{J}^{\alpha}$ and that $P^{\alpha}M_{q,p}P^{\alpha}$ commutes with $P^{\alpha}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}$, so $\gamma^{\alpha}(T)$ is invertible with inverse

$$\gamma^{\alpha}(T)^{-1} = P^{\alpha}M_{-q,-p}P^{\alpha} +$$

$$(I - P^{\alpha}M_{-q,-p}P^{\alpha})P^{\alpha}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}.$$

We also have

$$\gamma^{\beta}(T) = \gamma^{\beta}(T)^{-1} = I.$$

Therefore, $\gamma(T)$ is invertible, so T is a Fredholm operator. We will now use our index formula to compute the index of T.

$$\rho(\gamma(T)) = \rho^{\alpha}(\gamma^{\alpha}(T)) + \rho^{\beta}(\gamma^{\beta}(T)) - \xi^{\alpha,\beta}(\sigma^{\alpha,\beta}(T))$$

$$= P^{\alpha}P^{\beta}M_{q,p}P^{\alpha}P^{\beta} +$$

$$(I - P^{\alpha}P^{\beta}M_{q,p}P^{\alpha}P^{\beta})P^{\alpha}P^{\beta}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}P^{\beta},$$

and

$$\rho(\gamma(T)^{-1}) = P^{\alpha}P^{\beta}M_{-q,-p}P^{\alpha}P^{\beta} +$$

$$(I - P^{\alpha}P^{\beta}M_{-q,-p}P^{\alpha}P^{\beta})P^{\alpha}P^{\beta}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}P^{\beta}.$$

Next, direct computation yields that

$$\rho(\gamma(T)^{-1})\rho(\gamma(T)) = I$$

$$\rho(\gamma(T))\rho(\gamma(T)^{-1}) = I - AB,$$

where

$$A = P^{\alpha}P^{\beta}M_{q,p}(I - P^{\alpha}P^{\beta})M_{-q,-p}P^{\alpha}P^{\beta}$$

$$B = P^{\alpha}P^{\beta}M_{s,r}(I - P^{\alpha})M_{-s,-r}P^{\alpha}P^{\beta}.$$

Therefore,

$$\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T)) = - AB.$$

It is easy to check that this is a finite rank projection, and $e_{m,n}$ is in the range if and only if the pair of integers (m,n) satisfies the inequalities

$$0 \leq -\alpha m + n < -\alpha s + r$$

$$0 \geq -\beta m + n > -\beta q + p.$$

We therefore have from our index formula that the index of T is minus the number of pairs of integers (m,n) that satisfy the inequalities above.

To make this example more concrete, let q = 1, p = -1, r = 1, and s = 2. Then $\alpha = -1$. Also, let $\beta = \sqrt{2}$. The above inequalities become

$$0 \leq m + n < 3$$

$$0 \geq - m\sqrt{2} + n > - \sqrt{2} - 1.$$

Direct calculation yields that the only pairs of integers that are solutions to the inequalities are (0,0), (1,0), and (1,1). Therefore, the index of T in this case is -3.

EXAMPLE 2

Let α be rational and negative, and let β be any positive number, rational or irrational. Write $\alpha = \frac{p}{q}$, with p and q relatively prime and $p < 0$, $q > 0$. Also, choose positive integers r and s. Then let

$$T = P^{\alpha}P^{\beta}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}P^{\beta} + \tfrac{1}{2}P^{\alpha}P^{\beta}M_{-q,-p}P^{\alpha}P^{\beta}.$$

Now,

$$\gamma^{\alpha}(T) = P^{\alpha}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha} + \tfrac{1}{2}P^{\alpha}M_{-q,-p}P^{\alpha}$$

$$\gamma^{\alpha}(T)^{-1} = 2P^{\alpha}M_{q,p}P^{\alpha} +$$

$$\left[\sum_{n=0}^{\infty} (-2)^{-n}\ P^{\alpha}M_{-nq,-np}P^{\alpha}\right]P^{\alpha}M_{s,r}P^{\alpha}M_{-s,-r}P^{\alpha}.$$

Also,

$$\gamma^{\beta}(T) = I + \tfrac{1}{2}P^{\beta}M_{-q,-p}P^{\beta}$$

$$\gamma^{\beta}(T)^{-1} = \sum_{n=0}^{\infty} (-2)^{-n} P^{\beta} M_{-nq,-np} P^{\beta}.$$

Both $\gamma^{\alpha}(T)$ and $\gamma^{\beta}(T)$ are invertible, so T is Fredholm. Next,

$$\rho(\gamma(T)) = P^{\alpha} M_{s,r} P^{\alpha} M_{-s,-r} P^{\alpha} + \tfrac{1}{2} P^{\alpha} M_{-q,-p} P^{\alpha}$$

$$\rho(\gamma(T)^{-1}) = 2(I - P^{\alpha}P^{\beta} M_{s,r} P^{\alpha} M_{-s,-r} P^{\alpha}P^{\beta}) P^{\alpha}P^{\beta} M_{q,p} P^{\alpha}P^{\beta}$$

$$+ \left[\sum_{n=0}^{\infty} (-2)^{-n} P^{\alpha}P^{\beta} M_{-nq,-np} P^{\alpha}P^{\beta}\right] P^{\alpha}P^{\beta} M_{s,r} P^{\alpha} M_{-s,-r} P^{\alpha}P^{\beta}.$$

Therefore,

$$\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T)) = AB,$$

where

$$A = P^{\alpha}P^{\beta} M_{q,p} (I - P^{\alpha}P^{\beta}) M_{-q,-p} P^{\alpha}P^{\beta}$$

$$B = P^{\alpha}P^{\beta} M_{s,r} (I - P^{\alpha}) M_{-s,-r} P^{\alpha}P^{\beta}.$$

Just as in the Example 1, we have a finite rank projection, and $e_{m,n}$ is in its range if and only if the pair of integers (m,n) satisfies the inequalities

$$0 \leq -\alpha m + n < -\alpha s + r$$

$$0 \geq -\beta m + n > -\beta q + p.$$

We therefore have from our index formula that the index of T is the number of pairs of integers (m,n) that satisfy the inequalities above.

EXAMPLE 3

As we mentioned earlier, we can also consider skew quarter-plane Toeplitz operators with matrix-valued symbols. Let $\alpha = \frac{p}{q}$, with $p < 0$ and $q > 0$, and let $\beta = \frac{r}{s}$ with r and s both positive. We also assume that p and q and r and s are relatively prime. Now, define

$$T = \begin{bmatrix} P^{\alpha}P^{\beta}M_{mq,mp}P^{\alpha}P^{\beta} & -P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta} \\ P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta} & P^{\alpha}P^{\beta}M_{-mq,-mp}P^{\alpha}P^{\beta} \end{bmatrix}.$$

Then

$$\gamma^{\alpha}(T) = \begin{bmatrix} P^{\alpha}M_{mq,mp}P^{\alpha} & P^{\alpha}M_{ns,nr}P^{\alpha} \\ P^{\alpha}M_{-ns,-nr}P^{\alpha} & P^{\alpha}M_{-mq,-mp}P^{\alpha} \end{bmatrix},$$

and $\gamma^{\alpha}(T)^{-1}$ =

$$\begin{bmatrix} \frac{1}{2}P^{\alpha}M_{-mq,-mp}P^{\alpha} & \frac{1}{2}P^{\alpha}M_{ns,nr}P^{\alpha} \\ (\frac{1}{2}P^{\alpha}M_{ns,nr}P^{\alpha}M_{-ns,-nr}P^{\alpha} - I)P^{\alpha}M_{-ns,-nr}P^{\alpha} & \frac{1}{2}P^{\alpha}M_{mq,mp}P^{\alpha} \end{bmatrix}$$

Also,

$$\gamma^{\beta}(T) = \begin{bmatrix} P^{\beta}M_{mq,mp}P^{\beta} & - P^{\beta}M_{ns,nr}P^{\beta} \\ P^{\beta}M_{-ns,-nr}P^{\beta} & P^{\beta}M_{-mq,-mp}P^{\beta} \end{bmatrix},$$

and $\gamma^{\beta}(T)^{-1}$ =

$$\begin{bmatrix} (I - \frac{1}{2}P^{\beta}M_{ns,nr}P^{\beta}M_{-ns,-nr}P^{\beta})P^{\beta}M_{-mq,-mp}P^{\beta} & \frac{1}{2}P^{\beta}M_{ns,nr}P^{\beta} \\ - \frac{1}{2}P^{\beta}M_{-ns,-nr}P^{\beta} & \frac{1}{2}P^{\beta}M_{mq,mp}P^{\beta} \end{bmatrix}$$

Therefore, T is a Fredholm operator. To find the index of T, we compute

$$\rho(\gamma(T)) = \begin{bmatrix} P^{\alpha}P^{\beta}M_{mq,mp}P^{\alpha}P^{\beta} & -P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta} \\ P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta} & P^{\alpha}P^{\beta}M_{-mq,-mp}P^{\alpha}P^{\beta} \end{bmatrix},$$

and

$$\rho(\gamma(T)^{-1}) = \begin{bmatrix} X & \frac{1}{2}P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta} \\ Y & \frac{1}{2}P^{\alpha}P^{\beta}M_{mq,mp}P^{\alpha}P^{\beta} \end{bmatrix},$$

where

$$X = (I - \tfrac{1}{2}P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta})P^{\alpha}P^{\beta}M_{-mq,-mp}P^{\alpha}P^{\beta}$$

$$Y = (\tfrac{1}{2}P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta} - I)P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta}.$$

Then $\rho(\gamma(T))\rho(\gamma(T)^{-1}) =$

$$\begin{bmatrix} I - AB & 0 \\ 0 & I \end{bmatrix},$$

where

$$A = (I - P^{\alpha}P^{\beta}M_{mq,mp}P^{\alpha}P^{\beta}M_{-mq,-mp}P^{\alpha}P^{\beta})$$

$$B = (I - P^{\alpha}P^{\beta}M_{ns,nr}P^{\alpha}P^{\beta}M_{-ns,-nr}P^{\alpha}P^{\beta}).$$

Also, $\rho(\gamma(T)^{-1})\rho(\gamma(T)) = I$. Therefore,

$$\text{Trace}\left[\rho(\gamma(T))\rho(\gamma(T)^{-1}) - \rho(\gamma(T)^{-1})\rho(\gamma(T))\right]$$

$$= -\text{ Trace } AB.$$

In particular, if we let $\alpha = 0$ and $\beta = \infty$, then the trace of AB is easily seen to be mn. Therefore, in this case, the index of T is -mn.

V. Open questions

In this chapter, we consider some open questions and discuss in greater detail some of the points brought up in the previous chapters.

1. Is there a more direct way to show that $\mathfrak{R}^{\alpha,\beta}$ and $\mathfrak{T}^{\alpha,\beta}$ are the same algebra? When we first defined $\mathfrak{R}^{\alpha,\beta}$, we believed that it properly contained the skew quarter-plane Toeplitz algebra, inasmuch as there are many operators in $\mathfrak{R}^{\alpha,\beta}$ that do not seem to be in $\mathfrak{T}^{\alpha,\beta}$. For example, as we mentioned in Chapter 1, operators of the form $P^{\alpha}P^{\beta}M_{k,\ell}P^{\alpha}M_{m,n}P^{\alpha}P^{\beta}$ are in $\mathfrak{R}^{\alpha,\beta}$, but it is not at all obvious why these operators are in $\mathfrak{T}^{\alpha,\beta}$, especially in the cases where α and β are both irrational. In any case, the fact that these two algebras are the same is somewhat mysterious, and it would be desirable to have a more direct explanation of this fact.

2. Is there another description of $\mathcal{J}_{\beta}^{\alpha}$ and $\mathcal{J}_{\alpha}^{\beta}$? It seems to us that these two ideals should be represented as commutator ideals of some sort. Such a description would perhaps be helpful in understanding $\mathcal{J}^{\alpha,\beta}$ in the cases where α and β are irrational. It also seems likely that understanding the connection between foliations and skew quarter-plane Toeplitz algebras will hinge upon a better understanding of these ideals.

3. What is the index theory of $\mathcal{J}^{\alpha,\beta}$ when α and β are irrational? Understanding the index picture in this case amounts to identifying the index map between $K_1(\mathcal{J}^{\alpha,\beta})$ and $K_0(\mathcal{K})$. As we mentioned at the end of Chapter 2, it seems likely that the index map here is either an isomorphism or the zero homomorphism. Perhaps it is even possible for the index map to be something else, but this possibility can almost be discarded on metamathematical grounds. It may also turn out that the index theory of $\mathcal{J}^{\alpha,\beta}$ is not the same for all irrational numbers α and β. For example, we might get different index results when α and β are rationally dependent than when they are not.

4. Is the algebra $\mathcal{J}_{\infty}^{\alpha,\beta}$ closed under inverses? As it now stands, our index formula requires one to check that a Fredholm operator T in $\mathcal{T}^{\alpha,\beta}$ has the property that both $\gamma(T)$ and $\gamma(T)^{-1}$ are in $\mathcal{J}_{\infty}^{\alpha,\beta}$. While this is not terrible drawback, we would prefer to know that if X is in $\mathcal{J}_{\infty}^{\alpha,\beta}$ and X is invertible in $\mathcal{J}^{\alpha,\beta}$, then X^{-1} is in $\mathcal{J}_{\infty}^{\alpha,\beta}$ as well. To show closure under inverses, it would suffice to show that $\mathcal{J}_{\infty}^{\alpha,\beta}$ is closed under the holomorphic functional calculus. Knowing this would also imply that the inclusion map from $\mathcal{J}_{\infty}^{\alpha,\beta}$ to $\mathcal{J}^{\alpha,\beta}$ induces an isomorphism in K-theory between these algebras [8].

5. Is the construction of the cyclic cocyle on $\mathcal{J}_{\infty}^{\alpha,\beta}$ a special case of a more general construction? To our knowledge, no one has constructed a cyclic cocycle in the manner which we have. All other cyclic cocycles arise in connection with elliptic operators or in some other geometric context, but in our case, these connections seem to be missing.

6. Is there another representation of $\mathcal{T}^{\alpha,\beta}$ which allows one to answer some of the questions above? In the case of the half-plane Toeplitz algebra, one can define a "real valued" index by representing the algebra as a

Toeplitz algebra of translations on the real line. The success of this approach suggests that it would be fruitful for us to find an alternate representation for $\mathfrak{T}^{\alpha,\beta}$. We suspect that the index theory of $\mathfrak{T}^{\alpha,\beta}$ when α and β are irrational could be better analyzed this way. We also believe that an alternate representation of $\mathcal{T}^{\alpha,\beta}$ would allow us to either show that $\mathcal{T}^{\alpha,\beta}_{\infty}$ is closed under the holomorphic functional calculus, or else find another dense subalgebra of $\mathcal{T}^{\alpha,\beta}$ that is closed under this calculus and for which our cyclic cocycle is defined.

7. How can the skew quarter-plane Toeplitz algebra be used to study foliations on the torus? The relationship between the half-plane Toeplitz algebra and foliations by lines on the two-torus is discussed in [9]. The existence of this relationship suggests that perhaps the skew quarter-plane Toeplitz algebra provides a method of studying the case of two transverse foliations on the torus.

8. Can the skew quarter-plane Toeplitz algebra be generalized to study foliations on manifolds other than the torus? In the case of the torus, the subspaces onto

which we project to get the half-plane Toeplitz algebras arise from the positive eigenspaces of differentiation along the leaves of a foliation. It would be interesting to try to define Toeplitz algebras for foliations on other manifolds and see if the algebras provide invariants for these foliations. If so, the skew quarter-plane Toeplitz algebra should generalize in the above manner for manifolds that admit two transverse foliations.

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