On the Global Geometry of Complete Open Surfaces of Nonnegative Curvature

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Abstract of the Dissertation

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In this dissertation we will work with a complete open manifold M of nonnegative curvature. We present a detailed study involving various quantitative aspects of the global geometry of rays and arbitrary geodesics in M, primarily in the case of surfaces. Our results extend and build on the pioneering work of S. Cohn-Vossen as well as the basic ideas in the qualitative structure theory for such spaces given by J. Cheeger, W. T. Meyer, and D. Gromoll.

Our most important tool is the <u>Busemann function</u> $B: M \to \mathbb{R}$ associated with a ray r, $B(Q) := \lim_{t\to\infty}[t-\rho(r(t),Q)]$,

where ρ is the distance function on M. B is convex (hence continuous), but not necessarily differentiable. However, the singularities have an interesting geometry.

On a surface, it is a basic problem to understand how an arbitrary geodesic g behaves near infinity. We introduce a concept of asymptotic "winding" and discuss various results in this direction. For example, all geodesics have finite winding for total curvature less than 2π. If the total curvature equals 2π, the situation is more subtle.

Given r and B, we obtain a family of rays associated with B, called B-rays, which pass through every point of M. We develop the relationship between the B-rays and arbitrary geodesics. Restricting our attention to surfaces, it seems very important to analyze what happens at singularities of B. We introduce the notion of a B-wedge, i.e., a region W bounded by two B-rays which meet at their common initial point. We discuss the total curvature of W. Given that $B^a := \{P \in M | B(P) \leq a\}$, we study its boundary ∂B^a (which is called a "horosphere") by using the geometry of B-rays and B-wedges. Making strong use of all the preceding work we prove a main result: The B^a are compact if and only if the total curvature of M is greater than π. In fact, we arrive at more delicate conclusions.

Finally, we consider two rays $r, \tilde{r}$ and their associated Busemann functions $B, \tilde{B}$, respectively, in the case of total curvature equal to 2π. We show that B and $\tilde{B}$ are "asymptotically equal," i.e., the angles between the B-rays and $\tilde{B}$-rays become arbitrarily small far enough out.

This dissertation is lovingly dedicated to my parents, Barbara and Charles York, and respectfully dedicated to my advisor, Professor Detlef Gromoll.

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Chapter 0. Introduction and Preliminary Observations

Introduction

This dissertation deals with manifolds M (primarily surfaces) which are complete, noncompact, and have non-negative curvature. Recall that a ray is a unit speed geodesic $r : [0,\infty) \to M$ such that r minimizes distance between any two of its points. On a complete, noncompact manifold M there is at least one ray through any point $P \in M$.

Associated with any ray r is a Busemann function $B : M \to \mathbb{R}$ given by $B(P) := \lim_{t\to\infty}[t - \rho(r(t),P)]$, $P \in M$, where ρ is the distance function on M. Such a function B has many nice properties, but it is not necessarily differentiable. Several authors have studied M by "smoothing out" B: C^∞ Approximations of Convex, Subharmonic, and Plurisubharmonic Functions (1979), by R. Greene and H. Wu. But as we shall see, some interesting geometry occurs at the singularities.

We begin by developing results about two types of rays associated with a given ray r: the flow lines of the Busemann function B associated with r, and rays "induced" by r through a type of variation. It is seen that these two types of rays quite often coincide, and the dual interpretation of these rays is exploited throughout the paper.

An important geometric question concerns the asymptotic behavior of arbitrary geodesics. A notion of "winding" is introduced for surfaces, and a necessary condition for the existence of "infinitely winding" geodesics is established.

The level sets $B^{-1}(a)$, $a \in \mathbb{R}$, are known as _horospheres._ One encounters horospheres in Lobachevski's _Geometrical Researches on the Theory of Parallels_ (19840), where they occur as flat planes isometrically imbedded in hyperbolic 3-space. An important recent paper employing the sets $B^a = \{P \in M | B(P) \leq a\}$, $a \in \mathbb{R}$, is _On the Structure of Complete Manifolds of Nonnegative Curvature_ (1972), by J. Cheeger and D. Gromoll. It is proved there that the sets B^a are "totally convex sets," i.e. given any two points in the set, and any geodesic segment joining these two points, the geodesic segment is also contained in the set.

One of our main results in the case of a surface M, is to establish a connection between compactness of the sets B^a and the total curvature C(M) of M. Specifically, the B^a are compact if and only if $C(M) > \pi$. (See the appendix for an intuitive motivation in the case of smoothed-out cones.)

Finally, in the case of a surface with $C(M) = 2\pi$, we compare the flow lines associated with two different Busemann functions, and find that they are "asymptotically equal," i.e. the angle between the two families of flow lines becomes arbitrarily small sufficiently far out.

Preliminary Observations

We will freely use facts from Riemannian Geometry and Surface Theory (see [GKM] and [CE], as well as [S]). We shall use Toponogov's Theorem [cf. CE]:

Let M be complete, its sectional curvature $K_M \geq H$, and let g_1, g_2 be geodesic segments in M such that

$$g_1(l_1) = g_2(0) \text{ and } \measuredangle(-\dot{g}_1(l_1), \dot{g}_2(0)) = \alpha.$$

Let g_1 be minimal, and if $H > 0$, $L[g_2] \leq \pi \cdot H^{-\frac{1}{2}}$. Let $\bar{g}_1, \bar{g}_2 \subseteq M^H$, the simply connected 2-dimensional space of constant curvature H, be such that $\bar{g}_1(l_1) = \bar{g}_2(0)$, $L[g_i] = L[g_i] = l_i$, and

$$\measuredangle(-\dot{g}_1 \; (l_1), \dot{g}_2(0)) = \alpha.$$

Then

$$\rho(g_1(0), g_2(l_2)) \leq \rho(\bar{g}_1(0), \bar{g}_2(l_2)).$$

We note that the Busemann function is defined, i.e. the limit in its definition exists: For fixed P, the quantity $t - \rho(r(t), P)$ is nondecreasing in t, and is bounded above by $\rho(r(0), P)$.

PROOF: If $0 \leq s < t$, then $\rho(r(t), r(s)) = t - s$ (r is a ray). By the triangle inequality,

$$\rho(r(t),P) \leqq \rho(r(t),r(s)) + \rho(r(s),P) = (t-s) + \rho(r(s),P).$$

Thus

$$t - \rho(r(t),P) \geqq s - \rho(r(s),P).$$

Also,

$$t = \rho(r(t),r(0)) \leqq \rho(r(t),P) + \rho(P,r(0)),$$

so $t - \rho(r(t),P)$ is bounded above by $\rho(P,r(0))$.

We next show that for all $P,Q \in M$,

$$|B(P) - B(Q)| \leqq \rho(P,Q).$$

$$B(P) - B(Q) = \lim_{t\to\infty}[t - \rho(r(t),P)] - \lim_{t\to\infty}[t - \rho(r(t),Q)]$$

$$= \lim_{t\to\infty}[\rho(r(t),Q) - \rho(r(t),P)] \leqq \lim_{t\to\infty}[\rho(P,Q)]$$

$$= \rho(P,Q).$$

Interchanging P and Q gives

$$-\rho(P,Q) \leqq B(P) - B(Q).$$

We now interpret B geometrically. Suppose $B(P) = a$, and consider (for $t > a$) the open ball $B_{r(t)}(t-a)$ with center $r(t)$ and radius $t - a$. As we have seen, $t - \rho(r(t),P)$ increases to $B(P) = a$ (in the limit), so $t - a \leqq \rho(r(t),P)$. Thus P is not in the open ball $B_{r(t)}(t-a)$ for any t, but it is an accumulation point of the union, $t > a$, of these (nested) balls. Thus the level sets of B may be interpreted as spheres of infinite radius.

For example, in the case of the flat plane, the level sets of any Busemann function associated with any ray r are the lines perpendicular to the line containing r.

We shall use the following facts, first show by Cohn-Vossen (see [CV], 1936):

A complete open surface M with curvature $K \geq 0$ is either a flat cylinder or diffeomorphic to $\mathbb{R}^2$. Furthermore, such a surface has total curvature $(C(M) \leq 2\pi$.

We shall sometimes consider a ray $b : [0,\infty) \to M$ and its restriction $b : [s,\infty) \to M$, $s > 0$, to be the same. I.e., we shall, when convenient, delete a finite initial segment.

Chapter 1. Induced rays and B-rays

In this chapter we shall see that any noncompact complete manifold can be fibrated by the rays (B-rays) which form the flow-lines of a "Busemann function" B. If the sectional curvature $K \geqq 0$, then we obtain information as to how arbitrary geodesics behave relative to these B-rays (see Proposition 1.6). Also, for surfaces, a notion of "infinite winding" is developed, and a necessary condition for its existence shown.

In this chapter, unless otherwise stated, $\dim(M) = n$ is arbitrary. Recall that a <u>normal</u> geodesic is a geodesic parameterized by arc-length. Throughout we fix a ray r and its associated Busemann function B.

<u>Definition 1.1</u> A <u>B-ray</u> is a normal geodesic $b : [0,\infty) \to M$ such that $B(b(s)) - B(b(0)) = s$ for all $s \geqq 0$. We shall see (Corollary 1.7) that a "B-ray" is a ray.

We shall gain most of our information about B-rays from our study of the closely related "induced rays."

<u>Construction 1.2</u> Fix a point P. Pick an increasing sequence $t_k \to \infty$ of real numbers, and normal minimal connection μ_k from P to $r(t_k)$. Then the $v_k := \dot{\mu}_k(0)$ have an accumulation point v at P. Choosing a subsequence if necessary, we have $v_k \to v$. Let $\tilde{r}(t) := \exp(tv)$, $\tilde{r} : [0,\infty) \to M$. Then $\tilde{r}$ is a ray, and we say that r induces the ray $\tilde{r}$ at P, or that $\tilde{r}$ is an induced ray.

The following lemma, using induced rays will be generalized in Proposition 1.6, where we shall use B-rays.

<u>Lemma 1.3</u> Given a normal geodesic segment $g : [0,d] \to M$, induced ray $\tilde{r}$ at $g(0)$, induced ray $\tilde{\tilde{r}}$ at $g(d)$, $\Theta_0 := \measuredangle(\dot{g}(0),\dot{\tilde{r}}(0))$, $\Theta_d := \measuredangle(\dot{g}(d),\dot{\tilde{\tilde{r}}}(0))$, and $\Delta B := B(g(d))-B(g(0))$, we have

$$d \cos \Theta_0 \leqq \Delta B \leqq d \cos \Theta_d.$$

PROOF: Step 1: We show this is true for a minimal segment. Choose the t_k and μ_k which induce $\tilde{r}$ at $g(0)$. Let $\tilde{\mu}_k$ be minimal from $g(d)$ to $r(t_k)$, and let $\Theta_k := \measuredangle(\dot{g}(0),\mu_k(0))$. Then since $\dot{\mu}_k(0) = v_k \to v = \dot{\tilde{r}}(0)$, $\Theta_k \to \Theta_0$. Let $m_k := L[\tilde{\mu}_k]$, $n_k := L[\mu_k]$, $b_k := n_k-m_k$. Then $b_k = (t_k-m_k)-(t_k-n_k) \to B(g(d))-B(g(0)) = \Delta B$. In particular, the b_k are bounded. Also, $m_k \to \infty$. Toponogov's theorem now implies

$$m_k^2 \leqq n_k+d^2-2n_k d \cos \Theta_k = (b_k+m_k)^2+d^2-2(b_k+m_k) d \cos \Theta_k.$$

Thus $2(b_k+m_k) d \cos \Theta_k \leqq b_k^2 + d^2 + 2b_k m_k$,

or $\quad [(b_k/m_k)+1] d \cos \Theta_k \leqq [(b_k+d^2)/2m_k]+b_k$.

Let $k \to \infty$: $[0+1] d \cos \Theta_0 \leqq [0] + \Delta B$, or $d \cos \Theta_0 \leqq \Delta B$.

A similar argument yields $\Delta B \leqq d \cos \Theta_d$, or we can obtain this by reversing the orientation on g and applying the last result to get

$$d\cos(\pi-\Theta_d) \leqq B(g(0)) - B(g(d)),$$

or $\quad \Delta B = B(g(d)) - B(g(0)) \leqq d\cos\Theta_d.$

Step 2: If $g_{[0,d]}$ is not minimal, then divide $[0,d]$ into subintervals on which g is minimal, say

$0 = a_0 < a_1 < \ldots < a_n = d$, $g_{[a_{k-1}a_k]}$ is minimal. Let r induce r_k at $g(a_k)$, $\Theta_k := \measuredangle(\dot{g}(a_k), \dot{r}_k(0))$, $d_k := a_k - a_{k-1}$, and $(\Delta B)_k := B(g(a_k)) - B(g(a_{k-1}))$. Then

$$\begin{array}{l} d_1\cos\Theta_0 \leqq (\Delta B)_1 \leqq d_1\cos\Theta_1, \text{ so } \cos\Theta_0 \leqq \cos\Theta_1 \\ d_2\cos\Theta_1 \leqq (\Delta B)_2 \leqq d_2\cos\Theta_2, \text{ so } \cos\Theta_1 \leqq \cos\Theta_2 \\ \quad\vdots \qquad\qquad \vdots \\ d_n\cos\Theta_{n-1} \leqq (\Delta B)_n \leqq d_n\cos\Theta_n, \text{ so } \cos\Theta_{n-1} \leqq \cos\Theta_n. \end{array}$$

Thus $\cos\Theta_0 \leqq \cos\Theta_1 \leqq \ldots \leqq \cos\Theta_{n-1} \leqq \cos\Theta_n$.

Thus
$$\begin{array}{l} d_1\cos\Theta_0 \leqq (\Delta B)_1 \leqq d_1\cos\Theta_n \\ d_2\cos\Theta_0 \leqq (\Delta B)_2 \leqq d_2\cos\Theta_n \\ \quad\vdots \qquad\qquad \vdots \\ d_n\cos\Theta_0 \leqq (\Delta B)_n \leqq d_n\cos\Theta_n. \end{array}$$

Adding this last set of inequalities, and noting that $d_1+\ldots+d_n = d$ and that $(\Delta B)_1+\ldots+(\Delta B)_n = \Delta B$, we obtain

$$d\cos\Theta_0 \leqq \Delta B \leqq d\cos\Theta_d, \text{ where } \Theta_d = \Theta_n. \quad \text{QED}$$

<u>Corollary 1.4</u> An induced ray $\tilde{r} : [0,\infty) \to M$ is a B-ray.

PROOF: Fix $d > 0$. In the hypothesis of Lemma 1.3, let g be $\tilde{r}_{[0,d]}$. Then $\Theta_0 = 0$. Thus

$$d = d \cos 0 \leqq \Delta B \leqq d \cos \Theta_d \leqq d, \text{ i.e. } B(\tilde{r}(d)) - B(\tilde{r}(0)) = d$$

for all $d \geqq 0$, since the case $d = 0$ is also true. Thus $\tilde{r}$ is a B-ray. QED

Thus at any point of M we have at least one B-ray, since by Construction 1.2 there is an induced ray at any point, and by Corollary 1.4 every induced ray is a B-ray.

The next corollary, which shows uniqueness of induced rays under certain circumstances, generalizes to the case of uniqueness of B-rays in Corollary 1.8.

<u>Corollary 1.5</u> Given a B-ray $b : [0,\infty) \to M$, if r induces a ray $\tilde{r}$ at $b(s)$, $s > 0$, then $\tilde{r} = b_{[s,\infty)}$, i.e. $\tilde{r}(t) = b(t+s)$, $\tilde{r} : [0,\infty) \to M$.

PROOF: $s = L[b_{[0,s]}]$. Let $\Theta = \measuredangle(\dot{b}(s), \dot{\tilde{r}}(0))$. Then Lemma 1.3 implies that $\Delta B \leqq s \cos \Theta$. By assumption, $\Delta B = s$

(b is a B-ray), so therefore $s \leqq s \cos \Theta$, $s > 0$. Therefore, $1 \leqq \cos \Theta$, so $\Theta = 0$, i.e. $\dot{b}(s) = \dot{\tilde{r}}(0)$. Therefore, $\tilde{r} = b_{[s,\infty)}$. QED

<u>Proposition 1.6</u> Given a normal geodesic segment $g : [0,d] \to M$, let b_1 be a B-ray at $g(0)$, b_2 be a B-ray at $g(d)$, $\Theta_0 = \measuredangle(\dot{b}_1(0), \dot{g}(0))$, $\Theta_d = \measuredangle(\dot{b}_2(0), \dot{g}(d))$, and $\Delta B = B(g(d)) - B(g(0))$. Then

$$d \cos \Theta_0 \leqq \Delta B \leqq d \cos \Theta_d.$$

PROOF: Step 1: Assume first that $g : [0,d] \to M$ uniquely minimizes distance between $g(0)$ and $g(d)$. Choose a decreasing sequence of real numbers $s_k \to 0$, and let μ_k be a normal minimal connection between $b_1(s_k)$ and $b_2(s_k)$, $l_k = L[\mu_k]$, $\Theta_{1,k} = \measuredangle(\dot{b}_1(s_k), \dot{\mu}_k(0))$, $\Theta_{2,k} = \measuredangle(\dot{b}_2(s_k), \dot{\mu}_k(l_k))$, and $(\Delta B)_k = B(b_2(s_k)) - B(b_1(s_k))$. Then $b_1(s_k) \to g(0)$, $b_2(s_k) \to g(d)$, $(\Delta B)_k \to \Delta B$, and $l_k \to d$. Also, it is clear that "$\mu_k \to g$", i.e. if $v_k = \dot{\mu}_k(0)$ and $v = \dot{g}(0)$, then $v_k \to v$ (PROOF Suppose there exists a subsequence of the v_k (also labeled v_k) such that $v_k \to \tilde{v} \neq v$. Then

$$\exp(d\tilde{v}) = \exp(\lim_{k\to\infty} l_k v_k) = \lim_{k\to\infty} \exp(l_k v_k) = \lim_{k\to\infty} b_2(s_k) = b_2(0) = g(d).$$

Thus if $\tilde{g} : [0,d] \to M$ is defined by $\tilde{g}(t) := \exp(t\tilde{v})$, then $\tilde{g}(0) = g(0)$ and $\tilde{g}(d) = g(d)$. Therefore, since $L[\tilde{g}] = d$, $\tilde{g} = g$ by the assumption of unique minimality. Therefore, $v = \dot{g}(0) = \dot{\tilde{g}}(0) = \tilde{v}$, contradiction). Therefore, $\dot{\mu}_k(0) \to \dot{g}(0)$ and $\dot{\mu}_k(l_k) \to \dot{g}(d)$. Thus

$$\Theta_{1,k} = \measuredangle(\dot{b}_1(s_k), \dot{\mu}_k(0)) \to \measuredangle(\dot{b}_1(0), \dot{g}(0)) = \Theta_0$$

and $\Theta_{2,k} = \measuredangle(\dot{b}_2(s_k), \dot{\mu}_k(l_k)) \to \measuredangle(\dot{b}_2(0), \dot{g}(d)) = \Theta_d$.

By Corollary 1.5, there is a unique induced ray at $b_1(s_k)$, namely $b_{1[s_k,\infty)}$. Similarly for b_2. We can therefore apply Lemma 1.3 to get

$$l_k \cos \Theta_{1,k} \leqq (\Delta B)_k \leqq l_k \cos \Theta_{2,k}. \quad \text{Letting } k \to \infty,$$

$$d \cos \Theta_0 \leqq \Delta B \leqq d \cos \Theta_d.$$

Step 2 To prove the proposition in general, we divide $[0,d]$ into subintervals on which g uniquely minimizes distance between endpoints, and proceed exactly as in Step 2 of Lemma 1.3. QED

<u>Corollary 1.7</u> A B-ray $b : [0,\infty) \to M$ is a ray.

PROOF: Fix $d > 0$, and let "g" in Proposition 1.6 be $b_{[0,d]}$. Let "b_1" in Proposition 1.6 be b. Then $\Theta_0 = 0$, so $d = d \cos 0 \leqq \Delta B$. But for all P,Q in M,

$$|B(P) - B(Q)| \leqq \rho(P,Q)$$

where ρ is the distance function in M. Therefore,

$$L[b_{[0,d]}] = d \leqq B(b(d))-B(b(0)) \leqq \rho(b(d),b(0)).$$

Therefore,

$L[b_{[0,d]}] = \rho(b(d),b(0))$, i.e. $b_{[0,d]}$ is minimal for all $d \geqq 0$. QED

<u>Corollary 1.8</u> (Uniqueness of B-rays) Suppose that $b : [0,\infty) \to M$ is a B-ray. Then for all $s > 0$, there exists a unique B-ray at $b(s)$, namely $b_{[s,\infty)}$.

PROOF: Suppose we have a B-ray $\tilde{b}$ at b(s) making an angle Θ_s with b. Then by Proposition 1.6, $\Delta B \leqq s \cos \Theta_s$, where $\Delta B = B(b(s))-B(b(0))$. Since b is a B-ray, $B(b(s))-B(b(0)) = s$. Therefore, $s \leqq s \cos \Theta_s$, so $s > 0$ implies that $1 \leqq \cos \Theta_s$. Therefore, $\Theta_s = 0$, and $\dot{\tilde{b}}(0) = \dot{b}(s)$. QED

The following corollary shows an important property, which we shall use often.

<u>Corollary 1.9</u> Given a normal geodesic g, $a_1 < a_2$, B-rays b_k at $g(a_k)$, and $\Theta_k = \measuredangle(\dot{b}_k(0), \dot{g}(a_k))$, $k = 1,2$. Then $\Theta_1 \geqq \Theta_2$.

PROOF: By Proposition 1.6, $(a_2-a_1)\cos \Theta_1 \leqq \Delta B \leqq (a_2-a_1)\cos \Theta_2$. Therefore, since $a_2 - a_1 > 0$, $\cos \Theta_1 \leqq \cos \Theta_2$, so $\Theta_1 \geqq \Theta_2$. QED

The next proposition shows the only advantage B-rays have over induced rays, since induced rays do not have a "closure property."

<u>Proposition 1.10</u> (Closure property of B-rays) Given B-rays b_k with $v_k = \dot{b}_k(0)$ at P_k such that $v_k \to v$ at P, let $b(t) := \exp(tv)$, $b : [0,\infty) \to M$. Then b is also a B-ray.

PROOF: Fix $s \geqq 0$. Then $b_k(s) = \exp(sv_k) \to \exp(sv) = b(s)$. Also, $b_k(0) = P_k \to P = b(0)$. Therefore,

$$B(b(s))-B(b(0)) = B(\lim_{k\to\infty} b_k(s))-B(\lim_{k\to\infty} b_k(0))$$

$$= \lim_{k\to\infty}[B(b_k(s))-B(b_k(0))] = \lim_{k\to\infty}[s] = s. \quad \text{QED}$$

The function ω introduced in the next definition will be used in Chapters 2 and 5.

Definition 1.11 Given any (normal) geodesic g, define $\Theta_g : \mathbb{R} \to [0,\pi]$ by $\Theta_g(t) := \min\{\measuredangle(\dot{g}(t),\dot{b}(0)) \mid b$ is a B-ray at $g(t)\}$. Notice that by Proposition 1.10 this minimum exists. Also, by Corollary 1.9, Θ_g is nonincreasing. (This easily implies that Θ_g is continuous from the right, but we will see an example where it is not continuous.) We now define a function

$$\omega : \{\text{oriented normal geodesics}\} \to [0,\pi]$$

by $\quad \omega(g) := \lim_{t\to\infty} \Theta_g(t).$

Since Θ_g is bounded below by 0, and is nonincreasing, this limit exists.

The next result will be used to simplify the proof of Lemma 3.6.

Corollary 1.12 Given a normal geodesic $g : [a,b] \to M$, let $\Theta : [a,b] \to [0,\pi]$ be $\Theta(t) := \Theta_g(t)$. Since Θ is nonincreasing,

$\int_a^b \cos\Theta(t)\,dt$ is defined. Furthermore,

$$\int_a^b \cos\Theta(t)\,dt = B(g(b))-B(g(a)).$$

PROOF: Divide [a,b] into subintervals $a = t_0 < t_1 < \ldots < t_n = b$, i.e. the k-th subinterval is $[t_{k-1},t_k]$. Let $(\Delta t)_k = t_k - t_{k-1}$, $l_k = t_{k-1}$, and $r_k = t_k$. Proposition 1.6 implies that

$$\cos\Theta(l_k)\cdot(\Delta t)_k \leq B(g(r_k))-B(g(l_k)) \leq \cos\Theta(r_k)(\Delta t)_k.$$

Summing:

$$\sum_{k=1}^{n}\cos\Theta(l_k)\cdot(\Delta t)_k \leq B(g(b))-B(g(a)) \leq \sum_{k=1}^{n}\cos\Theta(r_k)\cdot(\Delta t)_k.$$

But both of these sums become, in the limit, $\int_a^b \cos\Theta(t)\,dt$. QED

The following lemma will be needed later.

<u>Lemma 1.13</u> Given a normal geodesic g, a sequence t_k of real numbers increasing to ∞, a B-ray b_k at $g(t_k)$, and $\Theta_k := \measuredangle(\dot{g}(t_k),\dot{b}_k(0))$, then $\lim_{k\to\infty}\Theta_k = \omega(g)$.

PROOF: By Corollary 1.9, $\Theta_g(t_{k-1}) \geqq \Theta_k$, and $\Theta_k \geqq \Theta_g(t_k)$ by the definition of Θ_g. Therefore, $\Theta_g(t_{k-1}) \geqq \Theta_k \geqq \Theta_g(t_k)$. Therefore since

$$\lim_{k\to\infty} \Theta_g(t_{k-1}) = \lim_{k\to\infty} \Theta_g(t_k) = \omega(g),$$

we have $\lim_{k\to\infty} \Theta_k = \omega(g)$. QED

We summarize some of these results:

A) Through any point there exists an induced ray (Construction 1.2).

B) All induced rays are B-rays (Corollary 1.4).

C) Through any point of a B-ray $b : [0,\infty) \to M$ there passes a unique B-ray, except possibly at b(0) (Corollary 1.8),

These unique B-rays are therefore induced rays.

D) All B-rays are rays (Corollary 1.7).

E) The "closure property" of B-rays (Proposition 1.10).

There are examples of B-rays which are not induced rays. For example, on the paraboloid $S = [z=x^2+y^2]$, let r be the meridian through P = (0,0,0) with $\dot{r}(0) = \langle 1,0,0 \rangle$.

Since only meridians of S are rays, these form the set of B-rays. But the only induced ray through P is r. Furthermore, all meridians starting at P are B-rays (by uniqueness of rays through points $\neq$ P on this surface, and the closure property of B-rays in Proposition 1.10).

This surface S also shows that Θ_g (Definition 1.11) is not continuous: Let ray r be as in the last paragraph, and let g be the geodesic which extends r to all reals. Then

$$\Theta_g(t) = \begin{cases} \pi & \text{if } t < 0 \\ 0 & \text{if } t \geqq 0. \end{cases}$$

As an application of the results of this section, we introduce a notion of infinite winding on a surface. For example, on the paraboloid of revolution S any geodesic g, other than the meridians of M, will "wind" infinitely often. I.e., if r is any meridian, then g will meet r infinitely often in the following way: There are increasing sequences s_k, t_k of real numbers such that $g(s_k) = r(t_k)$, and if $g_k = g|_{[s_k, s_{k+1}]}$, $r_k = r|_{[t_k, t_{k+1}]}$, and R_k is the bounded region bound by the curves g_k and r_k, then the R_k form a nested sequence $R_k \subseteq R_{k+1}$ which exhausts S ($S = \bigcup_{k=1}^{\infty} R_k$) (see Figure 1.1, 1.2). This situation differs from an infinitely oscillating curve (see Figure 1.3).

Why such a g on S winds infinitely often is indicated in the appendix. Also indicated in the appendix is why (for the case of a surface of revolution M) if the total curvature $C(M) < 2\pi$, then there can be no such infinite winders. In the case $C(M) = 2\pi$, infinite winders may exist (as in the above case), or may not exist.

Extending this concept to arbitrary surfaces, we have

<u>Definition 1.14</u> Fix a ray r in a surface M, M homeomorphic to the plane. A geodesic $g : [0,\infty) \to M$ is an ∞-winder if there exist increasing sequences s_k, t_k of real numbers such that $g(s_k) = r(t_k)$, and if $g_k = g|_{[s_k,s_{k+1}]}$, $r_k = r|_{[t_k,t_{k+1}]}$, and R_k is the bounded region bound by the simple curve $g_k \cup r_k$, then R_k are nested, $R_k \subseteq R_{k+1}$, and exhaust M.

With this definition, we now have the following theorem.

Theorem 1.15 If M has ∞-winders, then $C(M) = 2\pi$. Equivalently, if $C(M) < 2\pi$, then M can have no ∞-winders.

PROOF: Θ_g is nonincreasing (Corollary 1.9). Therefore, $\Theta_g(s) \leqq \Theta_g(0) < \pi/2$ for all $s \geqq 0$. Then by Proposition 1.6, if $s < t$, then

$$(t-s) \cos \Theta_g(s) \leqq B(g(t))-B(g(s)).$$

Then since $t - s > 0$ and $\cos \Theta_g(s) > 0$,

$$0 < B(g(t))-B(g(s))$$

i.e., B is increasing along g.

Let $B^a := \{P \varepsilon M | B(P) \leqq a\}$. By assumption there is an increasing sequence $t_k \to \infty$ and a sequence s_k such that $g(t_k) = r(s_k)$. Thus the bounded region R^k which has as its boundary the geodesic segments $g_{[t_k, t_{k+1}]}$ and $r_{[s_k, s_{k+1}]}$ contains B^{a_k}, where $a_k = B(g(t_k))$. By Gauss-Bonnet,

$C(R^k) = (2\pi - \Theta_g(t_k)) + \Theta_g(t_{k+1})$. The B^a are nested, and

$M = \bigcup_{t \in \mathbb{R}} B^t$. Thus $C(M) = \lim_{t\to\infty} C(B^t) = \lim_{k\to\infty} C(B^{a_k})$. Since $R^{k-1} \subseteq B^{a_k} \subseteq R^k$,

$2\pi - (\Theta_g(t_{k-1}) - \Theta_g(t_k)) \leqq C(B^{a_k}) \leqq 2\pi - (\Theta_g(t_k) - \Theta_g(t_{k+1}))$. But

$\lim_{k\to\infty} \Theta_g(t_{k-1}) = \lim_{k\to\infty} \Theta_g(t_k) = \lim_{k\to\infty} \Theta_g(t_{k+1}) = \omega(g)$. Therefore,

$2\pi \leqq \lim_{k\to\infty} C(B^{a_k}) \leqq 2\pi$, i.e. $C(M) = 2\pi$. QED

Note: The appendix contains an independent proof for the case of a surface of revolution.

We shall end this chapter with a simple proof of a well-known result. See [CG].

Given a Busemann function B, let $B^a := \{P \in M \mid B(P) \leqq a\}$. Also, recall that a set U is called a totally convex set (t.c.s.) if any geodesic segment joining any two points of U is itself contained in U.

Corollary 1.16 If $\dim(M) = n$ and $K \geqq 0$, then the sets B^a are all totally convex sets.

PROOF: This follows from the fact that the function B is convex, which locally means that given any geodesic μ, then the graph of $B \circ \mu$ lies above some line through $(0, B(\mu(0)))$, i.e. there is some constant k such that $B(\mu(t)) \geqq B(\mu(0)) + kt$ for all t. For $t \geqq 0$, this is precisely Proposition 1.6, with $k = \cos \theta_0$. This inequality holds with $k = \cos \theta_0$, again by Proposition 1.6, for $t < 0$ by reversing the orientation of μ: for $s < 0$, $(-s) \cdot \cos(\pi - \theta_0) \leqq B(\mu(s)) - B(\mu(0))$. Now a property of convex functions is that they can have no interior maximum. To prove the corollary, suppose that $P, Q \in B^a$, and that $\mu : [0,d] \to M$ is a (normal) geodesic segment such that $\mu(0) = P$ and $\mu(d) = Q$, and that there exists $t \in (0,d)$ such that $B(\mu(t)) > a$. But then since $B(P), B(Q) \leqq a$, there must be an interior maximum of B on $[0,d]$, contradiction. QED

Chapter 2. B-wedges and Their Total Curvature

In this chapter we shall consider certain noncompact regions which are bounded by B-rays, and develop formulas for their total curvature. In this way we are extending the Gauss-Bonnet Theorem to triangles with a "vertex at infinity," which in hyperbolic space is an "ideal triangle."

Throughout assume that $\dim(M) = 2$, and that M is not flat (hence is diffeomorphic to the plane).

The following construction will allow us to determine the precise formula for the total curvature of our "ideal triangle." The full generality ($\tilde{r}$ arbitrary) will not be used until Chapter 5, but this generality is proved here to avoid reproducing the proofs in Chapter 5.

Construction 2.1 Fix a ray r and its associated Busemann function B. Let $\tilde{r}$ be a ray such that $B \circ \tilde{r}$ is eventually increasing and which does not meet r. Let $\mu : [0,d] \to M$ be a normal minimal connection from $\tilde{r}(0)$ to $r(0)$. Choose a real number t_1 such that

$$B(\tilde{r}(t_1)) > \max\{B(\mu(t)) \mid t \in [0,d]\}.$$

This can be done, since the assumption "$B \circ \tilde{r}$ is eventually increasing" implies that $B \circ \tilde{r} \to \infty$ by Proposition 1.6, since $\cos \theta_0 > 0$ at any point where $B \circ \tilde{r}$ has increased.

By Construction 1.2, we get an induced ray b_1 at $P_1 := \tilde{r}(t_1)$ (b_1 is a B-ray by Corollary 1.4). Recall that in Construction 1.2, we have points on r and normal minimal connections whose initial vectors converge to $\dot{b}_1(0)$. Thus choose such a point Q_1 and r and $\mu_1 : [0,d] \to M$ a normal minimal connection from P_1 to Q_1 such that

$$\theta_1 := \measuredangle(\dot{\mu}_1(0), \dot{b}_1(0)) < 1, \text{ and such that if}$$

$$\tilde{\theta}_1 := \measuredangle(\dot{\tilde{r}}(t_1), \dot{b}_1(0)) \neq 0, \text{ then } \theta_1 < \tilde{\theta}_1.$$

Now define t_k, P_k, b_k, μ_k, Q_k, θ_k, and $\tilde{\theta}_k$, $k = 1,2,\ldots,$ inductively as follows: Assuming that these are defined for $k - 1$, let t_k be a real number such that

$$B(\tilde{r}(t_k)) > \max\{B(Q_{k-1}), B(P_{k-1})+1\}. \quad \text{let } P_k = \tilde{r}(t_k).$$

Use Construction 1.2 as before to get an induced B-ray b_k at P_k and a (normal) minimal connection $\mu_k : [0,d_k] \to M$ from P_k to a point Q_k on r such

$$\theta_k := \measuredangle(\dot{\mu}_k(0), \dot{b}_k(0)) < 1/k, \text{ and such that if}$$

$$\tilde{\theta}_k := \measuredangle(\dot{\tilde{r}}(t_k), \dot{b}_k(0)) \neq 0, \text{ then } \theta_k < \tilde{\theta}_k.$$

Notice that the curves $\tilde{r}, \mu$, and r divide M into two regions, say R_1 and R_2, each homeomorphic to a half-plane, and which intersect only along $\tilde{r}, \mu$, and r.

Claim: For each k, μ_k is contained entirely in either R_1 or R_2. (PROOF: $\theta_k \leq 1/k \leq 1 < \pi/2$ for all k. Thus, by Proposition 1.6, for all $t \in (0,d_k]$,

$$0 < t\cdot\cos\theta_k \leq B(\mu_k(t))-B(\mu_k(0)), \text{ so}$$

$$B(\mu_k(t)) > B(\mu_k(0)) = B(P_k) \geq B(P_1) > \max\{B(\mu(s))|s \in [0,d]\}.$$

Therefore, μ_k does not meet μ.

Also, since μ_k is minimal, and r and $\tilde{r}$ are rays, μ_k can meet r and $\tilde{r}$ only once, i.e. at the endpoints of μ_k (μ_k cannot coincide with r or $\tilde{r}$ since we assume that r and $\tilde{r}$ do not meet).

Therefore the endpoints of μ_k are the only points of μ_k to meet $\tilde{r} \cup \mu \cup r$.)

<u>Definition 2.2</u> With the notation of Construction 2.1, call one of the regions $R \in \{R_1,R_2\}$ a <u>good</u> region if an infinite number of the μ_k are contained in R. If R is a good region, then we call the other region $\overline{M-R}$ a bad region.

We shall represent this situation (see Figure 2.1) by drawing a small arrow at some point of $\tilde{r}$ pointing into the good region R, thus representing the μ_k which enter R.

The important thing to note here is that least one of the regions R_1,R_2 is good. It is possible that both are good. For example, letting S and r be as in the examples after Lemma 1.13, and letting $\tilde{r}$ be a portion of the meridian opposite r, $\tilde{r}$ is a B-ray. By the symmetry of S, we can replace any μ_k by its mirror image in the xz-plane.

Denote by C(R) the curvature integral $\int_R K$ of a region R.

<u>Proposition 2.3</u> Fix ray r and its associated Busemann function B. Let ray $\tilde{r}$ and μ be as in Construction 2.1. Let R be a good side of $\tilde{r} \cup \mu \cup r$, with angles

α at $\tilde{r} \cap \mu$ and β at $r \cap \mu$ (relative to R). Then

$$C(R) = \alpha+\beta-\pi-\omega(\tilde{r}) \quad \text{(R good)}.$$

Furthermore, letting $\tilde{R} = \overline{M-R}$ (i.e. $\tilde{R}$ is a bad region), we have $C(\tilde{R}) = \tilde{\alpha}+\tilde{\beta}-\pi+\omega(\tilde{r})-D$ ($\tilde{R}$ bad), where $\tilde{\alpha} := 2\pi - \alpha$ and $\tilde{\beta} := 2\pi - \beta$ are the angles at $\tilde{r} \cap \mu$ and $r \cap \mu$ as measured in $\tilde{R}$, and $D := 2\pi - C(M)$ is the difference between C(M) and its maximal possible value 2π.

PROOF: Choose a subsequence such that all $\mu_k \subseteq R$ (R good). Let R_k be the subset of R bounded by $\tilde{r}$, μ, r, and μ_k. Recalling $\tilde{\theta}_k$ from Construction 2.1, "if $\tilde{\theta}_k \neq 0$, then $\theta_k < \tilde{\theta}_k$" implies that b_k, like μ_k, points into R. The R_k are nested:

$$\min\{B(\mu_k(s)) \mid s \in [0,d_k]\} = B(\tilde{r}(t_k)) > B(Q_{k-1})$$
$$= \max\{B(\mu_{k-1}(s)) \mid s \in [0,d_{k-1}]\}.$$

Also, the R_k exhaust R: we choose $B(P_k) > B(P_{k-1})+1$.

Set $\delta_k = \measuredangle(\dot{\mu}_k(d_k), \dot{r}(s_k))$, where $Q_k = r(s_k)$. Thus

$$C(R) = \lim_{k\to\infty} C(R_k) = \lim_{k\to\infty}[\alpha+\beta+\delta_k+[\pi-(\Theta_k+\tilde{\Theta}_k)] - 2\pi] = \lim_{k\to\infty}[\alpha+\beta-\pi+\delta_k-\Theta_k-\tilde{\Theta}_k].$$

By Corollary 1.9, since b_k and r are B-rays, $\delta_k \leqq \Theta_k$. Thus $0 \leqq \delta_k \leqq \Theta_k < 1/k$, so $\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \Theta_k = 0$. Finally, $\tilde{\Theta}_k = \measuredangle(\dot{\tilde{r}}(t_k), b_k), \dot{b}_k(0))$, where b_k is a B-ray at $\tilde{r}(t_k)$. Therefore, by Lemma 1.13, $\lim_{k\to\infty} \tilde{\Theta}_k = \omega(r)$. Therefore, $C(R) = \alpha+\beta-\pi-\omega(\tilde{r})$. Finally, for a bad side $\tilde{R}$, $\tilde{R} = \overline{M-R}$, where R is good. Thus

$$\begin{aligned} C(\tilde{R}) &= C(M) - C(R) = C(M) - [\alpha+\beta-\pi-\omega(\tilde{r})] \\ &= (2\pi-\alpha) + (2\pi-\beta) - \pi - (2\pi-C(M)) + \omega(\tilde{r}) \\ &= \tilde{\alpha} + \tilde{\beta} - \pi + \omega(\tilde{r}) - D. \end{aligned}$$ QED

For our immediate purposes the following weaker version of Proposition 2.3 is sufficient:

<u>Corollary 2.4</u> Suppose in addition to the hypothesis of Proposition 2.3 that $\tilde{r}$ is a B-ray. Then

$$C(R) = \begin{cases} \alpha + \beta - \pi & \text{if } R \text{ is good} \\ \alpha + \beta - \pi - D & \text{If } R \text{ is bad.} \end{cases}$$

PROOF: If $\tilde{r}$ is a B-ray, then (Definition 1.11) $\Theta_{\tilde{r}}(t) = 0$ for all $t \geqq 0$. Therefore,

$$\omega(r) = \lim_{t\to\infty} \Theta_{\tilde{r}}(t) = 0.$$ QED

Notice that if $C(M) = 2\pi$, then since $D = 0$, the above integral equals $\alpha + \beta - \pi$ whether R is good or not.

The above notion of a good region will be used to prove Proposition 4.1, namely "if $C(M) > \pi$, then $B^a = \{P\epsilon M | B(P) \leq a\}$ is compact for all a." Its converse is also true, and to prove this we shall use the following notion:

<u>Definition 2.5</u> A B-wedge is a region W bounded by two B-rays $b_1 \neq b_2$ which meet at the common point $b_1(0) = b_2(0)$. $(\overline{W}, \epsilon)$ will indicate a B-wedge W and the angle of its vertex as measured in W.

$b_1 \cup b_2$ divides M into two regions, each one a B-wedge. We distinguish between "good" and "bad" B-wedges in the following way:

A. If $r \subseteq b_1$ or b_2 (say $r \subseteq b_1$), then the wedge into which the arrow in Figure 2.1 mentioned after Definition 2.2 (for $\tilde{r} = b_2$) is a good B-wedge.

B. If $r \nsubseteq b_1$ or b_2, let R_1 denote the region (bounded by the b_k) which contains r, and let R_2 denote the other. Then

   i) If both of the arrows (mentioned after Definition 2.2) associated with b_1 and b_2 point into R_1, then R_1 is good;

   ii) otherwise, R_2 is good.

If one of the regions R_1, R_2 is good, we shall then call the other region bad.

Notice that given any pair of distinct B-rays such that $b_1(0) = b_2(0)$, at least one of the two B-wedges which they bound is good.

Proposition 2.6 Given a B-wedge (W,ε),

$$C(W) = \begin{cases} \varepsilon \text{ if } W \text{ is good} \\ \text{or} \\ \varepsilon - D \text{ if } W \text{ is bad,} \end{cases}$$

where agin $D = 2\pi - C(M)$.

PROOF: Case A, say $r \subseteq b_1$ (see Figure 2.2): Using Corollary 2.4, let $\mu = b_2|_{[0,1]}$ and $\tilde{r} = b_2|_{[1,\infty)}$.

Thus $\beta = \varepsilon$ and $\alpha = \pi$.

Therefore, $C(W) = \pi + \varepsilon - \pi = \varepsilon$.

Case B.i (see Figure 2.3) Suppose r is contained in W and both arrows associated with b_1 and b_2 point into W. Let μ be a (normal) minimal connection from $P = b_1(0)$ to $r(0)$. Then $\mu \cup r$ divides W into two regions W_1, W_2, as shown. Therefore, by Corollary 2.4,

$$C(W) = C(W_1) + C(W_2) = (\varepsilon_1+\beta_1-\pi) + (\varepsilon_2+\beta_2-\pi)$$
$$= (\varepsilon_1+\varepsilon_2) + (\beta_1+\beta_2) - 2\pi = \varepsilon + 2\pi - 2\pi = \varepsilon.$$

Case B.ii (see Figure 2.4) Letting W be a good B-wedge, we have that r is contained in M - W, and that at least one of the arrows (say associated with b_1) points into W (see Figure). Claim: If R_1 denotes the good region bounded by $b_1 \cup \mu \cup r$, then $\overline{R_1 - W}$ is the good region bounded by $b_2 \cup \mu \cup r$. PROOF: The μ_k from Construction 2.1 associated with b_1 must meet b_2, say at the points $\mu_k(s_k)$. The angles they make with b_2 are $\leqq$ the corresponding angles they make with b_1 since b_1 and b_2 are B-rays (Corollary 1.9). Thus the restrictions of these μ_k to $\mu_{k[s_k,d_k]}$ satisfy the criteria of Construction 2.1 for b_2, i.e. R_2 is a good region.

Thus, we have

$$C(W) = C(R_1) - C(R_2) = [(\varepsilon+\alpha)+\beta-\pi - [\alpha+\beta-\pi] = \varepsilon.$$

Finally, if (W,ε) is bad, then $(\overline{M-W}, 2\pi-\varepsilon)$ is good. Therefore,

$$C(\overline{M-W}) = 2\pi - \varepsilon, \text{ so}$$
$$C(W) = C(M) - C(\overline{M-W}) = C(M) - (2\pi-\varepsilon) = \varepsilon - (2\pi - C(M))$$
$$= \varepsilon - D$$

QED

Chapter 3. Projections on Horospheres

In this chapter we introduce a function which projects onto the "horospheres" $B^{-1}(a)$. This function gives us that the horospheres are path-connected. We also see that if one horosphere is compact, then they all are. Throughout this chapter we shall assume that $\dim(M) = 2$.

Recall that $B^a := \{P \in M | B(P) \leq a\}$. Let B^{min} denote the B^a such that $B^b = \emptyset$ for all $b < a$, if it exists.

We shall make use of the following fact:

If $B^a \neq B^{min}$, then the boundary ∂B^a is a rectifiable curve without boundary (see [CG]). Notice that

$$\partial B^a = \{P \in M | B(P) = a\} \text{ if } B^a \neq B^{min}.$$

<u>Lemma 3.1</u> Suppose that the B-rays $b_1 \neq b_2$ have $P = b_1(0) = b_2(0)$, and that $P \notin B^{min}$. Then $\varepsilon := \measuredangle(\dot{b}_1(0), \dot{b}_2(0)) < \pi$, and the B-wedge (W, ε) bounded by b_1 and b_2 is such that

$$W \cap B^a = \{P\}.$$

PROOF: $0 \leq \varepsilon \leq \pi$. Suppose that $\varepsilon = \pi$. If $Q \in M$ $(Q \neq P)$, let $\mu: [0,d] \to M$ be a (normal) minimal connection from P to Q. Then μ makes an angle $\Theta \leq \pi/2$ with at least one of the b_1, b_2 (say b_1). Thus by Proposition 1.6, $0 \leq d \cdot \cos\Theta \leq B(Q) - B(P)$, so $B(P) \leq B(Q)$ for all $Q \in M$.

Therefore $P \in B^{min}$, contradiction. Therefore $\varepsilon < \pi$.

Now let (W, ε) be the B-wedge bounded by b_1 and b_2 with vertex angle ε (measured in W) less than π. If $Q \in W - \{P\}$, let $\mu : [0,d] \to M$ be a (normal) minimal connection from P to Q. Since μ, b_1, and b_2 are all minimal, μ W. Therefore, it makes an angle $\theta < \pi/2$ with at least one of the b_1, b_2 (say b_1). Therefore by Proposition 1.6,

$$0 < d \cdot \cos\theta \leqq B(Q) - B(P), \text{ i.e.}$$

$B(Q) > B(P)$ for all $Q \in W - \{P\}$. Therefore

$W \cap B^a = \{P\}$. QED

The following proposition, which classifies B-rays through horosphere $B^{-1}(a)$, is used many times.

<u>Proposition 3.2</u> Fix $B^a \neq B^{min}$ and $Q \notin B^a$. Let $P \in \partial B^a$ be a point of B^a closest to Q (B^a is closed). Then either Q lies on a B-ray through P or there exists a B-wedge (W, ε), $\varepsilon < \pi$, with vertex P such that Q is in the interior of W.

PROOF: Suppose there is no B-ray through P meeting Q. Parametrize ∂B^a near P by $c : [-1,1] \to \partial B^a$ such that $c(0) = P$. Let $m_k = c(-1/k)$ and $n_k = c(1/k)$. Let b_k^m be a B-ray through m_k and b_k^n be a B-ray through n_k. By Proposition 1.10, if v_m is an accumulation point of the $\dot{b}_k^m(0)$ and v_n is an accumulation point of the $\dot{b}_k^n(0)$, then

$$\dot{b}^m, b^n : [0,\infty) \to M,\ b^m(t) := \exp(tv_m)$$

and

$$b^n(t) := \exp(tv_n) \text{ are B-rays at P.}$$

Now let $\mu : [0,d] \to M$ be a (normal) minimal connection from P to Q. Since we have assumed that Q does not lie on a B-ray through P, μ differs from b^m and b^n. It is clear that if both b^m and b^n lie on the same side of μ in $M - B^a$, then by continuity one of the B-rays (say b_k^m) meets μ, say at the point $R = \mu(s) = b_k^m(s_k) \neq Q$. Our assumption of minimality implies that $\mu_{[0,s]}$ is minimal from R to B^a. But since b_k^m is a B-ray, $b^m_{k[0,s_k]}$ is minimal from R to B^a (PROOF: given $\tilde{Q}$ on ∂B^a,

$$\rho(\tilde{Q},R) \geq |B(R) - B(\tilde{Q})| = |B(R) - B(m_k)| = s_k).$$

Therefore,

$$b^m_{k[0,s_k]} \text{ and } \mu_{[0,s]}$$

have the same length, so that the broken geodesic $b_k^m|_{[0,s_k]} \cup \mu_{[s,d]}$ is minimal from P to m_k, contradiction. Therefore, the b^m, b^n lie on either side of μ in $M - B^a$, and thus $b^m \neq b^n$. Furthermore, it is clear that μ lies in the wedge W for which $W \cap B^a = \{P\}$ (else b_m and b_n are on the same side of μ in $M - B^a$, which we have just ruled out), and the angle at the vertex of this wedge is $<\pi$. QED

The following function is used to prove Proposition 3.5.

<u>Definition 3.3</u> For a fixed $B^a \neq B^{min}$, define $pr_a : M \to B^a$ as follows: Given $P \in M$, let $pr_a(P)$ be the unique point of B^a closest to P.

<u>Proposition 3.4</u> The function pr_a defined above is well-defined and continuous.

PROOF. Well-defined: If $P \in B^a$, then $pr_a(P) = P$. Therefore, assume that $P \notin B^a$. By Proposition 3.2, either P is on a B-ray b through a point Q of ∂B^a, or P lies in the interior of a B-wedge W with vertex Q on ∂B^a such that $W \cap B^a = \{Q\}$. In the first case, $pr_a(P) = Q$ (PROOF. It was noted in the proof of Proposition 3.2 that a B-ray b minimizes the distance from any point on it to B^a, where $a = B(b(0))$. Therefore, the only way that there could be another point $\tilde{Q} \in B^a$ closest to P would be if there were a B-ray $\tilde{b}$ through $\tilde{Q}$ meeting P. But then we have two B-rays through P, contradicting Corollary 1.8). Thus suppose we have the second case. Denote the B-rays which form the boundary of W by

b_1, b_2. Since M is diffeomorphic to $\mathbb{R}^2$, the curve $b_1 \cup b_2$ divides M into two regions. P is in the interior of W. Let $\mu : [0,d] \to M$ be a minimal connection from P to B^a. If $\mu(d) \neq Q$, then since $W \cap B^a = \{Q\}$ μ must meet $b_1 \cup b_2$ at a point $R \neq Q$, say at $R = \mu(s) = b_1(t)$. But then, as noted in the proof of Proposition 3.2, the broken geodesic $\mu_{[0,s]} \cup b_{1[0,t]}$ is minimal from P to Q, contradiction. Therefore, any minimal connection $\mu : [0,d] \to M$ from P to B^a has $\mu(d) = Q$.

Finally we show that pr_a is continuous: Fix $P \in M$, and suppose that $P_k \to P$. Suppose that $pr_a(P_k) \not\to pr_a(P)$, say that they have an accumulation point $Q \neq pr_a(P)$. Choose a subsequence of the P_k such that $\lim_{k\to\infty} pr_a(P_k) = Q$. By continuity of the distance function ρ of M,

$$\rho(P,B^a) = \lim_{k\to\infty} \rho(P_k,B^a) = \lim_{k\to\infty} \rho(p_k,pr_a(P_k)) = \rho(P,Q).$$

Therefore, by the uniqueness of the closest point of B^a to P, we have that $Q = pr_a(P)$, contradiction. Therefore, $pr_a(P_k) \to pr_a(P)$. QED

<u>Proposition 3.5</u> If $B^a \neq B^{min}$, then ∂B^a is path connected.

PROOF. Recall that the ray which gives us B is denoted $r : [0,\infty) \to M$. If $B(r(0)) \leq a$, then r meets ∂B^a, say at

$P := r(s)$. If $B(r(0)) > a$, then let $\mu:[0,d] \to M$ be a (normal) minimal connection between $r(0)$ and B^a, $\mu(0) := P \in B^a$, $\mu(d) = r(0)$. We shall show that in either case there is a (continuous) path in ∂B^a between any point $\tilde{P}$ of ∂B^a and P. Thus given any point $\tilde{P} \in \partial B^a$, let r induce a B-ray $\tilde{r}$ at $\tilde{P}$ (Construction 1.2). Thus we have an increasing sequence t_k of real numbers diverging to ∞ and minimal connections $\mu_k : [0,d_k] \to M$ from $\tilde{P}$ to $r(t_k)$ such that $\dot{\mu}_k(0) \to \dot{\tilde{r}}(0)$. In particular, choose a k such that

$$\measuredangle(\dot{\mu}_k(0), \dot{\tilde{r}}(0)) < \pi/2.$$

Therefore the function $B \circ \mu_k : [0,d_k]$ is increasing by Proposition 1.6. I.e.,

$$B(\mu_k(t)) \geq B(\mu(0)) = B(P) = a \text{ for all } t \geq 0.$$

We thus have a continuous curve c given by $r_{[s,t_k]} \cup \mu_{k[d_k,0]}$ (i.e. μ_k with the reversed orientation) or $\mu_{[0,d]} \cup r_{[0,t_k]} \cup \mu_{k[d_k,0]}$ from P to $\tilde{P}$. Furthermore, the value of B along this curve is $\geq a$. But it is clear that $pr_a(Q) \in \partial B^a$ for all $Q \in \overline{M-B^a}$ since B is continuous. Therefore, the function $pr_a \circ c$ is a continuous curve (pr_a is continuous by Proposition 3.4) from P to $\tilde{P}$ which lies in ∂B^a. QED

Whereas pr_a projects $\overline{M-B^a}$ onto $B^{-1}(a) = \partial B^a$, the following lemma shows what happens when we project onto ∂B^a from a point $P, B(P) < a$.

Lemma 3.6 If $P \varepsilon M$ has $B(P) < a$, and $\mu : [0,d] \to M$ is a minimal connection from P to ∂B^a, then μ is part of a B-ray through P.

PROOF. There exists a B-ray b through P, and it crosses ∂B^a at a distance $\tilde{d}$ from P. By Definition 1.1 of a B-ray, $\tilde{d} = a - B(P)$. Now suppose that μ is not part of a B-ray through P. Then there is an $\varepsilon > 0$ such that the function $\Theta = \Theta_\mu$ of Definition 1.11 is nonzero on $[0,\varepsilon]$. (This is true because Θ is nonincreasing (Corollary 1.9), and if $\Theta(t) = 0$ for all $t > 0$, i.e., μ is a B-ray starting at $\mu(t)$ for all $t > 0$, then μ is a B-ray starting at $\mu(0)$ by the closure property Proposition 1.10). Therefore by Corollary 1.12, $B(\mu(d)) - B(\mu(0)) = \int_0^d \cos \Theta(t)dt < d$, contradicting minimality since $\tilde{d} = a - B(P) = B(\mu(d)) - B(\mu(0)) < d$. Therefore μ is the initial portion of a B-ray through P. QED

The next lemma shows that to prove all horospheres are compact, we need only to show one horosphere compact.

<u>Lemma 3.7</u> If $B^a \neq B^{min}$ and ∂B^a is compact, then ∂B^b is compact for all $b \geqq a$.

PROOF: $\partial B^b = B^{-1}(b)$ is closed. Therefore, we are done if we show that ∂B^b is bounded. Since ∂B^a is compact, hence bounded, this will follow if we show that all points of ∂B^b are within a fixed distance of some point of ∂B^a. Thus let $Q \in \partial B^b$, and let P be the point of B^a closest to Q. By Proposition 3.2, either Q lies on a B-ray through P (in which case $\rho(P,Q) = b - a$), or Q lies in the interior of a B-wedge (W,ε), $\varepsilon < \pi$, with vertex P such that $W \cap B^a = \{P\}$.

Claim. $S = \{\varepsilon \mid (W,\varepsilon)$ is a B-wedge with vertex (say R) in ∂B^a and $W \cap B^a = \{R\}\}$ is bounded away from π. Proof. By Proposition 2.6,

$$= \begin{cases} C(W) \\ \text{or} \\ (C(W) + (2\pi - C(M)). \end{cases}$$

Since all the ε in S are less than π, S can approach π only if there is a sequence (W_k, ε_k) as in the definition of S such that $\varepsilon_k \to \pi$. We can assume that the W_k are distinct and disjoint. At most a finite number of the W_k can have $C(W_k) = \varepsilon_k$ since the W_k are disjoint and the total curvature of M is $\leqq 2\pi$. Therefore, we have a sequence of disjoint

(W_k, ε_k) for which $\varepsilon_k = C(W_k) + (2\pi - C(M))$ and $\varepsilon_k \to \pi$. Since the W_k are disjoint, $\sum_{k=1}^{\infty} C(W_k) \leq 2\pi$, so therefore $C(W_k) \to 0$. But $C(W_k) = (\varepsilon_k - \pi) + (C(M) - \pi) \to C(M) - \pi$. Therefore, $C(M) - \pi = 0$. But if $C(M) = \pi$, then $0 \leq C(W_k) = \varepsilon_k - \pi$, i.e., $\varepsilon_k \geq \pi$, contradiction. Therefore the set S is bounded away from π, say $\sup(S) = k < \pi$.

Returning now to the case of Q in (W, ε), $\varepsilon \leq k < \pi$, with vertex P. Letting $\mu : [0,d] \to M$ be a (normal) minimal connection from P to Q, μ' makes an angle $\theta \leq \varepsilon/2$ with at least one of the B-rays which form the sides of W. Thus $\theta \leq \varepsilon/2 \leq k/2 < \pi/2$. Therefore by Proposition 1.6, $d \cdot \cos\theta \leq \Delta B = b - a$, i.e. $d \leq (b-a)/\cos\theta \leq (b-a)/\cos(k/2)$. I.E., every point of ∂B^b is within $(b-a)/\cos(k/2)$ of ∂B^a. QED

<u>Lemma 3.8</u> If $B^a \neq B^{min}$, ∂B^a is compact, and M is not a flat cylinder, then B^a is compact.

PROOF: Since M is not a flat cylinder, it is diffeomorphic to the plane. Since ∂B^a is a compact, connected, rectifiable curve without boundary, it is homeomorphic to the circle S^1. By the Jordan curve theorem, this curve divides M into a bounded and an unbounded region. Since $M - \overline{B^a}$ is unbounded (it contains any B-ray through any point on ∂B^a), B^a is bounded. Therefore, since it is closed, it is compact. QED

Lemma 3.9 If $B^{-1}(a)$ is locally at P a geodesic segment g, then any B-ray through P is perpendicular to g at $g(0) = P$.

PROOF: If there is a B-ray b through P which makes an angle $\Theta < \pi/2$ with g, then by Proposition 1.6, (assuming g is normal) $B(g(d)) - B(g(0)) \geqq d \cdot \cos \Theta > 0$ for all sufficiently small $d > 0$. Therefore, $a = B(g(d)) > B(P) = a$, contradiciton. QED

Chapter 4. Compactness Criterion for Horospheres

All results in this chapter are used to prove Theorem 4.9, which essentially says that the horospheres are compact if and only if $C(M) > \pi$ or M is a flat cylinder.

Again in this chapter, we assume that $\dim(M) = 2$, $K \geq 0$, and that M is not a flat cylinder (hence M is diffeomorphic to the plane).

Given a region R, define $C(R) := \int_R K$. This first proposition will be part of Theorem 4.9.

<u>Proposition 4.1</u> If $C(M) > \pi$, then B^a is compact for all a.

PROOF: Choose $a \in \mathbb{R}$ sufficiently small such that $C(M-B^a) > \pi$, and such that $B^a \neq B^{min}$ (B^{min} has no interior). We break up the proof that this ∂B^a is compact into two steps. Thus: Suppose that ∂B^a is not compact.

Step 1: The closed set ∂B^a is thus a rectifiable curve $c : (-\infty,\infty) \to M$ without boundary which is path-connected (Proposition 3.5), so the sets $c((-\infty,0])$ and $c([0,\infty))$ are unbounded. Choose an increasing sequence t_k of real numbers, $t_k \to \infty$, and define $m_k := c(-t_k)$ and $n_k := c(t_k)$. Let b_k^m, b_k^n be B-rays through m_k, n_k, respectively, and let $\tilde{R}_k$ be the region bounded by the curve $b_k^m \cup c_{[-t,t]} \cup b_k^n$ contained in $\overline{M-B^a}$. Clearly the $\tilde{R}_k$ are nested. Claim: $\overline{M-B^a} = \bigcup_{k=1}^{\infty} \tilde{R}_k$.

Proof: As required above, $\tilde{R}_k \subseteq \overline{M-B^a}$ for all k, so $\overline{M-B^a} \supseteq \bigcup_{k=1}^{\infty} \tilde{R}_k$. Conversely, if $Q \in \overline{M-B^a}$, i.e. $B(Q) \geq a$, then (recall Definition 3.3) let μ be a minimal connection from $pr_a(Q) \in \partial B^a$ to Q. If $pr_a(Q) = c(s)$, choose k sufficiently large such that $s \in (-t_k, t_k)$. μ can not cross the b_k^m, b_k^n (for as noted in the proof of Proposition 3.2, this would imply two points on ∂B^a closest to Q, contradiction), and it cannot meet ∂B^a a second time. Thus $\mu \subseteq \tilde{R}_k$, so $Q \in \tilde{R}_k \subseteq \bigcup_{l=1}^{\infty} R_l$.

Now choose k_0 sufficiently large such that $C(\tilde{R}_{k_0}) > \pi$, and also such that r (the ray with which B is associated) is such that

$r \cap (M-B^a) \subseteq \tilde{R}_{k_0}$. Let $T = m_{k_0}$, $S = n_{k_0}$, $b_1 = b_{k_0}^m$, $b_2 = b_{k_0}^n$ and let g be a minimal connection from T to S (thus $g \subseteq B^a$ by Corollary 1.15). Let R denote the region bounded by the curve $b_1 \cup g \cup b_2$ which contains $\tilde{R}_{k_0}$. Therefore, $C(R) \geq C(\tilde{R}_{k_0}) > \pi$.

Let α and β be the angles at $b_1 \cap g$ and $g \cap b_2$ as measured in R. Claim: $C(R) = \alpha + \beta - \pi$. Proof: Choose a point $\tilde{M}$ on r in the interior of R, and let g_1, g_2 be minimal connections from T,S, respectively, to $\tilde{M}$. Let the angles $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \delta_3$ be as shown in Figure 4.1. Also, let the regions R_1, R_2, R_3 be as shown. We now show that the regions R_1, R_2 are "good" regions in the sense of Definition 2.2. Letting $U_k = b_1(t_k)$ for all k, eventually we have $B(U_k) > B(\tilde{M})$ for all $k \geq N$ for some N. Construct (normal) minimal connections $\mu_l^k : [0,d_l] \to M$ from U_k to $r(t_l)$ for all l. By Construction 1.2, the $\dot{\mu}_l^k(0)$ have a vector v as accumulation point such that $t \to \exp(tv)$ $t \geq 0$, is a B-ray (Corollary 1.4). Therefore by uniqueness (Corollary 1.8), this B-ray must be $b_1|_{[t,\infty)}$. Thus the $\dot{\mu}_l^k(0) \underset{\ell\to\infty}{\to} b_1(t_k)$. Thus choose (for each k) a l such that $\measuredangle(\dot{b}_1(t_k), \dot{\mu}_l^k(0)) \leq 1/k$. These μ_ℓ^k are as stipulated in Construction 2.1, so it remains to determine which "side" of b_1 they enter. The claim is that they are contained in R_1. Otherwise, they would enter the region contained in $M - B^a$ and bounded by the curve $b_1 \cup c_{(-\infty,-t_{k_0}]}$. But since the angle μ_ℓ^k makes with b_1 is less thant $\pi/2$, $B\circ\mu_k$ is increasing (by Proposition 1.6), so the μ_k cannot cross ∂B^a. They also cannot cross b_1 a second time, so they would be trapped outside of R. But R contains r, contradiction. Therefore, R_1 is good, as asserted, and similarly R_2 is good. Therefore by Corollary 2.4,

$$C(R_1) = \alpha_1 + \delta_1 - \pi$$

$$C(R_2) = \beta_1 + \delta_2 - \pi,$$ and by Gauss-Bonnet

$$C(R_3) = \alpha_2 + \beta_2 + \delta_3 - \pi.$$

Therefore,

$$C(R) = \sum_{n=1}^{3} C(R_n) = (\alpha_1+\alpha_2) + (\beta_1+\beta_2) + (\delta_1+\delta_2+\delta_3) - 3\pi$$

$$= \alpha + \beta + 2\pi - 3\pi = \alpha + \beta - \pi.$$

Therefore we have $\alpha + \beta - \pi = C(R) > \pi$, i.e., $\alpha + \beta > 2\pi$.

Step 2: We now construct a new ray at T: let μ_k be (normal) minimal connections from T to the points $c(-t_k)$, and let v be an accumulation point of the $\dot{\mu}_k(0)$. Then $\tilde{r} : [0,\infty) \to M$, defined by $\tilde{r}(t) := \exp(tv)$, is a ray (but not a B-ray). Associated to $\tilde{r}$ is a Busemann function $\tilde{B}$. Since T and the $c(-t_k)$ are in B^a, the μ_k are contained in B^a also, by Corollary 1.15. Therefore $\tilde{r} \subseteq B^a$. Now use Construction 1.2 with the ray $\tilde{r}$ to obtain the induced ray $\tilde{\tilde{r}}$ at S. Again, since $\tilde{\tilde{r}}$ is the limit of minimal geodesic segments which are contained in B^a (again by Corollary 1.15), $\tilde{\tilde{r}} \subseteq B^a$. Furthermore, since $\tilde{\tilde{r}}$ is induced by $\tilde{r}$, it is a $\tilde{B}$-ray by Corollary 1.4.

Let the angles a, b, α, β, be as in Figure 4.2, and let R^* denote the region bounded by the curve $\tilde{r} \cup g \cup \tilde{\tilde{r}}$ contained in B^a. Therefore by Corollary 2.4 (applied to $\tilde{r}$ and $\tilde{B}$), $a + b - \pi = C(R^*)$ or $C(R^*) + (2\pi - C(M))$, so in either case, $a + b - \pi \geqq C(R^*) \geqq 0$. Therefore, $a + b \geqq \pi$. Now let A,B be the angles of $b_1 \cup \tilde{r}$, $b_2 \cup \tilde{\tilde{r}}$ as measured in $R \cup R^*$. Thus

$A = \alpha + a$, $B = \beta + b$. Therefore, $A + B = (\alpha+\beta) + (a+b) > 2\pi + \pi = 3\pi$. Therefore, at least one of the angles A,B is greater than $3\pi/2$. Suppose w.l.o.g. that $A > 3\pi/2$. Thus $\Theta := \measuredangle(\dot{b}_1(0), \dot{\tilde{r}}(0)) < \pi/2$. But then by Proposition 1.6, applied to $\tilde{r}$, if $d > 0$, then

$$0 < d \cdot \cos\Theta \leq B(\tilde{r}(d)) - B(\tilde{r}(0)), \text{ so } B(\tilde{r}(d)) > B(\tilde{r}(0)) = a,$$

so that $\tilde{r}(d) \notin B^a$. But as noted earlier, $\tilde{r} \subseteq B^a$ contradiction. Therefore, the initial assumption that ∂B^a is noncompact was false.

Therefore, ∂B^a is compact for this value of a. Therefore, ∂B^b is compact for all $b \geq a$ by Lemma 3.7. Therefore, by Lemma 3.8, B^b is compact for all $b \geq a$. Finally, if $b < a$, then since B^b is closed and contained in the compact set B^a, B^b is compact. QED

The rest of this chapter will be used to prove the converse of Proposition 4.1, i.e. if $C(M) \leq \pi$, then the B^a are all noncompact. The proof will be by contradiction. Thus we shall assume that the B^a are all compact. This implies that a minimal set B^{min} exists. Thus we shall first establish several lemmas about B^{min} under the hypothesis that it exists.

<u>Lemma 4.2</u> If B^{min} exists, then either B^{min} is a single point, or there is a unique (up to parametrization) normal geodesic g such that $B^{min} = g(I)$, where I is either $[0,d]$ for some $d > 0$, or $I = [0,\infty)$.

PROOF: If $B^{min} = B^m$ is a single point, then we are done. Therefore, assume that there are at least two points $P \neq Q$ in B^{min}. Let g be a geodesic which passes through these two points. Claim: Any point of B^{min} must lie on g. Proof: Suppose not, say $R \in B^{min}$ does not lie on g. Let μ be a minimal connection from P to R, and $\tilde{\mu}$ be a minimal connection from R to Q. Since P,Q and R are in B^{min}, and B^{min} is a totally convex set (Corollary 1.15), the geodesic segments g, μ, and $\tilde{\mu}$ are contained in B^{min}. But they bound a bounded region U, so pick a point T in the interior of U, and let b be a B-ray through T, $T = b(0)$. Since b is unbounded, it meets one of the g, μ, $\tilde{\mu}$ at a point $b(d)$, $d > 0$. Therefore,

$$B(b(d)) = B(b(0)) + d > B(b(0)), \text{ so } B(b(d)) > m,$$

contradiction.

Therefore, any point of B^{min} is on g. Also, since B^{min} is totally convex, it is connected. Suppose that a real sequence $a_k \to a$, and that $g(a_k) \in B^{min}$ for all k. Therefore, $g(a_k) \to g(a)$, so therefore by the continuity of B, "min" $= \lim_{k\to\infty} B(g(a_k)) = B(g(a))$. Thus the set I of reals for which $B^{min} = g(I)$ is closed and connected. Finally, we show that $I \neq \mathbb{R}$. For if it were, then B^{min} would divide M into two unbounded regions R_1, R_2 (Jordan curve theorem). Letting $A_k \in R_k - B^{min}$, and b_k be a B-ray through A_k, $k = 1,2$, we note that since $B \circ b_k$ is increasing, the b_k do not meet B^{min}. Therefore, by going out far enough along the b_k, we have points such that $B(b_1(a)) = B(b_2(b)) = m$, $a,b > 0$. I.e., these points are both on B^m but in different R_k. Since B^{min} disconnects M, B^m is disconnected, contradicting Proposition 3.5. QED

<u>Proposition 4.3</u> If B^{min} exists and contains $g([a,b])$, where g is a (normal) geodesic and $a < b$, then for all $c \in [a,b]$, both geodesics perpendicular to g at $g(c)$ are B-rays, and the region R bounded by the B-rays perpendicular to g at $g(a)$ and $g(b)$ is flat.

PROOF: Given $c \in (a,b)$, choose an open ball $U = B_r(g(c))$ of radius r about $g(c)$, diffeomorphic to an open disk, such that $r < \min\{b-c, c-a\}$. Letting $a' = c - r$, $b' = c + r$, it is easy to see that $U \cap B^{min} = g((a'b'))$. Thus $U \cap B^{min}$ divides $U - (U \cap B^{min})$ into two open regions U_1 and U_2. Let a sequence of points P_k in U_1 have limit $g(c)$, and let b_k be B-rays through the P_k. Then $U \cap b_k \subseteq U_1$ for all k. The $\dot{b}_k(0)$ have an accumulation point v at $g(c)$ and by Proposition 1.10,

$b_c^1 : [0,\infty) \to M$, $b_c^1(t) := \exp(tv)$, is a B-ray, and therefore by Lemma 3.9, b_c^1 is perpendicular to B^{min} at g(c). By continuity, $b_c^1 \cap U \subseteq U_1$. Similarly, we get a B-ray b_c^2 perpendicular to B^{min} at g(c) such that $b_c^2 \cap U \subseteq U_2$. Therefore, both geodesics perpendicular to B^{min} at g(c) for all $c \in (a,b)$ are B-rays. The result for c = a,b follows from this and Proposition 1.10.

To show that the region R is flat, let R_1 and R_2 be as shown in Figure 4.3.

Case 1: R_1 and R_2 are good (B-wedges, as in Definition 2.5), then by Proposition 2.6

$$C(R) = C(M) - C(R_1) - C(R_2) \leq 2\pi - \pi - \pi = 0$$

so $$C(R) = 0$$

Case 2: R_1 bad, R_2 good (and similarly for R_1 good, R_2 bad): $\pi = C(R \cup R_2) = C(R) + C(R_2)$ and $C(R_2) = \pi$. Therefore, C(R) = 0.

Case 3: R_1, R_2 bad: $C(R \cup R_1) = C(R \cup R_2) = \pi$ (hence $C(R_1) = C(R_2)$). Given $c \in [a,b]$, let the regions R_1^c, R_2^c be as shown in Figure 4.4. At c = a, $R \cup R_2 = R_2^a$ is good, so $C(R_2^c) = \pi$ for c = a. Letting the value of c increase, if R_2 remains good all the way to c = b, then $C(R_2^a) = C(R_2^b) = \pi$, so $C(R_1) = C(R_2) = \pi$, and $C(R) = 0$. If, on the other hand, R_2^c becomes bad at c, then $C(R_2^c) = \pi - (2\pi - C(M)) = C(M) - \pi$. But by continuity, $\pi = \lim_{s \to c^-} C(R_2^s) = C(R_2^c)$. Thus $C(M) = 2\pi$. Therefore,

$$2\pi = C(M) = C(R \cup R_1) + C(R \cup R_2) - C(R) = \pi + \pi - C(R),$$

so $C(R) = 0$.

Thus in every case, $C(R) = 0$. Therefore, $K(P) = 0$ for all $P \in R$. QED

The hypotheses of the next lemma can never be true (see Theorem 4.9), but the lemma simplifies the proof of Corollary 4.7.

<u>Lemma 4.4.</u> If B^{min} exists and $C(M) < \pi$, then B^{min} is one point.

PROOF: Suppose there are two points $P \neq Q$ in B^{min}. Therefore, the (unique) minimal geodesic segment μ joining them is contained in B^{min} by Corollary 1.15. Therefore, by Proposition 4.3, both geodesics b_1, b_2 perpendicular to μ at P are B-rays. b_1 and b_2 thus form a B-wedge at P with vertex angle $\varepsilon = \pi$. Since at least one side W of $b_1 \cup b_2$ is a good B-wedge (Definition 2.5), we have by Proposition 2.6 that $C(W) = \varepsilon = \pi$. Therefore $\pi = C(W) \leqq C(M) \leqq \pi$, contradiction. QED

The next lemma generalizes Lemma 3.7 to the case $B^a = B^{min}$.

Lemma 4.5. If $B^{min} = B^m = g([a,b])$ for a geodesic g and $a \leqq b$, then the ∂B^s approach B^{min} $(s > m)$ in the following way: given $d > 0$, there exists $\varepsilon > 0$ such that for all $s \in (m, m+\varepsilon)$, $\partial B^s \subseteq B_d(B^{min})$, where $B_d(B^{min}) = \bigcup_{P \in B^{min}} B_d(P)$, where $B_d(P)$ is the open metric ball of radius d about P.

PROOF: Suppose not, i.e. there is a $d > 0$ and a sequence $s_k \downarrow m$ such that there is a

$$\tilde{P}_k \in \partial B^{a_k} - B_d(B^{min}).$$

Define $S = \overline{B_d(B^{min})} - B_{d/2}(B^{min})$.

Since B^{min} is bounded, $\overline{B_d(B^{min})}$ is bounded, as well as closed, hence compact. Therefore, S is compact, and $S \neq \emptyset$. Now a B-ray b at g(a) meets all the ∂B^{a_k}, so therefore, since $B_{d/2}(B^{min})$ is a neighborhood of g(a), $B_{d/2}(B^{min})$ will meet all the ∂B^{a_k} for k sufficiently large. Therefore, since all the ∂B^a are path-connected (by Proposition 3.5), and since $\tilde{P}_k \in \partial B^{a_k} - B_d(B^{min})$ for all k, ∂B^{a_k} meets S for all large k, say at $P_k \in S \cap \partial B^{a_k}$. Since S is compact, there exists an accumulation point P of the P_k. Choose a subsequence of the a_k such that $P_k \to P$. Since S is closed, $P \in S$. Also,

$$B(P) = B(\lim_{k\to\infty} P_k) = \lim_{k\to\infty} B(P_k) = \lim_{k\to\infty} a_k = m, \text{ i.e.}$$

$P \in B^{min} \cap S$. But since $B^{min} \subseteq B_{d/2}(B^{min})$,

$$B^{min} \cap S = B^{min} \cap (\overline{B_d(B^{min})} - B_{d/2}(B^{min})) = \emptyset.$$

Therefore, $P \in B^{min} \cap S$ is a contradiction. QED

The following lemma contains the technical aspects of two subsequent results.

<u>Lemma 4.6.</u> If $C(M) \leqq \pi$, and $B^{min} = B^m$ is one point, P, then there exists a B-wedge at P with angle $\varepsilon = \pi$.

PROOF: There is at least one B-ray b at P. Let (W,ε) denote the "maximal" good B-wedge at P, i.e., that B-wedge with vertex P which is good (in the sense of Definition 2.5 and Proposition 2.6) and has maximal angle. If there are no B-wedges at P, let this W simply be b (and $\varepsilon = 0$). Thus we have (by Proposition 2.6),

$$\varepsilon = C(W) \leqq C(M) \leqq \pi, \text{ so } W \neq M.$$

Suppose that $\varepsilon < \pi$. We will show that in this case W is not maximal, so that $\varepsilon = \pi$. Thus let a_k be a decreasing sequence of real numbers, $a_k \downarrow m$, where $B^{min} = B^m$. Thus the ∂B^{a_k} approach P in the sense of Lemma 4.5. Since $W \neq M$, fix $Q \notin W$ such that $B(Q) > a_k$ for all k. Take the minimal connection μ_k from Q to ∂B^{a_k}, say to $P_k \in \partial B^a$, for all k. By Proposition 3.2, either μ_k can be extended (from P_k to Q) to form a B-ray, or μ_k lies in the interior of a B-wedge (W_k, ε_k) such that $W_k \cap \partial B^a = \{P_k\}$. If the first case occurs an infinite number of times, then the limit of these B-rays through Q is a B-ray through Q (since the $P_k \to P$ by Lemma 4.5, so the initial vectors of the B-rays have an accumulation point at P which generates a B-ray by proposition 1.10) i.e., there is a B-ray through P which is not contained in W. If the first case occurs only a finite number of times, then the second case occurs an infinite number of times. Since the (W_k, ε_k) have $\varepsilon_k < \pi$, they must be good (for if they were not, then their complements, with vertex angles $2\pi - \varepsilon_k$, would have to be good. But then $2\pi - \varepsilon_k = C(M-W_k) \leqq C(M) \leqq \pi$, so $\varepsilon_k \geqq \pi$, contradiction). Therefore by Propositin 2.6, $C(W_k) = \varepsilon_k$. Claim: These W_k are nested, i.e., if $k_1 < k_2$, then $W_{a_{k_1}} \subseteq W_{a_{k_2}}$. Proof: If $k_2 > k_2$, then $a_{k_2} < a_{k_1}$. Therefore, μ_{k_2} from Q to P_{k_2} meets the boundary of W_{k_1}. The B-rays which bound W_{k_2} (and also lie on either side of μ_{k_2}) cannot meet the B-rays bounding W_{k_1}, and cannot meet μ_{k_2} a second time. Therefore, W_{k_1} lies entirely in the region bounded by the boundary of W_{k_2} which contains Q, namely W_{k_2}.

Now if $k_2 > k_1$, then $a_{k_2} < a_{k_1}$, so $W_{k_1} \subseteq W_{k_2}$. Therefore, $0 \leqq C(W_{k_2} - W_{k_1}) = C(W_{k_2}) - C(W_{k_1}) = \varepsilon_{k_2} - \varepsilon_{k_1}$, i.e. $\varepsilon_{k_2} \geqq \varepsilon_{k_1} > 0$.

Since the $P_k \to P$ (by Lemma 4.5), the B-rays bounding the W_k (b_k^1, b_k^2) have accumulation points at P, which are B-rays by Proposition 1.10. Thus choose a subsequence such that $b_k^1 \to b^1$, $b_k^2 \to b^2$. Let $\varepsilon_0 = \measuredangle(\dot{b}^1(0), \dot{b}^2(0))$. Then $\varepsilon_0 = \lim_{k\to\infty} \varepsilon_k$, $\varepsilon_0 \in (0,\pi]$ ($\varepsilon_0 \neq 0$ since the ε_k form a nondecreasing sequence). Since $\varepsilon_0 \neq 0$, $b^1 \neq b^2$. Let $(\tilde{W}, \varepsilon_0)$ denote the B-wedge bounded by b^1, b^2. Since in the original (W,ε) we have assumed $\varepsilon < \pi$, and since $\tilde{W} \subseteq \overline{M-W}$, we have at least one more B-ray through P which lies outside W. Therefore, in either case,

There is a B-ray through P not contained in W. We now show that this gives a contradiction.

Case 1: $W = b$ ($\varepsilon=0$). Then the new B-ray gives us at least one good B-wedge with angle $\varepsilon_0 > 0$, contradicting the maximality of $\varepsilon = 0$.

Case 2: $\varepsilon > 0$, i.e. we have a (non-degenerate) B-wedge. We have a B-ray at P not contained in W, denoted $\tilde{b}$. $\tilde{b}$ makes an angle $\theta < \pi$ with at least one of the B-rays bounding W, say b_1. Let (W^*,θ) be the B-wedge bounded by b_1 and $\tilde{b}$. Then since $\theta < \pi$, W is good (again since its complement has an angle too large for $C(M) \leqq \pi$). Therefore $C(W^*) = \theta$. Then $C(W \cup W^*) = C(W) + C(W^*) = \varepsilon + \theta =$ the angle of the B-wedge $W \cup W^*$, so that the B-wedge $W \cup W^*$ is good (for if it were bad, then its curvature integral would be $\varepsilon + \theta - D$ by Proposition 2.6, where here $D \neq 0$). This contradicts the maximality of W. Therefore, $\varepsilon = \pi$. QED

<u>Corollary 4.7</u>. If $C(M) < \pi$, then B^{min} does not exist.

PROOF: Suppose B^{min} does exist. Therefore by Lemma 4.4, B^{min} is a single point, say $B^{min} = \{P\}$. Then Lemma 4.6 implies that there is a B-wedge at P with angle π. But at least one side of this B-wedge is good (Definition 2.5), say M, so by Proposition 2.6

$$C(W) = \pi > C(M) \geqq C(W), \text{ contradiction.} \quad \text{QED}$$

<u>Proposition 4.8.</u> If $B^{min} = B^m$ exists, and $C(M) = \pi$, then there exists a (normal) geodesic g such that $B^{min} = g([0,\infty))$.

PROOF: By Lemma 4.2, B^{min} is either a point P or $g(I)$, where I is either $[0,b]$, $b > 0$, or $[0,\infty)$. We therefore want to rule out the first two possibilities:

I) If $B^{min} = \{P\}$, then Lemma 4.6 implies that there is a B-wedge at P with angle $\varepsilon = \pi$. Let R_2 denote the good side, and R_1 the other side. Then $C(R_2) = \pi$ so

$$C(R_1) = C(M) - C(R_2) = \pi - \pi = 0,$$

i.e., R_1 is flat.

II) If $B^{min} = g([0,b])$, $b > 0$, then Proposition 4.3 implies that there is a B-wedge at all $g(c)$, $c \in [0,b]$, with vertex angle π, and the region R (see Figure 4.5) is flat. Claim: One of the R_1, R_2 is good and the other is bad. Proof: If both are good, then $C(R_1) = C(R_2) = \pi$, so

$$2\pi = \pi + \pi = C(R_1) + C(R_2) = C(R_1 \cup R_2) \leq C(M) = \pi,$$

contradiction.

If both R_1 and R_2 are bad, then

$$C(R \cup R_2) = C(R \cup R_1) = \pi$$

so since $C(R) = 0$,

$$\pi = C(R) + C(R_2) = C(R_2)$$

and $\pi = C(R) + C(R_1) = C(R_1)$,

which as before is a contradiction. Thus one of the R_1, R_2 is good and the other is bad. Thus suppose (w.l.o.g.) that R_2 is good and R_1 is bad, and let $P = g(0)$. Hence

$$C(R_2) = \pi, \text{ so } C(R_1) = C(M) - C(R_2) = \pi - \pi = 0$$

so R_1 is flat.

Therefore, in both of these cases we have the same situation:

We have a point $P \in B^{min}$, a B-wedge R_1 at P with angle π and B-rays for boundaries (call them b_1, b_2), where R_1 is flat and $R_1 \cap B^{min} = \{P\}$. Claim: No geodesic starting at P and contained in R_1 (other than b_1 and b_2) can be a B-ray. Proof: If there is a B-ray b as described, it makes an angle $\theta \in (0,\pi)$ with b_1. Therefore the B-wedge (W,θ) bounded by b_1 and b is good (since its complement cannot be), so

$$0 < \theta = C(W) \leq C(R_1) = 0, \text{ contradiction.}$$

Now let $g : [0,\infty) \to M$ be the geodesic with $g(0) = P$ which is perpendicular to $b_1 \cup b_2$ and is contained in R_1. As noted in the last paragraph, g is not a B-ray. Fix $s > 0$, and choose a decreasing sequence $a_k \downarrow m$, where $a_k < B(g(s))$ for all k. Let $\mu_k : [0,d_k] \to M$ be minimal connections between $g(s)$ and the ∂B^{a_k}, say from $P_k \in \partial B^{a_k}$ to $g(s)$. Claim: The μ_k cannot meet $b_1 \cup b_2$. Proof: If

μ_k meets b_1 (say) at $\mu_k(t_k) = b_1(s_k)$, where $t_k \in (0,d_k)$, then since b_1 is a B-ray through $P \in B^{min}$, b_1 is minimal from $b_1(s_k)$ to ∂B^{a_k} (say at $b_1(u_k) \in \partial B^{a_k}$). Therefore the broken geodesic $b_1|_{[u_k,s_k]} \cup \mu_k|_{[t_k,d_k]}$ is minimal between g(s) and ∂B^{a_k}, contradiction. If on the other hand $\mu_k(0) = b_1(s_k) \in \partial B^{a_k}$, then since g(s) does not lie on the B-ray b_1, we have (by Proposition 3.2) that μ_k lies in the interior of a B-wedge at $b_1(s_k)$. But there is no B-wedge at $b_1(s_k)$ since there is a unique B-ray there (by Corollary 1.8). Therefore, the μ_k never meet b_1 (and similarly for b_2).

Letting $a_k \downarrow m$, then since the $P_k \in Int(R_1)$, and $R_1 \cap B^{min} = \{P\}$, and the ∂B^{a_k} approach B^{min} as in Lemma 4.5, we have $P_k \to P$.

If an infinite number of the μ_k are the initial portion of B-rays (as in Proposition 3.2), then since the $P_k \to P$, they will give us a B-ray at P (by Proposition 1.10), and this, too, will pass through $g(s) \in R_1$. But this possibility was ruled out (no B-rays through P in R_1 except b_1,b_2). Therefore, by Proposition 3.2 an infinite number of the k are such that μ_k is contained in the interior of a B-wedge (W,Θ) with vertex P_k, $0 < \Theta < \pi$. Fix one of these, and let $\tilde{b}_1,\tilde{b}_2$ denote the B-rays which form its sides. $P_k \in Int(R_1)$. Let L be the line through P_k parallel to $b_1 \cup b_2$. Then since R_1 is flat, and the $\tilde{b}_1,\tilde{b}_2$ cannot meet the b_1,b_2, the

$\tilde{b}_1, \tilde{b}_2$ must lie in the region bounded by L and contained in R_1. Since we have noted that $\Theta < \pi$, the B-wedge (W, Θ) is good (again since the angle of its compliment is too big for it to be good). But then $0 < \Theta = C(W) \leqq C(R_1) = 0$, contradiction. Therefore, Cases I and II are not possible. QED

Note: The third Case, where B^{min} exists and is of the form $g([0,\infty))$, can occur. For example, modify the cone $z^2 = x^2 + y^2$, $z \geqq 0$, by removing a neighborhood of the singularity and replacing it smoothly with a compact, convex cap. (See the appendix.)

The following theorem characterizes the B^a, ∂B^a and B^{min} (whether they are compact or noncompact) according to $C(M)$:

<u>Theorem 4.9</u> If M is open, complete, $\dim(M) = 2$, and $K \geqq 0$, then

i) If $C(M) < \pi$, then each B^a is noncompact (because B^{min} does not exist), and ∂B^a is compact if and only if M is a flat cylinder.

ii) If $C(M) = \pi$, then:

If B^{min} exists, then there is a (normal) geodesic g such that $B^{min} = g([0,\infty))$, and the $B^a \neq \emptyset$ are noncompact. Furthermore, the geodesic perpendicular to g at g(0) divides M into two regions, and the one containing B^{min} is flat.

If B^{min} does not exist, then all the B^a are noncompact, and all the ∂B^a are noncompact.

iii) If $C(M) > \pi$, then all the B^a (and ∂B^a) are compact, and therefore B^{min} exists.

PROOF: (i) Corollary 4.7 implies that B^{min} does not exist, so clearly the B^a are noncompact for all a. If there is an a such that ∂B^a is compact, then Lemma 3.8 implies that M is the flat cylinder. Conversely, if M is a flat cylinder, then the ∂B^a are compact.

(ii): If B^{min} exists, then the first assertion follows from Proposition 4.8 and Proposition 4.3. If B^{min} does not exist, then clearly the B^a are all noncompact. Therefore, the ∂B^a are all noncompact (else Lemma 3.8 implies that M is a flat cylinder, contradiction since $C(M) \neq 0$).

(iii): This is Proposition 4.1. QED

Chapter 5. Asymptotic Behavior of Geodesics when $C(M) = 2\pi$

In this chapter we consider a surface M with total curvature 2π. Under these hypotheses we see that all rays are asymptotically B-rays, and that arbitrary geodesics asymptotically behave the same way for any two Busemann functions.

We now generalize Construction 1.2 (induced rays) and show that their main feature (Corollary 1.4) still holds.

Lemma 5.1 Given ray r and its associated Busemann fuction B, let t_k be an increasing sequence, $t_k \to \infty$. Choose a sequence of points $P_k \to P \in M$, and let $\mu_k : [0,d_k] \to M$ be (normal) minimal connections from P_k to $r(t_k)$. Let the $v_k := \dot{\mu}_k(0)$ have accumulation point v at P, and let $g : [0,\infty) \to M$ be $g(t) = \exp(tv)$. Then g is a B-ray.

PROOF: Choose a subsequence such that $v_k \to v$. Fix $d > 0$. By continuity of exp and the distance function ρ on M, we have

$$\rho(g(d),g(0)) = \rho(\exp(dv),P) = \rho(\lim_{k\to\infty} \exp(dv_k), \lim_{k\to\infty} P_k)$$

$$= \lim_{k\to\infty} \rho(\exp(dv_k),\exp(0\cdot v_k)) = \lim_{k\to\infty} d = d.$$

Therefore, g is a ray. Again fix $d > 0$. (See Figure 5.1.) Let $b_k = n_k - m_k$. Then $\Delta B := B(g(d)) - B(P) = \lim_{k\to\infty} b_k$.

As in the proof of Lemma 1.3, we arrive at (replacing d by δ_k)

$$[(b_k/m_k) + 1]\cdot\delta_k \cos\Theta_k \leqq [(b_k^2+\delta_k^2)/2m_k] + b_k.$$

Claim: $\Theta_k \to 0$. Proof: By assumption, $v_k \to v$. We therefore need only show that $\tilde{v}_k \to v$. Suppose not, i.e. we can choose a subsequence such that $\tilde{v}_k \to \tilde{v} \neq v$. Let $\tilde{\mu} : [0,d] \to M$ be $\tilde{\mu}(t) = \exp(t\tilde{v})$. Then $\tilde{\mu}(0) = \lim_{k\to\infty} \exp(0\cdot\tilde{v}_k) = \lim_{k\to\infty} P_k = P$, and $\tilde{\mu}(d) = \exp(d\tilde{v}) = \lim_{k\to\infty} \exp(\delta_k v_k) = \lim_{k\to\infty} g(d) = g(d)$ since the $\delta_k \to d$. Therefore, $\tilde{\mu}$ is a minimal connection from $P = g(0)$ to $g(d)$ other than g, contradicting the fact that g is a ray. Therefore, $\lim_{k\to\infty} \Theta_k = 0$.

Taking the limit in the last inequality, we get

$$[0+1]d\cdot\cos\Theta \leqq 0 + \Delta B, \text{ or } d \leqq \Delta B.$$

Since $\Delta B \leqq d$, we have

$$d = \Delta B = B(g(d))-B(g(0)) \text{ for all } d > 0.$$

Therefore g is a B-ray. QED

In the rest of this chapter, we shall assume that M is a surface. Recall that $C(R) = \int_R K$ for a region R.

In the following theorem, we see that a given ray r_1 will "asymptotically approach" any family of $\tilde{B}$-rays, i.e. the angle it makes with these $\tilde{B}$-rays goes to zero.

<u>Theorem 5.2</u> Suppose that $C(M) = 2\pi$. Given rays r and $\tilde{r}$, we have their associated Busemann functions B and $\tilde{B}$, respectively, and the functions (Definition 1.11) ω and $\tilde{\omega}$. If r_1 is any B-ray, then

$$\tilde{\omega}(r_1) = \tilde{\omega}(r) = 0.$$

PROOF: If $r_1 = r$, then $\tilde{\omega}(r_1) = \tilde{\omega}(r)$. Thus suppose $r_1 \neq r$. Let μ be a (normal) minimal connection from $r_1(0)$ to $r(0)$, and let R be the region bounded by the curve $r_1 \cup \mu \cup r$ which contains an unbounded portion of $\tilde{r}$. (See Figure 5.2.) Let $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$. By Corollary 2.4, $C(R) = \alpha + \beta - \pi$ (since $D = 0$). Since $r, \tilde{r}$, and r_1 are rays, they intersect each other at most once. Therefore, suppose that we start $\tilde{r}$ at a point such that $\tilde{r}$ does not meet r or r_1. By the Gauss-Bonnet Theorem,

$$C(R_3) = \alpha_2 + \beta_2 + \gamma_3 - \pi.$$

Using the "arrow" notation of Definition 2.2 to denote "good" regions (with respect to the ray $\tilde{r}$), we have two cases.

Case 1: The arrow on r_1 and r both point into R. Therefore, by Proposition 2.3,

$C(R_1) = \alpha_1 + \gamma_1 - \tilde{\omega}(r_1) - \pi$, and

$C(R_2) = \beta_1 + \gamma_2 - \tilde{\omega}(r) - \pi$. Therefore,

$$\alpha + \beta - \pi = C(R_1) + C(R_2) + C(R_3)$$

$$= (\alpha_1+\gamma_1-\tilde{\omega}(r_1)-\pi) + (\beta_1+\gamma_2-\tilde{\omega}(r)-\pi) + (\alpha_2+\beta_2+\gamma_3-\pi)$$

$$= (\alpha_1+\alpha_2) + (\beta_1+\beta_2) + (\gamma_1+\gamma_2+\gamma_3) - \tilde{\omega}(r_1) - \tilde{\omega}(r) - 3\pi$$

$$= \alpha + \beta + 2\pi - \tilde{\omega}(r_1) - \tilde{\omega}(r) - 3\pi = \alpha + \beta - \pi - \tilde{\omega}(r_1) - \tilde{\omega}(r).$$

Therefore,

$$\tilde{\omega}(r_1) + \tilde{\omega}(r) = 0, \text{ so } \tilde{\omega}(r_1) = \tilde{\omega}(r) = 0.$$

Case 2: One of the arrows from r_1, r points out of R, say the arrow from r_1 points out of R. Therefore R_1 is a bad region. But as noted in the proof of Proposition 2.6 (the claim in Case B.ii), this implies that R_2 is a good region. Thus the formulas for $C(R_2)$ and $C(R_3)$ are as before, but $C(R_1) = \alpha + \beta - \pi + \tilde{\omega}(r_1)$ by Proposition 2.3. Therefore, by a calculation as before, $\alpha + \beta - \pi = C(R) = \alpha + \beta - \pi + \tilde{\omega}(r_1) - \tilde{\omega}(r)$ so $\tilde{\omega}(r_1) = \tilde{\omega}(r)$ for any B-ray r_1. Therefore, we need only show that $\tilde{\omega}(r) = 0$.

We know by Theorem 4.9 that the $\tilde{B}^a$ are compact, so fix a $\tilde{B}^a \neq \tilde{B}^{min}$. We have the function $\tilde{pr}_a : M \to \tilde{B}^a$ (Definition 3.3), where $\tilde{pr}_a(\overline{M-\tilde{B}^a}) \subseteq \partial\tilde{B}^a$. Since r is a ray, it leaves the compact set $\tilde{B}^a$. Therefore, the set $\{P_n = \tilde{pr}_a(r(n)) \mid n = 1,2,\ldots\}$

has an accumulation point P on $\partial\tilde{B}^a$. Letting $\mu_n : [0,d_n] \to M$ be (normal) minimal connections from P_n to $r(n)$, the $v_n := \dot{\mu}_n(0)$ have an accumulation point v at P. Let $g : [0,\infty) \to M$ be $g(t) = \exp(tv)$. By Lemma 5.1, g is a B-ray.

By Proposition 3.2, the μ_k are either the beginnings of $\tilde{B}$-rays, or there are $\tilde{B}$-wedges W_k at P_k such that μ_k is in the interior of W_k and $W_k \cap \partial\tilde{B}^a = \{P_k\}$. If an infinite number of the μ_k are $\tilde{B}$-rays, then so is their limit g (by Proposition 1.10), so $\tilde{\omega}(g) = 0$. Since $\tilde{\omega}(r_1) = \tilde{\omega}(r)$ for all B-rays r_1, and since g is a B-ray, we have $\tilde{\omega}(r) = \tilde{\omega}(g) = 0$.

If, on the other hand, only a finite number of the μ_k are beginnings of $\tilde{B}$-rays, then we have an infinite number of the $\tilde{B}$-wedges W_k as noted above.

There are two cases: an infinite number of the P_k are distinct, or an infinite number of the P_k coincide. In the first case (taking a subsequence if necessary) the $\tilde{B}$-wedges (W_k,ε_k) are mutually disjoint. Since $C(W_k) = \varepsilon_k$ (by Proposition 2.6), $\sum_{k=1}^{\infty} \varepsilon_k = C(\bigcup_{k=1}^{\infty} W_k) \leq C(M) = 2\pi$. Since the $\varepsilon_k > 0$, $\lim \varepsilon_k = 0$. If $\tilde{\varepsilon}_k$ denotes the minimum of angles between $\dot{\mu}_k(0)$ and the sides of W_k, then $0 < \tilde{\varepsilon}_k < \varepsilon_k$, so $\lim_{k\to\infty} \tilde{\varepsilon}_k = 0$. Since the sides of the W_k are $\tilde{B}$-rays which have a $\tilde{B}$-ray $\tilde{b}$ at P as accumulation point (Proposition 1.10), and since $\tilde{\varepsilon}_k \to 0$, $g = b$, i.e., g is a $\tilde{B}$-ray. Therefore, as before,

$$\tilde{\omega}(r) = \tilde{\omega}(g) = 0.$$

Finally, the last case is when an infinite number of the P_k coincide (hence the $P_k = P$). The W_k thus are a single wedge (W, ε). Since $g \subseteq W$, W is a $\tilde{B}$-wedge, and g is a B-ray, we have $\tilde{\omega}(r) = \tilde{\omega}(g) \leq \tilde{\Theta}_g(0) \leq \varepsilon/2$. But $B^a \neq B^{min}$ is arbitrary, so we may choose a as large as we like. Since $\varepsilon = C(W)$, and $C(W) \to 0$ as $a \to \infty$, we have $\tilde{\omega}(r) = 0$. QED

The following corollary generalizes Proposition 2.6 in the case $C(M) = 2\pi$.

<u>Corollary 5.3</u> If $C(M) = 2\pi$, and a wedge (W, ε) has two rays r, r_1 for boundaries, then $C(W) = \varepsilon$.

PROOF: In Proposition 2.3, choose $\tilde{r}$ and μ so that $r_1 = \tilde{r} \cup \mu$, hence $\alpha = \pi$ and $\beta = \varepsilon$. By assumption $D = 0$, and $\omega(\tilde{r}) = 0$ by Theorem 5.2. Therefore, by Proposition 2.3,

$$C(W) = \alpha + \beta - \pi - \omega(\tilde{r}) = \varepsilon. \quad \text{QED}$$

Finally, our last result shows that, in the case $C(M) = 2\pi$, any geodesic will behave asymptotically the same for any two Busemann functions.

<u>Corollary 5.4</u> If $C(M) = 2\pi$, r and $\tilde{r}$ are any rays (with their associated Busemann functions $B,\tilde{B}$ and the functions $\omega,\tilde{\omega}$ (Definition 1.11), and g is any geodesic, then $\omega(g) = \tilde{\omega}(g)$.

PROOF: By Theorem 4.9, the B^a and $\tilde{B}^a$ are all compact. Therefore, since g is unbounded (unless it is periodic, in which case $\omega(g)$ and $\tilde{\omega}(g)$ are $\pi/2$), g leaves all of these B^a and $\tilde{B}^a$. Since the B^a and $\tilde{B}^a$ exhaust M, given $\varepsilon > 0$, we can find a number a such that

$$C(B^a) > 2\pi - (\varepsilon/2) \text{ and } C(\tilde{B}^a) > 2\pi - (\varepsilon/2).$$

Hence

$$C(\overline{M-B^a}) < \varepsilon/2 \text{ and } C(\overline{M-\tilde{B}^a}) < \varepsilon/2.$$

Choose $t \in \mathbb{R}$ such that if $P := g(t)$, then $d := B(g(t)) \geq a$ and $c := \tilde{B}(g(t)) \geq a$, and $B\circ g$ and $\tilde{B}\circ g$ are increasing at P. Thus,

$$C((\overline{M-B^d}) \cup (\overline{M-\tilde{B}^c})) \leq C(\overline{M-B^d}) + C(\overline{M-\tilde{B}^c}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Let $b,\tilde{b}$ be B-, $\tilde{B}$-, rays at P of minimal angle with $\dot{g}(t)$. If (W,Θ) is the wedge bounded by $b \cup \tilde{b}$ such that $\Theta < \pi$, then $W \subseteq (\overline{M-B^d}) \cup (\overline{M-\tilde{B}^c}) := S$. (Proof: We first note that Θ is less than π, i.e., $\Theta \neq \pi$. Since B and $\tilde{B}$ increase along g at P, it follows from Proposition 1.6 that

$$\alpha := \measuredangle(\dot{g}(t),\dot{b}(0)) < \pi/2 \text{ and } \beta := \measuredangle(\dot{g}(t),\dot{\tilde{b}}(0)) < \pi/2.$$

Thus, $\Theta \leq \alpha + \beta < \pi/2 + \pi/2 = \pi$.

To show that $W \subseteq S$, let $Q \in W$, and let $\mu : [0,k] \to M$ be a (normal) minimal connection from P to Q. This makes an angle $\tilde{\theta} < \pi/2$ with at least one of the sides of W, say b. Therefore

$$0 < k \cdot \cos \tilde{\theta} \leq B(Q) - B(P), \text{ so } B(Q) > B(P0 = d$$

$$\text{so } Q \notin B^{d}.$$

If $\tilde{\theta} < \pi/2$ is the angle between g and $\tilde{b}$, a similar calculation shows that $Q \notin \tilde{B}^{c}$. Therefore $Q \in S$.

By Corollary 5.3, $\theta = C(W) \leq C(S) < \varepsilon$. $B = \pm \alpha \pm \theta$. Since ε can be made arbitrarily small by going out sufficiently far along g, the term $\theta \to 0$. Therefore,

$$\tilde{\omega}(g) = \lim_{t \to \infty} \beta(t) = \lim_{t \to \infty} \alpha(t) + \lim \theta(t) = \omega(g). \quad \text{QED}$$

Chapter 6. Further Questions

The concept of ∞-winders can perhaps be extended to a measurement of "finite winding." For example, perhaps an upper bound for the amount of winding of an arbitrary geodesic on a surface M can be established from a knowledge of the total curvature C(M). Such an upper bound exists in the case of a surface of revolution $z = f(r)$. Using the formulas in the appendix, one sees that if $C(M) < 2\pi$, then $L := \lim_{r\to\infty}[1 + (f'(r))^2]^{\frac{1}{2}}$ is finite, and $L = 2\pi / [2\pi - C(M)]$. Using the formula for geodesics, and using $[1 + (f'(r))^2]^{\frac{1}{2}} \leqq L$, we have

$$\lim_{r\to\infty}[\theta(r)-\theta(c)] = \int_C^{\infty}\frac{C[1+(f'(r))^2]^{\frac{1}{2}}}{\sqrt{r^2-c^2}}dr \leq \int_c^{\infty}\frac{c\cdot L}{r\sqrt{r^2-c^2}}dr = L\cdot\pi/2.$$

This being half of the geodesic, the total change in θ is $\Delta\theta \leqq \pi L$. Therefore $\Delta\theta \leqq 2\pi^{2/}[2\pi - C(M)]$.

When $C(M) = 2\pi$, there may or may not be ∞-winders (see the appendix). In the case of an arbitrary surface with $C(M) = 2\pi$, perhaps a necessary condition for the existence of ∞-winders exists.

The general idea and many results about B-rays are valid in dimension n. Perhaps the concept of a B-wedge can be extended to a "B-cone," and the projeciton map pr_a (Definition 3.3) defined in higher dimensions. The question of the compactness of the horospheres in higher dimensions might similarly be classified by the curvature of M.

In the case of a surface with $C(M) = 2\pi$, can we characterize any goedesic g as either an ∞-winder or "asymptotic" to some ray? If the asymptotic angle $\omega(g) > 0$, is g necessarily an ∞-winder?

Finally, suppose a surface M has $C(M) < 2\pi$. Initial results indicate that there are at most four values for $\tilde{\omega}(b)$, where $\tilde{b}$ is any $\tilde{B}$-ray, $\tilde{B}$ a fixed Busemann function.

Appendix

We give here an intuitive look into why the compactness of a set B^a is determined by the total curvature C(S) of a surface S by considering a special case.

Let S be a cone which has been modified by removing a neighborhood of the vertex and replacing it smoothly with a convex cap. We thus realize S as a surface of revolution $z = f(r)$, $r^2 = x^2 + y^2$, where the smooth function $z = f(r)$ is part of the line $z = mr$ for all r greater than some r_0. (See Figure A.1.)

It is easily seen (see [0] Section 5.6) that the total curvature of this kind of surface of revolution is

$$C(S) = 2\pi[1 - (1+m^2)^{-\frac{1}{2}}].$$

Now let us imagine that we have cut off the non-flat cap, and have then sliced the cone along a meridian $\tilde{r}$. We unroll this cone, and place it into the flat plane. (See Figure A.2.) (For convenience complete the cone by replacing the neighborhood of the vertex.) Let Θ denote the angle at the vertex as measured outside of the cone.

Claim: $C(S) = \Theta$.

PROOF: We need to show that $\Theta = 2\pi[1 - (1+m^2)^{-\frac{1}{2}}]$.

Consider the portion of the cone at distance less than or equal to some fixed r (see Figure A.3). Let R denote the radius of the circle forming the edge of this set, and let L denote its length. Thus $L = 2\pi R$. R and r are related by

$r^2 = R^2 + (mR)^2$, so $R = r(1+m^2)^{-\frac{1}{2}}$. Thus

$L = 2\pi r(1+m^2)^{-\frac{1}{2}}$. But L is related to Θ:

$L = r(2\pi - \Theta)$. Thus

$r(2\pi - \Theta) = L = 2\pi r(1+m^2)^{-\frac{1}{2}}$, so

$\Theta = 2\pi[1 - (1+m^2)^{-\frac{1}{2}}]$. QED

In the flat plane R^2, fix a ray r, and let B be its associated Busemann function. Then the level sets $B^{-1}(a)$ are lines perpendicular to the line containing r. We now cut out part of this plan to construct a cone S. Let $\tilde{r}$ be the meridian along which the two sides of S are joined, and let the angle Θ be as before. Depicted are typical level sets $B^{-1}(a)$ which do not enter the non-flat cap. Clearly, for any $\Theta > \pi$ (Figure A.4), the $B^{-1}(a)$ will meet $\tilde{r}$, and hence will be bounded. But if $\Theta \leq \pi$ (Figure A.5, A.6), then the $B^{-1}(a)$ will not meet $\tilde{r}$, and will thus be unbounded.

We shall now consider some examples concerning infinite winding. In [0], page 333, one finds the formula for a pre-geodesic (i.e., a curve which is a geodesic upon reparametrization) in a surface of revolution M, $z = f(r)$, $r^2 = x^2 + y^2$. Namely, a curve

$$g(r) = (r\cdot\cos\theta(r), r\cdot\sin\theta(r), f(r))$$

in M is a pre-geodesic if and only if

$$d\theta/dr = \pm c[1 + [f'(r)]^2]^{\frac{1}{2}}/r[r^2 - c^2]^{\frac{1}{2}},$$

where c is a constant. The total curvature C(M) of the above surface of revolution $z = f(r)$ is

$$C(M) = 2\pi\cdot[1 - \lim_{r\to\infty}[1 + [f'(r)]^2]^{-\frac{1}{2}}].$$

(This follows from pages 243 and 281 of [0].) Thus if $C(M) < 2\pi$, then $\lim_{r\to\infty} f'(r) = L < \infty$, so the function $\theta(r)$ above will be finite as $r \to \infty$.

In the case of $f(r) = r^2$, $C(M) = 2\pi$, and $\lim_{r\to\infty} f'(r) = \infty$, as can be seen by noting that

$$[1 + [f'(r)]^2]^{\frac{1}{2}} \geq |f'(r)|.$$

However, in the case $f(r) = [1 + r^2]^{3/4}$, $C(M) = 2\pi$, but the formula for $\theta(r)$ remains bounded as $r \to \infty$. Thus we see that there can be infinite winding only if $C(M) = 2\pi$, but that there are surfaces M with $C(M) = 2\pi$ on which there are no ∞-winders.

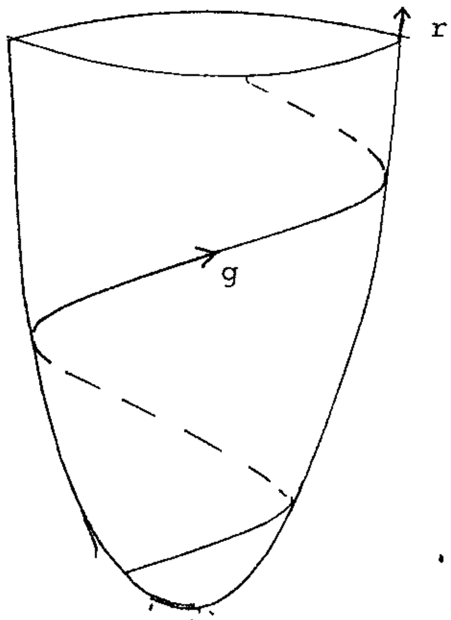

Fig. 1.1

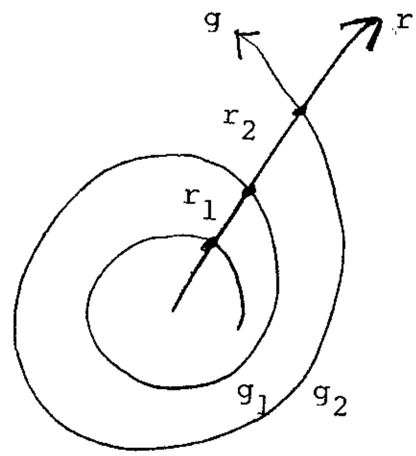

Fig. 1.2

Fig. 1.3

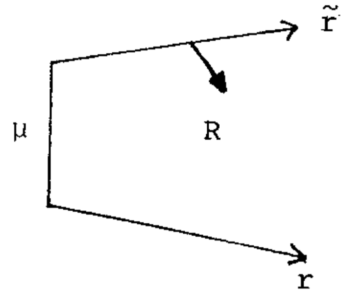

Fig. 2.1

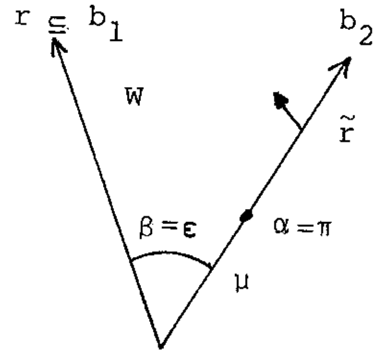

Fig. 2.2

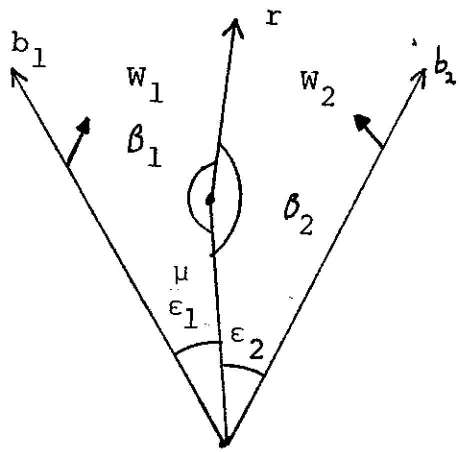

Fig. 2.3

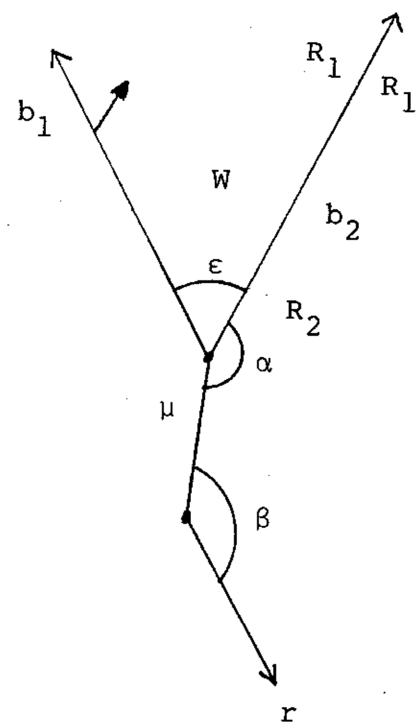

Fig. 2.4

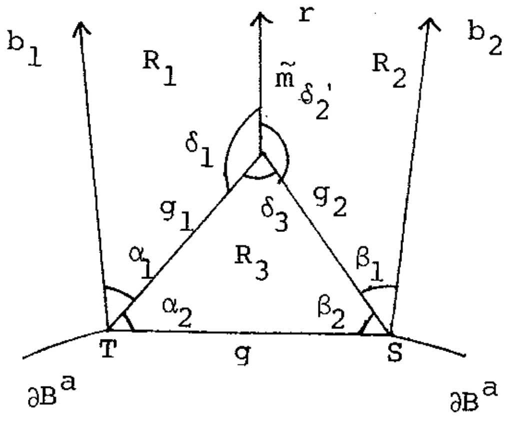

Fig. 4.1

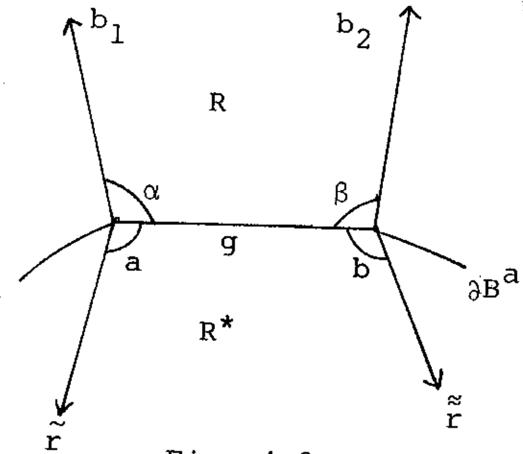

Fig. 4.2

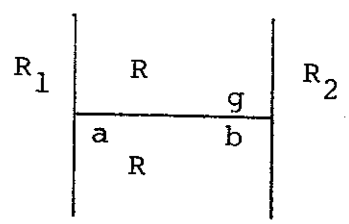

Fig. 4.3

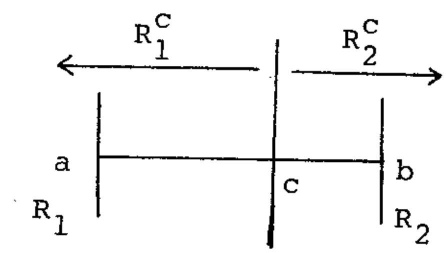

Fig. 4.4

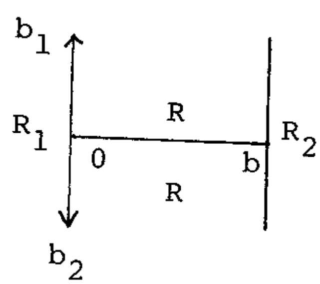

Fig. 4.5

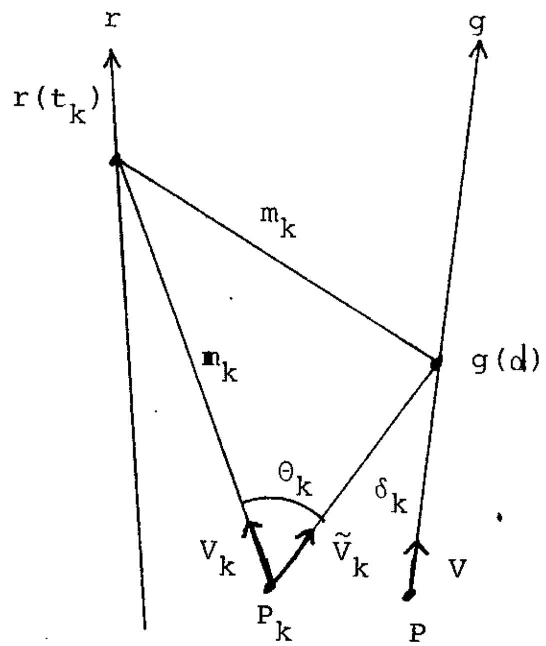

Fig. 5.1

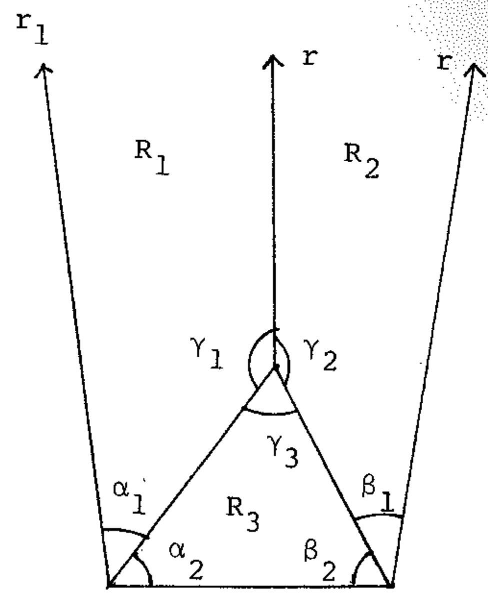

Fig. 5.2

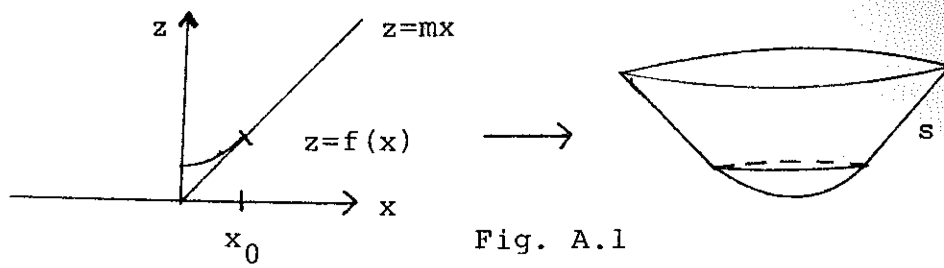

Fig. A.1

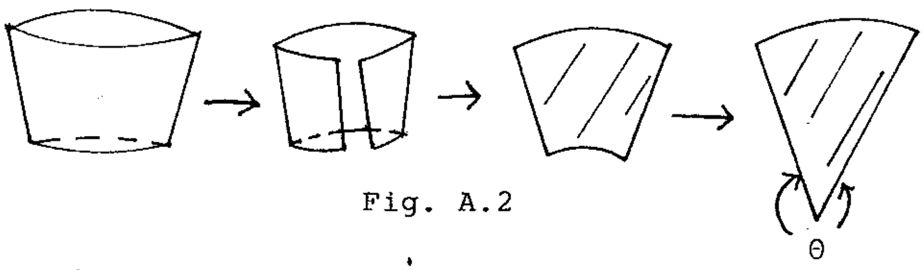

Fig. A.2

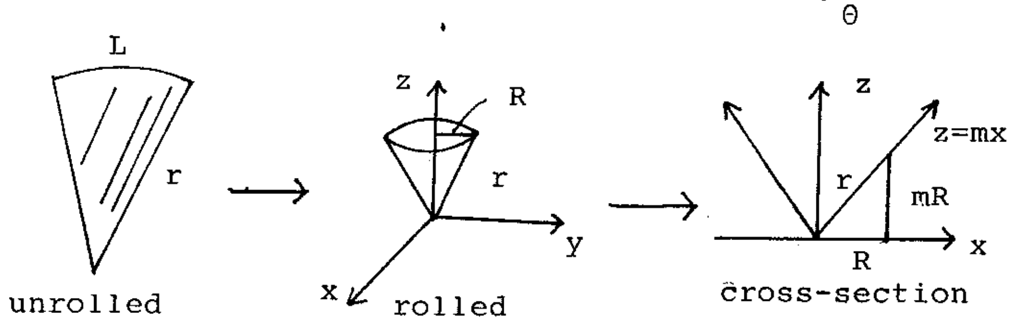

Fig. A.3

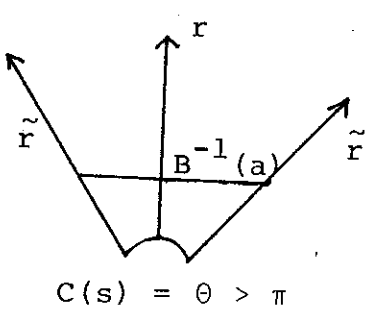

Fig. A.4

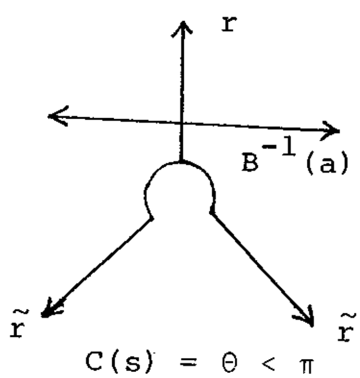

Fig. A.6

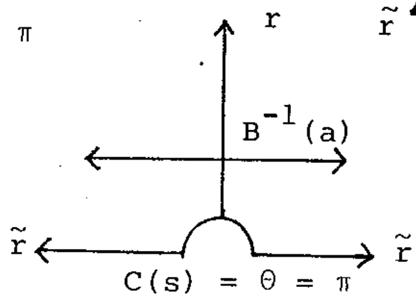

Fig. A.5

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