ARITHMETIC CLASSIFICATION OF FAMILIES OF ABELIAN VARIETIES OF QUATERNION TYPE.

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Abstract of the Dissertation

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A family of Abelian varieties of quaternion type, $A \xrightarrow{f} V$ is a fiber space whose fibers are Abelian varieties and which is parametrized by a Hilbert modular variety $\Gamma \backslash \mathbb{H}^t$.

We present a classification of such families with a given endomorphism ring and Hodge structure. The main result is that the bottom field of A is an abelian extension of the bottom field of V. An example of family of "Satake" type is constructed to show that the bottom field of A can be different from the bottom field of V.

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Introduction

A family of Abelian varieties of quaternion type $A \xrightarrow{f} V$ is a fiber space whose fibers are Abelian varieties and which is parameterized by a Hilbert modular variety $V = \Gamma\backslash\mathbf{H}^t$.

The aim of this dissertation is to study the arithmetic of the family. The main result is that, assuming the family is characterized by its Hodge structure, the bottom field of A is an abelian Galois extension of the bottom field of V. Definitions will be given later, but the best known example is the bottom field of a polarized Abelian variety (field of moduli), the central notion in the theory of complex multiplication.

The construction of $\Gamma\backslash\mathbf{H}^t$ is classical. Let k be a totally real algebraic number field B a quaternion algebra over k. Then we have an isomorphism

$$B \underset{\mathbf{Q}}{\otimes} \mathbf{R} \cong M_2(\mathbf{R}) \times \ldots \times M_2(\mathbf{R}) \times \mathbf{K} \times \ldots \times \mathbf{K}$$

where $M_2(\mathbf{R})$ is the total matrix algebra of degree two and $\mathbf{K}$ the algebra of real quaternions. Let $t > 0$ be the number of copies of $M_2(\mathbf{R})$ and $G = \mathrm{Res}_{k/\mathbf{Q}} \{a \in B | aa^i = 1\}$ where i is the canonical involution.

Any arithmetic subgroup Γ of G is a discontinuous group of transformations of the product of upper half planes $\mathbf{H}^t$. It defines a Hilbert modular variety $\Gamma\backslash\mathbf{H}^t$.

A family over $\Gamma\backslash\mathbf{H}^t$ is constructed from a symplectic representation. In particular we consider only rigid families, i.e., G is isomorphic to the Hodge group of the generic fiber of A.

The first Chapter is dedicated to the classification of the families induced by a given rigid symplectic representation ρ. We will prove

<u>Theorem</u> # $\begin{pmatrix}\text{isomorphism classes of families} \\ \text{defined by } \rho \text{ over } \Gamma\backslash\mathbf{H}^t\end{pmatrix} = h(\tilde{k})\cdot\nu$

where $\tilde{k}$ is the smallest Galois extension containing k and ν is a constant.

In the second Chapter we prove our main result via the study of the analytic structure of $A^\sigma \xrightarrow{f^\sigma} V^\sigma$, where σ is an automorphism of $\mathbf{C}$. This proof will proceed in several steps.

In Section 2.3, assuming $V^\sigma \cong V$ we prove that $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ can be defined by the same analytic data as $A \xrightarrow{f} V$ with the exception of the lattice defining the fibers. We will have, at that point, reduced the problem to the classification in Chapter 1 that establishes a correspondence between possible lattices and ideal classes of $\tilde{k}$. It is via the lattices that we construct the homomorphism

$$1 \to \mathrm{Gal}(K_A/K_V) \to \text{Ideal class group } (\tilde{k}).$$

In the third chapter we construct an example to show that $[K_A:K_V]$ can be different from 1. We apply the

Shimura-Taniyama [Sh-T] theory of complex multiplication to the fibers of CM type.

Notation and Conventions. All algebraic varieties are assumed to be connected and smooth. If a variety is defined over a field k, and K is an extension of k, we write X_K for the set of K-rational points of X. A vector space or algebra W over a field k determines an algebraic variety, again called W, such that $W_K = W \otimes_k K$ for any field K containing k. If $K > k$ and $[K:k] < \infty$ then $\mathrm{Res}_{K/k}$ denotes the restriction to the category of k-varieties.

Chapter 1.

1.1 Group Theoretic Families of Abelian Varieties

Kuga developed the general theory of families of Abelian varieties parameterized by a projective variety V, i.e. holomorphic fiber spaces, $A \xrightarrow{f} V$, whose fibers are Abelian varieties, and where A and V are projective varieties. Kuga's construction is purely analytic and makes use of Kodaira's Embedding Theorem to show the existence of an algebraic structure for A and V. We shall summarize this construction following Kuga [K] and Satake [S].

The starting point is an arithmetic variety V that we construct from a connected semisimple linear algebraic group G, defined over $\mathbf{Q}$ and with finite center. Let K be a maximal compact subgroup of G and Γ an arithmetic torsion free subgroup of G such that $\Gamma\backslash G$ is compact. If we also assume that $X = G_{\mathbf{R}}/K$ has a $G_{\mathbf{R}}$-invariant complex structure, then $V = \Gamma\backslash X$ with the induced complex structure is a compact complex manifold. V is known to be holomorphically equivalent to an algebraic submanifold of a complex projective space and is called an arithmetic variety.

It is over V that Kuga constructs families of Abelian varieties and uses a long list of ingredients to do so:

Let G, K, X, Γ and V be given as above. Let F be an even dimensional vector space over $\mathbf{Q}$ and $\beta: F\times F \to \mathbf{Q}$ a non degenerate bilinear form on F so that $Sp(F,\beta) = \{g \in GL(F_{\mathbf{R}}) \mid \beta(gx,gy) = \beta(x,y)\}$, the symplectic group is an algebraic group defined over $\mathbf{Q}$.

Let $\rho: G \to Sp(F,\beta)$ be an algebraic representation of G defined over $\mathbf{Q}$, which we will call a "symplectic representation" of G.

Let L be a lattice in F satisfying $\rho(\Gamma)L \subset L$ and $\beta(L,L) \subset L$.

A lattice satisfying the first condition exists since Γ is an arithmetic subgroup of G and the second condition is always satisfied replacing L by nL if necessary.

Via ρ we can define the semidirect product $G \times F_{\mathbf{R}}$ with the following multiplication law: $(g,w)\cdot(g',w') = (gg',\rho(g)w' + w)$.

Furthermore consider the product space $X \times F_{\mathbf{R}}$ on which the group $G \times F_{\mathbf{R}}$ acts:

$$\left.\begin{array}{l}(g,w) \in G\times F_{\mathbf{R}} \\ (x,u) \in X\times F_{\mathbf{R}}\end{array}\right\} \longrightarrow (g,w)(x,u) = (g(x),\rho(g)u + w)$$

The condition $\rho(\Gamma)L \subset L$ makes $\Gamma \times L$ into a discrete subgroup of $G \times L$ acting of $X \times F_{\mathbf{R}}$ freely and discontinuously. So finally we can define the fiber space $A \xrightarrow{f} V$, where $A = \Gamma \times L\backslash X\times F_{\mathbf{R}}$ and f is defined as follows:

$$\begin{array}{ccc} A & \xleftarrow[\nu]{\qquad} & X \times (F_{\mathbf{R}}/L) \\ \big\downarrow & & \big\downarrow \ proj_X \\ V = \Gamma\backslash X & \longleftarrow & X \end{array}$$

$A \xrightarrow{f} V$ is a smooth fiber bundle with fiber $F_{\mathbf{R}}/L$ and structure group $\rho(\Gamma)$. To make it into a holomorphic fiber space we need one more object.

We define the Eichler map τ associated to a symplectic representation ρ as a holomorphic map from X into the Siegel space,

$\tau : X \to X' = \mathfrak{S}(F_{\mathbf{R}},\beta) = \{J \in GL(F_{\mathbf{R}}) | J^2 = -1, \beta(x,Jy) >> 0$ and symmetric$\}$

which is weakly equivalent with respect to ρ, i.e. $\tau(g(x)) = \rho(g)(\tau(x))$. Note that neither the existence nor the uniqueness of τ is assured.

We identify $X' = \mathfrak{S}(F_{\mathbf{R}},\beta)$ with $SO(F,\beta)\backslash Sp(F,\beta)$ via the transitive action $Sp(F,\beta) \times X' \to X'$, $(g,J) \to gJg^{-1}$.

If $J_0 \in X'$ and $H_0' = 1/2J_0$ it is known (Satake [S], Chapter 2.§7), that adH_0' is a complex structure on $T_{J_0}(X')$ and there is a unique Hermitian complex structure on X' which induces the complex structure $J' = adH_0'$ on $T_{J_0}(X')$. We shall always consider X' as a complex manifold with this complex structure, and this structure is independent of J_0. Returning now to $\tau : X \to X'$ we recall that τ is ρ-equivariant. Therefore if we choose $0 \in X = G/K$ for every $x \in X$ we have $g(0) = x$ for some $g \in G$ and $\tau(x) = \tau(g(0)) = \rho(g)(\tau(0)) = \rho(g)\tau(0)\rho(g^{-1})$. Thus τ is completely determined by ρ and $\tau(0)$. We set $\tau(0) = J_0 \in X'$.

Let now J be the complex structure on $T_{\tau^{-1}(J_0)}(X) \cong \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of $G_{\mathbf{R}}$. Then $J = \mathrm{ad}H_0$, H_0 uniquely determined. We define the following conditions :

Condition (H1) $[d\rho H_0 - H_0', d\rho(Y)] = 0$, for all $Y \in$, $H_0' = 1/2J_0$

Condition (H2) $d\rho H_0 = H_0'$

Condition (H2) implies (H1) and (H1) is equivalent to τ being holomorphic (Satake [S], chapter 11.§8). τ makes $X \times (F_{\mathbf{R}}/L) \to X$ into a fiber space whose fibers are Abelian varieties with polarization β; in fact:

Lemma 1.1.1. $\tau(x)$ is a complex structure on $F_{\mathbf{R}}$ such that $(F_{\mathbf{R}}/L, \tau(x))$ is an Abelian variety with polarization β.

Moreover if $\gamma \in \Gamma$ we have:

Lemma 1.1.2. $(F_{\mathbf{R}}/L, \tau(x), \beta)$ is isomorphic to $(F_{\mathbf{R}}/L, \tau(\gamma x), \beta)$.

So also $A \xrightarrow{f} V$ is a fiber space whose fibers are Abelian varieties with a fixed polarization. We have:

Theorem 1.1.3. (Kuga [K] Theorem 11.6.3) Let $A \xrightarrow{f} V$ be the fiber space constructed above from the data: $(G, K, X, \Gamma, F, \beta, \rho, \tau)$. Then, if τ is holomorphic, or equivalently if the (H1) condition is satisfied, A has a unique structure J_A such that:

1. J_A restricted to the zero section coincides with the given structure on V (recall that $V = \Gamma\backslash X$ where X is a Hermitian symmetric space).
2. f: $A \xrightarrow{f} V$ is a holomorphic map.
3. J_A restricted to each fiber A_x coincides with $\tau(x)$.

We will call the holomorphic fiber space $A \xrightarrow{f} V$ a group theoretic family of Abelian varieties or a Kuga fiber variety, since using Kodaira's theorem we have the following.

Theorem 1.1.4. (Kuga [K] Theorem 11.6.8) Let $A \xrightarrow{f} V$ be a group theoretic family of Abelian varieties. If V is compact then A has a projective embedding.

Throughout this work we shall always consider a fixed projective realization of $A \xrightarrow{f} V$.

Remark: A natural problem arises from Theorem 1.1.3 and was posed by Kuga in the 1960's: Classify all of the representations of G, G a **Q**-simple algebraic group of hermitian type, into a symplectic group and investigate the existence of corresponding Eichler maps. Let $G_{\mathbf{R}}$ be the set of real points of G and K a maximal compact subgroup. G of Hermitian type means that $G_{\mathbf{R}}/K$ is a Hermitian symmetric space. So, given such a symplectic representation the existence of a corresponding Eichler map is the only obstruction

to the construction of Kuga fiber varieties over $\Gamma\backslash G/K$.
In the case when $G_{\mathbf{R}}$ has no compact factors the problem was solved by Satake. In the case when $G_{\mathbf{R}}$ has compact factors and the corresponding symmetric domains are of type II or III the solution is due to Addington ([Ad2],[Ad1]). We shall describe the solution of the quaternion case in the next Section.

1.2 Families arising from quaternion algebras.

In this paper we will investigate the arithmetic structure of families of Abelian varieties parameterized by a quaternion Hilbert modular variety.

Definition 1.2.1. A quaternion Hilbert modular variety V is the quotient space $\Gamma\backslash X$, where X is the symmetric space associated to $G = Res_{k/\mathbf{Q}}(SL_1(B))$, B a quaternion algebra over k, and Γ is an arithmetic subgroup of G.

Addington [Ad1] classified the representations of G into $Sp(F,\beta)$ which define families of Abelian varieties over V, equivalently for which there exist Γ, L, τ such that $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ satisfy the assumptions of Section 1.1. These symplectic representations of $G = Res_{k/\mathbf{Q}}(Sl_1(B))$ are describable by a combinatorial scheme called chemistry. Since they are our main tools we well give an explicit description, following Addington.

Let k be a totally real number field and B a division quaternion algebra over k, i.e. a central simple algebra of dimension 4 over k. We will denote by $\nu: B \to k$, the reduced norm of B. Assuming k of degree n over $\mathbf{Q}$, let $S = \{\phi_1, \ldots, \phi_n\}$ be the distinct embeddings of k into $\mathbf{R}$. We define $S_0 = \{\alpha \in S \mid B \otimes_{\alpha} \mathbf{R} \cong M_2(\mathbf{R})\}$ and $S_1 = S - S_0$.

If we now consider $SL_1(B) = \{x \in B \mid \nu(x) = 1\}$, it is an algebraic group defined over k and so $G = \mathrm{Res}_{k/\mathbf{Q}}(Sl_1(B))$ is a $\mathbf{Q}$-simple algebraic group defined over $\mathbf{Q}$ (Weil [W]).

Proposition 1.2.2. (a) $G_{\mathbf{R}} \cong \Pi_{\phi \in S} SL_1(B \otimes_{\mathbf{Q}} \mathbf{R}) \cong SL_2(\mathbf{R})^{|S_0|} \times \mathbf{K}^{1|S_1|}$

$$\cong SL_2(\mathbf{R})^{|S_0|} \times SU(2)^{|S_1|}$$

$$G_{\mathbf{C}} \cong SL_2(\mathbf{C})^{|S|}$$

(b) Identifying $G_{\mathbf{R}}$ and $SL_2(\mathbf{R})^{|S_0|} \times SU(2)^{|S_1|}$, a maximal compact subgroup is

$$K = SO(2)^{|S_0|} \times SU(2)^{|S_1|}.$$

So $G_{\mathbf{R}}$ is a semisimple Lie group and

$$G/K \cong (SL_2(\mathbf{R})/SO(2))^{|S_0|} \times \{p\}^{|S_1|} \cong \mathbf{H}^{|S_0|}$$

is a Hermitian symmetric space of non compact type as it is a product of Hermitian symmetric spaces.

Henceforth <u>G will always mean</u> $\mathrm{Res}_{k/\mathbf{Q}}(SL_1(B))$.

Put $\tilde{k} = \phi_1(k) \cdots \phi_n(k)$, then $\tilde{k}$ is a totally real Galois extension of $\mathbf{Q}$. $\Omega = \mathrm{Gal}(\tilde{k}/\mathbf{Q})$ acts transitively on S via $g(\alpha) =$

$g\circ\alpha$. The triple (Ω,S,S_0) is called chemistry. Elements of S are called atoms, subsets of S are called molecules and finite sums ΣM_j of molecules are called polymers.

Since G acts on S, G acts on the set of all molecules and polymers.

Definition 1.2.3. We say that a molecules is stable if $|M\cap S_0| \leq 1$ and rigid if $|M\cap S_0| = 1$. Analogously a polymer $P= \Sigma M_j$ is stable (resp. rigid) if P is G-invariant and each M_j is stable (resp. rigid).

For any polymer P we will construct a representation ρ_P of the algebraic group G.

We have seen in proposition 1.2.1 that

$$G = G_{\mathbf{C}} = \Pi_{\phi\varepsilon S}\, SL_1(B \otimes_{\phi} \mathbf{C}) = SL_2(\mathbf{C})^{|S|}$$

Let $proj_\phi$ be the projection map of G into its simple factor

$$G_\phi = SL_1(B \otimes_{\phi} \mathbf{C}) \cong SL_2(\mathbf{C}).$$

For every atom α let ρ_α be the map $proj_\alpha$ considered as a representation of G on the vector space $\mathbf{C}^2$.

For a molecule $M = \{\alpha_1,\ldots,\alpha_r\}$, set $\rho_M = \rho_{\alpha_1}\otimes\cdots\otimes\rho_{\alpha_r}$ For a polymer $P = \Sigma_i M_i$, set $\rho_P = \rho_{M^1} \oplus\ldots\oplus\rho_{M^t}$.

We are finally able to state:

Theorem 1.2.4. (Addington [Ad1]). Let G be the algebraic group $Res_{k/\mathbf{Q}}(SL_1(B))$.

1. Suppose that ρ is a symplectic representation of G defined over **Q** that defines a group theoretic family of Abelian varieties. Then there exists a stable polymer P such that ρ is equivalent to ρ_P over **C**.

2. Let P be a stable polymer. Then some multiple of ρ_P is an algebraic group representation of G defined over **Q** and defines a group theoretic family of Abelian varieties.

Proof. We will only describe in detail the representation and the construction for part 2. In fact it is these families that we will use throughout this paper.

Let P be an invariant polymer and M any molecule of P. The smallest orbit of M under the action of the Galois group is a subpolymer of P and we can write P as $P = \Sigma P_i$, P_i generated by any single molecule.

Definition 1.2.5. We call a minimal G-invariant polymer <u>prime</u> when it is generated by a single molecule.

For a prime polymer P there exists an integer $\mu \geq 1$ such that

$$\mu P = \sum_{g \in \Omega} gM$$

In general ρ_P is an algebraic group homomorphism. We shall prove that there exists and integer μ such that either $\rho_{\mu P}$ or $\rho_{\mu P} \oplus \rho_{\mu P}$ is an algebraic group representation defined over **Q**. Since $\rho_P = \Sigma \rho_{P_i}$, it is indeed enough to consider ρ_P when P is prime.

For any $\alpha \in \Omega = \mathrm{Gal}\,(\tilde{k}/\mathbf{Q})$ we define $B^\alpha = B \underset{\alpha}{\otimes} \tilde{k}$; Similarly, for any molecule $M = \{\alpha_1,\ldots,\alpha_r\}$, $B^M = B^{\alpha_1} \otimes\cdots\otimes B^{\alpha_r}$, and for any polymer $P = \Sigma M_i$, $B^P = B^{M_1} \oplus\cdots\oplus B^{M_t}$. B^α is a central simple algebra over $\tilde{k}$ for any α, as is B^M for any M.

Let F_M be a minimal left ideal in B^M . B^M acts on F_M by left multiplication so that we have a representation $\rho : B^M \to \mathrm{End}_{\tilde{k}}(F_M)$ and ρ is defined over $\tilde{k}$.

Lemma 1.2.6. Let P be a prime polymer, $P = \Sigma\, gM$. Them B^M is a central simple algebra over $\tilde{k}$ and $B^P = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(B^M)$.

B^P and B^M are isomorphic as algebras and $F_P = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(F_M)$ is a minimal left ideal in B^P.

Recalling that $\mathrm{Res}_{k/\mathbf{Q}}(B) \cong \Pi_{\sigma\in S} B^\sigma$ we call $\mathrm{proj}_\alpha : \mathrm{Res}_{k/\mathbf{Q}}(B) \to B^\alpha$ the natural projections.

For any molecule M we define $\mathbf{i}_M : \mathrm{Res}_{k/\mathbf{Q}}(B) \to B^M$ as $\mathbf{i}_M = \otimes_{\alpha\in M} \mathrm{proj}_\alpha$ and for any polymer $P = \Sigma M_i$, $\mathbf{i}_P = \oplus \mathbf{i}_{Mi}$.

Lemma 1.2.7. Assuming that P is as in Lemma 1.2.3, the map $\mathbf{i}_P : \mathrm{Res}_{k/\mathbf{Q}}(B) \to B^P$ is defined over $\mathbf{Q}$.

We are now prepared to define a representation of G. From the previous results we have that

$$\mathrm{Res}_{k/\mathbf{Q}}(B) \xrightarrow{\ \mathbf{i}_P\ } B^P \xrightarrow{\ \rho\ } \mathrm{End}_{\mathbf{Q}}(F_P)$$

is an algebra homomorphism defined over $\mathbf{Q}$, so restructuring to $G_{\mathbf{Q}}$:

$$G_{\mathbf{Q}} = \mathrm{Res}_{k/\mathbf{Q}}(SL_1(B)) \xrightarrow{i_P} B^P \xrightarrow{\rho} \mathrm{Aut}_{\mathbf{Q}}(F_P)$$

is an <u>algebraic group representation defined over $\mathbf{Q}$</u> that we will denote again by ρ. Extending ρ to $G_{\mathbf{C}}$ we get

$$\rho_{\mathbf{C}} : G_{\mathbf{C}} \to \mathrm{Aut}_{\mathbf{C}}(F_P \otimes \mathbf{C})$$

Returning to ρ_P we have:

Lemma 1.2.8. If B^M is a trivial simple algebra then $\rho = \rho_P$ over $\mathbf{C}$. Otherwise $\rho = \rho_{2P}$ over $\mathbf{C}$.

Proof. By definition if $M = \{\alpha_1, \ldots, \alpha_r\}$, $B^M = B^{\alpha_1} \otimes \cdots \otimes B^{\alpha_r}$. Recalling that a tensor product of division quaternion algebras is either trivial or equivalent to a division quaternion algebra, we can write:

$$B^M \cong M_N(B_0) \qquad B_0 = \begin{cases} \tilde{k} \text{ and } N = 2^r \\ \text{or} \\ \text{division quaternion algebra over } \tilde{k} \text{ and} \\ N = 2^{r-1} \end{cases}$$

It follows that $F_M = B_0{}^N$ viewed as column vectors. Now

$$\rho_{\mathbf{C}}: G_{\mathbf{C}} \to \otimes_{g\in\Omega} \otimes_{\alpha\in gM} M_2(\mathbf{C}) \subset M_{2^r m}(\mathbf{C})$$

if $m = |\Omega|$, $r = |M|$ and the representation space is $\mathbf{C}^{2^r m}$.

On the other hand $\rho : G_{\mathbf{C}} \to (\mathrm{Res}_{\tilde{k}/\mathbf{Q}}(B^M)) \otimes \mathbf{C}$ is the same map but the representation space may differ. In fact

$$\dim_{\mathbf{C}}(F_M \otimes \mathbf{C}) = \dim_{\mathbf{Q}} F_M = \begin{cases} 2^r m \text{ if } N = 2^r \\ \\ 2^{r+1} m \text{ if } N = 2^{r-1} \end{cases}$$

Then $\rho_{\mathbf{C}} = \rho_P$ if $B^M \cong M_N(\tilde{k})$ and $\rho_{\mathbf{C}} = 2\rho_P$ if $B^M \cong M_N(B_0)$. ∎

Now to complete the proof of part 2 (of 1.2.2) Addington considers $\Gamma \subset G_{\mathbf{Q}}$ an arithmetic subgroup of G and constructs a family of Abelian varieties over $V = \Gamma\backslash\mathbf{H}^{|S_0|}$ using

$\rho_{\mathbf{R}} : G_{\mathbf{R}} \to \mathrm{Aut}(F_P \otimes \mathbf{R})$.

For this purpose she defines β^P a non degenerate bilinear form on F_P such that $Sp(F_P, \beta^P)$ and τ^P a holomorphic Eichler map associated to ρ. Moreover she chooses L^P a $\rho(G)$-invariant lattice in F_P. The data $(G,K,X,\Gamma,F_P,L^P,\beta^P,\rho,\tau^P)$ satisfy the hypotheses of theorem 1.1.3 and give a Kuga fiber variety $A_P \xrightarrow{f} V$.

To extend the construction to any stable polymer $P = \Sigma P_i$ is immediate. In fact we can define the following data:

$(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ where $\rho = \rho_P$ or $\rho_P \oplus \rho_P$ whichever is indicated by lemma 1.2.5

$F = \oplus F_{Pi}$ or $F = \oplus(F_{Pi} \oplus F_{Pi})$

$L = \oplus L^{Pi}$ or $L = \oplus(L^{Pi} \oplus L^{Pi})$

$\beta = \oplus \beta^{Pi}$ or $\beta = \oplus(\beta^{Pi} \oplus \beta^{Pi})$

$\tau = \oplus \tau^{Pi}$ or $\tau = \oplus(\tau^{Pi} \oplus \tau^{Pi})$

The family defined by ρ will then be the fiber product over V of families:

$$\begin{array}{l} A = A_{P_1} \underset{V}{\times} \cdots \underset{V}{\times} A_{Pd} \\ \downarrow \\ V = \Gamma\backslash\mathbf{H}^{|S_0|} \end{array}$$

This completes the sketch of the proof of 1.2.4, part 2. ∎

Remark: A family of Abelian varieties, defined by a polymer representation ρ_P will not necessarily be the fiber product of families defined by the irreducible components of ρ_P. In general it is only isogeneous to such a product.

Definition 1.2.9 Let G be an algebraic group defined over a field F. A representation (V,ρ) of G defined over F is F-primary if for any F-irreducible invariant subspaces W and W' of V, $\rho|W' = (\rho|W)^\sigma$ for some $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$.

In this paper we will restrict ourselves to Kuga fiber varieties defined by **Q**-primary symplectic representations of G, or in other words by prime polymers $P = \Sigma gM$.

Proposition 1.2.10 (Satake [S]) Let $A \xrightarrow{f} V$ be a Kuga fiber variety defined by a symplectic representation $\rho\colon G \to Sp(F,\beta)$. Then if $F = \oplus_i F^{[i]}$ is the primary decomposition we have:

$$\rho = \oplus_i \ \rho^{[i]}, \quad \beta = \oplus_i \ \beta^{[i]}, \quad \tau = \oplus_i \ \tau^{[i]}$$

It follows that we can find an integer n such that:

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{id}\ \times nI\ } & A^{[1]} \times \cdots \times A^{[i]} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\ \mathrm{id}\ } & V \end{array}$$

where the map $nI\colon A_x \to A_x{}^{[1]} \times \ldots \times A_x{}^{[i]}$ is an isogeny.

1.3 Rigid Kuga varieties and Hodge Kuga varieties.

Let $A \xrightarrow{f} V$ be a Kuga fiber variety defined by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$.

Definition. We say that $A \xrightarrow{f} V$ is rigid if τ is uniquely determined by the rest of the data: G,K,X,Γ,F,L,β, and ρ.

Rigid families were introduced by Abdulali[Ab1] and we follow his beautiful exposition.

We recall that τ is completely defined by ρ and $\tau(0)$, where $0 \in X$ is an arbitrary point. In fact $\tau : X \to \mathfrak{S}(F_{\mathbf{R}},\beta)$ is equivariant with respect to ρ, so that we have $\tau(g(0)) = \rho(g)\tau(0)\rho(g)^{-1}$ for all $g \in G_{\mathbf{R}}$.

Definition 1.3.1. Let X_ρ be the set of possible $\tau(0)$'s i.e. the set of $J_0 \in \mathfrak{S}(F_{\mathbf{R}},\beta)$ such that the map $g(0) \to \rho(g)J_0\rho(g)^{-1}$ is well defined and satisfies the (H1)-condition (see 1.1).

Lemma 1.3.2. $A \xrightarrow{f} V$ is rigid if and only if X_ρ reduces to a point.

Theorem 1.3.3. (Satake [S], chapter 4, prop. 4.1). X_ρ is a complex submanifold of $\mathfrak{S}(F_{\mathbf{R}},\beta)$. Furthermore, G_ρ the Zariski connected component of the centralizer of $\rho(G)$ in $Sp(F,\beta)$ is a reductive subgroup which acts transitively on X_ρ.

Theorem 1.3.4. (Abdulali [Ab1], proposition 1.2.2). If the (H2)-condition is satisfied then the fiber variety $A \xrightarrow{f} V$ is rigid.

We return to families arising from quaternion algebras. Recall that for the chemistry (S, S_0, Ω) a polymer $P = \Sigma M_i$ is rigid if $|M_i \cap S_0| = 1$ for every i.

Theorem 1.3.5. (Abdulali [Ab1], theorem 1.3.7) Let P be a rigid stable polymer and suppose that μ is an integer such that $\rho_{\mu P}$ is defined over **Q**. Then any family of Abelian varieties induced by $\rho_{\mu P}$ is rigid and in particular satisfies the (H2)-condition.

Proof. The idea is to construct an Eichler map τ associated to $\rho_{\mu P}$ and show that τ satisfies the (H2)-condition and is therefore unique by theorem 1.3.3. We shall define τ after recalling some notation from section 1.2.

Let B be a division algebra over a totally real number field k. Define $G = \mathrm{Res}_{k/\mathbf{Q}}(SL_1(B))$ and identify $G_{\mathbf{R}}$ with $SL_1(\mathbf{R})^{|S_0|} \times SU(2)^{|S_1|}$, $K_{\mathbf{R}}$ maximal compact subgroup with $SO(2)^{|S_0|} \times SU(2)^{|S_1|}$,

and $X = G_{\mathbf{R}}/K_{\mathbf{R}}$ with $\mathbf{H}^{|S_0|}$. The element $H_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}^{|S_0|} \times 0^{|S_1|}$

of the Lie algebra of $G_{\mathbf{R}}$ determines a complex structure on X.

Set $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{|S_0|} \times I^{|S_1|}$ and $J_0 = \rho_{\mu P}(j)$, $\rho_{\mu P}\colon G_{\mathbf{R}} \to Sp(F_{\mathbf{R}}, \beta)$

By definition $\rho_M = \rho_{\alpha_1} \otimes \cdots \otimes \rho_{\alpha_r}$, $M = \{\alpha_1, \ldots, \alpha_r\}$. Therefore if $M \in P$, $|M \cap S_0| = 1$ and $\rho_M(j) = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ with respect to a suitable basis, so $J_0 = \rho_{\mu P}(j) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to a suitable basis of $F_{\mathbf{C}}$ and it is a complex structure on $F_{\mathbf{R}}$.

We are ready to define a map $\tau : X \to \mathcal{S}(F_{\mathbf{R}}, \beta)$ equivariant with respect to ρ, as $\tau(g(0)) = \rho(g) J_0 \rho(g)^{-1}$ where $0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in X = \mathbf{H}^{|S_0|}$. Abdulali shows that τ satisfies the (H2)-condition, $d\rho(H_0) = 1/2 J_0$. ∎

Let X be an Abelian variety given by (F, J, L, β):

- F an even dimensional vector space,
- J complex structure on F,
- $L \subset F$ a $\mathbf{Z}$-lattice,
- β a non degenerate bilinear form on F, such that
  - i. $\beta(L, L) \subset \mathbf{Z}$
  - ii. $\beta(u, Jv)$ is symmetric and positive definite.

Consider ϕ: $T = \{z \in \mathbf{C} \mid |z| = 1\} \to GL(F)$ with $\phi(\Theta) = \cos\Theta I + \sin\Theta J$.

Definition 1.3.7. The <u>Hodge group of X</u>, $Hg(X)$, is the smallest algebraic subgroup of $GL(V)$ defined over $\mathbf{Q}$ and containing $\phi(T)$.

Definition 1.3.8. A group theoretic family of Abelian varieties $A \xrightarrow{f} V$ defined by a symplectic representation ρ is of <u>Hodge type</u> if $\rho(G) = Hg(A_x)$ where A_x is a generic fiber.

Theorem 1.3.8. (Addington [Ad2]). Let P be a polymer. Then the families defined by $\rho_{\mu P}$ are of Hodge type if and only if P is rigid.

Definition 1.3.9. We say that an Abelian variety X has <u>complex multiplication</u> (is of <u>CM type</u>) if $Hom_{\mathbf{Q}}(X) = Hom(X,X)$ contains a commutative semisimple **Q**-algebra R such that $[R:\mathbf{Q}] = 2 \dim X$.

Theorem 1.3.10. (Mumford [M2]). a) Every family of Abelian varieties of Hodge type contains Abelian varieties of CM type.
b) If a family contains an Abelian variety of CM type then it is isomorphic to a family of Hodge type.

Corollary 1.3.11. (Addington [Ad1]). A family defined by a polymer representation contains a fiber with complex multiplication if and only if the polymer is rigid.

We will give an explicit description of the endomorphism ring of a CM fiber in the next section.

1.4. Endomorphism Ring of the fibers.

Let $A \xrightarrow{f} V$ be a rigid Kuga fiber variety given by the data: $(G,K,\Gamma,F,L,\beta,\rho,\tau)$, where as in 1.2.: k is an algebraic number field, $\tilde{k}$ the smallest Galois extension containing k, $\Omega = Gal(\tilde{k}/\mathbf{Q})$, B a quaternion algebra over k, $B \neq M_2(k)$,

$G = Res_{k/\mathbf{Q}}(SL_1(B))$

$\rho = \rho_{\mu P}$ symplectic representation defined over $\mathbf{Q}$ associated to a rigid polymer P.

In this section we compute the endomorphism ring of the generic fiber of $A \xrightarrow{f} V$ and the endomorphism ring of the fibers of CM type.

Let $\rho = \oplus_i \rho^{[i]}$ be the primary decomposition. Any $\rho^{[i]}$ determines naturally a family of Abelian varieties $A^{[i]} \to V$ and we know that $A \xrightarrow{f} V$ is isogeneous to $A^{[1]}\times..\times A^{[n]} \to V$ (see proposition 1.2.6). So since we are concerned with the endomorphism ring we may consider ρ to be $\mathbf{Q}$-primary.

Let ρ be a $\mathbf{Q}$-primary representation ; $\rho = \rho_P$, with $P = \Sigma gM$ and we will also assume $gM \neq \gamma M$ if $g \neq \gamma$.

Remembering the definition: $\rho_P = Res_{\tilde{k}/\mathbf{Q}}\rho_M$

with $\rho_M : Res_{k/\mathbf{Q}}(SL_1(B)) \to B^M \subset End_{\tilde{k}}(F^M)$ the regular representation.

Let A_x be the generic fiber of A, x in V, isomorphic to $(F,L,J_x = \tau(x))$. Any endomorphism of A_x has a natural rational representation:

$\mathrm{Hom}\ (A_X) \cong \{g \in \mathrm{End}_{\mathbf{Q}}(F) \mid gL = L\ ,\ gJ_X = J_X g\}$

$\mathrm{Hom}_{\mathbf{Q}}\ (A_X) \cong \{g \in \mathrm{End}_{\mathbf{Q}}(F) \mid gJ_X = J_X g\}$

Since $Hg(A_X)$, the Hodge group of A_X, is generated by $\cos\theta I + \sin\theta J_X$ and their conjugates over **Q**, we have:

$\mathrm{Hom}_{\mathbf{Q}}\ (A_X) \cong \{g \in \mathrm{End}_{\mathbf{Q}}(F) \mid gg' = g'g \text{ for any } g' \in Hg\ (A_X)\ \}.$

But $A \xrightarrow{f} V$ is a rigid family so that $Hg(A_X) = \rho(G)$ and

$\mathrm{Hom}_{\mathbf{Q}}\ (A_X) \cong \{g \in \mathrm{End}_{\mathbf{Q}}(F) \mid gg' = g'g \text{ for any } g' \in \rho(G)\ \}.$

Proposition 1.4.1. Let $A \xrightarrow{f} V$ be a Kuga fiber variety of quaternion type defined by a symplectic representation ρ. Let A_X be a generic fiber, $\tilde{k}$ the smallest Galois extension of **Q** containing k. If ρ is a polymer **Q**-primary representation , $\rho = \rho_P$, $P = \Sigma gM$, and $gM \neq \gamma M$ for $g \neq \gamma$, then

$\mathrm{End}_{\mathbf{Q}}\ (A_X) = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(B_0)$

where $B_0 = \tilde{k}$ if ρ is **C**-irreducible and B_0 is a division quaternion algebra over $\tilde{k}$ otherwise.

Proof. We assumed $P = \sum_{g\in\Omega} gM$, $\Omega = \mathrm{Gal}(\tilde{k}/\mathbf{Q})$, $M = \{\alpha_1, \ldots, \alpha_r\} \subset \Omega$

Therefore

$$\begin{array}{ccc} \mathrm{Res}_{k/\mathbf{Q}}(B) & \longrightarrow & B^M \subset \mathrm{End}_{\tilde{k}}(F_M) \\ \uparrow & & \downarrow \\ G = \mathrm{Res}_{k/\mathbf{Q}}SL_1(B) & \xrightarrow[\rho_P\ =\ \mathrm{Res}_{\tilde{k}/\mathbf{Q}}\rho_M]{} & \mathrm{Res}_{\tilde{k}/\mathbf{Q}}B^M \subset \mathrm{End}_{\mathbf{Q}}(F) \end{array}$$

F_M is a minimal left ideal and $F = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}F_M$ is a minimal left ideal in $\mathrm{Res}_{\tilde{k}/\mathbf{Q}}B^M$.

As we have seen in the proof of Lemma 1.3. we have: $B^M \cong M_N(B_0)$,

$F_M \cong B_0{}^N$, B_0 a simple division algebra over $\tilde{k}$ (B_0 is either $\tilde{k}$ or a division quaternion algebra). Under these identifications:

$$\{g \varepsilon \operatorname{End}_{\mathbf{Q}} F \mid gg' = g'g \quad g' \varepsilon \rho(G)\}$$

$$= \operatorname{Res}_{\tilde{k}/\mathbf{Q}}\{g \varepsilon \operatorname{End}_{\tilde{k}} F_M \mid gg' = g'g, g' \varepsilon \rho_M(SL_1(B))\}.$$

In fact for $M = \{\alpha_1, \ldots, \alpha_r\}$,

$$\rho_p = \underset{\Omega}{\otimes} \rho_{gM} = \underset{\Omega}{\otimes} \rho^{g\alpha_1} \otimes \ldots \otimes \rho^{g\alpha_r} \text{ and } F = \underset{\Omega}{\otimes} F_{gM}$$

by construction. Since $gM \neq \gamma M$ if $g \neq \gamma$, we have that $id_{gM} \varepsilon \rho(G)$ for any g.

Therefore $\{g \varepsilon \operatorname{End}_{\mathbf{Q}} F \mid gg' = g'g \text{ , } g' \varepsilon \rho(G)\}$

$$\subset \{g \mid g(F_{\alpha M}) = F_{\alpha M} \text{ for any } \alpha\} = \operatorname{Res}_{\tilde{k}/\mathbf{Q}} \operatorname{End}_{\tilde{k}} F_M.$$

We must look closely at $\operatorname{End}_{\tilde{k}} F_M$ and $\rho_M(SL_1(B))$.

We distinguish two cases.

(i) $B^M \cong M_N(\tilde{k})$, $F_M \cong \tilde{k}^N$; then $\operatorname{End}_{\tilde{k}} F_M = B^M$.

(ii) $B^M \cong M_N(B_0)$, $F_M \cong B_0{}^N$, B_0 division quaternion algebra over $\tilde{k}$; then $\operatorname{End}_{\tilde{k}} F_M = B^M \underset{\tilde{k}}{\otimes} B_0$.

In fact $\operatorname{End}_{\tilde{k}}(F_M)$ contains $B^M \otimes B_0$ where B_0 is identified with $B_0 I$ in the dual $(B^M)^*$ and acts on F_M by right multiplication.

Moreover $\dim_{\tilde{k}}(B^M \otimes B_0) = N^2 4^2 = \dim_{\tilde{k}} F_M$.

So we can write $\operatorname{End}_{\tilde{k}}(F_M) \cong B^M \otimes B_0$ with the convention that B_0 is either $\tilde{k}$ or a division quaternion algebra, depending on the class of B^M in the Braner group.

Finally we can compute $\operatorname{End}(A_x) \otimes \mathbf{Q}$.

i) $B^M \cong M_N(\tilde{k})$, $F_M \cong \tilde{k}^N$.

Then $\rho_M : Res_{k/\mathbf{Q}}(SL_1(B)) \to B^M = End_{\tilde{k}}(F_M)$ is an irreducible representation over $\tilde{k}$ as well as over $\mathbf{C}$, since $B_M \otimes \mathbf{C} \cong M_N(\mathbf{C}) = End_{\mathbf{C}}(F_M \otimes \mathbf{C})$.

So by Schur's lemma, $\tilde{k} = \{g \in End_{\tilde{k}}(F_M) | gg' = g'g \quad g' \in \rho(SL_1(B))\}$.

ii) $B^M \cong M_N(B_0)$, $F_M \cong B_0{}^N$.

Then $\rho_M : Res_{k/\mathbf{Q}}(SL_1(B)) \to B^M \subset End_{\tilde{k}}(F_M)$ is irreducible over $\tilde{k}$ but reducible over $\mathbf{C}$. In fact $B^M \otimes \mathbf{C} \cong M_{2N}(\mathbf{C})$, $F_M \otimes \mathbf{C} \cong \mathbf{C}^{4N}$. So by Schur's lemma we can only conclude that

$\{g \in End_{\tilde{k}}(F_M) | gg' = g'g \quad g' \in (SL_1(B))\}$ is a division algebra.

We need to define a new representation

$$\tilde{\rho} = \rho_M \otimes id : Res_{k/\mathbf{Q}}(SL_1(B)) \otimes SL_1(B_0) \to B^M \underset{\tilde{k}}{\otimes} B_0 = End_{\tilde{k}}(F_M).$$

$\tilde{\rho}$ is an irreducible representation over $\tilde{k}$ as a tensor product of irreducible representations and is moreover irreducible over $\mathbf{C}$, since $(B_M \underset{\tilde{k}}{\otimes} B_0) \otimes \mathbf{C} = End_{\mathbf{C}}(F_M \otimes \mathbf{C})$. We can now conclude as in (i), that $\tilde{k} = \{g \in End_{\tilde{k}}(F_M) | gg' = g'g \quad g' \in \rho(SL_1(B)) \otimes SL_1(B_0))\}$.

Now let $H = \{g \in End_{\tilde{k}}(F_M) = B^M \underset{\tilde{k}}{\otimes} B_0 | gg' = g'g \quad g' \in \rho_M(G)\}$.

$H \supset 1 \otimes B_0$ and any h in H can be written as $h = \Sigma h_i \otimes b_i$, where $B_0 = \langle b_1, \ldots, b_4 \rangle_{\tilde{k}}$, $h_i \in B^M$.

Let $g \otimes 1$ be an element of $\rho(SL_1(B)) \subset B_M \subset End_{\tilde{k}}(F)$.

We have: $h \cdot g \otimes 1 = g \otimes 1 \cdot h$

and $\Sigma\, h_i g \otimes b_i = \Sigma g h_i \otimes b_i$

if and only if $h_i g \otimes b_i = g h_i \otimes b_i$ for any i, i.e.

$(h_i \otimes 1)(g \otimes b_i) = (g \otimes b_i)(h_i \otimes 1)$ for any i and therefore

$h_i \otimes 1 \in \text{centralizer}_{\text{End}_{\tilde{k}}F_M}(\rho(SL_1(B)) \otimes SL_1(B_0)) = \tilde{k}.$

We have shown that $H = 1 \otimes B_0$. ■

Remark 1. $\text{End}(A_x) = O$, an order in $\text{Res}_{\tilde{k}/\mathbf{Q}} B_0$

If O is a maximal order, then

$$\text{End}(A_x) = \text{Res}_{\tilde{k}/\mathbf{Q}} O' \text{ , an } O_{\tilde{k}} \text{ order in } B_0.$$

Remark 2. If $A \xrightarrow{f} V$ is a Kuga fiber variety of quaternion type induced by a $\mathbf{Q}$-primary symplectic representation then the generic fiber is a simple Abelian variety.

The study of the fibers of CM type is far more complicated than the study of the generic fiber.

Let $A \xrightarrow{f} V = \Gamma\backslash X$ be a rigid Kuga variety of quaternion type induced by a polymer representation irreducible over $\mathbf{Q}$.

We have seen in 1.2 that $X = \mathbf{H}^t$. We shall construct an elliptic point λ in V, i.e. $\lambda = (\lambda_1,\ldots,\lambda_t)$ with λ_i elliptic for every i, such that $A_\lambda = f^{-1}(\lambda)$ is of CM type.

We need to consider the quaternion algebra B. We defined in Section 1.2.:

$S = \{\phi_1, \cdots, \phi_n\}$ the set of real embeddings of k and

$S_0 = \{\phi \in S \mid B \otimes_\phi \mathbf{R} \cong M_2(\mathbf{R})\}$, $|S_0| = t$.

For every $\phi \in S_0$ we consider the natural immersion i_ϕ:

$B \to B^\phi_{\mathbf{R}} = B \otimes_\phi \mathbf{R} \cong M_2(\mathbf{R})$.

Recalling that $\det \circ i_\phi = \nu$, where ν is the reduced norm, we have:

$$\{\alpha \in B \mid \nu(\alpha)^\phi > 0,\ \phi \in S_0\} \overset{i_{S_0}}{\rightarrow} GL_2^+(\mathbf{R})^t$$

Proposition 1.4.2. [Sh4, Proposition 9.4] Let L be a totally imaginary quadratic extension of k and q a k-linear isomorphism of L into B. Then $q(L^\times)$ is contained in $\{\alpha \in B|\nu(\alpha)^\phi>0,\ \phi \in S_0\}$ and every element of $q(L^\times)$ not contained in k has a unique fixed point λ on $\mathbf{H}^t$, which is common to all such elements of $q(L^\times)$. Moreover, $q(L^\times) = \{\gamma \in B|\nu(\gamma)^\phi>0,\ \phi \in S_0,\ \gamma(\lambda) = \lambda\}$.

Conversely if an element $\alpha \in B$, $\nu(\alpha)^\phi>0$ for any element in S_0, has a fixed point in $\mathbf{H}^t$ then $k(\alpha)$ is isomorphic to a totally imaginary number field.

Returning to the base space of our family:

$$V = \Gamma\backslash X \quad , \quad X = G_{\mathbf{R}}/K_{\mathbf{R}}$$

We recall the identification of X with $\mathbf{H}^t$:

$$G = \mathrm{Res}_{k/\mathbf{Q}}(SL_1(B)) \cong \prod_S SL_1(B^\phi)$$

$$G_{\mathbf{R}} \cong \prod_S SL_1(B^\phi) \otimes \mathbf{R} = SL_2(\mathbf{R})^{|S_0|} \times SU(2)^{|S_1|}$$

$$K_{\mathbf{R}} \cong SO(2)^{|S_0|} \times SU(2)^{|S_1|}$$

$$X = G_{\mathbf{R}}/K_{\mathbf{R}} \cong SL_2(\mathbf{R})^{|S_0|}/SO(2)^{|S_0|} \cong \mathbf{H}^t$$

So we can consider λ the fixed point of $q(L^{\times})$ as a point in X and therefore in V. We shall show that under suitable hypotheses A_λ is of CM type.

We need to construct a field R contained in $\mathrm{End}_{\mathbf{Q}}(A_\lambda)$ and $[R:\mathbf{Q}] = 2 \dim A_\lambda$.

As in the case of the generic fiber A_λ is isomorphic to (F,L,J_λ) and the rational representation for the endomorphism ring gives:

$$\mathrm{End}_{\mathbf{Q}}(A_\lambda) = \{g' \varepsilon \mathrm{End}_{\mathbf{Q}}F,\ g'J_\lambda = J_\lambda g'\},$$

where $g(\sqrt{-1}\times\cdots\times\sqrt{-1}) = \lambda$, $J_\lambda = \tau(\lambda) = \rho(g)\ J_0\ \rho(g)^{-1}$.

From the previous considerations λ is a fixed point in $\mathbf{H}^t$ for the group $\prod_S (L^{\times})^{\phi}$, which group we will denote by $(L^{\times})^S$. Let α be in $(L^{\times})^S$, $\alpha = (\alpha^{\phi_1}, \cdots, \alpha^{\phi_n})$. Since $\nu(\alpha)^{\phi} > 0$ for any ϕ we can define $\bar{\alpha} = \left(\left(\frac{\alpha^{\phi_1}}{\sqrt{\nu(\alpha)}^{\phi_1}}\right), \cdots, \left(\frac{\alpha^{\phi_n}}{\sqrt{\nu(\alpha)}^{\phi_n}}\right)\right)$ an element of G, so that $\bar{\alpha}$ belongs to the isotropy group of λ, K_λ. ρ and τ form an equivariant pair so it follows that $\rho(\bar{\alpha}) \varepsilon \mathrm{End}_{\mathbf{Q}}(A_\lambda)$. Moreover, $\rho = \oplus\ \rho_{gM}$ and $J_\lambda = \oplus\ J_{gM}$ by construction and so also $\rho(\alpha)$ belongs to $\mathrm{End}_{\mathbf{Q}}(A_\lambda)$ and $\rho((L^{\times})^S) \subset \mathrm{End}_{\mathbf{Q}}(A_\lambda)$. If we put $L^{\phi} = L \otimes_{\phi} \tilde{k}$ and $L^M = L^{\alpha_1} \otimes \cdots \otimes L^{\alpha_r}$ then we can state the previous result as

$$\mathrm{Res}_{\tilde{k}/\mathbf{Q}} L^M \subset \mathrm{End}_{\mathbf{Q}}(A_\lambda).$$

On the other hand J_0 and $J_\lambda = \rho(g)\ J_0 \rho(g)^{-1}$ are in $\rho(G_{\mathbf{R}})$ so that J_λ commutes with $\mathrm{Res}_{\tilde{k}/\mathbf{Q}}(1 \otimes B_0)$.

Finally: $\mathrm{End}_{\mathbf{Q}}(A_x) \supset \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(L^M \otimes B_0)$.

We shall look closely at L^M and show that:

Proposition 1.4.2. There exist L in B and K in B_0 such that:

$L^{\phi_1} \ldots L^{\phi_d} \circ K \cong L^M \otimes B_0$

so that: A_{λ_L} is of CM type

We will break the proof into several steps:

Lemma 1. Let B a quaternion algebra over k, k totally real algebraic number field. There exists $L \subset B$, L totally imaginary quadratic extension of k such that: $L^{\phi_1}, \ldots, L^{\phi_n}$ are linearly disjoint where $L^{\phi_i} = L \otimes_{\phi_i} k$ and $S = \{\phi_1, \ldots, \phi_n\}$ is the set of real embeddings of k.

Lemma 1a. Let k be an algebraic number field, and $\alpha_1, \ldots, \alpha_n$, γ elements of k. If $\sqrt{\gamma} \in k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$ then

$$\gamma \cdot \alpha_1^{\varepsilon_1} \cdots \alpha_n^{\varepsilon_n} \in k^2 \quad \text{where } \varepsilon_i = \{\begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix}$$

Proof. $k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$ is a Galois extension of k of degree 2^n,

$\text{Gal}\,(k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})/k) = Z_2{}^n$

We define $\alpha_I = \alpha_1^{\varepsilon_1} \cdots \alpha_n^{\varepsilon_n}$, ε_i either 0 or 1, so that we can write $k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) = \bigoplus_I k \sqrt{\alpha_I}$.

$\{\alpha_I\}_I$ are the 2^n eigenvectors of the Galois group.

If $\sqrt{\gamma} \in k(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$ then $\sqrt{\gamma}$ is an eigenvector of the Galois group and so

$\sqrt{\gamma} = a\sqrt{\alpha_I}$ for some $a \in k$ and I. Therefore $\gamma\alpha_I \in k^2$. ∎

Lemma 1.b. ([v], Ch.3, lemma 3.8)

Suppose B and k are as in lemma 1. L a quadratic extension of k. L is contained in B if and only if L_v is a quadratic extension of k_v for every v place of k such that $B_v = B \otimes k_v$ is a division algebra.

Lemma 1.c. ([v]), Ch.3, lemma 3.6) 1

There exists a quadratic extension L of k such that $L_v = L \otimes k_v$ is a given quadratic extension of k_v for a finite set of places of k.

We are finally able to prove lemma 1.

Proof. Let $d(B) = \prod_I p_i$ be the discriminant of B. Let p be a prime ideal of k, $p \neq p_i^{\phi}$ for any i in I and ϕ in S and char $(O_k/p) \neq 2$. Set L_{p_i} an unramified quadratic extension of k_{p_i} for any $i \in I$.

L_p a ramified quadratic extension of k_p

$L_{p^\phi} = k_{p^\phi}$ if $\phi \neq id$

$L_q = \mathbf{C}$ for any q infinite prime of k

By lemma 1.c. there exists an L, imaginary quadratic extension of k, $L = k(\sqrt{-d})$ that coincides locally with the given extensions. By lemma 1.b. L is contained in B. From our construction

$d(L) = p \cdot b$, b ideal of k, $(b, p^{\phi}) = 1$ for any ϕ in S, so $(d) = d(L) \cdot a^2$, a an ideal of k.

Now $L^{\phi} = k(\sqrt{-d^{\phi}})$ so $\prod_S (d)^{\phi} = \prod p^{\phi} \prod b^{\phi} \prod (a^2)^{\phi}$ with $(p^{\phi}, b^{\psi}) = 1$ for any ϕ, ψ in S.

The lemma is then proved applying lemma 1.a. ∎

Moreover we can show:

Lemma 2. Let B, B_0 be two quaternion algebras over k. Let $L \subset B$ be as in lemma 1. There exists $K \subset B_0$ such that $L^{\phi_1}, \cdots, L^{\phi_n}$, K are linearly disjoint.

Proof. In fact if we assume $L = k(\sqrt{-\alpha})$ we can choose, in the above notation, a prime ideal q of k such that $((\alpha), q) = 1$, $q \neq p^{\phi}$ for any ϕ in S and char $(O_k/q) \neq 2$. Then we can construct as in lemma 1, $K \subset B_0$ an imaginary quadratic extension of k such that $d(K) = qb'$, $(b', q^{\phi}) = 1$ for any ϕ in S. $K = k(\sqrt{-\beta})$, $(\beta) = d(K)(a')^2$ and

$$\beta\, \alpha^{\phi_1} \ldots \alpha^{\phi_n} = p^{\phi_1} \ldots p^{\phi_n} q\, \tilde{a}\ , \quad (\tilde{a}, p^{\phi}) = 1,\ (\tilde{a}, q) = 1$$

So $\beta\, \alpha^{\phi_1} \ldots \alpha^{\phi_n}$ is not in k^2 and again by lemma 1.a. the lemma follows ∎

Proof of Proposition 1.4.2.

It is now easy to extend L and K to $B \otimes_{id} \tilde{k}$ and $B_0 \otimes_{id} \tilde{k}$ respectively, so that they will satisfy proposition 1.4.2.

For L we just need to choose p, see proof of Lemma 1 such that $p \neq p_i^{\phi}$ for any $p_i | d(B)$, $p_i | d(B \otimes \tilde{k})$ and ϕ in S. Proceed analogously for K. We obtain then $L^{\phi_1}, \ldots, L^{\phi_n}$, K linearly disjoint where $L^{\phi} = L \underset{p}{\otimes} \tilde{k}$ so that $L^{\phi_1} \ldots L^{\phi_n} K \cong L^M \otimes K$ ∎

1.5 Lattices in F

Let $\mathfrak{A} = M_n(k)$ be a trivial central simple algebra over k. k an algebraic number field. Let F be a minimal left ideal in $\mathfrak{A}$, and O an order in $\mathfrak{A}$. In this section we clarify the O-invariant lattices of F. The main result will be:

Theorem. The number of isomorphism classes of O-invariant lattices in F is up to a constant, equal to the class number of k.

All the lattices are assumed to be O_k-lattices, O_k the ring of integers of k.

The main reference for this chapter is I. Reiner [R]

F is a minimal ideal of and since is trivial we have $\mathfrak{A} = \text{End}_k F$. Let L be a lattice in $\mathfrak{A}$, we define:

$$O_r(L) = \{x \in \mathfrak{A} \mid Lx \subset L\}$$

$$O_\ell(L) = \{x \in \mathfrak{A} \mid xL \subset L\}$$

$O_r(L)$ and $O_\ell(L)$ are orders in $\mathfrak{A}$ and called right order of L and left order of L respectively.

Proposition 1.5.1. There is a natural correspondence between lattices in F and lattices in $\mathfrak{A}$ with right order $M_n(O_k)$.

Proof. Let L be a lattice in F. $\mathfrak{A}$ is isomorphic to the direct sum of n copies of F so we can define the lattice $M = L \oplus \ldots \oplus L$, n copies of L.

We have $O_r(M) = M_n(O_k)$.

Viceversa given M a lattice in $\mathfrak{A}$ such that $O_r(M) = M_n(O_k)$ then $M = L \oplus \ldots \oplus L$. ∎

Lemma 1.5.2. Let L be lattice in F.

Then $O_\ell(L) = \{x \in \mathfrak{A} \mid xL \subset L\}$ is a maximal order of $\mathfrak{A}$.

Proof. Let $M = L \oplus \ldots \oplus L$ be the associated lattice to L in $\mathfrak{A}$. We have $O_r(M) = M_n(O_k)$ a maximal order and therefore $O_\ell(M)$ is a maximal order. But $O_\ell(L) = O_\ell(M)$ so $O_\ell(L)$ is maximal. ∎

Let L_1 and L_2 be O-invariant lattices in F, O any order of $\mathfrak{A}$. Put $\underline{a} = \{x \in \mathfrak{A}$ such that $xL_1 \subset L_2\}$. $\underline{a}$ is a two sided O-ideal

Proposition 1.5.3. $L_2 = \underline{a}L_1$

Proof. $\mathfrak{A}$ coincide with $\mathrm{End}_k F$, therefore there is ψ in $\mathfrak{A}$ such that $L_2 = \psi L_1$. ∎

Corollary 1.5.4. The action of the set of two sided O-ideals on the O-invariant lattices of F is transitive.

Proposition 1.5.5. Let L_1, L_2 be as above. Then L_1 is isomorphic to L_2 as O-modules if and only if $L_1 = \alpha L_2$ where α is in $k^\times$.

Proof. Suppose $L_1 = \phi L_2$, ϕ isomorphism of O-modules. Then $\phi x = x\phi$ for any x in O, but O is an order so ϕ belongs to center of $\mathfrak{A}$, i.e. $k^\times$. ■

Corollary 1.5.6. Let L_1, L_2 be as above. If L_1 is isomorphic to L_2 as O-module, then: $O_\ell(L_1) = O_\ell(L_2)$.

By lemma 1.5.2 we know that if L is a lattice in F then $O_\ell(L)$ is a maximal order, this allows us to classify the O-invariant lattices. Put $C(F,O) = \{L \text{ lattice in } F\}/\cong_O$,

where $L_1 \underset{O}{\cong} L_2$ means that L_1 and L_2 are isomorphic as O-modules.

Theorem 1.5.7. $C(F,O) = \bigcup_i C(F,O_i)$,

O_i are the maximal orders of $\mathfrak{A}$ containing O.

Proof. Let L_1, L_2 be two O-invariant lattices in the same isomorphism class. Then $L_1 = \alpha L_2$, $\alpha \in k^\times$, and $O_\ell(L_1) = O_\ell(L_2)$ is a maximal order by lemma 1.5.2. , containing O. The theorem is proved. ∎

To estimate the number of isomorphism classes of O-invariant lattices in F we need:

Lemma 1.5.6. The number of maximal orders of $\mathfrak{A}$ containing a given order O is finite.

Proof. $\mathfrak{A}$ is a trivial simple algebra then: $\mathfrak{A}^1 = \{a \in \quad | \ \nu(a) = 1\}$ is a k simple group. O^1 is an arithmetic subgroup of A^1. Let $\tilde{O}$ be a maximal order in $\mathfrak{A}$ containing O. Then $\tilde{O}^1$ is a subgroup of $\mathfrak{A}^1$ containing O^1 as a subgroup of finite index. By Borel's result [B] there ar only finitely many subgroups of $\mathfrak{A}^1$ containing O^1 as a subgroup of finite index. But since a maximal order in a trivial algebra is generated by its units elements the lemma is proved. ∎

Theorem 1.5.9. Let $\mathfrak{A}$ be a trivial central simple algebra over k.

$$|C(F,O)| = ch(k)$$

Where c is the number of maximal orders containing O and h(k) is the class number of k.

Proof. Let $\tilde{O}$ be a maximal order in $\mathfrak{A}$. Since $\mathfrak{A}$ is a trivial central simple algebra we have that the non zero prime ideals p of k and the prime ideals β of $\tilde{O}$ are in one to one correspondence:

$p = O_k \cap \beta \quad , \quad p\tilde{O} = \beta.$

Therefore by proposition 1.5.3 we have $|C(F,\tilde{O})| = h(k)$ and applying lemma 1.5.8 the theorem is proved. ■

Let L_1 and L_2 be two O-invariant lattices in F as above.

Definition 1.5.10. We say that L_1 and L_2 have the same genus if $L_{1,p}$ is isomorphic as O_p-module to $L_{2,p}$ for any prime p.

Theorem 1.5.10. Let $\mathfrak{A}$ be a trivial central simple algebra over k. The number of genus classes of O-invariant lattices in F is equal to the class number of k.

Proof. Let L_1 and L_2 be lattices in F, O-invariant. If L_1 and L_2 have the same genus then:

$O_\ell(L_{1,p}) = O_\ell(L_{2,p})$ for every p prime.

But $O_\ell(L_p) = (O_\ell(L))_p$ and two orders are equal if they coincide at every localization. So $O_\ell(L_1) = O_\ell(L_2)$ and it is a maximal order. The theorem follows. ■

<u>1.6 Classification of Kuga fiber varieties arising from quaternion algebras.</u>

Let $A \xrightarrow{f} \Gamma\backslash\mathbf{H}^t$ be a Kuga fiber variety defined, as in 1.2, by a symplectic representation ρ of $G = \mathrm{Res}_{k/\mathbf{Q}}SL_1(B)$. By Addington's

Theorem we can assume ρ to be a polymer representation. In this section we will classify, up to isomorphism, the rigid Kuga fiber varieties defined by a given polymer representation.

Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be Kuga fiber varieties given by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ and $(G,K,X,\Gamma,F',L',\beta',\rho',\tau')$ respectively.

Definition 1.6.1. We say that $A \xrightarrow{f} V$, $B \xrightarrow{g} V$ are isomorphic if there existes a **Q**-linear isomorphism $\psi:F \to F'$ such that:

$$\rho = \psi^{-1}\rho'\psi$$
$$\beta(x,y) = \beta'(\psi(x,y)), \qquad x,y \in F$$
$$\tau = \psi^{-1}\tau'\psi$$
$$L = \psi^{-1}L'.$$

The fiber of $A \xrightarrow{f} V$ as a real torus is isomorphic to F/L where F is a minimal left ideal in a **Q**-algebra and L is a lattice in F. We will show that the classification of the families induced by a given representation can be reduced to the classification of lattices in a given central simple algebra. Therefore this classification is easily obtained from the results of section 1.5.

Lemma 1.6.2. Let $\psi:F \to F'$ be a **Q**-linear isomorphism. Then $\rho(g) = \psi^{-1}\rho'(g)\psi$ and $L = \psi^{-1}L'$ if and only if L is isomorphic via ψ to L'

as Γ-modules.

Proof. Sufficiency is obvious so we want to prove necessity. Suppose $\psi: L \to L'$ is an isomorphism of Γ-modules; then $\psi(\rho(\gamma)z) = \rho'(\gamma)\psi(z)$ for any z in L and γ in Γ. Since Γ is an arithmetic subgroup of G, it is Zariski dense in G (Borel [B]) so that $\psi(\rho(g)z) = \rho'(g)\psi(z)$ for any g in G and z in L. Therefore $\rho = \psi^{-1}\rho'\psi$ and the lemma is proved. ∎

In most cases the isomorphism of L and L' as $\rho(\Gamma)$ modules determines the isomorphism of the two families. We have the following:

Proposition 1.6.3. (Satake [S]). Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be Kuga fiber varieties defined by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ and $(G,K,X,\Gamma,F',L',\beta',\rho',\tau')$. If ρ satisfies the H2 - condition and ρ' is equivalent to ρ over **Q**, then $B \xrightarrow{g} V$ can be defined by $(G,K,X,\Gamma,F,L',\beta,\tau)$.

Rigid families of quaternion type are defined by polymer representation satisfying the H2 - condition, so we have:

Corollary 1.6.4. Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be rigid Kuga fiber varieties defined by the same polymer representation.
Then $A \xrightarrow{f} V$ is isomorphic to $B \xrightarrow{g} V$ in the sense of definition

1.5.1 if and only if L and L' are isomorphic as Γ-molules.

Proof. Lemma 1.5.2 and proposition 1.5.3 give this result. ∎

We can now find a classification.

Suppose the data (G,K,X,F,β,ρ) is given.

Notation: We will denote by $C(\rho,\Gamma)$ the set of isomorphism classes of families of Abelian varieties defined by ρ over $V = \Gamma\backslash X$.

Definition 1.6.5. Let R be an algebra over **Q** and O_R a **Z**-order in R. An Abelian variety of $\underline{O_R \text{ type}}$ is a pair (X,ι) formed by an Abelian variety X and an isomorphism ι of O_R into End(X) such that $\iota_X(1) = 1_X$. We say that a Kuga fiber variety is of $\underline{O_R \text{ type}}$ if the fibers are of O_R type.

Definition 1.6.6. We say that two Kuga fiber varieties $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ of O_R type are $\underline{\text{isomorphic as } O_R \text{ type}}$ if:

(i) They are isomorphic (in the sense of 1.6.1), and

(ii) If $\psi:F \to F'$ is the **Q**-linear map inducing the isomorphism then $\psi\iota_A = \iota_B\psi$.

Notation: Let $C(\rho,\Gamma,O_R)$ be the set of classes of isomorphism of families of Abelian varieties of O_R-type defined by ρ.

Main Theorem 1.6.7. Let ρ be a rigid polymer representation of $G = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(SL_1(B))$, irreducible over $\mathbf{Q}$. B a quaternion algebra over k, $B \neq M_2(k)$. k a totally real number field, $\tilde{k}$ the smallest Galois extension of $\mathbf{Q}$ containing k. $\Omega = \mathrm{Gal}(\tilde{k}/\mathbf{Q})$.
Let R be a $\mathbf{Q}$-algebra of dimension n if $\rho = \rho_P$ and 4n otherwise.
Let O_R be a maximal order in R.
Then we have:

$$|C(\rho,\Gamma,O_R)| = c \cdot h(\tilde{k}).$$

Where c is a constant, $h(\tilde{k})$ is the class number of $\tilde{k}$ and Γ is an arithmetic subgroup of G.

Proof. Let $A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ be two Kuga fiber varieties defined by ρ. By proposition 1.6.3 we may assume:
$A \xrightarrow{f} V$ defined by $(G,K,X,\Gamma,F,L_1,\beta,\rho,\tau)$
and $B \xrightarrow{g} V$ defined by $(G,K,X,\Gamma,F,L_2,\beta,\rho,\tau)$
Let $P = \Sigma gM$, $gM \neq \gamma M$ if $\gamma \neq g$, the polymer such that $\rho = \rho_P$. In the notation of 1.2 we have:
(i) $\rho = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}\rho_M$, $\rho_M i_M : \mathrm{Res}_{k/\mathbf{Q}}(SL_1(B)) \to \mathrm{End}_{\tilde{k}} F_M$.
(ii) F_M minimal left ideal in B^M, $F = \mathrm{Res}_{\tilde{k}/\mathbf{Q}} F_M$.
(iii) The endomorphism ring of the generic fiber of any family defined by ρ depends on the class of B^M in the Brauer group. In

fact by proposition 1.4.1:

Let A_x be the generic fiber

if $B^M \cong M_N(B_0)$ $\quad \begin{cases} \mathrm{End}_{\mathbf{Q}} F \supset \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(B^M \otimes B_0) \\ \mathrm{End}_{\mathbf{Q}} A_x = 1 \otimes \mathrm{Res}_{\tilde{k}/\mathbf{Q}} B_0 \end{cases}$

if $B^M \cong M_N(\tilde{k})$ $\quad \begin{cases} \mathrm{End}_{\mathbf{Q}} F \supset \mathrm{Res}_{\tilde{k}/\mathbf{Q}} B^M \\ \mathrm{End}_{\mathbf{Q}} A_x = \mathrm{Res}_{\tilde{k}/\mathbf{Q}} \tilde{k} \end{cases}$

If A_x is any fiber we have only inclusions.

We can identify O_R with a maximal order in the endomorphism ring of the generic fiber, since $\dim_{\mathbf{Q}} R = 2\dim A_x$.

Therefore O_R contains $O_{\tilde{k}}$ (prop. 1.4.1) and we can consider O_R as an $O_{\tilde{k}}$-order, L_1 and L_2 as $O_{\tilde{k}}$-lattices.

$A \xrightarrow{f} V$ and $B \xrightarrow{g} V$ are isomorphic as families of O_R type if and only if L_1 and L_2 are isomorphic as $\mathbf{Z}[\rho(\Gamma)] \otimes O_R$-modules.

Therefore if and only if L_1 and L_2 are isomorphic as

$$O_{\tilde{k}}[\rho_M(\Gamma)] \otimes O_R\text{-modules.}$$

If ρ_M is a $\mathbf{C}$-irreducible representation of G, then, since Γ is Zariski dense in G, we have that $O_{\tilde{k}}[\rho_M(\Gamma)]$ is an order.

If ρ_M is a $\mathbf{C}$-reducible representation of G, then we have to consider $\rho_M \otimes \mathrm{id}\colon G \times SL_1(B_0) \to \mathrm{End}_{\tilde{k}}(F_M)$. $\rho_M \otimes \mathrm{id}$ is defined over $\tilde{k}$ and $\mathbf{C}$-irreducible.

$\Gamma \times O_R^1$ is an arithmetic subgroup of $G \times SL_1(B_0)$ so as above $O_{\tilde{k}}[\rho(\Gamma) \otimes O_R^1]$ is an order in $\mathrm{End}_{\tilde{k}}(F_M)$.

We can conclude: $C(\rho,\Gamma,O_R) \cong C(F_M,O)$,

$$\text{where } O = \begin{cases} O_{\tilde{k}}[\rho_M(\Gamma)] \\ \text{or} \\ O_{\tilde{k}}[\rho_M(\Gamma)] \otimes O_R \end{cases} \quad \text{is an order in } \mathrm{End}_{\tilde{k}} F_M$$

By theorem 1.5.9. $|C(\rho,\Gamma,O_R)| = c\ h(\tilde{k})$ where c is the number of maximal orders containing O.

Corollary 1.6.8. Suppose that O is a maximal order. Then we have a bijection:

$$C(\rho,\Gamma,O_R) \to \text{Ideal class group } (\tilde{k}).$$

Chapter 2.

2.0 Definitions

Let V be a projective variety defined over a subfield of **C**. Let $\sigma \in \mathrm{Ant}(\mathbf{C})$ be an automorphism of the complex numbers. If V is the variety determined by I(V) we define V^σ as the variety determined by $I(V)^\sigma$.

Definition. A subfield h of **C** is called the bottom field of V if an automorphism σ of **C** is the identity mapping on h if and only if V^σ is biregularly equivalent to V.

The definition of bottom field is due to Shimura, as are all the results quoted in this section [Sh1,2]. The best known example is the bottom field of a polarized Abelian variety (moduli field), the central notion in the theory of complex multiplication [Sh-T]. The moduli field is interpreted in terms of class field theory Shimura described also the bottom field of the Hilbert modular variety $\Gamma\backslash\mathbf{H}^t$ in terms of class field theory. In particular he constructed a Hilbert modular variety with bottom field different from **Q**.

Shimura's original definition is weaker: it requires V and V^σ to be only birationally equivalent. But in the case of Hilbert modular varieties the two definitions coincide. Let D be a bounded symmetric domain and Γ a properly discontinuous group of transformations of D with compact quotient and without fixed points. Then $\Gamma\backslash D$ is known to be isomorphic (analytically) to a non singular projective variety V. By Igusa V is a minimal model, i.e. every rational mapping of a variety W into V is defined at every simple point of W.

Therefore for V the biregular equivalence coincides with the birational equivalence.

Following Shimura we have:

2.0.0 If W is biregularly equivalent to V then the bottom field of W and V coincide.

2.0.1 Every field of definition of V contains the bottom field of V, if the latter exists.

2.0.2 Assume V is defined over an algebraic number field, then the bottom field of V exists and is an algebraic number field of finite degree.

Let $V = \Gamma\backslash \mathbb{H}^t$ be as above then the existence of the bottom field of V is assured:

$\Gamma\backslash D$ is rigid if the dimension is bigger then 1. (Calabi-Vesentini). $\Gamma\backslash D$ is therefore defined over an algebraic number field [Sh3] so that its bottom field exists by 2.0.2.

If $\Gamma\backslash D = V$ has dimension 1 then the argument is different.

By Torelli's theorem the bottom field of V coincides with the field of moduli of (J,), J the Jacobian variety of the curve V and the canonical polarization.

Let $A \xrightarrow{f} V$ be a family of Abelian varieties parametrized by a Hilbert modular variety. As we mentioned A and V are biregularly equivalent to a projective variety [K] ($V = \Gamma\backslash\mathbf{H}^t$ is always compact if the quaternion algebra is a division algebra).
If we assume that V and A have models defined over an algebraic number field then by 2.0.0 and 2.0.1 the bottom fields of A and V are well defined, exist and are algebraic number fields.
The main result of this chapter relates the two fields:

Theorem. The bottom field of A is a Galois extension of the bottom field of V.

From now on we will assume that $A \xrightarrow{f} V$ has a model defined over an algebraic number field.

2.1 $A^\sigma \xrightarrow{f^\sigma} V^\sigma$

Let $\sigma \in \text{Aut}(\mathbf{C})$ be an automorphism of the complex numbers. Let $A \xrightarrow{f} V$ be a family of Abelian varieties parametrized by a compact arithmetic variety. We shall identify A and V with their embedded images in the complex projective space so that f is a morphism of algebraic varieties and f^σ is defined as in 2.0.

Theorem 2.1.1. (M.H. Lee [L]) Let $f: A \longrightarrow V$ be a Kuga fiber variety over a compact arithmetic variety V. Then $f^\sigma: A^\sigma \to V^\sigma$ is a Kuga fiber variety over V^σ.

$A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is therefore defined analytically by the data:

$(G^{(\sigma)}, K^{(\sigma)}, X^{(\sigma)}, \Gamma^{(\sigma)}, F^{(\sigma)}, L^{(\sigma)}, \beta^{(\sigma)}, \rho^{(\sigma)}, \tau^{(\sigma)})$

and

$$\begin{array}{c} A^\sigma \cong (\Gamma^{(\sigma)} \underset{\rho^{(\sigma)}}{\times} L^{(\sigma)}) \backslash (X^{(\sigma)} \times F^{(\sigma)}) \\ \Big\downarrow f^{(\sigma)} \\ V^{(\sigma)} = \Gamma^{(\sigma)} \backslash X^{(\sigma)} \end{array}$$

Corollary 2.1.1. If $A \xrightarrow{f} V$ is of Hodge type then $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is of Hodge type.

Proof. A family of Abelian varieties $A \xrightarrow{f} V$ is of Hodge type if and only if it contains A_x, a fiber of CM type. But then $A_x{}^\sigma$ is contained in $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ and $A_x{}^\sigma$ is of CM type. ∎

Corollary 2.1.3. Let L_p and $L_p^{(\sigma)}$ be the p-adic completions of L and $L^{(\sigma)}$ respectively. Let $\hat{\Gamma}$ and $\hat{\Gamma}^\sigma$ be the completions of Γ and $\Gamma^{(\sigma)}$ respectively in the topology of the subgroups of finite index. Then L_p and $L_p^{(\sigma)}$ are isomorphic as $\hat{\Gamma}$ modules for any prime p. In other words

$$\hat{\Gamma} \underset{\hat{\rho}_p}{\times} L_p \overset{\sigma}{\cong} \hat{\Gamma}^{(\sigma)} \underset{\hat{\rho}_p^{(\sigma)}}{\times} L_p^\sigma$$

for any prime p.

Proof. Let $\{\Gamma_i\}$ be a cofinal system of subgroups of Γ, $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$ Then for any $i, X_i = \Gamma_i \backslash X$ is a finite unramified covering manifold of V and $\{X_i\}$ is a cofinal system of covering manifolds. For each i we consider $\Gamma_i \times p^i L$, so that $A_i \cong (\Gamma_i \underset{\rho}{\times} p^i L)\backslash(X \times F)$ is a finite unramified covering of $A \cong (\Gamma \underset{\rho}{\times} L)\backslash(X \times F)$. Moreover $A_i \xrightarrow{f_i} V_i$ is a family of Abelian varieties.

Applying σ to every A_i, we obtain a cofinal system $\{A_i^\sigma\}$ of unramified covering manifolds of A^σ. We have the following diagrams:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
| & & | \\
A_2 & \to & V_2 \\
| & & | \\
A_1 & \to & V_1 \\
| & & | \\
A & \to & V
\end{array}
\qquad\qquad
\begin{array}{ccc}
\vdots & & \vdots \\
| & & | \\
A_2^\sigma & \to & V_2^\sigma \\
| & & | \\
A_1^\sigma & \to & V^\sigma \\
| & & | \\
A^\sigma & \to & V^\sigma
\end{array}
$$

By theorem 2.1.1 $A_i^\sigma \to V_i^\sigma$ is a family of Abelian varieties for any i. Let $J_\ell(A)$ be the ℓ-adic points of any fiber of A; it doesn't matter which fiber as the ℓ-adic points of any two fibers are isomorphic. σ induces an isomorphism of $J_\ell(A)$ onto $J_\ell(A^\sigma)$ and therefore:

$$A_i^\sigma \cong (\Gamma_i^{(\sigma)} \underset{\rho^{(\sigma)}}{\times} p^i L^{(\sigma)})\backslash(X^{(\sigma)} \times F^{(\sigma)})$$

We proved that $\sigma\colon \hat{\Gamma} \times \Gamma_p \to \hat{\Gamma}^\sigma \times L_p^{(\sigma)}$ is a bijection. To show that σ is a homomorphism of groups we need the definition of $\hat{\rho}^\sigma$ as given in the proof of theorem 2.1.1. If $\hat{\gamma} \in \hat{\Gamma}^\sigma$, $\hat{\gamma} = \varprojlim \gamma_i^\sigma$ then $\rho(\hat{\gamma}) = \varprojlim (\rho(\gamma_i))^\sigma$.

So the conclusion follows at once:

$$\hat{\Gamma} \underset{\hat{\rho}_p}{\times} L_p \cong \hat{\Gamma}^\sigma \underset{\hat{\rho}_p^{(\sigma)}}{\times} L_p^{(\sigma)} \quad \text{as groups.} \blacksquare$$

2.2 Bottom fields.

Let $A \xrightarrow{f} V$ be a Kuga fiber variety parametrized by an arithmetic variety $V = \Gamma\backslash X$. We assume that $A \xrightarrow{f} V$ has a projective model defined over a number field. We define K_A as the bottom field of A and K_V as the bottom field of V.

Theorem 2.2.1. K_A and K_V exist and $K_A \supset K_V$.

Proof. The existence follows from 2.0.2 and the hypothesis . Let σ be an element of Aut $(\mathbf{C})$. By theorem 2.1.1 we know that $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is a Kuga fiber variety over an arithmetic variety $V^\sigma = \Gamma^{(\sigma)}\backslash X^{(\sigma)}$.

Assume that $\sigma|_{K_A} = \text{id}$, i.e., there exists a biholomorphic map $b(\sigma)\colon A^\sigma \to A$. If we show that $b(\sigma)$ induces a biholomorphic map between V and V^σ the theorem is proved. In fact then $\sigma|_{K_A} = \text{id}$ implies $\sigma|_{K_V} = \text{id}$ so that $K_A \supset K_V$.

Consider the fiber $A^\sigma{}_\lambda = (f^\sigma)^{-1}(\lambda)$, $\lambda \in V^\sigma$, and the holomorphic map $fb(\sigma): A^\sigma_\lambda \to V$.

A^σ_λ is an Abelian variety and V an arithmetic variety, $V = \Gamma\backslash X$.

So there exists a lifting $\tilde{f}_{\sigma,\lambda}$ such that the following diagram commutes:

$$\begin{array}{ccc} A^\sigma_\lambda & \xrightarrow{\;fb(\sigma)\;} & V \\ \Big\downarrow{\scriptstyle\pi} & & \Big\uparrow{\scriptstyle\bar{\eta}} \\ \mathbf{C}^n & \xrightarrow[\tilde{f}_{\lambda,\sigma}]{} & X \end{array}$$

Then $\tilde{f}_{\lambda,\sigma}$ is a bounded holomorphic function and therefore a constant, i.e. $fb(\sigma)(A^\sigma_\lambda) = x$ for some $x \in V$. This implies that $b(\sigma)$ sends fibers of A^σ to fibers of A and induces in the natural way, identifying V^σ with the zero section, $\eta(\sigma) : V^\sigma \to V$, $\eta(\sigma)$ holomorphic.

Replacing $b(\sigma)$ with $b(\sigma)^{-1}$ we conclude that $\eta(\sigma)$ is biholomorphic and $\sigma|_{K_V} = \text{id}$. ■

2.3 $V^\sigma \cong V$.

Let σ be an element of Ant($\mathbf{C}$)

We examined in 2.2 the structure of $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ as a family of Abelian varieties.

In this section we show that in the rigid quaternion case, if $V \cong V^\sigma$ then $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ can be defined by the data $(G,K,X,\Gamma,F,L^{(\sigma)},\beta,\rho,\tau)$. So it is the lattice $L^{(\sigma)}$ that characterizes the new family. We need some preliminary results:

Let Y_1 and Y_2 be Abelian varieties and $\sigma:Y_1 \to Y_2$ a morphism. We will denote by $\alpha_{\mathbf{Q}}$ the rational representation of α and by α_{ℓ} its ℓ-adic representation. The following is well known:

Lemma 2.3.1. The representations $\alpha_{\mathbf{Q}}$ and α_{ℓ} are equivalent.

Lemma 2.3.2. Given the morphism α, apply σ to get $\alpha^{\sigma}:Y_1^{\sigma} \to Y_2^{\sigma}$. Then $\alpha_{\mathbf{Q}} \sim (\alpha^{\sigma})_{\mathbf{Q}}$.

Proof. Let $J_{\ell}(Y_1)$, $J_{\ell}(Y_2)$ be the groups of ℓ-adic points. Then σ induces the isomorphisms:

$$J_{\ell}(Y_i) \cong J_{\ell}(Y_i{}^{\sigma})$$

and the commutative diagram

$$\begin{array}{ccc} J_{\ell}(Y_1) & \xrightarrow{\alpha_{\ell}} & J_{\ell}(Y_2) \\ \| \wr & & \| \wr \\ J_{\ell}(Y_1^{\sigma}) & \xrightarrow{\alpha_{\ell}^{\sigma}} & J_{\ell}(Y_2^{\sigma}) \end{array}$$

Therefore $\alpha_{\ell} \sim \alpha_{\ell}^{\sigma}$ and $\alpha_{\mathbf{Q}} \sim (\alpha^{\sigma})_{\mathbf{Q}}$. ∎

We assume that $A \xrightarrow{f} V$ is defined by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$ where $G = \mathrm{Res}_{k/\mathbf{Q}}(SL_1(B))$, and $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ is defined by $(G^{(\sigma)}, K^{(\sigma)}, X^{(\sigma)}, \Gamma^{(\sigma)}, F^{(\sigma)}, L^{(\sigma)}, \beta^{(\sigma)}, \rho^{(\sigma)}, \tau^{(\sigma)})$.

Following Lee [L] we analyze the representation $\rho^{(\sigma)}: G^{(\sigma)} \to Sp(F^{(\sigma)}, \beta^{(\sigma)})$ that defines $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$. Let $X^{(\sigma)}$ be the universal covering of V^{σ}, then:

$\Gamma^{(\sigma)}$ is the fundamental group of V^{σ},

$G^{(\sigma)}$ is the connected component of the identity in $\mathrm{Aut}(X^{(\sigma)})$.

We define $C(\Gamma^{(\sigma)})$ the commensurability subgroup of $\Gamma^{(\sigma)}$ in $G^{(\sigma)}$

as: $C(\Gamma^{(\sigma)}) = \{g^{\sigma} \in \mathrm{Aut}(X^{(\sigma)}) \mid [\Gamma^{(\sigma)} : g^{\sigma}\Gamma^{(\sigma)}(g^{\sigma})^{-1} \cap \Gamma^{(\sigma)}] < \infty\}$

We have:

$\Gamma^{(\sigma)} \subset G^{(\sigma)}$ and Zariski dense,

$C(\Gamma^{(\sigma)}) \to \hat{G}^{(\sigma)} = \mathrm{Aut}(\hat{X}^{(\sigma)})$

The representation $\rho^{(\sigma)}$ is actually defined on $C(\Gamma^{(\sigma)}) \to \hat{G}^{(\sigma)}$. Let $\Gamma^{(\sigma)} \supset \Gamma_1^{(\sigma)} \supset \ldots$ be a cofinal sequence of groups. Then the corresponding system of coverings of $V^{(\sigma)}$ is a cofinal system $\ldots \to V_2^{(\sigma)} \to V_1^{(\sigma)} \to V^{(\sigma)}$. In the same way we can consider the cofinal system $\{\Gamma_i^{(\sigma)} \underset{\rho(\sigma)}{\times} L^{(\sigma)}\}$. We then have

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
A_i^{(\sigma)} = {}_{(\Gamma_i^{(\sigma)} \times L^{(\sigma)})}\backslash X^{(\sigma)} \times F^{(\sigma)} & \to & V_i^{(\sigma)} \\
\downarrow & & \downarrow \\
A_{i-1}^{(\sigma)} & \longrightarrow & V_{i-1}^{(\sigma)} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
A^{(\sigma)} & \longrightarrow & V^{(\sigma)}
\end{array}
$$

Applying σ^{-1} to these constructions yields

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
A_i & \longrightarrow & V_i \\
\downarrow & & \downarrow \\
A_{i-1} & \longrightarrow & V_{i-1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
A & \longrightarrow & V
\end{array}
$$

where $\{V_i\}$ corresponds to a cofinal sequence $\{\Gamma_i\}$ and for each i $A_i \to V_i$ is a family of Abelian varieties. Therefore $A_i = (\Gamma_i \underset{\rho}{\times} L)\backslash(X \times F)$.

Let γ be an element of $\Gamma^{(\sigma)}$. Since $\Gamma^{(\sigma)} \to \hat{G}^{(\sigma)}$ we have $\gamma = \underset{\longleftarrow}{\lim}\, \gamma_i$ and for each i, $(\gamma_i, \rho^{(\sigma)}(\gamma)): A_i \to A_i$ is a covering transformation over A.

Applying σ^{-1} yields the set $\{\gamma_i^{\sigma^{-1}}, [\rho^{(\sigma)}(\gamma)]^{\sigma^{-1}}\}$ of covering transformations for the cofinal system $\to A_i \to A_{i-1} \to \dots \to A$.

Hence $\gamma_i^{\sigma^{-1}} = \gamma_i'$ for some $\gamma_i' \in \Gamma_i$ and the rational representation $([\rho^{(\sigma)}(\gamma)]^{\sigma^{-1}})_{\mathbf{Q}} = \rho(\gamma_i')$ is independent of the fiber.

This implies

Lemma 2.3.3. (i) $\sigma|\Gamma : \Gamma \to \Gamma^{(\sigma)}$ and

(ii) $\sigma|\Gamma$ is an isomorphism.

By lemma 2.3.2 we have $([\rho^{(\sigma)}]^{\sigma^{-1}})_{\mathbf{Q}} \sim \rho^{(\sigma)}$.

Thus there exists $\psi: F \to F^{(\sigma)}$ such that $\rho^{(\sigma)}(\gamma') \subset \psi\rho(G)\psi^{-1}$ for any $\gamma' \in \Gamma^{(\sigma)}$. Since $\Gamma^{(\sigma)}$ is dense in $G^{(\sigma)}$,

$$\rho^{(\sigma)}(G^{(\sigma)}) \subset \psi\rho(G)\psi^{-1}.$$

By interchanging $\rho^{(\sigma)}$ and ρ we can proceed in the same way to conclude:

Lemma 2.3.4. $\rho(G) = \psi^{-1}\rho^{(\sigma)}(G^{(\sigma)})\psi$

This result is the crucial step in the proof of the following:

Theorem 2.3.5. Let $A \xrightarrow{f} V$ be a rigid Kuga fiber variety of quaternion type defined by the data $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$. If σ is an element of Aut(**C**) and $V \cong V^{\sigma}$, then the Kuga fiber variety $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ can be defined by the data $(G,K,X,\Gamma,F,L^{(\sigma)},\beta,\rho,\tau)$.

Proof. We assume that $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ is defined by the data $(G^{(\sigma)},K^{(\sigma)},X^{(\sigma)},\Gamma^{(\sigma)},F^{(\sigma)},L^{(\sigma)},\beta^{(\sigma)},\rho^{(\sigma)},\tau^{(\sigma)})$. Let ϕ be the biregular isomorphism $\phi: V \to V^{\sigma}$. ϕ induces an isomorphism of the fundamental group $\phi_* : \pi_1(V) \to \pi_1(V^{\sigma})$.

Γ and $\Gamma^{(\sigma)}$ are arithmetic subgroups of G and $G^{(\sigma)}$ (respectively) so that $\pi_0(V) = \{e\}$, $\pi_0(V^{(\sigma)}) = \{e\}$ and $\pi_1(V) = \Gamma$, $\pi_1(V^{(\sigma)}) = \Gamma^{(\sigma)}$ (see Kajdan [Kj]).

Therefore ϕ induces and isomorphism $\phi_* : \Gamma \to \Gamma^{(\sigma)}$. Since Γ is Zariski dense in G we can extend ϕ_* to a **Q**-morphism $\phi_* : G \to G^{(\sigma)}$.

By lemma 2.3.3 we have $\rho(G) = \psi^{-1}\rho^{(\sigma)}(G^{(\sigma)})\psi$ so that:

$$\begin{array}{ccc} G & \xrightarrow{\quad\rho\quad} & Sp(F,\beta) \\ \phi_* \Big| & & \Big| \; I_{\psi} \\ G^{(\sigma)} & \xrightarrow{\quad\rho^{(\sigma)}\quad} & Sp(F^{(\sigma)},\beta^{(\sigma)}) \end{array}$$

The map of $\rho^{-1}I_{\psi}\rho^{(\sigma)}\phi_{*} : G \to G$ is an automorphism defined over $\mathbf{Q}$.$G = \mathrm{Res}_{k/\mathbf{Q}}SL_1(B)$ so that any $\mathbf{Q}$-automorphism is inner and thus there exists $x \in G$ such that any $\rho^{-1}I_{\psi}\rho^{(\sigma)}\phi_{*} = I_x$.

Therefore $\rho^{(\sigma)}\phi_{*} \sim \rho$.

Now the two representations ρ and $\rho^{(\sigma)}\phi_{*}$ satisfy the hypotheses of proposition 1.6.3. We conclude that $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ can be defined by $(G,K,X,\Gamma,F,L^{(\sigma)},\beta,\rho,\tau)$. ∎

Corollary 2.3.6. Let $L_p = L \otimes \mathbf{Z}_p$ and $L_p^{(\sigma)} = L^{(\sigma)} \otimes \mathbf{Z}_p$ for any prime p. Then L_p and $L_p^{(\sigma)}$ are isomorphic as Γ-modules :

$$\Gamma \underset{\rho_p}{\times} L_p \cong \Gamma \underset{\rho_p}{\times} L_p^{(\sigma)} .$$

Proof. The map $\psi : F \to F^{(\sigma)}$ introduced before Lemma 2.3.4 locally coincides with σ, and therefore:

by lemma 2.1.3 $$\hat{\Gamma} \underset{\rho_p}{\times} L_p \overset{\sigma\times\psi_p}{\cong} \hat{\Gamma}^{(\sigma)}{}_{\rho} \underset{\hat{\rho}_p}{\times} L_p^{(\sigma)} .$$

by lemma 2.3.3 $$\sigma|\Gamma : \Gamma \longrightarrow \Gamma^{(\sigma)}$$

so that $$\Gamma \underset{\rho_p}{\times} L_p \overset{\sigma\times\psi_p}{\cong} \Gamma^{(\sigma)} \underset{\rho_p^{(\sigma)}}{\times} L_p^{((\sigma)} .$$

Moreover in the proof above (2.3.5) we have shown that there exists $x \in G$ such that

$$\Gamma \underset{\rho}{\times} L_p^{(\sigma)} \overset{\phi_{*}\times\rho_p(x)}{\cong} \Gamma^{(\sigma)} \underset{\hat{\rho}^{(\sigma)}}{\times} L_p^{(\sigma)} .$$

where, for simplicity, we write $L^{(\sigma)}$ for $[\rho(x)]^{-1}L^{(\sigma)}$. $\phi_*^{-1} \times \sigma$ is an automorphism of Γ, so, as we have seen, it is given by conjugation by an element of G. The corollary is then proved.■

2.4 Gal ($\mathbf{C}$/Bottom field of V) → Ideal class group ($\tilde{k}$).

Let σ be an element of Gal($\mathbf{C}$/Bottom V), let $A \xrightarrow{f} V$ be a rigid family of quaternion type defined as usual by $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$. We proved that $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is defined by $(G,K,X,\Gamma,F,L^{(\sigma)},\beta,\rho,\tau)$.

In this section we will show that $L^{(\sigma)} = a(\sigma)L$, $a(\sigma)$ an ideal of $\tilde{k}$, and show that the map $\mathrm{Gal}(\mathbf{C}/K_V) \to IC(\tilde{k})$ is a group homomorphism. We first need to make some assumptions, as in Section 1.6.

i) $\rho = \rho_P$ is rigid, $\mathbf{Q}$-primary, $\mathbf{Q}$-irreducible

i.e., $P = \sum_{g\in\Omega} gM$; $gM \neq \gamma M$ when $g \neq \gamma$

ii) $A \xrightarrow{f} V$ is of type O_R maximal order in the $\mathbf{Q}$-algebra R,

$\dim_{\mathbf{Q}} R = \dim_{\mathbf{Q}}[\mathrm{End}_{\mathbf{Q}} A_x]$ where A_x is a generic fiber.

Lemma 2.4.1. If $A \xrightarrow{f} V$ is of type O_R then so is $A^\sigma \xrightarrow{f^\sigma} V^\sigma$.

Proof. In the notation of proposition 1.4.1 we may identify O_R with a $O_{\tilde{k}}$ maximal order in $\mathrm{Res}_{\tilde{k}/\mathbf{Q}}\, B_0 = \mathrm{End}_{\mathbf{Q}}(A_x)$. As before B_0 is either $\tilde{k}$ or a division algebra over $\tilde{k}$ and A_x is isomorphic to (F,L,J_x) where $F = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}\, B_0^N$.

Let b be an element of B_0 and r_b: $B_0 \to B_0$ the right multiplication by b. Put $R_b = \text{Res}_{\tilde{k}/\mathbf{Q}}\, r_b$. The rational representation of B_0 is N copies of R_b:

$i: B_0 \to GL(F)$

$b \to \text{Res}_{\tilde{k}/\mathbf{Q}}\, Nr_b = NR_b$

We can therefore write F as a sum of invariant subspaces $F = (\text{Res}_{\tilde{k}/\mathbf{Q}}\, B_0)^N$. If $S: O_R \to \text{End}\, A_x$ is the analytic representation then the generic fiber $A^\sigma{}_{x^\sigma}$ of $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is of type O_R via the representation $S^\sigma: O_R \to \text{End}(A_x{}^\sigma)$. We show that $S^\sigma = S$. Let $\text{Res}_{\tilde{k}/\mathbf{Q}}\, B_0$ be an i-invariant subspace of F. Then $J_x(\text{Res}_{\tilde{k}/\mathbf{Q}}\, B_0)$ is i-invariant and therefore coincides with a copy of $\text{Res}_{\tilde{k}/\mathbf{Q}}\, B_0$.

We can conclude that $S(b) = \frac{N}{2} R_b$ and

$S^\sigma(b) = \frac{N}{2} R_b^\sigma = \frac{N}{2} R_b$, as R_b is defined over $\mathbf{Q}$.

The choice of A_x is not restrictive and the lemma is proved. ∎

Lemma 2.4.2. (i) $L, L^{(\sigma)}$ can be considered as $O_{\tilde{k}}$ lattices in F_M,

$F = \text{Res}_{\tilde{k}/\mathbf{Q}} F_M$.

(ii) $L, L^{(\sigma)}$ are $O_{\tilde{k}}\, [\rho_M(\Gamma)] \otimes O_R$ - lattices of the same genus, so that $O_\ell(L) = O_\ell(L^{(\sigma)})$.

(iii) $L = a(\sigma)L^{(\sigma)}$, where $a(\sigma)$ is the $O_{\tilde{k}}$-ideal in $\tilde{k}$:

$a(\sigma) = [L^{(\sigma)}:L] = \{g \in \text{End}_{\mathbf{Q}} F_M \mid gL^{(\sigma)} \subset L\}$.

Proof. (i) is an obvious consequence of lemma 2.4.1, recalling that since O_R is a maximal order, $O_R \supset O_{\tilde{k}}$.

ii) follows from corollary 2.3.6 since all the local isomorphisms considered commute with the elements of O_R.

(iii) is just Proposition 1.5.3. ■

We will denote the order $O_{\tilde{k}}[\rho_M(\Gamma)] \otimes O_R$ by 0.

Lemma 2.4.3. Let L and $L^{(\sigma)}$ be $O_{\tilde{k}}$-lattices associated to $A \xrightarrow{f} V$ and $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ respectively.

Then $L^{(\sigma)} = a(\sigma)L$,

the ideal class of $a(\sigma)$ is independent of L and $L^{(\sigma)}$, and the map Gal(**C**/Bottom field (V)) → Ideal class group ($\tilde{k}$) is well defined.

Proof. We must show that the ideal class of $a(\sigma)$ is independent of the choice of L and $L^{(\sigma)}$. Suppose $A \xrightarrow{f} V$ has two analytic models as a family of O_R-type. Let L and L' be the lattices associated to the two models. Since the two models are biregularly isomorphic as O_R-type, L and L' are isomorphic as 0-modules, i.e., $L = \alpha L'$, $\alpha \in \tilde{k}^\times$.

Suppose $L = [L^{(\sigma)}:L]\, L^{(\sigma)}$ and $L' = [L^{(\sigma)}:L']L^{(\sigma)}$, then $\alpha[L^{(\sigma)}:L'] = [L^{(\sigma)}:L]$ and the ideal class of $a(\sigma)$ is independent on the choice of L. We can proceed analogously for $L^{(\sigma)}$. ■

Lemma 2.4.4. Let $A \xrightarrow{f} V$ and $A' \xrightarrow{f'} V$ be two families of quaternion type defined by $(G,K,X,\Gamma,F,\beta,\rho,\tau)$. Let L and L' be the corresponding lattices and let $\sigma \in Gal(\mathbf{C}/K_V)$. If we assume that $L = [L':L]L'$ where $[L':L]$ is an $O_{\tilde{k}}$ ideal, then $[L':L] \equiv [L'^{(\sigma)}:L^{(\sigma)}]$ as ideal classes, here $L'^{(\sigma)}$ and $L^{(\sigma)}$ are the $O_{\tilde{k}}$-lattices associated to $A'^{\sigma} \to V$ and $A^{\sigma} \to V$.

Proof. Since we are concerned with ideal classes, we can assume: $[L':L] \subset O_{\tilde{k}}$ so that $L' \subset L$. Let x be in V and π_x the natural projection $\pi_x\colon A_x \to A'_x$. Let Λ be Ker $\pi_x \cong (Res_{\tilde{k}/\mathbf{Q}}L)/Res_{\tilde{k}/\mathbf{Q}}L'$. Now for any α in $O_{\tilde{k}}$ we have the following diagrams:

$$\begin{array}{ccc} L' & \xrightarrow{Nr\alpha} & L' \\ \downarrow & & \downarrow \\ Res_{\tilde{k}/\mathbf{Q}}L' & \xrightarrow{NR_\alpha} & Res_{\tilde{k}/\mathbf{Q}}L' \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & A'_x & \longrightarrow & A_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \frac{N}{2}R_\alpha & & \downarrow \frac{N}{2}R_\alpha & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & A'_x & \longrightarrow & A_x & \longrightarrow & 0 \end{array}$$

Then $[L':L] = \{ \alpha \in O_{\tilde{k}} \mid \frac{N}{2}R_\alpha \big|_\Lambda \equiv 0 \}$

Applying σ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^\sigma & \longrightarrow & A'^{\sigma}_{x^\sigma} & \longrightarrow & A^{\sigma}_{x^\sigma} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda^\sigma & \longrightarrow & A'^{\sigma}_{x^\sigma} & \longrightarrow & A^{\sigma}_{x^\sigma} & \longrightarrow & 0 \end{array}$$

where $\Lambda^\sigma \cong \quad Res_{\tilde{k}/\mathbf{Q}}L'^{(\sigma)}/Res_{\tilde{k}/\mathbf{Q}}L'^{(\sigma)}$ and $\left(\frac{N}{2}R_\alpha\right)^\sigma = \frac{N}{2}R_\alpha$

So $$[L'^{\sigma}:L^{\sigma}] = \{ \alpha \mid \tfrac{N}{2} R_{\alpha} \Big|_{\Lambda^{\sigma}} \equiv 0 \}$$

$$= \{ \alpha \mid \left(\tfrac{N}{2} R_{\alpha}\right)^{\sigma} \Big|_{\Lambda^{\sigma}} \equiv 0 \}$$

$$= \{ \alpha \mid \tfrac{N}{2} R_{\alpha} \Big|_{\Lambda} \equiv 0 \} = [L':L]$$

Lemma 2.4.5.

The map $Gal(\mathbf{C}/K_V) \to$ Ideal class group $(\tilde{k})$ is an homomorphism of groups.

Proof. Let σ,δ be elements of $Gal(\mathbf{C}/K_V)$.

Consider the following ideal classes:

$$a(\sigma) = [L^{(\sigma)}:L]$$

$$a(\delta) = [L^{(\delta)}:L]$$

$$a(\delta\sigma) = [L^{(\sigma\delta)}:L]$$

By definition

$$L^{(\sigma\delta)} = [L^{(\sigma\delta)}:L^{(\delta)}]^{-1} [L^{(\sigma)}:L]^{-1} L$$

and $$L^{(\sigma\delta)} = [L^{(\sigma\delta)}:L]^{-1} L.$$

By the uniqueness of the associated ideal:

$$[L^{(\sigma\delta)}:L] = [L^{(\sigma\delta)}:L^{(\delta)}][L^{(\sigma)}:L]$$

and we just prove that $[L^{(\sigma\delta)}:L^{(\delta)} \equiv [L^{(\delta)}:L]$ as ideal classes. So $a(\sigma\delta) = a(\delta) \cdot a(\sigma)$. ∎

2.5 $1 \to \mathrm{Gal}(K_{A,0}/K_V) \to$ Ideal class group $(\tilde{k})$

Let $A \xrightarrow{f} V$ be a rigid Kuga fiber variety. In Section 2.2. we proved that the bottom field of A is an extension of the bottom field of V. Our discussion of $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ in the previous section suggests that the definition of bottom field of A must be modified to give an interpretation of the extension.

In fact if $V \cong V^\sigma$ then $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is a Kuga fiber variety defined by the same data as $A \xrightarrow{f} V$ and with the same endomorphisms ring of the generic fiber [same type]. We will therefore define the bottom field of a family with a given ring of endomorphisms $K_{A,0}$, and prove that $K_{A,0}$ is abelian over K_V.

Main Theorem 2.5.1. Let $A \to V$ be a Kuga fiber variety defined by: $(G,K,X,\Gamma,F,L,\beta,\rho,\tau)$.

We assume:

(i) B is a quaternion algebra over k, k a totally real number field, $B \neq M_2(k)$ and $G = \mathrm{Res}_{\tilde{k}/\mathbf{Q}}(SL_1(B))$.
$\tilde{k}$ is the smallest Galois extension of $\mathbf{Q}$ containing k.

(ii) $\rho = \rho_P$ rigid, $\mathbf{Q}$ irreducible, $\mathbf{Q}$ primary polymer representation

(iii) $A \xrightarrow{f} V$ is of 0-type, 0 a maximal order in $\mathrm{End}_{\mathbf{Q}}(A_x)$. A_x is a generic fiber.

Then $1 \to \mathrm{Gal}(K_{A,0}/K_V) \to$ Ideal class group $(\tilde{k})$

We need the definition of $K_{A,0}$.

We showed that $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ is of the same 0-type as $A \xrightarrow{f} V$, naturally we can modify the definition of bottom field of A.

Definition 2.5.2. We call the subfield $K_{A,0}$ of **C** the bottom field of A respect to 0 if an automorphism σ of **C** is the identity mapping on $K_{A,0}$ if and only if $A^{\sigma} \xrightarrow{f^{\sigma}} V^{\sigma}$ is isomorphic to $A \xrightarrow{f} V$ as a family of 0-type.

Proof. Let $L \supset K_{A,0} \supset K_V$ be a Galois extension of **Q**. We already constructed the homomorphism:

$$\mathrm{Gal}(L/K_V) \to \text{Ideal class group } (\tilde{k})$$

What is yet to be determined is the kernel.

Let $\sigma \in \mathrm{Gal}\ (L/K_V)$ such that $a(\sigma) = 1$, i.e. $L^{\sigma} = (q)L$, $q \in k^{\times}$. But $L^{(\sigma)} = (q)L$ if and only if A and A^{σ} are isomorphic. So we have

$$1 \to H \to \mathrm{Gal}(L/K_V) \to \text{Ideal class group } (\tilde{k}).$$

and $L^H = K_{A,0}$.

Chapter 3.

In this chapter we will construct a Kuga fiber variety of quaternion type, $A \xrightarrow{f} V$, with bottom field of A strictly bigger then the bottom field of V.

In particular we consider a totally real number field k of class number 2. Any rigid Kuga fiber variety, $A \xrightarrow{f} V$, constructed from a quaternion algebra over k has $[K_A:K_V] \leq 2$. We will show that with a suitable choice of the data we have $[K_A:K_V] = 2$. To do so we need to state several results of Complex Multiplication Theory. In fact a rigid Kuga fiber variety contains fibers of CM type (Section 1.3.) and the action of an automorphism of the complex numbers on such varieties is determined by the main theorem of Complex Multiplication.

The universal reference for this chapter is Shimura-Taniyama "Complex multiplication of Abelian varieties and its application to number theory".

3.1 The Endomorphism Ring of an Abelian variety

Let A be an Abelian variety defined over the complex numbers. We denote Hom(A) the set of all homomorphism of A into A. Hom(A) is a finitely generated free **Z**-module, we put $\mathrm{Hom}_{\mathbf{Q}}(A) = \mathrm{Hom}(A) \otimes \mathbf{Q}$.

Theorem 3.1.1. Let A be an Abelian variety of dimension n and F be a subfield of $Hom_{\mathbf{Q}}(A)$. Then we have:

(i) $[F:\mathbf{Q}]$ divides 2n

Suppose $[F:\mathbf{Q}] = 2n$

(ii) F is totally imaginary

(iii) The commutant of F in $Hom_{\mathbf{Q}}(A)$ is equal to F.

(iv) A is isogeneous to a product $B\times\cdots\times B$ where B is a simple Abelian variety.

(v) For every $\alpha \in F$, we have:

$$\nu(\alpha) = N_{F/\mathbf{Q}}(\alpha) \quad , \quad tr(\alpha) = Tr_{F/\mathbf{Q}}(\alpha)$$

Theorem 3.1.2. Let B a simple Abelian variety defined over **C**. Let K be the center of $Hom_{\mathbf{Q}}(B)$

(i) $K = Hom_{\mathbf{Q}}(B)$

(ii) $[K:\mathbf{Q}] = 2 \dim (B)$

(iii) K is totally imaginary

Let R be an algebra over **Q**, with identity element 1.

Definition 3.1.3. An Abelian variety of type (R) is a pair (A,i) formed by an Abelian variety A and an isomorphism i of R into $Hom_{\mathbf{Q}}(A)$

3.2 CM type

Let F be an algebraic number field of degree 2n. Let $\phi_1, \cdots, \phi_n$ be n distinct isomorphisms of F into **C**, we say that (F, $\{\phi_i\}$) is of CM type if: F contains to subfields K and K_0 satisfying the following conditions:

(CM1) K_0 is totally real and K is a totally imaginary quadratic extension of K_0.

(CM2) There are no two isomorphisms among the ϕ_i which are complex conjugate to each other on K.

Theorem 3.2.1. In order that (F,$\{\phi_i\}$) be a CM type, it is necessary and sufficient that there exists an Abelian variety of dimension n of type (F,$\{\phi_i\}$).

Theorem 3.2.2. Let A be an Abelian variety of type (F,$\{\phi_i\}$), $A \sim B\times\cdots\times B$. Let K be the subfield of F defined above, then: $\mathrm{Hom}_{\mathbf{Q}}(B) = K$.

3.3 The Reflex of a CM type.

Proposition 3.3.1. Let F be an extension of **Q** of degree 2n and $\{\phi_1, \cdots, \phi_n\}$ be a set of n distinct isomorphisms of F into **C**. (F,$\{\phi_i\}$) of CM type.

Let E be a Galois extension of **Q** containing F and Ω be the Galois group of E over **Q**. Denote by ρ the element of Ω such that ξ^ρ is the complex conjugate of ξ for every $\xi \in E$, and by S the set of all elements of Ω inducing some ϕ_i on F.

Put: $S^* = \{\sigma^{-1} | \sigma \in S\}$, $H^* = \{\gamma | \gamma \in \Omega, \gamma S^* = S^*\}$.

Let K^* be the subfield of E corresponding to H^* and $\{\psi_i\}$ the set of all the isomorphisms of K^* into **C** obtained from the elements of S^*. Then $(K^*,\{\psi_i\})$ is a CM type and we have

$$K^* = \mathbf{Q}(\Sigma_i \xi^{\psi_i} | \xi \in E) \quad .$$

$(K^*,\{\psi_i\})$ is determined only by $(F,\{\phi_i\})$ and independent on the choice of E.

Corollary 3.3.2. The Abelian varieties of $(K^*,\{\psi_i\})$ type are simple.

We call the CM type $(K^*,\{\psi_i\})$ of the above proposition <u>the reflex</u> of $(F,\{\phi_i\})$

For every CM type $(F,\{\phi_i\})$ we can find two subfields K and K_0 of F satisfying the conditions CM1 and CM2. Let $\{\tilde{\phi}_i\}$ be the set of distinct isomorphisms of K into **C** induced by the ϕ_i. Then it is easy to see that $(F,\{\phi_i\})$ and $(K,\{\tilde{\phi}_i\})$ have the same reflex.

Proposition 3.3.3. Let $(F,\{\phi_i\})$ be a CM type and $(K^*,\{\psi_i\})$ the reflex of $(F,\{\phi_i\})$. Let α be an element of K^*; put $\beta = \Pi\, \alpha^{\psi_i}$.

Then β is an element of F and we have $\beta\beta^{\rho} = N_{K^*/\mathbf{Q}}(\alpha)$. Set $\underline{a}$ be an ideal of K^*.

Put $\underline{b} = \Pi\, \underline{a}^{\psi_i}$, then $\underline{b}$ is an ideal of F; and we have

$\underline{b}\ \underline{b}^{\rho} = N_{K^*/\mathbf{Q}}(\underline{a})$.

3.4 The Main Theorem of Complex Multiplication.

Let $(K,\{\phi_i\})$ be a field of CM type, $[K:\mathbf{Q}] = 2n$, and $(K^*,\{\psi_i\})$ be the reflex of $(K,\{\phi_i\})$.

Let (A,i) be an Abelian variety of type $(K,\{\phi_i\})$. Let k_0 be the field of moduli of (A,) where is a polarization of A.

Put $k_0{}^* = K^* \cdot k_0$, $k_0{}^*$ is an abelian extension of K^*.

We assume Hom(A) = 0_K, 0_K ring of integers in K. Then there exists $\underline{a}$ an ideal of K such that A is isomorphic to the complex torus $\mathbf{C}^n/D(\underline{a})$. We say that (A,i) is of type $(K,\{\phi_i\}, \underline{a})$.

Let σ be an automorphism of **C** such that $\sigma \equiv$ id on K^*. Then (A^{σ},i^{σ}) is again of $(K,\{\phi_i\})$ type and isomorphic to the complex torus $\mathbf{C}^n/D(\underline{b})$ for some $\underline{b}$ ideal of K.

Theorem 3.4.1. Let (A,i) be an Abelian variety of type $(K,\{\phi_i\},\underline{a})$ as above. Let $\sigma \in \mathrm{Aut}(\mathbf{C}/K^*)$ and let $\underline{s}$ an ideal of K^* such that $\sigma = (\underline{s},K^*)$ on K^{ab}.

Then (A^{σ},i^{σ}) is of type $(K,\{\phi_i\}, N_{\Psi}(\underline{s}^{-1})\ \underline{a})$, where

$N_{\Psi}(\underline{s}^{-1}) = \prod_i (\underline{s}^{-1})^{\psi_i}$.

Theorem 3.4.2. Let (A,i) be an Abelian variety of type $(K,\{\phi_i\})$ as above. Let H_0 be the group of all ideals $\underline{a}$ of K^* such that there exists an element $\mu \in K$ for which we have

$\prod_i \underline{a}^{\psi_i} = (\mu)$, $N(\underline{a}) = \mu\,\bar{\mu}$.

Let k_0 be the field of moduli of (A,). Then H_0 is an ideal group of K^* and $k_0{}^*$ is the unramified class field over K^* corresponding to the ideal group H_0.

3.5 The Example

Let $A \xrightarrow{f} V$ be a Kuga fiber variety defined by B a division quaternion algebra over k a totally real Galois number field.

If we assume the defining representation rigid and irreducible over **Q**, we proved the following:

(i) The bottom field of A, K_A, is an abelian Galois extension of the bottom field of V, K_V.

(ii) If $\sigma \in Gal(K_A/K_V)$, then $A^\sigma \xrightarrow{f^\sigma} V^\sigma$ is defined by the same data as $A \xrightarrow{f} V$ except the lattice $L^{(\sigma)}$. In fact $L^{(\sigma)} = \underline{a}(\sigma)L$, $\underline{a}(\sigma)$ an ideal in k and we have:

$$1 \to Gal(K_A/K_V) \to \text{Ideal class group } (k)$$

If k has class number 1, then $K_A = K_V$. If k has class number bigger than 1 then be can apply the theory of complex multiplication to determine $\underline{a}(\sigma)$.

Let $A \xrightarrow{f} V$ be defined by the data:

$(G = Res_{k/\mathbf{Q}}\ SL_1(B), K, X, \Gamma, F, L, \beta, \rho, \tau)$

We make the following assumptions:

1. k is Galois over $\mathbf{Q}$, $\Omega = Gal(k/\mathbf{Q})$.

2. $\rho = \rho_P$; $P = \Sigma gM$, P rigid, $gM \neq \gamma M$ if $g \neq \gamma$.

3. $B^M = M_N(k)$, $F = Res_{k/\mathbf{Q}} F_M$.

4. $A \xrightarrow{f} V$ is of type O_k, O_k ring of integers in k.

5. Let A_λ be the fiber with complex multiplication constructed in Section 1.4 of type $(K, \{\phi_i\})$. Then we assume $Hom(A_\lambda) = O_K$, O_K the ring of integers of K.

A_λ as a complex torus is isomorphic to $(F/L, J_\lambda = \tau(\lambda))$, and if $\sigma \in Aut(\mathbf{C}/K_V)$, then $(A_\lambda)^\sigma = A^\sigma_{\lambda^\sigma}$ is isomorphic to $(F/L^{(\sigma)}, J_{\lambda^\sigma} = \tau(\lambda^\sigma))$.

From assumption 4 it follows that A_λ is of type $(K, \{\phi_i\}, \underline{a})$ where $\underline{a}$ is an O_K ideal. Let $(K^*, \{\psi_i\})$ be the reflex of K and σ an element of Aut $(\mathbf{C}/K^*)$. If $\sigma = (\underline{s}, K^*)$ on K^{ab} then $(A_\lambda)^\sigma$ is of type $(K, \{\phi_i\}, N_\Psi(\underline{s}^{-1})\underline{a})$. Moreover if we assume $\sigma \in Aut(\mathbf{C}/K^* K_V)$, we have:

$$N_\Psi(\underline{s}^{-1}) = \{x \in K \mid xL \subset L^{(\sigma)}\}.$$

On the other hand:

$$[L:L^{(\sigma)}] = \{x \in End_k F_M \mid xL \subset L^{(\sigma)}\}.$$

So we can conclude:

$$[L:L^{(\sigma)}] \cap k = N_\Psi(\underline{s}^{-1}) \cap k = \underline{a}(\sigma)$$

and we proved the following lemma:

Lemma 3.5.1 If σ is an element of $Aut(\mathbf{C}/K^*K_V)$ and $\sigma = (\underline{s},K^*)$ on K^{ab}, then:

$$[N_\Psi(\underline{s}^{-1})] \cap k = \underline{a}(\sigma).$$

We want to use this result to show that $\underline{a}(\sigma)$ may not be principal.

As a first step we need to analyze the CM type $(K,\{\phi_i\})$. K was constructed in Section 1.4 as:

$$K = L^{\alpha_1} \ldots . L^{\alpha_r} \cong L^{\alpha_1} \otimes \ldots \otimes L^{\alpha_r}.$$

$L = k(\sqrt{-\zeta})$ is a totally imaginary extension of k, contained in B and ζ is totally positive.

K is of CM type together with a set of embeddings into $\mathbf{C}$: $\phi_1,\ldots,\phi_n$. The analytic representation of K is equivalent to the direct sum of the ϕ_i. So to determine $\phi_1,\ldots,\phi_n$ we need to discuss the embeddings of K into $\mathbf{C}$ and the analytic representation.

Let $\Omega = Gal(k/\mathbf{Q}) = \{\alpha_1,\ldots,\alpha_g\}$, then $K^\Omega = L^{\alpha_1} \ldots . L^{\alpha_g}$ is the smallest extension of $\mathbf{Q}$ containing K. We can assume, by Proposition 1.4.2, that $[K^\Omega : \mathbf{Q}] = 2^g \cdot g$.

Lemma 3.5.2 The Galois group of K^Ω over $\mathbf{Q}$ is isomorphic to $\Omega \times \mathbf{Z}_2{}^\Omega$, where the product is defined in the following way:

$\psi, \phi \in \Omega \times \mathbf{Z}_2^{\Omega}$; $\phi = (\alpha,(\sigma_i))$, $\psi = (\gamma,(\delta_i))$

$\phi \cdot \psi = (\alpha\gamma,(\sigma_i \cdot \delta_{i\alpha}))$

Proof. Let ϕ be an element of $\mathrm{Gal}(K^{\Omega}/\mathbf{Q})$

Then ϕ restricted to k coincides with an element of Ω that we call α. For any $i \in \Omega$, $\phi: L^i \to L^{i\alpha}$ is an extension of α to L^i. So either $\phi(\sqrt{-\zeta^i}) = \sqrt{-\zeta^{i\alpha}}$ or $\phi(\sqrt{-\zeta^i}) = -\sqrt{-\zeta^{i\alpha}}$.

Put $\iota : \mathrm{Gal}(K^{\Omega}/\mathbf{Q}) \to \Omega \times \mathbf{Z}_2^{\Omega}$, $\iota(\phi) = (\alpha,(\sigma_i))$, where $\sigma_i = 0$ if $\phi(\sqrt{-\zeta^i}) = \sqrt{-\zeta^{i\alpha}}$ and $\sigma_i = 1$ if $\phi(\sqrt{-\zeta^i}) = -\sqrt{-\zeta^{i\alpha}}$.

ι is a bijection and it is easy to see that it is also an homomorphism of groups. ∎

Corollary 3.5.2 The embeddings of K into **C** are the restrictions of the elements of $\Omega \times \mathbf{Z}_2{}^{\Omega}$ to K.

Lemma 3.5.3 In the above notations. A_λ is of type $(K,\{\phi_{\alpha,S_0}\})$. Where $\phi_{\alpha,S_0}: K \to \mathbf{C}$ is defined in the following way:

$$\phi_{\alpha,S_0} = (\alpha,(\sigma_i)) \ , \ \sigma_{M \cap \alpha^{-1}S_0} = 0$$

Proof. A_λ as complex torus is isomorphic to $(F/L \ , \ J_\lambda = \tau(\lambda))$. J_λ is defined in the following way. Let g be an element of $G_{\mathbf{R}}$ such that $\lambda = \rho(g)\ (\sqrt{-1},\ldots\ldots,\sqrt{-1})$ and J the element of $G_{\mathbf{R}}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{|S_0|} \times I^{|S_1|}$.

Then $J_\lambda = \rho(g)\rho(J)\rho(g)^{-1}$. In other words:

$F = \oplus F_{M^i}$ and $J_\lambda = \oplus J_{\lambda,i}$, where

$J_{\lambda,i} = \rho_{M^i}(g)\ I \times \ldots \times \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \times \ldots \times I\ \rho_{M^i}(g)^{-1}$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ occurs at the place $gM \cap S_0$.

The rational representation of K is just $\rho\colon K \to \mathrm{End}_{\mathbf{Q}}(F)$ and $\rho = \oplus \rho_{M^i}$. Now $\rho_{M^i} : K \to \mathrm{End}_{\mathbf{C}}(F_{M^i})$ is equivalent to $\oplus\phi_i$ where the ϕ_i are all the extension of $i \in \Omega$ to K. So by the construction of J_{λ,M^i} we obtain that the analytic representation of K is equivalent to $\oplus\phi_{i,S_0}$. ■

A field of CM type

Let k be $\mathbf{Q}(\sqrt{10})$ and $\Omega = \mathrm{Gal}\ (k/\mathbf{Q}) = \{id,\gamma\}$.

k has class number 2 and its ideal classes are (1), $(2,\sqrt{10})$.

Let ζ be in k, ζ totally positive, for instance we will consider $\zeta = \dfrac{4-\sqrt{10}}{2}$.

Put $K = k(\sqrt{-\zeta})$ and $K^\gamma = k(\sqrt{-\zeta^\gamma})$. K and K^γ are linearly disjoint.

The smallest Galois extension of $\mathbf{Q}$ containing K is $E = K\cdot K^\gamma = k(\sqrt{-\zeta}, \sqrt{-\zeta^\gamma})$, $[E:\mathbf{Q}] = 8$. As we have seen in Lemma 3.5.2. $\mathrm{Gal}(L/\mathbf{Q}) = \Omega \times \mathbf{Z}_2{}^{\Omega}$.

K is of CM type with respect to $\{\phi_1,\phi_2\}$ where $\phi_1 = id$ and $\phi_2\colon k \to k$, $\phi_2 = \gamma$ on k and $\phi_2(\sqrt{-\zeta}) = (\sqrt{-\zeta^\gamma})$. L is also of CM type respect to $\{\phi_i\}$ where:

$\phi_1 = (id,0,0)$, $\phi_2 = (id,0,1)$, $\phi_3 = (\gamma,0,0)$, $\phi_4 = (\gamma,0,1)$.

We want to compute the reflex of E. Note that K and k are the fields satisfying condition CM1 and CM2. Therefore the reflex of E coincide with the reflex of K.

For E, in the notation of proposition 3.3.1, we have:

$$S = \{\phi_i\} \quad , \quad S^* = \{\sigma^{-1} \mid \sigma \in S\} = S$$

$$H^* = \{\gamma \mid \gamma \in \mathrm{Gal}\,(E/\mathbf{Q}) \quad \gamma S^* = S^*\} = \Omega \times \{0\}$$

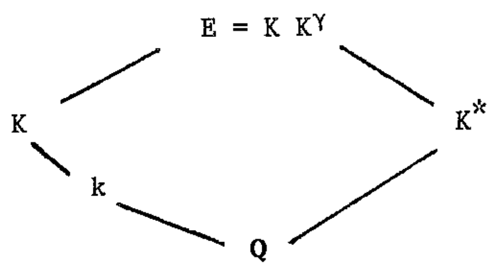
$$K^* = E^{H^*} = \mathbf{Q}\,(\sqrt{\zeta} + \sqrt{\zeta}^{\gamma}) \quad , \quad [K^*:\mathbf{Q}] = 4$$

K^* is a CM type respect to $\{\psi_1,\psi_2\}$:

$$\psi_1\,(\sqrt{\zeta} + \sqrt{\zeta}^{\gamma}) = \sqrt{\zeta} + \sqrt{\zeta}^{\gamma}$$

$$\psi_2\,(\sqrt{\zeta} + \sqrt{\zeta}^{\gamma}) = \sqrt{\zeta} - \sqrt{\zeta}^{\gamma} .$$

The diagram is as follows:



Let $\underline{a}$ be an ideal of K^* then $N_\Psi(\underline{a}) = \underline{a}^{\psi_1}\cdot\underline{a}^{\psi_2}$ is an ideal of K. In particular any ideal class of k contains an ideal of the form $N_\Psi(\underline{a}) \cap k$ for some $\underline{a}$ in K^*.

Since there are only two ideal classes in k: (1) and $(2,\sqrt{10})$ we need to verify that there exists $\underline{a}$ such that $N_\Psi(\underline{a}) = (2,\sqrt{10})$

Put $\underline{a} = (2, \sqrt{\zeta} + \sqrt{\zeta}^{\gamma})$, then:

$N_\Psi(\underline{a}) = \underline{a}^{\psi_1}\ \underline{a}^{\psi_2} = (2,\sqrt{10},\ 4\sqrt{\zeta}) = (2,\sqrt{10})$.

A Kuga fiber variety

Let k be $\mathbf{Q}(\sqrt{10})$ and $\Omega = \mathrm{Gal}(k/\mathbf{Q}) = \{id,\gamma\}$.

Let B be a quaternion algebra containing $K = k(\sqrt{-\zeta})$, $\zeta = \frac{4-\sqrt{10}}{2}$.

For instance we consider. $B = \left(\frac{-\zeta,1}{k}\right) \cong \left(\frac{1,\zeta}{k}\right)$

so that we have: $B \supset K$, $B \supset K^\gamma$ and

$$B \underset{\mathbf{Q}}{\otimes} \mathbf{R} = M_2(\mathbf{R}) \times M_2(\mathbf{R})$$

In the notation of Section 1.2.: $S_0 = S = \Omega$.

Put $G = \mathrm{Res}_{k/\mathbf{Q}}\, SL_1(B)$ and $\rho = \rho_P$, $P = \{id\} + \{\gamma\}$.

Note that ρ is the only rigid polymer representation for the chemistry (Ω,S,S_0). On the other hand to have a representation defined over $\mathbf{Q}$ we need to increase the multiplicity and consider $\rho_{2P} = \rho_P \oplus \rho_P$

We want to construct a family of Abelian varieties, $A \xrightarrow{f} V$, associated to ρ_{2P}. We need many ingredients.

1. The space of parameters V. Let 0 be a maximal order of B.

   Put $\Gamma = \mathrm{Res}_{k/\mathbf{Q}} 0^1$. Then Γ is an arithmetic subgroup of G.

   Γ defines a Hilbert modular variety: $V = \Gamma\backslash\mathbf{H}^2$.

2. The vector space. In our case $P = \{id\} + \{\gamma\}$ is the simple polymer generated by $\{id\}$. So $F = Res_{k/\mathbf{Q}}F_1$.
   F_1 minimal left ideal in B, therefore B itself.

3. The lattice. Let $\tilde{0}$ be a maximal order of $B \otimes B$ containing 0 and 0_E the ring of integers of E. Let L be a lattice in F_1, $\tilde{0}$ variant.

So we can define:

$$\begin{array}{l} A = \Gamma \times L\backslash X \times F_{\mathbf{R}} \\ \downarrow f \\ V = \Gamma\backslash \mathbf{H}^2 \end{array}$$

$A \xrightarrow{f} V$ is a family of Abelian varieties, Sec. 1.2.

Since $\Gamma = 0^1$, 0 a maximal order, $V = \Gamma\backslash\mathbf{H}^2$ has a projective model defined over $\mathbf{Q}$, Shimura [Sh3]. Therefore the bottom field of V is $\mathbf{Q}$ and K_A is an Abelian extension of $\mathbf{Q}$.

We want to show that $[K_A:K_V = \mathbf{Q}] = 2$

Let A_λ be the fiber of $A \xrightarrow{f} V$, of type $(E,\{\phi_i\})$ constructed in Section 1.4.. A_λ is isomorphic to F/L as real torus.

In our case $B^M = B$ is not a trivial algebra, but it is easy to generalize Lemma 3.5.3 and show that $\{\phi_i\}$ is the set of embeddings of E we discussed previously.

Let K^* be the reflex of E. Let $\underline{a}$ be the ideal of K^*, $\underline{a} = (2, \sqrt{\zeta} + \sqrt{\zeta}^{\gamma})^{-1}$, and $\sigma \in Aut(\mathbf{C}/K^*)$ such that $\sigma = (\underline{a}, K^*)$ on K^{ab}.

We note then that σ is the identity map on K_V so $(A_\lambda)^\sigma = A^\sigma_{\lambda^\sigma}$ is isomorphic to $F/L^{(\sigma)}$.

By Lemma 3.5.1. $L^{(\sigma)} = \underline{a}(\sigma)L = (N_\Psi(\underline{a}^{-1}) \cap k)L$ and as computed previously:

$$N_\Psi(\underline{a}^{-1}) \cap k = (2, \sqrt{10})$$

So we can conclude:

$$1 \to Gal(K_A/K_V) \to \text{Ideal class group } (k) \to 1$$

and $[K_A:K_V] = 2$.

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