ZETA-FUNCTION OF SUBELLIPTIC DIFFERENTIAL OPERATORS

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Abstract of the Dissertation

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On a compact contact manifold of dimension $2n+1$ the complex powers of non-negative self-adjoint second order differential operators doubly characteristic on the contact line bundle are considered.

Via the symbolic calculus on the group $\mathbb{R}\times\mathbb{H}^n$ ($\mathbb{H}^n$ is a Heisenberg group), the asymptotic expansion for the trace of the heat Kernel has been obtained. This allows us to get the analytic continuation for the zeta-function to the whole complex plane excluding the finite number of points $z_j=-(n+1)+j$, $j=0,\ldots,n,$ at which the zeta-function has simple poles.

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INTRODUCTION

Suppose M is a compact manifold of dimension $2n+1$ with a contact structure. It is defined by a line bundle $\Lambda \subset T^*M$ of codimension $2n$ which is symplectic in $T^*M\setminus 0$ or the symplectic form on $T^*M\setminus 0$ is nondegenerate acting on tangent vectors to Λ. We study the complex powers of self-adjoint non-negative differential operators of second order on M doubly characteristic on Λ.

Another characterization of contact structure is as follows: if λ is a local section of Λ, then $\lambda\wedge d\lambda\wedge\ldots\wedge d\lambda \neq 0$ where there are n factors of $d\lambda$. A choice of such λ provides M with a local volume form $\lambda\wedge d\lambda^n$. Two 1- forms associated with the same contact structure differ by a smooth nonvanishing multiple. It follows from the Darboux's theorem that any two contact manifolds of the same dimension are locally diffeomorphic via a map preserving the contact structure.

A CR- manifold with non-degenerate Levi form is an example of a contact manifold. A CR- structure is given by a complex n-dimensional subbundle $T_{1,0} \subset CTM$ satisfying $T_{1,0} \cap \overline{T}_{1,0} = \{0\}$ and assumed to be integrable (i.e., the Lie bracket $[T_{1,0}, T_{1,0}] \subset T_{1,0}$).

The Levi form is given by

$$\langle v, u \rangle_\lambda = -i\,d\lambda(v \wedge \bar{u}), \quad v, u \in T_{1,0}.$$

The nondegeneracy of the Levi form is equivalent to the condition $\lambda \wedge d\lambda^n \neq 0$. Dual to the Levi form is the norm $|\omega|_\lambda$ on real 1-forms ω given by

$$|\omega|_\lambda^2 = \langle \omega, \omega \rangle_\lambda = \sum_{j=1}^{n} |\,\omega(Z_j)|^2$$

where $(Z_1, \ldots, Z_n)$ is an orthonormal basis for $T_{1,0}$ with respect to the Levi form. Since $|\lambda|_\lambda = 0$ the norm $|\omega|_\lambda$ is degenerate. It follows that the sublaplacian operator Δ_b defined on functions by

$$\int_M (\Delta_b u)\, v \lambda \wedge d\lambda^n = \int_M \langle du, dv \rangle_\lambda \, \lambda \wedge d\lambda^n,$$

$$v \in C_0^\infty(M),$$

is subelliptic.

Folland and Stein [1] introduced function spaces S_r^K on M analogous to the Sobolev spaces. For example, $f \in S_1^2(M)$ if

$$\|f\|^2 = \int_M (|df|_\lambda^2 + f^2)\, \lambda \wedge d\lambda^n < \infty .$$

In the survey [2] it was noted that the embedding theorems also follow from [1]: $S_1^2(M) \subset L_1^r(M)$ if $1/r \geq 1/2 - 1/2n+2$ and the inclusion is compact if $1/r > 1/2 - 1/2n+2$.

The Heisenberg group $\mathbb{H}^n$ can be used as a standard model for a contact manifold as an Euclidean space for a Riemannian manifold (see [1] and [3]). If a point in $\mathbb{H}^n$ is denoted by (t,q,p) the contact structure on $\mathbb{H}^n$ is the line bundle invariant by right translations, whose fiber over the identity on $\mathbb{H}^n$ is spanned by dt.

In Taylor's book [3] a symbolic calculus has been developed to study the classes of pseudodifferential subelliptic operators. The symbols of convolution operators on $\mathbb{H}^n$ are their images under the basic representation of the Heisenberg group which are operators in the Weyl functional calculus. Methods of [3] are extensively used in this work.

In Section 1.1 of Chapter I a symbolic calculus is introduced for the convolution operators on the group $R\times\mathbb{H}^n$. Based on that a parametrix for the heat equation on $R\times\mathbb{H}^n$ is obtained in Section 1.2. In Section 1.3 complex powers of the right invariant differential operators on the group $\mathbb{H}^n$ are studied. Note that the complex powers of right invariant operators on Lie groups were considered by Folland [4].

In Chapter II subelliptic differential operators on compact contact manifolds are investigated. In Section 2.1 a class of operators with variable coefficients is obtained from the class of convolution operators on the group $\mathbb{R}\times\mathbb{H}^n$ using methods of [3]. This allows further in Section 2.2 to get an asymptotic expansion for the theta-function in which the coefficients of the non-integer powers of the time parameter cancel out. Such expansion was obtained by Beals, Greener, and Stanton [5] using a different approach. Based on results of Section 2.2 the behavior of the zeta-function is studied in Section 2.3.

In the case of subelliptic differential operators of second order, the poles of the zeta-function occur only at integer points. This implies that the zeta-function has a finite number of poles on the complex z-plane, and there are no poles for $\mathrm{Re}\,z>0$, which would not be the case if the order of operators was other than two. The analogous behavior of the zeta-function of the special class of elliptic self-adjoint positive definite differential operators of second order on the compact Rimanannian manifold of an even dimension and without boundary follows from [6]. The zeta-function of the harmonic oscillator Hamiltonian is considered in the Appendix.

CHAPTER I. RIGHT INVARIANT OPERATORS

Section 1.1. Convolution Operators on the Group $R\times \mathbb{H}^n$.

We will consider convolution operators on the group $G=R\times\mathbb{H}^n$, where $\mathbb{H}^n$ is the Heisenberg group.

As a C^∞- manifold, G is R^{2n+2}. A point of R^{2n+2} and its dual will be denoted by

$$(t,z)=(t,s,q,p),\quad t\in R,\ s\in R,\ q\in R^n,\ p\in R^n;$$

and

$$(\sigma,\zeta)=(\sigma,\tau,y,\eta),\ \sigma\in R,\ \tau\in R,\ y\in R^n,\ \eta\in R^n;$$

respectively. The group law is

$$(t_1,s_1,q_1,p_1)\cdot(t_2,s_2,q_2,p_2)=(t_1+t_2,\ s_1+s_2+\tfrac{1}{2}p_1q_2-\tfrac{1}{2}p_2q_1,q_1+q_2,p_1+p_2).$$

The dilation is defined for $r\in R\setminus 0$ by

$$\begin{aligned} r\cdot(t,s,q,p)&=(r^2t,\ r^2s,rq,\ rp),\\ r\cdot(\sigma,\tau,y,\eta)&=(r^2\sigma,\ r^2\tau,\ ry,\ r\eta). \end{aligned} \tag{1}$$

Let $\| \ \|$ be a Euclidean norm on R^{2n}; a "homogeneous norm" is defined on G by

$$|(t,z)| = \left[|t| + |s| + \|(q,p)\|^2\right]^{1/2}.$$

For $\lambda \in (0,\infty)$ irreducible unitary representations of $\mathbb{H}^n$ on L^2R^n are

$$\pi_{\pm\lambda}(s,q,p) = e^{i(\pm\lambda s I \pm \lambda^{1/2} q\cdot X + \lambda^{1/2} p\cdot D)}.$$

The infinite dimensional irreducible unitary representations of the group G are given by

$$\pi_{\sigma,\pm\lambda}(t,z) = e^{i\sigma t}\,\pi_{\pm\lambda}(z), \quad \sigma \in R, \ \lambda \in (0,\infty).$$

For a representation $\pi_{\sigma,\pm\lambda}$ to a function u on G we associate

$$\pi_{\sigma,\pm\lambda}(u) = \int_G u(g)\,\pi_{\sigma,\pm\lambda}(g)\,dg.$$

For a compactly supported function (or distribution) k on G let $\hat{k}(\sigma,\tau,y,\eta)$ denote the Euclidean space Fourier transform.

We have

$$\mathcal{T}_{\sigma,\pm\lambda}(k) = \hat{k}(\sigma, \pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D) = \sigma_k(\sigma, \pm\lambda)(X, D),$$

where

$$\sigma_k(\sigma, \pm\lambda)(x, \xi) = \hat{k}(\sigma, \pm\lambda, \pm\lambda^{1/2}x, \lambda^{1/2}\xi) \qquad (2)$$

and the operator $a(X, D)$ is defined by the Weyl functional calculus:

$$a(X, D) = \int \tilde{a}(q, p) e^{i(q\cdot X + p\cdot D)} dq\, dp$$

($\tilde{a}(q,p)$ is the inverse Fourier transform of a).

Formula (2) implies that

$$\hat{k}(\sigma, \pm\tau, y, \eta) = \sigma_k(\sigma, \pm\tau)(\pm\tau^{-1/2}y, \tau^{-1/2}\eta), \ \tau > 0.$$

Definition 1. The class $\Psi_0^m(G)$ ($\Psi_+^m(G)$) consists of functions $\hat{k}(\sigma, \tau, y, \eta)$, smooth except at O, and homogeneous of degree m with respect to the dilation (1), i.e.,

$$\hat{k}(r\cdot(\sigma, \zeta)) = r^m \hat{k}(\sigma, \zeta) \qquad (3)$$

for $r \in R\setminus 0$ $(r > 0)$.

If $\hat{k} \in \Psi_0^m(G)(\Psi_+^m(G))$, we say, that the convolution operator $K\,(Ku = K * u)$ belongs to the class $OP\Psi_0^m(G)(OP\Psi_+^m(G))$.

Let $S^m_{\rho\#}$ be the Frechet space with the seminorm:

$$[p]_{\alpha,m,\rho} = \sup_x \{[1+|x|^2]^{1/2}\}^{-m+\rho|\alpha|} |D_x^\alpha p(x)|.$$

Neglecting the singularity at the origin, the elements of $\Psi_0^m(G)$ belong to $S^m_{1/2\#}$ if $m \geq 0$ and to $S^{m/2}_{1/2\#}$ if $m < 0$.

Note that (3) is equivalent to

$$\sigma_K(r\sigma, \pm r\tau)(x,\xi) = r^m \sigma_K(\sigma, \pm\tau)(x,\xi).$$

In order to characterize the class $\Psi_0^m(G)$, we will consider an auxiliary classes of functions on R^{2n+1}.

Let Σ be a union of rays through the origin in the complex plane.

Definition 2. We say $a(\sigma,x,\xi) \in S^m_{1,\Sigma}$, m real, $(x,\xi) \in R^{2n}$, $\sigma \in \Sigma$, if $a(\sigma,x,\xi) \in C^\infty(R^{2n})$ for each fixed σ and for each multi-index α there is a constant C_α such that

$$|D^\alpha_{x,\xi}\, a(\sigma,x,\xi)| \leq C_\alpha (1+|x|+|\xi|+|\sigma|^{1/2})^{m-|\alpha|} \quad (4)$$

As usual $S_{\Sigma}^{-\infty} = \cap S_{1,\Sigma}^{m}$. For $a(\sigma, x, \xi) \in S_{1,\Sigma}^{m}$, the family of operators $a(\sigma, X, D)$ is defined by the Weyl functional calculus:

$$a(\sigma, X, D)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(\sigma, \tfrac{1}{2}(x+y), \xi) u(y)\, dy\, d\xi.$$

If $a(\sigma, x, \xi) \in S_{1,\Sigma}^{m}$, we say, that the operator $a(\sigma, X, D)$ belongs to the class $OPS_{1,\Sigma}^{m}$.

The classes $OPS_{1,0}^{m}$ were considered by A.Voros [7], Grossman, Loupias and Stein [8], and, in a much more general case by Hormander [9]. The symbolic calculus can be extended from classes $OPS_{1,0}^{m}$ to $OPS_{1,\Sigma}^{m}$. For example, the multiplication law is written as follows. If $a(\sigma, X, D) \in OPS_{1,\Sigma}^{m}$, $b(\sigma, X, D) \in OPS_{1,\Sigma}^{\mu}$, then

$$C(\sigma, X, D) = a(\sigma, X, D)\, b(\sigma, X, D) \in OPS_{1,\Sigma}^{m+\mu} \tag{5}$$

and

$$c(\sigma, x, \xi) \sim \sum_{j \geq 0} (1/j!) \{a, b\}_j (\sigma, x, \xi), \tag{6}$$

where

$$\{a, b\}_0 (\sigma, x, \xi) = a(\sigma, x, \xi)\, b(\sigma, x, \xi),$$

$$j \geq 1,\ \{a, b\}_j (\sigma, x, \xi) = \tag{7}$$

$$= (-i/2)^j \sum_{k=1}^{n} \left(\partial^2/\partial y_k \partial \xi_k - \partial^2/\partial x_k \partial \eta_k \right)^j a(\sigma, x, \xi)\, b(\sigma, y, \eta) \Big|_{\substack{y=x \\ \eta=\xi}}.$$

The meaning of (6) is that the difference between $c(\sigma, x, \xi)$ and the sum of the right side of (6) over $0 \le j \le N$ belongs to $S^{m+\mu-2N}_{1,\Sigma}$.

Following Voros [7], we introduce a class of comparison operators: powers of the harmonic oscillator. Let $h(x,\xi) = |x|^2 + |\xi|^2$, $H = h(X, D)$; H_K be the operator $(I+H)^K \in OPS^{2K}_{1,0}$ for each integer K. For each integer K, let W_K be the Hilbert space obtained by completion of the domain of H_K in $L^2(R^n)$ for the inner product

$$(u, v)_K = (H_K u, H_K v).$$

We obtain the sequence of the spaces

$$\ldots \subset W_K \subset \ldots \subset W_1 \subset W_0 \subset W_{-1} \subset \ldots \subset W_{-K} \subset \ldots$$

for $K > 0$, W_K and W_{-K} are dual of each other for the inner product of W_0. Also, $S(R^n) = \bigcap_K W_K$ and its topology is given by the directed family of seminorms $\| \ \|_K$, and $S'(R^n) = \bigcup_K W_K$ [10].

It was shown by Voros [7] that if $a(X,D) \in OPS^m_{1,0}$, then for any integers k, ℓ with $\ell \geq m/2$, $a(X,D)$ is a continuous operator $W_K \to W_{K-\ell}$. Let $\|a(\sigma, X, D)\|_{K,K-\ell}$ be the norm of the operator

$$a(\sigma, X, D) : W_K \to W_{K-\ell}.$$

To determine how $\|a(\sigma, X, D)\|_{K,K-\ell}$ depends on σ for $|\sigma|$ sufficiently large, we use the Calderon-Vaillancourt theorem, which states the following:

Calderon-Vaillancourt theorem.

If $a(X,\xi)$ satisfies the estimate $|D^{\alpha}_{X,\xi}a(X,\xi)| \leq A$ for $|\alpha| \leq K(n)$, then

$$\|a(X,D)\| \leq C(n)A .$$

The Calderon-Vaillancourt theorem and Definition 2 imply that if $m \geqslant 0$, then any operator $a(\sigma,X,D) \in OPS^{-m}_{1,0}$ is bounded as an operator from $L^2(R^n)$ to $L^2(R^n)$, and

$$\|a(\sigma,X,D)\| \leqslant C(1+|\sigma|^{1/2})^{-m}. \tag{8}$$

Estimate (8) yields the more general

Proposition 1. An operator $a(\sigma,X,D) \in OPS^{m}_{1,\Sigma}$ is bounded as an operator from W_K to $W_{K-\ell}$ for k,ℓ integers, $\ell \geqslant m/2$, and

$$\|a(\sigma,X,D)\|_{K,K-\ell} \leqslant \begin{cases} C_{K,\ell}\,(1+|\sigma|)^{m/2-\ell}, & \ell \leqslant 0, \\ C'_{K,\ell}(1+|\sigma|)^{m/2}, & \ell \geqslant 0, \end{cases} \tag{9}$$

$|\sigma|$ - is sufficiently large.

Proof.

Denote as $H_\ell(\sigma)$ the operator-function $(I+H+|\sigma|)^{\ell}, \sigma \in \Sigma$.

It suffices to show that $a(\sigma,X,D)$ is bounded on the domain of H_K in W_0 . Denote as $b(\sigma,X,D)$ the operator $H_{K-\ell}\,a(\sigma,X,D)H_{-K}$. For any vector u in the domain of H_K $a(\sigma,X,D)u =$ $= H_{\ell-K}\,b(\sigma,X,D)\,H_K u$. The operator H_K is an isometry of W_K into W_0 , operator $H_{\ell-K}$ is an isometry of W_0 into $W_{K-\ell}$. The operator $b(\sigma,X,D)$ belongs to the class $OPS^{m-2\ell}_{1,\Sigma}$, hence it is a bounded operator on W_0 .

In order to proof (9), we check it at first for the operator $H_{m/2}(\sigma)$. We have

$$\|H_{m/2}(\sigma)\|_{K,K-\ell} = \|H_{K-\ell}H_{m/2}(\sigma)H_{-K}\| ,$$

and

$$H_{K-\ell}H_{m/2}(\sigma)H_{-K} = H_{m/2}(\sigma)H_{\ell} .$$

The estimate

$$\|H_{m/2}(\sigma)H_{-\ell}\| \leq C\begin{cases}(1+|\sigma|)^{m/2}, & \ell \geq 0,\\ (1+|\sigma|)^{m/2-\ell}, & \ell \leq 0,\end{cases}$$

can be deduced from the formula (6), Calderon-Vaillancourt theorem, and the formula

$$\sup_{y\geq 0}\,(1+y+t)^{m}(1+y)^{-\ell} = \begin{cases}(1+t)^{m}, & \ell \geq 0 ,\\ C_{m,\ell}\,t^{m-\ell}, & \ell \leq 0,\ t \geq R_{m,\ell} .\end{cases}$$

Now, if $a(\sigma,X,D) \in OPS^{m}_{1,\Sigma}$, then

$$\|a(\sigma,X,D)\|_{K,K-\ell} = \| H_{m/2}(\sigma)(H_{-m/2}(\sigma)a(\sigma,X,D)\|_{K,K-\ell} \leq$$

$$\leq \| H_{m/2}(\sigma)\|_{K,K-\ell} \; \| H_{-m/2}(\sigma)\, a(\sigma,X,D)\|_{K,K} .$$

We have to show that

$$\| H_{-m/2}(\sigma)a(\sigma,X,D)\|_{K,K} \leq C, \tag{10}$$

where C does not depend on σ.

Again,

$$\| H_{-m/2}(\sigma)\, a(\sigma,X,D)\|_{K,K} = \| H_K H_{-m/2}(\sigma)\, a(\sigma,X,D)\, H_{-K} \| .$$

Denote as $h(\sigma,x,\xi)$ the symbol of the operator $H_{-m/2}(\sigma)a(\sigma,X,D)$.
The function $h(\sigma,x,\xi) \in S^{0}_{1,\Sigma}$, so $h(\sigma,x,\xi) \in S^{0}_{1,0}$
uniformly for σ or

$$\sup_{\substack{x,\xi,\sigma \\ \sigma\in\Sigma}} \left[D^{\alpha}_{x,\xi}\, h(\sigma,x,\xi)(1+|x|+|\xi|)^{|\alpha|} \right] \leq C .$$

and the same is true for the symbol of the operator

$$H_K H_{-m/2}(\sigma) a(\sigma, X, D) H_{-K} ,$$

which proves the estimate (10).

Definition 3. The class H^m consists of functions $a(\sigma, x, \xi)$, $\sigma \in R$, $(x,\xi) \in R^{2n}$, which are smooth on R^{2n+1} and satisfy the following condition:

$$a(\sigma, x, \xi) \sim \sum_{j \geq 0} \varphi_j(\sigma, x, \xi), \quad |\sigma| + |x|^2 + |\xi|^2 \to \infty, \tag{11}$$

where $\varphi_j(\sigma, x, \xi)$ is smooth off $(0,0,0)$ and satisfies the homogeneity condition:

$$\varphi_j(r^2\sigma, rx, r\xi) = r^{m-2j} \varphi_j(\sigma, x, \xi), \quad r > 0.$$

Definition 4. We say that the function $a(\sigma, x, \xi)$ belongs to the class H_0^m if $a(\sigma, x, \xi) \in H^m$ and

$$a(\sigma, -x, -\xi) = (-1)^m a(\sigma, x, \xi). \tag{12}$$

Definition 5. We say that the pair $a_{\pm}(\sigma,x,\xi)$ belongs to the class $H^m_{\pm}\,(H^m_{\pm,0})$ if both $a_+(\sigma,x,\xi)$ and $a_-(\sigma,x,\xi)$ belong to $H^m(H^m_0)$, and if their expantions are compatible in the following sense;

$$a_{\pm}(\sigma,x,\xi) \sim \sum_{j\geq 0} (\pm 1)^j \varphi_j(\sigma,\pm x,\xi). \tag{13}$$

Proposition 2. The function

$$\hat{K}(\sigma,\pm\tau,y,\eta) = \tau^{m/2} a_{\pm}(\tau^{-1}\sigma,\ \pm\tau^{-1/2}y,\ \tau^{-1/2}\eta),\quad \tau>0, \tag{14}$$

belongs to the class $\Psi^m_+(G)\ (\Psi^m_0(G))$ with $\sigma_K(\sigma,\pm 1)(x,\xi) = a_{\pm}(\sigma,x,\xi)$, if and only if $a_{\pm}(\sigma,x,\xi)$ belong to $H^m_{\pm}(H^m_{\pm,0})$.

Proof. It is needed to show that the function $\hat{K}(\sigma,\pm\tau,y,\eta)$ defined by (14) is smooth at $\tau=0$, $(\sigma,y,\eta)\neq 0$. From the formula (13) we have

$$a_{\pm}(\tau^{-1}\sigma,\ \pm\tau^{-1/2}y,\ \tau^{-1/2}\eta) \sim \sum_{j\geq 0} \tau^{-m/2}(\pm\tau)^j \varphi_j(\sigma,y,\eta), \tag{15}$$

as $\tau\to 0$, $|\sigma|+|y|^2+|\eta|^2=1$.

It follows from (15) that if $\tau\to 0$, $|\sigma|+|y|^2+|\eta|^2=1$,

$$\hat{K}(\sigma,\pm\tau,y,\eta) \sim \sum_{j\geq 0} (\pm\tau)^j \varphi_j(\sigma,y,\eta).$$

Assume that $\hat{k}(\sigma,\tau,y,\eta)$ belongs to the class $\Psi_o^m(G)$. By the homogeneity relation (3) we have

$$\hat{k}(\sigma,\pm\tau,y,\eta) = \tau^{m/2}\hat{k}(\tau^{-1}\sigma, \pm 1, \tau^{-1/2}y, \tau^{-1/2}\eta), \quad \tau>0. \tag{16}$$

Denote the function $\hat{k}(\sigma,\pm 1,y,\eta)$ as $a_{\pm}(\sigma,y,\eta)$. It is known that the function $\hat{k}(\sigma,\tau,y,\eta)$ is smooth at $\tau\to 0$, $|\sigma|+|y|^2+|\eta|^2=1$ or

$$\hat{k}(\sigma,\pm\tau,y,\eta)\sim\sum_{j\geq 0}(\pm\tau)^j\varphi_j(\sigma,y,\eta).$$

It follows that

$$a_{\pm}(\tau^{-1}\sigma,\pm\tau^{-1/2}y,\tau^{-1/2}\eta)\sim\sum_{j\geq 0}\tau^{-m/2}(\pm\tau)^j\varphi_j(\sigma,y,\eta)$$

or

$$a_{\pm}(r^2\sigma,ry,r\eta)\sim\sum_{j\geq 0}r^{m-2j}(\pm 1)^j\varphi_j(\sigma,\pm y,\eta), \tag{17}$$

$r\to+\infty$.

Note that

$$D_\tau^i D_\sigma^j D_{y,\eta}^\alpha\,\hat{k}(\sigma,\tau,y,\eta)\in\Psi_o^{m-2i-2j-|\alpha|}(G), \tag{18}$$

which follows from differentiation of the relation (3). In particular,

$$D^{\alpha}_{y,\eta}\hat{k}(\sigma,\tau,y,\eta)\in\Psi_0^{m-|\alpha|}(G),$$

Denote the function $D^{\alpha}_{y,\eta}a_{\pm}(\sigma,y,\eta)$ as $b^{\pm}_{\alpha}(\sigma,y,\eta)$.

From the formula (16) we have

$$D^{\alpha}_{y,\eta}\hat{k}(\sigma,\tau,y,\eta)=\tau^{\frac{m-|\alpha|}{2}}b^{\pm}_{\alpha}(\tau^{-1}\sigma,\pm\tau^{-1/2}y,\tau^{-1/2}\eta).$$

It can be shown similarly to (17) that

$$b^{\pm}_{\alpha}(r^2\sigma,ry,r\eta)\sim\sum_{j\geq 0}r^{m-2j-|\alpha|}(\pm 1)^j\varphi_{\alpha,j}(\sigma,y,\eta)$$

as $r\to+\infty$.

It is clear that the function $\hat{k}(\sigma,\pm\tau,y,\eta)$ defined by (14) is homogeneous with respect to dilation (3) for $r>0$. Assume that (13) is satisfied. If $r<0$ we have by (14) and (12)

$$\hat{k}(r^2\sigma,\pm r^2\tau,ry,r\eta)=$$

$$=|r|^m\tau^{m/2}a_{\pm}(\tau^{-1}\sigma,\pm(-)\tau^{-1/2}y,-\tau^{-1/2}\eta)=$$

$$=|r|^m(-1)^m\tau^{m/2}a_{\pm}(\tau^{-1}\sigma,\pm\tau^{-1/2}y,\tau^{-1/2}\eta)=$$

$$=r^m\hat{k}(\sigma,\pm\tau,y,\eta).$$

Note that if the functions $a_{\pm}(\sigma,x,\xi)$ belong to the $S(R^{2n+1})$ (Schwartz space of rapidly decreasing functions), then (14) defines an element of $OP\Psi^{m}_{+}(G)$.

If the functions $a_{\pm}(\sigma,x,\xi) \in H^{m}_{\pm}$ $(H^{m}_{\pm,o})$, we say that $a_{\pm}(\sigma,X,D) \in OPH^{m}_{\pm}$ $(OPH^{m}_{\pm,o})$.

Proposition 2 is similar to Proposition 2.2 (Chapter I) in [3]. This allows to consider products, adjoints, and hypoellipticity of convolution operators from the class $OP\Psi^{m}_{o}(G)$ in the manner it was performed in [3] for the similar class $OP\Psi^{m}_{o}(\mathbb{H}^{n})$.

Proposition 3. If $a_{\pm}(\sigma,X,D) \in OPH^{m}_{\pm}$ $(OPH^{m}_{\pm,o})$, $b_{\pm}(\sigma,X,D) \in OPH^{\mu}_{\pm}$ $(OPH^{\mu}_{\pm,o})$, then

$$a_{\pm}(\sigma,X,D)\, b_{\pm}(\sigma,X,D) = c_{\pm}(\sigma,X,D) \in OPH^{m+\mu}_{\pm}(OPH^{m+\mu}_{\pm,o}).$$

Proof. Assume that $a(\sigma,X,D) \in OPH^{m}$; $b(\sigma,X,D) \in OPH^{\mu}$. The class H^{m} is a subset of the class $S^{m}_{1,\Sigma}$, so the product $c(\sigma,X,D)$ belongs to the class $OPS^{m+\mu}_{1,\Sigma}$ and $c(\sigma,x,\xi)$ has the asymptotic expansion by the formulas (6) and (7).

It follows from formula (11) that if $a(\sigma,x,\xi) \in H^{m}(H^{m}_{o})$, $b(\sigma,x,\xi) \in H^{\mu}$ (H^{μ}_{o}), then $\{a,b\}_{j}(\sigma,x,\xi) \in H^{m+\mu-2j}(H^{m+\mu-2j}_{o})$ and $a(\sigma,X,D)\, b(\sigma,X,D) \in OPH^{m+\mu}(OPH^{m+\mu}_{o})$.

Now we have that $c_+(\sigma, X, D)$ and $c_-(\sigma, X, D)$ belong to $OPH^{m+\mu}$ $(OPH_0^{m+\mu})$ and

$$c_\pm(\sigma, x, \xi) \sim \sum_{j\geq 0} (1/j!) \{a_\pm, b_\pm\}_j (\sigma, x, \xi). \tag{19}$$

The $j^{\underline{th}}$ term of (19) belongs to the class $H_\pm^{m+\mu-2j}(H_{\pm,0}^{m+\mu-2j})$, so the series of (19) asymptotically sum to the element of $H_\pm^{m+\mu}$ $(H_{\pm,0}^{m+\mu})$. As a consequence of Proposition 3, we have

Proposition 4. If $K_1 \in OP\Psi_0^m(OP\Psi_+^m)$, $K_2 \in OP\Psi_0^\mu(OP\Psi_+^\mu)$, then $K_1K_2 \in OP\Psi_0^{m+\mu}(OP\Psi_+^{m+\mu})$, and

$$\sigma_{K_1K_2}(\sigma, \pm\lambda)(X, D) = \sigma_{K_1}(\sigma, \pm\lambda)(X, D)\sigma_{K_2}(\sigma, \pm\lambda)(X, D).$$

Assume that the operator K belongs to the class $OP\Psi_0^m(G)$ $(OP\Psi_+^m(G))$. It follows from Proposition 2 that the operators $\sigma_K(\sigma, \pm 1)(X, D) = a_\pm(\sigma, X, D)$ belong to the class $OPH_{\pm,0}^m$ $(OPH_\pm^m)$. It is known from the Weyl calculus that if $a(\sigma, X, D) \in OPS_{1,\Sigma,1}^m$ then $a(\sigma, X, D)^* = a^*(\sigma, X, D)$, and

$$a^*(\sigma, x, \xi) = \overline{a(\sigma, x, \xi)}.$$

It follows from the last formula that $a_\pm(\sigma, X, D)^* \in OPH_{\pm,0}^m$ $(OPH_\pm^m)$.

This implies

Proposition 5. If $K \in OP\Psi_0^m(G)(OP\Psi_+^m(G))$, then $K^* \in OP\Psi_0^m(G)$ $(OP\Psi_+^m(G))$, and

$$\sigma_{K^*}(\sigma,\pm\lambda)(X,D) = \sigma_K(\sigma,\pm\lambda)(X,D)^*.$$

Consider the case when the operators $\sigma_K(\sigma,\pm 1)(X,D)$ are elliptic.

Definition 6. We say, operator $a(\sigma,X,D) \in OPH^m$ is elliptic with parameter σ, if $\varphi_0(\sigma,x,\xi) \neq 0$ for $|\sigma| + |x|^2 + |\xi|^2 \neq 0$.

Let R_M be the set $\{\sigma \in R; |\sigma| \geq M\}$.

Proposition 6. If the operator $a(\sigma,X,D) \in OPH^m$ and it is elliptic with parameter σ, then there exists an operator $a(\sigma,X,D)^{-1}$ $\in OPS^m_{1,R_M}$ for some $M > 0$.

Proof. At first we show that $a(\sigma,X,D)$ has a parametrix $b(\sigma,X,D)$ $\in OPH^{-m}$. Let $\beta(\sigma,X,D)$ be the operator with Weyl symbol $\beta(\sigma,x,\xi)$ $= \varphi_0(\sigma,x,\xi)$ for $|\sigma| + |x|^2 + |\xi|^2$ large. By the formulas (6) and (7):

$$\beta(\sigma,X,D)a(\sigma,X,D) = I + r(\sigma,X,D),$$

where $r(\sigma,x,\xi) \sim \sum r^j(\sigma,x,\xi)$, $r^j(\sigma,x,\xi)$ are homogeneous of degree $m-2j-1$ in $(|\sigma|^{1/2},x,\xi)$. Therefore, the operator $r(\sigma,X,D) \in OPH^{m-1}$. By (6) and (7) $r^i(\sigma,X,D)$, $i \geq 1$, belongs to the class OPH^{m-i}, so the operator

$$b(\sigma,X,D) = (I - r(\sigma,X,D) + r^2(\sigma,X,D) - \dots)\beta(\sigma,X,D)$$

is a parametrix for the operator $a(\sigma,X,D)$ and belongs to the class OPH^{-m}.

Now it has to be shown that $a(\sigma,X,D)$ is invertible as an operator on W_k for all integer k for $|\sigma|$ sufficiently large. The product

$$b(\sigma,X,D)\,a(\sigma,X,D) = I + R_\sigma,$$

where the operator $R_\sigma \in OPS^{-\infty}_{1,\Sigma}$. It follows from Proposition 1, that there is $M>0$, such that $\|R_\sigma\|_{k,k-\ell} < 1/2$ for $|\sigma| \geq M$, so the operator $I + R_\sigma$ is invertible as an operator on W_k for any integer k. Denote as Q_σ the operator $(I+R_\sigma)^{-1} - I$. We have $Q_\sigma : S(R^n) \to S(R^n)$ is continuous, hence by the relation $Q_\sigma = -R_\sigma - Q_\sigma R_\sigma$ $Q_\sigma : S'(R^n) \to S(R^n)$ is continuous. It follows that

$$a(\sigma,X,D)^{-1} - b(\sigma,X,D) \in OPS^{-\infty}_{1,R_M}.$$

Assume that the operator $K \in OP\Psi^m_0(G)$ and the operators $\sigma_k(\sigma,\pm 1)(X,D)$ are elliptic with parameter σ, $\sigma \in R$.

Assume also that the operators $\sigma_K(\sigma,\pm i)(X,D)$ have the left inverses $[\sigma_K(\sigma,\pm i)(X,D)]^{-1}\in OPH^{-m}_{\pm}$ $(OPH^{-m}_{\pm,0})$. Then the operator L, such that

$$\sigma_L(\sigma,\pm\lambda)(X,D)=\lambda^{-m/2}[\sigma_K(\sigma,\pm i)(X,D)]^{-1},$$

is a left inverse for the operator K and $L\in OP\Psi^{-m}_{+}(G)$ $(OP\Psi^{-m}_{0}(G))$.

The next proposition is an extension for the case of the group $R\times IH^n$ of the Proposition 2.10[3].

Proposition 7. If $K\in OP\Psi^{m}_{+}(G)$ and $\sigma_K(\sigma,\pm i)(X,D)$ are elliptic with parameter σ, then K has a left inverse $L\in OP\Psi^{-m}_{+}(G)$ if and only if $\sigma_K(\sigma,\pm i)(X,D)$ are injective on $S(R^n)$, and such a right inverse if and only if $\sigma_K(\sigma,\pm i)(X,D)^*$ are injective on $S(R^n)$.

Corollary 1. If $K\in OP\Psi^{m}_{+}(G)$ and $\sigma_K(\sigma,\pm i)(X,D)$ are elliptic with parameter σ and injective on $S(R^n)$, then K is hypoelliptic.

Denote as A_σ the operator $a(X,D)-i\sigma$ where $a(X,D)$ is a differential operator of second order with the symbol

$$a(x,\xi) = \sum_{i,j=1}^{2n} a_{ij} \mathcal{X}_i \mathcal{X}_j \ , \quad \mathcal{X}_i = x_i, \ \ \mathcal{X}_{i+n} = \xi_i, \quad 1 \le i \le n .$$

The matrix $\{a_{ij}\}$ is strictly positive definite and symmetric. The operator A_σ satisfies the condition of Proposition 6.

Note that in this case the Weyl symbol for parametrix of A_σ:

$$B(\sigma,x,\xi) \sim \sum_{j\geq 0} \varphi_j(\sigma,x,\xi), \quad |\sigma| + |x|^2 + |\xi|^2 \to \infty,$$

where the functions $\varphi_j(\sigma,x,\xi)$ are solutions of the following equations:

$$(a(x,\xi) - i\sigma)\,\varphi_o(\sigma,x,\xi) = 1$$

$$\{a - i\sigma, \ \varphi_{2j-2}\}_2 + (a - i\sigma)\varphi_{2j} = 0 \ , \quad j = 1, 2, \ldots$$

The function $\varphi_{2j}(\sigma,x,\xi)$ is homogeneous of degree $-2-4j$ in $(|\sigma|^{1/2}, x, \xi)$ so the operator $B(\sigma, X, D) \in OPH^{-2}$.

As a consequence of the Propositions 6 and 7, we have

Corollary 2. Assume that $K \in OP\Psi_+^m(G)$ and $\sigma_K(\sigma,\pm 1)(X,D) = a_\pm(X,D) - i\sigma = A_\sigma$. If there exist $a_\pm(X,D)^{-1}$, then the operator K is hypoelliptic.

Definition 7. We say a convolution operator K belongs to $OP\Psi^m(G)$, if

$$K \sim \sum_{j \geq 0} K_j \,, \qquad K_j \in OP\Psi_0^{m-j}(G) \,,$$

in the sense, that the difference $K - \sum_{j=0}^{N} K_j$ is arbitrary smoothing for any sufficiently large N.

Section 1.2. Parametrix for the Heat Equation on the Group $\mathbb{R}\times\mathbb{H}^n$.

Let C_- be the half-plane $\{\operatorname{Im}\sigma<0\}$ with the closure $\bar{C}_-$.

Definition 8. The class $\Psi_h^m(G)$ is the subclass of $\Psi_0^m(G)$, consisting of functions which extend to $(\bar{C}_-\times R^{2n+1})\setminus 0$ in such a way to be C^∞ in all variables and holomorphic with respect to σ, $\sigma\in C_-$.

We will consider the operator $\partial/\partial t - L_\alpha$ on G where

$$L_\alpha=\sum_{j=1}^n (L_j^2+M_j^2)+i\alpha T$$

on $\mathbb{H}^n$. Note that

$$\pi_{\sigma,\pm\lambda}(\partial/\partial t-L_\alpha)=i\sigma+\lambda\Big\{\sum_{j=1}^n\big(-\partial^2/\partial x_j^2+x_j^2\big)\mp\alpha\Big\}.$$

Obviously, $(\partial/\partial t-L_\alpha)\in OP\Psi_0^2$ and

$$\sigma_{(\partial/\partial t-L_\alpha)}(\sigma,\pm 1)(x,\xi)=i\sigma+|x|^2+|\xi|^2\mp\alpha.$$

The operator $\sigma_{(\partial/\partial t-L_\alpha)}(\sigma,\pm 1)(X,D)$ is elliptic with parameter $i\sigma\mp\alpha$ for all $\sigma\in R$ and invertible on $L^2(R^n)$ if and only if $-n\mp\alpha\notin\{0,2,4,\ldots\}$.

Denoting as $b_\alpha(\sigma,x,\xi)$ the Weyl symbol of the inverse operator, we have the following equation

$$(|x|^2+|\xi|^2+i\sigma\mp\alpha)b_\alpha(\sigma,x,\xi)-1/8\sum_{k=1}^{n}\left(\partial^2/\partial y_k\partial\xi_k-\partial^2/\partial\eta_k\partial x_k\right)^2$$

$$\cdot\left(|x|^2+|\xi|^2+i\sigma\mp\alpha\right)b_\alpha(\sigma,y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}}\equiv 1.$$

After differentiation it becomes

$$-1/4\sum_{j=1}^{n}\left(\partial^2/\partial x_j^2+\partial^2/\partial\xi_j^2\right)b_\alpha+(|x|^2+|\xi|^2+i\sigma\mp\alpha)b_\alpha\equiv 1. \quad (20)$$

Denote by H operator $-\Delta+|x|^2$, the resolvent of H by

$$R_\gamma(X,D)=(H+\gamma I)^{-1}.$$

A solution of the equation (20) could be obtained as a Weyl symbol for the operator $R_\gamma(X,D)$ for $\gamma=i\sigma\mp\alpha$. In turn,

$$R_\gamma(x,\xi)=\int_0^\infty e^{-\gamma t}h_t(x,\xi)\,dt\ ,$$

where $h_t(x,\xi)$ is the well known Weyl symbol of the operator e^{-tH} (see [3]):

$$h_t(x,\xi) = (\cosh t)^{-n} exp\left[-(|x|^2+|\xi|^2)\tanh t\right]. \qquad (21)$$

Using formula (21) and changing the variables, we have

$$R_\gamma(x,\xi) = C_n \int_0^1 (1-t)^{-1+\frac{(\gamma+n)}{2}} (1+t)^{-1-\frac{(\gamma-n)}{2}} e^{-t(|x|^2+|\xi|^2)} dt. \qquad (22)$$

The integral (22) converges for $Re\,\gamma > -n$. Using the Taylor series expansions for the functions $(1+t)^{-1-(\gamma-n)/2}$ and $e^{-t(|x|^2+|\xi|^2)}$, the integral (22) can be rewritten as the convergent series:

$$R_\gamma(x,\xi) = C_n \sum_{j\geq 0} \int_0^1 (1-t)^{-1+(\gamma+n)/2} C_j\, t^j,$$

where C_j depends on x, ξ, γ, n, j. For each j the integral

$$\int_0^1 (1-t)^{-1+(\gamma+n)/2} t^j\, dt = \frac{(j-n)!}{\left(\frac{\gamma+n}{2}\right)\cdots\left(\frac{\gamma+n}{2}+j\right)}.$$

It follows that the integral (22) can be continued analytically to all complex γ excluding $-n-2j$, $j=0,1,\dots$ Therefore, the solution of the equation (20) can be written as follows:

$$b_\alpha(\sigma,x,\xi) = \int_0^\infty e^{-i\sigma t \mp \alpha t} h_t(x,\xi)\,dt$$

for $|Re\,\alpha| < n$.

Consider now an operator $\partial/\partial t + P_\alpha$ on G, where P_α is a more general second order differential operator on H^n:

$$P_\alpha = \sum_{j,K=1}^{n} a_{jK} X_j X_K + i\alpha T, \tag{23}$$

where $X_j = L_j$, $X_{j+n} = M_j$, $1 \le j \le n$, and $\{a_{jK}\}$ is a symmetric, positive definite matrix of real numbers. We have

$$\pi_{\sigma,\pm\lambda}(\partial/\partial t + P_\alpha) = i\sigma \mp \lambda\alpha - \lambda Q(X,D),$$

where

$$Q(x,\xi) = \sum_{j,K=1}^{n} a_{jK}\chi_j\chi_K, \quad \chi_j = x_j, \quad \chi_{j+n} = \xi_j, \quad 1 \le j \le n. \tag{24}$$

The operator $Q(\pm X, D)$ is a positive self-adjoint differential operator of the second order. Let $\mathfrak{S}$ be the symplectic form on R^{2n}:

$$\mathfrak{S}((x,\xi),(x',\xi')) = x\cdot\xi' - x'\cdot\xi .$$

If $Q(u,v)$ is the symmetric bilinear form on R^{2n} polarizing the quadratic form $Q(u)$ $(Q(u) = Q(u,u))$, the Hamilton map of $Q(x,\xi)$ is defined to be the linear map $\tilde{F}$ on R^{2n}:

$$\mathfrak{S}(u, \tilde{F}v) = Q(u,v), \qquad u,v \in R^{2n}. \tag{25}$$

$\tilde{F}$ is positive definite (if Q is positive definite) so its eigenvalues are of the form $\pm i\mu_j$, $1 \leq j \leq n$, $\mu_j > 0$.

It was shown in [3] that $Q(X,D)$ is unitarily equivalent to the operator

$$\sum_{j=1}^{n} \mu_j \left(-\partial^2/\partial x_j^2 + x_j^2\right),$$

so the spectrum of $Q(X,D)$ is of the form:

$$\left\{ \sum_{j=1}^{n} (2k_j+1)\mu_j , \ k_j \in Z^+ \cup \{0\} \right\}.$$

If $Q(x,\xi)$ takes the form

$$Q(x,\xi)=\sum_{j=1}^{n}\mu_j(x_j^2+\xi_j^2), \quad \mu_j>0,$$

then the equation (20) changes to

$$-\tfrac{1}{4}\sum_{j=1}^{n}\mu_j\left(\partial^2/\partial x_j^2+\partial^2/\partial\xi_j^2\right)b+\sum_{j=1}^{n}\mu_j(x_j^2+\xi_j^2)b \tag{26}$$

$$+(i\sigma\mp\alpha)b\equiv 1.$$

The formula for the solution of the equation (26) can be written as

$$\int_0^\infty e^{-i\sigma t\mp\alpha t}\, h_t^Q(x,\xi)\,dt, \tag{27}$$

where

$$h_t^Q(x,\xi)=\prod_{j=1}^{n}(\cosh t\mu_j)^{-1}$$

$$\cdot\exp\left\{-\sum_{j=1}^{n}(x_j^2+\xi_j^2)\tanh t\mu_j\right\}$$

and it is necessary to take $Re\,\alpha$ small enough in order to avoid the spectrum of the operator $Q(X,D)$ i.e.,

$$|Re\,\alpha| < \sum_{j=1}^{n} \mu_j \tag{28}$$

The formula (27) will be expressed in an invariant form as it was done in [3]. Denote by F_Q the map $\frac{1}{i}\tilde{F}_Q$. We have

$$\prod_{j=1}^{n} (\cosh t\mu_j)^2 = \det \cosh tF_Q$$

so

$$\prod_{j=1}^{n} (\cosh t\mu_j)^{-1} = (\det \cosh tF_Q)^{-1/2}.$$

Now let

$$A_Q = (F_Q^2)^{1/2}$$

be the unique square root of the matrix F_Q^2 with positive spectrum, and γ is a quadratic form on R^{2n} defined by

$$\gamma(A_Q z, z) = Q(z,z).$$

In the symplectic coordinate system on R^{2n} such that

$$Q(z,z)=\sum_j \mu_j\,(x_j^2+\xi_j^2), \quad z=(x,\xi),$$

we have

$$\gamma(z,z)=\sum_{j=1}^{n}(x_j^2+\xi_j^2),$$

so

$$\gamma(f(A_Q)z,z)=\sum_{j=1}^{n}f(\mu_j)(x_j^2+\xi_j^2)$$

and

$$\sum_{j=1}^{n}(x_j^2+\xi_j^2)\tanh(t\mu_j)=\gamma(\tanh tA_Q z,z)=$$

$$=Q(A_Q^{-1}\tanh tA_Q z,z).$$

Thus the formula (27) can be written invariantly as

$$b_\alpha(\sigma,x,\xi)=\int_0^\infty e^{-i\sigma t\mp\alpha t}\,\Phi_Q(t,x,\xi)\,dt, \tag{29}$$

where

$$\Phi_Q(t,z)=(\det\cosh tF_Q)^{-1/2}\,\cdot$$

$$\cdot\exp\{-Q(A_Q^{-1}\tanh tA_Q z,z)\}, \quad z=(x,\xi), \tag{30}$$

and

$$|\,Re\,\alpha\,| < \sum_{j=1}^{n} \mu_j .$$

The function b_α belongs to the Schwartz space $S(R^{2n+1})$. So, we can define an element of Ψ_h^{-2} by formula

$$\sigma_K(\sigma, \pm\tau)(x,\xi) = \tau^{-1} b_\alpha(\tau^{-1}\sigma, \pm\tau^{-1/2}x, \tau^{-1/2}\xi) .$$

Using the formula (29) we obtain

$$\hat{K}(\sigma, \pm\tau, y, \eta) = \int_0^\infty e^{-i\sigma t \mp \alpha\tau t} (\det\cosh\tau F_Q)^{-1/2}$$

$$\cdot \exp\{-Q(A_Q^{-1}\tanh(t\tau)A_Q\tau^{-1/2}z, \tau^{-1/2}z)\}dt ,$$

$$z = (y, \eta) .$$

The function $\hat{K}(\sigma, \pm\tau, y, \eta) = \sigma_K(\sigma, \pm\tau)(\pm\tau^{-1/2}y, \tau^{-1/2}\eta)$ belongs to the class Ψ_h^{-2}. It follows from the Proposition 1.17 of Beals, Griener, and Stanton [5] that the function k (inverse Fourier transform of $\hat{k}$) vanishes for $t \leq 0$.

Furthermore,

$$K_\alpha(t,s,q,p) = t^{-(n+1)} K_{1,\alpha}^{Q}(s/t, q/\sqrt{t}, p/\sqrt{t}), \quad t>0, \tag{31}$$

where the function $k^{Q}_{1,\alpha}$ belongs to the Schwartz space $S(R^{2n+1})$. If $\alpha=0$ the invariant form for the function $K^{Q}_{1}(s,q,p)$ is

$$K^{Q}_{1}(s,q,p)=c_n\int_{-\infty}^{+\infty}e^{is\tau}\,\Psi_{Q}(\tau,q,p)\,d\tau \tag{32}$$

with

$$\Psi_{Q}(\tau,q,p)=(-\tau^{-2n}\det\sinh\tau F_{Q})^{-1/2}$$

$$\cdot\exp\{-\tau Q(A_{Q}^{-1}\coth\tau A_{Q}z,z)\},\quad z=(q,p).$$

If $\alpha\neq 0$, $|\operatorname{Re}\alpha|<\sum_{j=1}^{n}\mu_j$, then

$$K^{Q}_{1,\alpha}(s,q,p)=K^{Q}_{1}(s/t+i\alpha,\,q/\sqrt{t},\,p/\sqrt{t}) \tag{33}$$

where $K^{Q}_{1}(s/t+i\alpha,q/\sqrt{t},p/\sqrt{t})$ is defined from (32) by an analytic continuation.

Section 1.3. Complex Powers on the Heisenberg Group

We consider complex powers of the right invariant differential operator $(-P)=(-P_0)$ defined by (23). If $\beta\in R$, then

$$\sigma_{(-P)^\beta}(\pm\lambda)(X,D)=\lambda^\beta Q^\beta(X,D),$$

where the Weyl symbol of the operator $Q(X,D)$ defined by (24). In accordance with the last formula we will analyze the complex powers of the operator $(-P)=(-P_0)$ as an operator $(-P)_z$, such that

$$\sigma_{(-P)_z}(\pm\lambda)(X,D)=\lambda^z Q^z(X,D).$$

Let $q_{-z}(x,\xi)$ be the Weyl symbol of the complex power $-z$, $Re z>0$, of the operator $Q(X,D)$. It connects with the Weyl symbol of the operator e^{-tQ} (function $\Phi_Q(t,x,\xi)$ from (30)) by the formula:

$$q_{-z}(x,\xi)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\Phi_Q(t,x,\xi)\,dt,\quad Re z>0. \tag{34}$$

The integral (34) converges for $Re\, z > 0$ and the function $q_{-z}(x, \xi)$ belongs to the class $H^{-2z}(R^{2n})$. If $Q = H(X, D)$, then by (21)

$$h_{-z}(x,\xi) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} (\cosh t)^{-n} \exp[-r^2 \tanh t]\, dt, \tag{35}$$

$$r^2 = |x|^2 + |\xi|^2.$$

Since $q_{-z}(x,\xi)$ belongs to the class $H^{-2z}(R^{2n})$, we can define an element $\hat{k}_{-z}$ of $\Psi^{-2z}(IH^n)$ by the formula

$$\sigma_{K_{-z}}(\pm\tau)(x,\xi) = \tau^{-z} q_{-z}(\pm \tau^{-1/2} x, \tau^{-1/2}\xi). \tag{36}$$

By (34)

$$\sigma_{K_{-z}}(\pm\tau)(x,\xi) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \Phi_Q(\tau t, \tau^{-1/2} x, \tau^{-1/2}\xi)\, dt. \tag{37}$$

If $Q = H(X,D)$, then the formula (37) obtains a simpler form:

$$\sigma_{K_{-z}}(\pm\tau)(x,\xi)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}(\cosh\tau t)^{-n}\exp\left\{-\frac{r^2}{\tau}\tanh\tau t\right\}dt.$$

It follows from the above considerations that the operator $(-P)_{-z}$ belongs to the class $OP\Psi_0^{-2z}(\mathbb{H}^n)$, $Re z>0$.

Note that

$$\Phi_Q(\tau t, \tau^{-1/2}x, \tau^{-1/2}\xi)=\sigma_{e^{tP}}(\pm\tau)(x,\xi)$$

So the formula (37) can be rewritten as follows:

$$\sigma_{(-P)_{-z}}(\pm\tau)(x,\xi)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\sigma_{e^{tP}}(\pm\tau)(x,\xi)\,dt. \qquad (38)$$

If $0<Re z<n+1$ then an explicit formula for the convolution kernel K_{-z} of the operator $(-P)_{-z}$ can be found from the formula (37):

$$K_{-z}(s,q,p)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}K_0(t,s,q,p)\,dt, \qquad (39)$$

where the function $K_0(t,s,q,p)$ defined by (31) with $\alpha=0$. Similarly, we will analyze complex powers of the operator

$(-P)_{-z,\alpha}$, $\alpha \neq 0$, on $\mathbb{H}^n$ as convolution operators $K_{-z,\alpha}u = K_{-z,\alpha} * u$ where

$$K_{-z,\alpha}(s,q,p) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} K_\alpha(t,s,q,p)\,dt. \tag{40}$$

The function $K_\alpha(t,s,q,p)$ was defined by (31).

For each (s,q,p), the function $K_\alpha(t,s,q,p) = O(t^{-(n+1)})$ as $t \to \infty$, if $t \to 0$, $(s,q,p) \neq 0$, $K_\alpha(t,s,q,p) = O(t^N)$ for any $N>0$. It follows that the integral (40) converges for $0 < \operatorname{Re} z < n+1$ and $(s,q,p) \neq 0$. It can be shown similarly that the function $K_{-z,\alpha}(s,q,p)$ is C^∞ on $\mathbb{H}^n \setminus 0$. Note that

$$K_{-z,\alpha}(r(s,q,p)) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} K_\alpha(t,r^2 s, rq, rp)\,dt =$$

$$= r^{2(z-(n+1))} K_{-z,\alpha}(s,q,p),$$

which shows that the operator $K_{-z,\alpha}$ belongs to the class $Op\Psi_0^{-2z}(\mathbb{H}^n)$.

In case of the operator L_α the formula for the function $\ell_{-z,\alpha}$ can be written explicitly.

The heat Kernel in this case is given by the formula

$$\ell_\alpha(t,s,q,p) = \int_{-\infty}^{\infty} e^{i\tau[s/t+i\alpha]} t^{-(n+1)} (\tau/\sinh\tau)^n$$

$$\cdot \exp\left\{-\tau\coth\tau \frac{(|q|^2+|p|^2)}{t}\right\} d\tau .$$

Substituting $\ell_\alpha(t,s,q,p)$ in (40), we have

$$\ell_{-z,\alpha}(s,q,p) = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} \left\{ \int_{-\infty}^{\infty} e^{i\tau[s/t+i\alpha]} t^{-(n+1)}\right.$$

$$\left. \cdot \left(\tau/\sinh\tau\right)^n \exp\left\{-\tau\coth\tau \frac{(|q|^2+|p|^2)}{t}\right\} d\tau\, dt .\right.$$

We will integrate at first with respect to t. Apparently,

$$\frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-(n+2)} e^{t^{-1}(i\tau s-\tau\coth\tau)(|q|^2+|p|^2)} dt =$$

$$= (i\tau s - (\tau\coth\tau)(|q|^2+|p|^2))^{z-(n+1)} .$$

We have

$$\ell_{-z,\alpha}(s,q,p) = \int_{-\infty}^{\infty} e^{-\alpha\tau} (\tau/\sinh\tau)^{z-1}$$

$$\cdot \{ i s \sinh\tau - \cosh\tau (|q|^2 + |p|^2)\}^{z-(n+1)} . \tag{41}$$

This formula is valid for $0 < \operatorname{Re} z < n+1$ and $|\operatorname{Re}\alpha| < n$. In the case of $z = k \in Z^+$ formula (41) can be continued analytically (by integration by parts) to $\alpha \in C$ such that $\mp\alpha$ avoids the set $\{n+2j,\ j = 0, 1, \ldots\}$.

CHAPTER II. OPERATORS WITH VARIABLE COEFFICIENTS

Section 2.1. Operator class

Let M be a compact contact manifold of dimension $2n+1$ We aim to obtain the class of operators with variable coefficients on M from the operator class $OP\Psi^m(R\times \mathbb{H}^n)$ in the way it was developed by Taylor [3]. In order to have symbolic operator calculi for this class we need to verify certain hypotheses that were stated in Chapter I of [3]. These hypotheses for the class $OP\Psi^m(G)$ can be written as follows:

$$\Psi_0^m(G) \subset \begin{cases} S^m_{1/2\#}, & m \geq 0, \\ S^{m/2}_{1/2\#}, & m \leq 0. \end{cases} \tag{42}$$

$$K_1 \in OP\Psi_0^m(G),\ K_2 \in OP\Psi_0^\mu(G) \rightarrow K_1K_2 \in OP\Psi_0^{m+\mu}(G). \tag{43}$$

$$\hat{K}(\sigma,\tau,y,\eta) \in \Psi_0^m(G) \rightarrow D_\sigma^i D_\tau^j D_{y,\eta}^\gamma \hat{K} \in \Psi_0^{m-2i-2j-|\gamma|}(G). \tag{44}$$

$$K \in OP\Psi_0^m(G) \rightarrow K^* \in OP\Psi_0^m(G). \tag{45}$$

If $K_j \in OP\Psi_0^{m-j}(G)$, $j=0,1,\ldots$, then there exists $K \in OP\Psi^m(G)$ such that

$$K \sim K_0 + K_1 + \cdots \tag{46}$$

The property (42) follows from Definition 1; properties (43) and (45) were stated as propositions 4 and 5, respectively; property (44) was stated by formula (18), and property (46) follows from Definition 7.

By Darboux's theorem an open set $\mathcal{U} \subset M$ can be mapped diffeomorphically to an open set $\Omega \subset \mathbb{H}^n$, preserving the contact form. For each $v \in \Omega$, if $\widehat{K}(v,\sigma,\tau,y,\eta)$ is a smooth function of v with values in $\Psi^m(G)$ then $K(v)$ defined by

$$K(v)(u) = K(v;\cdot) * u$$

is a smooth function of v taking values in $OP\Psi^m(G)$. Then we say that the operator K defined by

$$(Ku)(v) = K(v)u(v)$$

belongs to the class $OP\widetilde{\Psi}^m(R\times M)$. The symbol of the operator K we denote by

$$\sigma_K(v,\sigma,\pm\lambda)(X,D) = \pi_{\sigma,\pm\lambda}(K(v)).$$

As a consequence of Proposition 1.2 [3] we have the following:

Proposition 8. If $A \in OP\widetilde{\Psi}^{m}(R\times M)$, $B \in OP\widetilde{\Psi}^{\mu}(R\times M)$, then $AB \in OP\widetilde{\Psi}^{m+\mu}(R\times M)$. If $C \in OP\widetilde{\Psi}^{m+\mu}(R\times M)$ is defined by

$$\sigma_C(v,\sigma,\pm\lambda)(X,D) = \sigma_A(v,\sigma,\pm\lambda)(X,D)\sigma_B(v,\sigma,\pm\lambda)(X,D),$$

then

$$AB - C \in OP\widetilde{\Psi}^{m+\mu-1}(R\times M).$$

Section 2.2. Parametrix for the Heat Equation

Suppose P is a negative self adjoint second order differential operator, its principal symbol $p_2 \geq 0$ and vanishes to exactly second order on $\Lambda \subset T^*M \setminus 0$, the span of the contact form on M. Denote by F the Hamilton map of p_2 and by $tr^+ F$ the sum of the positive eigenvalues of $\frac{1}{i}F$. It was shown in [3] that if the condition

$$|\mathrm{sub}\, \sigma(P)| < tr^+ F \qquad \text{on } \Lambda \tag{47}$$

is satisfied, then P is hypoelliptic. The operator P also has a discrete spectrum, since the embedding $S_1^2(M) \subset L^2(M)$ is compact.

For $v \in \Omega$, we assume that

$$(Pu)(v) = P_\alpha(v) u(v),$$

where

$$P_\alpha(v) = \sum_{j,k=1}^{2n} a_{jk}(v) X_j X_k + i\alpha(v) T,$$

the matrix $\{a_{jk}(v)\}$ is symmetric and positive definite for each v, the functions $a_{jk}(v)$, $\alpha(v)$ are smooth functions of v.

The symbol of the operator $P_\alpha(v)$ is $\sigma_{P_\alpha}(v, \pm 1)(X, D)$, where

$$\sigma_{P_\alpha}(v, \pm 1)(x, \xi) = \sum a_{jK}(v)\, \chi_j \chi_K \mp \alpha(v) ,$$

$$\chi_j = x_j, \quad \chi_{j+n} = \xi_j, \quad 1 \leq j \leq n .$$

The operators $\sigma_{P_\alpha}(v, \pm 1)(X, D)$ are elliptic and invertible on $L^2(R^n)$ if the following condition is satisfied:

$$\mp\alpha(v) \neq \left\{ \sum_j (2K_j + 1)\mu_j(v), \quad K_j \in Z^+ \cup \{0\} \right\}, \tag{48}$$

$$v \in \Omega ,$$

$\mu_j(v)$ is the eigenvalue of the Hamilton map $\frac{1}{i} F_Q(v)$.

For $v \in \Omega$, we consider the operator $\partial/\partial t + P_\alpha(v)$ with the symbol $\sigma_{\partial/\partial t + P_\alpha}(v, \sigma, \pm\lambda)(X, D)$, where

$$\sigma_{\partial/\partial t + P_\alpha}(v, \sigma, \pm 1)(x, \xi) = \sum_{j,K} a_{jK}(v)\chi_j \chi_K \mp \alpha(v) + i\sigma,$$

The operators $\sigma_{\partial/\partial t+P_\alpha}(v,\sigma,\pm 1)(X,D)$ are elliptic with parameter $i\sigma$ and invertible on $L^2(R^n)$ if condition (48) is satisfied.

Proposition 9. If condition (48) is satisfied, then the operator $\partial/\partial t + P_\alpha$ is hypoelliptic on Ω with parametrix K_α in the class $OP\widetilde{\Psi}_h^{-2}$.

Proof. If the function $K_{1,\alpha}^{Q}$ is defined by (29), we consider the function

$$K_{\alpha(v)}(t,s,q,p)=t^{-(n+1)}K_{1,\alpha(v)}^{Q(v)}(s/t,\ q/\sqrt{t},\ p/\sqrt{t}), \tag{49}$$

for each v, and the corresponding operator

$$(K_\alpha u)(v) = (K_{\alpha(v)}u)(v), \qquad K_{\alpha(v)}u = K_{\alpha(v)}(\cdot) * u.$$

The operator K_α belongs to the class $OP\widetilde{\Psi}_h^{-2}(R\times M)$. It follows from Proposition 8 that

$$(\partial/\partial t + P_\alpha)K_\alpha = I + R, \qquad R \in OP\widetilde{\Psi}^{-1}(R\times M).$$

So, the operator $\partial/\partial t + P_\alpha$ has a left parametrix

$$K \sim K_\alpha - K_\alpha R + K_\alpha R^2 - \dots \equiv K_0 + K_1 + K_2 + \dots$$

It follows from Proposition 8 that $K_j \in OP\widetilde{\Psi}^{-2-j}(R\times M), j=0,1,\ldots$ Similarly it can be shown that K is a right parametrix. After the rearrangement, we can write

$$K \sim \sum_{j\geq 0} K_j' , \qquad K_j' \in OP\widetilde{\Psi}_h^{-2-j}(R\times M).$$

The function $\widehat{K}_j'(v,\sigma,\tau,y,\eta)$ belongs to the class $\widetilde{\Psi}_h^{-2-j}$. It is homogeneous with respect to dilation, i.e.,

$$\widehat{K}_j'(v, r^2\sigma, r^2\tau, ry, r\eta) = r^{-2-j}\widehat{K}_j'(v,\sigma,\tau,y,\eta).$$

It follows from propositions 1.9 and 1.17 [5] that $K_j'(v,t,s,q,p)$ is homogeneous of degree $2+j-2n-4$ and it vanishes for $t\leq 0$.

Substitution of $r=-1$ shows that $\widehat{K}_j'(v,\sigma,\tau,y,\eta)$ is an odd function of (y,η) if j is odd. So $K_j'(v,t,0)=0$ if j is odd, and it is homogeneous of degree $\frac{1}{2}(2+j-2n-4)$ in t when j is even.

Therefore,

$$K(v,t,0) \sim t^{-(n+1)} \sum_{i\geq 0} t^i K_i(v), \quad t\to 0. \tag{50}$$

If

$$Ku(t,v) = k(t,v,\cdot) * u(t,v), \quad (t,v) \in R \times IH^n,$$

then the kernel of the operator K is the function $K(t,v,t',v')$ independent on t and

$$tr\, e^{tP} = \int_M K(0,v,t,0)\, dvol(v) + A(t), \tag{51}$$

where $A(t) \in C^\infty(\bar{R}^+)$.

From formula (50) it follows

Proposition 10. If P is a negative self-adjoint differential operator of a second order on a compact contact manifold, and its principal symbol vanishes to exactly second order on $\Lambda \subset T^*M \setminus 0$, and the hypothesis (47) is satisfied, then

$$tr\, e^{tP} \sim t^{-(n+1)}(C_0 + C_1 t + \dots), \quad t \to 0. \tag{52}$$

Section 2.3. Analytical Continuation for Zeta-Function

Let λ_j, $j=0,1,\dots$ be the eigenvalues of the operator $(-P)$, $\lambda_j \geq 0$,

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1 .$$

The result on the eigenvalue asymptotics for $(-P)$ is known [3]; it follows from the asymptotic expansion (52) and Karamata's Tauberian theorem:

$$\lim_{\lambda \to \infty} \lambda^{-(n+1)} N(\lambda) = C . \qquad (53)$$

Denote by $\zeta_{(-P)}(z)$ the zeta-function of the operator $(-P)$:

$$\zeta_{(-P)}(z) = \sum_{j \geq 0} \lambda_j^{-z} , \qquad z \in C .$$

The formula (53) implies that $\zeta_{(-P)}(z)$ is a holomorphic function of z for $Re\, z > n+1$. We aim to continue analytically $\zeta_{(-P)}(z)$ for $Re\, z \leq n+1$.

Proposition 11. If P is a differential operator of second order on a compact contact manifold satisfying the above hypotheses, then the function $\zeta_{(-P)}(-z)$ has a finite number of simple poles at the points $z = 1, 2, \ldots, n+1$.

Proof. The zeta-function of the operator $(-P)$ and $tr\, e^{tP}$ are connected by the formula:

$$\zeta_{(-P)}(-z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, tr\, e^{tP} dt, \qquad Re\, z > n+1. \tag{54}$$

Note that the integral (54) converges and defines a function holomorphic for $Re\, z > n+1$ $\left(tr\, e^{tP} \underset{t \to 0}{\sim} c_0 t^{-(n+1)}\right.$ and $tr\, e^{tP} = O(t^{-N})$ as $t \to \infty$ for any $\left.N > 0\right)$. The function

$$\frac{1}{\Gamma(z)} \int_1^\infty t^{z-1}\, tr\, e^{tP} dt$$

is a holomorphic function of z. Consider separately for $Re\, z > (n+1) - i$, $0 \le i < n+1$, the integral

$$\frac{1}{\Gamma(z)} \int_0^1 t^{z-1} \left\{ \int_M K_i(x, t, 0)\, dvol(x) \right\} dt \tag{55}$$

The function $K_i(x,t,0)$ for each x is homogeneous in t of degree $-(n+1)+i$ or $K_i(x,t,0)=t^{-(n+1)+i}K_i(x,1,0)$.

Let γ be the contour consisting of the real axis from 1 to ρ, $0<\rho<1$, the circle $|S|=\rho$, and the real axis from ρ to 1. Denote by $I_i(z)$ the function

$$\int_\gamma S^{z-1-(n+1)+i}\left[\int_M K_i(x,1,0)\,d\mathrm{vol}(x)\right]ds.$$

If $\mathrm{Re}\, z>(n+1)-i$ then the integral over the circular part of γ tends to zero with ρ. It follows that

$$I_1(z)=-\int_0^1 t^{z-1}\left[\int_M K_i(x,t,0)\,d\mathrm{vol}(x)\right]dt \; +$$

$$+\int_0^1 (te^{2\pi i})^{z-1}\left[\int_M K_i(x,te^{2\pi i},0)\,d\mathrm{vol}(x)\right]dt.$$

So we have

$$\frac{1}{\Gamma(z)} \int_0^1 t^{z-1} \left\{ \int_M K_i(x,t,0)\, dvol(x) \right\} dt =$$

$$= \frac{\Gamma(1-z)}{2\pi i} e^{-i\pi z} I_i(z) . \tag{56}$$

The integral $I_i(z)$ converges uniformly in any finite region of the z -plane and so defines an entire function of z. Hence, the formula (56) gives the analytic continuation of (55) over the complex z -plane. The possible singularities are the poles of the function $\Gamma(1-z)$: points $z = 1, 2, \dots$

The function $I_i(z)$ at the points $z = (n+j)-i$, $j = 2, 3, \dots$, vanishes by Cauchy's theorem so the integral (55) does not have poles at these points. As a consequence, (55) has a finite number of simple poles at the points $z = 1, 2, \dots, (n+1)-i$, $0 \leq i < n+1$.

In accordance with (51) and (50) now we have to continue analytically for $Re\, z \leq n+1$ the expression

$$\frac{1}{\Gamma(z)} \left\{ \int_0^1 t^{z-1} \left\{ \sum_{i \geq n} \int_M K_i(x,t,0)\, dvol(x) + B(t) \right\} dt \right\},$$

where $B \in C^\infty(\bar{R}^+)$. By integration by parts it continues analytically to a holomorphic function for $z \in C$.

APPENDIX

ZETA-FUNCTION OF THE HARMONIC OSCILLATOR HAMILTONIAN

Consider the case when $a(X,D) = -\Delta + |x|^2 \equiv H$. The Weyl symbol of operator e^{-tH} ([3]) is equal to

$$h_t(x,\xi) = C_n (\cosh t)^{-n} \exp\{-(|x|^2 + |\xi|^2)\tanh t\}.$$

For $Re\, z > 0$, using the formula

$$h^{-z} = 1/\Gamma(z) \int_0^\infty t^{z-1} e^{-th} dt,$$

we define the operator $H^{-z}(X,D)$ as the operator with Weyl symbol

$$h_{-z}(x,\xi) = \frac{1}{\Gamma(z)} \left\{ \int_0^\infty t^{z-1} (\cosh t)^{-n} \exp\{-(|x|^2 + |\xi|^2)\tanh t\} dt \right\}.$$

Denote $|x|^2 + |\xi|^2$ as r^2. Using polar coordinates in (X,ξ)-space, we obtain

$$tr\, H^{-z}(X,D) = \iint h_{-z}(x,\xi)\, dx\, d\xi =$$

$$= \int_0^\infty \left\{ \int_0^\infty t^{z-1} (\cosh t)^{-n} \exp(-r^2 \tanh t) r^{2n-1} dt \right\} dr \frac{|S_{2n}|}{\Gamma(z)},$$

where $|S_{2n}|$ is the measure of the unit sphere in R^{2n}.
Inverting the order of integration by r and by t, we get

$$tr\, H^{-z}(X, D) = \frac{|S_{2n}|\,\Gamma(n)}{2\,\Gamma(z)} \int_0^\infty t^{z-1} (\sinh t)^{-n} dt. \qquad \text{(a.1)}$$

Proposition A1. The $tr\, H^{-z}(X, D)$ extends from $Re\, z > n$ to a meromorphic function on the complex plane with finite number of simple poles at the points $z_j = n - 2j$, $0 \leq j < n/2$.

Proof. Consider the integral

$$I(z) = \int_\gamma \frac{s^{z-1}}{(e^s - e^{-s})^n}, \qquad z = \sigma + i\tau, \qquad \text{(a.2)}$$

with contour γ consisting of the real axis from ∞ to ρ, $0 < \rho < \pi$, the circle $|s| = \rho$, and the real axis from ρ to ∞.

Assume that $\sigma > n$. On the circle $|s| = \rho$ we have

$$|s^{z-1}| \leq |s^{\sigma-1}| e^{2\pi|\tau|}$$

and

$$|(e^{s} - e^{-s})|^{n} > C|s|^{n}$$

so the integral over the circular part of γ tends to zero with ρ if $\sigma > n$. We have

$$I(z) = -\int_0^\infty \frac{t^{z-1}\,dt}{(e^{t} - e^{-t})^{n}} + \int_0^\infty \frac{(te^{2\pi i})^{z-1}}{(e^{t} - e^{-t})^{n}}\,dt$$

so

$$\int_0^\infty t^{z-1}(\sinh t)^{-n}\,dt = 2^{n}\left[(e^{2\pi i})^{z} - 1\right] I(z)$$

and

$$\operatorname{tr} H^{-z}(X, D) = \frac{2^{n}\,|S_{2n}|\,\Gamma(n)\,\Gamma(1-z)}{2\pi i}\,e^{-i\pi z}\,I(z)\,. \qquad \text{(a.3)}$$

The integral $I(z)$ converges uniformly in any finite region of the z-plane and so defines an entire function of z. Hence, the formula (a.3) gives the analytic continuation of $tr\, H^{-z}(X,D)$ over the complex z-plane. The possible singularities are the poles of function $\Gamma(1-z)$, points $z=1,2,\ldots$. The aim now is to show that the function $I(z)$ vanishes at the points $z=n+1, n+2,\ldots$ and $z_j=n-(2j+1),\ 0\leq j<(n-1)/2$. The integral (a.2) after the change of variable $u=e^{2z}-1$ can be written as

$$I(z)=\frac{1}{2^z}\int_{\gamma'}\frac{(u+1)^{n/2-1}}{u^n}\ln^{z-1}(u+1)\,du .$$

By Cauchy's theorem

$$I(z)=\frac{d^{(n-1)}}{du^{(n-1)}}\left\{(u+1)^{n/2-1}\ln^{z-1}(u+1)\right\}_{u=0} . \qquad \text{(a.4)}$$

It follows from (a.4) that $I(z)$ for $z=n+1,n+2,\ldots$ Assume that n is an odd number: $n=2m+1,\ s-1=k,\ k$- integer, $0\leq k\leq 2m$. To find $I(k+1)$ we use the Taylor series for the function

$$f(u)=(u+1)^{m-1/2}\ln^k(u+1) .$$

Note that

$$f(u) = \frac{d^K}{dt^K}\left[(1+u)^t\right]\Big|_{t=m-\frac{1}{2}} .$$

If $\binom{t}{j}$ is the coefficient of u^j in the Taylor series for the function $(1+u)^t$, then

$$(1+u)^t = \sum_{j\geq 0} \binom{t}{j} u^j$$

and

$$f(u) = \sum_{j\geq 0} \frac{d^K}{dt^K}\binom{t}{j} u^j .$$

So $I(K+1)$ is equal to $j!\,d^K\binom{t}{j}/dt^K$ for $j=2m,\ t=m-\frac{1}{2}$. Let $\sigma = t-m+\frac{1}{2}$,

$$\frac{d^K}{dt^K}\binom{t}{2m}\Big|_{t=m-\frac{1}{2}} = \frac{1}{(2m)!}\,\frac{d^K}{d\sigma^K}\,Q(\sigma)\Big|_{\sigma=0} ,$$

where

$$Q(\sigma) = \left[\sigma^2 - \frac{(2m-1)^2}{4}\right]\left[\sigma^2 - \frac{(2m-3)^2}{4}\right]\cdots\left[\sigma^2 - \frac{1}{4}\right] .$$

It follows that $I(K+1)$ coincides with the coefficient of σ^K in the polynomial $Q(\sigma)$, and

$$I(2r+1) \geqslant 0, \qquad I(2r) = 0, \qquad r = 0, 1, \ldots, m.$$

The case of even n can be considered similarly. Note that, if $n=1$, $tr\, H^{-z}(X, D)$ has one simple pole at the point $z=1$ with residue $1/2$; in fact, $tr\, H^{-z}(X, D) = (1/2)^z \zeta(z; 1/2)$, where

$$\zeta(z; 1/2) = \sum_{n \geqslant 0} \frac{1}{(n+1/2)^z}$$

If $n=2$, $tr\, H^{-z}(X, D)$ has one simple pole at the point $z=2$.

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