ON FOUR-DIMENSIONAL MANIFOLDS OF NONNEGATIVE CURVATURE

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Gerard Walschap

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Gerard Walschap

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Detlef Gromoll, Professor of Mathematics
Dissertation Director

Lowell Jones, Professor of Mathematics
Chairman of Defense

Blaine Lawson, Professor of Mathematics

Martin Rocek, Associate Professor, Institute for Theoretical Physics
Outside member

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Abstract of the Dissertation

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Noncompact manifolds of nonnegative curvature have been classified in dimensions ≤ 3. In dimension 4, the only case unaccounted for occurs when the soul S has codimension 2. By considering the holonomy of the normal bundle $\nu(S)$ of S in M, we show:

<u>Theorem</u>: If M^n has a soul S of codimension 2, then

a) There is a Riemannian submersion from M onto S

or

b) $\nu(S)$ is flat.

If both a) and b) occur, then M is a locally isometrically trivial fibration over S.

Next, we examine case a) above. In dimension 4, when S is diffeomorphic to a 2-sphere, one has:

Theorem: M^4 splits metrically iff. M is diffeomorphic to $S^2 \times \mathbb{R}^2$ and $M \to S$ has totally geodesic fibers.

Some results on the total curvature of both M and the fibers of $M \to S$ follow.

The standard examples with nonflat normal bundle are of the type $G \times_H P_2$. Here G is a Lie group with biinvariant metric, P_2 is $\mathbb{R}^2$ together with a metric of nonnegative curvature, and H is a closed subgroup of G acting on P_2 by isometries. Thus there is a unique metric of nonnegative curvature on the quotient, for which the projection π: $G \times P_2 \to G \times_H P_2$ becomes a Riemannian submersion. By explicitely calculating the sectional curvature, we show:

Theorem: Consider $M = S^3 \times_{S^1} \mathbb{R}^2$ with the standard submersion metric, and let $dr^2 + f^2 d\vartheta^2$ denote the fiber metric in polar coordinates. If h is a smooth function with compact support in $M \setminus S$, and bounded derivatives, then the metric on M obtained by deforming f to $f + \varepsilon h$ has nonnegative curvature for small enough ε.

In particular, choosing h adequately yields a metric that does not originate from the above construction.

Finally, we show:

__Theorem__: Let S denote the 2-sphere together with some metric of positive curvature. Then any 2-dimensional vector bundle over S admits a family of metrics of nonnegative curvature, with soul isometric to S, and totally geodesic fibers.

To Yoana

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0. Introduction

One of the central areas in contemporary Riemannian geometry is the study of complete manifolds of nonnegative sectional curvature K. While many questions still remain unanswered in both the compact and noncompact cases, the general structure in the noncompact situation is better understood at least as far as its topology is concerned. The first results are due to Cohn-Vossen [C-V] who dealt with the 2-dimensional case. More recently, Gromoll and Meyer [GM] studied the complete noncompact manifolds M with $K > 0$ in arbitrary dimensions, relying extensively on the crucial notion of convexity. One amazing result is that M is diffeomorphic to some Euclidean space. Finally, Cheeger and Gromoll [CG] extended the previous work to the case $K \geq 0$. They found that M is diffeomorphic to the normal bundle of a certain submanifold S, called a soul of M, which is compact, totally geodesic, and totally convex. When $K > 0$, S is a point, yielding the above theorem as a special case. Among the many other results they obtained, one in particular led them to a classification up to isometry in dimensions ≤ 3. It relies on the fact that if $\dim S = 1$, or $\operatorname{codim} S = 1$, then M is a locally isometrically trivial bundle over S.

The next step would be to obtain a classification in dimension 4, and here the only possibility as yet unaccounted for occurs when dimS = codimS = 2. In this work, we examine some properties of the case codimS = 2. But first we briefly recall some well known facts about complete manifolds of nonnegative curvature and Riemannian submersions. The proofs can be found in [CG], [CE], and [O'N]. The notation throughout this work follows [GKM]. Thus, if c: [a,b] → M is a smooth curve, X a vector field along c, then $\dot{c} := c_*D$, and $X' := \nabla_D X$. ΓE will denote the space of sections of the vector bundle E, and $\mathfrak{X}M := \Gamma TM$.

Let M be a connected Riemannian manifold. C ⊂ M is said to be *convex* if any 2 points in C can be joined by some minimal geodesic lying entirely in C. A convex set is *strongly convex* if any 2 of its points are connected by exactly one minimal geodesic in M. There is a continuous function conv: $M \to \mathbb{R}_+ \cup \{\infty\}$ such that any metric ball $B_\varepsilon(p)$ is strongly convex whenever $\varepsilon < \text{conv}(p)$. A weaker notion is that of local convexity: C is *locally convex* if every point in the closure of C has a relative neighborhood in C that is strongly convex. Finally, C is *totally convex* if any geodesic of M with endpoints in C is entirely contained in C. The relationship between the different types of convexity is summarized below, for complete M:

strongly convex $\Rightarrow$ convex $\Leftarrow$ totally convex

$\Downarrow$

locally convex

One has the following structure theorem for convex sets:

Theorem 0.1 *Let* C *be a closed connected locally convex subset of an arbitrary Riemannian manifold* M^n. *Then* C *carries the structure of an imbedded k-dimensional submanifold of* M *with smooth totally geodesic interior* intC = N *and (possibly nonsmooth) boundary* $\partial C = \overline{N} - N$, *which may be empty,* $0 \le k \le n$.

In what follows, M will denote a complete noncompact Riemannian manifold of nonnegative curvature. To carry out the basic soul construction, we need a totally convex set (t.c.s.) to begin with. It can be constructed as follows: choose $p \in M$, and a ray $\gamma: [0,\infty] \to M$ originating at p (i.e. γ is a geodesic with $t = d(\gamma(0), \gamma(t))$). Such a ray exists because M is noncompact. Define the open half-space $B_\gamma = \cup_{t>0} B_t(\gamma(t))$. Then $M - B_\gamma$ is a t.c.s. To obtain a compact t.c.s., let $C_t = \cap_\gamma(M - B_{\gamma_t})$, where the intersection is taken over all rays γ originating at p, and $\gamma_t(s) = \gamma(t+s)$. Then $\{C_t\}_{t\ge 0}$ is a family of compact t.c.s. such that:

(i) $p \in \partial C_0$

(ii) $M = \cup_{t\ge 0} C_t$

(iii) $t_2 \ge t_1$ implies $C_{t_2} \supset C_{t_1}$ and

$$C_{t_1} = \{q \in C_{t_2} \,/\, d(q, \partial C_{t_2}) \geq t_2 - t_1\}$$
$$\partial C_{t_1} = \{q \in C_{t_2} \,/\, d(q, \partial C_{t_2}) = t_2 - t_1\}.$$

A soul of M is a minimal t.c.s. obtained from C_0, as follows. For C closed and convex with $\partial C \neq \emptyset$, we let

$$C^a = \{p \in C \,/\, d(p, \partial C) \geq a\},$$
$$C^{max} = \cap_{C^a \neq \emptyset} C^a.$$

Theorem 0.2 *Let* M *have nonnegative curvature and let* C *be closed and locally convex, resp. totally convex,* $\partial C \neq \emptyset$. *Then*

(1) for any a, C^a *is locally convex, resp. totally convex,*

(2) $\dim C^{max} < \dim C$.

Theorem 0.2 is actually a corollary of the following more general result.

Theorem 0.3 *With the hypothesis of 0.2 let* $f: C \to \mathbb{R}$ *be defined by* $f(p) = d(p, \partial C)$. *Then* f *is (weakly) concave, i.e.*

$$f \circ c(\alpha t_1 + \beta t_2) \geq \alpha f \circ c(t_1) + \beta f \circ c(t_2)$$

for any normal geodesic c *contained in* C, *and* $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. *Moreover, suppose* $f \circ c \equiv d$ *is constant on some interval* $[a,b]$. *Let* Z *denote the parallel vector field along* $c|_{[a,b]}$ *such that* $Z(a) = \dot{c}_a(0)$, *where* c_a *is any minimal geodesic from* $c(s)$ *to* ∂C. *Then for any* s, $\exp_{c(s)} tZ(s)|_{[0,d]}$ *is a minimal geodesic from* $c(s)$

to ∂C *and the rectangle* $V: [a,b]\times[0,d] \to M$ *defined by* $V(s,t) = \exp_{c(s)} tZ(s)$ *is flat and totally geodesic.*

Now choose a compact t.c.s. as in the basic construction. By applying Theorem 0.2 repeatedly if necessary, we obtain a compact totally geodesic submanifold S without boundary, which is totally convex, $0 \leq \dim S < \dim M$. Such a manifold is called a *soul* of M. The following 2 properties of souls will be used later on, and contain basically all the information needed for a classification of manifolds of nonnegative curvature in dimension ≤ 3.

Theorem 0.4

(a) M *is diffeomorphic to the normal bundle* $\nu(S)$ *of* S *in* M.

(b) If $\dim S = 1$ *or* $\dim M - 1$, *then* M *is a locally isometrically trivial bundle over* S.

We next look briefly at Riemannian submersions. Consider Riemannian manifolds E^{n+k}, M^n, and a submersion $\pi: E \to M$. Each tangent space E_q splits uniquely as $E_q{}^v \oplus E_q{}^h$, where $E_q{}^v$ is the space tangent to the fiber through q, and $E_q{}^h$ its orthogonal complement. π is called a *Riemannian submersion* if $\pi_*|E_q{}^h$ is isometric onto $M_{\pi(q)}$ for each $q \in E$. The bundle TE^v (resp. TE^h) is called the *vertical* (resp. *horizontal*) bundle, and induces a decomposition of $Z \in \mathfrak{X}E$ as $Z = Z^v + Z^h$. The *horizontal lift* of $X \in \mathfrak{X}M$ is the unique horizontal $\overline{X} \in \mathfrak{X}E$ that is π-related to X. $\overline{X}$ is also called a *basic* vector field.

Proposition 0.5

(1) The Levi-Civita connections on E, M *are related by*

$$(\nabla_{\overline{X}}\overline{Y})^h = \overline{\nabla_X Y}, \quad X,Y \in \mathfrak{X}M$$

$$(\nabla_{\overline{X}}\overline{Y})^v = \frac{1}{2}\,[\overline{X},\overline{Y}]^v, \quad X,Y \in \mathfrak{X}M$$

This last expression is $\mathfrak{F}(E)$- *linear, so that*

$$A_U V := (\nabla_U V)^v, \quad U,V \in \Gamma TE^h$$

defines a skew-symmetric tensor field on TE^h, *called the* *__O'Neill tensor__*.

(2) Let $\overline{c}$ *be a horizontal lift of a curve* c: [a,b] → M. *Then* c *is a geodesic if and only if* $\overline{c}$ *is one. Consequently,*

$$\pi \circ \exp^E |{E_q}^h = \exp^M \circ \pi_* |{E_q}^h$$

(3) For local orthonormal $X,Y \in \mathfrak{X}M$,

$$K_{X,Y} = K_{\overline{X},\overline{Y}} + \frac{3}{4}\,\|\,[\overline{X},\overline{Y}]^v\|^2$$

The following facts, whose proofs can be found in [H] and [V], will also be needed.

Theorem 0.6 *Let* $\pi: E \to M$ *be a Riemannian submersion, with* E *complete.*

(1) Each curve γ *in* M *induces a map between the fibers over the endpoints, by means of horizontal lifts of* γ. *These maps are isometries if and only if the fibers of* π *are totally geodesic.*

(2) If the fibers are totally geodesic, then $A_{\bar{X}}\bar{Y}$ *restricted to a fiber is a Killing vector field on that fiber,* $X, Y \in \mathfrak{X}M$.

(3) If the fibers are totally geodesic and if the O'Neill tensor is identically zero, then π *is a locally isometrically trivial fibration over* M.

1. Basic results

M will denote a complete noncompact manifold of nonnegative curvature with soul S.

Lemma 1.1 *Let* c: [0,a] → S *be a piecewise smooth curve joining* p *and* q *in* S, *and suppose* $\gamma : [0,\infty) \to M$ *is a ray originating at* p. *If* $u \in M_q$ *denotes the parallel translate of* $\dot{\gamma}(0) \in M_p$ *along* c, *then* $t \mapsto \exp_q(tu)$ *is a ray originating at* q.

Proof Since any piecewise smooth curve is a limit of broken geodesics, we may assume that c is a geodesic, and thus extendable to $c: \mathbb{R} \to S$. Carry out the basic soul construction at p, so that $M = \bigcup_{t \geq 0} C_t$, with $\gamma(t) \in \partial C_t$. Now $c(\mathbb{R})$ lies in the compact set S, and is therefore contained in some C_{t_0}, hence in every C_t for $t \geq t_0$. By 0.3, the distance function $s \mapsto d(c(s), \partial C_t)$ is concave. Being bounded from below and defined on all of $\mathbb{R}$, it must be constant. Consider the parallel field X along c with $X(0) = \dot{\gamma}(0)$, and set $c_s(t) := \exp_{c(s)} tX(s)$.

Again by 0.3, c_s is a minimal geodesic from $c(s)$ to ∂C_t. Since this is true for all $t \geq t_0$, c_s is a ray. □

Recall that M is diffeomorphic to the normal bundle $\nu(S)$ of S in M. The following result was already known to D. Gromoll in the case dimM = 4.

Theorem 1.2 *Suppose codim* S = 2. *Then one of the following holds:*

(a) The normal bundle of S *is flat (with respect to the induced connection)*

(b) There is a Riemannian submersion $\pi: M \to S$.

<u>Remark:</u> (a) and (b) are not mutually exclusive. In fact, their intersection consists precisely of those M which are locally isometrically trivial bundles over S, cf. 1.4.

Proof of 1.2. Since the fibers of ν are 2-dimensional, the reduced holonomy group $\Phi_0(p)$ of the connection is either trivial or isomorphic to $SO(2) \cong S^1$. The trivial case corresponds to (a). Assume then that $\Phi_0(p)$ is isomorphic to S^1 for each $p \in S$. The remaining part of the proof is divided into several steps. First, notice that every direction in the normal bundle yields a ray, i.e. given $v \in \nu(S)$, $\|v\|=1$, $t \mapsto \exp(tv)$ is a ray. Indeed, since M is noncompact, there is at least one ray emanating from any one point of M. Fix $p \in S$, and choose $v \in M_p$ so that $t \mapsto \exp_p(tv)$ is a ray. By [CG] Theorem 5.1, $v \in \nu(S)$. Since S is totally geodesic in M, a parallel section of ν along a curve will be parallel in M. By 1.1, $t \mapsto \exp(tu)$ is a ray for any u in $\Phi_0(p)v$. Since $\Phi_0(p)$ is S^1, the result follows. Next, let $p \in S$, and carry out the basic soul construction at p. Then $S = C_0 = \partial C_0$, and the closure of $B_t(S)$ equals C_t, where $B_t(S) := \{q \in M \,/\, d(q,S) < t\}$. To see this, consider a minimal connection γ' from a given $q \in M-S$ to S. Then $\gamma := -\gamma'$ is a ray with $\gamma(t_0)=q$, where $t_0 := d(q,S)$. Let X denote the parallel vector field along some minimal geodesic $c: [0,a] \to S$ from $\gamma(0)$ to p, with $X(0) = \dot\gamma(0)$. Then $t \mapsto \tilde\gamma(t) := \exp tX(a)$ is a ray at p, $\tilde\gamma(t_0) \in \partial C_{t_0}$, and by 0.3, $s \mapsto \exp_{c(s)} t_0 X(s)$ is a curve in ∂C_{t_0} from q to $\tilde\gamma(t_0)$. In particular, $q \in \partial C_{t_0}$. Thus $\partial \overline{B}_{t_0}(S) \subset \partial C_{t_0}$, $t_0 > 0$. This also shows that $C_0 \subset S$. Now assume q is in S, and choose a minimal geodesic c from p to q. By the argument in 1.1, $c(\mathbb{R})$ is contained in some ∂C_t. Then $p = c(0)$ belongs to $\partial C_0 \cap \partial C_t$, so $t = 0$. Hence $S \subset \partial C_0$. The inclusion $\partial C_t \subset \partial \overline{B_t}(S)$ now follows easily.

Finally, we show that $\exp_\nu : \nu(S) \to M$ is a diffeomorphism. Since every q in M has a minimal connection to S, $\exp_\nu$ is onto. Suppose there are 2 minimal connections $\gamma_i: [0,t_0] \to M$ from S to q, i = 1, 2. This would contradict $\gamma_1(t_0+\delta) \in \partial C_{t_0+\delta}$, since the composite curve

$\gamma_2|[0,t_0] * \gamma_1|[t_0,t_0+\delta]$ is a connection of length $t_0+\delta$ from S to $\gamma_1(t_0+\delta)$ which can be shortened. Thus $\exp_\nu$ is 1-1.

To complete the proof of 1.2, recall that if K denotes the connection map of $\nu(S)$, then

$$\langle\langle a , b \rangle\rangle := \langle Ka , Kb \rangle + \langle \pi_{\nu_*}a , \pi_{\nu_*}b \rangle, \quad a, b \in (T\nu)_v$$

defines a metric on $\nu(S)$, called the connection metric, such that the projection π_ν: $\nu(S) \to S$ becomes a Riemannian submersion. Define $\pi := \pi_\nu \circ \exp_\nu^{-1} : M \to S$. Then π is a submersion, and to show π is Riemannian, it suffices to establish the following:

(1) $\exp_{\nu_*}$ maps the horizontal and vertical subspaces of π_ν onto mutually orthogonal subspaces.

(2) $\exp_{\nu_*}$ is isometric on the horizontal subspaces.

So let $0 \neq z \in \nu(S)$, $\pi_\nu(z) =: p$, $a \in (T\nu)_z$ horizontal, $b \in (T\nu)_z$ vertical. Since exp is radially isometric, we may assume $\langle\langle b,A_z z \rangle\rangle = 0$, where $A_z : \nu_p \to (\nu_p)_z$ denotes the canonical isomorphism between the fiber through p and its tangent space at z. Set $u := \exp_{\nu_*} b$, $w := \exp_{\nu_*} a$, and let γ denote the ray $\gamma(t) = \exp(tz/\|z\|)$. u determines a variation of γ through rays emanating from p, and thus a Jacobi field X along γ, with $X(0)=0$, $X'(0)=(A_z^{-1}b)/\|z\|$, and $X(\|z\|)=u$. Consider the geodesic c: $\mathbb{R} \to S$ with $\dot c(0)=\pi_{\nu_*} a=\pi_* w$. c and γ determine a flat totally geodesic rectangle $V(t,s) = \exp_{c(t)} sW(t)$, where W is the parallel vector field along c with $W(0) = z/\|z\|$. Thus the Jacobi field Y along γ, $Y(s) := V_* \partial_{t|0,s}$ is parallel along γ.

Moreover, by uniqueness of horizontal lifts, $\|z\|\widehat{\dot W(0)} = a$, so that $w = \exp_* a = Y(\|z\|)$. Then $\|w\| = \|Y(\|z\|)\| = \|Y(0)\| = \|\pi_{\nu_*} a\| = \|a\|$, which proves (2).

Finally, since X and Y are Jacobi and Y is parallel, $\langle X',Y\rangle - \langle Y',X\rangle = \langle X',Y\rangle$ is constant, and $\langle X',Y\rangle = \langle X',Y\rangle|_0 = \langle A_z{}^{-1}b,\pi_{\nu_*} a\rangle/\|z\| = 0$. Therefore, $\langle X,Y\rangle$ is constant, and $\langle u,w\rangle = \langle X,Y\rangle|_{\|z\|} = \langle X,Y\rangle|_0 = 0$, which proves (1). □

We now examine the submersion case in more detail. For the sake of simplicity, M and S will be assumed oriented, even though this hypothesis is often unnecessary . In any case, local results carry through to nonorientable M, while similar global results can be obtained by considering the orientation covering.

Denote by J the canonical complex structure on $\nu(S)$, i.e. $JU = V$ for (local) oriented orthonormal sections $\{U,V\}$ of ν. Define vector fields $\tilde\partial_r$, $\tilde\partial_\vartheta$ on $\nu(S)-S$ as follows:

$$\tilde\partial_r|_z := A_z z/\|z\|,\quad \tilde\partial_\vartheta|_z := A_z Jz,\ z \in \nu(S)-S,$$

where A is the isomorphism defined in 1.2. ($\tilde\partial_r$, $\tilde\partial_\vartheta$, when restricted to a fiber, are just the standard polar coordinates vector fields). Let ∂_r and ∂_ϑ denote the corresponding $\exp_\nu$-related vector fields on M-S, with dual 1-forms dr and $d\vartheta$. Observe that $\partial_r = \nabla d_S$, where d_S is the distance function from the soul, while ∂_ϑ, when restricted to a ray originating at S, is a Jacobi field Y with initial conditions $Y(0)=0$, $\|Y'(0)\|=1$. Moreover,

$[\partial_r,\partial_\theta] = 0$, and if $\overline{X}$ is the horizontal lift of $X \in \mathfrak{X}S$, then $[\overline{X}, \partial_r] = [\overline{X}, \partial_\theta] = 0$, since $[\tilde{X}, \tilde{\partial}_r] = [\tilde{X}, \tilde{\partial}_\theta] = 0$ in $\nu(S)$, for the horizontal lift $\tilde{X}$ of X to $\nu(S)$. This can be seen as follows. First notice that the fibers of ν are totally geodesic: if c is a curve in the zero section, then a horizontal lift of c is a parallel section U of ν along c. Since S is totally geodesic, U is parallel in M along c, and thus $\langle U,U \rangle$ is constant. By 0.6(1), ν has totally geodesic fibers. Therefore

$$\begin{aligned} 0 = \langle \tilde{\nabla}_{\tilde{\partial}_\theta} \tilde{X}, \tilde{\partial}_\theta \rangle &= \frac{1}{2} \tilde{X}\langle \tilde{\partial}_\theta, \tilde{\partial}_\theta \rangle + \langle [\tilde{\partial}_\theta, \tilde{X}], \tilde{\partial}_\theta \rangle \\ &= \frac{1}{2} \tilde{X}(r^2) + \langle [\tilde{\partial}_\theta, \tilde{X}], \tilde{\partial}_\theta \rangle \\ &= r\langle \tilde{X}, \tilde{\partial}_r \rangle + \langle [\tilde{\partial}_\theta, \tilde{X}], \tilde{\partial}_\theta \rangle \\ &= \langle [\tilde{\partial}_\theta, \tilde{X}], \tilde{\partial}_\theta \rangle \end{aligned}$$

On the other hand,

$$\langle [\tilde{\partial}_\theta, \tilde{X}], \tilde{\partial}_r \rangle = [\tilde{\partial}_\theta, \tilde{X}](r) = \tilde{\partial}_\theta \langle \tilde{X}, \tilde{\partial}_r \rangle - \tilde{X}\langle \tilde{\partial}_\theta, \tilde{\partial}_r \rangle = 0$$

Hence $[\tilde{\partial}_\theta, \tilde{X}]^v = 0$. Since $\tilde{\partial}_\theta$ is vertical, $[\tilde{\partial}_\theta, \tilde{X}] = [\tilde{\partial}_\theta, \tilde{X}]^v = 0$.

Proposition 1.3

(i) Let Ω *denote the curvature form of* $\nu(S)$ *, viewed as a 2-form on* S, *i.e.* $\Omega(X,Y) := \langle R(X,Y)U, JU \rangle$, *for* $X,Y \in \mathfrak{X}S$, $U \in \Gamma\nu$ *of norm 1. If* $\overline{X}, \overline{Y} \in \mathfrak{X}M$ *are the horizontal lifts of* $X,Y \in \mathfrak{X}S$, *then*

$$[\overline{X},\overline{Y}]^v = -\Omega(X,Y)\ \partial_\theta.$$

In particular, if the O'Neill tensor is zero (resp. nonzero) at some point q, then it is identically zero (resp. nowhere zero) on the fiber through q.

(ii) Set $G^2 := \langle \partial_\vartheta, \partial_\vartheta \rangle$, *so that the fiber metric is* $dr^2 + G^2 d\vartheta^2$. *If the O'Neill tensor is nonzero on a fiber, then* G *is bounded on that fiber. The intrinsic sectional curvature of a fiber equals the one induced by* M,

$$K_{fiber} = -G^{-1}G_{rr}.$$

(iii) Consider $\nu(S)$ *with the connection metric, and replace the standard flat fiber metric* $dr^2+r^2d\vartheta^2$ *by* $dr^2+(G\circ \exp_\nu)^2 d\vartheta^2$. *Then* $\exp_\nu: \nu(S) \to M$ *is an isometry.*

Proof. As before, $\tilde{X}$ and $\overline{X}$ are the horizontal lifts of $X \in \mathfrak{X}S$ to $\nu(S)$ and M respectively. Since $\exp_\nu$ preserves the orthogonal splitting,

$$[\overline{X},\overline{Y}]^v|_{\exp z} = \exp_* [\tilde{X},\tilde{Y}]^v|_z\,,\; z \in \nu(S).$$

If R and K denote the curvature tensor and the connection map of $\nu(S)$, then

$$R(X,Y)z = -K[\tilde{X},\tilde{Y}]|_z\,, \text{ or equivalently,}$$

$$[\tilde{X},\tilde{Y}]^v|_z = -A_z R(X,Y)z = -\Omega(X,Y)A_z Jz = -\,\Omega(X,Y)\,\tilde{\partial}_\vartheta|_z\,.$$

Applying $\exp_{\nu_*}$ to the last equation now yields (i).

By O'Neill's formula 0.5(3) and (i),

$$(3/4)\Omega^2(X,Y)G^2 = (3/4)\|\,[\overline{X},\overline{Y}]^v\|^2 = K_{X,Y} - K_{\overline{X},\overline{Y}} \le K_{X,Y}\,,$$

hence G is bounded if Ω is nonzero.

Consider a horizontal $u \in TM$. Since $\nabla_u \partial_r = 0$, we have $\ell_u(\partial_r,\partial_r) = \ell_u(\partial_r,\partial_\vartheta) = 0$, where ℓ_u is the second fundamental form of the fiber with respect to u; by the Gauss equations, the intrinsic sectional curvature of the fiber equals the one induced by M. The 2-dimensional Jacobi equation then yields $K_{fiber} = -G^{-1}G_{rr}$. Finally, (iii) is implicitely contained in the proof of 1.2. □

For any horizontal unit-speed geodesic $c\colon \mathbb{R} \to M$, $T := \partial_\vartheta \circ c$ is a Jacobi field along c. Let $\mu(t)$ denote the principal curvature of the fiber through $c(t)$ with corresponding principal curvature direction $G^{-1}T$. Thus $S_{\dot c}\, \partial_\vartheta = \mu T$, (S is the second fundamental tensor of the fiber), and

$$\mu = (G\circ c)^{-1}\,(G\circ c)' = (G\circ c)^{-2}\,\langle T',T\rangle.$$

Differentiating this equation yields:

$$\mu' = -\,K_{\dot c,T} + (G\circ c)^{-2}\|T'\|^2 - 2\mu^2.$$

Suppose now that $\nu(S)$ is flat, or equivalently, that the O'Neill tensor is identically zero. Then $T'^h = 0$, for if X is horizontal,

$$\langle T'^h, X\circ c\rangle = -\langle T,(X\circ c)'\rangle = -\langle T, A_{\dot c}X\rangle = 0.$$

Thus $\|T'\|^2 = \|T'^v\|^2 = \mu^2(G\circ c)^2$, and

$$\mu' = -\,\mu^2 - K_{\dot c,T}$$

This in turn implies that $\mu \equiv 0$. For if φ is an antiderivative of μ, then

$$(e^\varphi)'' = e^\varphi(\mu' + \mu^2) \le 0$$

Thus e^{φ} is concave and bounded from below, hence constant, and $\mu \equiv 0$. Therefore, the fibers are totally geodesic. Together with the fact that ν is flat, this implies (cf. 0.5(3)):

Theorem 1.4 *Assume* S *has codimension 2. If* $\nu(S)$ *is flat and if every normal direction represents a ray, then* M *is locally isometrically a product* .

One should take care, when dealing with flat normal bundles, to distinguish them from trivial ones. Of course, if S is topologically a 2-sphere, then $\nu(S)$ is trivial whenever it is flat. The converse is not true in general. Consider for example the free $\mathbb{R}$-action Γ on $S^2 \times \mathbb{R}^2 \times \mathbb{R}$ given by the formula $(q,u,t_0) \mapsto (\varphi_t q, e^{it}u, t_0+t)$, where φ_t denotes rotation by angle t in S^2 about the z-axis, and e^{it} is rotation by angle t in $\mathbb{R}^2$ around the origin. Γ acts freely by isometries on the Riemannian product $S^2 \times \mathbb{R}^2 \times \mathbb{R}$, and there is a unique metric of nonnegative curvature on $M = S^2 \times \mathbb{R}^2 \times \mathbb{R} \,/\Gamma$ for which the projection $S^2 \times \mathbb{R}^2 \times \mathbb{R} \to M$ becomes a Riemannian submersion, cf. section 2. We claim that M is diffeomorphic to $S^2 \times \mathbb{R}^2$, and that under this identification, the soul S turns out to be $S^2 \times 0$, while the submersion π: M $\to$ S becomes the projection π_1: $S^2 \times \mathbb{R}^2 \to S^2 \times 0$. Nevertheless, the metric on M is not a Riemannian product, hence the normal bundle of S is not flat even though it is trivial. To see that M is diffeomorphic to $S^2\times\mathbb{R}^2$, consider ρ: $S^2\times\mathbb{R}^2\times\mathbb{R} \to S^2\times\mathbb{R}^2$, $\rho(q,u,t):= (\varphi_{-t}q,e^{-it}u)$. Then $\rho^{-1}(q,u) = \Gamma$-orbit of $(q,u,0)$, and therefore f: $S^2\times\mathbb{R}^2 \to M$ is a diffeomorphism, where $f(q,u):= [(q,u,0)]$. Identify M with $S^2\times\mathbb{R}^2$. Next, notice that if U is the Killing field on S^2 with flow φ, $\partial/\partial\vartheta$ the usual polar coordinate vector field on $\mathbb{R}^2$, and D the standard vector field on $\mathbb{R}$, then the vertical subspace of ρ at $(q,u,0)$ is spanned by

$$\begin{cases}(U,\partial/\partial\vartheta,D), \text{ if } u \neq 0\\ (U,0,D), \text{ if } u = 0\end{cases}$$

while the horizontal subspace is spanned by $(V,0,0)$, where $V \in \mathfrak{X}S^2$ is orthogonal to U of unit length, $(U,0,-\|U\|^2D)$, $(0,\partial/\partial r,0)$, and $(-r^2U,(1+\|U\|^2)\partial/\partial\vartheta,-r^2D)$. If q is the North or South pole, the first two vectors are to be replaced by a basis for $(S^2)_q$, while if $u = 0$, the last two vectors are to be replaced by a basis for $(\mathbb{R}^2)_0$. In particular, the normal bundle of $S^2\times 0$ has fiber over $(q,0)$ equal to $0_q\times(\mathbb{R}^2)_0$. Since $(0,\partial/\partial r,0)$ is horizontal, $t \mapsto \rho(q,tv,0) = (q,tv)$ is a geodesic in M for any $q \in S^2$, $v \in \mathbb{R}^2$. Consequently, every direction in the normal bundle yields a ray. By the basic soul construction, it follows that $S^2\times 0$ is the soul of M, and the submersion $M \to S$ is the projection π_1: $S^2\times\mathbb{R}^2 \to S^2\times 0$. In fact, it is easily seen that the ρ-horizontal vector fields $(U,0,-\|U\|^2D)$ and $(V,0,0)$ on $S^2\times\mathbb{R}^2\times\mathbb{R}$ are ρ-related to the vector fields $X:= ((1+\|U\|^2)U,\|U\|^2\partial/\partial\vartheta)$ and $Y:= (V,0)$ on

$S^2 \times \mathbb{R}^2$. Moreover, $\pi_{1*}X = X \circ \pi_1$, $\pi_{1*}Y = Y \circ \pi_1$, and $\|X_{q,u}\| = \|(U,0,-\|U\|^2 D)_{q,u,0}\| = \|(U,0,-\|U\|^2 D)_{q,0,0}\| = \|X_{q,0}\| = \|\pi_{1*}X_{q,u}\|$. Similarly, $\|Y_{q,u}\| = \|\pi_{1*}Y_{q,u}\|$, and so X,Y are basic for π_1. It follows that the normal bundle of S is not flat. For if T denotes the unit vector field in direction $(-r^2 U,(1+\|U\|^2)\partial/\partial\vartheta,-r^2 D)$, then

$$\rho_* T = \frac{1}{G}\partial_\theta = (0, \frac{(\varepsilon^2+r^2)^{1/2}}{\varepsilon r}\partial/\partial\vartheta) \qquad (*)$$

where $\varepsilon := (1+\|U\|^2)^{1/2}$. Thus

$$\begin{aligned} \| [X,Y]^V \circ \rho \| &= | \langle [X,Y] \circ \rho, \rho_* T \rangle | \\ &= | \langle \rho_* [(U,0,-\|U\|^2 D),(V,0,0)], \rho_* T \rangle | \\ &= | \langle [(U,0,-\|U\|^2 D),(V,0,0)], T \rangle | \\ &= | \langle ([U,V],0,0) + V\|U\|^2 (0,0,D) , T \rangle | \\ &= |(V\|U\|^2) \langle (0,0,D) , T \rangle| \neq 0, \end{aligned}$$

except on equator and poles. Therefore $\nu(S)$ is not flat. Notice that the fibers are not totally geodesic. In fact, by (*),

$$G = \frac{r\varepsilon}{(\varepsilon^2+r^2)^{1/2}}$$

and the principal curvature of the fibers in direction Y is $(YG)/G \neq 0$, since $V\|U\| \neq 0$. This feature is the key obstruction here. Indeed, one has

Theorem 1.5 *If* M^4 *is a trivial bundle over* S, *and* $\pi : M \to S$ *has totally geodesic fibers, then* π *is a locally isometrically trivial fibration*

Together with 1.4, this result immediately implies

Corollary 1.6 *Suppose* M^4 *has soul* S *diffeomorphic to a 2-sphere, and every direction in* $\nu(S)$ *is a ray direction. Then the following statements are equivalent:*

(i) $\nu(S)$ *is flat*

(ii) M *is diffeomorphic to* $S \times \mathbb{R}^2$ *and* $\pi : M \to S$ *has totally geodesic fibers*

(iii) $M = S \times P_2$ *isometrically, where* P_2 *is* $\mathbb{R}^2$ *together with some metric of nonnegative curvature.*

To prove 1.5, we need

Lemma 1.7

(i) $\operatorname{div} \partial_\theta = \partial_\theta \ln G$. *If* ∂_θ *is divergence-free, then it is a Killing field on* M.

(ii) If $\nu(S)$ *is not flat and* $\pi : M \to S$ *has totally geodesic fibers, then* ∂_θ *is a Killing field.*

Proof of 1.7 If $\{X_i\}$ is a local orthonormal basis of basic vectors fields, then

$$\begin{aligned}\operatorname{div} \partial_\vartheta &= G^{-2}\langle \nabla_{\partial_\vartheta} \partial_\vartheta, \partial_\vartheta\rangle + \langle \nabla_{\partial_r} \partial_\vartheta, \partial_r\rangle + \sum_i \langle \nabla_{X_i} \partial_\vartheta, X_i\rangle \\ &= \partial_\vartheta \ln G - \langle \partial_\vartheta, \nabla_{\partial_r} \partial_r\rangle - \sum_i \langle \partial_\vartheta, (\nabla_{X_i} X_i)^v\rangle \\ &= \partial_\vartheta \ln G.\end{aligned}$$

Assume $\operatorname{div} \partial_\vartheta = 0$. Then

$$\begin{aligned}&\langle \nabla_{X_i} \partial_\vartheta, X_j\rangle + \langle \nabla_{X_j} \partial_\vartheta, X_i\rangle = -\langle \partial_\vartheta, (\nabla_{X_i} X_j)^v + (\nabla_{X_j} X_i)^v\rangle = 0 , \\ &\langle \nabla_{X_i} \partial_\vartheta, \partial_\vartheta\rangle + \langle \nabla_{\partial_\vartheta} \partial_\vartheta, X_i\rangle = \langle [X_i, \partial_\vartheta], \partial_\vartheta\rangle = 0 , \\ &\langle \nabla_{X_i} \partial_\vartheta, \partial_r\rangle + \langle \nabla_{\partial_r} \partial_\vartheta, X_i\rangle = - \langle \partial_\vartheta, \nabla_{X_i} \partial_r + \nabla_{\partial_r} X_i\rangle = 0 , \\ &\langle \nabla_{\partial_\vartheta} \partial_\vartheta, \partial_r\rangle + \langle \nabla_{\partial_r} \partial_\vartheta, \partial_\vartheta\rangle = \langle [\partial_r, \partial_\vartheta], \partial_\vartheta\rangle = 0.\end{aligned}$$

Thus ∂_ϑ is a Killing field. To prove (ii), choose $p \in S$ so that $\Omega_p \neq 0$. Since the fibers are totally geodesic, $[\overline{X}, \overline{Y}]^v = -\Omega(X,Y)\ \partial_\vartheta$ is Killing on the fiber through p, cf. 0.6(2), implying $\partial_\vartheta G = 0$ on this fiber. But for any basic X, $X\partial_\vartheta G = \partial_\vartheta X G = 0$, so that $\partial_\vartheta G \equiv 0$ on M. By (i), ∂_ϑ is Killing on M. □

Proof of 1.5 If π is not locally isometrically trivial, then $\nu(S)$ cannot be flat by 1.4. By 1.7, ∂_ϑ is a Killing field. Fix some positive r, and consider the set N of points of M at distance r from S. N has nonnegative curvature by the Gauss equations, is diffeomorphic to $S\times S^1$, and thus admits a parallel vector field Z by basic harmonic theory or [CG]. Then $\langle Z,\partial_\vartheta\rangle$ is a harmonic function on the compact N, and is therefore constant. Since $G = \|\partial_\vartheta\|$ is also constant on N, the same must be true for the angle between Z and ∂_ϑ. Choose $p \in S$ so that $\Omega_p \neq 0$, and let $q \in N \cap \pi^{-1}(p)$. If $\overline{X}$, $\overline{Y}$ are basic orthonormal, equation (2.2) in section 2 yields:

$$\nabla_{\overline{X}_q}\partial_\vartheta = \frac{1}{2}\Omega_p(X,Y)G^2\overline{Y}_q \neq 0,\ \nabla_{\overline{Y}_q}\partial_\vartheta = -\frac{1}{2}\Omega_p(X,Y)G^2\overline{X}_q \neq 0.$$

But $0 = \overline{X}\langle Z,\partial_\vartheta\rangle = (1/2)\Omega(X,Y)G^2\langle\overline{Y},Z\rangle$, so that $Z \perp \overline{Y}$ on the fiber over p. Similarly $Z \perp \overline{X}$, and Z is then vertical on this fiber. Hence Z is vertical everywhere, and so ∂_ϑ, being a constant multiple of Z, is a parallel vector field, contradicting $\nabla_{\overline{X}_q}\partial_\vartheta \neq 0$. Thus π is locally isometrically trivial. □

Recall that the total curvature of an oriented complete even-dimensional manifold M is defined as $\int_M \chi$ (if it exists), where χ is the Chern-Euler form of M. When dim M = 2, $\chi = (1/2\pi)\,K$ (K is the sectional curvature), and for $K \geq 0$, it is known that the total curvature is bounded between 0 and 1, cf. [C-V].

Lemma 1.8 *Suppose* $\Omega \neq 0$ *at some* $p \in S$. *Then the fiber through* p *has total curvature 1. In particular, if* $\pi : M \to S$ *has totally geodesic fibers and is not locally a Riemannian product, then every fiber has total curvature 1.*

Proof By 1.3(ii), G is bounded on the fiber through p. Since $G_{rr} \leq 0$, $r \mapsto G_r(r,\vartheta)$ is a nonincreasing function of r and admits a limit as $r \to \infty$. This limit must be 0, for otherwise, G would grow without bound. Thus

$$\int_0^\infty -G_{rr}\big|_{r,\vartheta}\, dr = \lim_{r \to 0} G_r\big|_{r,\vartheta} = 1$$

and the total curvature of the fiber through p is:

$$\frac{1}{2\pi}\int_{\text{fiber}} K_{\text{fiber}} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty - G_{rr}\, dr\, d\vartheta = 1 \text{ , by 1.3(ii).}$$

If M is not locally isometrically a product, then Ω is nonzero at some $p \in S$ by 1.4. Thus the fiber through p has total curvature 1. Since the fibers are totally geodesic, they are all isometric to one another (cf. 0.6(1)), and the statement follows. □

It is known that the total curvature of any 4-dimensional oriented manifold of nonnegative curvature exists, and is bounded between 0 and the Euler characteristic of M, cf. [P].

Assume dim M = 4. Under our additional assumptions, namely dim S = 2 and every normal direction represents a ray, we can prove a stronger result:

Theorem 1.9 *Let* $\kappa(p)$ *denote the total curvature of the fiber through* $p \in S$, $\kappa: S \to [0,1]$. *Then the total curvature of* M^4 *equals:*

$$\frac{1}{2\pi}\int_S \kappa K_S$$

where K_S *is the sectional curvature of* S.

Assume furthermore that S *is diffeomorphic to the 2-sphere (the only other possibility is* S = *flat torus, in which case the total curvature of* M *is* 0.*), and that* $\pi: M \to S$ *has totally geodesic fibers. Then the total curvature of* M *is* 2, *unless* M = $S \times P_2$ *isometrically, in which case it is* 2κ.

Proof Let $M^r := \{q \in M \,/\, d(q,S) \leq r\}$. Thus each ∂M^r is diffeomorphic via $\exp_\nu^{-1}$ to the sphere bundle of radius r over S, and admits the restriction of $\nabla d_S = \partial_r$ as unit normal vector field. ω_r and ω_s will denote the volume forms of ∂M^r and S respectively. The Gauss Bonnet theorem for manifolds with boundary then yields:

$$\int_{M^r} \chi = \chi(S) + \int_{\partial M^r} g_r \omega_r$$

where $\chi(S)$ is the Euler characteristic of S, and

$g_r(q) = (-1/4\pi^2)\{\lambda_1 K_{23} + \lambda_2 K_{13} + \lambda_3 K_{12} + \lambda_1\lambda_2\lambda_3\}$, cf. [P].

Here the λ_i are the principal curvatures of ∂M^r at q, with principal curvature direction u_i, and K_{ij} is the sectional curvature of the plane spanned by u_i and u_j. Now $\nabla_u \partial_r = 0$ for horizontal u, and $\nabla_{(1/G)\partial_\vartheta} \partial_r = G^{-2}G_r\, \partial_\vartheta$. Thus

$$\int_{M^r} \chi = \chi(S) - \frac{1}{4\pi^2} \int_{\partial M^r} K_h G^{-1} G_r \omega_r ,$$

where $K_h(q)$ is the sectional curvature of the unique horizontal 2-plane contained in $(\partial M^r)_q$. Since the restriction of π to ∂M^r is a Riemannian submersion, Fubini's theorem yields:

$$\int_{\partial M^r} K_h G^{-1} G_r \omega_r = \int_{\partial M^r} \{K_S - \frac{3}{4} f^2 G^2\} G^{-1} G_r \omega_r$$

$$= \int_S K_S (\int_0^{2\pi} G_r \, d\vartheta)\omega_s - \frac{3}{4}\int_S f^2 (\int_0^{2\pi} G^2 G_r \, d\vartheta)\omega_s$$

Here, f is defined by the equation $\Omega = f\omega_s$. Now

$$\int_S f^2\Big(\int_0^{2\pi} G^2 G_r\, d\vartheta\Big)\omega_s = \int_{\{f\neq 0\}} f^2\Big(\int_0^{2\pi} G^2 G_r\, d\vartheta\Big)\omega_s$$

$$\leq \frac{4}{3}\int_{\{f\neq 0\}} K_S\Big(\int_0^{2\pi} G_r\, d\vartheta\Big)\omega_s\ , \text{ by 1.3(ii). Thus}$$

$$\lim_{r\to\infty}\int_S f^2\Big(\int_0^{2\pi} G^2 G_r\, d\vartheta\Big)\omega_s = 0\ , \text{ by monotone convergence, and}$$

$$\int_M \chi = \lim_{r\to\infty}\int_{M^r}\chi = \chi(S) - \frac{1}{4\pi^2}\lim_{r\to\infty}\int_S K_S\Big(\int_0^{2\pi} G_r\, d\vartheta\Big)\omega_s$$

$$= \frac{1}{4\pi^2}\lim_{r\to\infty}\int_S\Big(\int_0^{2\pi} 1 - G_r\, d\vartheta\Big)K_S\omega_s$$

$$= \frac{1}{2\pi}\int_S \kappa K_S\omega_s .$$

The last statement of the theorem now follows from 1.6. □

2. Some metrics on vector bundles over spheres

Theorem 1.4 shows that the flat bundle case is rigid. The standard examples of nonnegative curvature in the nonflat case are found in [CG] and [C]. We briefly recall this construction, for fibers diffeomorphic to $\mathbb{R}^2$.

Let G be a Lie group with biinvariant metric, and let P_2 denote $\mathbb{R}^2$ together with a metric of nonnegative curvature. Suppose H is a closed subgroup of G which acts on P_2 by isometries. Then H acts freely on the Riemannian product $G \times P_2$ via $(g,m) \mapsto (gh,h^{-1}m)$, and there is a metric of nonnegative curvature on the quotient $M = G \times_H P_2$ with respect to which the projection π: $G \times P_2 \rightarrow M$ becomes a Riemannian submersion. For example, let $G = S^3$, and $H = S^1$ acting on $\mathbb{R}^2$ by rotations around the origin, so that M is topologically the 2-dimensional vector bundle over S^2 associated with the Hopf fibration. It is straightforward to check that with the above metric, the soul (= the zero section) of M is isometric to the 2-sphere of constant curvature 4. The fibers are totally geodesic, and with the notation of section 1, $G = r/(1+r^2)^{1/2}$, while $f \equiv 2$. To be specific, view S^3 as the Lie group of quaternions of norm 1, and S^1 as the subgroup $S^3 \cap \mathbb{C}$. The homogeneous space S^3/S^1 is diffeomorphic to the 2- sphere via φ: $S^3/S^1 \rightarrow S^2 \subset \mathbb{R}^3 =$

span{i,j,k}, $\varphi(qS^1) := qiq^{-1}$. Consider the projection $\pi\colon S^3 \to S^3/S^1$. Let X_1, X_2, X_3 denote the left-invariant vector fields on S^3 whose values at e are $A_e i$, $A_e j$, $A_e k$ respectively. Thus the fiber qS^1 of π through q is the image of the integral curve of X_1 through q. The standard biinvariant metric on S^3 is just the usual metric $\langle X_i, X_j\rangle = \delta_{ij}$, and we consider that metric on S^2 for which π becomes a Riemannian submersion. If $\mathfrak{h}$ denotes the Lie algebra of S^1, then $\mathfrak{m} := \mathfrak{h}^\perp$ is spanned by $\{X_2, X_3\}$, and by 0.5(3), the sectional curvature of S^2 is

$$K = K_{S^3} + \frac{3}{4}\,\|[X_2,X_3]^v\|^2 = 1 + \frac{3}{4}\,\|2X_1\|^2 = 4.$$

Thus S^3/S^1 is isometric to the 2-sphere of constant curvature 4. Next, we look at the associated bundle $M = S^3 \times_{S^1} \mathbb{R}^2$. One has the commutative diagram

$$\begin{array}{ccc} S^3 \times \mathbb{R}^2 & \xrightarrow{\ \pi_1\ } & S^3 \\ \rho \downarrow & & \downarrow \pi \\ M & \xrightarrow[\ \pi_M\ ]{} & S^2 \end{array}$$

Here π_1 is projection onto the first factor, $\rho(q,u) := [(q,u)]$, where $(q,u) \sim (qz, z^{-1}u)$, $q \in S^3$, $z, u \in \mathbb{R}^2 = \mathbb{C}$, $\|z\| = 1$, and π_M is defined by the equation $\pi_M \circ \rho = \pi \circ \pi_1$, cf. [P2]. π_M is a fiber bundle with fiber $\mathbb{R}^2$, and it is known that if M is given the ρ-submersion metric, then the soul S of M is S^2 and the submersion $M \to S$ from 1.2 is π_M, after identification of S^2 with the zero section, cf. [CG], [C], and [CE]. Since the proof of this does not appear in the litterature, we shall carry it out in some detail. $\partial/\partial r$ and $\partial/\partial\vartheta$ will denote the standard polar coordinate vector fields on $\mathbb{R}^2$. For $(q,u) \in S^3\times\mathbb{R}^2$, $\rho^{-1}(\rho(q,u)) = \{(qz, z^{-1}u)/\ z \in S^1\}$, and therefore $(S^3\times\mathbb{R}^2)_{q,u}{}^{v}$ is spanned by $(X_{1|q}, -\partial/\partial\vartheta_{|u}) = \dot{c}(0)$, where $c(t) := (qe^{it}, e^{-it}u)$. The horizontal subspace at (q,u) is thus spanned by $(X_2,0)_{|q,u}$, $(X_3,0)_{|q,u}$, $(0,\partial/\partial r)_{|q,u}$, $(1+r^2)^{-1/2}(rX_1, r^{-1}\partial/\partial\vartheta)_{|q,u}$, for $u \neq 0$. Here, $r := \|u\|$. By the proof of 1.2, it suffices to establish that $\rho(S^3\times 0)$ is totally convex and that every direction in its normal bundle is a ray direction. For the first part, consider a geodesic $c\colon [0,b] \to M$ joining $\rho(p,0)$ to $\rho(q,0)$. Its horizontal ρ-lift $\bar{c}$ with $\bar{c}(0) = (p,0)$ is a geodesic in $S^3\times\mathbb{R}^2$ from $(p,0)$ to $(qz,0)$ for some $z \in S^1$. Then the projection $\pi_2\circ\bar{c}$ is a geodesic in $\mathbb{R}^2$ with $\pi_2\circ\bar{c}(0) = \pi_2\circ\bar{c}(b) = 0$. Thus $\pi_2\circ\bar{c} \equiv 0$, and $c = \rho\circ\bar{c}$ is contained in $\rho(S^3\times 0)$. This shows that $\rho(S^3\times 0)$ is totally convex. For the second part, notice that the tangent space of $\rho(S^3\times 0)$ at $\rho(q,0)$ is spanned by $\rho_*(X_2,0)_{|q,0}$, $\rho_*(X_3,0)_{|q,0}$. Therefore $\gamma\colon [0,\infty) \to M$, $\gamma(t) := \rho(q,tu)$, $u \in \mathbb{R}^2$ of norm 1, is a ρ-horizontal geodesic with $\dot\gamma(0) \in$ normal bundle of

$\rho(S^3\times 0)$. Let $E := S^3\times\mathbb{R}^2$. For any $t > 0$,

$$t = L(\gamma|_{[0,t]}) \geq d_M(\rho(q,0),\rho(q,tu)) = d_E(\rho^{-1}\rho(q,0),\rho^{-1}\rho(q,tu)) = t.$$

Thus γ is a ray. In other words, every direction in the normal bundle yields a ray, and $\rho(S^3\times 0)$ is the soul of S. Finally, it is clear that the submersion $M \to S$ from 1.2 is π_M, since the fibers of π_M coincide with the fibers of the normal bundle of S.

Next, we compute G and Ω (notation as in section 1). Consider the diffeomorphism ρ: $q\times\mathbb{R}^2 \to \rho(q\times\mathbb{R}^2)$ = fiber of π_M over $\pi(q)$. Since

$$(0,\partial/\partial\vartheta) = -\frac{r^2}{r^2+1}(X_1,-\partial/\partial\vartheta) + \frac{r}{r^2+1}(rX_1,r^{-1}\partial/\partial\vartheta),$$

we have

$$G = \|\rho_*(0,\partial/\partial\vartheta)\| = \frac{r}{(r^2+1)^{1/2}}\|\rho_*(r^2+1)^{-1/2}(rX_1,r^{-1}\partial/\partial\vartheta)\|$$

$$= \frac{r}{(r^2+1)^{1/2}}$$

In particular, the fibers are totally geodesic. To compute Ω, we proceed as follows: let $v := \pi_{M*}\rho_*(X_2,0)_{|q,u}$, $w := \pi_{M*}\rho_*(X_3,0)_{|q,u}$. Then $\{v,w\}$ is an orthonormal basis of $S^2{}_{\pi(q)}$. Recall that $[.,.]^v$ is a tensor field on the horizontal subbundle, and as such is defined pointwise. If $T := (r^2+1)^{-1/2}(rX_1,r^{-1}\partial/\partial\vartheta)$ ($\Rightarrow$ $\rho_*T = G^{-1}\partial_\vartheta$), then by 1.3,

$$
\begin{aligned}
|\Omega(v,w)| &= \frac{1}{G\circ\rho(q,u)}\,|\langle[\rho_*(X_2,0)_{q,u},\ \rho_*(X_3,0)_{q,u}],\ \rho_*T_{q,u}\rangle| \\
&= \frac{1}{G\circ\rho(q,u)}\,|\langle([X_2,X_3],0),\ T\rangle_{|q,u}| \\
&= \frac{2}{G\circ\rho(q,u)}\,|\langle X_1,T\rangle_{|q,u}| \\
&= 2.
\end{aligned}
$$

In contrast to the rigidity when $\nu(S)$ is flat, one has

Theorem 2.1 *Consider* $M = S^3\times_{S^1}\mathbb{R}^2$ *with the standard submersion metric. Let* h *denote an arbitrary real valued function with compact support in* $M-S$ *and with bounded derivatives up to order 2. Then for small enough* $\varepsilon > 0$, *the metric on* M *obtained by deforming* G *to* $\tilde{G} = G + \varepsilon h$ *has nonnegative sectional curvature.*

Notice that if one chooses h so that $h_\vartheta \neq 0$, then the resulting metric on M cannot originate from the construction described above, i.e. M is not isometrically a quotient $S^3\times_{S^1}\mathbb{R}^2$ for any metrics on S^3 and $\mathbb{R}^2$, since in such a quotient, ∂_ϑ must be a Killing field, implying $G_\vartheta = 0$. To see this, notice that if φ denotes the S^1-action on $\mathbb{R}^2$, then the corresponding Killing field is $\pm\partial/\partial\vartheta$. For any $s \in [0,2\pi]$, $(q,u) \mapsto (q,\varphi_s u)$ is an isometry of $S^3\times\mathbb{R}^2$ that preserves the vertical subspaces, and is therefore isometric on the horizontal ones. It then induces an isometry $\rho(q,u) \mapsto \rho(q,\varphi_s u)$ of M with corresponding Killing field $\rho_*(0,\pm\partial/\partial\vartheta) = \pm\partial_\vartheta$.

Before proceeding to the proof of the theorem, we include for future reference some results that are valid for any 4-dimensional manifold M in the context of 1.2(b). X,Y will denote a local oriented orthonormal basis of vector fields on S, as well as their horizontal lifts. $\mu := (XG)/G$ and $\lambda := (YG)/G$ are the principal curvatures of the fibers of $\pi : M \to S$ in directions X and Y respectively. Then straightforward computations yield:

(2.2)

$\nabla_X\partial_r = \nabla_{\partial_r}X = 0;\quad \nabla_X\partial_\vartheta = \nabla_{\partial_\vartheta}X = \mu\,\partial_\vartheta + (1/2)fG^2Y;$

$\nabla_Y\partial_\vartheta = \nabla_{\partial_\vartheta}Y = \lambda\,\partial_\vartheta - (1/2)fG^2X;$

$\nabla_{\partial_r}\partial_\vartheta = \nabla_{\partial_\vartheta}\partial_r = G^{-1}G_r\,\partial_\vartheta\,;\quad \nabla_{\partial_r}\partial_r = 0;$

$\nabla_{\partial_\vartheta}\partial_\vartheta = G^{-1}G_\vartheta\,\partial_\vartheta - GG_r\,\partial_r - G^2\mu X - G^2\lambda Y\,.$

These equalities in turn imply:

(2.3)

$R(X,\partial_r)\partial_r = R(\partial_r,X)X = 0\;;$

$2R(X,\partial_r)Y = R(X,Y)\partial_r = fG^{-1}G_r\,\partial_\vartheta\,;$

$K_{X,\partial_\vartheta} = G^{-1}(\nabla_X X - XX)G + (1/4)f^2G^2\,;$

$K_{Y,\partial_\vartheta} = G^{-1}(\nabla_Y Y - YY)G + (1/4)f^2G^2;$

$K_{fiber} = -G^{-1}G_{rr}\,;$

$\langle R(\partial_\vartheta,X)X,Y\rangle = -(1/2)\{(Xf)G^2 + 3fG(XG)\}$;

$\langle R(\partial_\vartheta,Y)Y,X\rangle = (1/2)\{(Yf)G^2 + 3fG(YG)\}$;

$\langle R(\partial_r,\partial_\vartheta)\partial_\vartheta,X\rangle = -G\,\partial_r XG$; $\langle R(\partial_r,\partial_\vartheta)\partial_\vartheta,Y\rangle = -G\,\partial_r YG$;

$\langle R(X,\partial_\vartheta)\partial_\vartheta,Y\rangle = G(\nabla_X Y - XY)G$;

$K_{X,Y} = K_S - (3/4)f^2G^2$.

Proof of (2.2) By the proof of 1.1, $\nabla_X\partial_r = \nabla_{\partial_r}X = 0$, $\nabla_{\partial_r}\partial_r = 0$. The other covariant derivatives are computed with the help of the local orthonormal basis $\{X,Y,\partial_r,G^{-1}\partial_\vartheta\}$. Thus, for example,

$$\langle\nabla_X\partial_\theta,\ G^{-1}\partial_\theta\rangle = \frac{1}{2G}X\langle\partial_\theta,\partial_\theta\rangle = XG = G\mu$$

$$\langle\nabla_X\partial_\theta,\partial_r\rangle = X\langle\partial_\theta,\partial_r\rangle - \langle\partial_\theta,\nabla_X\partial_r\rangle = 0$$

$$\langle\nabla_X\partial_\theta,X\rangle = \langle\nabla_{\partial_\theta}X,X\rangle = \frac{1}{2}\partial_\theta\langle X,X\rangle = 0$$

$$\langle\nabla_X\partial_\theta,Y\rangle = -\langle\partial_\theta,\nabla_X Y\rangle = -\frac{1}{2}\langle\partial_\theta,[X,Y]^V\rangle = \frac{1}{2}fG^2$$

Therefore, $\nabla_X\partial_\theta = \nabla_{\partial_\theta}X = \mu\partial_\theta + \frac{1}{2}fG^2Y$, and similarly,

$\nabla_Y\partial_\theta = \nabla_{\partial_\theta}Y = \lambda\partial_\theta - \frac{1}{2}fG^2Y.$

$$\langle\nabla_{\partial_r}\partial_\theta,G^{-1}\partial_\theta\rangle = \frac{1}{2G}\partial_r\langle\partial_\theta,\partial_\theta\rangle = G_r$$

$$\langle\nabla_{\partial_r}\partial_\theta,\partial_r\rangle = -\langle\partial_\theta,\nabla_{\partial_r}\partial_r\rangle = 0$$

$$\langle\nabla_{\partial_r}\partial_\theta,X\rangle = -\langle\partial_\theta,\nabla_{\partial_r}X\rangle = 0$$

so that $\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = G^{-1}G_r\partial_\theta$.

$$\langle\nabla_{\partial_\theta}\partial_\theta, G^{-1}\partial_\theta\rangle = \frac{1}{2G}\,\partial_\theta\langle\partial_\theta,\partial_\theta\rangle = G_\theta$$

$$\langle\nabla_{\partial_\theta}\partial_\theta,\partial_r\rangle = -\,\langle\partial_\theta,\nabla_{\partial_\theta}\partial_r\rangle = -GG_r$$

$$\langle\nabla_{\partial_\theta}\partial_\theta,X\rangle = -\,\langle\partial_\theta,\nabla_{\partial_\theta}X\rangle = -G^2\mu$$

$$\langle\nabla_{\partial_\theta}\partial_\theta,Y\rangle = -G^2\lambda$$

implying $\nabla_{\partial_\theta}\partial_\theta = G^{-1}G_\theta\partial_\theta - GG_r\partial_r - G^2\mu X - G^2\lambda Y$.

Proof of (2.3)

$$\nabla_X\partial_r = \nabla_{\partial_r}X = 0 \Rightarrow R(X,\partial_r)\partial_r = R(\partial_r,X)\partial_r = 0$$

$$\nabla_X X \text{ basic} \Rightarrow \nabla_{\partial_r}\nabla_X X = 0 \Rightarrow R(\partial_r,X)X = R(X,\partial_r)X = 0$$

$$R(X,\partial_r)Y = -\nabla_{\partial_r}\nabla_X Y = -\nabla_{\partial_r}(\nabla_X Y)^v = -\frac{1}{2}\nabla_{\partial_r}[X,Y]^v$$

$$= -\frac{1}{2}\nabla_{[X,Y]^v}\partial_r \quad ([X,Y]^v = f\partial_\theta,\ f_r = 0 \Rightarrow [\partial_r,[X,Y]^v] = 0)$$

$$= -\frac{1}{2}\nabla_{[X,Y]}\partial_r = \frac{1}{2}R(X,Y)\partial_r.$$

$$R(X,Y)\partial_r = -\nabla_{[X,Y]}\partial_r = f\nabla_{\partial_\theta}\partial_r = fG^{-1}G_r\partial_\theta$$

$$\Rightarrow R(X,Y)\partial_r = 2R(X,\partial_r)Y = fG^{-1}G_r\partial_\theta$$

$$R(\partial_\theta,X)X = \nabla_{\partial_\theta}\nabla_X X - \nabla_X\nabla_{\partial_\theta}X$$

$$= \nabla_{\partial_\theta}(-\langle X,\nabla_X Y\rangle Y) - \nabla_X\nabla_{\partial_\theta}X$$

$$(\text{using } \nabla_X X = (\nabla_X X)^h = \langle\nabla_X X,Y\rangle Y = -\langle X,\nabla_X Y\rangle Y)$$

$$= -\langle X,\nabla_X Y\rangle\nabla_{\partial_\theta}Y - \nabla_X\{\mu\partial_\theta + \frac{1}{2}fG^2Y\}$$

$$= -\langle X,\nabla_X Y\rangle\{\lambda\partial_\theta - \frac{1}{2}fG^2X\} - (X\mu)\partial_\theta - \mu\{\mu\partial_\theta + \frac{1}{2}fG^2Y\}$$

$$-\frac{1}{2}X(fG^2)Y - \frac{1}{2}fG^2\nabla_X Y$$

$$= -\{X\mu + \mu^2 - \frac{1}{4}f^2G^2 + \lambda\langle X,\nabla_X Y\rangle\}\partial_\theta - \frac{1}{2}\{X(fG^2) + \mu fG^2\}Y$$

$$(\text{using } \nabla_X Y = (\nabla_X Y)^h + (\nabla_X Y)^v = \langle\nabla_X Y,X\rangle X - \frac{1}{2}f\partial_\theta)$$

$$= -\{X\mu + \mu^2 - \frac{1}{4}f^2G^2 - G^{-1}(\nabla_X X)G\}\partial_\theta - \frac{1}{2}\{X(fG^2) + \mu fG^2\}Y.$$

$$\Rightarrow K_{X,\partial_\theta} = -X\mu - \mu^2 + \frac{1}{4}f^2G^2 + G^{-1}(\nabla_X X)G$$

$$= G^{-1}(\nabla_X X - XX)G + \frac{1}{4}f^2G^2$$

and $\langle R(\partial_\theta,X)X,Y\rangle = -\frac{1}{2}\{X(fG^2) + \mu fG^2\} = -\frac{1}{2}\{G^2Xf + 3fGXG\}$.

The equations for K_{Y,∂_θ} and $\langle R(\partial_\theta,Y)Y,X\rangle$ are similar.

$$R(\partial_r,\partial_\theta)\partial_\theta = \nabla_{\partial_r}\nabla_{\partial_\theta}\partial_\theta - \nabla_{\partial_\theta}\nabla_{\partial_r}\partial_\theta$$

$$= \nabla_{\partial_r}\{G^{-1}G_\theta\partial_\theta - GG_r\partial_r - G^2\mu X - G^2\lambda Y\} - \nabla_{\partial_\theta}\{G^{-1}G_r\partial_\theta\}$$

$$= G^{-2}(-G_rG_\theta + GG_{r\theta})\partial_\theta + G^{-1}G_\theta G^{-1}G_r\partial_\theta - (G_r^2 + GG_{rr})\partial_r$$

$$- \partial_r(GXG)X - \partial_r(GYG)Y - G^{-2}(GG_{r\theta} - G_rG_\theta)\partial_\theta$$

$$- G^{-1}G_r(G^{-1}G_\theta\partial_\theta - GG_r\partial_r - G^2\mu X - G^2\lambda Y)$$

$$= -GG_{rr}\partial_r - G(\partial_r XG)X - G(\partial_r YG)Y$$

$$\Rightarrow K_{\partial_r,\partial_\theta} = -G^{-1}G_{rr}, \quad \langle R(\partial_r,\partial_\theta)\partial_\theta, \left\{\begin{matrix}X\\Y\end{matrix}\right. \rangle = -G\partial_r\left\{\begin{matrix}X\\Y\end{matrix}\right. G.$$

$$\langle R(X,\partial_\theta)\partial_\theta, Y\rangle = \langle \nabla_X\nabla_{\partial_\theta}\partial_\theta, Y\rangle - \langle \nabla_{\partial_\theta}\nabla_X\partial_\theta, Y\rangle$$

$$= \langle \nabla_X\{G^{-1}G_\theta\partial_\theta - GG_r\partial_r - G^2\mu X - G^2\lambda Y\}, Y\rangle - \langle \nabla_{\partial_\theta}\{\mu\partial_\theta + \frac{1}{2}fG^2Y\}, Y\rangle$$

$$= G^{-1}G_\theta\langle \nabla_X\partial_\theta, Y\rangle - G^2\mu\langle \nabla_X X, Y\rangle - X(G^2\lambda) - \mu\langle \nabla_{\partial_\theta}\partial_\theta, Y\rangle - \frac{1}{2}\partial_\theta(fG^2)$$

$$= G^{-1}G_\theta(\frac{1}{2}fG^2) + G(\nabla_X Y)^h G - G^2\lambda\mu - G^2X\lambda + \mu\lambda G^2 - fGG_\theta$$

$$= -\frac{1}{2}fGG_\theta + G(\nabla_X Y)^h G - GXYG = G(\nabla_X Y - XY)G$$

$$\Rightarrow \langle R(X,\partial_\theta)\partial_\theta, Y\rangle = G(\nabla_X Y - XY)G. \quad \square$$

Proof of 2.1. Let C:= supp h. Since $\tilde{\tilde{G}} = G$ outside C, it suffices to check that $K \geq 0$ on C. Notice that (2.2) and (2.3) remain valid in the $\tilde{\tilde{G}}$-metric. Moreover, the G-metric is of the type described in the proof of 2.4, with $\varepsilon = 1$. Thus the only planes of zero curvature in the G-metric are those spanned by ∂_r and a horizontal vector. Planes close by are spanned by {U,V}, where after normalization, we may assume $\|U^h\| = 1$, and $\langle V, \partial_r \rangle = 1$. If X is the horizontal lift of $\pi_* U$, and Y is chosen so that {X,Y} is a local orthonormal basis of basic vector fields, then

$$U = X + \alpha\partial_r + \beta\partial_\theta,\ V = \partial_r + \gamma X + \delta Y + \xi\partial_\theta,\ \alpha,\beta,\gamma,\delta,\xi \in \mathbb{R}.$$

A different plane will in general yield a different X, but this will turn out to be irrelevant because of the boundedness assumptions.

Expanding $\langle R(U,V)V,U\rangle$ and making use of the first two equations of (2.3) as well as the symmetries of the curvature tensor, yields:

$$\begin{aligned}\langle R(U,V)V,U\rangle = {} & \delta^2 K_{X,Y} + (\beta\gamma-\xi)^2\tilde{\tilde{G}}^2 K_{X,\partial_\theta} + (\beta\delta)^2\tilde{\tilde{G}}^2 K_{Y,\partial_\theta} + \\ & (\beta-\alpha\xi)^2\tilde{\tilde{G}}^2 K_{\text{fiber}} + 2\delta(\xi-\beta\gamma)\langle R(\partial_\theta,X)X,Y\rangle + \\ & 3\delta(\beta-\alpha\xi)\langle R(X,Y)\partial_r,\partial_\theta\rangle + \\ & 2(\beta\gamma-\xi)(\beta-\alpha\xi)\langle R(\partial_r,\partial_\theta)\partial_\theta,X\rangle + \\ & 2\beta\delta(\beta-\alpha\xi)\langle R(\partial_r,\partial_\theta)\partial_\theta,Y\rangle + 2\beta\delta(\beta\gamma-\xi)\langle R(X,\partial_\theta)\partial_\theta,Y\rangle \\ & + 2\beta\delta^2\langle R(\partial_\theta,Y)Y,X\rangle.\end{aligned}$$

Set $x_1:=\delta$, $x_2:=\beta-\alpha\xi$, $x_3:=\beta\gamma-\xi$, $x_4:=\beta\delta$. Then the above expression is a quadratic function of $x = (x_1,x_2,x_3,x_4)$, which will be shown to be positive definite for small ε. First we show that $Q = Q_1 + \varepsilon Q_2$, where Q_1 is the corresponding quadratic function for the G-metric, and the matrix of Q_2 has its entries bounded on C for all ε say, less than 1. For the sake of simplicity, the term "bounded" will be used to refer to any expression that is bounded on C for all $\varepsilon < 1$. Finally, recall that $f \equiv 2$ in the G-metric, and therefore also in the $\tilde{G}$-metric. (2.3) now yields:

$$K_{X,Y} = K_S - \frac{3}{4}f^2(G+\varepsilon h)^2 = K_S - \frac{3}{4}f^2G^2 + \varepsilon\{\text{bounded}\}$$

$$\tilde{G}^2K_{\text{fiber}} = -\tilde{G}\tilde{G}_{rr} = -(G+\varepsilon h)(G_{rr}+\varepsilon h_{rr}) = -GG_{rr} + \varepsilon\{\text{bounded}\}$$

$$\tilde{G}^2K_{X,\partial_\theta} = \tilde{G}(\nabla_X X - XX)\tilde{G} + \frac{1}{4}f^2\tilde{G}^4 = \tilde{G}(\nabla_X X - XX)(\varepsilon h) + \frac{1}{4}f^2(G+\varepsilon h)^4$$
$$= \frac{1}{4}f^2G^4 + \varepsilon\{\text{bounded}\}$$

and similarly, $\tilde{G}^2K_{Y,\partial_\theta} = \frac{1}{4}f^2G^4 + \varepsilon\{\text{bounded}\}$.

$$\langle R(X,Y)\partial_r,\partial_\theta\rangle = f\tilde{G}\tilde{G}_r = f(G+\varepsilon h)(G_r+\varepsilon h_r) = fGG_r + \varepsilon\{\text{bounded}\}$$

$$\langle R(Y,X)X,\partial_\theta\rangle = -\frac{1}{2}((Xf)\tilde{G}^2 + 3f\tilde{G}X\tilde{G}) = -\frac{3}{2}f\tilde{G}X(\varepsilon h) = \varepsilon\{\text{bounded}\}$$

and similarly, $\langle R(X,Y)Y,\partial_\theta\rangle = \varepsilon\{\text{bounded}\}$.

$$\langle R(\partial_r,\partial_\theta)\partial_\theta,X\rangle = -\tilde{G}\partial_r X\tilde{G} = -\tilde{G}\partial_r X(\varepsilon h) = \varepsilon\{\text{bounded}\},$$

and similarly, $\langle R(\partial_r,\partial_\theta)\partial_\theta,Y\rangle = \varepsilon\{\text{bounded}\}$.

Finally, recall that $XG = YG = G_\theta = 0$, implying $(\nabla_X Y)G = 0$. Thus

$$\langle R(X,\partial_\theta,\partial_\theta,Y\rangle = \tilde{G}(\nabla_X Y - XY)\tilde{G} = \tilde{G}(\nabla_X Y - XY)(\varepsilon h) = \varepsilon\{\text{bounded}\}.$$

Therefore $Q = Q_1 + \varepsilon Q_2$, and by hypothesis there exists $\eta > 0$ such that $|Q_2(x)| \le \eta\|x\|^2$ for all $\varepsilon < 1$. Now Q_1 has matrix

$$\begin{bmatrix} K_S - \frac{3}{4}f^2 \cdot G^2 & \frac{3}{2}fGG_r & 0 & 0 \\ \frac{3}{2}fGG_r & -GG_{rr} & 0 & 0 \\ 0 & 0 & \frac{1}{4}f^2G^4 & 0 \\ 0 & 0 & 0 & \frac{1}{4}f^2G^4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4+r^2}{1+r^2} & \frac{3r}{(1+r^2)^2} & 0 & 0 \\ \frac{3r}{(1+r^2)^2} & \frac{3r^2}{(1+r^2)^3} & 0 & 0 \\ 0 & 0 & \frac{r^4}{(1+r^2)^2} & 0 \\ 0 & 0 & 0 & \frac{r^4}{(1+r^2)^2} \end{bmatrix}$$

and is therefore positive definite for $r \neq 0$, since the upper left corner submatrix has positive trace and determinant. Let $\Theta > 0$ be a lower bound for the eigenvalues of Q_1 on $C := \operatorname{supp} h$, and choose $0 < \varepsilon < \min\{\Theta/\eta, 1\}$. Then $(Q_1 + \varepsilon Q_2)(x) \ge Q_1(x) - \varepsilon|Q_2(x)| \ge 0$. Thus the $\tilde{G}$-metric has nonnegative curvature. Uniform boundedness in ε is crucial here, and the reader may want to compare this construction with the one given in [BDS]. □

The associated bundle construction in [C] shows that any $\mathbb{R}^2$-bundle over S^n admits a metric of nonnegative curvature. Actually, a somewhat stronger result is true:

Theorem 2.4 *Let* S *denote the n-sphere together with some metric of positive curvature, and let* $\pi: E \to S$ *be a 2-dimensional vector bundle over* S*. Then there exists a family of metrics of nonnegative curvature on* E*, each of which has soul isometric to* S *, with totally geodesic fibers.*

Proof. For $n > 2$, E is a trivial bundle (cf. [S]), and one then takes the isometric product $S \times P_2$, where P_2 is $\mathbb{R}^2$ together with any metric of nonnegative curvature. Assume then that $n=2$ and that E is nontrivial. By the classification theorem of bundles over spheres, every vector bundle over the 2-sphere is orientable, cf. [S]. Choose an orientation of E. As before, given a Riemannian connection on E with curvature tensor R, the corresponding curvature form Ω will be identified with $f\omega$, where ω is the volume form of S, and $f: S \to \mathbb{R}$ is given locally by $f = \Omega(X,Y)(U,JU)$, X, Y local oriented orthonormal vector fields on S, U local section of E with $\|U\| = 1$, and J the canonical complex structure on E.

Fix any Riemannian connection on E, and let Ω denote its curvature form. Set $c := (1/\mathrm{vol}\, S)\int_S \Omega = (\int_S \Omega)/(\int_S \omega)$, and $\tilde{\Omega} = c\omega$.

We claim there exists a Riemannian connection $\tilde{\nabla}$ on E with curvature form $= \tilde{\Omega}$. To see this, notice that $\int_S (\tilde{\Omega} - \Omega) = 0$, so that $\tilde{\Omega} = \Omega + d\Theta$, for some 1-form Θ on S. Now define $\tilde{\nabla}$ by $\tilde{\nabla}_X U = \nabla_X U + \Theta(X)JU$, $X \in \mathfrak{X}S$, $U \in \Gamma E$. J is parallel with respect to ∇: if U is parallel of unit-length, then

$$\nabla_X(JU) = \langle \nabla_X(JU), U\rangle U = -\langle JU, \nabla_X U\rangle U = 0.$$

It is easily verified that $\tilde{\nabla}$ is a Riemannian connection. If $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$, then

$$\begin{aligned}
\tilde{R}(X,Y)U &= \tilde{\nabla}_X(\nabla_Y U + \Theta(Y)JU) - \tilde{\nabla}_Y(\nabla_X U + \Theta(X)JU) - \tilde{\nabla}_{[X,Y]} U \\
&= R(X,Y)U + \Theta(X)J\nabla_Y U + \nabla_X(\Theta(Y)JU) + \Theta(X)\Theta(Y)JU \\
&\quad - \Theta(Y)J\nabla_X U - \nabla_Y(\Theta(X)JU) - \Theta(Y)\Theta(X)JU - \Theta([X,Y])JU \\
&= R(X,Y)U + d\Theta(X,Y)JU.
\end{aligned}$$

Thus the curvature form of $\tilde{\nabla}$ is $\tilde{\Omega}$. Now choose a Riemannian connection ∇ as above, so that $\Omega = c\omega$. Given $u \in E$, let A_u denote the canonical vector space isomorphism between the fiber through u and its tangent space at u. One has the vector fields ∂_r, ∂_ϑ on E-S given by

$$\partial_r|_u = A_u u/\|u\|, \quad \partial_\vartheta|_u = A_u Ju, \quad u \in E\text{-}S.$$

Next define a Riemannian metric on E as follows: $\pi: E \to S$ is to be a Riemannian submersion, where the horizontal subspaces are those determined by the connection ∇, and the metric on the fibers is taken to be $dr^2 + G^2 d\theta^2$, with $G := \varepsilon r/(\varepsilon^2 + r^2)^{1/2}$, for some fixed $\varepsilon > 0$ satisfying $\varepsilon^2 < (4/3c^2) \min K_S$ (K_S = sectional curvature of S). Notice that replacing the connection ∇ by $\tilde{\nabla}$, $\tilde{\nabla}_X U := \nabla_X U + dh(X)JU$, for $h: S \to \mathbb{R}$, changes the horizontal distribution and therefore the metric, even though the curvature form remains unchanged. Thus $\Omega = c\omega$ actually determines a family of metrics on E. A standard argument shows that (2.2) and (2.3) remain valid, with G as above, $f \equiv c$, $\mu = \lambda \equiv 0$. The details follow. Recall from section 1 that if X is basic, then $[\partial_r, \partial_\theta] = [X, \partial_r] = [X, \partial_\theta] = 0$.

Lemma 2.5 *Under the above hypotheses, let* X,Y *be basic. Then*

(1) $[X,Y]^v = -\Omega(X,Y)\partial_\theta$ ($\Rightarrow$ $[X,Y]^v = -c\partial_\theta$ *for orthonormal oriented* X,Y)

(2) $\nabla_X \partial_r = \nabla_{\partial_r} X = 0$

(3) *The fibers are totally geodesic and* $\nabla_{\partial_r} \partial_r = 0$.

Proof of 2.5 (1) was proved in 1.3. Now

$$\langle \nabla_X \partial_r, \partial_r \rangle = \frac{1}{2} X \langle \partial_r, \partial_r \rangle = 0$$

$$2\langle\nabla_X\partial_r,\partial_\theta\rangle = X\langle\partial_r,\partial_\theta\rangle + \partial_r\langle\partial_\theta,X\rangle - \partial_\theta\langle X,\partial_r\rangle + \langle\partial_\theta,[X,\partial_r]\rangle$$

$$+ \langle\partial_r,[\partial_\theta,X]\rangle - \langle X,[\partial_r,\partial_\theta]\rangle$$

$$= 0$$

$$\langle\nabla_X\partial_r,X\rangle = -\langle\partial_r,\nabla_X X\rangle = 0$$

$$\langle\nabla_X\partial_r,Y\rangle = -\langle\partial_r,\nabla_X Y\rangle \overset{(1)}{=} -\frac{1}{2}\Omega(X,Y)\langle\partial_r,\partial_\theta\rangle = 0$$

$\Rightarrow \nabla_X\partial_r = 0$. Finally, $[X,\partial_r] = 0 \Rightarrow \nabla_{\partial_r} X = \nabla_X\partial_r = 0$.

To prove (3), consider the second fundamental form ℓ_X of the fibers with respect to X. Then by (2),

$$\ell_X(\partial_r,\partial_r) = \ell_X(\partial_r,\partial_\theta) = 0.$$

$$\ell_X(\partial_\theta,\partial_\theta) = \langle\nabla_{\partial_\theta} X,\partial_\theta\rangle = \langle\nabla_X\partial_\theta,\partial_\theta\rangle = \frac{1}{2}X\langle\partial_\theta,\partial_\theta\rangle = \frac{1}{2}X\frac{\varepsilon^2 r^2}{1+r^2} = 0$$

Thus the fibers are totally geodesic. Now,

$$\langle\nabla_{\partial_r}\partial_r,\partial_r\rangle = \frac{1}{2}\partial_r\langle\partial_r,\partial_r\rangle = 0, \text{ and}$$

$$\langle\nabla_{\partial_r}\partial_r,\partial_\theta\rangle = -\langle\nabla_{\partial_r}\partial_\theta,\partial_r\rangle = -\langle\nabla_{\partial_\theta}\partial_r,\partial_r\rangle = -\frac{1}{2}\partial_\theta\langle\partial_r,\partial_r\rangle = 0$$

$\Rightarrow (\nabla_{\partial_r}\partial_r)^V = 0 \Rightarrow \nabla_{\partial_r}\partial_r = 0$, since the fibers are totally geodesic. □

Notice that by 2.5(3) and the Gauss equations, the induced curvature of the fibers equals the intrinsic curvature,

$$K_{fiber} = -\frac{G_{rr}}{G} = \frac{3\varepsilon^2}{(\varepsilon^2+r^2)^2}.$$

Lemma 2.6 *Let* R *denote the curvature tensor of the Riemannian manifold* E, X, Y $\in \mathfrak{X}S$ *basic. Then*

(a)

$$R(X,\partial_r)\partial_r = R(\partial_r,X)X = 0 \qquad (2.6.1)$$

$$R(X,Y)\partial_r = 2R(X,\partial_r)Y = \Omega(X,Y)\,\frac{G_r}{G}\,\partial_\theta \qquad (2.6.2)$$

$\Rightarrow \langle R(X,Y)\partial_r,\partial_\theta\rangle = \Omega(X,Y)G_rG.$

Let $\{X,Y\}$ *be a local orthonormal oriented basis of* $\mathfrak{X}S$. *Then*

(b)

$$\nabla_{\partial_\theta}X = \nabla_X\partial_\theta = \frac{1}{2}\,cG^2Y \qquad (2.6.3)$$

$$\nabla_{\partial_\theta}Y = \nabla_Y\partial_\theta = -\frac{1}{2}\,cG^2X \qquad (2.6.4)$$

$$\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{G_r}{G}\,\partial_\theta \qquad (2.6.5)$$

$$\nabla_{\partial_\theta}\partial_\theta = \frac{G_\theta}{G}\,\partial_\theta - GG_r\,\partial_r = -GG_r\partial_r \qquad (2.6.6)$$

$$K_{X,\partial_\theta} = K_{Y,\partial_\theta} = \frac{1}{4}\,c^2G^2 \qquad (2.6.7)$$

$$\langle R(\partial_\theta, X)X, Y\rangle = \langle R(\partial_\theta, Y)Y, X\rangle = 0 \qquad (2.6.8)$$

$$\langle R(X,\partial_\theta)\partial_\theta Y\rangle = \langle R(Y,\partial_\theta)\partial_\theta, X\rangle = 0 \qquad (2.6.9)$$

$$\langle R(\partial_r,\partial_\theta)\partial_\theta, X\rangle = \langle R(\partial_r,\partial_\theta)\partial_\theta, Y\rangle = 0 \qquad (2.6.10)$$

$$K_{X,Y} = K_S\circ\pi - \frac{3}{4}c^2G^2 \qquad (2.6.11)$$

Proof of 2.6 (a)(2.6.1) follows from 2.5(1). Next,

$$R(X,\partial_r)Y = -\nabla_{\partial_r}\nabla_X Y = -\nabla_{\partial_r}(\nabla_X Y)^v = -\frac{1}{2}\nabla_{\partial_r}[X,Y]^v$$
$$= -\frac{1}{2}\nabla_{[X,Y]^v}\partial_r = -\frac{1}{2}\nabla_{[X,Y]}\partial_r = \frac{1}{2}R(X,Y)\partial_r$$

and $R(X,Y)\partial_r = -\nabla_{[X,Y]^v}\partial_r = \Omega(X,Y)\nabla_{\partial_\theta}\partial_r = \Omega(X,Y)\frac{G_r}{G}\partial_\theta \Rightarrow$ (2.6.2)

(b) $\nabla_{\partial_\theta}X$ is horizontal since the fibers are totally geodesic, and

$\langle\nabla_{\partial_\theta}X,X\rangle = 0$, while $\langle\nabla_{\partial_\theta}X,Y\rangle = \langle\nabla_X\partial_\theta, Y\rangle = -\langle\partial_\theta,\nabla_X Y\rangle = \frac{1}{2}cG^2$.

$\Rightarrow$ (2.6.3), and the argument for (2.6.4) is similar.

$\nabla_{\partial_r}\partial_\theta$ is vertical since the fibers are totally geodesic, and

$\langle\nabla_{\partial_r}\partial_\theta,\partial_r\rangle = -\langle\nabla_{\partial_r}\partial_r,\partial_\theta\rangle = 0$, while $\langle\nabla_{\partial_r}\partial_\theta,\partial_\theta\rangle = \frac{1}{2}(G^2)_r = GG_r$.

$\Rightarrow$ (2.6.5), and the argument for (2.6.6) is similar.

(c) is a consequence of (b):

$$R(X,\partial_\theta)\partial_\theta = \nabla_X\nabla_{\partial_\theta}\partial_\theta - \nabla_{\partial_\theta}\nabla_X\partial_\theta = -\nabla_{\partial_\theta}\{\frac{1}{2}cG^2Y\} = \frac{1}{4}cG^4X$$

⇒ (2.6.7), (2.6.9), and (2.6.10). (2.6.11) follows from O'Neill's formula and 2.5(1). Finally:

$$\langle R(\partial_\theta,X)X,Y\rangle = \langle \nabla_{\partial_\theta}\nabla_X X,Y\rangle - \langle \nabla_X \nabla_{\partial_\theta} X,Y\rangle$$

$$= \partial_\theta\langle \nabla_X X,Y\rangle - \langle \nabla_X X,\nabla_{\partial_\theta} Y\rangle - \frac{1}{2}\, c G^2 \langle \nabla_X Y,Y\rangle$$

Since $\nabla_X X$ and Y are basic, $\partial_\theta\langle \nabla_X X,Y\rangle = 0$. Moreover,

$$\langle \nabla_X X,\nabla_{\partial_\theta} Y\rangle = -\frac{1}{2}\, c G^2 \langle \nabla_X X,X\rangle = 0.\ \langle \nabla_X Y,Y\rangle = \frac{1}{2}\, X\langle Y,Y\rangle = 0.$$

⇒ (2.6.8). This concludes the proof of 2.6. □

We now return to the proof of 2.4. To see that E has nonnegative curvature, consider $u, w \in E_q$. If $q \in E-S$, then there exist local basic X,Y, $\{X,Y\}$ oriented orthonormal, such that

$$u = (\eta X+\alpha\partial_r+\beta\partial_\theta)|_q\ ,\ w = (\Theta\partial_r+\gamma X+\delta Y+\zeta\partial_\theta)|_q,$$

for some $\alpha, \beta, \gamma, \delta, \zeta, \eta, \Theta \in \mathbb{R}$. Simplifying and grouping terms with the help of 2.6,

$$\langle R(u,w)w,u\rangle = (\eta\delta)^2 K_{X,Y} + (\beta\Theta-\alpha\zeta)^2 G^2 K_{\partial_r,\partial_\theta} + (\beta\gamma-\eta\zeta)^2 G^2 K_{X,\partial_\theta}$$

$$+ (\beta\delta)^2 G^2 K_{Y,\partial_\theta} + 3\eta\delta\,(\beta\Theta-\alpha\zeta)\langle R(X,Y)\partial_r,\partial_\theta\rangle,$$

where the right side is evaluated at q.

Thus $\langle R(u,w)w,u\rangle = Q(\eta\delta,\beta\Theta-\alpha\zeta,\beta\gamma-\eta\zeta,\beta\delta)$, where $Q\colon \mathbb{R}^4 \to \mathbb{R}$ is the quadratic function with matrix

$$A = \begin{bmatrix} K_{X,Y} & \frac{3}{2}\langle R(X,Y)\partial_r, \partial_\theta\rangle & 0 & 0 \\ \frac{3}{2}\langle R(X,Y)\partial_r, \partial_\theta\rangle & G^2 K_{\partial_r, \partial_\theta} & 0 & 0 \\ 0 & 0 & G^2 K_{X,\partial_\theta} & 0 \\ 0 & 0 & 0 & G^2 K_{Y,\partial_\theta} \end{bmatrix}$$

Now $K_{X,\partial_\theta} = K_{Y,\partial_\theta} = (1/4)c^2G^2 > 0$, so by 2.6, A is positive definite iff. its upper left corner

$$B = \begin{bmatrix} K_S\circ\pi - \frac{3}{4}c^2G^2 & \frac{3}{2}cGG_r \\ \frac{3}{2}cGG_r & -GG_{rr} \end{bmatrix}$$ is positive definite.

But

$$K_S - \frac{3}{4}c^2G^2 > K_S - \frac{3}{4}c^2\varepsilon^2 > 0$$

by choice of ε, while

$$-GG_{rr} = \frac{3\varepsilon^2G^2}{(\varepsilon^2+r^2)^2} > 0.$$

Thus the trace of B is positive. Finally,

$$\frac{1}{G^2}\det B = \{K_S - \frac{3}{4}\frac{c^2\varepsilon^2 r^2}{\varepsilon^2+r^2}\}\frac{3\varepsilon^2}{(\varepsilon^2+r^2)^2} - \frac{9}{4}\frac{c^2\varepsilon^6}{(\varepsilon^2+r^2)^3}$$

$$= \{K_S - \frac{3}{4}c^2\varepsilon^2\}\frac{3\varepsilon^2}{(\varepsilon^2+r^2)^2} > 0.$$

Therefore B is positive definite, and $\langle R(u,w)w,u\rangle \geq 0$. It is worth mentioning that the only nontrivial solutions for $\langle R(u,w)w,u\rangle = 0$ are span$\{u,w\}$ = span{horizontal vector, ∂_r}, a fact that was used in the proof of 2.1. The proof consists of a straightforward case by case analysis: assume $\langle R(u,w)w,u\rangle = 0$. Since Q is positive definite, we have

$$\eta\delta = 0 \qquad (1)$$

$$\beta\delta = 0 \qquad (2)$$

$$\beta\Theta - \alpha\xi = 0 \qquad (3)$$

$$\beta\gamma - \eta\xi = 0 \qquad (4)$$

Assume first $\delta \neq 0$. Then (1), (2) $\Rightarrow \eta = \beta = 0 \Rightarrow u = \alpha\partial_r$. If $\alpha = 0$, we obtain a trivial solution $u = 0$. Otherwise, (3) $\Rightarrow \xi = 0$ and $w = \Theta\partial_r + \gamma X + \delta Y$, so span$\{u,w\}$ = span$\{\partial_r, \gamma X+\delta Y\}$, where $\gamma X+\delta Y$ is horizontal. Next assume $\delta = 0$. We distinguish 3 cases:

(i) $\beta = \xi = 0$. Then $u = \eta X + \alpha\partial_r$, $w = \Theta\partial_r + \gamma X$, so span$\{u,w\}$ = span$\{\partial_r, X\}$.

(ii) One of β or ξ, say ξ is 0. Then (3) $\Rightarrow \Theta = 0$, (4) $\Rightarrow \gamma = 0$, so $w = 0$. The case $\beta = 0$ is similar.

(iii) $\beta, \varsigma \neq 0$. Then (3) $\Rightarrow \alpha = (\beta/\varsigma)\Theta$, (4) $\Rightarrow \eta = (\beta/\varsigma)\gamma$, and thus $u = (\beta/\varsigma)w$, again a trivial solution. This concludes the proof of the claim.

When $q \in S$, one replaces $\partial_r, \partial_\vartheta$ by an orthonormal basis of $S_q^{\perp}$. The matrix A then becomes

$$\begin{bmatrix} K_S & \frac{3}{2}c & 0 & 0 \\ \frac{3}{2}c & 3\varepsilon^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which again is nonnegative. □

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