BRANCHED SUPERMINIMAL SURFACES IN S^4

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Abstract of the Dissertation

Branched Superminimal Surfaces in S^4

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We show that branched superminimal surfaces in S^4 can be classified by pairs of meromorphic functions of the same degree with the same ramification divisors. We use this to show that the space of harmonic maps of degree d from S^2 to S^4 is connected. We also construct examples of unbranched superminimal surfaces of genus 0 with area $4\pi d$ where $d \geq 3$.

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PREFACE

In a study of minimal surfaces in euclidean spheres, Calabi showed that every minimal immersion of S^2 in S^n arises from an isotropic map to projective space. An immediate corollary of this was that the image of S^2 lies "fully" in an even dimensional sphere. This work was used in dimension 4 by Robert Bryant to show that every compact Riemann surface can be superminimally immersed in S^4. In this thesis, we study the "moduli" of branched superminimal immersions of compact Riemann surfaces into S^4.

In Chapter 1, we briefly discuss minimal immersions in general. We also define superminimal surfaces in terms of the vanishing of a holomorphic quartic form which is constructed from the second fundamental form B (where trace $B \equiv 0$). Then we give an outline of Calabi's construction of minimal surfaces in euclidean spheres using what he calls "pseudo-holomorphic maps."

In Chapter 2, we develop the machinery for studying the structure of the space of branched superminimal surfaces in S^4. We begin by discussing holomorphic contact structures. We then observe that by removing a pair of skew projective lines in $\mathbb{P}^3$, we obtain a projection map to $\mathbb{P}^1 \times \mathbb{P}^1$. This enables us to represent branched superminimal surfaces by pairs of meromorphic functions. Next, we examine the space $\mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ which is "similar" to $\tilde{\mathbb{P}}^3$ ($\mathbb{P}^3$ blown up along the pair of skew lines). We then discuss a "contact map" between the two manifolds. Next, we return to $\mathbb{P}^3$ and analyze the ramification divisors and the degrees of the pair of meromorphic functions corresponding to a superminimal surface.

In Chapter 3, we consider the case of S^2. By using the fact that a meromorphic function on S^2 is just a rational function, we reduce our problem to studying the Grassmannian $G(2,d+1)$ and a map to projective space. We then prove that the space H_d of harmonic maps of degree d from S^2 to S^4 is connected for $d \geqq 1$. In the last section, we give examples of superminimal immersions of S^2 in S^4.

In Chapter 4, we tackle the case of a compact Riemann surface of positive genus. We give conditions under which a pair of degree d meromorphic functions with the same ramification divisor can give rise to a branched superminimal immersion into S^4.

CHAPTER 1

Preliminaries

The intent of this chapter is to provide a brief survey of minimal surfaces. Most proofs of the statements mentioned here are omitted (but references are supplied). The first section concerns some general facts about minimal immersions. The second section deals mainly with Calabi's work on minimal immersions of S^2 in euclidean spheres. Bryant's result on superminimal surfaces in S^4 is also mentioned.

§1.1. Minimal immersions

Let $\psi : M \to \bar{M}$ be an isometric immersion, where M and $\bar{M}$ are Riemannian manifolds of dimension n and $\bar{n}$ respectively. Consider the induced bundle $\psi^*(T\bar{M})$ equipped with the connection $\bar{\nabla}$ induced from the Riemannian connection on $\bar{M}$. The bundle decomposes orthogonally into $TM \oplus NM$ where TM and NM are the tangent and normal bundles of M respectively. The second fundamental form of the immersion ψ is a section B of $\mathrm{Hom}(TM\otimes TM, NM)$ defined by $B(V,W) := (\bar{\nabla}_V W)^N$ where V,W are vector fields tangent to M and $(\ )^N$ denotes the orthogonal projection to the normal bundle. The mean curvature of ψ is the normal vector field $H :=$ trace B. The immersion ψ is said to be minimal iff $H \equiv 0$.

For $n = 2$, minimal immersions are just the conformal harmonic immersions. More generally, ψ is a branched minimal

<u>immersion</u> if it is minimal away from the set of isolated singular points (where $d\psi$ vanishes). These are precisely the nonconstant conformal harmonic maps. (cf. [EL_1], [ES]). Observe that since a Riemann surface of genus 0 admits no holomorphic differentials, any harmonic map $\psi : S^2 \to \overline{M}$ is automatically conformal. Thus, branched minimal immersions of S^2 in $\overline{M}$ are just the nonconstant harmonic maps from S^2 to $\overline{M}$.

Let $\sum$ be a 2-dimensional manifold. Let $\psi : \sum \to \mathbb{R}^n$ be a conformal immersion. Choose isothermal coordinates (x,y). Set $z = x+iy$ and $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Then the induced metric has the form $ds^2 = 2F|dz|^2 = 2F(dx^2+dy^2)$ where $F = \frac{1}{2}|\psi_x|^2 = \frac{1}{2}|\psi_y|^2 = \langle\partial\psi, \bar{\partial}\psi\rangle$. Note that $\langle,\rangle$ denotes the complex bilinear extension to $\mathbb{C}^n$ of $\langle,\rangle$ in $\mathbb{R}^n$. The Laplace-Beltrami operator is given by $\Delta = \frac{2}{F}\partial\bar{\partial}$. The map ψ is <u>harmonic</u> if $\Delta\psi = 0$.

Now consider a conformal immersion $\psi : \sum \to S^n$. We may view ψ as an R^{n+1}-valued function satisfying $\langle\psi,\psi\rangle \equiv 1$. The minimal surface equation is then

$$\partial\bar{\partial}\psi = -F\psi \tag{3.1}$$

Observe that ψ is a branched minimal immersion iff it satisfies (3.1) with F having at most a finite number of zeroes. The conformality condition $\langle\partial\psi,\partial\psi\rangle = 0$ together with the condition $\langle\psi,\psi\rangle \equiv 1$ imply

$$\langle\partial^i\psi, \partial^j\psi\rangle = 0 \qquad i + j \leq 3 \tag{3.2}$$

Let $\mathbb{C}N$ denote the complexified normal bundle of ψ. Define a local section of $\mathbb{C}N$ by

$$\varphi := \frac{1}{2}\{B(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) - iB(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\} = \partial^2\psi - \frac{1}{F}\langle\partial^2\psi,\bar{\partial}\psi\rangle\partial\psi \tag{3.3}$$

It follows from (3.2) that $\langle\varphi,\varphi\rangle = \langle\partial^2\psi,\partial^2\psi\rangle$ and thus from (3.1) and (3.2)

$$\bar{\partial}\langle\varphi,\varphi\rangle = 2\langle\partial^2\bar{\partial}\psi,\partial^2\psi\rangle = 2\langle\partial(-F\psi),\partial^2\psi\rangle = 0. \tag{3.4}$$

Define $\Phi := \langle\varphi,\varphi\rangle dz^4$. It is straightforward to verify that Φ is a well defined section of $\otimes^4_{\mathbb{C}}T^{1,0*}\Sigma$. Thus by (3.4), Φ is a holomorphic quartic form on Σ, i.e. $\Phi \in H^0(\Sigma;(\Omega^1)^4)$.

Definition. A (branched) minimal immersion $\psi : \Sigma \to S^n$ is (branched) superminimal if the holomorphic quartic form Φ vanishes identically.

Observe that since S^2 has no nontrivial holomorphic quartic differentials, every (branched) minimal immersion of S^2 in S^n is automatically (branched) superminimal.

Note that Φ is constructed from the second fundamental form B where $H = \text{trace } B \equiv 0$.

§1.2. The Calabi Construction

In this section, we outline Calabi's construction of minimal immersions of S^2 in euclidean spheres. His main result is that the image of S^2 lies "fully" in an even dimensional sphere with area a multiple of 2π. This result was sharpened by Barbosa who showed that the area is a multiple of 4π.

Let $\psi : \sum \to S^n \subset \mathbb{R}^{n+1}$ be an isometric minimal immersion of a compact Riemann surface in S^n. Consider the holomorphic form $\Lambda_j dz^{2j} \in H^0(\sum, (\Omega^1)^{2j})$ where $\Lambda_j = \langle \partial^j\psi, \partial^j\psi\rangle$. Note that $\Lambda_2 dz^4$ is nothing other than our quartic form Φ discussed in the last section. Calabi observes that if $\Lambda_j \equiv 0$ for all $1 \leqq j \leqq k-1$ (with $k \geqq 2$), then

$$\langle \partial^i\psi, \partial^j\psi\rangle = 0 \text{ for all } 1 \leqq i+j \leqq 2k-1,\ i,j \geqq 0 \text{ (where } \partial^0\psi = \psi)$$
$$\text{and } \langle \partial^i\psi, \partial^{2k-i}\psi\rangle = (-1)^i\Lambda_k \text{ for } 0 \leqq i \leqq 2k.$$

Such an immersion is indexed by k. Calabi calls an immersion of infinite index a <u>pseudoholomorphic immersion</u>. In other words, ψ is pseudoholomorphic if $\langle \partial^i\psi, \partial^j\psi\rangle = 0$ for all $i + j \geqq 1$. Note that if genus$(\sum) = 0$, ψ is automatically pseudoholomorphic since $H^0(S^2, (\Omega^1)^{2j}) = 0$ for all $j > 0$.

The complex osculating space of order k at a point $p \in \sum$ is the pull back of the subspace $O(\psi)$ of $\mathbb{C}^{n+1}$ spanned by all the derivities $\partial^i\bar{\partial}^j\psi$ with $0 \leqq i+j \leqq k$. Using the minimal surface equation $\partial\bar{\partial}\psi = -F\psi$, we find that $O(\psi)$ is spanned by the $2k + 1$ vectors $\psi, \partial\psi, \ldots, \partial^k\psi, \bar{\partial}\psi, \ldots, \bar{\partial}^k\psi$ evaluated at p.

We say that a subspace $W \subset \mathbb{C}^{n+1}$ is <u>isotropic</u> if $\langle v,w\rangle = 0$ for all $v,w \in W$. This means that W is orthogonal to $\overline{W}$, and thus $2\cdot\dim(W) \leqq n$. The pseudoholomorphic condition $\langle \partial^i\psi, \partial^j\psi\rangle = 0$ for $i + j > 0$ means geometrically that the subspace $V(\psi)$ of $\mathbb{C}^{n+1}$ spanned by $\partial\psi, \partial^2\psi, \ldots, \partial^k\psi$ at a point

$p \in \sum$ is isotropic and orthogonal to ψ. Let $\Psi_k := \partial\psi \wedge \ldots \wedge \partial^k\psi$. Then the plane $\mathrm{Span}(\Psi_{k(p)})$ is isotropic.

Let m be the largest integer such that $\Psi_m \not\equiv 0$ but $\Psi_{m+1} \equiv 0$. This implies that $\partial^{m+1}\psi = \sum_{i=1}^{m} a_i \partial^i \psi$, $a_i \in C^\infty(\sum)$ and $\partial\Psi_m = a_m \Psi_m$. Thus $\partial^{m+k}\psi = \sum_{i=1}^{m} b_i^k \partial^i \psi$ (with $b_i^1 = a_i$). Suppose $\Lambda_k = \langle\partial^k\psi, \partial^k\psi\rangle \equiv 0$ for $1 \leqq k \leqq m$. Then since $\langle\partial^i\psi, \partial^j\psi\rangle = 0$ for $1 \leqq i+j \leqq 2m+1$, we in fact have $\langle\partial^i\psi, \partial^j\psi\rangle = 0$ for all $i + j \geqq 1$. Thus the condition $\Lambda_k \equiv 0$ for $1 \leqq k \leqq m$ is equivalent to ψ being pseudoholomorphic.

Assume now that $\psi : S^2 \to S^n \subset \mathbb{R}^{n+1}$ is a minimal immersion which is <u>linearly full</u>, i.e. $\psi(S^2)$ is not contained in any hyperplane of R^{n+1}. Set $\Psi := \psi \wedge \Psi_m \wedge \overline{\Psi}_m$ (where m is as in the previous paragraph). Observe that $|\Psi| = |\Psi_m|^2 \neq 0$. Minimality implies that $\partial\Psi = a_m\Psi$ and $\overline{\partial}\Psi = \overline{a}_m\Psi$. Thus, the class $[\Psi] \in \mathbb{P}(\Lambda^{2m+1}\mathbb{C}^{n+1})$ is constant. Since the image of ψ is contained in $\mathrm{Span}(\Psi)$ which is constant, real and of dimension $2m + 1$, and since ψ is linearly full, $n + 1 = 2m+1$. Thus

<u>Theorem</u>. <u>Let</u> $\psi : S^2 \to S^n$ <u>be a (branched) minimal immersion</u>. <u>Then there is an integer</u> $m \leqq n/_2$ <u>so that</u> $\psi : S^2 \to S^{2m}$ <u>is a linearly full (branched) minimal immersion</u>.

<u>Note 1</u>. The theorem holds for $\psi : \sum \to S^n$ where $\sum$ is a compact Riemann surface and ψ is a pseudoholomorphic immersion. For the rest of this section, results for (branched) minimal immersions of S^2 in S^{2m} can be replaced by (branched)

pseudoholomorphic immersions of a compact Riemann surface Σ in S^{2m}.

Note 2. For m = 2, the condition that the immersion $\psi : \Sigma \to S^4$ be pseudoholomorphic is equivalent to the condition that it be superminimal.

Consider now a minimal immersion $\psi : S^2 \to S^{2m}$ which is linearly full. Since the plane $\Psi_m(p) = \partial\psi\wedge\ldots\wedge\partial^m\psi$ is isotropic, so is $\overline{\Psi}_m(p)$. Now, $\overline{\partial}\,\overline{\Psi}_m = \overline{a}_m\overline{\Psi}_m$, $\overline{\partial}\Psi = \overline{a}_m\Psi$ and $\Psi : S^2 \to \Lambda^{2m+1}(\mathbb{C}^{2m+1}) \cong \mathbb{C}$. Thus at the points where $\Psi \neq 0$, we have $\overline{\partial}(\frac{1}{\Psi}\,\overline{\Psi}_m) = 0$. By projectivizing, we can define $[\overline{\Psi}_m] : S^2\big|_{\{z|\overline{\Psi}_m(z)\neq 0\}} \to \mathbb{P}(\Lambda^m\mathbb{C}^{2m+1})$. The map $[\overline{\Psi}_m]$ is holomorphic $\Psi(z) = 0$ iff $\overline{\Psi}_m(z) = 0$.

Let $I_m := \{\xi \in G(m,2m+1)\,|\,\xi \text{ is isotropic}\}$

$= \{\text{isotropic m-planes in } \mathbb{C}^{2m+1}\}$.

Let $V \in I_m$ and let $z_1,\ldots,z_m$ be a hermitian orthonormal basis for V. Writing $z_k = \frac{1}{\sqrt{2}}(x_k+iy_k)$, we obtain an orthonormal set $\{x_1,y_1,\ldots,x_m,y_m\}$ in $\mathbb{R}^{2m+1}$. Given an orientation on $\mathbb{R}^{2m+1}$, there is a unit vector $u \in \mathbb{R}^{2m+1}$ so that $\mathcal{B} = \{x_1,y_1,\ldots,x_m,y_m,u\}$ is an orthonormal basis for $\mathbb{R}^{2m+1}$. Let V' be another isotropic m-plane. In a similar manner, we obtain $\mathcal{B}' = \{x_1'y_1',\ldots,x_1',y_1',u'\}$, an orthonormal basis for $\mathbb{R}^{2m+1}$. There is an element $g \in SO(2m+1)$ sending $\mathcal{B}$ to $\mathcal{B}'$. Thus, SO(2m+1) acts transitively on I_m. The subgroup U(m) fixes I_m. So I_m is the homogeneous space SO(2m+1)/U(m). Now given $V \in I_m$, we can decompose $\mathbb{C}^{2m+1}$ into $\overline{V} \oplus V \oplus \mathbb{C}\cdot u$. We thus have an SO(2m+1)-equivariant map $\pi : I_m \to S^{2m}$ so that $\pi\circ[\overline{\Psi}_m] = \psi\big|_{\mathrm{dom}[\overline{\Psi}_m]}$

where $\pi(V) = u$.

Proposition. π *is a Riemannian submersion. Furthermore, the map* $[\overline{\Psi}_m]$ *is horizontal with respect to the Riemannian submersion.*

Proof. (cf. [C1], [C2], [M], [L1]).

Note. We have a fibration

$$\frac{SO(2m)}{U(m)} \to \frac{SO(2m+1)}{U(m)} = I_m \xrightarrow{\pi} \frac{SO(2m+1)}{SO(2m)} = S^{2m}$$

where the fiber above a point $x \in S^{2m}$ is the space of all orthogonal almost complex structures compatible with the orientation on $T_x S^{2m}$. At a point $J \in I_m(x)$, the almost complex structure on the horizontal plane H_J is tautologically J itself (using the identification $\pi_* : H_J \xrightarrow{\approx} T_x S^{2m}$).

It is a fact that the zeroes of $\overline{\Psi}_m$ are isolated. Let p be a zero of $\overline{\Psi}_m$ and D a small disc centered at p. Since $[\overline{\Psi}_m]$ is horizontal, $[\overline{\Psi}_m]|_{D-p}$ is holomorphic and bounded. Thus $[\overline{\Psi}_m]$ extends to D and hence to all of S^2.

Theorem (Calabi). *Let* $\psi : S^2 \to S^{2m}$ *be a branched minimal immersion which is linearly full. Then* $[\overline{\Psi}_m]$ *extends to a holomorphic horizontal map on* S^2 *such that the diagram*

$$\begin{array}{ccc} & & I_m \\ & \overset{[\overline{\Psi}_m]}{\nearrow} & \downarrow \pi \\ S^2 & \xrightarrow{\psi} & S^{2m} \end{array}$$

commutes.

Calabi then goes on to show that $\psi(S^2)$ has area $2\pi k$ where $k \in \mathbf{Z}^+$ and $\frac{k}{2} \geq \binom{m+1}{2}$ where the lower bound area of $2\pi m(m+1)$ is attained. This result has been refined by Barbosa who showed that the area is actually $4\pi d$, $d \in \mathbf{Z}^+$. In this thesis, we shall often refer to a pseudoholomorphic map whose image has area $4\pi d$ as a map of degree d.

<u>Note</u>. It is a well known fact that a holomorphic immersion of a complex manifold M in a Kähler manifold X is always minimal. Suppose $\pi : X \to Y$ is a Riemannian submersion. If φ is a holomorphic immersion of M in X which is horizontal with respect to π, then $\pi \circ \varphi$ is minimal. Since I_m is Kähler, holomorphic horizontal curves in I_m project to minimal surfaces in S^{2m}.

We now consider the case when m = 2. observe that $I_2 = SO(5)/U(2) \cong \mathbb{P}^3(\mathbb{C})$. This identification gives us the Penrose fibration $\pi : \mathbb{P}^3(\mathbb{C}) \to S^4$. This fibration can be obtained via a quotient of 2 Hopf maps. Choose homogeneous coordinates (z_0, z_1, z_2, z_3) for $\mathbb{P}^3(\mathbb{C})$. Consider $\mathbb{C}^4 \cong \mathbb{H}^2$ as a quaternion vector space with left scalar multiplication. The identification is given by $(z_0, z_1, z_2, z_3) \to (z_0 + z_1 j, z_2 + z_3 j)$. The Kähler form of the Fubini-Study metric is given by $\omega = \partial\bar{\partial} \log\|z\|^2$. The Penrose fibration is then given by the quotient

$$\begin{array}{ccc} & \mathbb{C}^4 - \{0\} = \mathbb{H}^2 - \{0\} & \\ {\scriptstyle \text{Hopf}_{\mathbb{C}}} \swarrow & & \searrow {\scriptstyle \text{Hopf}_{\mathbb{H}}} \\ \mathbb{P}^3(\mathbb{C}) & \xrightarrow{\ \pi\ } & \mathbb{P}^1(\mathbb{H}) = S^4 \end{array}$$

with fiber $\mathbb{P}^1(\mathbb{C})$. The horizontal 2-plane field H for π is given by a 1-form whose lifting to $\mathbb{C}^4 - \{0\}$ is

$$\Omega := \frac{1}{\|z\|^2}(z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2).$$

Superminimal surfaces in S^4 are just the projections to S^4 of nonsingular holomorphic curves in $\mathbb{P}^3(\mathbb{C})$ which are integral curves of H. Unfortunately, it is difficult to find integral curves of H in $\mathbb{P}^3(\mathbb{C})$ directly. Our search for superminimal surfaces will be vastly simplified if we can find a "contact" manifold "similar" to $\mathbb{P}^3(\mathbb{C})$ where it is easy to find integral curves of the contact plane field. All we need do is to send the integral curves over to $\mathbb{P}^3(\mathbb{C})$ via a "contact" map. Robert Bryant has found a birational correspondence between $\mathbb{P}^3(\mathbb{C})$ and the projectivized tangent space of $\mathbb{P}^2(\mathbb{C})$ carrying H to the contact plane field. Using that, he was able to prove the following result.

Theorem (Bryant). Every compact Riemann surface admits a conformal minimal immersion in S^4.

In this thesis, I will be using another contact manifold – $\mathbb{P}T(\mathbb{P}^1\times\mathbb{P}^1)$ – which is "similar" to $\mathbb{P}^3(\mathbb{C})$. Also, I will let $\mathbb{P}^n$ denote $\mathbb{P}^n(\mathbb{C})$ from now on.

CHAPTER 2

Characterization of branched superminimal surfaces in S^4

In this chapter, we characterize branched superminimal surfaces in S^4 by pairs of meromorphic functions. We relate the bidegrees of such pairs of functions to the degree of the canonical lift of the surface in $\mathbb{P}^3$. The basic idea behind our construction is that given a pair of skew lines L_1, L_2 in $\mathbb{P}^3$, there is a well defined projection from $\mathbb{P}^3 - (L_1 \cup L_2)$ to $\mathbb{P}^1 \times \mathbb{P}^1$.

§2.1. Holomorphic contact structures

Let V be a complex (2n+1)-dimensional manifold. A holomorphic contact form on V is a holomorphic 1-form θ with values in a line bundle $L \to V$ and satisfying the nondegeneracy condition $\theta \wedge (\partial\theta)^n \neq 0$. A holomorphic contact structure is an equivalence class of contact forms under the relation that $\theta \in \Gamma\, \Omega^1(L)$ is equivalent to $\tilde{\theta} \in \Gamma(\Omega^1(\tilde{L}))$ iff there exists an isomorphism $\psi : L \to \tilde{L}$ such that $\psi^*\tilde{\theta} = \theta$. More geometrically, a contact structure is a nondegenerate holomorphic distribution $\mathcal{H}$ of hyperplanes (i.e. the orthogonal spaces of some twisted holomorphic 1-form). (cf. [A], [Le]).

Example 1. Let M be an n-dimensional complex manifold. Then $\mathbb{P}T^*M$ has a canonical holomorphic contact structure. Let $\pi : \mathbb{P}T^*M \to M$ be the projection map onto the base space. A point $\varphi \in \mathbb{P}T^*M$ defines a hyperplane P_φ in $T_{\pi(\varphi)}M$. The contact hyperplane at φ is given by $(\pi_*^{-1})_\varphi(P_\varphi)$.

<u>Example 2</u>. Let $\pi : \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the projectivized tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. The space $\mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ has a canonical holomorphic contact structure where the holomorphic 2-plane field K at a point $y \in \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ is given by $(\pi_*^{-1})_y(L_y)$ where L_y denotes the tangent line at $\pi(y)$ corresponding to y.

<u>NOTE</u>. The projectivized contangent space $\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to the projectivized tangent space $\mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ since an element of $\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1)$ defines a tangent line on $\mathbb{P}^1 \times \mathbb{P}^1$. In fact, we have

$$\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(\mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2)) = \mathbb{P}((\mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2)) \otimes \mathcal{O}(2,2))$$
$$= \mathbb{P}(\mathcal{O}(0,2) \oplus \mathcal{O}(2,0)) = \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1).$$

So the contact structure on $\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1)$ obtained by Example 1 is the same as that on $\mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ obtained by Example 2.

<u>Example 3</u>. Consider the Hopf fibration $p : \mathbb{P}^3 \to S^4$. Now $\mathbb{P}^3$ has a holomorphic contact 2-plane field H orthogonal to the fibers of p with respect to the Fubini-Study metric on $\mathbb{P}^3$ [cf. Bryant]. H can be described in local coordinates as follows. Let $[z_0, z_1, z_2, z_3]$ denote homogeneous coordinates on $\mathbb{P}^3$. The holomorphic horizontal 2-plane field H for p is given by a 1-form whose lifting to $\mathbb{C}^4 - \{0\}$ is $\Omega := \frac{1}{\|z\|^2}(z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2)$. Let $\omega := dz_0 \wedge dz_1 + dz_2 \wedge dz_3$

denote the standard holomorphic symplectic form on $\mathbb{C}^4$. Let $\xi := z_0{}^{\partial}/\partial z_0 + z_1{}^{\partial}/\partial z_1 + z_2{}^{\partial}/\partial z_2 + z_3{}^{\partial}/\partial z_3$. Then $\Omega = \frac{1}{\|z\|^2}\xi \lrcorner\, \omega$

§2.2. Projection to $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the 2 skew lines L_1, L_2 in $\mathbb{P}^3$ defined by

$L_1 := \{[0,0,z_2,z_3] \mid [z_2,z_3] \in \mathbb{P}^1\}$ and $L_2 := \{[z_0,z_1,0,0] \mid [z_0,z_1] \in \mathbb{P}^1\}$.

Note that the lines L_1 and L_2 are the fibers over the north and south poles respectively of S^4 under the Hopf fibration.

Lemma 2.1. There is a well-defined projection map $pr : \mathbb{P}^3 - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{P}^1$ as fiber.

Proof. It suffices to show that to each point $x \in \mathbb{P}^3 - (L_1 \cup L_2)$, there is a unique line through x which intersects L_1 and L_2.

Consider the 2 planes P_1 and P_2 in $\mathbb{P}^3$ defined by $P_1 := \mathrm{Span}(x, L_1)$ and $P_2 := \mathrm{Span}(x, L_2)$. Since L_1 and L_2 are skew, P_1 and P_2 intersect in a line L which contains the point x. The line L must intersect both L_1 and L_2 since 2 lines intersect in a plane. The intersection of L with L_1 and L_2 gives the projection of x to $\mathbb{P}^1 \times \mathbb{P}^1$ (identified with $L_1 \times L_2$).

Proposition 2.2. The fibers of $pr : \mathbb{P}^3 - (L_1 \cup L_2) \to \mathbb{P}^1 \times \mathbb{P}^1$ are horizontal with respect to p (i.e. the fibers of pr are integral curves of $\mathcal{H}$).

Proof. Let $(x,y) \in L_1 \times L_2$. (We identify $L_1 \times L_2$ with $\mathbb{P}^1 \times \mathbb{P}^1$.) Let $L \subset \mathbb{P}^3$ denote the line through x and y, i.e. $L = pr^{-1}(x,y)$.

Denote the inverse images of L, L_1, L_2, x and y to $\mathbb{C}^4 - \{0\}$ by P, P_1, P_2, ℓ_x and ℓ_y respectively.

Note 1. P_1 and P_2 are orthogonal with respect to ω. Let $A \in P_1$ and $B \in P_2$. Then $A = (0,0,a,b)$ and $B = (c,d,0,0)$ for some $a,b,c,d \in \mathbb{C}$. It is clear from the definition of ω that $\omega(A,B) = 0$. Since ω is skew, we also have $\omega(A,A) = 0$ and $\omega(B,B) = 0$.

Now pick nonzero vectors $X \in \ell_x \subset P_1$ and $Y \in \ell_y \subset P_2$.

Note 2. P is spanned by X and Y.

Now let $V_1 = aX+bY$ and $V_2 = cX+dY$ be 2 vectors in P. Then by Note 1, $\omega(V_1,V_2) = 0$. Thus ω vanishes on P. Let $\pi : \mathbb{C}^4 - \{0\} \to \mathbb{P}^3$. Since ξ is tangent to the fibers of π and $\Omega|_L = (\xi \lrcorner \omega)|_P$, we have that Ω vanishes on L. Thus L is horizontal with respect to p.

QED

§2.3. The blow up of $\mathbb{P}^3$

Let X denote the blow up of $\mathbb{P}^3$ along L_1 and L_2, i.e. $X := \{([z_0,z_1,z_2,z_3], [y_0,y_1], [y_2,y_3]) | z_0y_1 = z_1y_0, z_2y_3 = z_3y_2\}$. Observe that X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$:

$\tilde{\pi} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ where $\tilde{\pi}([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) = ([y_0,y_1],[y_2,y_3])$

Claim: $X = \mathbb{P}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))$.

Consider the Hopf bundle $\mathcal{O}(-1) \to \mathbb{P}^1$. (This is a subbundle of the trivial bundle $\underline{\mathbb{C}}^2 \to \mathbb{P}^1$.) Taking a Cartesian product, we have the bundle $\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$ which is a subbundle of the trivial bundle $\underline{\mathbb{C}}^4 \to \mathbb{P}^1 \times \mathbb{P}^1$ $(\underline{\mathbb{C}}^4 = \underline{\mathbb{C}}^2 \oplus \underline{\mathbb{C}}^2)$. Projectivizing, we obtain

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1)) & \subset & \underline{\mathbb{P}}^3 \\ \downarrow & \swarrow & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

Let $z := [z_0, z_1, z_2, z_3]$, $u := [y_0, y_1]$ and $v := [y_2, y_3]$. We can consider u and v as elements of $\mathbb{P}^3$ by writing $u = [y_0, y_1, 0, 0]$ and $v = [0, 0, y_2, y_3]$. We want to show that the triple (z,u,v) corresponds to an element of the bundle $\mathbb{P}(\mathcal{O}(-1,0)\oplus\ (0,-1))$. Let ℓ_u and ℓ_v denote the liftings of u and v to $\mathbb{C}^4 \doteq \mathbb{C}^2 \oplus \mathbb{C}^2$. It is clear from the definition of u and v that ℓ_u and ℓ_v are linearly independent lines in $\mathbb{C}^4$. Let $\ell := \tilde{\pi}^{-1}(u,v) \subset X$. Now ℓ is just the line in $\mathbb{P}^3$ uniquely determined by $u \in L_2$ and $v \in L_1$. Let P_ℓ denote the lifting of ℓ to $\mathbb{C}^4$. We see that P_ℓ is spanned by ℓ_u and ℓ_v. Now $z \in \ell$. Let ℓ_z denote the lift of z to $\mathbb{C}^4$. Thus $\ell_z \subset P_\ell$, i.e. z corresponds to an element of $\mathbb{P}(P_\ell)$. Hence, to each triple (z,u,v) in X, we obtain an element (ℓ_z, u, v) of the bundle $\mathbb{P}(\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$. The converse follows from a similar argument. Thus, the identification of X with $\mathbb{P}(\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1))$ as $\mathbb{P}^1$-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ is clear. Note that $\mathbb{P}(\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1) = \mathbb{P}((\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1)\otimes\mathcal{O}(1,1) = \mathbb{P}(\mathcal{O}(0,1)\oplus\mathcal{O}(1,0))$. However, $(\mathcal{O}(-2,0)\oplus\mathcal{O}(0,-2))\otimes\mathcal{O}(a,b) = \mathcal{O}(-2+a,b)\oplus\mathcal{O}(a,-2+b) \neq \mathcal{O}(-1,0)\oplus\ (0,-1)$ for any a,b. Consequently, X and $\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1)$ are different bundles over

$\mathbb{P}^1 \times \mathbb{P}^1$. From now on, for ease of notation, we shall let Y denote $\mathbb{P}T^*(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$.

§2.4. <u>The contact map</u>

Let $\psi : X \to Y$ be defined by

$$\psi([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) = ([y_0,y_1][y_2,y_3],[z_0dy_1-z_1dy_0,z_2dy_3-z_3dy_2]).$$

We have the following diagram:

$$\begin{array}{ccccc} \mathbb{P}^3 & \xleftarrow{\beta} & X & \xrightarrow{\psi} & Y \\ \downarrow{\scriptstyle P} & & & \searrow{\scriptstyle \tilde{\pi}} & \downarrow{\scriptstyle \pi} \\ S^4 & & & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where β is the blow down map.

Note that H extends to all of X and for $x \in X$, $\tilde{\pi}_*(H_x)$ is a tangent line in $T_{\tilde{\pi}(x)}(\mathbb{P}^1 \times \mathbb{P}^1)$, i.e. $\tilde{\pi}_*(H_x) \in \mathbb{P}T_{\tilde{\pi}(x)}(\mathbb{P}^1 \times \mathbb{P}^1)$. Also, observe that $\tilde{\pi} = \pi \circ \psi$ where π is the projection to the first two factors. Now let $\ell := \tilde{\pi}_*(H_x)$. Then $\pi_*^{-1}(\ell)$ is the contact plane at ℓ. Now $\ell = \tilde{\pi}_*(H_x) = (\pi \circ \psi)_*(H_x) = \pi_* \circ \psi_*(H_x)$. Thus $\pi_*^{-1}(\ell) = \psi_*(H_x)$. We thus get:

<u>Lemma 2.3</u>. ψ_* <u>sends the horizontal plane field</u> H <u>in</u> X <u>to the contact plane field</u> K <u>in</u> Y.

Recall that the two skew lines $L_1, L_2 \subset \mathbb{P}^3$ were defined by $L_1 := \{[0,0,z_2,z_3] \mid [z_2,z_3] \in \mathbb{P}^1\}$ and $L_2 := \{[z_0,z_1,0,0] \mid [z_0,z_1] \in \mathbb{P}^1\}$. The blow ups σ_1 of L_1 and σ_2 of L_2 are given by

$\sigma_1 := \{([0,0,z_2,z_3],[y_0,y_1],[z_2,z_3]) \mid [y_0,y_1]\in\mathbb{P}^1$ and $[z_2,z_3]\in\mathbb{P}^1\}$ and

$\sigma_2 := \{([z_0,z_1,0,0],[z_0,z_1],[y_2,y_3]) \mid [z_0,z_1]\in\mathbb{P}^1$ and $[y_2,y_3]\in\mathbb{P}^1\}$.

We observe that

$\psi(\sigma_1) = \{([y_0,y_1],[z_2,z_3][0,1]) \mid [y_0,y_1]\in\mathbb{P}^1, [z_2,z_3]\in\mathbb{P}^1\}$ and

$\psi(\sigma_2) = \{([z_0,z_1],[y_2,y_3],[1,0]) \mid [z_0,z_1]\in\mathbb{P}^1, [y_2,y_3]\in\mathbb{P}^1\}$.

Proposition 2.4. ψ is a branched 2-fold covering map. It is branched precisely along σ_1 and σ_2.

This proposition will be proved in the next section.

§2.5. The involution on X

We first define an involution $\alpha : X \to X$ by

$$\alpha([z_0,z_1,z_2,z_3],[y_0,y_1],[y_2,y_3]) = ([z_0,z_1,-z_2,-z_3],[y_0,y_1],[y_2,y_3]).$$

(Actually, α is an involution on $\mathbb{P}^3$ which is extended to X in a trivial manner.)

NOTE:

1. $\alpha|_{\sigma_1} = \mathrm{Id}$, $\alpha|_{\sigma_2} = \mathrm{Id}$ and $\alpha^*\Omega = \Omega$.
2. By Note 1, α_* maps the horizontal plane H_x at $x \in X$ to the horizontal plane $H_{\alpha(x)}$ at $\alpha(x)$.
3. Let $u \in L_1$ and $v \in L_2$. Denote by ℓ_{uv} the line in $\mathbb{P}^3$ uniquely determined by u and v. Since $\alpha(u) = u$ and $\alpha(v) = v$, we have $\alpha(\ell_{uv}) = \ell_{uv}$. Consequently $\tilde{\pi}\circ\alpha = \tilde{\pi}$. (Actually, this follows immediately from the

definition of α and $\tilde{\pi}$.)

4. Since $\tilde{\pi}_*(H_x) = \pi_* \circ \psi_*(H_x) = \psi(x)$, we have

$$\psi(\alpha(x)) = \tilde{\pi}_*(H_{\alpha(x)}) = \tilde{\pi}_*(\alpha_* H_x) \quad \text{by Note 2}$$
$$= (\tilde{\pi} \circ \alpha)_*(H_x)$$
$$= \tilde{\pi}_*(H_x) \quad \text{by Note 3}$$
$$= \psi(x)$$

Thus $\psi \circ \alpha = \psi$, i.e. ψ is α-invariant.

Notes 1-4 imply that ψ is at least 2 to 1 except along σ_1 and σ_2. Recall that $\psi([0,0,z_2,z_3],[y_0,y_1],[z_2,z_3]) = ([y_0,y_1],[z_2,z_3],[0,1])$ and $\psi([z_0,z_1,0,0],[z_0,z_1],[y_2,y_3]) = ([z_0,z_1],[y_2,y_3],[1,0])$. It is thus clear that ψ is 1 to 1 on σ_1 and σ_2. Let us now examine the map ψ explicitly in local coordinates. We assume that $x \notin \sigma_1 \cup \sigma_2$, We can then set $z_i = y_i$ for $i = 0,1,2,3$.

<u>Chart 1</u>. Suppose $z_0 = y_0 = 1$ and $z_2 \neq 0$. Set $s = y_1$ and $t = {}^{y_3}/y_2$. Then $ds = dy_1$, $dt = \frac{1}{z^2}[z_2 dy_3 - z_3 dy_2]$. Thus $z_2^2 dt = z_2 dy_3 - z_3 dy_2$. Hence $\psi([1,z_1,z_2,z_3],s,t) = (s,t,[ds,z_2^2 dt])$. We also have $\psi([1,z_1,-z_2,-z_3],s,t) = (s,t,[ds,z_2^2 dt])$

<u>Chart 2</u>. Suppose $z_1 = y_1 = 1$ and $z_3 \neq 0$. Set $u = y_0$ and $v = {}^{y_2}/y_3$. We get: $du = dy_0$ and $-z_3^2 dv = z_2 dy_3 - z_3 dy_2$. Thus $\psi([z_0,1,z_2,z_3],u,v) = (u,v,[du,-z_3^2 dv])$.

<u>Chart 3</u>. Suppose $z_0 = y_0 = 1$ and $z_3 \neq 0$. Set $s = y_1$ and

$v = {}^{y_2}/y_3$. We get: $\psi([1,z_1,z_2,z_3],s,v) = (s,v,[ds,-z_3^2 dv])$

<u>Chart 4</u>. Suppose $z_1 = y_1 = 1$, $z_2 \neq 0$. Set $u = y_0$, $t = {}^{y_3}/y_2$. Then $\psi([z_0,1,z_2,z_3],u,t) = (u,t,[-du,z_2^2 dt])$.

From the above local coordinate expressions for ψ, it is clear that ψ is 2 to 1 away from σ_1 and σ_2. Now ψ is a holomorphic map with finite fibers between compact complex 3-folds. Thus, it is a branched covering map of degree 2. This proves Proposition 2.4.

Let us now examine locally the inverse image of ψ. Pick a point $y \in Y - (S_1 \cup S_2)$. Locally, y has coordinates (s,t,a). Recall that $\psi([1,z_1,z_2,z_3],s,t) = (s,t,[ds,z_2^2 dt])$ where $s = z_1$ and $t = z_3/z_2$. Then

$$\psi^{-1}(y) = \psi^{-1}(s,t,a) = ([1,s,\sqrt{a}, \sqrt{a}\, t],s,t).$$

The involution α on X corresponds to a permutation of the roots. Thus,

<u>Proposition 2.5</u>. <u>The map</u> $\psi : X \to Y$ <u>is equivalent to the projection map</u> $P : X \to X/\mathbb{Z}_2$ <u>where the</u> $\mathbb{Z}_2$<u>-action on</u> X <u>is given by the involution</u> α.

§2.6. <u>The involution on S^4</u>

We shall now examine the action of α on S^4. Recall the identification of S^4 with $\mathbb{P}^1(\mathbb{H})$:

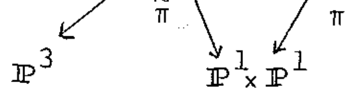
$$[q_1,q_2] \begin{array}{l} \xrightarrow{\varphi_1} \mathbb{R}^4 = \mathbb{H} \\ \xrightarrow[\varphi_2]{} \mathbb{R}^4 = \mathbb{H} \end{array}$$

where $\varphi_1([q_1,q_2]) = q_1^{-1}q_2$ and $\varphi_2([q_1,q_2]) = q_2^{-1}q_1$ with transition functions $q \to q^{-1}\ \frac{1}{\|q\|^2}\ \bar{q}$, where $q_1^{-1}q_2$ and $q_2^{-1}q_1$ correspond to the images in $\mathbb{R}^4$ of the stereographic projections from the south pole and the north pole respectively of the point in S^4. Now $p([z_0,z_1,z_2,z_3]) = [z_0+z_1 j, z_2+z_3 j] \in \mathbb{P}^1(\mathbb{H})$ where $[z_0,z_1,z_2,z_3] \in \mathbb{C}\mathbb{P}^3$ and "j" is the quaternion "j". Thus, $p(\alpha[z_0,z_2,z_2,z_3]) = p[z_0,z_1,-z_2,-z_3] = [z_0+z_1 j,-(z_2+z_3 j)]$. The involution α thus descends to an involution in $S^4 = \mathbb{P}^1(H)$ as follows: $\alpha([q_1,q_2]) = [q_1,-q_2]$ for all $[q_1,q_2] \in \mathbb{P}^1(\mathbb{H})$. (We denote by the same letter "α" both the involutions on X and S^4.)

Now $\varphi_1 \circ \alpha([q_1,q_2]) = \varphi_1([q_1,-q_2]) = -q_1^{-1}q_2$ and $\varphi_2 \circ \alpha([q_1,q_2]) = \varphi_2([q_1,-q_2]) = -q_2^{-1}q_1$. Hence the action of α on a point $x \in S^4$ is just the antipodal map on the $S^3 \subset S^4$ obtained by the intersection of the horizontal 4-plane through x with S^4. (This S^3 is the "latitudinal S^3"). Thus, the geodesic 3-sphere in S^4 passing through the North and South poles is invariant under α.

§2.7. Some degree computations.

In this section, we compute the degree of the total preimage in $\mathbb{P}^3$ of a holomorphic curve in Y. Recall the diagram:



Let ℓ_1 and ℓ_2 (ℓ_1' and ℓ_2') denote the preimages in X (Y) of the first and second factors of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively under the map $\tilde{\pi} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ ($\pi: Y \to \mathbb{P}^1 \times \mathbb{P}^1$). Let S_1 and S_2 denote the 2 distinguished sections of Y corresponding to lines tangent to the second and first factors of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Recall that $\psi_*(\sigma_1) = S_1$ and $\psi_*(\sigma_2) = S_2$. Note that $\psi_*(\ell_i) = 2\ell_i'$, $i = 1,2$. Let H be a hyperplane in $\mathbb{P}^3$. Then $\beta^*H = \sigma_1 + \ell_1 = \sigma_2 + \ell_2$. Thus $\sigma_1 - \sigma_2 = \ell_2 - \ell_1$. Also, $S_1 - S_2 = \psi_*(\sigma_1 - \sigma_2) = \psi_*(\ell_2 - \ell_1) = 2(\ell_2' - \ell_1')$. Hence, the Picard groups of X and Y are given by

$$\mathrm{Pic}(X) = \mathbf{Z}\{\ell_1, \ell_2, \sigma_1, \sigma_2\}/\langle \sigma_1 - \sigma_2 = \ell_2 - \ell_1 \rangle \quad \text{and}$$

$$\mathrm{Pic}(Y) = \mathbf{Z}\{\ell_1', \ell_2', S_1, S_2\}/\langle S_1 - S_2 = 2(\ell_2' - \ell_1')\rangle$$

Let $F = (f_1, f_2) : \sum \to \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic map of a compact Riemann surface of genus g to $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (n,m). Then the curve $C = F(\sum)$ is of class (m,n). Let $\tilde{F}$ denote the canonical lift of F to Y and let $C' := \tilde{F}(\sum)$. (The lift of a point $x \in C$ is the tangent line to C at x.) If we assume that C is smooth, then

$$\deg \tilde{F}^*(\ell_1') = m, \quad \deg \tilde{F}^*(\ell_2') = n$$

$$\deg \tilde{F}^*(S_1) = \text{number of "branch points" of } f_1 = 2g-2+2n$$

and $$\deg \tilde{F}^*(S_2) = \text{number of "branch points" of } f_2 = 2g-2+2m$$

where "deg" refers to the intersection number of $\tilde{F}(\sum)$ with the relevant generators. Let $\tilde{C} := \psi^{-1}(C') \subset X$ and $\gamma := \beta_*(\tilde{C}) \subset \mathbb{P}^3$.

Then for a generic hyperplane H in $\mathbb{P}^3$ we have

$$\begin{aligned}\deg\gamma &= H\cdot\beta_*(\tilde{C}) = \beta^*H\cdot\tilde{C} = (\sigma_1+\ell_1)\cdot(\psi^{-1}C') \\ &= \psi_*(\sigma_1+\ell_1)\cdot C' = (S_1+2\ell_1')\cdot\tilde{F}_*(\Sigma) \\ &= \deg\tilde{F}^*(S_1+2\ell_1') = 2g-2+2n+2m.\end{aligned}$$

I used the term "branch point" in the previous paragraph. Let me define it as follows. Let $\varphi : \Sigma \to \mathbb{P}^1$ be a holomorphic map of a compact Riemann surface to $\mathbb{P}^1$. A point $x \in \Sigma$ is a <u>ramification point</u> of φ if $d\varphi(x) = 0$ and its image $\varphi(x) \in \mathbb{P}^1$ is called a <u>branch point</u> of φ. If the map φ is of degree d and Σ has genus g, then the Riemann-Hurwitz Theorem tells us that the number of branch points of φ (counting multiplicities) is $2g + 2d - 2$. (cf. GH). The <u>ramification divisor</u> of φ is the formal sum $\sum a_i p_i$ where p_i is a ramification point of φ with multiplicity a_i, and where the sum is taken over all ramification points of φ. Let $\mathrm{Ram}(\varphi)$ denote the ramification divisor of φ.

Suppose $\deg f_1 = \deg f_2 = d$ and $\mathrm{Ram}\, f_1 = \mathrm{Ram}\, f_2$. Then the curve $C = F(\Sigma)$ has singular points with the property that $\deg \tilde{F}^*(S_1) = \deg \tilde{F}^*(S_2) = 0$. Consequently, $\deg\gamma = 2d$.

§2.8. <u>Conjugate branched superminimal surfaces</u>

Suppose $f : \Sigma \to S^4$ is a branched superminimal immersion of a compact Riemann surface in S^4. Generically, $f(\Sigma)$ misses a pair of antipodal points on S^4 (say the north and south poles.)Also, generically, $\alpha(f(\Sigma)) \neq f(\Sigma)$, i.e. $f(\Sigma)$ is not α-invariant. Let $\tilde{f} : \Sigma \to \mathbb{P}^3$ be the holomorphic horizontal lift of f to $\mathbb{P}^3$.

Proposition 2.6. A generic branched superminimal surface $f(\Sigma)$ in S^4 has the property that its lift $\tilde{f}(\Sigma)$ in $\mathbb{P}^3$ is not α-invariant.

Proof. The proposition follows immediately from the definition of the involution α (and the fact that α-invariance in $\mathbb{P}^3$ project to α-invariance in S^4).

QED

Note. The converse is not necessarily true, i.e. the fact that $f(\Sigma)$ is α-invariant does not imply that $\tilde{f}(\Sigma)$ is α-invariant. For example, consider the totally geodesic S^2 of area 4π contained in the equator of S^4 It is obviously α-invariant. However, its lift in $\mathbb{P}^3$ is a curve γ of degree one (and hence, a $\mathbb{P}^1$) which avoids the 2 skew lines L_1 and L_2, and hence it is not α-invariant. Observe that $\alpha(\gamma)$ projects down to the same geodesic S^2 (but with the opposite orientation).

Since a generic branched superminimal surface $f(\Sigma)$ in S^4 avoids the poles, it lift $\tilde{f}(\Sigma)$ avoids the 2 skew lines L_1 and L_2. Thus, $\tilde{f}(\Sigma)$ is diffeomorphic to its image in X under the blow up of $\mathbb{P}^3$ along L_1 and L_2. To avoid cumbersome notation, we shall denote the image of $\tilde{f}(\Sigma)$ in X by $\tilde{f}(\Sigma)$ also. Now, by notes 1-4 in §2.5, we have $\tilde{\pi}\circ f(\Sigma) = \tilde{\pi}\circ(\alpha\circ\tilde{f}(\Sigma))$ and that $\alpha\circ\tilde{f}(\Sigma)$ is holomorphic and horizontal in $\mathbb{P}^3$ and thus project to a branched superminimal surface in S^4, i.e., we get "conjugate" branched superminimal surfaces for free. Thus,

<u>Corollary 2.7</u>. <u>Given a generic branched superminimal surface</u> $f(\Sigma)$ <u>in</u> S^4, <u>we obtain a conjugate branched superminimal surface</u>, $\alpha\circ f(\Sigma)$, <u>in</u> S^4.

§2.9. <u>Bidegrees and ramification divisors</u>.

Let $f(\Sigma)$ be a generic branched superminimal surface in S^4. Its lift $\tilde{f}(\Sigma)$ is a holomorphic horizontal curve γ in $\mathbb{P}^3$. The homology degree of $\gamma\subset\mathbb{P}^3$ is its fundamental class in $H_2(\mathbb{P}^3;\mathbb{Z}) \cong \mathbb{Z}$. This degree is also the intersection number of γ with a generic $\mathbb{P}^2$ in $\mathbb{P}^3$. Let $\tilde{\pi} = (\tilde{\pi}_1,\tilde{\pi}_2)$ denote the projection map of $\mathbb{P}^3 - (L_1\cup L_2)$ to $\mathbb{P}^1\times\mathbb{P}^1$ ($\mathbb{P}^1\times\mathbb{P}^1$ is identified with $L_1\times L_2$).

<u>Proposition 2.8</u>. <u>Suppose that</u> $\deg(\gamma) = d$. <u>Then the holomorphic curve</u> $C = \tilde{\pi}\circ\tilde{f}(\Sigma)$ in $\mathbb{P}^1\times\mathbb{P}^1$ <u>has bidegree</u> (d,d).

<u>Proof</u>, Let $x_1 \in L_1$. The fiber $\tilde{\pi}_1^{-1}(x)\subset\mathbb{P}^3$ is the plane $P_1 = \mathrm{Span}(x_1,L_2)$. Since $\deg\gamma = d$, P_1 has d intersection points with γ. Similarly, for $x_2 \in L_2$, the plane $P_2 = \tilde{\pi}_2^{-1}(x_2)$

has d intersection points with γ. Thus $C = \tilde{\pi}(\gamma)$ has bidegree (d,d).

QED

Let $f, \tilde{f}, \tilde{\pi}_1$ and $\tilde{\pi}_2$ be as before. Define f_1 and f_2 by $f_1 := \tilde{\pi}_1 \circ f$ and $f_2 := \tilde{\pi}_2 \circ f$.

<u>Proposition 2.9</u>. <u>Suppose that</u> $\deg f_1 = \deg f_2$. <u>Then</u> $\mathrm{Ram}\, f_1 = \mathrm{Ram}\, f_2$.

<u>Proof</u>. Let $\gamma := \tilde{f}(\Sigma)$. Let z_0 be a ramification point of f_1. Let $p \in \gamma$ denote the point $\tilde{f}(z_0)$. Then the point $x := \tilde{\pi}_1(p)$ is a branch point of f_1. Let $y := \tilde{\pi}_2(p)$ and let L denote the line in $\mathbb{P}^3$ through x and y. Since x is a branch point of f_1, γ is tangent to L at p. Let $v \in T_p\gamma$. We thus have $\tilde{\pi}_{1*}(v) = 0$ and $\tilde{\pi}_{2*}(v) = 0$. Hence, y is a branch point of f_2. Thus z_0 is in the ramification locus of both f_1 and f_2.

QED

<u>Lemma</u> 2.10. <u>The holomorphic map</u> $F = (f_1, f_2) : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$ <u>has a canonical lift</u> $\tilde{F}$ to $Y = \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$.

<u>Proof</u>. Suppose $(df_1(z), df_2(z)) \neq (0,0)$. Then $\tilde{F}(z) = (f_1(z), f_2(z)[f_1'(z), f_2'(z)])$. We are thus left with a finite set of singular points. Suppose 0 is a singular point. Then $f_1'(z) = z^p g_1(z)$ and $f_2'(z) = z^q g_2(z)$ where $g_1(0) \neq 0$, $g_2(0) = 0$ and without loss of generality $1 \leq p \leq q$. So $\tilde{F}(z) = (f_1(z), f_2(z), [g_1(z), z^{q-p} g_2(z)])$ for z in a neighborhood of 0.

QED

Proposition 2.11. *Suppose* $f : \Sigma \to S^4$ *is a generic superminimal immersion*, i.e. $f(\Sigma)$ *avoids the north and south poles in* S^4 *and is not* α*-invariant*. *Let* $\tilde{f} : \Sigma \to \mathbb{P}^3$ *be the holomorphic horizontal lift of* f. *Let* $f_1 := \tilde{\pi}_1 \circ \tilde{f}$ *and* $f_2 := \tilde{\pi}_2 \circ \tilde{f}$. *Suppose that* $\deg f_1 = \deg f_2 = d \geq 2$. *Then* $f_2 \neq A \circ f_1$ *for any* $A \in PSL(2,\mathbb{C})$.

Proof. Suppose $f_2 = A \circ f_1$ for some $A \in PSL(2,\mathbb{C})$. Then $F = (f_1, f_2) = (f_1, A \circ f_1) : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through $\mathbb{P}^1$ as follows:

$$\Sigma \xrightarrow[f_1]{} \mathbb{P}^1 \xrightarrow[G=(Id,A)]{} \mathbb{P}^1 \times \mathbb{P}^1.$$

Since G has bidegree (1,1), it is nonsingular and its canonical lift $\tilde{G}$ (given by lemma 2.10) avoids the 2 sections S_1 and S_2. Since $\deg f_1 \geq 2$, f_1 is necessarily branched. Thus the canonical lift of F, $\tilde{F}$ is a branched covering map of Σ into $\tilde{G}(\mathbb{P}^1) \cong \mathbb{P}^1$, i.e. $\tilde{F}(\Sigma)$ is branched. Consequently, the lift $\tilde{\tilde{F}}(\Sigma)$ in $\mathbb{P}^3$ is branched and hence projects to a branched superminimal surface in S^4. This contradicts the assumption that $f(\Sigma) \subset S^4$ is unbranched.

QED

Note that for $d = 1$, Σ must have genus zero, and so $f(\Sigma)$ is totally geodesic in S^4.

We thus have:

Theorem A. *Every superminimal immersion* $f : \Sigma \to S^4$ *arises from a pair of meromorphic functions* (f_1, f_2) *on* Σ *such that*

1. $\deg f_1 = \deg f_2 = d$ *for some integer* $d \geq 1$.
2. $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$ *where* $\mathrm{Ram}(f_i)$ *denotes the ramification divisor of* f_i.
3. *For* $d \geq 2$, $f_1 \neq A \circ f_2$ *for any* $A \in PSL(2, \mathbb{C})$.

We would like to generate superminimal surfaces in S^4 by considering pairs of meromorphic functions on Σ which satisfy the 3 conditions in Theorem A. Suppose $F = (f_1, f_2)$ is such a pair. Let $\tilde{C} := \tilde{F}(\Sigma)$. Our degree computations in §2.7 show that the total preimage curve $\gamma := \beta \circ \psi^{-1}(\tilde{C})$ in $\mathbb{P}^3$ has degree $2d$. Suppose γ consists of 2 connected components, γ_1 and γ_2. Then $\alpha(\gamma_1) = \gamma_2$ and consequently $\deg \gamma_1 = \deg \gamma_2 = d$. Under suitable conditions, γ_1 and γ_2 will project down to a conjugate pair of superminimal surfaces in S^4. We shall examine the genus zero and higher genus cases in the next two chapters.

CHAPTER 3

Genus Zero

In this chapter, we analyze the space of branched superminimal surfaces of genus zero in S^4. We begin by studying rational functions and their ramification divisors. We show that given a generic meromorphic function f of degree d, there are $\frac{(2d-2)!}{d!(d-1)!}$ distinct $PSL(2,\mathbb{C})$-orbits of meromorphic functions of degree d with the same ramification divisor as f. This fact enables us to construct examples of superminimal surfaces of area $4\pi d$ in S^4 for $d \geqq 3$. We also show that the space of branched superminimal surfaces of genus 0 and degree d in S^4 is connected for each $d \geqq 1$.

§3.1. Meromorphic functions, Grassmannians and resultants

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map of degree d (i.e. f is a meromorphic function of degree d). Then f can be expressed as a rational function of the form $\frac{P(z)}{Q(z)}$ where $P(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$ and $Q(z) = b_d z^d + \dots b_1 z + b_0$ where a_i and b_i are complex numbers. The map f is of degree d if at least one of the two polynomials is of degree d, and that P(z) and Q(z) have no common root. In other words, $\deg(f) = d$ if and only if the resultant of P(z) and Q(z) does not vanish (cf. [VW]). Let $P = (a_d, a_{d-1}, \dots, a_1, a_0)$ and $Q = (b_d, b_{d-1}, \dots, b_1, b_0)$ denote the vectors in $\mathbb{C}^{d+1}$ corresponding to the coefficients of P(z) and Q(z) respectively. Then the resultant $R(P,Q)$ of P(z) and Q(z) is the determinant of the matrix

$$M = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} \quad \text{where}$$

$$A_1 = \begin{bmatrix} a_d & a_{d-1} \cdots a_1 \\ 0 & a_d \cdots\cdots a_2 \\ & \ddots \quad \vdots \\ \bigcirc & a_d \end{bmatrix} \qquad A_2 = \begin{bmatrix} a_0 & & \bigcirc \\ a_1 & a_0 & \\ \vdots & & \ddots \\ a_{d-1} & \cdots & a_0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} b_d & b_{d-1} \cdots b_1 \\ 0 & b_d \cdots\cdots b_2 \\ & \ddots \quad \vdots \\ \bigcirc & b_d \end{bmatrix} \qquad B_2 = \begin{bmatrix} b_0 & & \bigcirc \\ b_1 & b_0 & \\ \vdots & & \ddots \\ b_{d-1} & \cdots\cdots & b_0 \end{bmatrix}$$

The resultant is a homogeneous polynomial of bidegree (d,d) in the a_i and the b_j. Furthermore, $\mathcal{R}(P,Q)$ is irreducible over any arbitrary field (cf. [VW]). We thus require that $(P,Q) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}$, where $\mathcal{R}$ is the irreducible resultant divisor. Observe that $(\lambda P, \lambda Q)$ describes the same function as (P,Q) for any $\lambda \in \mathbb{C}^*$. Thus the space of meromorphic functions of degree d is

$$M_d := \mathbb{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \mathcal{R}) \subset \mathbb{P}^{2d+1}.$$

Define an action of GL(2,d+1) on $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ as follows: $g\cdot(P,Q) := (\alpha P+\beta Q, \gamma P+\delta Q)$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2,\mathbb{C})$. Let $N := \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \Delta$ where $\Delta = \{(P,Q) \mid P \wedge Q = 0\}$. Observe that for $(P,Q) \in N$, $g\cdot(P,Q) = (\alpha P+\beta Q, \gamma P+\delta Q) = (P_1, Q_1)$, and

$P_1 \wedge Q_1 = (\alpha P+\beta Q)\wedge(\gamma P+\delta Q) = (\alpha\delta-\beta\gamma)P\wedge Q \neq 0$. Thus, $GL(2,\mathbb{C})$ acts on N. In fact, we have a free action:

$g\cdot(P,Q) = (\alpha P+\beta Q,\gamma P+\delta Q) = (P,Q)$ implies that $g = I$ since $P\wedge Q \neq 0$.

Note that we can identify N with the Stiefel manifold of 2-frames in $\mathbb{C}^{d+1}$. For $(P,Q) \in N$, let $[P\wedge Q]$ denote the 2-plane in $\mathbb{C}^{d+1}$ spanned by P and Q. Let $P_1, Q_1 \in [P\wedge Q]$. Then $P_1 = \alpha P+\beta Q$, $Q_1 = \gamma P+\delta Q$ for some $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. If $P_1\wedge Q_1 \neq 0$, then $0 \neq P_1\wedge Q_1 = (\alpha\delta-\beta\gamma)P\wedge Q$, i.e. $\alpha\delta - \alpha\gamma \neq 0$. Thus, $GL(2,\mathbb{C})$ acts transitively on pairs of noncollinear vectors in $[P\wedge Q]$. It follows that $N/GL(2,\mathbb{C}) = G(2,d+1)$, and $\pi : N \to G(2,d+1)$ is a principal $GL(2,\mathbb{C})$ bundle (where $\pi(P,Q) = [P\wedge Q]$).

Let us now return to the resultant in $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$.

<u>Lemma 3.1</u>. $R(g\cdot(P,Q)) = (\det g)^d R(P,Q)$.

<u>Proof</u>. Let $(\tilde{P},\tilde{Q})$ denote $g\cdot(P,Q)$, and let the resultant of $(\tilde{P},\tilde{Q})$ be given by the determinant of the matrix

$$\tilde{M} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} .$$

Since $(\tilde{P},\tilde{Q}) = (\alpha P+\beta Q,\gamma P+\delta Q)$, we observe that

$$\tilde{A}_1 = \alpha A_1+\beta B_1 \qquad \tilde{A}_2 = \alpha A_2+\beta B_2$$

$$\tilde{B}_1 = \gamma A_1+\delta B_1 \qquad \tilde{B}_2 = \gamma A_2+\delta B_2$$

i.e. $\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{bmatrix} \cdot \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$ where I = identity matrix in $GL(d,\mathbb{C})$

It is straightforward to verify that $\det\begin{bmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{bmatrix} = (\alpha\delta-\beta\gamma)^d = (\det g)^d$.

Thus, $\det \tilde{M} = (\det g)^d \cdot \det M$, i.e. $R(g\cdot(P,Q)) = (\det g)^d R(P,Q)$.

QED

It follows that $R \subset \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ is fixed under the action of $GL(2,\mathbb{C})$. Let $\mathrm{Reg}(R)$ denote the regular part of R. Since R is irreducible, $\mathrm{Reg}(R)$ is connected. Note that $\Delta = \{(P,Q) \mid P \wedge Q = 0\} \subset R$ and Δ has codimension d in $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$. So, Δ cannot disconnect $\mathrm{Reg}(R)$ (which has dimension 2d + 1). Consequently $(\mathrm{Reg}(R)) \cap N$ is connected, i.e. $R \cap N$ is irreducible. For ease of notation, we shall let R to also denote $R \cap N$. By Lemma 3.1, $\dim(R/GL(2,\mathbb{C})) = \dim(\pi(R)) = 2d-3$, and since $\mathrm{Reg}(R)$ is connected and $\pi : N \to G(2,d+1)$ is a principal $GL(2,\mathbb{C})$-bundle, $\pi(\mathrm{Reg}(R)) = \mathrm{Reg}(\pi(R))$ is connected. Thus, $\pi(R)$ is an irreducible divisor in $G(2,d+1)$.

Observe that the space of meromorphic functions of deg d is $M_d = \mathbb{P}(N-R)$. We thus have a free action of $PSL(2,\mathbb{C})$ on M_d, and $M_d/PSL(2,\mathbb{C}) = G(2,d+1)$.

§3.2. <u>The ramification divisor</u>

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map of degree d. Recall that $z_0 \in \mathbb{P}^1$ is a ramification point of f if $f_*(v) = 0$ for all $v \in T_{z_0}\mathbb{P}^1$. Expressing f as a rational function $\frac{P(z)}{Q(z)}$, we have: $f'(z) = [Q(z)P'(z) - P(z)Q'(z)]/[Q(z)]^2$. Then the ramification points of f are given by the zero locus of $Q(z)P'(z) - P(z)Q'(z)$, a polynomial of degree 2d - 2. Note that if $\deg(Q(z)P'(z)-P(z)Q'(z)) = k < 2d - 2$, then ∞ is a ramification point of order 2d - 2 - k.

Define a map $\Psi^d : M_d = \mathbb{P}(N-R) \to \mathbb{P}^{2d-2}$ by $[(P,Q)] \mapsto$ [coefficients of $\{Q(z)P'(z) - P(z)Q'(z)\}$].

Note that $\Psi^d(\lambda P,\lambda Q) = [\lambda^2$.coefficients of $\{Q(z)P'(z)-P(z)Q'(z)\}]$

$= [$coefficients of $\{Q(z)P'(z)-P(z)Q'(z)\}]$.

Also, if $Q(z)P'(z) -P(z)Q'(z) = 0$, we have that

$$\frac{P'(z)}{P(z)} = \frac{Q'(z)}{Q(z)}, \text{ i.e. } \log P(z) = \log Q(z) + C = \log(\tilde{C}Q(z)),$$

i.e. $P(z) = \tilde{C}Q(z)$, thus $[(P,Q)] \notin M_d$. Thus the map Ψ^d is a well defined map. We shall refer to Ψ^d as the ramification map.

Lemma 3.2. PSL(2,$\mathbb{C}$) preserves the fibers of Ψ^d.

Proof. Let $g \in PSL(2,\mathbb{C})$. Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be a representative of g. Then $\Psi^d(g\cdot[(P,Q)]) = \Psi^d([(\alpha P+\beta Q,\gamma P+\delta Q)])$

$= [$coefficients of $\{(\gamma P(z)+\delta Q(z))(\alpha P'(z)+\beta Q'(z))$

$-(\alpha P(z)+\beta Q(z))(\gamma P'(z)+\delta Q'(z))\}]$

$= [$coefficients of $\{(\alpha\delta-\beta\gamma)(Q(z)P'(z)-P(z)Q'(z)\}]$

$= [$coefficients of $\{Q(z)P'(z)-P(z)Q'(z)\}]$

$= \Psi^d([(P,Q)])$.

QED

Corollary 3.3. $\mathbb{P}SL(2,\mathbb{C})$ acts freely on the fibers of Ψ^d.

Proof. $PSL(2,\mathbb{C})$ acts freely on $M_d = \mathbb{P}(\mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - R)$, and by Lemma 3.2, it preserves fibers.

QED

We thus have an induced map $\Psi_d : G(2,d+1) \to \mathbb{P}^{2d-2}$, where $[P\wedge Q] \to$ [coefficients of $\{Q(z)P'(z) - P(z)Q'(z)\}$]. This is a well defined map.

Note that for $d = 2$, $G(2,3) \cong G(1,3) = \mathbb{P}^2$, and so $\Psi_2 : \mathbb{P}^2 \to G(2,3) \cong \mathbb{P}^2$.

Proposition 3.4. Ψ_2 has degree 1 and is nonsingular everywhere. Hence Ψ_2 is a biholomorphism.

A consequence is that $\Psi_2 : M_2 \to \mathbb{P}^2$ has connected fibers. Thus given any pair of meromorphic functions (f_1, f_2) where $\deg f_1 = \deg f_2 = 2$, such that f_1 and f_2 have the same ramification divisor, we have $f_2 = g \circ f_1$ for some g, a Möbius transformation.

Proof of Proposition. Let $[P\wedge Q] \in G(2,3)$. Then $[P\wedge Q]$ can be represented by one of the following matrices:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{or} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where P and Q correspond to the row vectors. For the first matrix, we have $P(z) = z^2+a$, $Q(z) = z+b$. Then

$$\Psi_2([P\wedge Q]) = [\text{coefficients of } \{Q(z)P'(z) - P(z)Q'(z)\}]$$

$$= [\text{coefficients of } \{(z+b)(2z) - (z^2+a)\}] = [1,2b,-a]$$

i.e. $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \mapsto [1,2b,-a]$.

For the second matrix, $P(z) = z^2+az$, $Q(z) = 1$. Then

$\Psi_2([P\wedge Q]) = [\text{coefficients of } \{2z+a\}] = [0,2,a]$,

i.e. $\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto [0,2,a]$.

Note that ∞ is a ramification point in this case. Lastly, we have $\Psi_2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [0,0,1]$ since $P(z) = z$, $Q(z) = 1$. Observe that this is a degenerate case since $(P,Q) \in \mathcal{R}$. From the explicit computation of each of the three cases, it is clear that Ψ_2 is one-to-one and nonsingular everywhere.

QED

<u>Corollary</u> 3.5. <u>Let</u> f <u>be a meromorphic function of degree 2</u>. <u>Let</u> g <u>be any other meromorphic function of degree 2 with the property that</u> $\mathrm{Ram}(f) = \mathrm{Ram}(g)$. <u>Then</u> $g = A\circ f$ <u>for some</u> $A \in PSL(2,\mathbb{C})$.

<u>Corollary 3.6</u>. <u>There are no superminimal surfaces in</u> S^4 <u>whose lifting to</u> $\mathbb{P}^3$ <u>is a curve of degree 2</u>.

<u>Proof</u>. The genus 0 case follows immediately from Proposition 2.10 and Corollary 3.5. The following argument proves the general case. Let γ be a holomorphic horizontal curve in

$\mathbb{P}^3$ such that $\deg(\gamma) = 2$. Pick any 3 distinct points A,B,C on γ. Let L_1 and L_2 denote the lines through A and B, and A and C respectively. Let P denote the plane spanned by L_1 and L_2. Note that P contains the points A,B and C. Since $\deg(\gamma) = 2$, necessarily γ is contained in P, i.e. γ is planar. Since there are no horizontal planes in $\mathbb{P}^3$ (otherwise, that horizontal $\mathbb{P}^2$ would be diffeomorphic to S^4!), γ must be a projective line. Since $\deg(\gamma) = 2$, γ is necessarily branched. (Nevertheless, γ projects to a totally geodesic surface in S^4.)

QED

We now consider the case when $d = 3$: $\Psi_3 : G(2,4) \to P^4$. Let $[P \wedge Q] \in G(2,4)$. Generically, $[P \wedge Q]$ can be represented by a matrix of the form $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$. Then, we have $P(z) = z^3+az+b$ and $Q(z) = z^2+cz+d$. Now

$$Q(z)P'(z) - P(z)Q'(z) = (z^2+cz+d)(3z^2+a) - (z^3+az+b)(2z+c)$$
$$= z^4 + 2cz^3 + (3d-a)z^2 - 2bz + ad - bc.$$

Thus, $\Psi_3([P \wedge Q]) = [1,2c,3d-a,-2b,ad-bc]$. Now consider the point $[1,2,0,-2,2] \in \mathbb{P}^4$. This gives us $c = 1$ and $b = 1$. The other 2 equations yield $a = 3d$ and $ad - bc = 2$. These reduce to a single equation: $3d^2 - 1 = 2$ since $bc = 1$. Thus, $d = \pm 1$ and $a = \pm 3$. hence, $[1,2,0,-2,2]$ has 2 preimage points in $G(2,4)$. This leads to an example of 2 "distinct" meromorphic functions of degree 3 with the same ramification divisor.

§3.3. An example.

Consider the following 2 meromorphic functions:

$$f = \frac{z^3+3z+1}{z^2+z+1}, \quad g(z) = \frac{z^3-3z+1}{z^2+z-1}.$$

Now $f'(z) = \dfrac{z^4+2z^3-2z+2}{(z^2+z+1)^2}$ and $g'(z) = \dfrac{z^4+2z^3-2z+2}{(z^2+z-1)^2}$.

Thus, df and dg have the same zeroes, i.e. f and g have the same ramification divisor.

Claim. f and g belong to distinct orbits of $PSL(2,\mathbb{C})$.

Proof. Suppose instead that $g = A\circ f$ for some $A \in PSL(2,\mathbb{C})$. Let $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ be a representative of A. Then

$$A\circ f(z) = \frac{\alpha(z^3+3z+1)+\beta(z^2+z+1)}{\gamma(z^3+3z+1)+\delta(z^2+z+1)}$$

$$= \frac{\alpha z^3+\beta z^2+(3\alpha+\beta)z+(\alpha+\beta)}{\gamma z^3+\delta z^2+(3\gamma+\delta)z+(\gamma+\delta)} = g(z) = \frac{z^3-3z+1}{z^2+z-1}.$$

Equating coefficients in the numerator, we get $\alpha = 1$, $\beta = 0$, $3\alpha + \beta = -3$ and $\alpha + \beta = 1$, a contradiction.

QED

§3.4. The general formula.

First, let us examine the degree 3 case, i.e. $\Psi_3 : G(2,4) \to \mathbb{P}^4$. Let $[P\wedge Q] \in G(2,4)$ be represented by the matrix $\begin{pmatrix} a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 \end{pmatrix}$. Let P and Q denote the row vectors (a_3,a_2,a_1,a_0) and (b_3,b_2,b_1,b_0) respectively. Recall that the Plucker embedding $G(2,4) \to \mathbb{P}(\Lambda^2\mathbb{C}^4) = \mathbb{P}^5$ is given by $\binom{P}{Q} \to [P\wedge Q]$. Let ω denote the bivector $P\wedge Q$. Choose a basis $\{\ell_3,\ell_2,\ell_1,\ell_0\}$ for $\mathbb{C}^4$. Then

$$\omega = P\wedge Q = (a_3b_2-a_2b_3)\ell_3\wedge\ell_2 + (a_3b_1-a_1b_3)\ell_3\wedge\ell_1 + (a_3b_0-a_0b_3)\ell_3\wedge\ell_0$$

$$+ (a_2b_1-a_1b_2)\ell_2\wedge\ell_1 + (a_2b_0-a_0b_2)\ell_2\wedge\ell_0 + (a_1b_0-a_0b_1)\ell_1\wedge\ell_0$$

$$= (x_{32},x_{31},x_{30},x_{21},x_{20}x_{10})$$

where $x_{ij} = a_ib_j - a_jb_i$. The x_{ij}'s are called the Plücker coordinates of $[P\wedge Q]$. Since ω is a simple bivector, $\omega\wedge\omega = 0$. Thus

$$x_{32}x_{10} - x_{31}x_{20} + x_{30}x_{21} = 0. \qquad (*)$$

Hence, the image of $G(2,4)$ in $\mathbb{P}^5$ is a quadric hypersurface given by (*). Now let $P(z) = a_3z^3+a_2z^2+a_1z+a_0$ and $Q(z) = b_3z^3 + b_2z^2 + b_1z + b_0$. Then,

$$Q(z)P'(z)-P(z)Q'(z) = z^4(a_3b_2-a_2b_3)+z^3(2(a_3b_1-a_1b_3))+z^2(3(a_3b_0-a_0b_3)$$

$$+(a_2b_1-a_1b_2))+z(2(a_2b_0-a_0b_2))+(a_1b_0-a_0b_1)$$

$$\equiv z^4(x_{32})+z^3(2x_{31})+z^2(3x_{30}+x_{21})+z(2x_{20})+x_{10}.$$

Let G^4 denote the image of $G(2,4)$ in $\mathbb{P}^5$. Then the map Ψ_3 can be given in Plücker coordinates by $\Psi_3([P\wedge Q]) = [x_{32},2x_{31},3x_{30}+x_{21},2x_{20},x_{10}]$. The map Ψ_3 can thus be thought of as the restriction to G^4 of a "map" from $\mathbb{P}^5$ to $\mathbb{P}^4$. (Quotation marks had to be used as Ψ_3 obviously cannot extend to all of $\mathbb{P}^5$).

Let $L : \mathbb{C}^6 \to \mathbb{C}^5$ denote the linear map given by $(x_{32},x_{31},x_{30},x_{21},x_{20},x_{10}) \to (x_{32},2x_{31},3x_{30}+x_{21},2x_{20},x_{10})$. It is clear that L has maximal rank and so L is onto and has a 1-dimensional kernel, say K. Now, K is the lifting to $\mathbb{C}^6$ of some point $\varkappa \in \mathbb{P}^5$ ($\varkappa = \mathbb{P}K$). Thus Ψ_3 can be extended to a map from $\mathbb{P}^5 - \{\varkappa\}$ to $\mathbb{P}^4$. Let $\tilde{\mathbb{P}}^5$ denote the blow-up of $\mathbb{P}^5$ at $\varkappa$. That is, we obtain $\tilde{\mathbb{P}}^5$ by replacing each point $x \in \mathbb{P}^5$ with a pair (x,ℓ) where ℓ is a line through x and $\varkappa$. Then, we have a well-defined map $\tilde{\Psi}_3 : \tilde{\mathbb{P}}^5 \to \mathbb{P}^4$. Note that $\varkappa \notin G^4$ since Ψ_3 is well-defined on G^4.

Now let $q \in \mathbb{P}^4$. The number of points in $(\tilde{\Psi}_3)^{-1}(q)\big|_{G^4}$ is the degree of the map Ψ_3. Let ℓ_q denote the lifting of q to $\mathbb{C}^5$. Then $L^{-1}(\ell_q)$ is a 2-dimensional subspace Π_q of $\mathbb{C}^6$ containing K. Thus, $(\tilde{\Psi}_3)^{-1}(q)$ is just $\mathbb{P}(\Pi_q)$, which is a projective line Λ_q in $\mathbb{P}^5$ containing $\varkappa$. Since G^4 is a quadric hypersurface in $\mathbb{P}^5$ and $\varkappa \notin G^4$, Λ_q intersects G^4 at precisely 2 points, and so Ψ_3 has degree 2 as claimed earlier. This suggests that the degree of Ψ_d is given by the degree of the image of G(2,d+1) in $\mathbb{P}^N$ under the Plücker embedding. Let us now consider the general case.

Let $N = \frac{1}{2}(d+2)(d-1) = \binom{d+1}{2} - 1 = \dim(\mathbb{P}(\Lambda^2\mathbb{C}^{d+1}))$. Let $P = (a_d,\ldots,a_o)$ and $Q = (b_d,\ldots,b_0)$ be 2 vectors in $\mathbb{C}^{d+1}$

which span the plane $[P \wedge Q] \in G(2,d+1)$. Then the Plucker embedding $G(2,d+1) \to \mathbb{P}^N$ is given by $\binom{P}{Q} \mapsto [P \wedge Q]$. Choose Plucker coordinates x_{ij} on $\mathbb{P}^N$ where $i > j$, $i = d,\ldots,1$, $j = d-1,\ldots,0$. Let $P(z) = a_d z^d + a_{d-1}z^{d-1} + \ldots + a_1 z + a_0$ and $Q(z) = b_d z^d + \ldots + b_1 z + b_0$. Then

$$Q(z)P'(z) - P(z)Q'(z) = \alpha_{2d-2}z^{2d-2} + \ldots + \alpha_n z^n + \ldots + \alpha_1 z + \alpha_0$$

where $$\alpha_n = \sum_{\substack{i+j=n+1 \\ i>j}} (i-j)x_{ij}, \quad n = 2d-2,\ldots,0.$$

Consider the linear map $L : \mathbb{C}^{N+1} \to \mathbb{C}^{2d-1}$ given by $(x_{ij}) \mapsto (\alpha_{2d-2},\ldots,\alpha_n,\ldots,\alpha_0)$.

Observe that since α_n contains only the x_{ij}'s which satisfy the condition $i + j = n+1$, L has maximal rank. Let K denote the kernel of L. Then $\dim K = \frac{1}{2}(d^2+d) - 2d+1 = \frac{1}{2}(d-2)(d-1)$. Let $\varkappa := \mathbb{P}K$, a projective $\frac{1}{2}d(d-3)$-plane in $\mathbb{P}^N$. Note that the image of $G(2,d+1)$ in $\mathbb{P}^N$, G^{2d-2}, does not intersect $\varkappa$ by construction. Let $\tilde{\mathbb{P}}^N$ denote the blow-up of $\mathbb{P}^N$ along $\varkappa$. Let $q \in \mathbb{P}^{2d-2}$. Let $\tilde{\Psi}_d$ denote the map induced on $\tilde{\mathbb{P}}^N$. Then $\Lambda_q = (\tilde{\Psi}_d^{-1})(q)$ is a projective $\frac{1}{2}(d-2)(d-1)$-plane in $\mathbb{P}^N$, i.e. a plane of dimension complementary to that of G^{2d-2}. Consequently, the number of points of intersection of Λ_q with G^{2d-2} is the degree of G^{2d-2} in $\mathbb{P}^N$, which turns out to be $\frac{(2d-2)!}{(d-1)!d!}$ (cf. [K], [KL]). As a consequence, generically there are $\frac{(2d-2)!}{(d-1)!d!}$ distinct $PSL(2,\mathbb{C})$-orbits of holomorphic maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^1$ which have the same ramification divisor. We thus have

Theorem B. Let f be a generic meromorphic function of degree $d \geqq 2$. Let R denote the ramification divisor of f. Then under the action of $PSL(2,\mathbb{C})$, we have $\frac{(2d-2)!}{(d-1)!d!}$ distinct orbits of meromorphic functions of degree d with ramification divisor R.

§3.5. The space H_d

In Chapter 2, we showed that every branched superminimal surface in S^4 arises from a pair of meromorphic functions each of degree $d \geqq 1$ with the same ramification divisor. Furthermore, for $d \geqq 2$, if the surface is unbranched, the 2 functions do not differ by a Möbius transformation.

Now let $F = (f_1, f_2) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic map of bidegree (d,d) such that $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$. By the results of Chapter 2, the lifted curve $\tilde{F}(\mathbb{P}^1)$ in $Y = \mathbb{P}T(\mathbb{P}^1 \mathbb{P}^1)$ avoids the 2 distinguished sections S_1 and S_2 of Y. Since $\psi : \tilde{\mathbb{P}}^3 - (\sigma_1 \cup \sigma_2) \to Y - (S_1 \cup S_2)$ is a covering map of degree 2 and $\pi_1 \mathbb{P}^1 = 0$, the map $\tilde{F}$ lifts to a map $\tilde{\tilde{F}} : \mathbb{P}^1 \to \tilde{\mathbb{P}}^3 - (\sigma_1 \cup \sigma_2)$. Let $\gamma_1 = \beta \circ \tilde{\tilde{F}}(\mathbb{P}^1)$ and $\gamma_2 := \beta \circ \alpha \circ \tilde{\tilde{F}}(\mathbb{P}^1) = \alpha(\gamma_1)$. Then γ_1 and γ_2 project to a conjugate pair of branched superminimal surfaces, Σ_1 and Σ_2 in S^4. If $\tilde{F}$ is an immersion, then the pair of surfaces are unbranched. We also showed that for $d \geqq 2$, a necessary condition for Σ_1 and Σ_2 to be unbranched is that f_1 and f_2 belong to different orbits of $PSL(2,\mathbb{C})$. Our search for unbranched superminimal surfaces is thus aided by the following immediate consequence of Theorem B:

Theorem C. For each $d \geqq 3$, there is a branched superminimal surface of genus 0 in S^4 which arise from a pair of meromorphic functions (f_1, f_2) each of degree d such that $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$ and that f_1 and f_2 belong to distinct $PSL(2,\mathbb{C})$-orbits.

Proof. Theorem B tells us that there are $\frac{(2d-2)!}{(d-1)!d!}$ distinct orbits.

QED

Theorem D. Let H_d denote the space of branched superminimal surfaces of genus 0 in S^4 whose lifting to P^3 are holomorphic horizontal curves of degree d. For each $d \geqq 1$, H_d is parametrized by a space of complex dimension $2d + 4$.

Proof. A meromorphic function of degree d is determined by $(2d+1)$ complex parameters. The therom follows immediately from the fact that the fibers of Ψ_d are 3-dimensional.

QED

Note. Theorem D is in agreement with the results of Verdier [V2]. Verdier in fact shows that H_d is naturally equipped with the structure of a complex algebraic variety of pure dimension $2d + 4$, and for $d \geqq 3$, H_d possesses 3 irreducible components. We will show that in fact, H_d is connected.

§3.6. Connectivity of H_d

Recall that a meromorphic function of degree d is an element of $M_d = \mathbb{P}N - \mathcal{R}$ where $N = \mathbb{C}^{d+1} \times \mathbb{C}^{d+1} - \{(P,Q) | P \wedge Q = 0\}$ and where $\mathcal{R}$ is the resultant divisor. We have a ramification map $\Psi^d : M_d \to \mathbb{P}^{2d-2}$. The action of $PSL(2,\mathbb{C})$ induces a map $\Psi_d : G(2,d+1) - \pi(\mathcal{R}) \to \mathbb{P}^{2d-2}$ where $\pi(\mathcal{R}) = \mathcal{R}/PSL(2,\mathbb{C})$, an irreducible variety of codimension 1. For ease of notation, we will let $\mathcal{R}$ and $\mathcal{R}'$ denote $\pi(\mathcal{R})$ and $\Psi_d(\pi(\mathcal{R}))$ respectively, for the rest of this section. Now $\Psi_d : G(2,d+1) \to \mathbb{P}^{2d-2}$ is a branched covering map. Let $R :=$ ramification locus of Ψ_d and $B := \Psi_d(R) =$ branch locus of Ψ_d. Then

$$\Psi_d : G(2,d+1) - R - \mathcal{R} \to \mathbb{P}^{2d-2} - B - \mathcal{R}' \text{ is a covering map.}$$

Now consider the diagnol map $\delta : \mathbb{P}^{2d-2} \to \mathbb{P}^{2d-2} \times \mathbb{P}^{2d-2}$. Let $\mathcal{M}_d := G(2,d+1) - \mathcal{R}$. Then, modulo the action of $PSL(2,\mathbb{C})$, an element of $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ is a pair of meromorphic functions of degree d with the same ramification divisor. We will show that the space $\delta^*(\mathcal{M}_d \times \mathcal{M}_d)$ is connected.

Lemma 3.7. $\mathcal{R}$ is not a component of R. Consequently, $\dim(R \cap \mathcal{R}) \leqq 2d - 4$.

Proof. In §3.1, we showed that $\mathcal{R}$ is irreducible. Thus, it suffices to show that there exists an $x \in \mathcal{R}$ such that $x \notin R$. Let $P(z) = z^d + z^2$, $Q(z) = z$. Certainly $[P \wedge Q] \in \mathcal{R} \subset G(2,d+1)$.

For $\Psi^d(a_j,b_k) = (\ldots,c_m,\ldots)$, we have

$$\left.\frac{\partial c_m}{\partial a_j}\right|_{(P,Q)} = (2j-m-1)b_{m-j+1}\Big|_{(P,Q)} \neq 0 \quad \text{if } m \neq j,\ m \neq 1.$$

i.e. this derivative does not vanish for $j = m = 0,2,3,\ldots,d-1,d$. Also,

$$\left.\frac{\partial c_m}{\partial b_k}\right|_{(P,Q)} = (m-2k+1)a_{m-k+1}\Big|_{(P,Q)} \neq 0 \text{ if } m = d+k-1 \text{ or } m = k+1$$

i.e. this derivative does not vanish for $k = 0$, $m = 1$, $d - 1$ and for $k = 1$, $m = d$;...; $k = d - 1$, $m = 2d - 2$. Consequently, $d\Psi^d\big|_{(P,Q)}$ has maximal rank. Thus $[P\wedge Q] \notin R$.

QED

Observe that any diagonal point $(q,q) \in \delta^*(M_d \times M_d)$ is path connected to any other point $(q',q') \in \delta^*(M_d \times M_d)$ since M_d is connected. Thus, to show that $\delta^*(M_d \times M_d)$ is path connected, it suffices to show that $(x,y) \in \delta^*(M_d \times M_d)$ is path connected to the point (y,y) for any point (x,y). Now let $(x,y) \in \delta^*(M_d \times M_d)$. Let $\Psi_d(x) = \Psi_d(y) = * \in \mathbb{P}^{2d-2} - R'$. Without loss of generality, $* \in \mathbb{P}^{2d-2} - B - R'$, and hence, $x,y \notin R$. (If $* \in B$, we can find a path C in $\mathbb{P}^{2d-2} - R'$ so that $C(0) = *$ and $C(1) = *' \notin B$). Since $G(2,d+1) - R - R$ is connected, there is a path $\tilde{\gamma} \subset G(2,d+1) - R - R$ so that $\tilde{\gamma}(0) = x$, $\tilde{\gamma}(1) = y$. Then $\gamma := \Psi_d(\tilde{\gamma})$ is a based loop in $\mathbb{P}^{2d-2}-B-R$ i.e. $[\gamma] \in \pi_1(\mathbb{P}^{2d-2}-B-R',*)$. Thus $\gamma : S^1 \to \mathbb{P}^{2d-2}-B-R \subset P^{2d-2}$. Since $\mathbb{P}^{2d-2}$ is simply connected, we can extend γ to a map $\gamma' : D^2 \to \mathbb{P}^{2d-2}$. By Thom transversality and Lemma 3.7, we can make γ' transversal to Reg(B), Reg(R') and $\Psi_d(R\cap R) = B\cap R'$, i.e. $\gamma'(D^2) \cap \{\mathrm{Sing}(B) \cup \mathrm{Sing}(R') \cup \{B\cap R'\}\} = \emptyset$. Then $\gamma'(D^2)$ intersects Reg(B) and Reg(R') in a finite number of points, say $\gamma'(D^2) \cap \mathrm{Reg}(B) = \{z_1,\ldots,z_n\}$ and $\gamma'(D^2) \cap \mathrm{Reg}(R') = \{\zeta_1,\ldots,\zeta_m\}$ where $z_i \neq \zeta_j$ for any i,j. Let σ_i and τ_j be tiny based loops around z_i and ζ_j respectively. Then γ is homotopic to a composition of the σ_i's and τ_j's. Observe that the τ_j's act trivially on $F = \Psi_d^{-1}(*)$. Let $x_1 = x$ and $x_{n+1} = y$. Since $[\gamma](x) = y$, we have $[\sigma_1](x_1) = x_2$, $[\sigma_2](x_2) = x_3,\ldots,[\sigma_n](x_n) = x_{n+1} = y$ for some $x_2,\ldots,x_n \in F$. Let $\tilde{\sigma}_i$ be the lifting of σ_i so that $\tilde{\sigma}_i(0) = x_i$ and $\tilde{\sigma}_i(1) = x_{i+1}$. As σ_i traces along the boundary of a tiny disc D_i around the branch point z_i, $\tilde{\sigma}_i$ traces some path around some ramification point $y_i \in \Psi_d^{-1}(z_i)$. Let $\tilde{D}_i$ denote the contractible disc in $G(2,d+1) - R$ around y_i which projects to D_i. Suppose $\sigma_i(t)$ traces ∂D_i for $t \in [t_{\alpha_i}, t_{\beta_i}]$. Let $u_i = \tilde{\sigma}_i(t_{\alpha_i})$ and $v_i = \tilde{\sigma}_i(t_{\beta_i})$. Let $\tilde{\alpha}_i$ be a path from u_i to y_i and let $\tilde{\beta}_i$ be a path from y_i to v_i. Say $\tilde{\alpha}_i(t_{\alpha_i}) = u_i$, $\tilde{\beta}_i(t_{\beta_i}) = v_i$ and $\tilde{\alpha}_i(t_{\varepsilon_i}) = \tilde{\beta}_i(t_{\varepsilon_i}) = y_i$ for some $t_{\varepsilon_i} \in (t_{\alpha_i}, t_{\beta_i})$. Consider the modified path $\tilde{\sigma}_i'$ defined as follows:

$$\tilde{\sigma}_i'(t) = \begin{cases} \tilde{\sigma}_i(t) & t \in [0, t_{\alpha_i}] \\ \tilde{\alpha}_i(t) & t \in [t_{\alpha_i}, t_{\varepsilon_i}] \\ \tilde{\beta}_i(t) & t \in [t_{\varepsilon_i}, t_{\beta_i}] \\ \tilde{\sigma}_i(t) & t \in [t_{\beta_i}, 1] \end{cases} .$$

Let $\sigma_i' := \Psi_d(\tilde{\sigma}_i')$. Observe that σ_i' is a homotopically trivial loop in $\mathbb{P}^{2d-2} - R'$. Let $\tilde{\sigma}_i''$ denote the lifting of σ_i' so that $\tilde{\sigma}_i''(0) = \sigma_i''(1) = y$. Let γ_i denote the path $(\tilde{\sigma}_i', \tilde{\sigma}_i'')$ in $\delta^*(M_d \times M_d)$ from (x_i, y) to (x_{i+1}, y). We have thus constructed a path $\gamma_n \circ \gamma_{n-1} \circ \cdots \circ \gamma_1$ in $\delta^*(M_d \times M_d)$ from (x,y) to (y,y). Thus

Theorem E. For each $d \geq 1$, H_d, the space of branched superminimal surfaces of genus 0 and degree d in S^4 is connected.

§3.7. Examples.

Consider the map $F_d = (f_1, f_2) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ $(d>2)$ where

$$f_1(z) = \frac{P_1(z)}{Q_1(z)} = \frac{z^d + dz + 1}{z^{d-1} + z + (d-2)} \qquad \text{and}$$

$$f_2(z) = \frac{P_2(z)}{Q_2(z)} = \frac{z^d - dz + 1}{z^{d-1} + z - (d-2)} .$$

We will show that for $d > 2$, F_d gives rise to a conjugate pair of superminimal surfaces (unbranched) in S^4.

Note that f_1 and f_2 belong to different orbits under the $PSL(2,\mathbb{C})$ action.

Lemma 3.8. For $d > 2$, F_d has bidegree (d,d). Furthermore, $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$.

Proof. We must show that $P_i(z)$ and $Q_i(z)$ have no common zeroes $(i=1,2)$.

Suppose ζ is a common zero of $P_1(z)$ and $Q_1(z)$. Certainly ζ must be a zero of $P(z) = zQ_1(z) - P_1(z) = z^2 - 2z - 1$.

But $P(z)$ has roots $1 \pm\sqrt{2}$, which are certainly not roots of $P_1(z)$ or $Q_1(z)$. Thus $\deg(f_1) = d$. A similar argument shows that $\deg(f_2) = d$. Now

$$f_1'(z) = \frac{R(z)}{Q_1^2(z)} = \frac{z^{2d-2}+(d-1)z^d-(d-1)z^{d-2}+d(d-2)-1}{[z^{d-1}+z+(d-2)]^2} \quad \text{and}$$

$$f_2'(z) = \frac{R(z)}{Q_2^2(z)} = \frac{z^{2d-2}+(d-1)z^d-(d-1)z^{d-2}+d(d-2)-1}{[z^{d-1}+z-(d-2)]^2} .$$

Thus, $\text{Ram}(f_1) = \text{Ram}(f_2)$.

QED

<u>Proposition 3.9</u>. <u>The map F_d is generically one to one onto its image</u>. <u>Hence it is not a branched covering map</u>.

<u>Proof</u>. $F_d(0) = (f_1(0), f_2(0)) = (\frac{1}{d-2}, \frac{-1}{d-2})$. Note that 0 is not a ramification point of either f_1 or f_2. We shall compute $F_d^{-1}(\frac{1}{d-2}, \frac{-1}{d-2})$. This amounts to solving the simultaneous equations:

$$\frac{z^d+dz+1}{z^{d-1}+z+(d-2)} = \frac{1}{d-2} \quad \text{and} \quad \frac{z^d-dz+1}{z^{d-1}+z-(d-2)} = \frac{-1}{d-2} .$$

We obtain:

$$(d-2)(z^d+dz+1) - (z^{d-1}+z+d-2) = 0 \quad \text{and}$$

$$(d-2)(z^d-dz+1) - (z^{d-1}+z-(d-2)) = 0.$$

This reduces to solving the simultaneous equations

$$g_1(z) = (d-2)z^d - z^{d-1} + (d(d-2)-1)z = 0 \text{ and}$$

$$g_2(z) = (d-2)z^d + z^{d-1} - (d(d-2)-1)z = 0.$$

Observe that if ζ is a common zero of g_1 and g_2, then certainly it is a zero of $g_1 + g_2 = 2(d-2)z^d$ $(d>2)$. But $g_1 + g_2$ has 0 as its only solution. Thus $F_d^{-1}(\frac{1}{d-2}, \frac{-1}{d-2}) = \{0\}$, i.e. F_d is generically one to one onto its image.

QED

<u>Proposition</u> 3.10. <u>The map</u> $\tilde{F}_d : \mathbb{P}^1 \to \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ <u>is nonsingular</u>.

<u>Proof</u>. It suffices to show that $d\tilde{F}_d$ does not vanish at the ramification points. We shall split the proof into 3 cases.

<u>Case 1</u>. Assume that the poles of $Q_1(z)$ and $Q_2(z)$ are not ramification points. Then $\tilde{F}_d$ can be described locally by

$\tilde{F}_d(z) = (f_1(z), f_2(z), G(z))$ where

$$G(z) = \frac{f_1'(z)}{f_2'(z)} = \left[\frac{z^{d-1}+z-(d-2)}{z^{d-1}+z+(d-2)}\right]^2.$$

It suffices to show that G' does not vanish at the ramification points. Now

$$G'(z) = 2\,\frac{z^{d-1}+z-(d-2)}{(z^{d-1}+z+(d-2))^3}\cdot 2(d-2)h(z)$$

where $h(z) = (d-1)z^{d-2} + 1$. Observe that $h(z)$ vanishes when $z^{d-2} = \frac{-1}{d-1}$. Let ζ be a (d-2)th root of $\frac{-1}{d-1}$. Then

$$R(\zeta) = \zeta^{2d-2} + (d-1)\zeta^{d} - (d-1)\zeta^{d-2} + d(d-2) - 1$$

$$= \zeta^{2}(\zeta^{2(d-2)} + (d-1)\zeta^{d-2}) - (d-1)\zeta^{d-2} + d(d-2) - 1$$

$$= \zeta^{2}((\tfrac{1}{d-1})^{2}-1) + d(d-2) \neq 0.$$

Thus, the zeroes of G' do not coincide with the ramification points, i.e. $\tilde{F}_d$ is nonsingular.

<u>Case 2</u>. Suppose ζ is a common zero of $R(z)$ and $Q_1(z)$. Let $\tilde{f}_1(z) = Q_1(z)/P_1(z)$. Then locally,

$\tilde{F}_d(z) = (\tilde{f}_1(z), f_2(z), G(z))$ where $G(z) = \tilde{f}_1'/f_2' = -[Q_2(z)/P_1(z)]^2$.
Then $G'(z) = -2[Q_2(z)/P_1^3(z)]\cdot\Delta$ where

$$\Delta = P_1(z)Q_2'(z) - Q_2(z)P_1'(z) = -z^{2d-2} + (1-d)z^{d} + d(2d-4)z^{d-1} + (d-1)z^{d-2}$$
$$+ d(d-2) + 1.$$

Let $S = R + \Delta = d(2d-4)z^{d-1} + 2d(d-2)$.

First, observe that $Q_1(z)$ and $Q_2(z)$ have no common zeroes since $Q_1(z) + Q_2(z) = 2(d-2) \neq 0$ for $d > 2$. Thus $G'(\zeta) = 0$ if and only if $\Delta(\zeta) = 0$. Suppose that ζ is a common zero of Δ and R. Then ζ must be a zero of S. But $S(z)$ vanishes when $z^{d-1} = -2d(d-2)/d(2d-4) = -1$. Then ζ must be a (d-1)th root of -1. But $Q_1(\zeta) = -1 + \zeta + (d-2) = \zeta + d - 3 \neq 0$ for $d > 2$, contradicting our assumption that ζ was a zero of

$Q_1(z)$. Thus $G'(\zeta) \not\equiv 0$.

Case 3. Suppose ζ is a common zero of $R(z)$ and $Q_2(z)$. Let $\tilde{f}_2 = Q_2/P_2$. Then locally, $\tilde{F}_d(z) = (f_1(z), \tilde{f}_2(z), G(z))$

where $G(z) = f_1'/\tilde{f}_2' = -[P_2(z)/Q_1(z)]^2$. So

$G'(z) = -2[P_2(z)/Q_1^3(z)]\cdot\Delta$ where

$$\Delta = Q_1(z)P_2'(z) - P_2(z)Q_1'(z)$$
$$= z^{2d-2} + (d-1)z^d + d(2d-4)z^{d-1} + (d-1)z^{d-2} - d(d-2) - 1.$$

Let $S = R - \Delta = -d(2d-4)z^{d-1} + 2d(d-2)$.

If ζ is a common zero of Δ and R, certainly it is a zero of S. But S vanishes when $z^{d-1} = 2d(d-2)/d(2d-4) = 1$. i.e. ζ is a $(d-1)$th root of 1. But $Q_2(\zeta) = \zeta - d + 3 \not\equiv 0$ for $d > 2$, a contradiction. Thus $G'(\zeta) \not\equiv 0$.

QED

We thus see that the total preimage $\beta\circ\psi^{-1}(\tilde{F}_d(\mathbb{P}^1))$ is a conjugate pair of nonsingular holomorphic, horizontal curves in $\mathbb{P}^3$, which projects to a conjugate pair of superminimal surfaces, each of area $4\pi d$, in $S^4 (d \geq 3)$.

CHAPTER 4

Higher Genus

We now consider branched superminimal immersions of a compact Riemann surface Σ of genus greater than zero in S^4.

First, let us recall the basic facts from Chapter 2. Suppose $f : \Sigma \to S^4$ is a branched superminimal immersion where area $(f(\Sigma)) = 4\pi d$. Generically, $f(\Sigma)$ misses a pair of antipodal points on S^4 (say the north and south poles) and is not α-invariant, where α is the involution on $S^4 = \mathbb{P}^1(\mathbb{H})$ defined by $[q_1,q_2] \to [q_1,-q_2]$. We obtain a "conjugate" branched superminimal surface $\alpha \circ f(\Sigma)$. Let $\tilde{f} : \Sigma \to \mathbb{P}^3$ denote the canonical lift of f to $\mathbb{CP}^3$. Then $\tilde{f}(\Sigma)$ is a holomorphic curve of degree d which misses the 2 projective lines L_1 and l_2 corresponding to the fibers (of the Penrose fibration) above the north and south poles. Furthermore, $f(\Sigma)$ does not coincide with $\alpha \circ f(\Sigma)$. The branched covering map $\psi : \tilde{\mathbb{P}}^3 \to \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ sends both $\tilde{f}(\Sigma)$ and $\alpha \circ \tilde{f}(\Sigma)$ to some curve $\tilde{C}$ in $\mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ which projects to a curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d,d). The curve $\tilde{C}$ misses the 2 distinguished sections S_1 and S_2 corresponding to lines tangent to the second and first factors of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Let f_1 and f_2 denote the first and second factor projections of C. We see that deg f_1 = deg f_2 = d and $\mathrm{Ram}(f_1) = \mathrm{Ram}(f_2)$. Observe that f is totally geodesic for d = 1,2, and is linearly full provided $d \geqq 3$. Note that if $f_2 = A \circ f_1$ for some $A \in PSL(2,\mathbb{C})$, then the map $F = (f_1,f_2) : \Sigma \to \mathbb{P}^1\ \mathbb{P}^1$ factors through P^1 as follows:

$$\Sigma \xrightarrow{g} \mathbb{P}^1 \xrightarrow{(g_1, A\circ g_1)} \mathbb{P}^1 \times \mathbb{P}^1$$

where g is a holomorphic map of degree d and g_1 is a holomorphic map of degree 1. The result of Chapter 3 imply that the map $G = (g_1, A\circ g_1) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ gives rise to a conjugate pair of totally geodesic surfaces in S^4. Thus F gives rise to a conjugate pair of (branched) totally geodesic surfaces in S^4. Consequently, $f : \Sigma \to S^4$ is linearly full provided $f_2 \not\equiv A\circ f_1$ for any $A \in PSL(2,\mathbb{C})$. We will be interested in constructing linearly full branched superminimal immersions of Σ in S^4 from pairs of meromorphic functions f_1, f_2 on Σ, each of degree $d \geqq 3$ such that $\text{Ram}(f_1) = \text{Ram}(f_2)$ and $f_2 \not\equiv A\circ f_1$ for any $A \in PSL(2,\mathbb{C})$. With these conditions, the canonically lifted curve $\tilde{F}(\Sigma) \subset \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1)$ misses the 2 distinguished sections S_1 and S_2. Let $\tilde{C}$ denote the curve $\tilde{F}(\Sigma)$. We require that $\psi^{-1}(\tilde{C})$ consist of 2 connected components, γ_1 and γ_2, such that $\alpha(\gamma_1) = \gamma_2$ and $\psi(\gamma_1) = \psi(\gamma_2) = \tilde{C}$. If this is the case, then γ_1 and γ_2 project to a conjugate pair of (branched) superminimal surfaces in S^4.

Let $X := \tilde{\mathbb{P}}^3 - (\sigma_1 \cup \sigma_2) \cong \mathbb{P}^3 - (L_1 \cup L_2)$ and $Y := \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1) - (S_1 \cup S_2)$. Note that $\pi_1 X = 0$ and $\psi : X \to Y$ is a 2-1 covering map. The maps that we are considering, $F = (f_1, f_2) : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$, are such that $\tilde{F}(\Sigma) \subset Y$. Observe that $\tilde{F}$ lifts to a map $\tilde{\tilde{F}} : \sum \to X$ iff $\tilde{F}_*(\pi_1\sum) = 0$. If $\tilde{F}_*(\pi_1\sum) = 0$, we have 2 maps from $\sum$ to X : $\tilde{\tilde{F}}$ and $\alpha\circ\tilde{\tilde{F}}$. Thus

Theorem F. Let $F = (f_1, f_2) : \sum \to \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic map of a compact Riemann surface of genus g to $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose the map has bidegree (d,d) such that $\text{Ram}(f_1) = \text{Ram}(f_2)$ and $f_2 \neq A\circ f_1$ for any $A \in PSL(2,\mathbb{C})$. Let $\tilde{F} : \sum \to \mathbb{P}T(\mathbb{P}^1 \times \mathbb{P}^1) - (S_1 \cup S_2)$ be the canonical lift of F. Then F gives rise to a conjugate pair of branched superminimal surfaces of genus g in S^4 provided $\tilde{F}_*(\pi_1\sum) = 0$.

Note. The condition $\tilde{F}_*(\pi_1\sum) = 0$ is automatically satisfied if genus$(\sum) = 0$. However, if $\tilde{F}_*(\pi_1\sum) \neq 0$, then we don't have a lift of $\sum$ to X. Nevertheless, there is a 2-fold cover $\tilde{\sum}$ of $\sum$ which lifts to X, where genus$(\tilde{\sum}) = 2g - 1$. We then obtain a superminimal surface in S^4 of genus $2g - 1$.

An easy way to satisfy the condition is by factoring through $\mathbb{P}^1$. Let $\varphi : \sum \to \mathbb{P}^1$ be a holomorphic map of degree d' from a compact Riemann surface of genus g to $\mathbb{P}^1$. Let $(f_1, f_2) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic map of bidegree (d,d) which give rise to a branched superminimal immersion of $\mathbb{P}^1$ in S^4. (There are lots of these maps from the results of Chapter 3!) Let $F = (F_1, F_2) := (f_1\circ\varphi, f_2\circ\varphi) : \sum \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the map of bidegree (dd', dd') given by precomposing with φ. Certainly $\text{Ram}(F_1) = \text{Ram}(F_2)$. If we assume that $f_2 \neq A\circ f_1$ for any $A \in PSL(2,\mathbb{C})$, certainly $F_2 \neq A\circ F_1$ for any A $\varepsilon PSL(2,\mathbb{C})$. Let $\tilde{F} : \Sigma \to Y$ be the canonical lift of F. Then $\tilde{F}_*(\pi_1 \Sigma) = 0$ and by Theorem F, $\tilde{F}$ lifts to a holomorphic horizontal map to $\mathbb{P}^3$. Thus, we have lots of branched superminimal immersions of Σ in S^4.

There are many questions remaining in the case of a compact Riemann surface of genus $g > 0$. For instance, suppose F is a map of Σ into $\mathbb{P}^1 \times \mathbb{P}^1$ which factors through $\mathbb{P}^1$. Can we deform F to a map F' which does not factor through $\mathbb{P}^1$, but which gives rise to a branched superminimal surface in S^4? Can F' give rise to an unbranched superminimal surface? I hope to address these unanswered questions in the near future.

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