HYPERBOLIC INVARIANTS FOR INFINITELY GENERATED FUCHSIAN GROUPS

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Ara Basmajian

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Ara Basmajian

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Bernard Maskit

Bernard Maskit, Professor of Mathematics
Dissertation Director

Paul Kumpel

Paul Kumpel, Professor of Mathematics
Chairman of Defense

Irwin Kra

Irwin Kra, Professor of Mathematics

Laurence Baxter

Laurence Baxter, Associate Professor, Applied Mathematics & Statistics Department
Outside member

This dissertation is accepted by the Graduate School.

Alexander H. King

Graduate School

Abstract of the Dissertation

Hyperbolic Invariants for Infinitely Generated Fuchsian Groups

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A Fuchsian group is said to be of type (g;n;m) if its associated quotient surface is of genus g with n points and m discs removed. Let h be a hyperbolic element of PSL(2,R). We define a quantity $c(h)$, called the collar width of h, which only depends on the translation length of h. We prove the following theorem.

Suppose γ and β are hyperbolic elements oriented so that each of their axes lies to the right of the other. Let d be the hyperbolic distance between the axes of γ and β. Then (γ,β) form standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) if and only if

$$c(\gamma)+c(\beta)\leq d.$$

With equality holding if and only if the group is of type (0;1;2).

We consider surfaces constructed by gluing together infinitely many pairs of pants. The Fuchsian group associated with such a surface

will be an infinitely generated Fuchsian group having non-conjugate simple hyperbolics $\{g_i\}$ with axes $\{L_i\}$. These axes have the property that L_i separates L_{i-1} from L_{i+1}. We call such a sequence a nested sequence of axes.

We supply necessary and sufficient conditions, in the form of inequalities involving distances between geodesics, for an infinitely generated Fuchsian group G to be constructed from only (0;1;2) groups. Let d_i be the distance between the axes L_i and L_{i+1}. We show that G is of the first kind if $\sum d_i = \infty$.

It is well known (The Nielsen Isomorphism theorem) that every type preserving isomorphism between finitely generated Fuchsian groups is a topological deformation. Using the above constructions we show that the Nielsen Isomorphism theorem does not extend to infinitely generated groups.

It is evident that one can use the combination theorem a finite number of times to construct new surfaces out of simple ones. We investigate what happens when we iterate the combination theorem an infinite number of times.

To my father and mother

Vincent and Yvette Basmajian

TABLE OF CONTENTS

List of Figures

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CHAPTER I

0. This chapter will serve as a very brief introduction to Fuchsian groups; the main objective being to set up notation for the chapters to follow. For a more detailed account on the theory of Fuchsian groups and Riemann surfaces the reader is referred to the books of Maskit [M-1] and Beardon [B]. Our approach to the subject is for the most part geometric. An introduction to the analytic theory can be found in [F-K].

Other writers who have studied infinitely generated Fuchsian groups include Keen [K-1] and Purzitsky [P].

1. Two models of the hyperbolic plane H are the upper half-plane $\{z\in\mathbb{C}:\text{Im}(z)>0\}$ and the Poincaré disc $\{z\in\mathbb{C}:|z|<1\}$. Their line elements are $ds^2=\frac{|dz|^2}{(\text{Im}(z))^2}$ and $ds^2=\frac{2|dz|^2}{1-|z|^2}$, respectively. Both of these models have Euclidean boundaries which can be identified with the boundary of H, ∂H. Thus the hyperbolic boundary of the upper half-plane is the extended real numbers $\mathbb{R}\cup\infty$ and for the Poincaré disc it is the unit circle. We let $\rho(*,*)$ be the hyperbolic distance between two points of H.

The geodesics for the upper half-plane are vertical lines and arcs of circles which are orthogonal to the real line. For the Poincaré disc the geodesics are straight lines through the origin and arcs of circles which are orthogonal to the unit circle. We denote the geodesic with endpoints a and b on the boundary of H by [a,b].

The group of orientation preserving isometries of the upper half-plane (with the hyperbolic metric) is made up of the Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}$$

where $ad-bc=1$ and $a,b,c,d \in \mathbb{R}$. This group can be naturally identified with the projectived special linear group, PSL(2,R). Thus one can talk about the square of the trace of an orientation preserving isometry f, denoted $tr^2(f)$. Note that this quantity is conjugation invariant.

We classify the orientation preserving isometries of H by the square of their trace. Namely,

f is <u>hyperbolic</u> if and only if $tr^2(f)>4$;

f is <u>parabolic</u> if and only if $tr^2(f)=4$; and

f is <u>elliptic</u> if and only if $0 \leq tr^2(f)<4$.

A hyperbolic element h has a unique invariant geodesic line in H, called the <u>axis of h</u> and denoted $A(h)$. If z is a point on $A(h)$, the <u>translation length of h</u>, $T(h)$, is the distance $\rho(z,h(z))$. The translation length and the square of the trace are related by the following formula,

$$tr^2(f)=4\cosh^2(T(f)/2).$$

A <u>reflection</u> in the circle with center a and radius r is the unique mapping σ which satisfies,

$$|\sigma(z)-a||z-a|=r^2.$$

Observe that $\sigma^2=1$. A reflection in a circle orthogonal to the boundary (or equivalently in a hyperbolic geodesic) of H is an orientation reversing isometry of H. Hence the composition of two such reflections is an orientation preserving isometry of H.

2. A Fuchsian group is a discrete subgroup of PSL(2,R). A group $G \subset PSL(2,R)$ acts discontinuously at $z \in \hat{\mathbb{C}}$ if there exists an open neighborhood V of z so that $gV \cap V = \varnothing$ for all but finitely many elements $g \in G$. The set of all points in $\hat{\mathbb{C}}$ at which G acts discontinuously is called the regular set of G and is denoted $\Omega(G)$. The complement of $\Omega(G)$ in $\hat{\mathbb{C}}$ is called the limit set of G, denoted $\Lambda(G)$. We have the following fundamental theorem.

THEOREM. *Suppose G is a subgroup of PSL(2,R). Then the following are equivalent.*

(i). G is discrete;

(ii). G acts discontinuously at some point $z \in H$; and

(iii)G acts discontinuously at every point of H.

If $\Lambda(G) = \partial H$, we say that G is a Fuchsian group of the first kind. Otherwise, $\Omega(G) \cap \partial H$ is the union of open intervals and G is said to be of the second kind. These intervals are known as intervals of discontinuity. An open half-space $B \subset H$ which "bounds" an interval of discontinuity is called a boundary half-space for G. If there exists a hyperbolic element $g \in G$ whose axis "bounds" this interval of discontinuity, we say that g is a boundary hyperbolic element with boundary half-space B.

3. Suppose G is a torsion free Fuchsian group. Then H/G is an oriented hyperbolic surface of constant negative curvature whose fundamental group is isomorphic to G. The natural projection map $\pi: H \rightarrow H/G$ is a covering with group of deck transformations G.

A conjugacy class is parabolic (hyperbolic) if an element in the class, and hence every element, is parabolic (hyperbolic). Maximal parabolic conjugacy classes in G correspond to punctures on H/G. Maximal hyperbolic conjugacy classes in G correspond to closed geodesics on H/G.

A Fuchsian group G is of type (g;n;m) if H/G is a compact surface of genus g with n points and m conformal holes removed. If $g=0$ and $m+n=3$, we say that H/G (and likewise G) is a pair of pants.

4. The Nielsen (convex) region N(G) for a Fuchsian group G is the smallest non-empty G-invariant convex subset of H. Equivalently, N(G) is H−{boundary half-spaces of G}.

One of the esential ways of understanding the action of a Fuchsian group on H is by way of fundamental polygons.

A (convex) polygon D in H is the intersection of countably many open half-spaces, with the property that any compact set in H intersects only finitely many geodesics which define the half-spaces. The geodesic segments that make up the boundary of D are called <u>sides</u>. Two sides meet at a <u>vertex</u>.

<u>DEFINITION.</u> Let G be a Fuchsian group. A polygon D is a <u>(convex) fundamental polygon</u> for G if the following conditions hold.

(i). $gD \cap D = \varnothing$ for all non-trivial $g \in G$;

(ii) $\bigcup_{g \in G} g\overline{D} = H$;

(iii). The sides of D are paired by elements of G;

(iv). (local finiteness) Any compact set intersects only finitely many translates of D.

An important feature of a fundamental polygon lies in the fact that $\overline{D}/G$ is hyperbolically isometric to H/G.

At this point there arises a natural question. Which polygons with sides paired by elements of PSL(2,R) are fundamental polygons for Fuchsion groups? Poincaré's theorem supplies us with the answer. We content ourselves with a restricted form of the theorem. For the general form see [M-1].

<u>POINCARÉ'S POLYGON THEOREM.</u> *Suppose D is a finite sided (convex) polygon in H, satisfying the following conditions.*

(i). The sides of D are paired by elements of PSL(2,R);

(ii) $\gamma D \cap D = \varnothing$ for all the side pairing transformations γ;

(iii). The orbit of a vertex under the side pairing transformations is the set of all vertices of D;

(iv). The sum of the angles at the vertices is equal to 2π; and

(v). D has no sides that meet tangentially on the boundary of H.

Then D is a fundamental polygon for the Fuchsian group generated by the side pairing transformations.

5. A basic tool in the theory of Fuchsian groups is the combination theorem. It enables one to construct complicated Fuchsian groups out of simple ones.

Suppose h is a boundary hyperbolic in a Fuchsian group Γ. Call the component of $H-A(h)$ which contains $N(\Gamma)$, $B(h)$. Let $B'(h)$ be the other component.

COMBINATION THEOREM. *Let D_1 and D_2 be (convex) fundamental polygons for the Fuchsian groups Γ_1 and Γ_2, respectively. Suppose h is a primitive boundary hyperbolic in both Γ_1 and Γ_2 satisfying*

(i). $N(\Gamma_1)\cap N(\Gamma_2)=A(h)$;

(ii). $D_1\cap B(h)\subset D_2\cap B(h)$;

(iii). $D_2\cap B'(h)\subset D_1\cap B'(h)$; and

(iv). The sides of D_1 that are paired by h lie on the same geodesic lines as the sides of D_2 which are paired by h.

Then $\langle\Gamma_1,\Gamma_2\rangle$ is a Fuchsian group with fundamental polygon $D_1\cap D_2$.

6. Suppose G is a Fuchsian group. An elliptic element $g \in G$ is said to be a minimal rotation if $\mathrm{tr}^2(g)$ is maximal among non-trivial elements of $\langle g \rangle$. A group isomorphism $\Phi: G \to G'$ between two Fuchsian groups is said to be type preserving if the following conditions hold.

(i). $g \in G$ is parabolic if and only if $\Phi(g)$ is;

(ii) $g \in G$ is a minimal rotation if and only if $\Phi(g)$ is; and

(iii) $g \in G$ is a boundary hyperbolic element if and only if $\Phi(g)$ is.

Another central theorem in the theory of Fuchsian groups and Teichmüller spaces is the Nielsen Isomorphism theorem.

NIELSEN ISOMORPHISM THEOREM. *Suppose $\Phi: G \to G'$ is a type preserving isomorphism between two finitely generated Fuchsian groups. Then there exists a homeomorphism $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $fgf^{-1} = \Phi(g)$.*

CHAPTER II

0. In this chapter we supply necessary and sufficient conditions (theorem II.22) for two hyperbolic elements (γ,β) to form standard generators for a Fuchsian group of type (0;0;3) or (0;1;2). These conditions are in terms of the hyperbolic distance between the axes of γ and β and the "collar widths" of γ and β. In Section 7 we state our own version of a collar lemma. Other writers who have investigated collars about geodesics include Buser [Bu], Halpern [H], Keen [K-2], Maskit [M-2], Matelski [Ma], and Randol [R].

First we prove an intermediate result (theorem II.1) with a series of lemmas which will occupy most of this chapter.Throughout this chapter G is a Fuchsian group of type (0;0;3) or (0;1;2).

1. Let the map π: H→H/G be projection onto the quotient space, H/G. H/G is topologically a sphere with three holes. Choose oriented loops around each of the holes so that the other loops lie to the right(figure 1). Name these loops A,B, and C in some order where C goes around the puncture, if there is one. The ordered pair (γ,β) form standard generators for G if $G=\langle\gamma,\beta\rangle$, $\pi(A(\gamma))$ is freely homotopic to A, $\pi(A(\beta))$ is freely homotopic to B, and $\pi(A(\gamma^{-1}\beta^{-1}))$ is freely homotopic to C. In the case where $\gamma^{-1}\beta^{-1}$ is parabolic, $A(\gamma^{-1}\beta^{-1})$ is replaced by any

horocycle of $\gamma^{-1}\beta^{-1}$.

REMARK. The projection of the axis of any hyperbolic element comes equipped with an orientation induced by the orientation on the axis, namely the positive direction of the axis is from repelling to attracting fixed point.

We now state the main result of this chapter.

(II.1)THEOREM *Let γ and β be two hyperbolic elements of $PSL(2,R)$ oriented so that each of their axes lies to the right of the other . Then (γ,β) form standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) if and only if*

$$(II.2) \qquad d=\rho(A(\gamma),A(\beta))> Log\, \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}\,.$$

$$(II.3) \qquad T(\gamma) \geq 2\, Log\left(\frac{e^{\frac{T(\beta)}{2}}(e^{d}+1)-(e^{d}-1)}{e^{\frac{T(\beta)}{2}}(e^{d}-1)-(e^{d}+1)}\right).$$

Equality in (II.3) holds if and only if the group $\langle\gamma,\beta\rangle$ is of type (0;1;2).

We call the above theorem the "collar theorem" and we say that the quantity

$$c(\beta)=\mathrm{Log}\, \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}.$$

is the (one-sided) collar width of β.

2.This section is devoted to the lemmas needed to prove (II.1) .

<u>(II.4)LEMMA.</u> *Any group G with standard generators* (γ,β) *can be conjugated by an element* $f \in PSL(2,R)$ *so that* $f\gamma f^{-1}$ *has repelling fixed point* x *and attracting fixed point* 1, $f\beta f^{-1}$ *has repelling fixed point* 0 *and attracting fixed point* ∞, *and* $f\gamma^{-1}\beta^{-1}f^{-1}$ *has repelling fixed point* z *and attracting fixed point* y, *where* $1<x<y\leq z$ ($y=z$ *if* $\gamma^{-1}\beta^{-1}$ *is parabolic*).*(See figure* 2)

<u>REMARK.</u> A set of generators of this form is said to be normalized.

<u>PROOF.</u> Let $\eta=\gamma^{-1}\beta^{-1}$. Draw the axes of γ,β, and η (if η is not parabolic) and note that the orientations on A,B, and C induce orientations on $A(\gamma)$, $A(\beta)$, and $A(\eta)$ (a horocycle if η is parabolic) so that each axis lies to the right of all the others. Thus we must have the configurations described in figure 3 (or some cyclic permutation of it).

Notice that the other possibility (the configuration shown in figure 4) can not happen since $\eta\beta\gamma(0)\neq 0$ contradicts the relation $\eta\beta\gamma=1$.So we can assume that γ, β, and η are arranged as in figure 3.

Construct the element $f \in PSL(2,R)$ which takes the attracting fixed point of β to ∞, the repelling fixed point of β to 0, and the attracting fixed point of γ to 1. After conjugating G by f, we have the configuration pictured in figure 5.□

Suppose G is a normalized Fuchsian group with standard generators (γ,β). Let y be the fixed point of $\eta= \gamma^{-1}\beta^{-1}$ if G is of type

(0;1;2).

Draw the common orthogonals to each pair of axes (in the (0;1;2) case draw the unique geodesic orthogonal to the axis $A(\gamma)$ with endpoint y: also draw the geodesic orthogonal to $A(\beta)$ with endpoint y).

For the (0;1;2) case:

Call reflection in the geodesic orthogonal to $A(\gamma)$ with endpoint y, σ_1.

Call reflection in the common orthogonal to $A(\beta)$ and $A(\gamma)$, σ_2.

Call reflection in the geodesic orthogonal to $A(\beta)$ with endpoint y, σ_3.

For the (0;0;3) case:

Call reflection in the common orthogonal to $A(\gamma)$ and $A(\eta)$, σ_1.

Call reflection in the common orthogonal to $A(\beta)$ and $A(\gamma)$, σ_2.

Call reflection in the common orthogonal to $A(\eta)$ and $A(\beta)$, σ_3.

(refer to figures 6 and 7).With the above notation we have the following lemma.

(II.5)LEMMA. *G is equal to the orientation preserving subgroup of* $\langle\sigma_1,\sigma_2,\sigma_3\rangle$. *Furthermore* $\gamma=\sigma_2\sigma_1$, $\beta=\sigma_3\sigma_2$, *and* $\eta=\sigma_1\sigma_3$.

PROOF. We first show that $\gamma=\sigma_2\sigma_1$, $\beta=\sigma_3\sigma_2$, and $\eta=\sigma_1\sigma_3$. Since the σ_2 reflection circle is orthogonal to the axis $A(\beta)$, we can represent β as reflection in the σ_2 reflection circle followed by reflection in some other geodesic orthogonal to the axis $A(\beta)$. Similarly, since the σ_2 reflection circle is orthogonal to the axis $A(\gamma)$, we can represent γ as reflection in some geodesic orthogonal to the axis $A(\gamma)$ followed by reflection in the σ_2 reflection circle.

To this end, we let $\sigma_3' = \beta\sigma_2$ and $\sigma_1' = \sigma_2\gamma$. The σ_3' reflection circle is orthogonal to $A(\beta)$ and the σ_1' reflection circle is orthogonal to $A(\gamma)$. Consider the product

$\sigma_3'\sigma_1' = (\beta\sigma_2)(\sigma_2\gamma) = \beta(\sigma_2)^2\gamma = \beta\gamma = \eta^{-1}$. Thus the σ_3' and σ_1' reflection circles are orthogonal to $A(\eta)$. Since the σ_3' reflection circle is also orthogonal to $A(\beta)$ we conclude that the σ_3' reflection circle is the common orthogonal to $A(\eta)$ and $A(\beta)$. Hence $\sigma_3' = \sigma_3$. Similarly, $\sigma_1' = \sigma_1$ since the σ_1' reflection circle is the common orthogonal to $A(\eta)$ and $A(\gamma)$. So we have that $\gamma = \sigma_2\sigma_1$, $\beta = \sigma_3\sigma_2$, and $\eta = \gamma^{-1}\beta^{-1} = (\sigma_1\sigma_2)(\sigma_2\sigma_3)$ $= \sigma_1\sigma_3$.

Finally, note that the orientation preserving subgroup of $<\sigma_1,\sigma_2,\sigma_3>$ is $<\sigma_2\sigma_1,\sigma_3\sigma_1,\sigma_1\sigma_3>$ and by the above argument $<\sigma_2\sigma_1,\sigma_3\sigma_1,\sigma_1\sigma_3> = <\gamma,\beta,\eta> = G$. □

<u>(II.6) LEMMA.</u> *The common orthogonal to the geodesics* $[0,\infty]$ *and* $[1,x]$, $x>1$, *has endpoints* $-\sqrt{x}$ *and* $\sqrt{x}$.

<u>PROOF.</u> It is enough to find the reflection σ which interchanges 0 and ∞ ,and interchanges 1 and x. The reflection $\sigma(z) = \frac{a^2}{\bar{z}}$ interchanges 0 and ∞. Moreover, $x = \sigma(1) = a^2$ implies that $a = \sqrt{x}$. We conclude that the common orthogonal to the geodesics $[0,\infty]$ and $[1,x]$ has center 0 and radius $\sqrt{x}$. □

(II.7) LEMMA. *Suppose* $\beta(z)=\lambda z$, $\lambda>1$, *and* $\beta=\sigma_3\sigma_2$ *where* σ_2 *is reflection in the circle of radius* $\sqrt{x}$ *and center* 0. *Then* σ_3 *is reflection in the circle of radius* $\sqrt{\lambda x}$ *and center* 0.

PROOF. Note that σ_3 must be reflection in a circle orthogonal to the axis of β, since the axis of β is exactly the common orthogonal between the σ_2 reflection circle and the σ_3 reflection circle. Hence $\sigma_3=\frac{y^2}{\bar{z}}$, where y is the radius of the reflection circle. We must compute y in terms of λ and x. We know that $\sigma_2(z)=\frac{(\sqrt{x})^2}{\bar{z}}=\frac{x}{\bar{z}}$. Setting $\sigma_3=\beta\sigma^{-1}$, we find that $\frac{y^2}{\bar{z}}=\sigma_3(z)=\beta(\sigma_2^{-1}(z))=\beta(\sigma_2(z))=\beta(\frac{x}{\bar{z}})=\frac{\lambda x}{\bar{z}}$. Thus $y=\sqrt{\lambda x}$. □

(II.8) LEMMA. *Suppose G with standard generators* (γ,β) *is normalized so that* $\beta(z)=\lambda z$, $\lambda>1$, *and* $x>1$ *is the repelling fixed point of* γ. *Then* $\lambda>x$.

PROOF. We know that G is equal to the orientation preserving subgroup of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ where $\gamma=\sigma_2\sigma_1$, $\beta=\sigma_3\sigma_2$, and $\eta=\sigma_1\sigma_3$. Since $\beta(A(\gamma))\cap A(\gamma)=\varnothing$, we must have that the radius of the σ_3 reflection circle is strictly bigger than x. By (II.7), the radius of the σ_3 reflection circle is $\sqrt{\lambda x}$. Hence,

$$\sqrt{\lambda x}>x .$$

We conclude that $\lambda>x$. □

The following two lemmas allow us to express the inequality $\lambda>x$ in hyperbolic terms.

<u>(II.9)</u> <u>lemma.</u> *consider the normalized geodesics* $[0,\infty]$ *and* $[1,x]$. x *uniquely determines the hyperbolic distance* d *between these two geodesics and conversely* d *uniquely determines the endpoint* x . *Furthermore* , $x=\coth^2(\frac{d}{2})$.

<u>PROOF.</u> Let $B\in PSL(2,R)$ be the Möbius transformation which takes $-\sqrt{x}$ to ∞ , $\sqrt{x}$ to 0 , and x to 1 (Refer to figure 8). That is ,

$$B(z)=\left(\frac{z-\sqrt{x}}{z+\sqrt{x}}\right)\left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right) . \qquad \text{Then}$$

$$d=\rho([0,\infty],[1,x])=\text{Log } B(\infty) \qquad\qquad (B(\infty)>1)$$

$$=\text{Log }\left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right)=\text{Log}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) .$$

Exponentiating on both sides we obtain ,

$$\text{(II.10)} \qquad\qquad e^{d}=\frac{\sqrt{x}+1}{\sqrt{x}-1}$$

The formula $x=\coth^2(\frac{d}{2})$ for $x>1$, follows by inverting equation (II.10).

Finally , notice that d is monotonically decreasing . Hence x uniquely determines d and conversely d uniquely determines x . □

Substituting for λ and x , in terms of their hyperbolic equivalents, we arrive at the following lemma whose proof is a straight forward computation .

<u>(II.11)</u> <u>lemma.</u> *If* $\lambda=e^{T(\beta)}$ *and* $x=\left(\frac{e^{d}+1}{e^{d}-1}\right)^2$, *then* $\lambda>x$ *if and only if*

$$d>Log\ \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}.$$

To simplify notation and writing we let $\rho(\sigma_i,\sigma_j)$ denote the hyperbolic distance between the σ_i and σ_j reflection circles . No confusion should arise .

<u>(II.12)lemma.</u> *Fix* $y>x$. *Let* σ_2 *be reflection in the common orthogonal to* $[0,\infty]$ *and* $[1,x]$, $x>1$. *Let* σ_1 *be reflection in some geodesic* $[a,b]$ *which is orthogonal to* $[1,x]$, *where* $x<b\leq y$. *Let* a' *be the endpoint of the geodesic orthogonal to* $[1,x]$ *with right endpoint* y . *Denote reflection in this geodesic by* σ_1' . *Then*

$$\rho(\sigma_2,\sigma_1)\geq\rho(\sigma_2,\sigma_1') \qquad (II.13)$$

with equality in (*II*.13) *if and only if* $\sigma_1=\sigma_1'$ (*refer to figure* 9). *Furthermore,*

$$\rho(\sigma_2,\sigma_1')= Log\ (\sqrt{x}\frac{y-1}{y-x}) \ .$$

<u>PROOF.</u> Note that the σ_1 reflection circle and the σ_1' reflection circle are either disjoint or identical . We conclude that $a'\leq a$ with equality if and only if $\sigma_1=\sigma_1'$. Since the distances $\rho(\sigma_2,\sigma_1)$ and $\rho(\sigma_2,\sigma_1')$ are measured along the geodesic [1,x] , we must have that $\rho(\sigma_2,\sigma_1)\geq\rho(\sigma_2,\sigma_1')$ with equality if and only if $a'=a$. Moreover, by the above remark $a'=a$ if and only if $\sigma_1=\sigma_1'$. This verifies inequality (II.13) .

Next we compute $\rho(\sigma_2,\sigma_1')$. Normalize by the Möbius transformation $B\in PSL(2,R)$ which takes 1 to 0 , x to ∞ , and $\sqrt{x}$ to 1 (refer to figure 10). That is ,

$$B(z)=\left(\frac{z-1}{z-x}\right)\left(\frac{\sqrt{x}-x}{\sqrt{x}-1}\right)$$

Therefore , $\rho(\sigma_2,\sigma_1') = \rho(B\sigma_2 B^{-1}, B\sigma_1' B^{-1}) = \text{Log}\,(-B(y))$

$$= \text{Log}\left(-\frac{y-1}{y-x}\right)\left(\frac{\sqrt{x}-x}{\sqrt{x}-1}\right)$$

$$= \text{Log}\left(\sqrt{x}\frac{y-1}{y-x}\right) \qquad \square$$

3. We are now prepared to prove (II.1) . We first show the necessity of inequalities (II.2) and (II.3) .

Suppose G has standard generators (γ,β) . By (II.4) , we can normalize G so that the axis of γ is $[1,x]$, $x>1$, the axis of β is $[0,\infty]$, and the axis of $\eta = \gamma^{-1}\beta^{-1}$ lies to the right of x (or the fixed point of η lies to the right of x if G is of type (0;1;2)) , with the orientations as indicated in figure 11 .

Now $\beta(z) = \lambda z$, $\lambda > 1$. (II.8) implies that $\lambda > x$. Moreover , by (II.9) and (II.11), $\lambda > x$ is equivalent to the inequality

$$d = \rho(A(\beta), A(\gamma)) > \text{Log}\ \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}.$$

Thus we have verified inequality (II.2) .

Next , we will verify inequality (II.3) . We know that G is equal to the orientation preserving subgroup of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ (refer to figure 12) . Let σ_1' be reflection in the geodesic orthogonal to $A(\gamma)$ with right endpoint $\sqrt{\lambda x}$. Notice that the σ_1 reflection circle cannot intersect the

σ_3 reflection circle , for otherwise the element $\sigma_3\sigma_1$ would be elliptic . (We saw in (II.7) that $\sqrt{\lambda x}$ is the radius of the σ_3 reflection circle .) Hence , the right endpoint of the σ_1 reflection circle must be less than or equal to $\sqrt{\lambda x}$. Applying (II.12) with $y=\sqrt{\lambda x}$ we have

$$\text{(II.14)} \qquad \rho(\sigma_2,\sigma_1) \geq \rho(\sigma_2,\sigma_1') = \text{Log}\left(\sqrt{x}\frac{\sqrt{\lambda x}-1}{\sqrt{\lambda x}-x}\right)$$

hence,

$$\text{(II.15)} \quad T(\gamma) = 2\rho(\sigma_2,\sigma_1) \geq 2\rho(\sigma_2,\sigma_1') = 2\text{Log}\left(\sqrt{x}\frac{\sqrt{\lambda x}-1}{\sqrt{\lambda x}-x}\right).$$

Thus , factoring $\sqrt{x}$ from the denominator , we have

$$\text{(II.16)} \qquad T(\gamma) \geq 2\text{Log}\left(\frac{\sqrt{\lambda x}-1}{\sqrt{\lambda}-\sqrt{x}}\right).$$

Substituting $\lambda = e^{T(\beta)}$ and $x=\left(\frac{e^d+1}{e^d-1}\right)^2$ in the right side of (II.16) we obtain the following inequality ,

$$T(\gamma) \geq 2\text{Log}\,\frac{e^{\frac{T(\beta)}{2}}\left(\frac{e^d+1}{e^d-1}\right)-1}{e^{\frac{T(\beta)}{2}}-\left(\frac{e^d+1}{e^d-1}\right)}$$

which simplifies to

$$\text{(II.17)} \qquad T(\gamma) \geq 2\text{Log}\,\frac{e^{\frac{T(\beta)}{2}}(e^d+1)-(e^d-1)}{e^{\frac{T(\beta)}{2}}(e^d-1)-(e^d+1)}.$$

Note that by (II.12) we have equality in (II.14) , and hence in (II.17) , if and only if $\sigma_1=\sigma_1'$, which occurs if and only if it is of type (0;1;2) . Thus we have verified inequality (II.3) .

We will now demonstrate the sufficiency of inequalities (II.2) and (II.3) . Suppose γ and β are two hyperbolic elements satisfying inequalities (II.2) and (II.3) oriented so that each of their axes lies to the right of the other . Without loss of generality , normalize by an element of PSL(2,R) , so that the axis of β is $[0, \infty]$ (0 the repelling fixed point and ∞ the attracting) and the axis of γ is $[1,x]$ ($x>1$, x the repelling fixed point and 1 the attracting) . Construct the common orthogonal to $[0,\infty]$ and $[1,x]$ and call reflection in this geodesic σ_2 . Let $\sigma_3(z)=\beta\sigma_2(z)$ and $\sigma_1(z)=\sigma_2\gamma(z)$. Now , inequality (II.2) is equivalent to $\lambda > x$ where $\beta(z)= \lambda z$, $\lambda>1$.

$$\text{Thus} \qquad \lambda > x^2$$

$$\text{and therefore} \qquad \sqrt{\lambda x} > x \quad .$$

But $\sqrt{\lambda x}$ is the radius of the σ_3 reflection circle . We conclude that the σ_3 reflection circle does not intersect $A(\gamma)$. (Refer to figure 13)

Next let σ_1' be the geodesic orthogonal to $[1,x]$ with $\sqrt{\lambda x}$ as one of its endpoints . Since equation (II.15) is equivalent to equation (II.17) , inequality (II.3) is equivalent to

$$T(\gamma) \geq 2\rho(\sigma_2,\sigma_1') .$$

Hence , since $\gamma=\sigma_2\sigma_1$, we have

$$2\rho(\sigma_2,\sigma_1)\geq 2\rho(\sigma_2,\sigma_1')$$

and therefore , $$\rho(\sigma_2,\sigma_1)\geq \rho(\sigma_2,\sigma_1') .$$

Moreover , since the σ_1 and σ_1' reflection circles are either disjoint or identical we must have that the σ_1 reflection circle has right endpoint b where $x< b \leq \sqrt{\lambda x}$ (figure 14).

The group $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ acts discontinuously on H. To see this, observe that the convex region bounded by the σ_i reflection circles and the boundary of H is, by Poincare's theorem, a fundamental polygon for

$\langle\sigma_1,\sigma_2,\sigma_3\rangle$.

Now consider the orientation preserving subgroup G of $\langle\sigma_1,\sigma_2,\sigma_3\rangle$. G is a finite index subgroup of a discontinuous group on H , hence G acts discontinuously on H . Furthermore G is a Fuchsian group of type (0;0;3) or (0;1;2) and has standard generators (γ,β) , where $\eta = \gamma^{-1}\beta^{-1}$ is either a boundary hyperbolic or parabolic element. □

4. As an immediate consequence of (II.1) we have the following corollary .

<u>(II.18)COROLLARY.</u> *Given any triple of positive numbers* $(T(\gamma),T(\beta),d)$ *satisfying inequalities (II.2)and (II.3) there exists a Fuchsian group of type* (0;0;3) ((0;1;2) *if equality in (II.3)*) *with standard generators* (γ,β) *such that the translation length of* γ *equals* $T(\gamma)$, *the translation length of* β *equals* $T(\beta)$, *and the distance between the axes* $A(\gamma)$ *and* $A(\beta)$ *is* d .

<u>PROOF.</u> Let β be the hyperbolic element with axis $[0,\infty]$ and translation length $T(\beta)$. Orient β so that its attracting fixed point is ∞ . Let γ be the hyperbolic element with axis $[1,x]$ and translation length $T(\gamma)$ (we saw in (II.9) that d determines x uniquely). Orient γ so that its attracting fixed point is 1 . Then (γ,β) satisfy all the hypotheses of (II.1) . Whence (γ,β) form standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) . □

5. Inequality (II.3) has an equivalent formulation in terms of the collar widths about γ , β , and the hyperbolic distance , d , between the axes of γ and β . Recall that the collar width of β , $c(\beta)$, is

$$\operatorname{Log} \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}.$$

<u>(II.19)LEMMA.</u> *Suppose $d > c(\beta)$. Then inequality (II.3) is equivalent to*

$$(II.20) \qquad c(\beta) + c(\gamma) \leq d$$

where equality holds in (II.3) if and only if equality holds in (II.20)

.

<u>PROOF.</u> We start with $c(\beta) + c(\gamma) \leq d$, that is

$$Log\, \frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1} + Log\, \frac{e^{\frac{T(\gamma)}{2}}+1}{e^{\frac{T(\gamma)}{2}}-1} \leq d \quad .$$

Exponentiating on both sides and clearing denominators we have

$$(e^{\frac{T(\beta)}{2}}+1)(e^{\frac{T(\gamma)}{2}}+1) \leq e^{d}(e^{\frac{T(\beta)}{2}}-1)(e^{\frac{T(\gamma)}{2}}-1) \quad .$$

After multiplying and gathering terms, we have

$$(II.21) \ [-e^{d}(e^{\frac{T(\beta)}{2}}-1)+(e^{\frac{T(\beta)}{2}}+1)]e^{\frac{T(\gamma)}{2}}+(e^{\frac{T(\beta)}{2}}+1) \leq -e^{d}(e^{\frac{T(\beta)}{2}}-1)$$

Note that $[-e^{d}(e^{\frac{T(\beta)}{2}}-1)+(e^{\frac{T(\beta)}{2}}+1)] < 0$, since $d > c(\beta)$.

Solving for $T(\gamma)$ in (II.21) we find that

$$T(\gamma) \geq 2\mathrm{Log}\, \frac{-e^{d}(e^{\frac{T(\beta)}{2}}-1)-(e^{\frac{T(\beta)}{2}}+1)}{-e^{d}(e^{\frac{T(\beta)}{2}}-1)+(e^{\frac{T(\beta)}{2}}+1)}$$

and hence , by some straight forward rearranging we have

$$T(\gamma) \geq 2\mathrm{Log}\, \frac{e^{\frac{T(\beta)}{2}}(e^{d}+1)-(e^{d}-1)}{e^{\frac{T(\beta)}{2}}(e^{d}-1)-(e^{d}+1)}$$, which is

precisely

equation (II.3) . Obviously these computations are reversible . □

We remark that equation (II.20) , $c(\beta) + c(\gamma) \leq d$, implies that $d > c(\beta)$ since $c(\gamma) > 0$. Hence we can restate (II.1) in the following geometric manner.

<u>(II.22)THEOREM.</u> *Let γ and β be two hyperbolic elements of PSL(2,R) oriented so that each of their axes lies to the right of the other . Then (γ,β) form standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) if and only if*

(II.23) $\qquad c(\beta) + c(\gamma) \leq d$, *where* $d = \rho(A(\gamma)A(\beta))$.

Equality holds in (II.23) if and only if the group $\langle\gamma,\beta\rangle$ is of type (0;1;2) .

6. Suppose (γ,β) are standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) . Then $\pi(A(\gamma))$ and $\pi(A(\beta))$ are boundary geodesics (simple closed geodesics which bound an annulus) on a sphere with topologically three holes . Since the metric on H/G is complete , the hyperbolic distance , d′ , between these closed geodesics is realized by the length of a geodesic segment . Furthermore , this geodesic segment is orthogonal to both $\pi(A(\gamma))$ and $\pi(A(\beta))$. To see this , note that the distance between two geodesics on H is realized as the length of the common orthogonal line segment between them and hence since orthogonality is preserved by the map $\pi\colon H\to H/G$, the quotient H/G also shares this property .

Finally , since the distance between closed geodesics is realized by a common orthogonal segment , we must have $d' = \inf\rho(A(\gamma'),A(\beta'))$, where the infimum runs over all $\gamma',\beta' \in G$ which are conjugate to γ and β, respectively .

The next proposition shows that we know exactly what d′ is .

<u>(II.24)PROPOSITION.</u> *If* (γ,β) *are standard generators for a Fuchsian group of type* (0;0;3) *or* (0;1;2) , *then* $d' = \rho(A(\gamma),A(\beta))$. *Where d′ is the distance from* $\pi(A(\gamma))$ *to* $\pi(A(\beta))$.

<u>PROOF.</u> By the remarks preceding the proposition it is enough to show that the distance $\rho(A(\gamma),A(\beta))$ is less than or equal to the distance $\rho(A(\gamma'),A(\beta'))$, for any γ', $\beta' \in G$ which are conjugate to γ and β , respectively . In fact , we need only consider the distances $\rho(A(\gamma'),A(\beta))$

, where $\gamma' \varepsilon$ G runs through all the conjugates of γ .

Normalize (γ,β) as in figure 6 (or 7) . Define the interior of σ_i, denoted $\mathrm{int}(\sigma_i)$, to be the connected component of H — {the σ_i reflection circle } which does not contain the other two reflection circles . The exterior of σ_i , denoted $\mathrm{ext}(\sigma_i)$, is defined to be the other component .

(II.25)OBSERVATION $\mathrm{int}(\sigma_j) \subset \mathrm{ext}(\sigma_i)$ for $j \neq i$.

(II.26)OBSERVATION If S is any set in H so that $S \subset \mathrm{int}(\sigma_j)$, then by (II.25) $\sigma_i S \subset \mathrm{int}(\sigma_i)$ for $i \neq j$.

(II.27)OBSERVATION Suppose $\tau \varepsilon \langle\sigma_1,\sigma_2,\sigma_3\rangle$ so that $\tau = \sigma_{i_n} \dots \sigma_{i_1}$ and no two successive elements in the product are the same . Let $S \subset \mathrm{int}(\sigma_i)$ where $i \neq i_1$. Then $\tau S \subset \mathrm{int}(\sigma_{i_n})$ by (II.26) .

Using (II.27) we prove the following lemma whose proof is due to Bernard Maskit (personal communication) .The proposition is now an easy consequence of this lemma.

(II.28)LEMMA. *If* $\tau \varepsilon \langle\sigma_1,\sigma_2,\sigma_3\rangle$. *Then*

$$\rho(\tau A(\gamma),A(\beta)) \geq \rho(A(\gamma),A(\beta)) \quad (II.29)$$

PROOF. Since σ_1 and σ_2 fix the axis $A(\gamma)$ we can without loss of generality assume that τ begins with σ_3 . If the word τ has length one then (II.29) holds since $\sigma_3 A(\beta) = A(\beta)$ and hence ,

$$\rho(\sigma_3 A(\gamma),A(\beta)) = \rho(A(\gamma),\sigma_3 A(\beta)) = \rho(A(\gamma),A(\beta)) \quad .$$

We will induct on the length of τ . We assume that (II.29) holds for all words τ of length less than or equal to n . Let $\tau = \sigma_{i_n} \dots \sigma_3$, where no two successive elements of the product are the same . We will show that inequality (II.29) holds if we replace τ by $\sigma_i \tau$, for any $i=1,2,$or 3.

Now , if σ_i is either σ_2 or σ_3 then since $\sigma_i A(\beta) = A(\beta)$ we have

$$\rho(\sigma_i \tau A(\gamma), A(\beta)) = \rho(\sigma_i \tau A(\gamma), \sigma_i A(\beta)) = \rho(\tau A(\gamma), A(\beta)) \ .$$

The last equality follows from the fact that σ_i is an isometry of H. Since by assumption , $\rho(\tau A(\gamma), A(\beta)) \geq \rho(A(\gamma), A(\beta))$ we can conclude that $\rho(\sigma_i \tau A(\gamma), A(\beta)) \geq \rho(A(\gamma), A(\beta))$ where σ_i is either σ_2 or σ_3 . Thus we need only check the case where $\sigma_i = \sigma_1$.

If σ_{i_n} is σ_1 then $\sigma_1 \tau = \sigma_1 \sigma_1 \sigma_{i_{n-1}} \ldots \sigma_3 = \sigma_{i_{n-1}} \ldots \sigma_3$ which is a word of length $n-1$ and hence by assumption satisfies (II.29) .

If σ_{i_n} is either σ_2 or σ_3 then $\tau A(\gamma)$ is contained in either $int(\sigma_2)$ or $int(\sigma_3)$ .To see this , set $S = \sigma_3 A(\gamma)$ and apply (II.27) .

Draw the Euclidean line segment , T , passing through the center of the σ_1 reflection circle and tangent to $\tau A(\gamma)$ (figure 15). Notice that T lies below the line passing through 0 and tangent to $\tau A(\gamma)$ (this line measures $\rho(\tau A(\gamma), A(\beta))$) and furthermore T is fixed by the reflection σ_1 . Hence , since $\tau A(\gamma)$ is contained in either $int(\sigma_2)$ or $int(\sigma_3)$, we have that $\sigma_1 \tau A(\gamma) \subset int(\sigma_1)$ by (II.26) .Moreover , $\sigma_1 \tau A(\gamma)$ lies below T . Therefore , since T lies below the line through the origin which is tangent to $\tau A(\gamma)$ (figure 16) , we have

$$\rho(\sigma_1 \tau A(\gamma), A(\beta)) \geq \rho(\tau A(\gamma), A(\beta)) \geq \rho(A(\gamma), A(\beta)) \ .$$

The last inequality follows by assumption ; this completes the induction step and the proof of the lemma . □

7. Let g be a hyperbolic element in a Fuchsian group G . Define the <u>collar</u> <u>region</u> <u>for</u> <u>g</u> to be

$$\mathcal{N}_0(g) = \{ z \in H : \rho(z, A(g)) < c(g) \} \qquad \text{(figure 17)}$$

Throughout this section , (γ,β) are standard generators for a normalized Fuchsian group of type (0;0;3) or (0;1;2) . σ_1 , σ_2 , and σ_3 play their usual roles as they did in figure 6 .

Recall that the geometric condition in the collar theorem is the inequality $c(\beta) + c(\gamma) \leq d$. This simply says that the collar regions of γ and β do not intersect , that is $\mathcal{N}_0(\beta) + \mathcal{N}_0(\gamma) = \varnothing$. (figure 18)

A <u>collar</u> about a simple closed geodesic ω on a surface S is a subsurface which contains ω and is topologically an annulus . We will show that $\mathcal{N}_0(\gamma)$ and $\mathcal{N}_0(\beta)$ project to disjoint collars on $H/\langle\gamma,\beta\rangle$.

<u>(II.29)LEMMA.</u> $\mathcal{N}_0(\beta) \cap \{ \sigma_1$ *reflection circle* $\} = \varnothing$

and $\mathcal{N}_0(\gamma) \cap \{ \sigma_3$ *reflection circle* $\} = \varnothing$.

<u>PROOF.</u> We will show that $\rho(A(\beta),\sigma_1) > c(\beta)$. Recall (II.7) that the right endpoint of the σ_1 reflection circle is less than or equal to $\sqrt{\lambda x}$; where $\beta(z) = \lambda z$, $\lambda > 1$, and x is the right endpoint of the axis $A(\gamma)$ .Also recall (II.6) that $\sqrt{x}$ is the right endpoint of the σ_2 reflection circle .

Construct the geodesic , L , with left endpoint $\sqrt{x}$ and right endpoint $\sqrt{\lambda x}$ (figure 19) and observe that $\rho(A(\beta),\sigma_1) \geq \rho(A(\beta),L)$. We wish to compute the distance $\rho(A(\beta),L)$. Let $f \in PSL(2,R)$ be the Möbius transformation which fixes 0 and ∞ and takes $\sqrt{x}$ to 1 . Hence f(z)

$=\frac{1}{\sqrt{x}}z$ and thus $f(\sqrt{\lambda x}\ )=\sqrt{\lambda}$ (figure 20). Therefore , we have

$$\rho(A(\beta),L)=\rho(fA(\beta),fL) = \rho(A(\beta),fL)$$

which by (II.10) is ,

$$= \text{Log}\ \frac{\lambda^{\frac{1}{4}}+1}{\lambda^{\frac{1}{4}}-1} > \text{Log}\ \frac{\lambda^{\frac{1}{2}}+1}{\lambda^{\frac{1}{2}}-1} = c(\beta)\ .$$

The inequality follows from the fact that $\text{Log}\ \frac{s+1}{s-1}$ is a strictly decreasing function for $s>1$.

Whence we have ,

$$\rho(A(\beta),\sigma_1)\geq\rho(A(\beta),L) > c(\beta)\ .$$

Thus , $\mathcal{R}(\beta) \cap \{\ \sigma_1 \text{ reflection circle }\} = \varnothing$.

Similarly , $\mathcal{R}(\gamma) \cap \{\ \sigma_3 \text{ reflection circle }\} = \varnothing$. □

<u>(II.30)LEMMA.</u> *Suppose* $\tau\,\varepsilon\langle\sigma_1,\sigma_2,\sigma_3\rangle$ *and at least one of the factors of* τ *is* σ_1 *.Then* $\tau\mathcal{R}(\beta) \cap \mathcal{R}(\beta)=\varnothing$.

<u>PROOF.</u> τ can be written as a product $\sigma_{i_n}...\sigma_{i_1}$ and since σ_2 and σ_3 fix $\mathcal{R}(\beta)$, we can assume without loss of generality that $\sigma_{i_1} = \sigma_1$. Now , $\sigma_1(\mathcal{R}(\beta)) \subset \text{int}\ (\sigma_1)$ and hence by the previous lemma $\sigma_1\mathcal{R}(\beta) \cap \mathcal{R}(\beta)= \varnothing$.

We induct on the length of the word τ. Suppose $\tau\mathcal{R}(\beta) \subset \text{int}(\sigma_i)$ and $\tau\mathcal{R}(\beta)\cap\mathcal{R}(\beta)=\varnothing$. Consider $\sigma_j\tau$ where $j\neq i$ (the word τ ends with σ_i and hence if $j=i$, $\sigma_j\tau$ would be a word of length $n-1$).

If σ_j is either σ_2 or σ_3 then clearly

$$\sigma_j\tau\mathcal{R}(\beta) \cap \mathcal{R}(\beta)= \sigma_j\tau\mathcal{R}(\beta) \cap \sigma_j\mathcal{R}(\beta) = \sigma_j(\tau\mathcal{R}(\beta) \cap \mathcal{R}(\beta))$$

which by assumption , $=\sigma_j(\emptyset)$

and hence , $=\emptyset$

Furthermore , $\sigma_j \tau \mathcal{R}_0(\beta) \subset \mathrm{int}(\sigma_j)$, since $\tau \mathcal{R}_0(\beta) \subset \mathrm{ext}(\sigma_j)$.

Thus the only case to consider is $\sigma_j = \sigma_1$. Draw the Euclidean line segment, T, passing through the center of the σ_1 reflection circle and tangent to $\tau \mathcal{R}_0(\beta)$. Note that T lies below the collar line $\{z \varepsilon H : \rho(A(\beta),z) = c(\beta)\}$ (figure 21). Thus $\sigma_1 \tau \mathcal{R}_0(\beta) \subset \mathrm{int}(\sigma_1)$ and since T is invariant under σ_1 , we have that $\sigma_1 \tau \mathcal{R}_0(\beta) \cap \mathcal{R}_0(\beta) = \emptyset$. This completes the induction step . □

(II.31)PROPOSITION. $\mathcal{R}_0(\beta)$ *is precisely invariant under* β *in* $\langle \gamma,\beta \rangle$.

PROOF. Clearly $\beta^n \mathcal{R}_0(\beta) = \mathcal{R}_0(\beta)$ for all $n \varepsilon Z$. Suppose $g \varepsilon \langle \gamma,\beta \rangle$ and $g \notin \langle \beta \rangle$. Then g can be written as a product of the σ_i where one of the σ_i is σ_1 (otherwise an even product of σ_2's and σ_3's is just an element of $\langle \beta \rangle$) . Then by the previous lemma,

$$g \mathcal{R}_0(\beta) \cap \mathcal{R}_0(\beta) = \emptyset \quad .$$

(II.32) REMARK. Of course the same analysis could have been performed on γ to conclude that $\mathcal{R}_0(\gamma)$ is precisely invariant under $\langle \gamma \rangle$ in $\langle \gamma,\beta \rangle$.

The above proposition tells us that $\mathcal{R}_0(\beta)$ projects to a collar about $\pi(A(\beta))$ (similarly for $\mathcal{R}_0(\gamma)$). (figure 22)

In fact , the collars about $\pi(A(\gamma))$ and $\pi(A(\beta))$ are disjoint . To see this , we need to show that for any $g \in \langle \gamma,\beta \rangle$, $g \mathcal{R}_0(\beta) \cap \mathcal{R}_0(\gamma) = \emptyset$ where $\langle \gamma,\beta \rangle$ are standard generators for a Fuchsian group of type (0;0;3) or (0;1;2) .

Once again we appeal to the reflections σ_1 , σ_2 , and σ_3 . Note that σ_1 and σ_2 take $\mathcal{R}_0(\gamma)$ to itself and similarly σ_2 and σ_3 take $\mathcal{R}_0(\beta)$ to itself . Since $\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing$ and $\sigma_2\mathcal{R}_0(\beta)=\sigma_3\mathcal{R}_0(\beta)=\mathcal{R}_0(\beta)$, clearly we have

$$\sigma_2\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing$$

and

$$\sigma_3\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing \ .$$

Furthermore , since σ_1 leaves $\mathcal{R}_0(\gamma)$ fixed , we have

$$\sigma_1\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing \ .$$

We induct on the length of $\tau \in \langle\sigma_1,\sigma_2,\sigma_3\rangle$. Suppose $\tau\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing$ where τ has length less than or equal to n . Consider $\sigma_j\tau$. If τ does not contain σ_1 as a factor then $\sigma_j\tau\mathcal{R}_0(\beta)=\sigma_j\mathcal{R}_0(\beta)$ and hence ,

$$\sigma_j\tau\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\sigma_j\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing.$$

If τ does contain σ_1 as a factor , then without loss of generality , since σ_2 and σ_3 leave $\mathcal{R}_0(\beta)$ fixed , the word τ begins with σ_1 . Hence ,

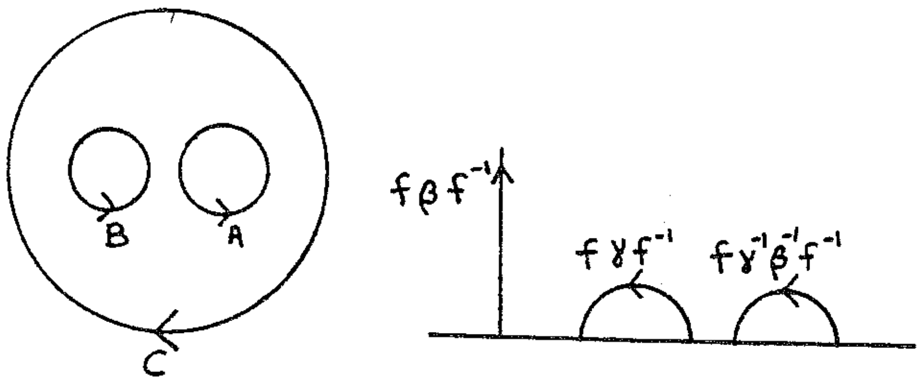
$$\tau\mathcal{R}_0(\beta)=\sigma_{i_n}\ldots\sigma_1\mathcal{R}_0(\beta)\subset\sigma_{i_n}\ldots\sigma_{i_2}\mathrm{int}(\sigma_1)\subset\mathrm{int}(\sigma_{i_n}) \ .$$

The last inclusion comes from (II.27) .

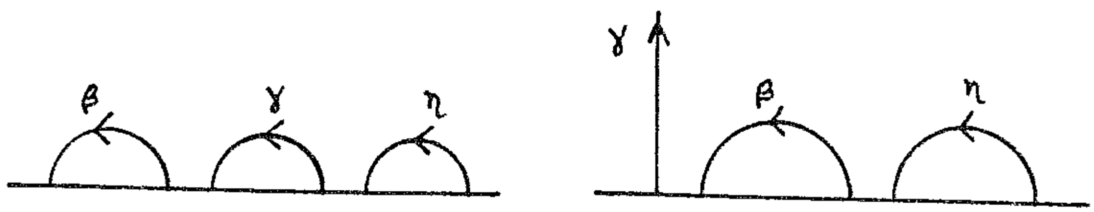
Now , if σ_j is σ_1 or σ_2 then since σ_1 and σ_2 leave $\mathcal{R}_0(\gamma)$ invariant clearly , $\sigma_j\tau\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing$. So we assume that $\sigma_j=\sigma_3$. By (II.29) the intersection of $\mathcal{R}_0(\gamma)$ and $\mathrm{int}(\sigma_3)$ is empty . Thus since $\sigma_3\tau\mathcal{R}_0(\beta)\subset\mathrm{int}(\sigma_3)$ we conclude that $\sigma_3\tau\mathcal{R}_0(\beta)\cap\mathcal{R}_0(\gamma)=\varnothing$. This competes the induction step. We have proven the following theorem .

<u>(II.23)THEOREM.</u> *Suppose (γ,β) are standard generators for a*

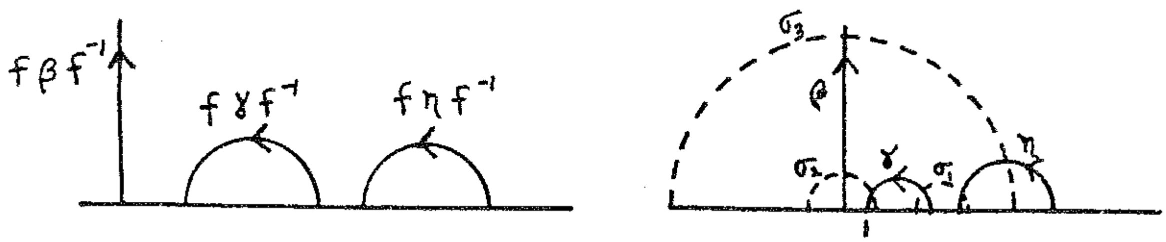
Fuchsian group G of type (0;0;3) *or* (0;1;2) . *Then* $\mathfrak{R}_{(\beta)}$ *and* $\mathfrak{R}_{(\gamma)}$ *project to disjoint collars about* $\pi(A_{(\beta)})$ *and* $\pi(A_{(\gamma)})$, *respectively* .



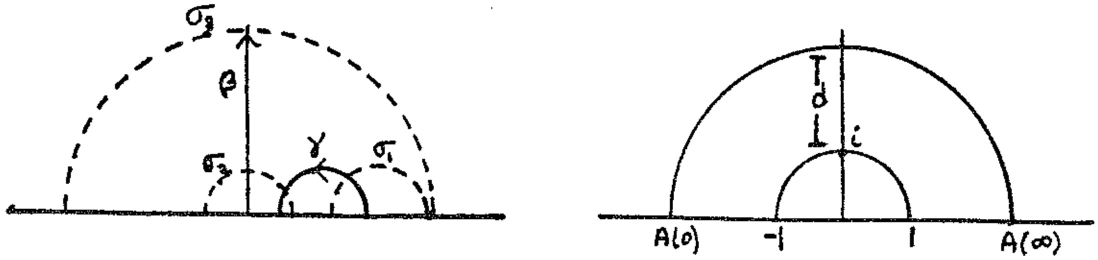
Figures 1 and 2.



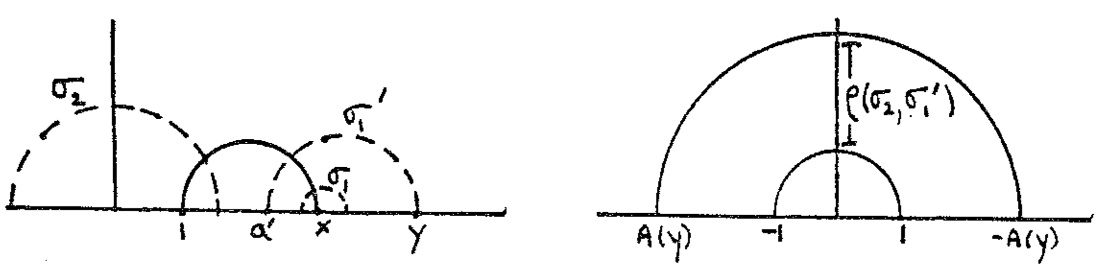
Figures 3 and 4.



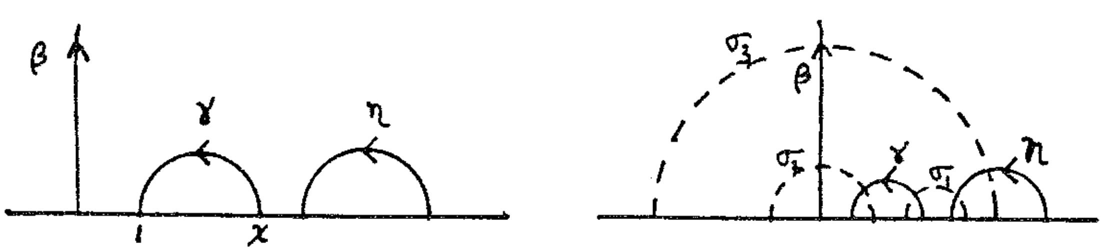
Figures 5 and 6.



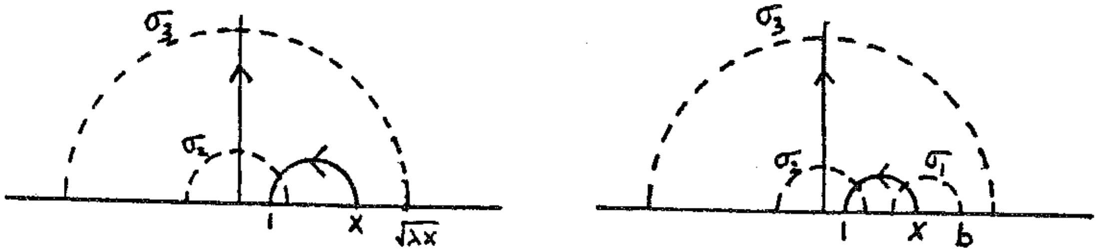
Figures 7 and 8.



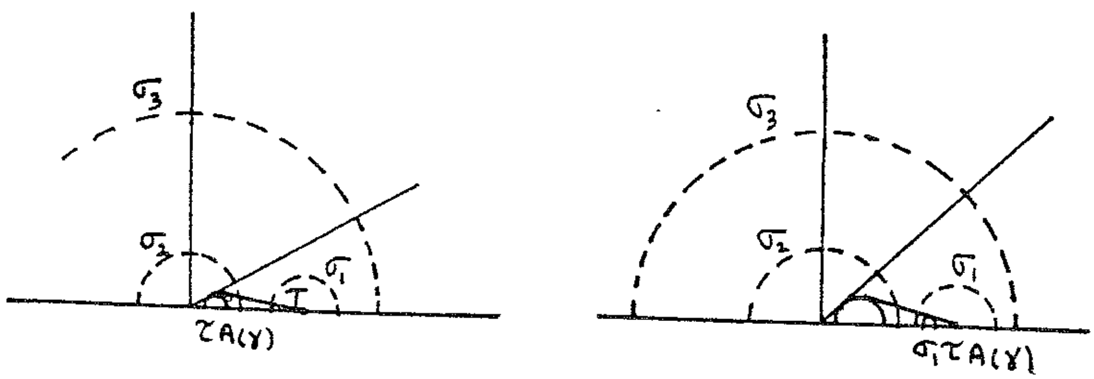
Figures 9 and 10.



Figures 11 and 12.

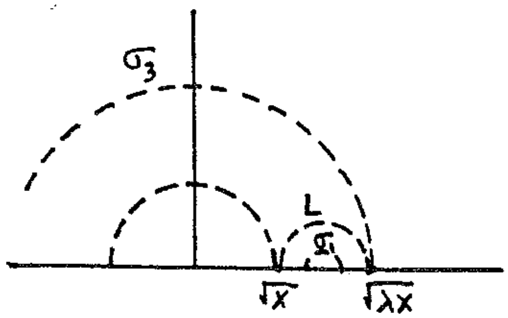


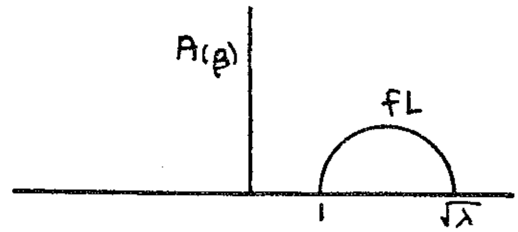
Figures 13 and 14.



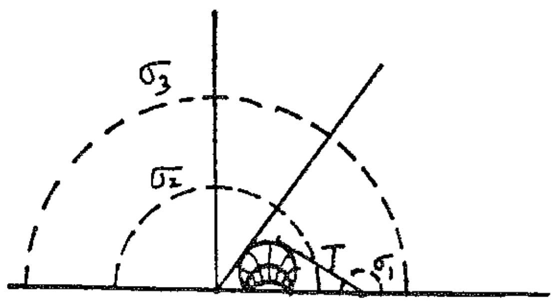
Figures 15 and 16.

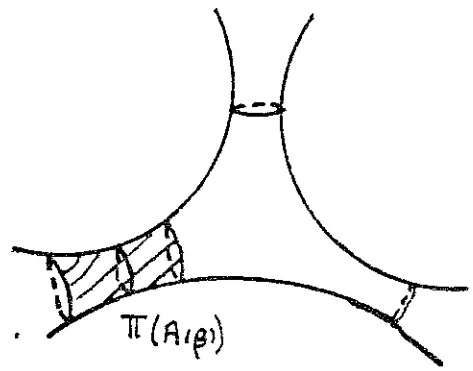
Figures 17 and 18.





Figures 19 and 20.





Figures 21 and 22.

CHAPTER III

0. In this section we develop a theory of nested sequences of geodesics .

1. Let $L_i = [x_i , y_i]$, for $i= 1,2,3...$, be a geodesic in H . We say that the sequence $\{L_i\}$ is a nested sequence of geodesics if L_{i-1} and L_{i+1} lie in different components of $H - L_i$, for $i = 2, 3, 4, ...$ and if the L_i are disjoint in $\overline{H}$. (figure 23) A nested sequence of geodesics converges to the geodesic $L= [x,y]$ $(x \neq y)$, denoted $L_i \to L$, if $x_i \to x$ and $y_i \to y$ in $\hat{R}$. If $x_i \to x$ and $y_i \to x$, we say that the nested sequence of geodesics converges to the point $x \in \hat{R}$ (or converges to the boundary of the hyperbolic plane). Clearly , the limit of a nested sequence is unique . The following proposition demonstrates that these are the only possibilities for convergence .

(III.1)PROPOSITION. *A nested sequence of geodesics either converges to a geodesic or to a point of the hyperbolic plane .*

PROOF. Let $\{L_i\}$ be a nested sequence of geodesics . Without loss of generality , by normalizing with an element of PSL(2,R) , we can assume that $L_1=[-1,1]$ and $L_i= [x_i,y_i]$, where

$$-1 < x_i < x_{i+1} < y_{i+1} < y_i < 1 \text{ for } i=2,3,4... .$$

(See figure 24.)

We consider the closed intervals in R of the form $I_i=[x_i,y_i]$ and we note that they are nested , that is , $I_1 \supset I_2 \supset I_3 \supset \ldots$. These closed intervals are of course in one to one correspondence with the geodesics $[x_i,y_i]$ and hence it follows that their behavior completely determines the behavior of the nested sequence $\{[x_i,y_i]\}$.

Now , the intersection $\cap I_i$ is either an interval , say $[x,y]$, or a point $x=y$. Therefore $x_i \to x$ and $y_i \to y$ (where x may equal y). Thus either $\{L_i\}$ converges to a geodesic or to a point . □

<u>(III.2)PROPOSITION.</u> *Suppose $\{L_i\}$ is a nested sequence of geodesics which converge to the geodesic L . Then*

$$\lim_{i \to \infty} \rho(L_i,L)=0 \ .$$

<u>PROOF.</u> Without loss of generality , suppose the $\{L_i\}$ are normalized as in the proof of proposition (III.1). Let $L=[x,y]$, $x<y$.

First we compute the distance between the geodesics L_i and L . Let B be the element of PSL(2,R) which takes y_i to ∞ , x to 1 , and x_i to 0 ; that is

$$B(z)= \frac{z-x_i}{z-y_i}\,\frac{x-y_i}{x-x_i} \ .$$

Then we have $\rho(L_i,L)=\rho(B(L_i),B(L))=\rho([0,\infty],[1,B(y)])$, (note :$B(y)>1$) and hence by substituting B(y) for x in formula (II.10) this equals ,

$$\text{(III.3)} \qquad =\text{Log}\ \frac{\sqrt{B(y)}+1}{\sqrt{B(y)}-1} \ .$$

Now , since

(III.4) $$B(y)=\frac{y-x_i}{y-y_i}\,\frac{x-y_i}{x-x_i}$$

we have that $B(y)\to\infty$ as $x_i\to x$ and $y_i\to y$, $(x\neq y)$ .Thus $\mathrm{Log}\,\frac{\sqrt{B(y)}+1}{\sqrt{B(y)}-1}$ approaches zero as $x_i\to x$ and $y_i\to y$ .We conclude that the distance $\rho(L_i,L)$ approaches zero as $i\to\infty$. □

The converse to (III.2) is not true . Take any nested sequence of geodesics $\{L_i\}=\{[x_i,y_i]\}$,where $-1<x_i<x_{i+1}<y_{i+1}<y_i<1$, which converge to a geodesic $[x,y]$. Construct the geodesic $L'=[x,\frac{x+y}{2}]$ (figure 25). Since the x_i converge to x , equations (III.3) and (III.4) imply that the distance

$$\rho(L_i,L')=\rho([x_i,y_i],[x,\tfrac{x+y}{2}])$$

goes to zero .

On the other hand , the y_i do not converge to $\frac{x+y}{2}$, hence the nested sequence $\{L_i\}$ does not converge to L' .

<u>(III.5)PROPOSITION.</u> *Let the sequence $\{L_i\}$ be a nested sequence of geodesics. Then the L_i converge to a geodesic if and only if*

$$\lim_{i\to\infty}\rho(L_1,L_i)<\infty\quad .$$

<u>PROOF.</u> Suppose L_i converges to a geodesic L . Since the L_i are nested we have

(III.6) $$\rho(L_1,L_i)\leq\rho(L_1,L)\quad\text{for all } i=1,2,3,\ \dots\ .$$

Taking the limit in (III.6) as $i\to\infty$ and noting that the distances $\rho(L_1,L_i)$ increase with i , we conclude that the limit,

$$\lim_{i\to\infty}\rho(L_1,L_i)<\infty\quad\text{exists .}$$

Next , suppose that the sequence $\{L_i\}$ coverges to a point , say x .

Without loss of generality , we normalize as in the proof of (III.1) , so that $L_i = [x_i, y_i]$, where

$$-1 < x_i < x_{i+1} < y_{i+1} < y_i < 1 \text{ for } i=2,3,4\ldots .$$

Then
$$\rho(L_1,L_i)=\rho([x_1,y_1],[x_i,y_i])=\mathrm{Log}\ \frac{\sqrt{B(y_1)}+1}{\sqrt{B(y_1)}-1}$$

where ,
$$B(y_1)= \frac{y_1-x_i}{y_1-y_i}\ \frac{x_1-y_i}{x_1-x_i} .$$

Now observe that $B(y_1)\to 1$ as $x_i \to x$ and $y_i \to x$. Thus ,

$$\lim_{i\to\infty} \rho(L_1,L_i)\to\infty \ . \ \square$$

Suppose $\{L_i\}$ is a nested sequence of geodesics . We introduce an orientation for each L_i by designating L_{i+1} to lie to the right of L_i . Let Σ_i be the unique common orthogonal to the geodesics L_i and L_{i+1} . The distance from Σ_i to Σ_{i+1} is measured by traversing L_i , hence the following definition makes sense .

$$\text{(III.8)}\quad s_i = \begin{cases} \rho(\Sigma_i,\Sigma_{i+1}) \text{ , if } L_i \text{ is traversed in the positive direction .} \\ -\rho(\Sigma_i,\Sigma_{i+1}) \text{ , if } L_i \text{ is traversed in the negative direction .} \end{cases}$$

We call s_i the <u>slide</u> <u>(twist)</u> <u>parameter</u> for L_{i+1} .

We would like to express nested sequences of geodesics in hyperbolic terms . The next proposition allows us to do exactly this . Set $d_i = \rho(L_i,L_{i+1})$.

<u>(III.9)PROPOSITION.</u> *The sequence of distances* $\{d_i\}$ *together with the sequence of slide parameters* $\{s_i\}$ *, denoted* $(\{d_i\},\{s_i\})$ *, uniquely*

determine (up to an element of PSL(2,R)) a nested sequence of geodesics .

<u>PROOF.</u> Suppose $\{L_i\}$ is a nested sequence of geodesics . Normalize by an element of PSL(2,R) so that $L_1 = [-1,1]$ and the unique common orthogonal between L_1 and L_2 is $[0,\infty]$. The orientations induced on $\{L_i\}$ are as in figure 26 . Clearly L_2 is determined as soon as d_1 is specified . Next , L_3 is determined once we specify s_1 and d_2 . In general , L_n is determined once $\{d_i\}_{i=1}^{n-1}$ and $\{s_i\}_{i=1}^{n-2}$ are chosen . □

In the sequel , we will need to identify when a nested sequence of geodesics converges to a geodesic or to a point . The following propositions ((III.10) and (III.12)) supply us with some tools .

<u>(III.10)PROPOSITION.</u> *Suppose $\{L_i\}$ is a nested sequence of geodesics which converge to a geodesic . Then*

$$\sum_{1}^{\infty} \rho(L_i,L_{i+1})<\infty \quad .$$

<u>PROOF.</u> First we claim that the following inequality holds ,

$$\sum_{1}^{m-1} \rho(L_i,L_{i+1}) \leq \rho(L_1,L_m) \ . \qquad \text{(III.11)}$$

Once we have verified (III.11) the theorem follows easily by letting m go to infinity and noting that the limit ,

$$\lim_{m\to\infty} \rho(L_1,L_m) \text{ exists}$$

since the L_i converge to a geodesic .

To verify (III.11), we observe that the common orthogonal, Σ, to L_1 and L_m passes through each of the geodesics L_2, L_3, ... L_{m-1}. Set $z_i=\Sigma \cap L_i$. Then,

$$\rho(L_1,L_m)=\rho(z_1,z_2)+\rho(z_2,z_3)+\dots+\rho(z_{m-1},z_m),$$

but the distance $\rho(L_i,L_{i+1}) \leq \rho(z_i,z_{i+1})$. This verifies (III.11). □

<u>(III.12)PROPOSITION.</u> *Suppose $\{L_i\}$ is a nested sequence of geodesics such that $\sum d_i < \infty$ and $\sum |s_i| < \infty$. Then the nested sequence $\{L_i\}$ converges to a geodesic.*

<u>PROOF.</u> Let Σ_i be the common orthogonal to the geodesics L_i and L_{i+1} and set $z_i = \Sigma_i \cap L_i$ (refer to figure 27). Then we have,

$$\rho(L_1,L_m) \leq \rho(z_1,z_m)$$

and hence since the geodesic joining z_1 and z_m is shorter than any curve joining z_1 and z_m we have,

(III.13) $\qquad \leq d_1+|s_1|+d_2+|s_2|+\dots+d_m+|s_m| = \sum_1^m d_i + \sum_1^m |s_i|$

Letting $m \to \infty$ in (III.13) we find that $\lim_{m\to\infty}\rho(L_1,L_m) < \infty$. Therefore the L_i converge to a geodesic. □

Since the distance between two geodesics is a continuous function (formula (III.3)) of the endpoints of the geodesics, we remark that given any two geodesics M and N we can construct a third geodesic T which separates M and N such that the distances $\rho(M,T)$ and $\rho(N,T)$ are arbitrarily small. In fact, the absolute value of the slide parameter (the distance from the common orthogonal of M and T to the common orthogonal of T and N) can be made arbitrarily large (refer to figure

28). Using this fact we next construct an example to show that the conclusion in proposition (III.10) is not sufficient to guarantee convergence of a nested sequence to a geodesic .

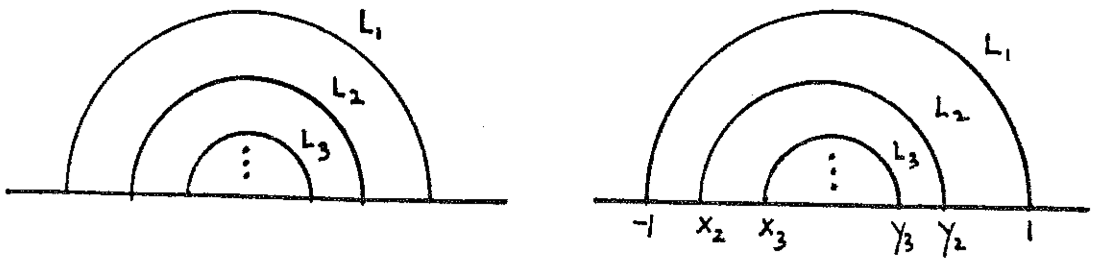
Consider a nested sequence of geodesics $\{L_i\}$ all having the same common orthogonal and converging to a point . Pair off all the geodesics of the sequence $\{L_i\}$ so that L_1 is paired with L_2 , L_3 is paired with L_4 , and in general L_{2n-1} is paired with L_{2n} . For the n^{th} pair in the sequence construct , as indicated above , a geodesic T which comes within a distance of $1/n^2$ to M and a distance of $\frac{1}{(n+1)^2}$ to N . Append all of these geodesics to the sequence $\{L_i\}$ in their appropriate places . Call the new sequence $\{L_i\}$ and note that $\rho(L_n,L_{n+1}) < \frac{1}{n^2}$. Hence ,

$$\sum \rho(L_n,L_{n+1}) < \infty \ .$$

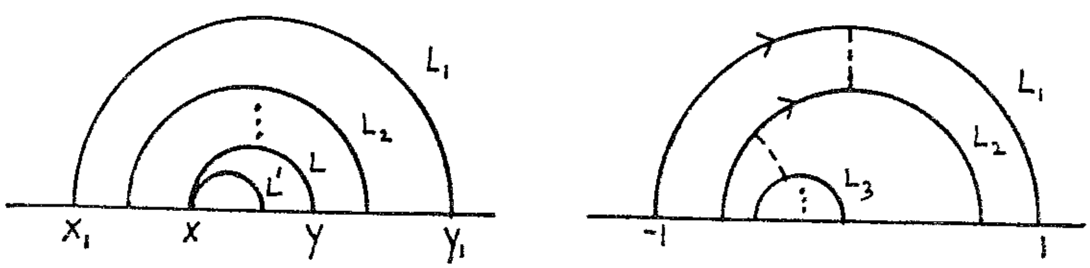
On the other hand , since the appended sequence $\{L_i\}$ has a subsequence which converges to a point , we conclude that the appended sequence converges to a point . Thus the conclusion in (III.10) is not sufficient to guarantee convergence to a geodesic .

Our next example illustrates the fact that the converse to (III.12) is not true . That is , we construct a nested sequence which converges to a geodesic and yet has $\sum |s_i| = \infty$.

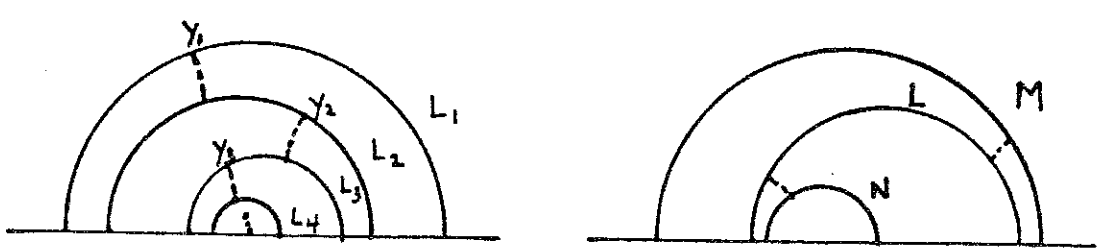
We start with a nested sequence of geodesics $\{L_i\}$ all having the same common orthogonal and converging to a geodesic . Pair off the geodesics as we did in the previous example . For each pair M and N construct the geodesic T which separates M and N so that the distance from the common orthogonal of M and T to the common orthogonal of T and N is bounded from below by some positive constant which does not depend on the pair . Once again , append all of these geodesics to the sequence and call the new sequence also $\{L_i\}$. We note that the appended sequence $\{L_i\}$ converges to a geodesic and yet $\sum |s_i|=\infty$. Thus we have constructed an example with the desired properties .



Figures 23 and 24.



Figures 25 and 26.



Figures 27 and 28.

CHAPTER IV

0. Our central aim in this chapter is to construct infinitely generated Fuchsian groups of the first kind by using the combination theorem an infinite number of times. Thus the two natural questions that this chapter is concerned with are; How do we construct infinitely generated Fuchsian groups using the combination theorem and when are these groups of the first kind?

1. The main purpose of this section is to prove the following theorem .

(IV.1)THEOREM. *Let* $\{G_i\}_{i=1}^{\infty}$ *be Fuchsian groups with* g_i *and* g_{i+1} *non-conjugate primitive boundary hyperbolics in* G_i *so that*

$$N(G_i) \cap N(G_{i+1}) = A(g_{i+1}) \text{ , for } i=1,2,3, \dots \text{ .}$$

Let $G = \langle G_i \rangle_{i=1}^{\infty}$ *. Then*

(*i*) *G is a Fuchsian group .*

(*ii*) g_i *is simple in G , for* $i=1,2,3, \dots$.

(*iii*) *H/G is an infinite sequence of surfaces each glued to the next along the simple closed dividing geodesics* $\{\pi A(g_i)\}$.

We will prove (IV.1) part (i) , by showing that the Nielsen region $N(G_1)$ is precisely invariant under G_1 in G . Throughout this section G and G_i are as in the hypotheses to (IV.1) . These hypotheses imply among other things , that the axes of the $\{g_i\}$ form a nested sequence of geodesics (figure 29) .

We next define some terms and set up some notation . Let I be an interval of discontinuity for a Fuchsian group Γ . Draw the geodesic L whose endpoints are the endpoints of I . The half-space bounded by L and I in $\overline{H}$ is said to be a boundary half-space for Γ . The geodesic L is called a boundary geodesic for Γ [M V.G].

let H be the boundary half-space of g_i in G_i and H′ the boundary half-space of g_{i+1} in G_i . Define R_i to be the complement of the set

$$N(G_i) \cup H \cup H' \quad .$$

Thus R_i is a disjoint union of boundary half-spaces of G_i (figure 30) . Note that R_i has infinitely many connected components . A component of R_i is called an R_i - region . We set $R = \cup R_i$. The following lemmas (IV.2 - IV.4) describe how different elements of G move the Nielsen region $N(G_1)$ around with respect to the R_i - regions .

(IV.2)LEMMA. *Fix a positive integer i which is not equal to one . Let $h \in G_i - G_1$ and $h \neq 1$. Then $h(N(G_1))$ is contained in an R_{i-1} - region if $h \in \langle g_i \rangle$, otherwise $h(N(G_1))$ is contained in an R_i - region.*

PROOF. If $h \in \langle g_i \rangle$,then h is also a member of G_{i-1} . Since g_i and g_{i-1} are non-conjugate boundary hyperbolics of G_{i-1} , we conclude that h takes the boundary half-space of g_{i-1} (in G_{i-1}) into R_{i-1} . Since $N(G_1)$ is contained in the boundary half-space of g_{i-1} , we have $h(N(G_1)) \subset R_{i-1}$ (figure 31).

Next , suppose $h \notin \langle g_i \rangle$. Then h takes the boundary half-space H of g_i (in G_i) into R_i , and hence (see figure 32)

$$h(N(G_1)) \subset hH \subset R_i \quad .\square$$

(IV.3)LEMMA. *Fix a positive integer i . Let $h \in G_k$, $k>i$, and $h \neq 1$. Then $h(R_i)$ is contained in an R_{k-1} -region , if $h \in \langle g_k \rangle$ (figure 33). Otherwise , $h(R_i)$ is contained in an R_k -region (figure 34).*

PROOF. Suppose $h \in \langle g_k \rangle$. Then h is also a member of G_{k-1} . Since g_k and g_{k-1} are non-conjugate boundary hyperbolics , we conclude that h takes the boundary half-space of g_{k-1} in G_{k-1} into R_{k-1} . Since R_i is contained in the boundary half-space of g_{k-1} , $hR_i \subset R_{k-1}$.

Next , suppose that $h \notin \langle g_k \rangle$. Then h takes the boundary half-space H of g_k in G_k into R_k , and therefore

$$hN(G_1) \subset hH \subset R_k \ . \square$$

(IV.4)LEMMA. *Fix a positive integer i . Let $h \in G_k$,$k<i$, and $h \neq 1$. Then $h(R_i)$ is contained in an R_{k+1} -region , if $h \in \langle g_{k+1} \rangle$ (figure 35). Otherwise , $h(R_i)$ is contained in an R_k -region (figure 36).*

PROOF. Suppose $h \in \langle g_{k+1} \rangle$, then h ia also a member of G_{k+1} . Since g_{k+1} and g_{k+2} are non-conjugate boundary hyperbolics , we conclude that h takes the boundary half-space of g_{k+2} in G_{k+1} into R_{k+1} . Hence $h(R_i) \subset R_{k+1}$, because R_i is contained in the boundary half-space of g_{k+2} .

Next , consider $h \notin \langle g_{k+1} \rangle$. Then h takes the boundary half-space H of g_{k+1} in G_k into R_k , and thus

$$h(R_i) \subset hH \subset R_k \ . \square$$

<u>(IV.5)LEMMA.</u> *Any element $g \in G$ can be written as a (finite) product so that no two successive elements are in the same G_i .*

<u>PROOF.</u> Since $G = \langle G_i \rangle_{i=1}^{\infty}$, we can write g as some finite product , say $h'_n \ldots h'_1$. Now starting from the right we consider the pair $h'_2 h'_1$. If they are not in the same group we move to $h'_3 h'_2$ and continue the process . If they are in the same group then we rewrite the product as one element , say h_1 , and then we start the process over with $h'_3 h_1$.

We continue this finite process until we reach h'_n . Clearly , any two successive elements now come from distinct G_i's . $\square$

Suppose $g \in G - G_1$. We write g as the product $h_{i_n} \ldots h_{i_1}$, $h_{i_k} \in G_{i_k}$, where no two successive elements are in the same G_i .Without loss of generality , the first element , h_{i_1} , is not in G_1 . Now (by IV.2), h_{i_1} takes $N(G_1)$ to either an R_{i_1}-region or an $R_{i_1 - 1}$ -region . Since h_{i_2} is not in the same group as h_{i_1} we can apply lemma (IV.3) or (IV.4) (whichever is applicable) to conclude that $h_{i_2} h_{i_1} N(G_1)$ is contained in a component of R. We continue this process , always noting that we can apply lemmas (IV.3) or (IV.4) because the next element h_{i_j} (for $j = 2,3,4,\ldots,n$) is never being applied to an R_{i_j} -region . Thus $gN(G_1)$ is contained in a component of R and we conclude that

$$gN(G_1) \cap N(G_1) = \varnothing \text{ for all } g \in G - G_1 \ .$$

We have proven the following proposition .

<u>(IV.6)PROPOSITION.</u> *The Nielson region $N(G_1)$ is precisely invariant under G_1 in G .*

<u>(IV.7)REMARK.</u> With a slight modification of lemma (IV.2) we could have proven that the Nielsen region of G_i , $N(G_i)$, is precisely invariant under G_i in G .

To illustrate how an element of G moves the Nielsen region of G_1, we take a specific example .

<u>(IV.8)EXAMPLE.</u> Let $g = g_3g_1g_3$. g_3 takes the Nielsen region of G_1 to an R_2-region (figure 37). Next , g_1 takes R_2 into R_1 (by IV.4) and hence $g_1g_3N(G_1)$ is an R_1-region (figure 38). Finally , g_3 (by IV.3) takes R_1 into R_2 . Thus $g_3g_1g_3N(G_1)$ lies in an R_2-region (figure 39).

We now prove theorem (IV.1).

<u>PROOF</u>(of IV.1). Let x be a point in the interior of $N(G_1)$. Since G_1 is a Fuchsian group , there exists an open set $U \subset N(G_1)$ so that

$$hU \cap U = \varnothing \text{ for all } h \in G_1 .$$

Furthermore , since $N(G_1)$ is precisely invariant under G_1 in G , we have that $gU \cap U = \varnothing$ for all $g \in G$. Thus G acts discontinuously at x , hence on all of H .This shows that G is a Fuchsian group .

To prove (ii) , we simply note that the axis $A(g_i) \subset N(G_i)$ and hence by remark (IV.7) , if $g \in G - G_i$ then $gA(g_i) \cap A(g_i) = \varnothing$. Also , since g_i is a boundary hyperbolic for G_i , we have for $g \in G_i$

$$gA(g_i) \cap A(g_i) \neq \varnothing \text{ if and only if } g \in \langle g_i \rangle .$$

Thus g_i is a simple hyperbolic in G .

Finally , we note that H/G is an infinite sequence of surfaces S_i

each glued to the next along the simple closed geodesic $\pi(A(g_i))$. The only thing left to show is that the geodesics $\pi A(g_i)$ are dividing.

We argue by contradiction. Assume one of the geodesics $\pi A(g_i)$ is not dividing. This can only happen if there exists a boundary hyperbolic element g in some G_j which is not a boundary hyperbolic in G and is not conjugate to any of the g_i. Thus clearly for large N, g is not a boundary hyperbolic in $G^N = \langle G_1,\ldots,G_N \rangle$. Consequently, its enough to show that a boundary hyperbolic element remains a boundary hyperbolic element after applying the combination theorem a finite number of times. In fact, its enough to show that it remains a boundary hyperbolic after one application of the combination theorem.

Suppose Γ_1 and Γ_2 are Fuchsian groups satisfying

$$N(\Gamma_1) \cap N(\Gamma_2) = A(h).$$

Furthermore, suppose that g is a boundary hyperbolic element in Γ_2 which is not conjugate (in Γ_2) to h. Then the axis of g bounds an interval of discontinuity I for the group Γ_2. We would like to show that I is also an interval of discontinuity for $\langle \Gamma_1, \Gamma_2 \rangle$.

We claim that I is precisely invariant under $\langle g \rangle$ in $\langle \Gamma_1, \Gamma_2 \rangle$. To see this, observe that since g and h are not conjugate in Γ_2, any Γ_2-translate of the boundary half-space of $h \in \Gamma_2$ is disjoint from the boundary half-space of $g \in \Gamma_2$. Hence, its easy to see that if we represent an element $\gamma \in \langle \Gamma_1, \Gamma_2 \rangle - \langle g \rangle$ in normal form we have $\gamma I \cap I = \emptyset$. This completes the argument for (iii).□

2. The machinery developed in chapters II, III, and the beginning of this chapter will allow us to construct infinitely generated Fuchsian groups with presribed distances between axes of simple hyperbolics . Geometrically , we start with two spheres with three holes, type (0;0;3) groups ,and glue them along a common boundary geodesic to obtain a sphere with four holes . Next , we take another sphere with three holes and glue along a common boundary geodesic to obtain a sphere with six holes ; we continue this process ad infinitum while controlling the distances between the boundary geodesics . Topologically , we obtain a surface , S , with an infinitely generated fundamental group (a surface of infinite type) (figure 40). The closed geodesics we glued along lift to H to form a nested sequence of geodesics . We will study the geometry of S through these nested sequences .

We remind the reader that

$$c(\beta)=\mathrm{Log}\,\frac{e^{\frac{T(\beta)}{2}}+1}{e^{\frac{T(\beta)}{2}}-1}\,,\quad \text{where } \beta \text{ is hyperbolic .}$$

Define $K(T(\beta),d)$ to be the right side of inequality (II.3) . First we need a lemma about spheres with three holes .

<u>(IV.9)LEMMA.</u> *Fix* $M>0$. *Let* $\beta\in PSL(2,R)$ *be a hyperbolic element with axis* $A(\beta)=[0,\infty]$ *oriented so that its repelling fixed point is* 0 . *Suppose we are given a geodesic* $L=[1,x]$,$x>1$, *where* $\rho(A(\beta),L)>c(\beta)$. *Then there exists a hyperbolic transformation* γ *with axis* L *and*

translation length $T(\gamma)>M$ *so that* (γ,β) *form standard generators for a Fuchsian group of type* (0;0;3).

PROOF. Construct the hyperbolic transformation γ with axis $A(\gamma)=[1,x]$, attracting fixed point 1, and translation length $T(\gamma)$ satisfying $T(\gamma)>\max\{K(T(\beta),d),M\}$.

Then γ and β are oriented so that each of their axes lies to the right of the other and they satisfy inequalities (II.2) and (II.3) . Hence by the collar theorem (II.1) , (γ,β) form standard generators for a Fuchsian group of either (0;0;3) or (0;1;2) type . However , since $T(\gamma)$ is strictly bigger than $K(T(\beta),d)$ we can exclude the (0;1;2) case and conclude that $\langle\gamma,\beta\rangle$ is of type (0;0;3) with $T(\gamma)>M$. □

We say that a hyperbolic element γ in a Fuchsian group Γ is dividing (in Γ) if $\pi(A(\gamma))$ separates H/Γ .

In the proof of the following theorem we will make use of the fact that $d>c(g)$ is equivalent to (by a straightforward computation)

$$T(g)>\mathrm{Log}\,\frac{e^{d}+1}{e^{d}-1} .$$

(IV.10)THEOREM. *Given a sequence of positive numbers* $\{d_i\}$ *and real numbers* $\{s_i\}$ *there exists an infinitely generated Fuchsian group with simple dividing hyperbolics* $\{g_i\}$ *such that*

(*i*) $\{A(g_i)\}$ *form a nested sequence of geodesics* .

(*ii*) $\rho(A(g_i)A(g_{i+1}))=d_i$ *for* $i=1,2,3,\ldots$.

(*iii*) $\rho(\Sigma_i,\Sigma_{i+1})=s_i$ *for* $i=1,2,3,\ldots$ *where* Σ_i *is the common orthogonal to* $A(g_i)$ *and* $A(g_{i+1})$.

PROOF. Construct the nested sequence of geodesics $\{L_i\}$ (Proposition (III.9)) with $\rho(L_i,L_{i+1})=d_i$ and $\rho(\Sigma_i,\Sigma_{i+1})=s_i$. If we can find an infinitely generated Fuchsian group G having simple dividing hyperbolics g_i with axes L_i then (i), (ii), and (iii) would automatically be satisfied . Thus the rest of this proof is devoted to constructing such a group .

Normalize the geodesics L_1 and L_2 by an element $f_1 \in PSL(2,R)$ so that $f_1(L_1)=[0,\infty]$ and $f_1(L_2)=[1,x_1]$, where $x_1>1$. Construct the hyperbolic transformation $\overline{g}_1$ with axis $[0,\infty]$, translation length, $T(\overline{g}_1)$, satisfying

$$T(\overline{g}_1)>\mathrm{Log}\,\frac{e^{d_1}+1}{e^{d_1}-1},$$

and oriented so that its repelling fixed point is 0. We apply (IV.9) with

$$M=\mathrm{Log}\left(\frac{e^{d_2}+1}{e^{d_2}-1}\right)^2,\ \beta=\overline{g}_1,$$

and $L=L_2$ to conclude that there exists a Fuchsian group of type (0;0;3) with standard generators $(\overline{g}_2,\overline{g}_1)$ where

(IV.11) $$T(\overline{g}_2)>\mathrm{Log}\left(\frac{e^{d_2}+1}{e^{d_2}-1}\right)^2 \quad \text{or equivalently}$$

(IV.12) $$d_2>c(\overline{g}_2)\ .$$

Now , conjugating this group by f_1^{-1} and letting $g_1=f_1^{-1}\overline{g}_1f_1$ and $g_2=f_1^{-1}\overline{g}_2f_1$ we obtain a Fuchsian group G_1 of type (0;0;3) with standard generators (g_2,g_1) satisfying $A(g_1)=L_1$, $A(g_2)=L_2$, and $d_2>c(g_2)$. This last property being the collar condition for g_2 (refer to figure 41).

Note that the Nielsen region of G_1 , $N(G_1)$, lies between the geodesics L_1 and L_2 because the Nielsen region of $f_1G_1f_1^{-1}$ lies between $[0,\infty]$ and $[1,x]$ (figure 42). We construct the infinitely generated group G by induction .

Suppose that we are given the Fuchsian group G_{n-1} having nonconjugate primitive boundary hyperbolics g_{n-1} and g_n with axes L_{n-1} and L_n , respectively . Furthermore , suppose that the translation length of g_n satisfies ,

$$\text{(IV.13)} \qquad T(g_n) > \mathrm{Log}\left(\frac{e^{d_n}+1}{e^{d_n}-1}\right)^2$$

with g_n oriented so that L_{n-1} lies to the right of $A(g_n)$. We will construct G_n .

Normalize the geodesics $A(g_n)$ and L_{n+1} by an element $f_n \in PSL(2,R)$ so that $f_nA(g_n)=[0,\infty]$ and $f_n(L_{n+1})=[1,x_n]$. Notice that $f_ng_nf_n^{-1}$ has its attracting fixed point at 0. So , we consider its inverse $f_ng_n^{-1}f_n^{-1}$. Note that since $d_n > c(g_n)$, we have that the distance

$$\rho(A(\beta),L)=d_n>c(g_n)=c(f_ng_n^{-1}f_n^{-1})=c(\beta) \; .$$

Letting $M=\mathrm{Log}\left(\frac{e^{d_{n+1}}+1}{e^{d_{n+1}}-1}\right)^2$, $\beta=f_ng_n^{-1}f_n^{-1}$, and $L=[1,x_n]$ we apply (IV.9) to conclude that there exists a hyperbolic transformation $\overline{g}_{n+1}$ with axis $[1,x_n]$ so that $(\overline{g}_{n+1},f_ng_n^{-1}f_n^{-1})$ are standard generators for a Fuchsian group of type (0;0;3) . Conjugating by f_n^{-1} and letting $g_{n+1}=f_n^{-1}\overline{g}_{n+1}f_n$ we have constructed a Fuchsian group G_n of type (0;0;3) with standard generators (g_{n+1}, g_n^{-1}) , and hence also nonconjugate primitive boundary hyperbolics , satisfying

$$T(g_{n+1}) > \operatorname{Log}\left(\frac{e^{d_{n+1}}+1}{e^{d_{n+1}}-1}\right)^2 .$$

Let $G = \langle G_i \rangle$ and note that

$$N(G_i) \cap N(G_{i+1}) = A(g_{i+1}) \text{ for } i = 1,2,3, \ldots .$$

Since G satisfies all the hypotheses of (IV.1) , we have that G is an infinitely generated Fuchsian group with simple dividing hyperbolics $\{g_i\}$. Clearly (i), (ii) , and (iii) are satisfied by construction .□

3. Our aim in this section is to derive conditions for constructing an infinitely generated Fuchsian group using only (0;1;2) groups with prescribed distances between simple hyperbolics . We remark that we have much less freedom here than in the previous construction using (0;0;3) groups . This arises because there is a two real parameter family of (0;1;2) groups whereas there is a three real parameter family of (0;0;3) groups ; in essence one of the parameters in the (0;0;3) case , namely the length of one of the boundary geodesics , has been set equal to zero .

Geometrically , we start with a sphere with two holes and a puncture . We take another sphere with two holes and a puncture where one of the holes of this new surface has the same "size" as one of the holes of the old surface . We glue the two surfaces along a common boundary geodesic . Next , we take this new surface and glue to it another sphere with two holes and a puncture where ,once again, one of the holes of this surface is of the same "size" as one of the holes in the previous surface . As before , these surfaces are glued along their common boundary geodesic . We continue this process ad infinitum . Topologically, we obtain a surface resembling the one in figure 43 .

We note that if the distances $\{d_i\}$ between the nested simple closed geodesics have been prescribed , then assigning a length (or equivalently a collar width) to the first simple closed geodesic determines the length of all the other simple closed geodesics in the nest. To see this , let $\{g_i\}$ be the simple hyperbolic elements representing the nested sequence of geodesics . Recall that the collar theorem applied to (0;1;2) groups says that $c(g_2)+c(g_1)=d_1$ where g_1 and g_2 are the lifts of the first two simple closed geodesics . Since d_1 and $c(g_1)$ are prescribed , $c(g_2)$ is determined . Moreover $c(g_n)+c(g_{n-1})=d_{n-1}$ for n=2,3,4,... .Thus all the other collar widths are determined . Since $c(g_n)$ and $T(g_n)$ are monotonic functions of each other , we conclude that prescribing the length of the first geodesic and the distances $\{d_i\}$ determines the lengths of all the other simple closed geodesics in the nested sequence .

We set $t_n = \sum_{i=1}^{n} (-1)^{i+1} d_i$.

<u>(IV.14)LEMMA.</u> *Let $G^m=\langle G_1,\ldots,G_m\rangle$ be a Fuchsian group where each G_i is of type* (0;1;2) *with standard generators (g_{i+1},g_i^{-1}) satisfying $N(G_i)\cap N(G_{i+1})=A(g_{i+1})$ for $i=1,2,3\ldots$. Let $d_i=\rho(A(g_i),A(g_{i+1}))$. Then for*

each $n=1,2,3,\ldots m+1$ *we have* ,

$$c(g_{n+1})=\begin{Bmatrix} -(t_n-c(g_1)) \text{ for } n \text{ even} \\ (t_n-c(g_1)) \text{ for } n \text{ odd} \end{Bmatrix} .$$

In particular ,

(*IV*.15) $$c(g_{n+1})=|t_n-c(g_1)| .$$

PROOF. Recall that the collar theorem (II.22) applied to G_k supplies us with the formula $c(g_{k+1})=d_k-c(g_k)$. Hence , using the collar theorem repeatedly , we have

$$c(g_{n+1})=d_n-c(g_n)=d_n-(d_{n-1}-c(g_{n-1}))$$

$$=d_n-d_{n-1}+(d_{n-2}-c(g_{n-2}))=\ldots=$$

$$=\begin{Bmatrix} d_n-d_{n-1}+d_{n-2}-\ldots-d_1+c(g_1) \text{ for } n=2k \\ d_n-d_{n-1}+\ldots+d_1-c(g_1) \text{ for } n=2k-1 \end{Bmatrix}$$

$$=\begin{Bmatrix} -(t_{2k}-c(g_1)) \text{ for } n=2k \\ (t_{2k-1}-c(g_1)) \text{ for } n=2k-1 \end{Bmatrix} .$$

Clearly $c(g_{n+1})=|t_n-c(g_1)|$. □

(IV.16)THEOREM. *Suppose* c *and* $\{d_i\}_{i=1}^{\infty}$ *are all positive real numbers . Then there exists an infinitely generated Fuchsian group* $G=\langle G_i\rangle_{i=1}^{\infty}$, *where each* G_i *is of type* (0;1;2) *with standard generators* (g_{i+1},g_i^{-1}) , *so that*

(*IV*.17) $$N(G_i)\cap N(G_{i+1})=A(g_{i+1}) \text{ for } i=1,2,3,\ldots$$

(*IV*.18) $$c(g_1)=c$$ *and*

(*IV*.19) $$\rho(A(g_i),A(g_{i+1}))=d_i \text{ for } i=1,2,3,\ldots$$

if and only if the following set of inequalities are satisfied ,

(*IV*.20) $t_n > c$ *for all odd* n *and*

(*IV*.21) $t_n < c$ *for all even* n

$$\textit{where } t_n = \sum_{i=1}^{n} (-1)^{i+1} d_i \ .$$

PROOF. First , we prove the necessity of inequalities (IV.20) and (IV.21) . Suppose $G = \langle G_i \rangle_{i=1}^{\infty}$, where each G_i is of type (0;1;2) with standard generators (g_{i+1}, g_i^{-1}) satisfying (IV.17), (IV.18), and (IV.19). By the previous lemma we have ,for $n = 2k$

$$0 < c(g_{n+1}) = -(t_{2k} - c(g_1))$$

and hence , $$c(g_1) > t_{2k} \ ;$$

but $c = c(g_1)$, we conclude that $c > t_{2k}$ verifying inequality (IV.21).

If $n = 2k - 1$, then we have by the previous lemma ,

$$0 < c(g_{n+1}) = t_{2k-1} - c(g_1)$$

and since $c = c(g_1)$ we have that $c < t_{2k-1}$, verifying inequality (IV.20). Thus inequalities (IV.20) and (IV.21) are satisfied for all n .

For the sufficiency , assume that c and $\{d_i\}_{i=1}^{\infty}$ are positive real numbers satisfying inequalities (IV.20) and (IV.21) for all n .

Construct the normalized nested sequence of geodesics $\{L_i\}$ as in figure 24 of chapter III with the property that the distance $\rho(L_i, L_{i+1})$ equals d_i for each i ; Note that there are many such sequences , pick one of them .

Normalize the geodesics L_1 and L_2 by an element $f_1 \in PSL(2,R)$ so that $f_1(L_1) = [0,\infty]$ and $f_1(L_2) = [1, x_1]$,where $x_1 > 1$ is determined by d_1 (refer to II.9). Construct the hyperbolic transformation $\overline{g}_1$ with axis $[0,\infty]$, collar width $c(\bar{g}_1)=c$, and oriented so that its repelling fixed point is 0. Next , construct the hyperbolic transformation $\bar{g}_2$ with axis $[1,x_1]$, collar width $c(\bar{g}_2)=d_1-c(\bar{g}_1)$ (note $d_1>c(\bar{g}_1)$ by IV.20 for n=1), and oriented so that its repelling fixed point is x_1. Then by the collar theorem , $(\bar{g}_2,\bar{g}_1)$ form standard generators for a Fuchsian group of type (0;1;2) . Conjugating this group by f_1^{-1} and letting $g_1=f_1^{-1}\bar{g}_1f_1$ and $g_2=f_1^{-1}\bar{g}_2f_1$ we obtain a Fuchsian group G_1 of type (0;1;2) with standard generators (g_2,g_1) satisfying $A(g_1)=L_1$ and $A(g_2)=L_2$. Notice that the Nielsen region of G_1 lies between L_1 and L_2 .Since inequality (IV.21) for n=2 says that $d_1-d_2<c(g_1)$, we conclude that $d_2>d_1-c(g_1)=c(g_2)$. The last equality is from (g_2,g_1) forming standard generators for a (0;1;2) group . Thus $d_2>c(g_2)$.

In general , suppose G_{n-1} is a Fuchsian group of type (0;1;2) with standard generators (g_n,g_{n-1}) where $A(g_{n-1})=L_{n-1}$ and $A(g_n)=L_n$. Furthermore , suppose that $d_n>c(g_n)$.

Normalize the geodesics $A(g_n)$ and L_{n+1} by an element $f_n\in PSL(2,R)$ so that $f_nA(g_n)=[0,\infty]$ and $f_nL_{n+1}=[1,x_n]$,where $x_n>1$ is determined by d_n. Note that the attracting fixed point of $f_ng_nf_n^{-1}$ is 0 .Hence we consider $f_ng_nf_n^{-1}$. Construct the hyperbolic element $\bar{g}_{n+1}$ with axis $[1,x_n]$, oriented so that x_n is its repelling fixed point, and with collar width

$$c(g_{n+1})=d_n-c(f_ng_n^{-1}f_n^{-1}) .$$

Since by assumption the distance, $d_n>c(g_n)$ we have

$$d_n>c(g_n)=c(g_n^{-1})=c(f_ng_n^{-1}f_n^{-1}) .$$

Thus $c(g_{n+1})$ makes sense . We conclude that $(\bar{g}_{n+1},f_ng_n^{-1}f_n^{-1})$ are standard generators for a Fuchsian group of type (0;1;2) . Conjugating by f_n^{-1} and setting $g_{n+1}=f_n^{-1}\bar{g}_{n+1}f_n$ we obtain a Fuchsian group G_n of type (0;1;2) with standard generators (g_{n+1},g_n^{-1}) such that $A(g_n)=L_n$ and $A(g_{n+1})=L_{n+1}$. To complete the induction step we need to show that $d_{n+1}>c(g_{n+1})$.

If n is even , then by (IV.20), we have

$$\sum_{i=1}^{n+1} (-1)^{i+1} d_i > c(g_1) \qquad \text{and therefore}$$

$$d_{n+1} > d_n - d_{n-1} + \dots - d_1 + c(g_1) = c(g_{n+1}).$$

This last equality comes from the previous lemma . Similarly , if n is odd then by inequality (IV.21) , we have

$$\sum_{i=1}^{n+1} (-1)^{i+1} d_i < c(g_1) \qquad \text{and hence}$$

$$d_{n+1} > d_n - d_{n-1} + \dots + d_1 - c(g_1) = c(g_{n+1}).$$

Once again the last equality follows from the previous lemma . This completes the induction step. Set $G=\langle G_i \rangle_{i=1}^{\infty}$. By (IV.1) ,G is a Fuchsian group satisfying (IV.17), (IV.18), and (IV.19) by construction . □

(IV.22)Remark. Since we never applied the combination theorem along the axis $A(g_1)$ or any of its conjugates , the group G just constructed has g_1 as a primitive boundary hyperbolic element . Furthermore, G has infinitely many nonconjugate parabolic elements .

One should note that the twist parameters never entered into the construction of G . Thus G can be constructed with whatever sequence of twist parameters one would like .

Before ending this section , we derive a corollary to the above theorem which guarantees the existence of an infinitely generated Fuchsian group made up of (0;1;2) groups .

(IV.22)COROLLARY. *If* $\{d_i\}_{i=1}^{\infty}$ *is a nonincreasing (or nondecreasing) sequence of positive numbers then there exists a Fuchsian group* $G=\langle G_i\rangle_{i=1}^{\infty}$ *, where each* G_i *is of type* (0;1;2) *with standard generators* (g_{i+1},g_i^{-1}) *satisfying* (IV.17) *and* (IV.19) *. In other words , H/G is the union of an infinite number of spheres with two holes and a puncture, glued along common boundary geodesics, where the distance between successive geodesics is* d_i *(figure* 43).

PROOF. Suppose the sequence $\{d_i\}_{i=1}^{\infty}$ is nonincreasing, that is $d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n \geq \dots$. If we can find a positive constant c satisfying (IV.20) and (IV.21) we will be done by (IV.16). Now either $d_i \to 0$ or $d_i \to K > 0$.

If $d_i \to 0$, then the alternating series $\sum_{i=1}^{\infty} (-1)^{i+1} d_i$ converges . Hence set $c = \sum_{i=1}^{\infty} (-1)^{i+1} d_i$ and note that c is bigger than all the odd partial sums and smaller than all the even partial sums, that is, c satisfies (IV.20) and (IV.21).

If $d_i \to K > 0$, then the alternating series $\sum_{i=1}^{\infty} (-1)^{i+1} d_i$ diverges . Furthermore, we have

$$t_{2k} < t_{2k+2} < \dots < t_{2k+3} < t_{2k+1} < t_{2k-1} \text{ for all } k,$$

where t_n is the n^{th} partial sum of the series $\sum_{i=1}^{\infty} (-1)^{i+1} d_i$. Hence, there exist an interval of c's which satisfy the inequalities (IV.20) and (IV.21).

For a nondecreasing sequence $\{d_i\}_{i=1}^{\infty}$ of positive numbers , simply take any real number c that satisfies $0 < c < d_1$. Note that since the d_i are nondecreasing the even partial sums are negative and decreasing, and the odd partial sums are bigger than d_1 and increasing . Thus c once again satisfies inequalities (IV.20) and (IV.21). □

4. For this section we assume that G is one of the infinitely generated Fuchsian groups constructed from the methods of Section 3. Thus $G=\langle G_i\rangle_{i=1}^{\infty}$ is a Fuchsian group where G_i is of type (0;1;2) with standard generators (g_{i+1},g_i^{-1}) satisfying (IV.17) and (IV.19). The following propositions relate the distances $\{d_i\}_{i=1}^{\infty}$ with the translation lengths of the $\{g_i\}_{i=1}^{\infty}$. Recall that $t_n=\sum_{i=1}^{n} (-1)^{i+1} d_i$.

(IV.22)PROPOSITION. *If $d_i\rightarrow 0$, then $T(g_i)\rightarrow\infty$.*

PROOF. We know by the collar theorem that for each i , $d_i > c(g_i)$. Hence as $d_i\rightarrow 0$, the collar width about g_i goes to 0. Since,

$$c(g_i)= \text{Log}\ \frac{e^{\frac{T(g_i)}{2}}+1}{e^{\frac{T(g_i)}{2}}-1},$$

we conclude that as $d_i\rightarrow 0$ the translation lengths of the g_i go to infinity.□

(IV.22)PROPOSITION. *If $\{d_i\}_{i=1}^{\infty}$ is a nondecreasing sequence such that the distances $d_i\rightarrow\infty$. Then the translation lengths $T(g_i)\rightarrow 0$.*

PROOF. Note that the even sums $t_{2k}\rightarrow -\infty$ and the odd sums $t_{2k-1}\rightarrow\infty$.

Hence, by the lemma preceding (IV.16), we have $c(g_{i+1})\to\infty$. Thus the translation lengths $T(g_i)\to 0$. □

(IV.22)PROPOSITION. *If* $\{d_i\}_{i=1}^{\infty}$ *is a constant sequence, then* $T(g_i)$ *takes on only two values, for all i. In particular* $\{T(g_i)\}$ *is bounded from above and below.*

PROOF. Set $d_i=d$. Then the even sums $t_{2k}=0$ and the odd sums $t_{2k-1}=d$. Hence since $c(g_{n+1})=|t_n-c(g_1)|$, we have

$$c(g_{i+1})=\begin{cases} c(g_1)\ , & \text{for i even} \\ d-c(g_1)\ , & \text{for i odd} \end{cases} .$$

We conclude that the translation lengths $\{T(g_i)\}$ take on only two values.□

5. In this section we supply sufficient conditions, in terms of the hyperbolic distances between simple closed geodesics, for an infinitely generated Fuchsian group to be of the first kind. Our construction will rely upon the techniques of Section 3 where we used (0;1;2) groups to construct surfaces of infinite type.

Throughout this section, we suppose that $G=\langle G_i\rangle_{i=0}^{\infty}$ is some infinitely generated Fuchsian group where G_0 is of type (0;2;1) and the G_i, for $i=1,2,3,\ldots$, are of type (0;1;2) having standard generators (g_{i+1},g_i^{-1}) and satisfying the condition, $N(G_i)\cap N(G_{i+1})=A(g_{i+1})$ for $i=0,1,2,3,\ldots$.

Under these hypotheses the axes of the hyperbolic transformations $\{g_i\}$ form a nested sequence of geodesics, $\{A(g_i)\}$. We remind the reader that in Section 3 we saw how to construct such a G.

Let B_i be the boundary half-space of g_i in G_{i-1}. Set $B=\bigcap_{i=1}^{\infty} B_i$ and $G^n=\langle G_0,\ldots,G_n\rangle$. We have the following lemma.

(IV.22)LEMMA. *Suppose that the nested sequence of geodesics $\{A(g_i)\}$ converge to a geodesic. Then B is precisely invariant under the identity in G. In particular, B is a boundary half-space for G (figure* 44).

PROOF. Let g be any non-trivial element of G. Certainly we can find a large enough positive integer n so that $g\in G^n=\langle G_0,\ldots,G_n\rangle$ and $g\notin\langle g_{n+1}\rangle$. Since g_{n+1} is a boundary hyperbolic element in G^n, we have $gB_{n+1}\cap B_{n+1}=\varnothing$ and thus, since $B\subset B_{n+1}$, $gB\cap B=\varnothing$.□

The intersection of B with the boundary of H is an interval of discontinuity for G. We say that it is an interval of discontinuity for the nested sequence of axes $\{A(g_i)\}$.

We define the set $\mathcal{L}=\mathcal{L}(G;\{g_i\})$ to be the set of all nested sequences of axes $\{A(g_i')\}$, where g_i' is conjugate to g_i in G. Observe that the elements that conjugate the g_i are not necessarily the same.

Recall that $d_i=\rho(A(g_i)A(g_{i+1}))$.

(IV.23)LEMMA. *Fix a positive integer i. (a)If $h\in G^{i-1}$ and $h\notin G_i$, then the axis $A(g_i)$ separates $A(hg_ih^{-1})$ from $A(g_{i+1})$. (b)If $h\in\langle G_k\rangle_{k=i+1}^{\infty}$, then* *the axis* $A(hg_{i+1}h^{-1})$ *separates* $A(hg_ih^{-1})$ *from* $A(g_{i+1})$.

PROOF. We first prove (a). Suppose $h \in G^{i-1}$ and $h \notin G_i$. Observe that g_i is a boundary hyperbolic element in G^{i-1}, hence h moves the axis $A(g_i)$ away from the boundary half-space of g_i in G^{i-1}. We also know that the axis $A(g_{i+1})$ lies in the boundary half-space of g_i. Thus the axis $A(g_i)$ separates $A(hg_ih^{-1})$ from $A(g_{i+1})$.

To prove (b), suppose $h \in \langle G_k \rangle_{k=i+1}^{\infty}$. Note that g_{i+1} is a boundary hyperbolic element for the group $\langle G_k \rangle_{k=i+1}^{\infty}$ and hence h moves the axis $A(g_{i+1})$ away from the boundary half-space of g_{i+1}. Furthermore, since the axis $A(g_i)$ lies in the boundary half-space of g_{i+1}, the axis $A(hg_{i+1}h^{-1})$ separates $A(hg_ih^{-1})$ from $A(g_{i+1})$. This verifies (b).□

The following proposition shows that the distance between two boundary geodesics of a pair of pants does not change after applying the combination theorem.

(IV.24)PROPOSITION. *Fix a positive integer i. Suppose G is as above and* $h \in G$. *Then*

$$(IV.25) \qquad \rho(hg_ih^{-1}, g_{i+1}) \geq \rho(g_i, g_{i+1}).$$

PROOF. We view G as the free product of the group G^{i-1} and the group $\langle G_k \rangle_{k=i}^{\infty}$ amalgamated over the subgroup $\langle g_i \rangle$. We write h in normal form $h_n \ldots h_1$ with respect to this free product decomposition.

First, suppose that $h_1 \notin G_i$. Then either (a) or (b) hold (depending on whether h_1 is contained in G^i or in $\langle G_k \rangle_{k=i}^{\infty}$). Applying the successive elements $h_2, h_3, \ldots, h_n$ to the axis $h_1A(g_i)$ and noting that the separation properties (a) or (b) continue to hold, we see that

$$\rho(hg_ih^{-1},g_{i+1})\geq\rho(g_i,g_{i+1}).$$

Next, if $h_1\in G_i$ then we replace the element g_i by $h_1g_ih_1^{-1}$ and proceed as above. If $h\neq h_1$, then the above argument shows that (IV.25) holds. Now if $h=h_1$, then inequality (IV.25) follows from the fact that $\rho(g_i,g_{i+1})$ is the distance between the closed geodesics which represent the elements g_i and g_{i+1}(II.24).□

(IV.26)LEMMA. *Suppose that the series $\sum d_i=\infty$. Then all nested sequences in $\mathcal{L}$ go to the boundary of H.*

PROOF. It is enough to show that the following inequality holds,

$$\rho(g_i',g_{i+1})\geq\rho(g_i,g_{i+1})=d_i$$

where g_i' is any conjugate of g_i. But this is exactly inequality (IV.25).□

(IV.27)LEMMA. *If I is an interval of discontinuity for G, then I is an interval of discontinuity for some nested sequence in $\mathcal{L}$.*

PROOF. We note that G has no boundary hyperbolics. If I is an interval of discontinuity for G then for all n, I is contained in an interval of discontinuity for G^n. But the only intervals of discontinuity in G^n are ones that are bounded by the axis of g_n or by a conjugate of g_n. This holds for each n. Hence all such axes bound I. We conclude that I is an interval of discontinuity for some nested sequence in $\mathcal{L}$.□

One can interpret the above lemma as saying that intervals of discontinuity come in two forms. Either they are intervals for boundary hyperbolics or they are limits of intervals of simple dividing hyperbolics.

Suppose we have that the sum of the distances between the simple closed geodesics $\{\pi(A(g_i))\}$ on H/G diverge, that is $\sum d_i = \infty$. Then all nested sequences in $\mathcal{L}$ would go to the boundary of H. We conclude that G is of the first kind. We state this formally in the following theorem.

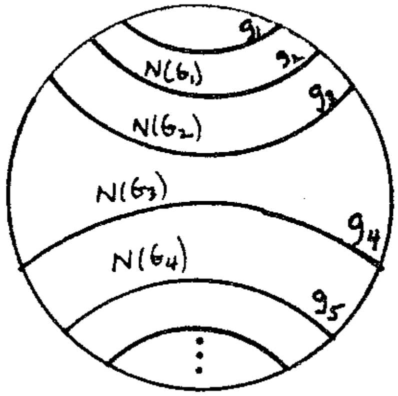
(IV.28)THEOREM. *Suppose that $G=\langle G_i\rangle_{i=0}^{\infty}$ is an infinitely generated Fuchsian group where G_0 is of type (0;2;1) and the G_i, for $i=1,2,3,\ldots$, are of type (0;1;2) having standard generators (g_{i+1}, g_i^{-1}) and satisfying the condition, $N(G_i)\cap N(G_{i+1})=A(g_{i+1})$ for $i=0,1,2,3,\ldots$. Let d_i, for positive i, be the distance between the axis of g_i and the axis of g_{i+1}. Then G is of the first kind if the series $\sum d_i$ diverges.*

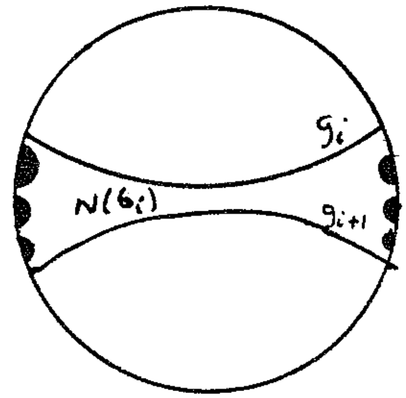
Recall that Σ_i denotes the common orthogonal to the geodesics $A(g_i)$ and $A(g_{i+1})$. The twist parameter which measures the oriented distance between the common orthogonals Σ_i and Σ_{i+1} is denoted by s_i.

Our next example illustrates the point that the nested sequence of axes $\{A(g_i)\}$ can converge to the boundary of H even though G is of the second kind. The crucial point here is that there are nested sequences of geodesics which converge to the boundary of H and yet the sum of the distances between the geodesics in the nest converge.

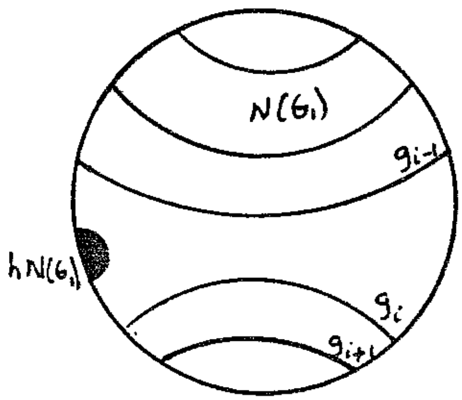
(IV.29)EXAMPLE. Construct the group $G=\langle g_1, g_2, \ldots, g_n, \ldots\rangle$ as indicated in the beginning of this section with $d_n = 1/n^2$ making sure that the twist parameters are all equal to zero, that is $s_n = 0$. Figure 46 illustrates what the nested sequence of axes $\{A(g_i)\}$ would look like. The accompanying quotient surface H/G is illustrated in figure 47. Since the nested sequence of axes converge to a geodesic, the half-space bounded by this geodesic is precisely invariant under the identity in G. In particular, G is of the second kind.

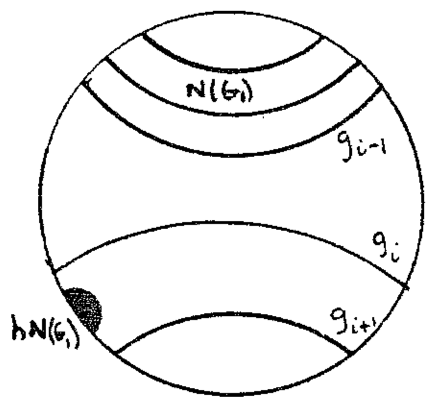
We observe that Dehn twisting about the simple closed geodesics $\pi(A(g_i))$ by a length $s_i = T(g_{i+1})$ corresponds to conjugating the group $\langle g_i, \ldots, g_n, \ldots \rangle$ by g_i. Hence we have $\langle g_i, g_i g_{i+1} g_i^{-1}, \ldots \rangle$. On the universal cover this amounts to looking at another nested sequence of axes $\{A(g_j')\}$ in $\mathcal{L}$, where the g_j' equal g_j if j is less than or equal to i and otherwise they are $g_i g_j g_i^{-1}$(figure 48). Notice that the pairs $(g_{j+1}', (g_j^{-1})')$ are a standard set of generators for G_j. Now, if we Dehn twist for each g_j by a large multiple of $T(g_{j+1})$ we will force the axes $\{A(g_j')\}$ to converge to the boundary of H (figure 49). See figure 50 for the accompanying picture on the surface H/G. Thus G is of the second kind and yet one of its nested sequences of axes of standard generators goes to the boundary of H.





Figures 29 and 30.





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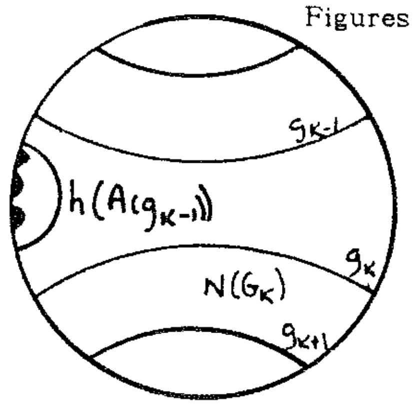

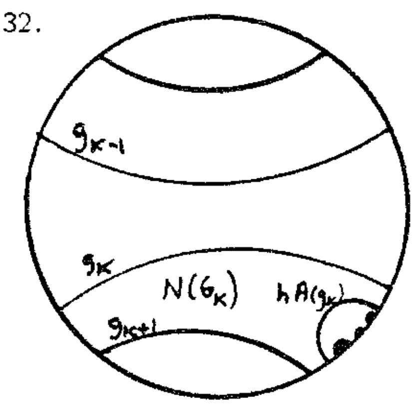

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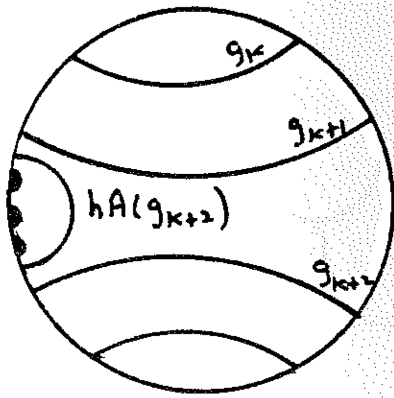

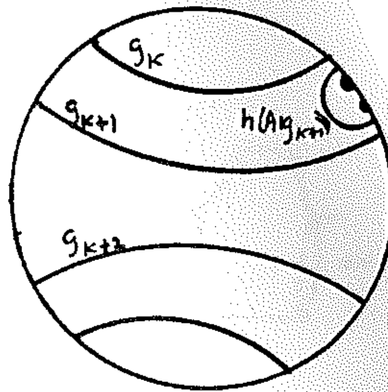

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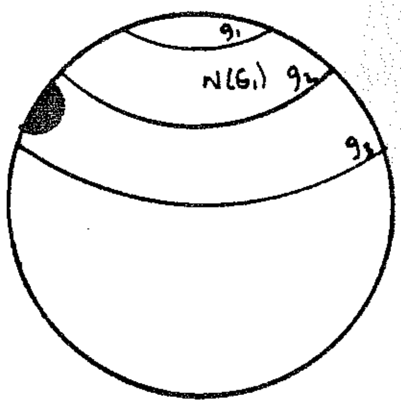

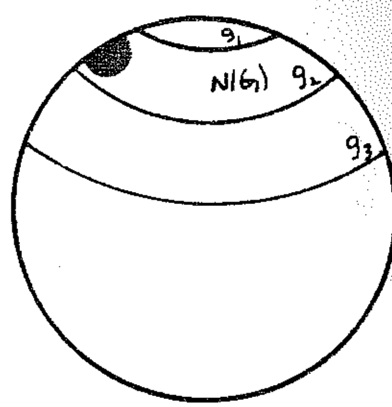

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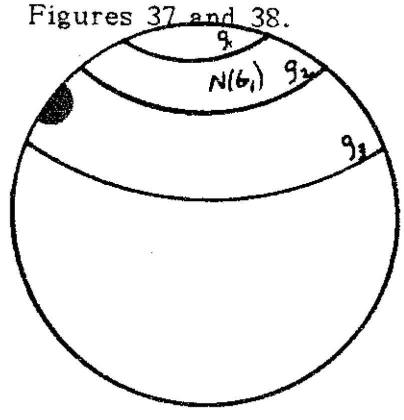

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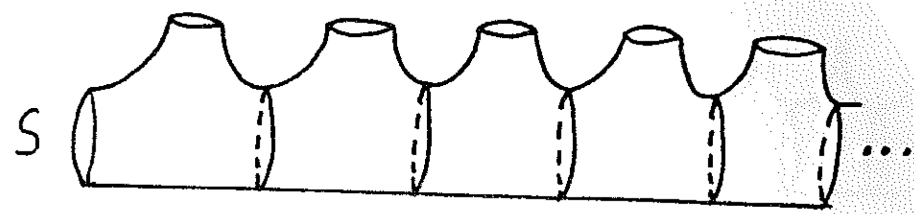

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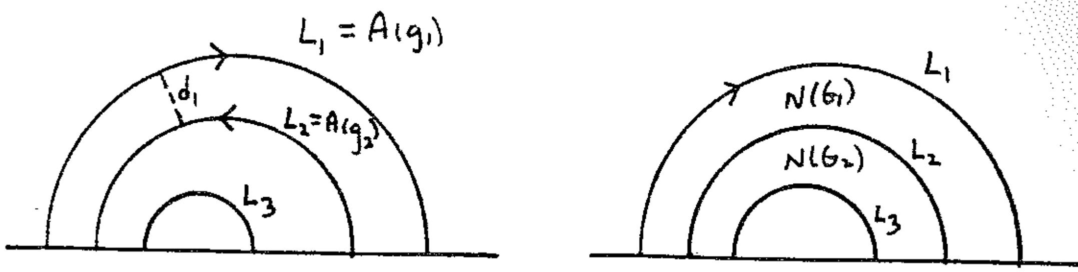

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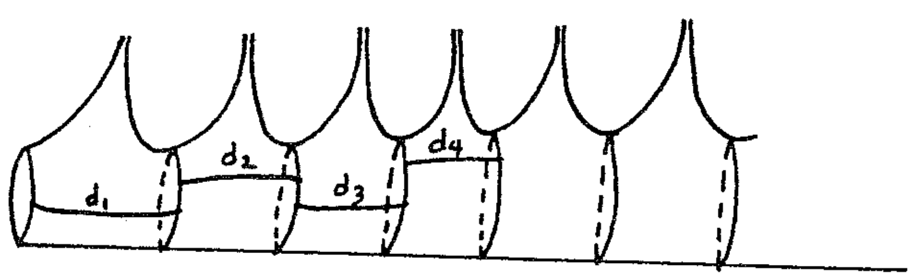
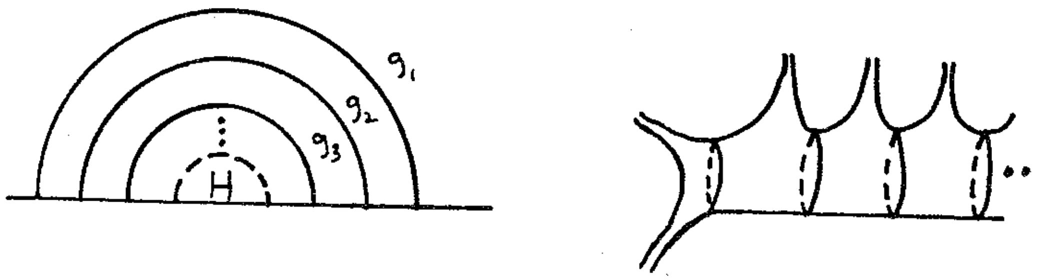
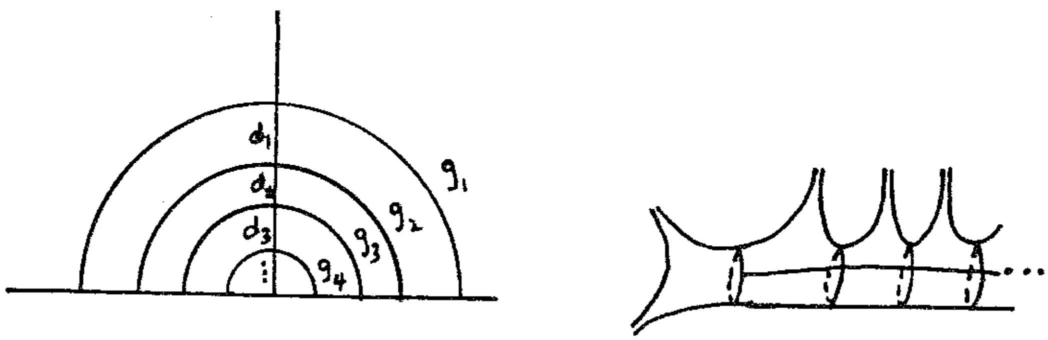


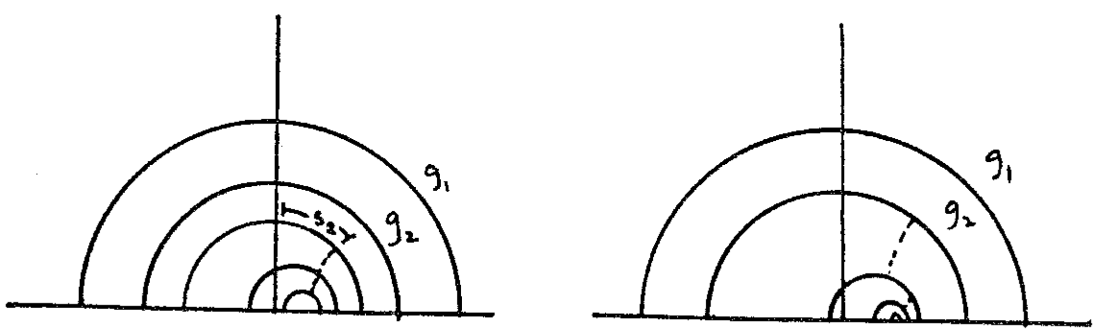
Figures 43.



Figures 44 and 45.



Figures 46 and 47.



Figures 48 and 49.

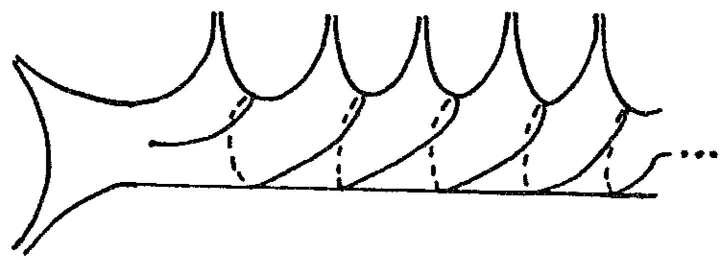


Figure 50.

CHAPTER V

0. It is evident that we can use the combination theorem a finite number of times to create more complicated surfaces from very simple ones. In fact, the combination theorem tells us exactly what this complicated surface would look like in terms of the fundamental polygons which represent the simpler surfaces. In Section 1, we investigate what happens when we iterate the combination theorem an infinite number of times. In Section 2, we show that the Nielsen isomorphism theorem does not extend to infinitely generated Fuchsian groups.

1. We start with a sequence of Fuchsian groups $\{G_i\}$ where g_i and g_{i+1} are nonconjugate primitive boundary hyperbolic elements in G_i satisfying $N(G_i) \cap N(G_{i+1}) = A(g_{i+1})$ for all i. The axes of the g_i form a nested sequence of geodesics. Set $d_i = \rho(A(g_i), A(g_{i+1}))$.

Our aim in this section is to show that the conclusion of the combination theorem, that the resulting polygon obtained by intersecting the previous polygons is a fundamental polygon, does not necessarily hold if we apply the combination theorem infinitely often along the nested sequence of axes $\{A(g_i)\}$. However, if we assume that the series $\sum d_i$ diverges then this conclusion does hold. Before we go on we need some notation.

Recall that G^n is defined to be $\langle G_1,\ldots,G_n\rangle$. Let D_i be a fundamental polygon for G_i and set $D^n = D_1 \cap \ldots \cap D_n$. Let $B(g_n)$ be the half-space which is the component of $H - A(g_n)$ containing the Nielsen region $N(G_{n-1})$. Denote the other component by $B'(g_n)$. We say that the $\{D_i\}$ pairwise satisfy the compatibility conditions in the combination theorem if for each i, D_i and D_{i+1} satisfy the hypotheses in the combination theorem; that is,

(a) $D_i \cap B(g_{i+1}) \subset D_{i+1} \cap B(g_{i+1})$;

(b) $D_{i+1} \cap B'(g_{i+1}) \subset D_i \cap B'(g_{i+1})$; and

(c) The sides of D_i that are paired by g_{i+1} lie on the same geodesic lines as the sides of D_{i+1} which are paired by g_{i+1}.

Note that after applying the combination theorem to G_1 and G_2, we have $D^2 \cap B(g_2) = D^1 \cap B(g_2)$ and in general after applying the combination theorem to G^{n-1} and G_n we have $D^n \cap B(g_n) = D^{n-1} \cap B(g_n)$ that is, D^{n-1} and D^n are the same on the half-space $B(g_n)$.

We let $\mathcal{B}(G^n)$ be the set of all boundary half-spaces of G^n and $\mathcal{B}'(G^n)$ be the set of all boundary half-spaces of G^n with the boundary half-space of g_{n+1} and its conjugates deleted. Observe that the set $N(G^n) \cup \mathcal{B}'(G^n)$ is invariant under G^n.

(V.1)THEOREM. *Let $G = \langle G_i \rangle_{i=1}^{\infty}$ be an infinitely generated Fuchsian group, where G_i is a Fuchsian group containing the nonconjugate primitive boundary hperbolics g_i and g_{i+1} satisfying $N(G_i) \cap N(G_{i+1}) = A(g_{i+1})$. Let $\{D_i\}$ be fundamental polygons for $\{G_i\}$*

which pairwise satisfy the compatability conditions in the combination theorem. Set $D=\bigcap_{i=1}^{\infty} D_i$.

Then D is a fundamental polygon for G if $\sum d_i=\infty$.

PROOF. Observe that the sides of D are paired by elements of G and that D is a convex hyperbolic polygon. Now, let g be a nontrivial element of G. Then g is contained in G^n for a sufficiently large n. Hence $gD^n \cap D^n=\emptyset$ and therefore, since $D \subset D^n$, we have that $gD \cap D=\emptyset$.

Let $x \in H$. Since the series $\sum d_i$ diverges, all nested sequences in $\mathcal{L}$ go to the boundary of H. Hence there exists a positive integer m so that $x \in N(G^m) \cup \mathcal{R}'(G^m)$. Since D^m is a fundamental polygon for G^m, there exists an element $g \in G^m$ so that $g(x) \in \bar{D}^m$. Furthermore, since $N(G^m) \cup \mathcal{R}'(G^m)$ is invariant under G^m and $\mathcal{R}'(G^m) \subset B(g_{m+1})$, we have $g(x) \in \bar{D}^m \cap B(g_{m+1}) = \bar{D} \cap B(g_{m+1})$. Thus $g(x) \in \bar{D}$.

Finally we would like to show that the tesselation is locally finite. Let K be a compact set in H. Once again, since all nested sequences in $\mathcal{L}$ go to the boundary of H, there exists a positive integer m so that

$$K \subset N(G^m) \cup \mathcal{R}'(G^m).$$

Since the translates of D^m by elements of G^m form a locally finite tesselation and since

$$D^m \cap B(g_{m+1}) = D \cap B(g_{m+1}),$$

we have that K meets only finitely many G-translates of D^m. Lastly, $D \subset D^m$ implies that K meets finitely many translates of D.□

Our next example will show that the above theorem is false if the series $\sum d_i < \infty$.

First, we construct a fundamental polygon for a normalized Fuchsian group Γ of type (0;0;3) or (0;1;2). Thus Γ has standard generators (γ,β) with axes and orientations as depicted in figure 6(7 if in the (0;1;2) case) so that $\beta\gamma$ is a boundary hyperbolic element(a parabolic element, if in the (0;1;2) case). Let Σ be the common orthogonal to the axes $A(\gamma)$ and $A(\beta)$. Consider the image of Σ under β, $\beta(\Sigma)$. The right endpoint of $\beta(\Sigma)$ lies to the right of the right endpoint of $A(\eta)$. We draw the common orthogonal to $A(\eta)$ and $\beta(\Sigma)$ and call the point of intersection between this common orthogonal and $\beta(\Sigma)$, $\beta(z)$. Hence z is a point on Σ which divides Σ into two geodesic line segments; call them s_2 and s_1'. $\beta(z)$ divides the common orthogonal to $A(\eta)$ and $\beta(\Sigma)$ into two geodesic segments; call the segment which intersects the axis $A(\eta)$, s_3. Set $s_2'=\beta(s_2)$ and $s_1=\gamma^{-1}(s_1')$. Finally, let s_3' be $\eta(s_3)$. Since $\eta\beta\gamma=1$, η takes the vertex point $\beta(z)$ to the point $\gamma^{-1}(z)$. We claim that the angle made at the vertex $\gamma^{-1}(z)$ is $\pi/2$. To see this, consider the (oriented) angle between the sides $\eta^{-1}(s_1)$ and $\eta^{-1}(s_3')$. Since $\eta^{-1}=\beta\gamma$ and $\eta^{-1}(s_3')=s_3$, this is the same as looking at the angle between $\beta\gamma(s_1)$ and s_3. But $\gamma(s_1)=s_1'$ and s_1' lies on Σ; hence $\beta\gamma(s_1)$ is the line segment on $\beta(\Sigma)$ which does not overlap s_2'. We conclude that the angle between the sides s_1 and s_3' is the same as the angle between $\beta(\Sigma)$ and s_3 which is $\pi/2$.

We consider the polygon P bounded by the sides constructed above having the identifications indicated in figure 51. Observe that the hypotheses to Poincare's theorem are all satisfied and thus P is a fundamental polygon for the group Γ.

If the group Γ is not normalized, then we normalize (conjugate) it by an element f of PSL(2,R) so that its standard generators are as in figure 6(7 if in the (0;1;2) case). Next, we construct the fundamental polygon as above and conjugate it by f^{-1}. Since f is an isometry of H, this new polygon is a fundamental polygon for Γ. Once again we name this polygon P.

Observe that two of the sides of P lie on the common orthogonal to the axes of the standard generators γ and β.

(V.2)EXAMPLE. Consider a Fuchsian group $G=\langle G_i \rangle$; where the G_i are (0;0;3) groups having standard generators (g_{i+1}, g_i^{-1}) and satisfying

$$N(G_i) \cap N(G_{i+1}) = A(g_{i+1}) \ .$$

Suppose further that the $\{A(g_i)\}$ form a nested sequence of geodesics; where $d_n = 1/n^2$ and $s_n = 0$. The important point to keep in mind is that the twist parameters have all been set equal to zero and hence all the axes of the g_i have the same common orthogonal, call it L.

We normalize each group G_i and construct the fundamental polygon D_i as we did above. We note that the $\{D_i\}$ pairwise satisfy the compatability conditions in the combination theorem. We set $D = \cap D_i$.

Note that the polygon D has infinitely many sides which lie on the same geodesic L. Furthermore, since the nested sequence of axes $\{A(g_i)\}$ converge to a geodesic, the infinitely many sides which lie on L must accumulate to a point of H. We conclude that the action of G on D cannot be locally finite. Thus this conclusion of the combination theorem does not in general hold.

2. Let $\Phi:G \to G'$ be an isomorphism between two torsion free Fuchsian groups. Recall that Φ is said to be a type preserving isomorphism if boundary hyperbolics correspond to boundary hyperbolics and parabolics correspond to parabolics. We say that Φ is a topological deformation if there exists a homeomorphism $f:\overline{H} \to \overline{H}$ so that

$$\Phi(g)(z) = fgf^{-1}(z) \text{ for all } z \in \overline{H} \text{ and } g \in G.$$

In this section, we construct a type preserving isomorphism Φ between two torsion free Fuchsian groups which is not a topological deformation. Thus showing that the Nielsen Isomorphism theorem does not hold for infinitely generated Fuchsian groups.

(V.3)EXAMPLE. In Section 5 of chapter IV we saw how to construct infinitely generated Fuchsian groups using only groups of type (0;1;2) (except for the initial group G_0 which is of type (0;2;1)). Recall that if $\{d_n\}$ is a sequence of non-increasing positive real numbers then there exists an infinitely generated Fuchsian group $G = \langle G_n \rangle_{n=0}^{\infty}$ where

$$d_n = \rho(A(g_n), A(g_{n+1})).$$

Let G be the group with length parameters $\{d_n = 1/n^2\}$ and twist parameters $\{s_n = 0\}$. Since the nested sequence of axes $\{A(g_n)\}$ converges to a geodesic L, the half-space defined by L is precisely invariant under the identity in G. We conclude that the interval on the boundary of H bounded by the endpoints of L is an interval of discontinuity for G.

Thus G is of the second kind.

Next, let $G'=\langle G'_n\rangle_{n=0}^{\infty}$ be the group with length parameter $\{d'_n=1/n\}$ and twist parameter $\{s'_n=0\}$. Note that the nested sequence of axes $\{A(g'_n)\}$ converge to the boundary of H and (by the arguments of Section 5, chapter IV) the group G' is of the first kind.

It is obvious by their construction that the groups G and G' are isomorphic by the map $\Phi: G \to G'$ which takes the generator g_n to the generator g'_n.

We would like to show that Φ is type preserving. By construction, we know that G and G' have no boundary hyperbolics. Thus we need only show that Φ preserves parabolics.

First we note that the elements $g_i g_{i+1}$ and their inverses $g_{i+1}^{-1} g_i^{-1}$ represent all the conjugacy classes of parabolic elements; that is, any parabolic element $g \in G$ is equal to $h g_i g_{i+1} h^{-1}$ or $h g_{i+1}^{-1} g_i^{-1} h^{-1}$ for some $h \in G$. Of course the same statement holds for G'. Since Φ preserves the type of a conjugacy class, it is enough to show that Φ takes the parabolic element $g_i g_{i+1}$ to a parabolic element of G' (note: we need not consider the inverses $g_{i+1}^{-1} g_i^{-1}$ because

$$\Phi(g_{i+1}^{-1} g_i^{-1}) = \Phi((g_i g_{i+1})^{-1}) = (\Phi(g_i g_{i+1}))^{-1}$$

and the inverse of a parabolic element is parabolic).

Now, $$\Phi(g_i g_{i+1}) = \Phi(g_i)\Phi(g_{i+1}) = g'_i g'_{i+1};$$

but $g'_i g'_{i+1}$ are standard generators for G'_i. Hence, $g'_i g'_{i+1}$ is a parabolic element of G. We conclude that Φ is a type preserving isomorphism.

On the other hand, Φ is not a topological deformation. To see this, suppose there exists a homeomorphism $f:\overline{H}\to\overline{H}$ so that $\Phi(g)(z)=fgf^{-1}(z)$ for all $z\in\overline{H}$ and $g\in G$; f would have to take the regular set of G to the regular set of G′. This is clearly impossible since G is of the second kind and G′ is of the first kind.

(V.4)REMARK. One could equally as well construct examples where all of the G_i and G_i' are of type (0;0;3), thus making G and G′ groups of the second kind.

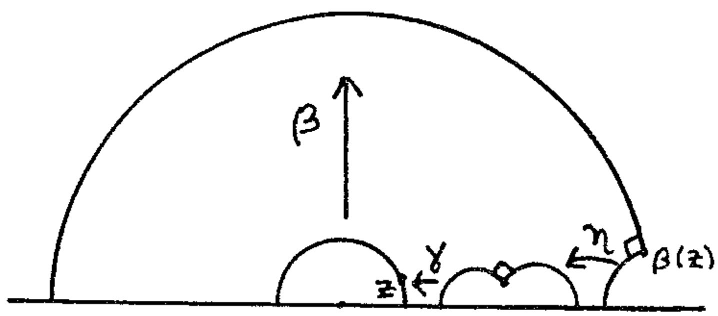


Figure 51.

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