On the Crossed Product C*-Algebras Associated with Furstenberg Transformations on Tori

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Abstract of the Dissertation

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In this thesis, the notion of "Furstenberg transformation" on an n-dimensional torus $\mathbb{T}^n$ is introduced. The corresponding crossed product C*-algebras are studied and some classes of these C*-algebras are classified up to $*$-isomorphism and up to strong Morita equivalence. Some further developments and conjectures are discussed. In addition, the strong Morita equivalence for C*-algebras associated with minimal rotations on tori is determined.

To my parents, brothers, and Yun Liang.

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Acknowledgement

The classification of C*-algebras associated with Furstenberg transformations on the 2-dimensional torus was basically completed by November, 1984 while I visited the Mathematical Sicences Research Institute, Berkeley, CA, along with Professor R.G. Douglas. Since then I have benefited from several helpful discussions with William Paschke and Jonathan Rosenberg concerning the computations of the range of a tracial state on K_0-group of the C*-algebra associated with a Furstenberg transformation on a higher dimensional torus, and the computation of the K-theory of this C*-algebra. I would like to thank William Paschke, Marc Rieffel, Judith Packer and Hongshen Yin for making the preprints [34], [35], [36], [41], and [59] available to me.

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Introduction

Topological dynamical systems have been studied by topologists as well as physicists for many years. Operator algebraists soon realized that topological dynamical systems can be viewed as a special case of C*-dynamical systems which yields interesting new examples of C*-algebras of many types. These examples raise the possibility of studying connections between the nature of a topological dynamical system and the structure of the corresponding crossed product C*-algebra. The success of the generalization of the topological K-theory of Atiyah and Hirzebruch (see [2] also [1]) to general non-commutative C*-algebras (see [56], [28], [5] and [29], etc.) gives a way of finding very important invariants for C*-algebras, and the work of Pimsner and Voiculescu [44], [45]; Connes [8]; Fack and Skandalis [19], as well as Pimsner [42] enables one to compute the K-groups as well as the KK-groups of crossed product C*-algebras. Thus the classifications of crossed product C*-algebras become possible and has been quite successful.

The work of Rieffel in [50] and Pimsner and Voiculescu in [43] taken together gives a complete classification of the crossed product C*-algebras $A_\theta = C(\mathbb{T}^1) \times_\theta \mathbb{Z}$ associated with irrational rotations, with rotation number θ on $[0,1]$, on the one dimensional torus $\mathbb{T}^1$. That is, for irrational θ

and θ' in $(0,1)$, $A_{\theta'}$ is $*$-isomorphic to A_θ if and only if $\theta' = \theta$ or $1 - \theta$. Since their work, irrational rotation C*-algebras have been received a great deal of attention by many mathematicians who studied these algebras as a first step for motivating general studies in different fields, e.g. A. Connes in [9], and Pimsner and Voiculescu in [40] and [44]. Soon after, Riedel in [48] generalized the classification of irrational rotation C*-algebras to the classification of C*-algebras associated with minimal rotations on compact metric abelian groups. In a different direction, Packer in [35] and [36] obtained a complete classification for twisted group C*-algebras of the 3-dimensional discrete Heisenberg group. The methods used in obtaining the classification are K-theory and the computation of ranges of tracial states on the K_0-groups of these C*-algebras, and also, the construction of the so-called "Rieffel Projections" [50].

In attempting to obtain a natural method for computing the K-theory of the irrational rotation C*-algebras, Pimsner and Voiculescu discovered the Pimsner-Voiculescu exact sequence for the crossed product C*-algebras $A \times_\alpha \mathbb{Z}$ of a C*-algebra A by an automorphism α of A. This yielded a new proof of the classification for irrational rotation C*-algebras [44].

Since the range of a tracial state on the K_0-group of a C*-algebras is useful in distinguishing C*-algebras, Pimsner (as well as others) formulated an exact sequence of traces and determinants on the K-groups for crossed product C*-algebras of a C*-algebra A by an automorphism α of A in [41]. This exact sequence turns out to be a very powerful tool for computing the range of a tracial state on the K_0-group of a crossed product C*-algebra.

Pursuing the ideas in [50], [44] and [41], we will consider in this thesis, the classification of the crossed product C*-algebras, denoted by $A_{F_{f,\theta}}$, associated with what we call "Furstenberg transformations," $F_{f,\theta}$ on tori for θ in [0,1), (see definition (1.1) in §1). This is interesting because $F_{f,\theta}$ is a natural generalization of the rotation ρ_θ on the 1-dimensional torus $\mathbb{T}^1$, and many $F_{f,\theta}$'s can be viewed as affine transformations on the n-torus $\mathbb{T}^n$. We will compute the K-groups of $A_{F_{f,\theta}}$ using the Pimsner-Voiculescu exact sequence, and determine the range of a tracial state on $K_0(A_{F_{f,\theta}})$ using the work of Pimsner [41]. It turns out that, in general $K_*(A_{F_{f,\theta}})$ has a torsion subgroup in contrast to the rotation algebras on tori considered by Riedel [48]. We will then classify the C*-algebras associated with some special classes of Furstenberg transformations and the results we obtain are similar to the classification for irrational rotation C*-algebras (see §3).

In [50], Rieffel showed that if θ and θ' are in the same orbit of the action of $GL(2;\mathbb{Z})$ on the irrational numbers in [0,1) which is defined by the corresponding linear fractional transformation, then $A_{\theta'}$ is strong Morita equivalent to A_θ despite the fact that $A_{\theta'}$ is $*$-isomorphic to A_θ if and only if $\theta' = \theta$ or $1 - \theta$. This answers the question of how different these non-isomorphic irrational rotation C*-algebras are. Rieffel further showed in [52] that all rational rotation C*-algebras, that is, A_θ for which θ is rational, are strong Morita equivalent to algebra $C(\mathbb{T}^2)$ of all continuous complex valued functions on $\mathbb{T}^2$.

Let $F_{k,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$ be the Furstenberg transformation defined by $F_{k,\theta}(z_1,z_2) = (z_1 z_2^k, e^{2\pi i\theta} z_2)$, $k \neq 0$. Then following the idea of Rieffel in [50], we will prove that if θ is irrational, then $A_{F_{m,\theta'}} = C(\mathbb{T}^2) \times_{F_{m,\theta'}} \mathbb{Z}$ is strong Morita equivalent to $A_{F_{k,\theta}}$ if and only if $|m| = |k|$ and θ' is in the same orbit of θ under the action of $GL(2;\mathbb{Z})$. If θ is rational, then $A_{F_{k,\theta}}$ is strong Morita equivalent to $A_{F_{f,0}}$. These results are similar to those obtained by Rieffel in [50] and [52] for rotation C*-algebras A_θ. As a byproduct, we also obtain the strong Morita equivalence for C*-algebras associated with minimal rotations on tori. Therefore, we are able to combine all our previous results into one uniform version.

The structure of this thesis is organized as follows. In Section 1 we give basic definitions and results on

C*-dynamical systems as well as on topological dynamical systems and on crossed product C*-algebras. Especially, we introduce the notion of "Furstenberg transformations" on tori and state and prove some results on these dynamical systems. Section 2 is devoted to the study and computation of the K-groups of the C*-algebras associated with a Furstenberg transformation and to computing the range of a tracial state on the K_0-group of such a C*-algebra. In Section 3, we classify the C*-algebras associated with some special classes of Furstenberg transformations on $\mathbb{T}^n$. In particular, we determine the strong Morita equivalence for the C*-algebras associated with affine transformations on the 2-torus and also determine the strong Morita equivalence for the C*-algebras associated with minimal rotations on tori as a by-product in §4. In Section 5, we simply combine all previous results into one uniform version. Section 6 is full of questions, conjectures and remarks on further developments. Finally, we give a proof for the integrality of the Chern character from $K^*(\mathbb{T}^n)$ into $H^*(\mathbb{T}^n;\mathbb{Q})$ in the Appendix.

§1. Preliminaries

In this section we will introduce some basic concepts and results concerning dynamical systems, crossed product C*-algebras and tracial states. We will determine when the crossed product C*-algebra of a C*-dynamical system has a unique tracial state and when a tracial state on a crossed product C*-algebra can be realized. Moreover, we are going to introduce the notion of a "Furstenberg transformation" on the n-dimensional torus $\mathbb{T}^n$ and state and prove results concerning these dynamical systems.

(1.1) Let G be a locally compact, Hausdorff, topological group with the left invariant Haar measure dt. A be a C*-algebra and $\alpha : G \to \mathrm{Aut}\, A$ be a strongly continuous representation, i.e. $\alpha_g(x)$ is continuous in g for each fixed x in A and in the norm on A. If, in addition, G and A are separable, then (A,α,G) is called a C*-dynamical system. If $A = C_0(X)$ is the commutative C*-algebra of all continuous functions on X with values in $\mathbb{C}$ which vanish at infinity, then any automorphism of A is induced from a homeomorphism of X and conversely. In this case, we denote a dynamical system by (X,α,G), where $\alpha : G \to \mathrm{Homeo}(X)$ is strongly continuous in the sense that the induced $\alpha : G \to \mathrm{Aut}(C_0(X))$ is strongly continuous.

(1.2) Let $K(G,A)$ denote the collection of all continuous maps from G into A with compact support and let $\Delta : G \to \mathbb{R}$ be the modular function. If we define involution and convolution on K(G,A) by

$$y^*(s) = \Delta(s)^{-1}\alpha_s(y(s^{-1})^*)$$

and

$$(y\times z)(t) = \int_G y(s)\alpha_s(z(s^{-1}t))dt$$

for all y, z in K(G,A), then K(G,A) becomes a $*$-algebra with convolution as product. For each y in K(G,A), define $\|y\|_1 = \int_G \|y(t)\| dt$. Then K(G,A) is a normed algebra with an isometric involution and we let $L^1(G,A)$ denote its norm completion.

As shown in [38, §7.6], there exists a non-degenerate separable $*$-representation of $L^1(G,A)$ into the algebra of bounded linear operators on a separable Hilbert space.

(1.3) Definition. The universal representation (π_u, H_u) of $L^1(G,A)$ is the direct sum of all non-degenerate separable representations of $L^1(G,A)$ and the crossed product of (A,α,G) is the norm closure of $\pi_u(L^1(G,A))$ in $B(H_u)$, which we denote by $A\times_\alpha G$.

(1.4) Theorem (see [38, 7.6.6 Theorem]). For each C*-dynamical system (A,G,α) there is a covariant representation (π,u,H) such that $A\times_\alpha G$ is contained in $C^*(\pi(A)\cup u_G)$, and for any other covariant representation (π',u',H'), there is a unique representation (ρ,H') of $A\times_\alpha G$ so that $\pi' = \rho\circ\pi$ and $u' = \rho\circ u$. In particular, if A is unital, G is compact or discrete, then $A\times_\alpha G$ is just the C*-algebra generated by $\pi(A) \cup u_G$, which we denote by $C^*(\pi(A)\cup u_G)$. (Note, a covariant representation (π,u,H) of (A,G,α) means that (π,H) is a representation of A and u is a representation of the Group G in the unitary operators on H which satisfies $u_g\pi(a)u_g^* = \pi(\alpha_g(a))$, for all g in G and a in A.)

(1.5) There is also a notion of the "reduced crossed product" C*-algebra for a C*-dynamical system (A,α,G). Since in this thesis we deal mainly with the case when $G = \mathbb{Z}$ or $\mathbb{R}$, the "crossed product" and the "reduced crossed product" C*-algebras coincide. In fact, this is true for any amenable group G (see [38; 7.7.7 Theorem]). We will use crossed product without distinguishing between the two for amenable groups.

(1.6) Theorem ([38, 7.7.9]). Let (A,α,G) be a C*-dynamical system and $\mathcal{B}$ be a G-invariant C*-subalgebra of A. Suppose G is amenable. Then $\mathcal{B}\times_\alpha G$ is naturally contained in $A\times_\alpha G$ and if $\mathcal{B} \neq A$, then $\mathcal{B}\times_\alpha G \neq A\times_\alpha G$.

(1.7) If G is discrete and A is unital, then $A\times_\alpha G$ is equal to $C^*(\pi(A)\cup u_G)$ as in Theorem (1.4). We define a map from the dense $*$-subalgebra $\mathcal{B} = \left\{ \sum_{g\in G} a_g u_g \,\middle|\, \begin{array}{l} a_g \in \pi(A) \text{ and } a_g \neq 0 \text{ for only} \\ \text{finitely many } g \text{ in } G \end{array} \right\}$ of $A\times_\alpha G$ onto A by $\Phi : \sum_{g\in G} a_g u_g \mapsto a_e$, where e is the unit of G.

(1.8) Proposition (Zeller-Meier). Φ is norm reducing and hence Φ extends to a $*$-linear map from $A\times_\alpha G$ onto A. Moreover,

Φ is positive.

This proposition is essentially due to Zeller-Meier in [60], and is explicitly proved by Itoh as Theorem 4.1 in [25].

(1.9) Corollary. Assume G is discrete. If φ is a G-invariant tracial state on A, then φ can be extended to a tracial state $\tilde{\varphi}$ on $A\times_{\alpha}G$.

<u>Proof</u>. Since A is unital and G is discrete, we consider the dense $*$-subalgebra $\mathcal{B}$ in $A\times_{\alpha}G$ as described in (1.8) and the map Φ from $\mathcal{B}$ onto A. Define $\tilde{\varphi}$ on $\mathcal{B}$ by

$$\tilde{\varphi}\Big(\sum_{g\in G} a_g u_g\Big) = \varphi(a_e).$$

Then

$$\Big|\tilde{\varphi}\Big(\sum_{g\in G} a_g u_g\Big)\Big| = \Big|\varphi\Big(\Phi\Big(\sum_{g\in G} a_g u_g\Big)\Big)\Big| \le \Big\|\Phi\Big(\sum_{g\in G} a_g u_g\Big)\Big\| \le \Big\|\sum_{g\in G} a_g u_g\Big\|,$$

by Proposition (1.8). Furthermore, $\tilde{\varphi}$ is tracial on $\mathcal{B}$ as is easily checked by the traciality of φ. Therefore, $\tilde{\varphi}$ extends to a tracial state on $A\times_{\alpha}G$.

(1.10) <u>Remark</u>. The positive linear map $\Phi : A\times_{\alpha}G \to A$ is usually called the "canonical conditional expectation" and it can be shown to satisfy the following properties:

i) $\Phi(ab) = a\cdot\Phi(b)$ and $\Phi(ba) = \Phi(b)\cdot a$ for a in A and b in $A\times_{\alpha}G$; and

ii) $\Phi(b^*) = \Phi(b)^*$ for b in $A\times_{\alpha}G$.

One can define the notion of a "conditional expectation" with the properties above from a C*-algebra onto a subalgebra. It turns out to be useful for different purposes, see [49], [55] and [25], etc.

In the next few sections, we mainly study the very special C*-dynamical systems which arise when $G = \mathbb{Z}$ and $A = C(X)$, where X is a compact Hausdorff space on which G acts by homeomorphisms, or a compact abelian group on which G acts by rotations or by affine transformations. Later on, in §6, we will discuss the possible generalization of this special case to the case when A is a "non-commutative torus" and G acts by "affine transformations".

(1.11) We now let X be a fixed compact Hausdorff space with a probability measure μ, G a countable discrete group and α be a homomorphism from G to the measure preserving homeomorphisms of X. Then (X,α,G) is a topological dynamical system.

We adopt some definitions from [39].

(1) α is said to be minimal if every orbit of α is dense in X.

(2) α is said to be ergodic with respect to the G-invariant measure μ, if the only α-invariant Borel subsets of X are those which have measure either zero or one.

(3) Two topological dynamical systems are said to be conjugate, denoted by $(X,\alpha,G) \sim (Y,\beta,G)$, if there is a homeomorphism $\phi : X \to Y$ so that $\phi \circ \alpha_g = \beta_g \circ \phi$ for g in G.

(4) α is said to be topologically transitive if there is a point x in X so that the orbit of G on x is dense in X.

(5) G acts uniquely ergodically on X if there is only one G-invariant probability measure on X.

(6) A complex number λ is said to be an eigenvalue of α_g for some g in G, if there is a complex valued <u>invertible</u> and <u>continuous</u> function f on X, so that $f(\alpha_g(x)) = \lambda f(x)$ for all x in X. For basic properties these concepts, see [39] or [58] .

(1.12) <u>Proposition</u>. Let (X,α,G) be a topological dynamical system. Assume for each g in G, there is an eigenvalue $\lambda_g \neq 1$ for α_g. Then tracial states on $C(X) \times_\alpha G$ are in one-one correspondence with G-invariant probability measures on X.

<u>Proof</u>. By Theorem (1.3), there is a covariant representation (π,u,H) of $(C(X),\alpha,G)$ such that $C(X)\times_\alpha G = C^*(\pi(C(X))Uu_G)$. Identify $\pi(C(X))$ with $C(X)$. We need only to show for any tracial state τ on $C(X)\times_\alpha G$, g in $G\backslash\{e\}$, and f in $C(X)$ that $\tau(f\cdot u_g) = 0$.

Since α_g has an eigenvalue $\lambda_g \neq 1$, one has a non-vanishing function f_g on X (see Definition (6) in (1.11)) so that

$$u_g \cdot f_g \cdot u_g^* = \alpha_g(f_g) = \lambda_g \cdot f_g.$$

This implies $\tau(f \cdot u_g) = \tau(f \cdot u_g \cdot f_g \cdot f_g^{-1}) = \tau(f \cdot u_g \cdot f_g \cdot u_g^* u_g \cdot f_g^{-1})$ $= \tau(f \cdot \lambda_g \cdot f_g \cdot u_g \cdot f_g^{-1}) = \lambda_g \tau(f \cdot u_g)$. Since $\lambda_g \neq 1$. It follows that $\tau(f \cdot u_g) = 0$.

(1.13) Remark. We have obtained a much more general statement for certain C*-dynamical systems which generalizes this proposition and which leads to the study of tracial states on crossed product C*-algebras for "homogeneous dynamical system," see [26].

(1.14) Corollary. If (X,α,G) satisfies the hypotheses of the previous proposition and G acts on X uniquely ergodically then $C(X)\times_\alpha G$ has a unique tracial state.

To give Proposition (1.11) and Corollary (1.14) some meaning we have to determine when the hypotheses are fulfilled.

(1.15) Let T be a homeomorphism of the compact metric space X. The dynamical system $(X,T,\mathbb{Z})$ is called equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$, so that $d(x,y) < \delta$ for x,y in X, implies $d(T^n x, T^n y) < \varepsilon$, for all n in $\mathbb{Z}$. $(X,T,\mathbb{Z})$ is called distal if $x \neq y$, x and y in X implies

$\inf_n d(T^n x, T^n y) > 0$, where d is the metric.

Clearly an equicontinuous system is distal.

(1.16) Theorem (see [39; Corollary 2.21]). A nontrivial minimal distal dynamical system $(X,T,\mathbb{Z})$ has a non-constant eigenfunction.

(1.17) Corollary. A nontrivial minimal distal system $(X,T,\mathbb{Z})$ with X connected has infinitely many eigenvalues not equal to 1.

Proof. Since T has a nontrivial eigenfunction f which is continuous, $T(f) = \lambda f$ for some λ in $\mathbb{C}|\{0\}$. We claim $\lambda \neq 1$, and λ is unimodular. We first prove λ is unimodular. If not, then $f \circ T^n = \lambda^n f$ would imply for any x in X, that $|f(x)| = 0$ when $|\lambda| > 1$ or $|f(x)| = \infty$ when $|\lambda| < 1$ since $0 < C_1 < |f(x)| < C_2 < +\infty$ for all x in X. Next we show $\lambda \neq 1$. Suppose to the contrary, then $f \circ T = f$, i.e. $f(Tx) = f(x)$ for all x in X, and hence $f(T^n x) \equiv f(x)$ for n in $\mathbb{Z}$. Since T is minimal, $\{T^n x | n \in \mathbb{Z}\}$ is dense in X, and therefore, f has to be constant. This contradicts the assumption that f is not constant. Now, if $\lambda^n = 1$, we define an equivalence relation on X so that $x \sim y$ if $f(x) = f(y)$. Then every $T^{n \cdot k} x$ is equivalent to x. Since f is continuous, each equivalence class is a closed subset of X. Since $\overline{\{T^{nk} x | k \in \mathbb{Z}\}}$ is contained in the equivalence class S_x of x, $T^i S_x$ contains $\overline{\{T^{nk+i} x | k \in \mathbb{Z}\}}$ for $i = 1, \cdots, n-1$.

Now since T is minimal, it follows that $\bigcup_{i=0}^{n-1} T^i S_x = X$. By the connectedness of X, X cannot be the disjoint union of finitely many closed subsets. Therefore, S_x must intersect some $T^i S_x$ and hence $T^i S_x \subset S_x$. It follows that $S_x = X$, i.e., f is a constant which contradicts the assumption. This shows that $\lambda^n \neq 1$ for any non-zero integer n. The corollary follows.

The proof above, in fact, proves the following corollary.

(1.18) <u>Corollary</u>. If $(X,T,\mathbb{Z})$ is topologically transitive and X is connected, then a non-trivial eigenvalue λ of T is not periodic and $|\lambda| = 1$.

(1.19) <u>Corollary</u>. Let $X = \mathbb{T}^n$ be the n-dimensional torus as a compact abelian group, $\mathbb{Z}$ be a dense subgroup generated by x_0 in X. If α is the representation of $\mathbb{Z}$ defined by left translation, then α_{x_0} has infinitely many eigenvalues.

<u>Proof</u>. Such a dynamical system is distal, uniquely ergodic and minimal, see [39; Theorem 2.11 and Proposition 2.10].

(1.20) When X is the unit circle $\mathbb{T}^1$ in $\mathbb{C}$, $\mathbb{Z} \cong \{\lambda^n | n\in\mathbb{Z}$ and $\lambda = e^{2\pi i\theta}$, with θ irrational$\}$ is represented as irrational rotations, the crossed product C*-algebra $C(\mathbb{T}^1)\times_\theta\mathbb{Z} = A_\theta$ which is called the irrational rotation C*-algebra with rotation number θ, has been intensively studied with much understanding, [50], [43], [44] and [52], etc. When X is abelian and $\mathbb{Z} \cong \{x_0^n | n\in\mathbb{Z}, x_0\in X\}$ is dense in X, the crossed product

C*-algebra $C(X)\times_\alpha \mathbb{Z}$ has also been studied, see [48]. From the topological point of view, rotations on a connected compact group are all homotopic to the identity map of the group as a topological space. To obtain some different examples, one seeks transformations on a connected space which are not homotopic to the identity. This simple observation leads us to study the following dynamical system which is based on the work of Furstenberg [20].

(1.21) Definition. A Furstenberg transformation $F_{f,\theta}$ on the n-torus $\mathbb{T}^n = \{(z_1,\cdots,z_n) | z_1,\cdots,z_n \in \mathbb{T}^1\}$ is defined as follows, $F_{f,\theta}(z_1,\cdots z_n) = (z_1 \cdot f_1(z_2,\cdots,z_n),\cdots,z_{n-1} \cdot f_{n-1}(z_n), e^{2i\pi\theta} \cdot z_n)$, where θ lies in $[0,1)$ and each f_j is a continuous function with $|f_j| = 1$, and $f_j(z_{j+1},\cdots,z_n)$ is homotopic to $z_{j+1}^{d_j}$ as a function $f_j(\cdot,z_{j+2},\cdots,z_n) : \mathbb{T}^1 \to \mathbb{T}^1$ for all $(z_{j+2},\cdots,z_n)$ in $\mathbb{T}^{n-(j+1)}$, where $d_j \neq 0$ is an integer.

We will say that a Furstenberg $F_{f,\theta}$ satisfies condition (A), if θ is irrational and each f_j satisfies a uniform Lipschitz condition in z_{j+1}, that is

$$|f_j(z_{j+1},\cdots,z_n) - f_j(z'_{j+1},z_{j+2},\cdots,z_n)| < M|z_{j+1}-z'_{j+1}|$$

for some constant M and all $z_{j+2},\cdots,z_n$ in $\mathbb{T}^1$.

A Furstenberg transformation $F_{f,\theta}$ is said to satisfy condition (B), if θ is irrational and $f_j(\overline{z}_{j+1},\cdots,\overline{z}_n) = \overline{f_j(z_{j+1},\cdots,z_n)}$ for $j = 1, 2,\cdots,n-1$.

(1.22) It is clear that a Furstenberg transformation $F_{f,\theta}$ is not homotopic to the identity. The following theorem can be found in [20].

(1.23) <u>Theorem</u> (Furstenberg). If $F_{f,\theta}$ satisfies condition (A), then it is minimal, uniquely ergodic and has infinitely many distinct eigenvalues.

(1.24) <u>Corollary</u>. If $F_{f,\theta}$ satisfies condition (A), then the crossed product C*-algebra $A_{F_{f,\theta}} = C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ is simple and has a unique tracial state.

<u>Remark</u>. The simplicity of the crossed product C*-algebra in this corollary follows from work of S. Power [46] or of R. Powers [47].

(1.25) In general, the crossed product C*-algebra $C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ is neither simple nor does it have a unique tracial state. The reason we are interested in these C*-algebras is as we indicated in (1.22), that Furstenberg transformations are not homotopic to the identity, which contrasts with the case of rotations on a connected compact group. In the later situation, questions concerning the structure of crossed product C*-algebras are easier to study since the rotations themselves are easier to understand. There is lots of nice work related to this subject, see e.g. [50], [43], [44], [52], [48],

[14], and [59]. On the other hand, when $n = 2$, θ is irrational and the map $F_{f,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$, $F_{f,\theta}(z_1,z_2) = (z_1 \cdot z_2, e^{2\pi i\theta} \cdot z_2)$, is also called the Anzai skew transformation on the 2-torus (see [39]). The structure of the associated crossed product C*-algebras has been studied by J. Packer in a quite complete fashion (see [34]). Moreover, since the C*-algebra associated with a Furstenberg transformation $F_{f,\theta}$ with θ in $[0,1)$ contains the rotation algebra A_θ even when θ is not irrational, in a natural way, it is natural to view the study of these C*-algebras as an extension of the work of Rieffel in [50] and Pimsner and Voiculescu in [43] and also Packer in [34]. Besides, when we specialize the functions which is used in the definition of Furstenberg transformation to be $f_i(z_{i+1},\cdots,z_n) = z_{i+1}^{k^i_{i+1}} \cdots z_n^{k^i_n}$, with $k^i_{i+1} \neq 0$ and where the k^i_j's are integers, $i = 1, 2, \cdots, n-1$, then $F_{f,\theta}$ becomes an affine transformation of the compact abelian group $\mathbb{T}^n$.

(1.26) <u>Definition</u>. An affine transformation $T = g \cdot A$ on a compact group G consists an automorphism A of the group G followed by a translation of a group element g in G.

Clearly, the affine transformation $\tilde{T} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \circ \begin{pmatrix} k^1_1 & \cdots & k^1_n \\ \vdots & \ddots & \vdots \\ k^n_1 & \cdots & k^n_n \end{pmatrix}$

on $\mathbb{R}^n$, where k^i_j's are integers, $\det((k^i_j)_{n\times n}) = \pm 1$, and $\theta_1, \cdots, \theta_n$ are in $\mathbb{R}^1$, can be viewed as the affine transformation

$T : \mathbb{T}^n \to \mathbb{T}^n$ defined by

$$T(z_1,\cdots,z_n) = (\lambda_1 z_1^{k_1^1}\cdots z_n^{k_n^1},\cdots,\lambda_n z_1^{k_1^n}\cdots z_n^{k_n^n}).$$

In particular, $\tilde{T} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \theta \end{bmatrix} \circ \begin{bmatrix} 1 & k_2^1 & \cdots & k_n^1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & k_n^{n-1} \\ & & & 1 \end{bmatrix}$ with $k_{i+1}^i \neq 0$, $i=2,\cdots n-1$,

gives a Furstenberg transformation T on $\mathbb{T}^n$ satisfying condition (A) and (B) if θ is irrational in $[0,1)$.

Our task is to study and classify the C*-algebras associated with these Furstenberg transformations.

(1.27) <u>Proposition</u>. The rotation C*-algebra $A_\theta = C(\mathbb{T}^1)\times_\theta \mathbb{Z}$, where θ is in $[0,1)$, is naturally contained in $C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z} = A_{F_{f,\theta}}$.

<u>Proof</u>. Let $C(\mathbb{T}^1)$ be the C*-subalgebra of $C(\mathbb{T}^n)$ generated by the last coordinate function z_n. Then $C(\mathbb{T}^1)$ is $F_{f,\theta}$ invariant and $F_{f,\theta} : \mathbb{T}^1 \to \mathbb{T}^1$ by $F_{f,\theta}(z_n) = e^{2\pi i\theta}\cdot z_n$ is just the rotation on $\mathbb{T}^1$ by θ. Hence, A_θ is naturally contained in $A_{F_{f,\theta}}$ by Theorem (1.4).

§2. K-theory and traces on K_0-groups

This section is devoted to computing the K-groups of the crossed product C*-algebras associated with Furstenberg transformations on $\mathbb{T}^n$. Unlike the case when crossed product C*-algebras are given by rotation on C*(G) determined by a character of G into $\mathbb{T}^1$, where G is a discrete torsion free group, see [48], [13] and [59], we usually have torsion subgroups in $K_*(A_{F_{f,\theta}})$. These are important factors in determining isomorphism classes of these C*-algebras. Another important invariant for these C*-algebras is the range of a tracial state on the K-group. We will compute it using results in [44] by looking at the "Rieffel projections" in irrational or rational rotation C*-algebras and by employing the techniques developed in [44] and [41], as well as by applying the theorem concerning the integrality of the Chern character on $K^*(\mathbb{T}^n)$ in the appendix. For references on K-theory, see [7], [3], and [56].

(2.1) In [44], Pimsner and Voiculescu established a six term exact sequence for crossed product C*-algebras associated with a C*-dynamical system $(A,\alpha,\mathbb{Z})$,

(2.2)
$$\begin{tikzcd}
K_0(A) \arrow[r, "\alpha_*^{-1} - I"] & K_0(A) \arrow[r, "i_*"] & K_0(A\times_\alpha \mathbb{Z}) \arrow[d, "\delta"] \\
K_1(A\times_\alpha \mathbb{Z}) \arrow[u, "\delta"] & K_1(A) \arrow[l, "i_*"'] & K_1(A) \arrow[l, "\alpha_*^{-1} - I"']
\end{tikzcd}$$

(2.3) The map δ involved in this sequence is described as follows. If S is the unilateral shift on $\ell^2(\mathbb{Z}^+)$, then S is a partial isometry, and the C*-algebra C*(S) generated by S on $\ell^2(\mathbb{Z}^+)$, contains all compact operators K. Define T to be the C*-subalgebra of $(A\times_\alpha\mathbb{Z}) \otimes C^*(S)$ generated by $A \otimes I$ and $u_\alpha \otimes S$, where I is the identity operator on $\ell^2(\mathbb{Z}^+)$ and u is the unitary operator in the covariant representation of $A\times_\alpha\mathbb{Z} = C^*(A,u_\alpha)$. Then as was shown in [44], the following sequence is exact.

(2.4) $$0 \to A\otimes K \xrightarrow{\psi} T \xrightarrow{\pi} A\times_\alpha\mathbb{Z} \to 0$$

for some maps ψ and π. If $e_{oo} = \begin{bmatrix} 1 & 0\cdots0\cdots \\ 0 & 0\cdots0\cdots \\ \vdots & \ddots \quad \vdots \\ \vdots & \quad \ddots \vdots \end{bmatrix}$ is the rank one projection in $K = \underset{n}{\lim}\, M_n(\mathbb{C})$ and i denotes the inclusion $i : A \to A\otimes K$ define to be $i : A \ni a \mapsto a\otimes e_{oo} \in A\otimes K$, then it is easy to see that i induces an isomorphism $K_\varepsilon(A) \to K_\varepsilon(A\otimes K)$ for $\varepsilon=0, 1$. Associated with the short exact sequence $(*)$, there is a long exact sequence for K-groups, see [56] or [3], $\cdots \to K_\varepsilon(A\otimes K) \xrightarrow{\psi_*} K_\varepsilon(T) \xrightarrow{\pi_*}$ $K_i(A\times_\alpha\mathbb{Z}) \xrightarrow{\partial} K_{i-1}(A\otimes K) \to \cdots$. We define $\delta = i_*^{-1} \circ \partial$, and all the other maps in the six term exact sequence (2.2) are clear.

(2.5) We are now going to prove a lemma concerning the naturality of the Pimsner-Voiculescu sequence in an appropriate sense. Although it is implicit in [44], there is no explicit statement of it.

(2.6) Definition. Given C*-dynamical systems (A,α,G) and $(\mathcal{B},\beta,G)$, a $*$-homomorphism $\phi : A \to \mathcal{B}$ is called covariant if $\phi(\alpha_g(a)) = \beta_g(\phi(a))$ for all a in A and g in G.

It is clear that a covariant $*$-homomorphism induces a map $\tilde{\phi} : K(G,A) \to K(G,\mathcal{B})$ defined by

$$\tilde{\phi}(f)(g) = \phi(f(g)).$$

One can check that

$$\begin{aligned}\tilde{\phi}(f^*)(g) &= \phi(f^*(g)) = \phi(\Delta(g)^{-1}\alpha_g(f(g^{-1})^*)) \\ &= \Delta(g)^{-1}\beta_g(\phi(f(g^{-1})^*)) = \Delta(g)^{-1}\beta_g([\tilde{\phi}(f)(g^{-1})]^*) \\ &= (\tilde{\phi}(f))^*(g),\end{aligned}$$

and therefore, $\tilde{\phi}$ is a $*$-homomorphism from $K(G,A)$ to $K(G,\mathcal{B})$. Hence $\tilde{\phi}$ extends to a $*$-homomorphism from $A\times_\alpha G$ into $\mathcal{B}\times_\alpha G$ by the universal property. The injectivity, surjectivity as well as bijectivity of $\tilde{\phi}$ are equivalent to those of ϕ and thus we have the following proposition.

(2.7) Proposition. Let $\phi : A \to \mathcal{B}$ be a covariant $*$-homomorphism with respect to C*-dynamical systems (A,α,G) and $(\mathcal{B},\beta,G)$. Then ϕ induces a $*$-homomorphism $\tilde{\phi} : A\times_\alpha G \to \mathcal{B}\times_\beta G$ such that ϕ is injective or surjective or bijective, iff the same is true for $\tilde{\phi}$.

(2.8) Proposition. The Pimsner-Voiculescu sequence (2.2) is natural in the sense that if $(A,\alpha,\mathbb{Z})$ and $(\mathcal{B},\beta,\mathbb{Z})$ are C*-dynamical systems and $\phi : A \to B$ is covariant, then the following diagram is commutative.

$$
\begin{tikzcd}
K_0(A) \arrow[rr, "\alpha_*^{-1} - I"] \arrow[dr, "\phi_*"] & & K_0(A) \arrow[rr, "i_*"] \arrow[d, "\phi_*"] & & K_0(A \times_\alpha \mathbb{Z}) \arrow[dl, "\widetilde{\phi}_*"'] \arrow[ddd, "\delta"] \\
& K_0(B) \arrow[r, "\beta_*^{-1} - I"] & K_0(B) \arrow[r, "j_*"] & K_0(B \times_\beta \mathbb{Z}) \arrow[d, "\delta'"] & \\
& K_1(B \times_\beta \mathbb{Z}) \arrow[u, "\delta'"'] & K_1(B) \arrow[l, "j_*"'] & K_1(B) \arrow[l, "\beta_*^{-1} - I"'] & \\
K_1(A \times_\alpha \mathbb{Z}) \arrow[uuu, "\delta"] \arrow[ur, "\widetilde{\phi}_*"'] & & K_1(A) \arrow[ll, "i_*"'] \arrow[u, "\phi_*"] & & K_1(A) \arrow[ll, "\alpha_*^{-1} - I"'] \arrow[ul, "\phi_*"]
\end{tikzcd}
$$

Proof. The only uncertain part of the diagram is that

$$
\begin{tikzcd}
K_\varepsilon(A \times_\alpha \mathbb{Z}) \arrow[r, "\widetilde{\phi}_*"] \arrow[d, "\delta"'] & K_\varepsilon(B \times_\beta \mathbb{Z}) \arrow[d, "\delta'"] \\
K_{\varepsilon-1}(A) \arrow[r, "\widetilde{\phi}_*"] & K_{\varepsilon-1}(B)
\end{tikzcd}
\tag{2.9}
$$

But the commutativity of this rectangle is a consequence of the commutativity of the following diagram.

(2.10)
$$\begin{tikzcd}
& K_{\varepsilon}(A\times_{\alpha}\mathbb{Z}) \arrow[rr, "\tilde{\phi}_*"] \arrow[dl, "\partial"'] & & K_{\varepsilon}(B\times_{\beta}\mathbb{Z}) \arrow[dr, "\partial'"] & \\
K_{\varepsilon-1}(A\otimes K) \arrow[rrrr, "(\phi\otimes 1)_*"] & & & & K_{\varepsilon-1}(B\otimes K) \\
& K_{\varepsilon-1}(A) \arrow[ul, "i_*"] \arrow[rr, "\phi_*"] & & K_{\varepsilon-1}(B) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

The commutativity of the upper rectangle is from the naturality of K-theory for the commutative diagram of the exact sequences,

(2.11)
$$\begin{tikzcd}
0 \arrow[r] & A\otimes K \arrow[r, "\psi_A"] \arrow[d, "\phi\otimes I"'] & T_A \arrow[r, "\pi_A"] \arrow[d, "\gamma"] & A\times_{\alpha}\mathbb{Z} \arrow[r] \arrow[d, "\tilde{\phi}"] & 0 \\
0 \arrow[r] & B\otimes K \arrow[r, "\psi_B"] & T_B \arrow[r, "\pi_B"] & B\times_{\beta}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where γ is induced by mapping $A\otimes I$ to $B\otimes I$ via $\phi\otimes I$ and mapping $u_{\alpha}\otimes S$ to $u_{\beta}\otimes S$. Of course, one has to check if γ is actually a $*$-homomorphism, but this is true since it is just the restriction of $\tilde{\phi}\otimes 1 : A\times_{\alpha}\mathbb{Z} \otimes C^*(S) \to B\times_{\beta}\mathbb{Z} \otimes C^*(S)$ to T_A. The commutativity of the diagram (2.11) is just the matter of looking at the maps ψ_A, ψ_B and π_A, π_B constructed in [44]. Next, the commutativity of the lower rectangle in (2.10) is trivial. Since the maps i_* and j_* are isomorphisms, we immediately see that the diagram (2.9) is commutative. This completes the proof.

(2.12) Let $F_{f,\theta}$ be a Fursterberg transformation on $\mathbb{T}^n$. Then we have the corresponding cyclic six term exact sequence for the crossed product C*-algebra $A_{F_{f,\theta}} = C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ as in (2.2). In order to compute $K_*(A_{F_{f,\theta}})$, one has to know exactly how $(F^{-1}_{f,\theta})_*$ acts on generators of $K_*(C(\mathbb{T}^n))$. To do this, we need the following theorem and lemma.

(2.13) Theorem. $K_*(C(\mathbb{T}^n))$, $K^*(\mathbb{T}^n)$, $\check{H}^*(\mathbb{T}^n;\mathbb{Z})$ and $\Lambda^*\check{H}^1(\mathbb{T}^n;\mathbb{Z})$ are all naturally isomorphic as rings, where $\check{H}^*$ denotes the Čech cohomology theory (see, the Appendix).

Remark. The first homomorphism $K_*(C(\mathbb{T}^n)) \cong K^*(\mathbb{T}^n)$ is the standard isomorphism given in [33].

(2.14) Lemma. Let $F_{f,\theta} : \mathbb{T}^n \to \mathbb{T}^n$ be a Furstenberg transformation, then $F_{f,\theta}$ is homotopic to an automorphism $K : \mathbb{T}^n \to \mathbb{T}^n$. We call K the representative of $F_{f,\theta}$, where K has the form $K(z_1,\cdots,z_n) = (z_1 z_2^{k_2^1}\cdots z_n^{k_n^1},\cdots,z_{n-1}z_n^{k_n^{n-1}},z_n)$ for integers k_j^i, $1 \le i < j \le n$.

Proof. Each continuous function $f_k : \mathbb{T}^k \to \mathbb{T}^1$ is homotopic to $z_1^{i_1}\cdots z_k^{i_k}$, where the z_i's are the coordinate functions of $\mathbb{T}^k$, since the cohomotopy group $\pi^1(\mathbb{T}^k) \cong \check{H}^1(\mathbb{T}^k) \cong \mathbb{Z}^k$ and the coordinate functions $z_1,\cdots,z_k$ give the canonical generators of $\check{H}^1(\mathbb{T}^k)$, (see [56, 3.9]). Therefore, $F_{f,\theta}$ is homotopic to K with $k_{i+1}^i \neq 0$, $i = 1,2,\cdots,n-1$.

(2.15) By this lemma, for the purpose of computing the K-theory of the C*-algebra $C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z} = A_{F_{f,\theta}}$, it is enough to consider the case when $F_{f,\theta} = K$. But, explicitly computing the K-groups of $C(\mathbb{T}^n)\times_K\mathbb{Z}$ is still not an easy matter. We have to restrict ourselves to the following special class of automorphisms.

(2.16) <u>Definition</u>. A sequence of non-zero integers $\{k_i,\cdots,k_{n-1}\}$ is said to be <u>descending</u> if $|k_{i+1}|\,\big|\,|k_i|$ for all i. A Furstenberg transformation $F_{f,\theta}$ with respective $K: \mathbb{T}^n \to \mathbb{T}^n$ of the form $K(z_1,\cdots,z_n) = (z_1 z_2^{k_1},\cdots,z_{n-1}z_n^{k_{n-1}},z_n)$, s.t. $\{k_1,\cdots,k_{n-1}\}$ is a descending sequence is called a <u>descending transformation.</u>

(2.17) <u>Proposition</u>. If $F_{f,\theta}$ is a descending transformation with representative $K : \mathbb{T}^n \to \mathbb{T}^n$ and descending sequence $\{k_1,\cdots k_{n-1}\}$, then the torsion subgroup of $K_*(C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus\cdots\oplus \mathbb{Z}_{k_{n-1}}^{(m_{n-1})}$, where $\mathbb{Z}_{k_i}^{(m_i)}$ is the group of m_i-copies of the cyclic group $\mathbb{Z}_{k_i} = \mathbb{Z}/k_i\mathbb{Z}$.

To prove this proposition we also need the following standard algebraic lemma, although it is more general than we need, (see for example: "Abstract Algebra," Theorem 7, p. 168, by C.H. Sah).

(2.18) <u>Lemma</u>. If $K = (k_{ij})_{n\times n}$ is an integral matrix, then $\mathbb{Z}^{(n)}/K\,\mathbb{Z}^{(n)} \cong \mathbb{Z}^{(n)}/S\cdot K\cdot T\,\mathbb{Z}^{(n)}$, for S and T in $GL(n;\mathbb{Z})$. Moreover, $\mathbb{Z}^{(n)}/K\mathbb{Z}^{(n)} \cong \mathbb{Z}^{(m)}\oplus \mathbb{Z}_{k_1} \oplus\cdots\oplus \mathbb{Z}_{k_i}$ for some $m \le n$ and integers

$k_1, \cdots, k_i$ depending on K, $m+i = n$.

We now turn to the proof of Proposition (2.17).

(2.19) Let $[z_{i_1}] \wedge [z_{i_1}] \wedge \cdots \wedge [z_{i_m}]$ be in $\Lambda^m \check{H}^1(\mathbb{T}^n)$ where z_i's are coordinate functions of $\mathbb{T}^n$, which gives a canonical basis of $\check{H}^1(\mathbb{T}^n)$. Then we have

$$K_*([z_{i_1}]\wedge\cdots\wedge[z_{i_m}]) = ([z_{i_1}]+k_{i_1}[z_{i_1+1}])\wedge\cdots\wedge([z_{i_m}]+k_{i_m}[z_{i_m+1}]).$$

We give an order to the basis of $\Lambda^*\check{H}^1(\mathbb{T}^n)$, which is the set $\{[z_{i_1}]\wedge\cdots\wedge[z_{i_k}] \mid i_1<\cdots<i_k,\ k=1,2,\cdots n\}$, by the following. $[z_{i_1}]\wedge\cdots\wedge[z_{i_k}] < [z_{j_1}]\wedge\cdots\wedge[z_{j_{k'}}]$ if $k < k'$ or $k = k'$ but the first $i_\ell \neq j_\ell$ satisfies $i_\ell < j_\ell$. With this ordering of the basis, K_* has a representation as a matrix:

See the following page.

$$
K_* \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_1 \wedge z_2 \\ \vdots \\ z_1 \wedge z_n \\ z_2 \wedge z_3 \\ \vdots \\ z_2 \wedge z_n \\ \vdots \\ z_{n-1} \wedge z_n \\ z_1 \wedge z_2 \wedge z_3 \\ z_1 \wedge z_2 \wedge z_4 \\ \vdots \\ z_1 \wedge z_2 \wedge z_n \\ \vdots \\ z_{n-2} \wedge z_{n-1} \wedge z_n \\ \vdots \\ \vdots \\ \vdots \\ z_1 \wedge \cdots \wedge z_n \end{pmatrix}
=
\begin{pmatrix}
1 & k_1 & & & & & & & & & & & & & & & & & & & \\
 & 1 & k_2 & \ddots & k_{n-1} & & & & & & & & & & & & & & & & \\
 & & \ddots & \ddots & 1 & 0 & & & & & & & & & & & & & & & \\
 & & & & 0 & 0 & & & & & & & & & & & & & & & \\
 & & & & & 1 & k_2 & * & \cdots & * & & & & & & & & & & & \\
 & & & & & & 1 & k_3 & * & \cdots & * & & & & & & & & & & \\
 & & & & & & & \ddots & \ddots & & & & & & & & & & & & \\
 & & & & & & & & k_{n-1} & * & \cdots & * & & & & & & & & & \\
 & & & & & & & & 1 & 0 & \cdots & 0 & & & & & & & & & \\
 & & & & & & & & 0 & 0 & \cdots & 0 & & & & & & & & & \\
 & & & & & & & & & & & & 1 & k_3 & * & \cdots & * & & & & \\
 & & & & & & & & & & & & & 1 & k_4 & * & \cdots & * & & & \\
 & & & & & & & & & & & & & & \ddots & \ddots & & & & & \\
 & & & & & & & & & & & & & & & k_{n-1} & * & \cdots & * & & \\
 & & & & & & & & & & & & & & & 1 & 0 & \cdots & 0 & & \\
 & & & & & & & & & & & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & & & & & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & & & & & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & & & & & & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & & & & & & & & & & 1 & k_{n-2} & * & * & \\
 & & & & & & & & & & & & & & & & & 1 & 0 & 0 & \\
 & & & & & & & & & & & & & & & & & 0 & 0 & 0 & \\
 & & & & & & & & & & & & & & & & & & & 1 &
\end{pmatrix}_{\ell \times \ell}
\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_1 \wedge z_2 \\ \vdots \\ z_1 \wedge z_n \\ \vdots \\ z_{n-1} \wedge z_n \\ z_1 \wedge z_2 \wedge z_3 \\ \vdots \\ z_1 \wedge z_2 \wedge z_n \\ \vdots \\ z_{n-2} \wedge z_{n-1} \wedge z_n \\ \vdots \\ \vdots \\ \vdots \\ z_1 \wedge \cdots \wedge z_n \end{pmatrix} .
$$

where all $*$'s following k_i can be devided by k_i.

To show this, let $a_{i_j} = [z_{i_j}]$, $a_{i_{j_1}\cdots i_{j_\ell}} = [z_{i_{j_1}}]\wedge\cdots\wedge[z_{i_{j_\ell}}]$ and let $k_{i_{j_1}}\cdots k_{i_{j_\ell}} = k_{i_{j_1}\cdots i_{j_\ell}}$, then

$$K_*([z_{i_1}]\wedge\cdots\wedge[z_{i_m}]) = (a_{i_1}+k_{i_1}a_{i_1+1})\wedge\cdots\wedge(a_{i_m}+k_{i_m}a_{i_m+1})$$

$$= a_{i_1\cdots i_m} + k_{i_m}a_{i_1\cdots i_{m-1}(i_m+1)} + k_{i_{m-1}}a_{i_1\cdots i_{m-2}(i_{m-1}+1)i_m}+\cdots+$$

$$+k_{j_1\cdots j_\ell}a_{i_1\cdots i_{(j_1-1)}(i_{j_1}+1)i_{(j_1+1)}\cdots i_{(j_\ell-1)}(i_{j_\ell}+1)i_{(j_\ell+1)}\cdots i_m}+\cdots+$$

$$+k_{i_1\cdots i_m}a_{(i_1+1)\cdots(i_m+1)}.$$

These $k_{j_1\cdots j_\ell}$'s are clearly divisible by k_{i_m} since $j_1<\cdots<j_\ell<i_m$, and $\{k_1,\ldots,k_{n-1}\}$ is a descending sequence.

Therefore,

$$K_* - I = \begin{pmatrix}
0 & k_1 & & & & & & & & & & & \\
 & 0 & k_2 & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & \ddots & k_{n-1} & & & & & & & & \\
 & & & & 0 & & & & & & & & \\
 & & & & & 0 & & & & & & & \\
 & & & & & & 0 & k_2 & *\cdots* & & & & \\
 & & & & & & & 0 & k_3 & *\cdots* & & & \\
 & & & & & & & & 0 \ddots & \ddots & \ddots & & \\
 & & & & & & & & \ddots & k_{n-1} & *\cdots* & & \\
 & & & & & & & & & 0 & 0\cdots0 & & \\
 & & & & & & & & & & \cdots\cdots & & \\
 & & & & & & & & & & 0 & k_{n-1} & \\
 & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & 0
\end{pmatrix}_{\ell\times\ell}$$

and $\mathbb{Z}^{(\ell)}/(K_*-I)\mathbb{Z}^{(\ell)} \cong \mathbb{Z}^{(\ell)}/S(K_*-I)T\,\mathbb{Z}^{(\ell)}$ for any S and T in $GL(\ell,\mathbb{Z})$. But clearly using S in $GL(\ell,\mathbb{Z})$ we can delete all entries denoted by $*$'s and $S(K_*-I)$ has the form

$$\begin{bmatrix}
0 & k_1 & & & & & & & & & & & \\
 & 0 & k_2 & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & \ddots & k_{n-1} & & & & & & & & \\
 & & & & 0 & & & & & & & & \\
 & & & & & 0 & 0 & & & & & & \\
 & & & & & & 0 & k_2 & & & & & \\
 & & & & & & & 0 & k_3 & & & & \\
 & & & & & & & & \ddots & \ddots & & & \\
 & & & & & & & & & \ddots & k_{n-1} & & \\
 & & & & & & & & & & 0 & 0 & \\
 & & & & & & & & & & & \ddots & \ddots \\
 & & & & & & & & & & & 0 & k_{n-1} \\
 & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & 0
\end{bmatrix}$$

Now, we can also choose S' and T in $GL(\ell;\mathbb{Z})$ so that $S'S(K_*-I)T$ has the form,

$$\begin{bmatrix}
0 & & & & & & & & & & & & \\
 & k_1 & & & & & & & & & & & \\
 & & k_2 & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & k_{n-1} & & & & & & & & \\
 & & & & & 0 & & & & & & & \\
 & & & & & & 0 & & & & & & \\
 & & & & & & & k_2 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & k_{n-1} & & & \\
 & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & & k_{n-1} \\
 & & & & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & & & & & 0
\end{bmatrix}$$

Hence, $\mathbb{Z}^{(\ell)}/(K_*-I)\mathbb{Z}^{(\ell)} \cong \mathbb{Z}^{(\ell)}/S'S(K_*-I)T\,\mathbb{Z}^{(\ell)} \cong$
$\cong \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus \cdots \oplus \mathbb{Z}_{k_{n-1}}^{(m_{n-1})}$. If instead of $(F^{-1}_{f,\theta})_*$ we use $(F_{f,\theta})_*$ in the Pimsner-Voiculescu exact sequence, we clearly have proved the proposition. But it is true that

$$\mathbb{Z}^{(\ell)}/((F^{-1}_{f,\theta})_*-I)\mathbb{Z}^{(\ell)} = \mathbb{Z}^{(\ell)}/(F^{-1}_{f,\theta})_*(I-(F_{f,\theta})_*)\mathbb{Z}^{(\ell)}$$

$$\cong \mathbb{Z}^{(\ell)}/(I-(F_{f,\theta})_*)\mathbb{Z}^{(\ell)} = \mathbb{Z}^{(\ell)}/((F_{f,\theta})_*-I)\mathbb{Z}^{(\ell)},$$

by Lemma (2.18). This completes the proof.

(2.20) <u>Corollary</u>. Let $n=2$, $F_{f,\theta}(z_1,z_2) = (z_1 \cdot f(z_2), e^{2\pi i\theta}\cdot z_2)$, and f be homotopic to z_2^k with $k \neq 0$. Then

$$K_0(C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

and $$K_1(C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/_{k\mathbb{Z}}).$$

<u>Proof</u>. Let K be the automorphism of the 2-dimensional torus, $K(z_1,z_2) = (z_1 \cdot z_2^k, z_2)$. Then since $K_0(C(\mathbb{T}^2))$ is generated by 1 in $\mathbb{Z} \cong \check{H}^0(\mathbb{T}^2)$ and $[z_1]\wedge[z_2]$ in $\check{H}^2(\mathbb{T}^2)$ we have

$K_*(1) = 1$ and $K_*([z_1]\wedge[z_2]) = ([z_1]+k[z_2]) \wedge [z_2] = [z_1]\wedge[z_2]$.

Since $K_1(C(\mathbb{T}^2))$ is generated by $[z_1]$ and $[z_2]$ in $\check{H}^1(\mathbb{T}^2)$, we have $K_*([z_1]) = [z_1] + k[z_2]$ and $K_*([z_2]) = [z_2]$. Hence we get, from the Pimsner-Voiculescu sequence in (2.2), that

$$K_0(C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

and $$K_1(C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/_{(k\mathbb{Z})}.$$

(2.21) We now turn our attention to computing the ranges of a tracial state on the K_0-group. For crossed product C*-algebras associated with affine transformations on $\mathbb{T}^n$, the range of a tracial state on the K_0-group has been calculated by several mathematicians, see e.g. [34], [18]. The methods used in these papers are essentially two, the Pimsner-Voiculescu sequence and the Connes' trace formula [8] for the special case. Both methods seem to require that one know exactly how many eigenvalues the affine transformations have. In our case, since $F_{f,\theta}$ is not, in general, an affine transformation on $\mathbb{T}^n$, we cannot directly apply those results. In the two-dimensional case, we had obtained results for $F_{f,\theta}$ similar to those for affine transformations, by looking at generators in the K-groups and using the Pimsner-Voiculescu sequence. More precisely, we used known results concerning rational and irrational rotation C*-algebras in [44, Appendix] combined with the naturality of the Pimsner-Voiculescu sequence, and showed that the range of any tracial state on the K_0-group of $A_{F_{f,\theta}} = C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}$ is exactly equal to that of the rotation algebra A_θ contained in $C(\mathbb{T}^2)\times_{F_{f,\theta}}\mathbb{Z}$, see (1.27). But in the case of higher dimension, the group seems to have difficulty taking account of all of the generators of $K_1(C(\mathbb{T}^n))$. Fortunately, some recent beautiful work of Pimsner [41] gives a way to systematize the

the proof we had, and enables us to "neglect" those extra generators in $K_1(C(\mathbb{T}^n))$. We state without proof, a theorem which is essentially Corollary 4 in [41].

(2.22) Let $(C(X),\alpha, \mathbb{Z})$ be a C*-dynamical system where X is a compact Hausdorff space. Let $\{x_1,\cdots,x_k\}$ be generators of the subgroup G of $K_1(C(X))$ which consists of all elements of $\bar{x}$ in the kernel of the map $\alpha_*^{-1} - I : K_1(C(X)) \to K_1(C(X))$ so that x is also represented by an invertible function in C(X). Choose projections $\{p_1,\cdots,p_k\}$ in $M_n(C(X))$ for some large integer n, such that in the Pimsner-Voiculescu sequence $\delta[p_i] = x_i$ for $i = 1,2,\cdots,k$. Then we have Pimsner's theorem,

<u>Theorem</u> (Pimsner). For any tracial state τ on $C(X)\times_\alpha\mathbb{Z}$, the range of τ on $K_0(C(X)\times_\alpha\mathbb{Z})$ is

$$\tau_*(K_0(C(X)\times_\alpha\mathbb{Z})) = \tau(K_0(C(X))) + \tau(p_1)\mathbb{Z}+\cdots+\tau(p_k)\mathbb{Z},$$

with the notations introduced above.

With this theorem, we can compute the range of any tracial state on $K_0(A_{F_{f,\theta}})$ for any higher dimensional Furstenberg transformation on $\mathbb{T}^n$.

(2.23) <u>Theorem</u>. Let $F_{f,\theta}$ be a Furstenberg transformation on $\mathbb{T}^n$, and τ be a tracial state on $A_{F_{f,\theta}}$. Then $\tau_*(K_0(A_{F_{f,\theta}}))$ $= \mathbb{Z} + \theta\mathbb{Z}$.

<u>Proof</u>. By Proposition (1.27), we have the inclusions

$$\begin{tikzcd}
C(\mathbb{T}^1) \arrow[r] \arrow[d, hook] & C^*(z_n) \arrow[r, hook] \arrow[d, hook] & C(\mathbb{T}^n) \\
A_\theta \arrow[r, hook] & A_{F_{f,\theta}} = C(\mathbb{T}^n)\times_{F_{f,\theta}} \mathbb{Z} &
\end{tikzcd}$$

where $C^*(z_n)$ is the C^*-subalgebra of $C(\mathbb{T}^n)$ generated by the last coordinate function z_n, on which $F_{f,\theta}$ acts as $F_{f,\theta}(z_n) = e^{2\pi i\theta}\cdot z_n$. By Proposition (2.8), the following diagram is commutative,

$$\begin{tikzcd}
K_o(A_\theta) \arrow[r, "j_*"] \arrow[d, "\delta'"] & K_o(A_{F_{f,\theta}}) \arrow[d, "\delta"] \\
K_1(C(\mathbb{T}^1)) \arrow[r, hook, "j_*"] & K_1(C(\mathbb{T}^n))
\end{tikzcd}$$

where δ and δ' are connecting homomorphisms in Pimsner-Voiculescu sequence, see (2.8). Hence, $\mathbb{Z} + \theta\mathbb{Z} = \tau_*(j_*(K_o(A_\theta)))$ is contained in $\tau_*(K_o(A_{F_{f,\theta}}))$ since projections in $A_\theta \otimes M_k$ are also the projections in $A_{F_{f,\theta}} \otimes M_k$ and $\tau_*\circ j_* = (\tau\circ j)_*$. Note by the results of Rieffel in [50] and Pimsner and Voiculescu in the appendix of [44] that the canonical tracial state on $K_o(A_\theta)$ has the range $\mathbb{Z} + \theta\mathbb{Z}$ for any θ in $[0,1)$, and a result of Elliot as Lemma 2.3 in [17] shows any tracial state on $K_o(A_\theta)$ gives the same map to $\mathbb{R}^1$. Hence, $\tau_*(j_*(K_o(A_\theta))) = \mathbb{Z} + \theta\mathbb{Z}$. On the other hand, any element $[X]$ in $K_1(C(\mathbb{T}^n))$ with x in $GL(C(\mathbb{T}^n))$ is an integral combination of $[z_1],\cdots,[z_n]$, the coordinate functions in $C(\mathbb{T}^n)$ considered as generators of $\check{H}^1(\mathbb{T}^n) \cong GL(C(\mathbb{T}^n))/GL^\circ(C(\mathbb{T}^n)) \hookrightarrow K_1(C(\mathbb{T}^n))$, where $GL^\circ(C(\mathbb{T}^n))$

is the connected component of the constant function 1 in $GL(C(\mathbb{T}^n, 1)$. If $(F^{-1}_{f,\theta})[x] = [x]$ in $K_1(C(\mathbb{T}^n))$, then $[x]$ is in the subgroup $[z_n]\cdot\mathbb{Z}$ of $\check{H}^1(\mathbb{T}^n) \hookrightarrow K_1(C(\mathbb{T}^n))$, since the only $[z_i]$ such that $(F^{-1}_{f,\theta})_*[z_i] = [z_i]$ is $[z_n]$. Now, using the result of Pimsner and Voiculescu we mentioned above, in the appendix of [44], that the projection p in $A_\theta \hookrightarrow A_{F_{f,\theta}}$ which is constructed by Rieffel in [50] and is such that $\delta([p]_\circ) = [z_n]$, where $[p]_\circ$ is in $K_0(A_{F_{f,\theta}})$ and $[z_n]$ is in $K_1(C(\mathbb{T}^n))$, satisfies $\tau_*([p]) = \tau(p) = \theta$. Now by Theorem in (2.22), we have

$$\tau_*(K_0(A_{F_{f,\theta}})) = \tau_*(K_0(C(\mathbb{T}^n))) + \theta\mathbb{Z} = \mathbb{Z} + \theta\mathbb{Z},$$

since for any compact connected space X and any tracial state τ on $C(X)$, $\tau_*(K_0(C(X))) = \mathbb{Z}$.

Note, we used Lemma 2.3 in [17], that any tracial state on A_θ gives the same map from $K_0(A_\theta)$ into $\mathbb{R}^1$, for any θ in $[0,1)$, in proving $\tau_*([p]) = \tau(p) = \theta$, where p is the "Rieffel" projection constructed in [50] (see also the appendix of [44]), for a tracial state τ on $A_{F_{f,\theta}}$, since p is in A_θ and $\tau \circ j$ is a tracial state on A_θ.

This completes the proof.

(2.24) The computation above of the range of any tracial state on $K_0(A_{F_{f,\theta}})$ can be generalized to give a proof of a similar result for C*-algebras associated with Furstenberg affine transformation on certain non-commutative tori. See the discussion on this in (6.4) of §6.

§3. Some classifications for $A_{F_{f,\theta}}$

In Section 2, we computed the K-groups of the crossed product C*-algebras $A_{F_{f,\theta}} = C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ for $F_{f,\theta}$ being a descending transformation (see Proposition (2.17)). We also computed the range of any tracial state on $K_0(C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z})$ for a Furstenberg transformation $F_{f,\theta}$ on $\mathbb{T}^n$ by means of the Pimsner-Voiculescu sequence and a theorem of Pimsner. Using these results, classification up to $*$-isomorphism of some of these crossed product C*-algebras is possible. Later on, we will also consider strong Morita equivalence for these C*-algebras.

(3.1) <u>Theorem</u>. Let $F_{f,\theta} : \mathbb{T}^n \to \mathbb{T}^n$ be a Furstenberg transformation satisfying condition (B) in (1.21). Then $C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ is $*$-isomorphic to $C(\mathbb{T}^n)\times_{F_{f,\theta'}}\mathbb{Z}$ if and only if $\theta' = \theta$ or $1-\theta$, assuming that both θ and θ' be in the interval $[0,1)$.

<u>Proof</u>. By Theorem (2.23), any tracial state on $A_{F_{f,\theta}}$ will induce a map on $K_0(A_{F_{f,\theta}})$ with range $\mathbb{Z} + \theta\mathbb{Z}$ in $\mathbb{R}^1$. Therefore, the statement that $C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z}$ is $*$-isomorphic to $C(\mathbb{T}^n)\times_{F_{f,\theta'}}\mathbb{Z}$ necessarily implies $\theta' = \theta$ or $\theta' = 1-\theta$, because θ is irrational. To see this, suppose $\mathbb{Z}+\theta\mathbb{Z} = \mathbb{Z}+\theta'\mathbb{Z}$. Then there exist m, n, k and ℓ in $\mathbb{Z}$ so that the following equations hold,

$$\begin{cases} m+n\theta = \theta' \\ k+\ell\theta' = \theta. \end{cases}$$

We then have, $\begin{cases} \theta = k+\ell\theta' \\ (\ell m+k) + (\ell n+1)\theta = 0 \end{cases}$, and hence,

$\ell n+1 = 0$, $\ell m+k = 0$ since θ is irrational. For $\ell n+1 = 0$ we must have $\ell n = -1$ and so that $\ell = \pm 1$ and $n = \pm 1$. Now θ' is in $[0,1]$ too, we must have $m = 0$ when $n = 1$ or $m = 1$ when $n = -1$.

On the other hand, if $\theta' = 1-\theta$, let $\varphi_o : \mathbb{T}^n \to \mathbb{T}^n$ be such that $\varphi_o(z_1,\cdots,z_n) = (\overline{z}_1,\cdots,\overline{z}_n)$ where $\overline{z}_i$ denotes the complex conjugation of z_i. Then φ_o is clearly a homeomorphism of $\mathbb{T}^n$. Moreover, we have

$$\varphi_o^{-1} F_{f,\theta'} \varphi_o(z_1,\cdots,z_n) = \varphi^{-1}(\overline{z}_1 \cdot f_1(\overline{z}_2,\cdots,\overline{z}_n),\cdots,\overline{z}_{n-1} \cdot f_{n-1}(\overline{z}_n), e^{2\pi i\theta'} \cdot \overline{z}_n)$$

$$= (z_1 \cdot \overline{f_1(\overline{z}_2,\cdots,\overline{z}_n)},\cdots,z_{n-1} \cdot \overline{f_{n-1}(\overline{z}_n)}, e^{-2\pi i\theta'} \cdot z_n)$$

$$= (z_1 \cdot f_1(z_2,\cdots,z_n),\cdots,z_{n-1} \cdot f_{n-1}(z_n), e^{2\pi i(1-\theta')} \cdot z_n)$$

$$= F_{f,1-\theta'}(z_1,\cdots,z_n),$$

for any θ' in $[0,1)$. This shows that $F_{f,\theta}$ and $F_{f,1-\theta}$ are topologically conjugate. Therefore, it follows that $A_{F_{f,\theta}}$ is $*$-isomorphic to $C(\mathbb{T}^n) \times_{F_{f,1-\theta}} \mathbb{Z}$, and the proof is complete.

(3.2) <u>Corollary</u>. $A_{F_{f,\theta}} \otimes M_m$ is $*$-isomorphic to $A_{F_{f,\theta'}} \otimes M_\ell$ if and only if $m = \ell$ and $\theta' = \theta$ or $1 - \theta$, where $F_{f,\theta}$ is a

Furstenberg transformation on $\mathbb{T}^n$ satisfying condition (B), note, by hypotheses θ is irrational, and where θ' is in $[0,1)$.

To prove this corollary we have to prove a lemma concerning tracial state on $A_{F_{f,\theta'}} \otimes M_\ell$.

(3.3) Lemma. For a C*-algebra A with unit, any tracial state ω on $A \otimes M_n$ is of the form $\tau \otimes (\frac{1}{n}\mathrm{tr})$, where τ is a tracial state on A and tr is the usual trace on M_n.

Proof. It is obvious that M_n has a unique tracial state $(\frac{1}{n}\mathrm{tr})$, which is faithful on positive elements. Now if we set $\tau(a) = \varphi(a\otimes 1)$ for a in A, then τ is a tracial state on A, we claim that $\varphi(a\otimes b) = \tau(a)\cdot(\frac{1}{n}\mathrm{tr}\, b)$ for all a in A and b in M_n. It is enough to check this for positive elements in A, so assume that a is positive. If $\tau(a) = 0$, then by the generalized Cauchy-Schwartz inequality for states, we have

$$|\varphi(a\otimes b)| = |\varphi(a^{\frac{1}{2}}\otimes 1)\cdot(a^{\frac{1}{2}}\otimes b)|$$

$$\leq \varphi((a^{\frac{1}{2}}\otimes 1)^2)\cdot\varphi((a^{\frac{1}{2}}\otimes b^*)(a^{\frac{1}{2}}\otimes b)) = \varphi(a\otimes 1)\varphi(a\otimes b^*b) = 0$$

and therefore, $\varphi(a\otimes b) = \tau(a)\cdot\frac{1}{n}\mathrm{tr}\, b = 0$. If $\tau(a) \neq 0$, then $\varphi(a\otimes b)/\tau(a)$ is a well-defined tracial state on M_n, and hence must be equal to $\frac{1}{n}\mathrm{tr}\, b$, by the uniqueness of the tracial state on M_n. In any case, $\varphi(a\otimes b) = \tau(a)(\frac{1}{n}\mathrm{tr}\, b)$. This proves the lemma.

(3.4) Remark. This lemma was known by many mathematicians;

especially it is used in [50]. The proof of the lemma clearly can be used to prove a generalized result for $A \otimes_{\min} \mathcal{B}$, where "min" denotes the injective C*-norm on the algebraic tensor product of A and $\mathcal{B}$, and where $\mathcal{B}$ has a unique tracial state, see e.g. [59].

(3.5) Lemma. If A is a unital C*-algebra such that all tracial states on A have the same range on $K_0(A)$, then all tracial stateson $A \otimes M_n$ have the same range on $K_0(A \otimes M_n)$.

Proof. Let p be a projection in $(A \otimes M_n) \otimes M_k$, and φ be a tracial state on $A \otimes M_n$. Then p is of the form $(A_{ij})_k$ for A_{ij} in $A \otimes M_n$ and φ has the form $\tau \otimes \frac{1}{n}\mathrm{tr}$, as shown in Lemma (3.3) where τ is a tracial state on A. Now by definition, the induced map of φ on $K_0(A \otimes M_n)$ is

$$\varphi_*(p) \overset{\text{def.}}{=\!=\!=} \varphi\Big(\sum_{i=1}^{k} A_{ii}\Big) = \sum_{i=1}^{k} \varphi(A_{ii}) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n} \tau(a^i_{jj}) \overset{\text{def.}}{=\!=\!=} \frac{1}{n}\tau_*(\tilde{p})$$

where $A_{ii} = (a^i_{jk})_{n\times n}$ for a^i_{jk} in A, and $\tilde{p} = p$ in $(A \otimes M_n) \otimes M_k$ is viewed as a projection in the matrix algebra $A \otimes M_{n\cdot k}$ of A, and τ_* is the induced map of τ on $K_0(A)$. Therefore, $\varphi_*(K_0(A \otimes M_n)) = \frac{1}{n}\tau_*(K_0(A))$. Since $\tau_*(K_0(A))$ does not depend on the choice of the tracial state τ, φ_* does not either. We have the conclusion.

Now we can prove Corollary (3.2).

Suppose $A_{F_{f,\theta}} \otimes M_n$ is *-isomorphic to $A_{F_{f,\theta}} \otimes M_\ell$. Then by Lemma (3.5), the range of any tracial state on $K_0(A_{F_{f,\theta}} \otimes M_n)$

is $\frac{1}{n}(\mathbb{Z}+\theta\mathbb{Z})$ and the range of any tracial state on $K_0(A_{F_{f,\theta'}} \otimes M_\ell)$ is $\frac{1}{\ell}(\mathbb{Z}+\theta'\mathbb{Z})$. It is easy to check this is the case if and only if $m = \ell$ and $\theta' = \theta$ or $1-\theta$ as the proof of Theorem (3.1). On the other hand, if $m = \ell$ and $\theta' = \theta$ or $1-\theta$, then $A_{F_{f,\theta}}$ is $*$-isomorphic to $A_{F_{f,\theta'}}$ hence $A_{F_{f,\theta}} \otimes M_n$ is $*$-isomorphic to $A_{F_{f,\theta'}} \otimes M_n$.

(3.6) We now consider another classification of the C*-algebras associated with descending transformations.

<u>Theorem</u>. Let $F_{f,\theta}$ be a descending transformation on $\mathbb{T}^n$ of the form $F_{f,\theta}(z_1,\cdots,z_n) = (z_1 \cdot z_2^{k_1},\cdots,z_{n-1} \cdot z_n^{k_{n-1}}, e^{2\pi i\theta} \cdot z_n)$, where θ is irrational. If $F_{f',\theta'}(z_1,\cdots,z_n) =$
$= (z_1 \cdot z_2^{k_1'},\cdots,z_{n-1} \cdot z_n^{k_{n-1}'}, e^{2\pi i\theta'} \cdot z_n)$, is another descending transformation, then $A_{F_{f,\theta}} \otimes M_m$ is $*$-isomorphic to $A_{F_{f',\theta'}} \otimes M_\ell$ if and only if $m = \ell$, and $\theta' = \theta$ or $1-\theta$ and also $|k_i'| = |k_i|$, $i = 1,2,\cdots,n-1$.

To prove this theorem we need a simple limma.

(3.8) <u>Lemma</u>. For $\theta' = \theta$ or $1-\theta$ and $|k_i'| = |k_i| = i = 1,2,\cdots,n-1$, the descending transformations $F_{f,\theta}$ and $F_{f',\theta'}$ as in the theorem, are topologically conjugate.

<u>Proof</u>. We may assume $\theta' = \theta$ since $F_{f',\theta'}$ is conjugate to $F_{f',1-\theta'}$. Let $\eta_1,\cdots,\eta_{n-1}$ be signs of $k_1,\cdots,k_{n-1}$, respectively, and let

$\varepsilon_1,\ldots,\varepsilon_n$ be 1 or -1, such that the following equations hold,

$$\varepsilon_i \cdot \varepsilon_{i+1} = \eta_i,\ i = 1,2,\ldots,n-1.$$

These equations have a solution, for instance, let $\varepsilon_n = \eta_n$, then $\varepsilon_{n-1} = \eta_{n-1}\cdot\varepsilon_n^{-1}$, $\varepsilon_{n-2} = \eta_{n-2}\cdot\varepsilon_{n-1}^{-1},\ldots,\varepsilon_1 = \eta_1\cdot\varepsilon_2^{-1}$. Define a homeomorphism $\varphi_\varepsilon(z_1,\cdots,z_n) = (z_1^{\varepsilon_1},\ldots,z_n^{\varepsilon_n})$, when $\varepsilon_i = 1$, $z_i^{\varepsilon_i} = z_i$ and when $\varepsilon_i = -1$, $z_i^{\varepsilon_i} = \overline{z_i}$. Then we have $\varphi_\varepsilon^2 = \mathrm{Id}_{\mathbb{T}^n}$ and $\varphi_\varepsilon^{-1}\, F_{f,\theta}\, \varphi_\varepsilon(z_1,\cdots,z_n) = \varphi_\varepsilon^{-1}(z_1^{\varepsilon_1}\cdot z_2^{\varepsilon_2 k_1}, z_2^{\varepsilon_2}\cdot z_3^{\varepsilon_3 k_2},\ldots,$ $z_{n-1}^{\varepsilon_{n-1}}\cdot z_n^{\varepsilon_n k_{n-1}}, e^{2\pi i\theta}\cdot z_n^{\varepsilon_n}) = (z_1\cdot z_2^{\varepsilon_1\varepsilon_2 k_1}, z_2\cdot z_3^{\varepsilon_2\varepsilon_3 k_2},\ldots,$ $z_{n-1}\cdot z_n^{\varepsilon_{n-1}\varepsilon_n k_{n-1}}, e^{-2\pi i\theta}\cdot z_n) = (z_1, z_2^{|k_1|},\ldots,z_{n-1}\cdot z_n^{|k_{n-1}|}, e^{-2\pi i\theta}\cdot z_n)$ $\overset{\text{def}}{=\!=\!=} F_{|f|,1-\theta}(z_1,\cdots,z_n)$. Therefore, $F_{f,\theta}$ is topologically conjugate to $F_{|f|,1-\theta}$. But $F_{|f|,1-\theta}$ is clearly topological conjugate to $F_{|f|,\theta}$ as proved in the last paragraph in Theorem (3.1). Hence the lemma follows.

We now prove Theorem (3.6).

If $A_{F_{f,\theta}}$ is $*$-isomorphic to $A_{F_{f',\theta'}}$, then θ' is necessarily equal to θ or $1-\theta$, as the proof for Theorem (3.1). Now by Proposition (2.17) in §2, the torsion subgroup of $K_*(C(\mathbb{T}^n)\times_{F_{f,\theta}}\mathbb{Z})$ is $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus\cdots\oplus \mathbb{Z}_{k_{n-1}}^{(m_{n-1})}$ and the torsion subgroup of $K_*(C(\mathbb{T}^n)\times_{F_{f',\theta'}}\mathbb{Z})$ is $\mathbb{Z}_{k_1'} \oplus \mathbb{Z}_{k_2'}^{(m_2)} \oplus\cdots\oplus \mathbb{Z}_{k_{n-1}'}^{(m_{n-1})}$, therefore, $|k_1'| = |k_1|,\ldots,|k_{n-1}'| = |k_{n-1}|$ since both $\{k_1\cdots k_{n-1}\}$ and

$\{k_1',\ldots,k_{n-1}'\}$ are descending. On the other hand, if $\theta' = \theta$ or $1-\theta$ and $|k_i'| = |k_i|$ for $i = 1,2,\ldots,n-1$, then by Lemma (3.7), $F_{f,\theta}$ is conjugate to $F_{f',\theta'}$, therefore, $A_{F_{f,\theta}}$ is $*$-isomorphic to $A_{F_{f',\theta'}}$. The argument for $A_{F_{f,\theta}} \otimes M_m$ is $*$-isomorphic to $A_{F_{f',\theta'}} \otimes M_\ell$ if and only if $m=\ell$, $|k_i'| = |k_i|$ and $\theta' = \theta$ or $1-\theta$ in the theorem is just the same as that of Corollary (3.2).

(3.8) Unlike the irrational case, if θ is rational, then the classification of the C*-algebras $A_{F_{f,\theta}}$ is more delicate. Let $\theta = \frac{q}{p}$ where q and p are positive integers such that $(q,p) = 1$. If $\frac{q'}{p'}$ is another such rational number with $p' \neq p$, then we see that $A_{F_{f,\frac{q}{p}}}$ is not $*$-isomorphic to $A_{F_{f,\frac{q'}{p'}}}$ by comparing the ranges of tracial states on them. The subtlety here is that we do not know if $A_{F_{f,\frac{q}{p}}}$ is $*$-isomorphic to $A_{F_{f,\frac{q'}{p}}}$ for $q' \neq p-q$ where both q and q' are assumed less than p. However, we will be able to determine when they are strongly Morita equivalent in the next section.

We are also able to obtain classifications of some other classes of C*-algebras associated with Furstenberg transformations. But we will not include it in this thesis.

§4. Strong Morita equivalence for $A_{F_{f,\theta}}$

The notion of "strong Morita equivalence" for C*-algebras was first introduced by M. Rieffel [49], [51] in analogy to the algebraic notion of "Morita equivalence" for rings. The word "strong" refers to the norm topology on C*-algebras. Later, it was shown in [6] that this equivalence relation for C*-algebras is just the same as that of "stable equivalence" which was introduced by L. Brown in [4]. Therefore, strongly Morita equivalente C*-algebras have the same K-groups as well as KK-groups. Thus this notion provides an efficient tool to study C*-algebras with the same K-groups. Readers who are interested in this topic are referred to [49], [51], [21], [7], and [12].

Recall that the rotation C*-algebra A_θ is the crossed product C*-algebra associated with a rotation on the 1-dimensional torus $\mathbb{T}^1$ with rotation number θ in $[0,1)$. In [50] and [52], Rieffel classified rotation algebras up to strong Morita equivalence. His result shows that some of these non-isomorphic rotation C*-algebras are not so different. In particular, Rieffel showed that if θ and θ' are irrational, then $A_{\theta'}$ is strong Morita equivalent to A_θ if and only if θ and θ' are in the same orbit of the action of $GL(2,\mathbb{Z})$ on the irrational numbers in $(0,1)$ defined by linear fractional transformation. He also showed with an explicit construction of the imprimitivity bimodules, that all rational rotation C*-algebras A_θ are strong Morita equivalent to $C(\mathbb{T}^2)$. Following Rieffel's idea and using a theorem in [7] or in [12], Packer obtained in [35] & [36] similar results for C*-algebras of projective representations of the Heisenberg group. The C*-algebras considered in [35], [36] are closely related to C*-algebras associated with Furstenberg transformations on $\mathbb{T}^2$.

As a special case of Theorem (3.6), if $F_{k,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $F_{k,\theta}(z_1,z_2) = (z_1 \cdot z_2^k, e^{2\pi i\theta} \cdot z_2)$, then $A_{F_{k,\theta}}$ is $*$-isomorphic to $A_{F_{m,\theta'}}$ if and only if $|m| = |k|$ and $\theta' = \theta$ or $1 - \theta$, where we are assuming that θ and θ' are irrational numbers in $[0,1)$. In order to study the problem of strong Morita equivalence for the C*-algebras associated with general Furstenberg transformations, we have to consider that problem for this special class of $A_{F_{k,\theta}}$'s, that is, we have to determine when $A_{F_{k,\theta}}$ and $A_{F_{m,\theta'}}$ are strong Morita equivalent, which is possible even though they are not $*$-isomorphic. We will solve the problem for this class by applying a result of P. Green which was described and proved in [51] together with a construction. We leave the general case for later study. We will also obtain, as a byproduct, the strong Morita equivalence for the C*-algebras considered by Riedel [48].

(4.1) <u>Definition</u> [51]. Let A and $\mathcal{B}$ be C*-algebras. By an A-$\mathcal{B}$-equivalence bimodule (also called imprimitivity A-$\mathcal{B}$-bimodule) we mean a left A and right B bimodule X on which there are defined an A-valued and a B-valued inner-product such that

1) $\langle x,y\rangle_A z = x\langle y,z\rangle_B$ for x,y and z in X.

2) The representation of A on X is a continuous *-representation by operators which are bounded for $\langle,\rangle_B$, that is $\langle ax,ax\rangle_B \leq \|a\|^2\langle x,x\rangle_B$ as positive elements of B, etc., and similarly for the right representation of B on X.

3) The linear span of the $\langle X,X\rangle_A$, which is an ideal in A, is dense in A, and similarly for the $\langle X,X\rangle_B$.

We say that two C*-algebras A and B are strongly Morita equivalent if there exists an A-B-equivalence bimodule.

It can be verified [49] that

$$\|x\| = \|\langle x,x\rangle_A\|^{\frac{1}{2}} \ (=\|\langle x,x\rangle_B\|^{\frac{1}{2}}),$$

defines a norm on X, and that all the structure extends to the completion of X. From now on we will assume that X is complete for this norm.

(4.2) Theorem [51, Proposition 2.2 and Corollary 2.6]. Let A, B and X be as in Definition (4.1). Suppose A and B are unital and τ is a tracial state on A. Then there is an induced positive tracial function τ_X on B, so that,

$$\tau_X(\langle x,y\rangle_B) = \tau(\langle x,y\rangle_A)$$

for all x, y in X. Moreover,

$$\tau_X(K_0(\mathcal{B})) = \tau(K_0(A)).$$

(4.3) Definition. A C*-algebra $\mathcal{B}$ has property (D) if all tracial states on $\mathcal{B}$ have same range on $K_0(\mathcal{B})$.

(4.4) Theorem. Let $A, \mathcal{B}$ and X be as above. Suppose A and $\mathcal{B}$ are unital and both A and $\mathcal{B}$ have property (D). Let τ and ϕ be tracial states on A and $\mathcal{B}$ respectively, such that

$$\tau(K_0(A)) = \mathbb{Z} + \theta_1\mathbb{Z} + \cdots + \theta_n\mathbb{Z}$$

and
$$\phi(K_0(\mathcal{B})) = \mathbb{Z} + \theta_1'\mathbb{Z} + \cdots + \theta_n'\mathbb{Z},$$

where θ_i and θ_i' are in (0,1) for $i = 1,2,\cdots,n$, and $(\theta_1,\cdots,\theta_n)$ are rationally independent. Then $\{\theta_1',\cdots,\theta_n'\}$ is also rationally independent and there exists an (n+1) by (n+1) matrix $(a_{i,j})$ in $GL(n+1;\mathbb{Z})$, so that

$$\begin{bmatrix}\theta_1'\\ \cdot\\ \cdot\\ \cdot\\ \theta_n'\end{bmatrix} = (a_{i,j}) * \begin{bmatrix}\theta_1\\ \cdot\\ \cdot\\ \cdot\\ \theta_n\end{bmatrix} \underline{\underline{\text{definition}}} \begin{bmatrix}\dfrac{a_{1,0}+a_{1,1}\theta_1+\cdots+a_{1,n}\theta_n}{a_{0,0}+a_{0,1}\theta_1+\cdots+a_{0,n}\theta_n}\\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{a_{n,0}+a_{n,1}\theta_1+\cdots+a_{n,n}\theta_n}{a_{0,0}+a_{0,1}\theta_1+\cdots+a_{0,n}\theta_n}\end{bmatrix}$$

To prove this theorem, we need some lemmas.

(4.5) <u>Lemma</u>. Let S_n be the set of n-tuples $(\theta_1,\cdots,\theta_n)$ where θ_i is an irrational number in $(0,1)$, $i = 1,2,\cdots,n$, and $(\theta_1,\cdots,\theta_n)$ are rationally independent. The action of $GL(n+1;\mathbb{Z})$ on S_n defined in the theorem (4.4) is a group action.

<u>Proof</u>. Clearly $I_{n+1} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix}$. We need to check that

if A and B are in $GL(n+1;\mathbb{Z})$, then

$$(A \cdot B) * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = A * \left(B * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix}\right).$$

In fact,

$$(A \cdot B) * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}\dfrac{C_{1,0} + C_{1,1}\theta_1 + \cdots + C_{1,n}\theta_n}{C_{0,0} + C_{0,1}\theta_1 + \cdots + C_{0,n}\theta_n}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \dfrac{C_{n,0} + C_{n,1}\theta_1 + \cdots + C_{n,n}\theta_n}{C_{0,0} + C_{0,1}\theta_1 + \cdots + C_{0,n}\theta_n}\end{bmatrix}$$

where $C_{i,j} = \sum_{k=0}^{n} a_{i,k} \cdot b_{k,j}$. On the other hand,

$$A*(B*\begin{pmatrix}\theta_1\\ \cdot\\ \cdot\\ \cdot\\ \theta_n\end{pmatrix}) = A*\begin{pmatrix}\dfrac{b_{1,0}+b_{1,1}\theta_1+\cdots+b_{1,n}\theta_n}{b_{0,0}+b_{0,1}\theta_1+\cdots+b_{0,n}\theta_n}\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{b_{n,0}+b_{n,1}\theta_1+\cdots+b_{n,n}\theta_n}{b_{0,0}+b_{0,1}\theta_1+\cdots+b_{0,n}\theta_n}\end{pmatrix}$$

$$\underset{\overline{\overline{\text{let } \theta_0=1}}}{} \begin{pmatrix}\dfrac{a_{1,0}+a_{1,1}\dfrac{\sum_{i=0}^{n} b_{1,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}+\cdots+a_{1,n}\dfrac{\sum_{i=0}^{n} b_{n,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}}{a_{0,0}+a_{0,1}\dfrac{\sum_{i=0}^{n} b_{1,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}+\cdots+a_{0,n}\dfrac{\sum_{i=0}^{n} b_{n,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}}\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{a_{n,0}+a_{n,1}\dfrac{\sum_{i=0}^{n} b_{1,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}+\cdots+a_{n,n}\dfrac{\sum_{i=0}^{n} b_{n,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}}{a_{0,0}+a_{0,1}\dfrac{\sum_{i=0}^{n} b_{1,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}+\cdots+a_{0,n}\dfrac{\sum_{i=0}^{n} b_{n,i}\theta_i}{\sum_{i=0}^{n} b_{0,i}\theta_i}}\end{pmatrix}$$

$$= \begin{pmatrix}\dfrac{\sum_{0\le j,i\le n} a_{1,j}b_{j,i}\theta_i}{\sum_{0\le i,j\le n} a_{0,j}b_{j,i}\theta_i}\\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{\sum_{0\le i,j\le n} a_{n,j}b_{j,i}\theta_i}{\sum_{0\le i,j\le n} a_{0,j}b_{j,i}\theta_i}\end{pmatrix} = \begin{pmatrix}\dfrac{\sum_{i=0}^{n}(\sum_{j=0}^{n} a_{1,j}b_{j,i})\theta_i}{\sum_{i=0}^{n}(\sum_{j=0}^{n} a_{0,j}b_{j,i})\theta_i}\\ \cdots\ \cdots\ \cdots\ \cdots\\ \dfrac{\sum_{i=0}^{n}(\sum_{j=0}^{n} a_{n,j}b_{j,i})\theta_i}{\sum_{i=0}^{n}(\sum_{j=0}^{n} a_{0,j}b_{j,i})\theta_i}\end{pmatrix} = (A\cdot B)*\begin{pmatrix}\theta_1\\ \cdot\\ \cdot\\ \cdot\\ \theta_1\end{pmatrix}$$

Of course, to make sense of the computation, one has to check that for any A in GL(n+1;$\mathbb{Z}$) and $\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ in S_n, $A*\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ is also in S_n. Suppose the contrary, that is, that $A*\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ is not in S_n. Then there exist $p_0,\cdots,p_n$ in $\mathbb{Z}$, so that

$$p_0 + \sum_{i=1}^{n} p_i \cdot \frac{\sum_{j=0}^{n} a_{i,j}\theta_j}{\sum_{j=0}^{n} a_{0,j}\theta_j} = 0.$$

Hence, $0 = \sum_{i=0}^{n} p_i \sum_{j=0}^{n} a_{i,j}\theta_j = \sum_{j=0}^{n}\left(\sum_{i=0}^{n} p_i a_{i,j}\right)\theta_j$. (Note, $\theta_0=1$).

Since $\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ is in S_n, $\sum_{i=0}^{n} p_i a_{i,j} = 0$, $j = 0,1,\cdots,n$. But $(a_{i,j}) = A$ is in GL(n+1;$\mathbb{Z}$) and hence we have $p_i = 0$, $i = 0,1,\cdots,n$, which is a contradiction. Therefore, $A*\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ is also in S_n. This completes the proof.

(4.6) Lemma. Let $\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix}$ and $\begin{pmatrix}\theta'_1\\ \vdots\\ \theta'_n\end{pmatrix}$ be in S_n. If r is a positive number, so that $r(\mathbb{Z}+\theta_1\mathbb{Z}+\cdots+\theta_n\mathbb{Z}) = (\mathbb{Z}+\theta'_1\mathbb{Z}+\cdots+\theta'_n\mathbb{Z})$, then there is A in GL(n+1;$\mathbb{Z}$) so that $A*\begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix} = \begin{pmatrix}\theta'_1\\ \vdots\\ \theta'_n\end{pmatrix}$.

<u>Proof</u>. When $n=1$, this lemma was known and used by many mathematicians. It was mentioned in [15], and later used in [16] and proved in [54]. We repeat the proof in [54] for $n=1$, and then get the general statement for $n > 1$.

Since $\theta_1 > 0$, $\theta_1' > 0$ are irrational numbers we may define two orders on $\mathbb{Z} \oplus \mathbb{Z}$, denoted by $P_{(1,\theta_1)}$ and $P_{(1,\theta_1')}$. That is, $P_{(1,\theta_1)}$ and $P_{(1,\theta_1')}$ are positive cones for $\mathbb{Z} \oplus \mathbb{Z}$, where

$$P_{(1,\theta_1)} = \{(m,n) \in \mathbb{Z} \oplus \mathbb{Z} | m + n\theta_1 > 0\}$$

and

$$P_{(1,\theta_1')} = \{(m,n) \in \mathbb{Z} \oplus \mathbb{Z} | m + n\theta_1' > 0\}.$$

Now, since $r(\mathbb{Z}+\theta_1\mathbb{Z}) = \mathbb{Z} + \theta_1'\mathbb{Z}$, we have an isomorphism $\phi : (\mathbb{Z}\oplus\mathbb{Z}, P_{(1,\theta_1)}) \to (\mathbb{Z}\oplus\mathbb{Z}, P_{(1,\theta_1')})$, that is, θ preserves the positive cones since r is positive and $r(\mathbb{Z}+\theta_1\mathbb{Z})^+ = (\mathbb{Z}+\theta_1'\mathbb{Z})^+$, where $(\mathbb{Z}+\theta\mathbb{Z})^+ = \{n+m\theta | n+m\theta \geq 0\}$. Since ϕ is an isomorphism of $\mathbb{Z} \oplus \mathbb{Z}$, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $GL(2;\mathbb{Z})$. Moreover, (n,m) is in $P_{(1,\theta_1)}$ if and only if $\phi(n,m) = (n,m)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (an+cm, bn+dm)$ is in $P_{(1,\theta_1')}$. This is equivalent to saying that $n + m\theta_1 > 0$ if and only if $(an+cm) + (bn+dm)\theta_1' > 0$, that is,

$(a+b\theta_1')n + (c+d\theta_1')m > 0$ or $n + \dfrac{c+d\theta_1'}{a+b\;_1'}\,m > 0$ since $a + b\theta_1' > 0$, (ϕ is order preserving isomorphism). If we set $\overline{\theta} = \dfrac{c+d\theta_1'}{a+b\theta_1'}$, then we have $n + m\theta_1 > 0$ if and only if $n + m\overline{\theta} > 0$. This implies $\theta_1 = \overline{\theta}$ and therefore $\theta_1 = \dfrac{c+d\theta_1'}{a+b\theta_1'}$ and $\theta_1' = \dfrac{c-a\theta_1}{-d+b\theta}$. But $\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$ is also in $GL(2;\mathbb{Z})$.

Now suppose $n \geq 1$, and that $\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$ and $\begin{bmatrix} \theta_1' \\ \vdots \\ \theta_n' \end{bmatrix}$ are as in the lemma. Then we can also define two orders on $\overbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}^{n+1}$, where $P_{(1,\theta_1',\ldots,\theta_n')}$ and $P_{(1,\theta_1,\ldots,\theta_n)}$ denote the positive cones. Then $P_{(1,\theta_1',\ldots,\theta_n')} = \{(m_0,\ldots,m_n) \in \mathbb{Z}^{n+1} | m_0+m_1\theta_1'+\cdots+m_n\theta_n'>0\}$, and $P_{(1,\theta_1,\ldots,\theta_n)} = \{(m_0,\ldots,m_n) \in \mathbb{Z}^{n+1} | m_0+m_1\theta_1+\cdots+m_n\theta_n > 0\}$. Since $r^{-1}(\mathbb{Z}+\theta_1'\mathbb{Z}+\cdots+\theta_n'\mathbb{Z}) = \mathbb{Z} + \theta_1\mathbb{Z}+\cdots+\theta_n\mathbb{Z}$ with $r > 0$, we have an order isomorphism $\phi : (\mathbb{Z}^{n+1}, P_{(1,\theta_1',\ldots,\theta_n')}) \to$ $\to (\mathbb{Z}^{n+1}, P_{(1,\theta_1,\ldots,\theta_n)})$. Thus ϕ determines a matrix $(a_{i,j})_{n+1}$ in $GL(n+1;\mathbb{Z})$. Moreover, $(m_0,m_1,\ldots,m_n)$ is in $P_{(1,\theta_1',\ldots,\theta_n')}$ if and only if $\phi(m_0,\ldots,m_n) = (m_0,\ldots,m_n)\cdot(a_{i,j})_{(n+1)}$ is in $P_{(1,\theta_1,\ldots,\theta_n)}$. This is equivalent to saying that $m_0+m_1\theta_1'+\cdots+m_n\theta_n' > 0$ if and only if

$$\sum_{i=0}^{n} m_i a_{i,0} + \sum_{i=0}^{n} m_i a_{i,1}\theta_1 + \cdots + \sum_{i=0}^{n} m_i a_{i,n}\theta_n > 0,$$

that is, if and only if $m_0 \sum_{j=0}^{n} a_{0,j}\theta_j + \cdots + m_n \sum_{j=0}^{n} a_{n,j}\theta_j > 0$, where $\theta_0 = 1$. Therefore, $m_0 + m_1\theta_1'+\cdots+m_n\theta_n' > 0$ if and only if $m_0 + m_1\overline{\theta}_1+\cdots+m_n\overline{\theta}_n > 0$ where $\overline{\theta}_i = \sum_{j=0}^{n} a_{i,j}\theta_j / \sum_{j=0}^{n} a_{0,j}\theta_j$, and especially, $m_0 + m_i\theta_i' > 0$ if and only if $m_0 + m_i\overline{\theta}_i > 0$ for all $i = 1,2,\cdots,n$. By the proof of this lemma for $n = 1$, we see that $\overline{\theta}_i = \theta_i'$ for $i = 1,2,\cdots,n$. Therefore

$$\begin{bmatrix}\theta_1' \\ \vdots \\ \theta_n'\end{bmatrix} = A * \begin{bmatrix}\theta_1 \\ \vdots \\ \theta_n\end{bmatrix}$$, where $A = (a_{i,j})_{n+1}$ is in $GL(n+1;\mathbb{Z})$, which completes the proof.

(4.7) We can now prove Theorem (4.4). By Theorem (4.2), τ_χ is a tracial positive functional on $\mathcal{B}$, where τ is a tracial state on A. Therefore, there is a positive real number $r > 0$, such that $\phi = \frac{1}{r}\tau_\chi$ is a tracial state on $\mathcal{B}$. Hence,

$$\phi(K_0(\mathcal{B})) = \mathbb{Z} + \theta_1'\mathbb{Z} + \cdots + \theta_n'\mathbb{Z},$$

by the assumptions in Theorem (4.4). Since $\tau_\chi(K_0(\mathcal{B})) = \tau(K_0(A))$ and both $\mathcal{B}$ and A have property (D) defined in (4.3), we have

(*) $$\frac{1}{r}(\mathbb{Z}+\theta_1\mathbb{Z}+\cdots+\theta_n\mathbb{Z}) = \mathbb{Z} + \theta_1'\mathbb{Z}+\cdots+\theta_n'\mathbb{Z}.$$

Since $\begin{bmatrix}\theta_1 \\ \vdots \\ \theta_n\end{bmatrix}$ and $\begin{bmatrix}\theta_1' \\ \vdots \\ \theta_n'\end{bmatrix}$ are both in S_n as defined in Lemma (4.5), the conclusion of theorem (4.4) follows from Lemma (4.6).

Note, since $\begin{bmatrix}\theta_1 \\ \vdots \\ \theta_n\end{bmatrix}$ is in S_n, and $\begin{bmatrix}\theta_1' \\ \vdots \\ \theta_n'\end{bmatrix}$ and $\begin{bmatrix}\theta_1 \\ \vdots \\ \theta_n\end{bmatrix}$ satisfy (*) by the proof of Lemma (4.5), we have $\begin{bmatrix}\theta_1' \\ \vdots \\ \theta_n'\end{bmatrix}$ is also in S_n.

(4.8) The following theorem of P. Green appears and is proved as situation 10 in [51].

<u>Theorem</u> (P. Green). Let H and K be locally compact groups, which act on a locally compact space P such that the two actions commute and both actions are <u>free</u> and <u>wandering</u>. Then $C^*(K,P/H) = C_0(P/H)\times K$ and $C^*(H,P/K) = C_0(P/K)\times H$ are strongly Morita equivalent.

<u>Note</u>. The action of a group G on P is called <u>free</u> if for any x in P and g in G, $g\cdot x = x$ implies $g = id_G$; and the action of G on P is called <u>wandering</u> if the set $\{g\varepsilon G \mid g\mathcal{S}\cap\mathcal{S} \neq \phi\}$ is precompact in G for any compact subset $\mathcal{S}$ of P.

(4.9) <u>Proposition</u>. Let $F_{k,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$ be the Furstenberg transformation $F_{k,\theta}(z_1,z_2) = (z_1\cdot z_2^k, e^{2\pi i\theta}\cdot z_2)$ where θ is in (0,1). Then $A_{F_{k,\theta}}$ is strongly Morita equivalent to $A_{F_{k,\frac{1}{\theta}}}$.

<u>Proof</u>. Let $P = \mathbb{T}^1\times\mathbb{R}^1$ and $H \cong \mathbb{Z}$ and $K \cong \mathbb{Z}$. Let H act on P by translation on the second coordinate, that is, $\alpha_n(z,s) = (z,s+n)$ for (z,s) in $\mathbb{T}^1\times\mathbb{R}^1$ and n in H, and K acts on P by $\beta_n(z,s) = (z\cdot e^{2\pi iks}, s+\theta)$, where α and β are the representations of the group H and K into the group of all homeomorphisms of P, respectively. Then both actions are free and wandering so that P/K and P/H are both compact Hausdorff spaces.

Moreover, $C^*(H,P/K)$ and $C^*(K,P/H)$ are strongly Morita equivalent by Green's Theorem. We claim that P/K and P/H are both homeomorphic to $\mathbb{T}^2$, the ordinary 2-torus. It is trivial that P/H is homeomorphic to $\mathbb{T}^2$. To show the other, we define a map ϕ from P/K to $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$ by

$$\phi((z,s)_\theta) = (z\cdot e^{2\pi i\gamma(s)}, e^{2\pi i\frac{s}{\theta}}),$$

where $(z,s)_\theta$ is a point in P/K and $\gamma(s)$ is some real valued function on $\mathbb{R}^1$, which will be constructed below. Since we want ϕ to be a map on P/K, we have to find for what kind of function γ on $\mathbb{R}^1$ the map ϕ is well defined. In other words, when can the question mark in the following rectangle be removed

$$\begin{array}{ccc}
(z,s)_\theta & \longrightarrow & (z\cdot e^{2\pi i\gamma(s)}, e^{2\pi i\frac{s}{\theta}}) \\
\| & & \|\,? \\
(z\cdot e^{2\pi i ks}, s+\theta)_\theta & \longrightarrow & (z\cdot e^{2\pi i ks}\cdot e^{2i\pi(s+\theta)}, e^{2\pi i\frac{s}{\theta}})\,?
\end{array}$$

Therefore, we want $e^{2\pi i\gamma(s)} = e^{2\pi ik\theta}\cdot e^{2\pi i\gamma(s+\theta)}$. For this purpose, one can easily find that $\gamma(s) = -\frac{k}{2}s(\frac{s}{\theta}-1)$. We get a map $\phi : P/K \to \mathbb{T}^2$ by defining

$$\phi((z,s)_\theta) = (z\cdot e^{-2\pi i\frac{k}{2}s(\frac{s}{\theta}-1)}, e^{2\pi i\frac{s}{\theta}}).$$

It is easy to check that ϕ is well-defined, one to one and onto. Moreover, ϕ is also a homeomorphism. Now we have to find that how ϕ relates the action of H on P/K to the action of $\mathbb{Z}$ on $\mathbb{T}^2$. We have, for α_n defined above, let $\tilde{\alpha} = \phi\alpha_1\phi^{-1}$, then we have

$$\tilde{\alpha}(z_1,z_2) = \phi\alpha_1\phi^{-1}(z_1,z_2) = \phi\alpha_1\phi^{-1}(z_1,e^{2\pi is})$$

$$= \phi\alpha_1((z_1\cdot e^{2\pi i\frac{k}{2}\theta s(s-1)},\theta s)_\theta) = \phi((z_1\cdot e^{2\pi i\frac{k}{2}\theta s(s-1)},\theta s+1)_\theta)$$

$$= (z_1\cdot e^{-2\pi i\frac{k}{2}(2s+\frac{1}{\theta}+1)},e^{2\pi i\frac{1}{\theta}}\cdot e^{2\pi is}) = (z_1\cdot z_2^{-k}\cdot e^{-2\pi i\frac{k}{2}(\frac{1}{\theta}+1)},e^{2\pi i\frac{1}{\theta}}\cdot z_2).$$

Let $\lambda = e^{2\pi i\frac{1}{\theta}}$ and $\eta = e^{-2\pi i\frac{k}{2}(\frac{1}{\theta}+1)}$, we have $\tilde{\alpha} : \mathbb{T}^2 \to \mathbb{T}^2$ by $\tilde{\alpha}(z_1,z_2) = (\eta\cdot z_1\cdot z_2^{-k},\lambda z_2)$. We claim that $\tilde{\alpha}$ is topologically conjugate to $F_{k,\frac{1}{\theta}} : \mathbb{T}^2 \to \mathbb{T}^2$. First, $\tilde{\alpha}$ is conjugate to $\overline{\alpha}$, where $\overline{\alpha} : \mathbb{T}^2 \to \mathbb{T}^2$ is defined by $\overline{\alpha}(z_1,z_2) = (\overline{\eta}_1\cdot z_1\cdot z_2^k,\lambda z_2)$. If we let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $F(z_1,z_2) = (z_1\cdot z_2,\rho\, z_2)$, where ρ satisfies $\rho^k = \eta\lambda$, then we have

$$\overline{\alpha}\,F(z_1,z_2) = \overline{\alpha}(z_1\cdot z_2,\rho z_2) = (\overline{\eta}\cdot(z_1\cdot z_2)\cdot(\rho z_2)^k,\lambda\rho z_2)$$

$$= (\overline{\eta}\cdot\rho^k z_1\cdot z_2^{k+1},\lambda\rho z_2) \quad \text{and}$$

$$F\,F_{k,\frac{1}{\theta}}(z_1,z_2) = F((z_1\cdot z_2^k,\lambda z_2)) = ((z_2 z_2^k)\cdot(\lambda z_2),\rho\lambda z_2) = (\lambda z_1 z_2^{k+1},\lambda\rho z_2).$$

Since $\overline{\eta}\rho^k = \lambda$, we have $FF_{k,\frac{1}{\theta}} = \overline{\alpha}F$. Therefore, $\tilde{\alpha}$ is conjugate to $F_{k,\frac{1}{\theta}}$. This shows that $C^*(H,P/K) = C(P/K)\times_\alpha H$ is $*$-isomorphic to $C(\mathbb{T}^2)\times_{F_{k,\frac{1}{\theta}}}\mathbb{Z}$. Since $C^*(K,P/H) = C(\mathbb{T}^2)\times_{F_{k,\theta}}\mathbb{Z}$, we finally obtain that $C(\mathbb{T}^2)\times_{F_{k,\theta}}\mathbb{Z}$ is strongly Morita equivalent to $C(\mathbb{T}^2)\times_{F_{k,\frac{1}{\theta}}}\mathbb{Z}$.

(4.10) <u>Remark</u>. The function γ used in this proof was first used by J. Packer in [35], in classifying the C*-algebras associated with projective representations of the Heisenberg group up to strong Morita equivalence. However, Packer's proof depends on the explicit construction of equivalence bimodules and then on the utilization of a result in [7] or in [12].

(4.11) <u>Theorem</u>. Let $F_{k,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$ be the Furstenberg transformation $F_{k,\theta}(z_1,z_2) = (z_1\cdot z_2^k, e^{2\pi i\theta}\cdot z_2)$ for θ being irrational in (0,1). Then $A_{F_{k,\theta}} \otimes M_n$ and $A_{F_{m,\theta'}} \otimes M_\ell$ are strongly Morita equivalent if and only if $|m| = |k|$ and θ and θ' are in the same orbit of the action of $GL(2;\mathbb{Z})$ on irrational numbers by linear functional transformations.

<u>Proof</u>. It is obvious that for any ρ in (0,1), the C*-algebra $A_{F_{k,\rho}}$ is $*$-isomorphic to $A_{F_{k,\rho+n}}$ for any integer n. Let $GL(2;\mathbb{Z})$ act on irrational numbers in (0,1) by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \theta = \frac{c+d\theta}{a+b\theta}$, then by Lemma (4.5), it is a group action. Since $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate $GL(2;\mathbb{Z})$ [30] and S and T carry θ to θ^{-1} and $\theta+1$ respectively, we conclude that $A_{F_{k,\theta}}$ is strongly Morita equivalent to $A_{F_{k,\theta'}}$ for all θ' in the orbit of $GL(2;\mathbb{Z})$.

On the other hand, by Theorem (4.3) if $A_{F_{k,\theta'}}$ and $A_{F_{k,\theta}}$ are strongly Morita equivalent, then θ' must be in the same orbit of θ by the action of $GL(2,\mathbb{Z})$. Now since strongly Morita equivalent C*-algebras have same K-groups, we have that if $A_{F_{k,\theta}} \otimes M_n$ and $A_{F_{m,\theta'}} \otimes M_\ell$ are strongly Morita equivalent, then $|k| = |m|$ by Corollary (2.19). Conversely, if $|k| = |m|$ and θ and θ' are in the same orbit of $GL(2;\mathbb{Z})$, then $A_{F_{k,\theta}}$ is *-isomorphic to $A_{F_{m,\theta}}$, and $A_{F_{m,\theta}}$ is strongly Morita equivalent to $A_{F_{m,\theta'}}$, we have $A_{F_{k,\theta}} \otimes M_n \overset{SME}{\sim} A_{F_{k,\theta}} \overset{SME}{\sim} A_{F_{m,\theta}} \overset{SME}{\sim} A_{F_{m,\theta'}} \overset{SME}{\sim} A_{F_{m,'\theta}} \otimes M_\ell$, where "$\underset{\sim}{SME}$" stands for the terminology "strongly Morita equivalent to."

(4.12) We now consider the rational case.

<u>Theorem</u>. If θ is rational in $(0,1)$, then $A_{F_{k,\theta}} \otimes M_n$ is strongly Morita equivalent to $A_{F_{k,0}}$.

We need a lemma before proving this theorem.

(4.13) <u>Lemma</u>. For $\theta = \frac{q}{p}$ in $(0,1)$, where p, q are positive integers with $(p,q) = 1$, there exists a sequence $\{W_i\}_{i=1}^{n}$ of matrices in $GL(2;\mathbb{Z})$ and each W_i is one of the three matrices: $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ so that

$$(W_n \cdot W_{n-1} \cdots W_1) * \theta = 0$$

and

$$(W_i \cdots W_1) * \theta \neq 0 \quad \text{for } 1 \leq i < n.$$

Proof. Since $0 < q < p$, we assume $\frac{p}{q} = n_0 + \frac{p_1}{q}$ for $0 < p_1 < q$, we also have $(p_1,q) = 1$. Then let $W_1 = S$, $W_2 = \cdots = W_{n_0+1} = T^{-1}$, we will have $(W_{n_0+1}\cdots W_2\cdot W_1)*\theta = (W_{n_0+1}\cdots W_2)*((\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix})*\theta)$

$= (W_{n_0+1}\cdots W_2)*(n_0 + \frac{p_1}{q}) = (W_{n_0+1}\cdots W_3)*(n_0 - 1 + \frac{p_1}{q}) = \cdots = \frac{p_1}{q}$.

Proceeding with this construction, we will finally have the lemma.

(4.14) Proof of Theorem (4.12).

By the lemma, for $\theta = \frac{q}{p}, 0 < q < p$ and $<p,q> = 1$, we have $\{W_1,\cdots,W_n\}$ in $GL(2;\mathbb{Z})$ with W_i is one of the three matrices S, T and T^{-1} so that $(W_n\cdots W_1)*\theta = 0$ and $(W_i\cdots W_1)*\theta \neq 0$ $(i<n)$. Clearly, W_n must be of the form $(\begin{smallmatrix}1 & 0\\ \pm 1 & 1\end{smallmatrix})$. Otherwise $W_n = (\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix})$, but $\theta' = (W_{n-1}\cdots W_1)*\theta \neq 0$, hence $W_n*\theta' \neq 0$, contradicts to the assumption. Since each $(W_i\cdots W_1)*\theta \neq 0$ for $1 \leq i < n$, we can apply Proposition (4.9) and the trivial fact that $A_{F_{k,\theta}}$ is $*$-isomorphic to $A_{F_{k,\theta\pm m}}$ for any integer m.

Therefore, $A_{F_{k,\theta}} \overset{SME}{\sim} A_{F_{k,W_1*\theta}} \overset{SME}{\sim} A_{F_{k,(W_2W_1)*\theta}} \overset{SME}{\sim}$

$\overset{SME}{\sim}\cdots\overset{SME}{\sim} A_{F_{k,(W_n\cdots W_1)*\theta}} = A_{F_{k,0}}$. Hence we have

$A_{F_{k,\theta}} \otimes M_n \overset{SME}{\sim} A_{F_{k,\theta}} \overset{SME}{\sim} A_{F_{k,0}}$.

(4.15) To conclude this section, we will consider the classification of C*-algebras associated with minimal rotations on tori up to strong Morita equivalence. The classification of these C*-algebras up to *-isomorphism is already solved by N. Ridel in [48].

(4.16) <u>Definition</u>. Let G be a topological group, and g be in G. The <u>rotation</u> ρ_g of the group G is defined by multiplication $\rho_g(g') = g\cdot g'$ for all g' in G. The rotation ρ_g is called <u>minimal</u>, if g generates a dense subgroup of G.

(4.17) <u>Example</u>. Let G be the ordinary torus $\mathbb{T}^n$, the group multiplication and inverse operation are defined as follows. Let $(z_1,\ldots,z_n)$ and $(z_1',\ldots,z_n')$ be in $\mathbb{T}^n$, then

$$(z_1,\ldots,z_n)\cdot(z_1',\ldots,z_n') = (z_1\cdot z_1',\ldots,z_n\cdot z_n'),$$

and $$(z_1,\ldots,z_n)^{-1} = (\bar{z}_1,\ldots,\bar{z}_n).$$

The rotation $\rho_{(\theta_1,\ldots,\theta_n)}$ on $\mathbb{T}^n$ is defined to be

$$\rho_{(\theta_1,\ldots,\theta_n)}(z_1,\ldots,z_n) = (e^{2\pi i\theta_1}\cdot z_1,\ldots,e^{2\pi i\theta_n}\cdot z_n),$$

where $\theta_1,\ldots,\theta_n$ are all in $[0,1)$.

(4.18) <u>Theorem [41, Theorem 5]</u>. For any tracial state τ on $A = C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$, the range of the trace τ on $K_0(A)$ is $\mathbb{Z} + \theta_1\mathbb{Z}+\ldots+\theta_n\mathbb{Z}$.

This is a generalization of Corollary 3.4 in [48], see also [17].

(4.19) <u>Theorem</u> [48]. If $(\theta_1,\dots,\theta_n)$ is rationally independent set in $[0,1)$, then $C(\mathbb{T}^n)\times_{\rho(\theta_1,\dots,\theta_n)}\mathbb{Z}$ is $*$-isomorphic to $C(\mathbb{T}^n)\times_{\rho(\theta_1',\dots,\theta_n')}\mathbb{Z}$ if and only if $\mathbb{Z}+\theta_1\mathbb{Z}+\dots+\theta_n\mathbb{Z}$ $=\mathbb{Z}+\theta_1'\mathbb{Z}+\cdots+\theta_n'\mathbb{Z}$, where $(\theta_1',\dots,\theta_n')$ is a subset of $[0,1)$.

(4.20) To determine the strong Morita equivalence for these minimal rotation algebras on $\mathbb{T}^n$, we need Theorem (4.4) and the following algebraic lemma.

(4.21) <u>Lemma</u>. Let C_{ij} be the matrix $I_{n+1}+E_{i,j}$ in $GL(n+1;\mathbb{Z})$, where I_{n+1} is the identity matrix and $E_{i,j}$ is the $(n+1)$ by $(n+1)$ matrix in $M_{n+1}(\mathbb{Z})$ with 1 at the $(i,j)^{th}$ entry and 0 at all the other entry. Let also that

$$I_{ij} = \begin{array}{rc} & \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & \cdots & \cdots & 0 & 1 & \\ & & & \vdots & 1 & & & 0 & \\ & & & \vdots & & \ddots & & \vdots & \\ & & & 0 & & & 1 & \vdots & \\ & & & 1 & 0 & \cdots & \cdots & 0 & \\ j^{th}\rightarrow & & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{array}\right] \\ & \qquad\qquad\uparrow i^{th} \end{array}$$

in $GL(n+1,\mathbb{Z})$ and

$$\varepsilon_j = \left[\begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{array}\right] \quad (j^{th} \nearrow -1).$$

Then $GL(n+1,\mathbb{Z})$ is generated by all C_{ij} and I_{ij} and ε_j for $0\le j<i\le n$.

<u>Proof</u>. Note first for $0\le j<i\le n$, $I_{ij}C_{ij}I_{ij}=C_{ji}$ since

$I_{ij}(I+E_{ij})I_{ij} = I_{ij}^2 + I_{ij}E_{ij}I_{ij} = I_{n+1} + E_{jj}I_{ij} = I_{n+1} + E_{ji} = C_{ji}$.

Therefore, it suffices to show that all elementary matrices (see the definition in page 13 of "Integral Matrices," by M. Newman), and this is just the Theorem II.7 in page 24 of "Integral Matrices" by M. Newman [32].

(4.22) Theorem. Let $(\theta_1,\ldots,\theta_n) = \theta$ and $(\theta_1',\ldots,\theta_n') = \theta'$ where θ_i and θ_i', $i = 1,2,\ldots,n$ are all in $(0,1)$ and $(\theta_1,\ldots,\theta_n)$ is rationally independent. Then the C*-algebras $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots\theta_n)}\mathbb{Z}$ and $C(\mathbb{T}^n)\times_{\rho(\theta_1',\ldots,\theta_n')}\mathbb{Z}$ are strongly Morita equivalent if and only if $(\theta_1',\ldots,\theta_n')$ and $(\theta_1,\ldots,\theta_n)$ are in the same orbit of $GL(n+1;\mathbb{Z})$ (see Lemma (4.5)).

Proof. Let $\mathcal{A} = C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$ and $\mathcal{B} = C(\mathbb{T}^n)\times_{\rho(\theta_1',\ldots,\theta_n')}\mathbb{Z}$ If $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$ and $C(\mathbb{T}^n)\times_{\rho(\theta_1',\ldots,\theta_n')}\mathbb{Z}$ are strongly Morita equivalent, there is an $\mathcal{A}-\mathcal{B}$-equivalence bimodule X. Let τ be the unique tracial state on $\mathcal{A}$ by Corollary (1.13) and (1.18), and τ_X be the induced positive travial functional on $\mathcal{B}$ (see Theorem (4.2)). Then by Theorem (4.2), $\tau_X(K_0(\mathcal{B})) = \tau(K_0(\mathcal{A})) = \mathbb{Z}+\theta_1\mathbb{Z}+\ldots+\theta_n\mathbb{Z}$. On the other hand, any tracial state ϕ on $\mathcal{B}$ gives the range on $K_0(\mathcal{B})$

$$\phi(K_0(\mathcal{B})) = \mathbb{Z} + \theta_1'\mathbb{Z}+\cdots+\theta_n'\mathbb{Z}$$

by Theorem (4.18). Let $\tau_X = \frac{1}{r}\phi$ for some $r > 0$ and ϕ a tracial state on $\mathcal{B}$. Then we have

$$\gamma(\mathbb{Z}+\theta_1\mathbb{Z}+\ldots+\theta_n\mathbb{Z}) = \mathbb{Z} + \theta_1'\mathbb{Z}+\ldots+\theta_n'\mathbb{Z}.$$

Since $(\theta_1,\ldots,\theta_n)$ is rationally independent, $(\theta_1',\ldots,\theta_n')$ is too, by Theorem (4.4). Again by Theorem (4.4), $(\theta_1,\ldots,\theta_n)$ and $(\theta_1',\ldots,\theta_n')$ are both in the same orbit of $GL(n+1;\mathbb{Z})$ on S_n (see Definition in Lemma (4.5)).

Conversely, if $(\theta_1,\ldots,\theta_n)$ and $(\theta_1',\ldots,\theta_n')$ are both in the same orbit of $GL(n+1;\mathbb{Z})$, then $(\theta_1',\ldots,\theta_n')$ is rationally independent since $(\theta_1,\ldots,\theta_n)$ is.

Since $GL(n+1;\mathbb{Z})$ is generated by $\{C_{ij},I_{ij}\}_{i>j\geq 0}^{n}$ and $\{\varepsilon_k, k=0,1,2,\ldots,n\}$, see Lemma (4.21), we need only to prove the converse for those $(\theta_1',\ldots,\theta_n')$'s which are images of $(\theta_1,\ldots,\theta_n)$ by actions of C_{ij}, I_{ij} and ε_k for all $n \geq i > j \geq 0$, and $k = 0,1,2,\ldots,n$.

It is trivial that the C*-algebras $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$ and $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_{k-1},-\theta_k,\theta_{k+1},\ldots,\theta_n)}\mathbb{Z}$ are *-isomorphic, and so the converse is true for ε_k, $k > 0$, since, if $k > 0$,

$$\varepsilon_k * \begin{pmatrix}\theta_1\\ \vdots\\ \theta_n\end{pmatrix} = \begin{pmatrix}\theta_1\\ \vdots\\ \theta_{k-1}\\ -\theta_k\\ \theta_{k+1}\\ \vdots\\ \theta_n\end{pmatrix}.$$

If $k = 0$, we have

$$\varepsilon_0 * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}_{n+1} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}-\theta_1\\ \vdots\\ -\theta_n\end{bmatrix}$$

. But clearly,

$C(\mathbb{T}^n)\times_{\rho(-\theta_1,\ldots,-\theta_n)}\mathbb{Z}$ is also $*$-isomorphic to $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$.

Next, for $i > 0$, $C_{i,0} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ \theta_{i-1}\\ \theta_i+1\\ \theta_{i+1}\\ \vdots\\ \theta_n\end{bmatrix}$, but $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_n)}\mathbb{Z}$

is $*$-isomorphic to $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_i+1,\ldots\theta_n)}\mathbb{Z}$ is obvious, we checked the converse for $C_{i,0}$, $n \geq i > 0$. For $i > j \geq 1$,

$C_{i,j} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ \theta_{i-1}\\ \theta_i+\theta_j\\ \theta_{i+1}\\ \vdots\\ \theta_n\end{bmatrix}$, since for the unique tracial state

ϕ on $C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_{i-1},\theta_i+\theta_j,\theta_{i+1},\ldots,\theta_n)}\mathbb{Z}$, we still have

$$\begin{aligned}&\phi(K_0(C(\mathbb{T}^n)\times_{\rho(\theta_1,\ldots,\theta_{i-1},\theta_i+\theta_j,\theta_{i+1},\ldots,\theta_n)}\mathbb{Z}))\\ &= \mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_{i-1}\mathbb{Z} + (\theta_i+\theta_j)\mathbb{Z} + \theta_{i+1}\mathbb{Z} + \ldots + \theta_n\mathbb{Z}\\ &= \mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_i\mathbb{Z} + \ldots + \theta_n\mathbb{Z}.\end{aligned}$$

By Theorem (4.19), $C(\mathbb{T}^n)\times_{\rho(\theta_1,\dots,\theta_{i-1}\theta_i+\theta_j,\theta_{i+1},\dots\theta_n)}\mathbb{Z}$ is $*$-isomorphic to $C(\mathbb{T}^n)\times_{\rho(\theta_1,\dots,\theta_n)}\mathbb{Z}$.

Now we consider $I_{ij} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix}$. For $0 < j < i \leq n$, I_{ij} has the form

$$I_{ij} = \begin{bmatrix}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 & \cdots & \cdots & \cdots & 0\,1 &&& \\
&&& \vdots & 1 & & & 0 &&& \\
&&& \vdots & & \ddots & & \vdots &&& \\
&&& 0 & & & 1 & \vdots &&& \\
&&& 1\,0 & \cdots & \cdots & \cdots & 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{bmatrix}$$

hence $$I_{ij} * \begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ \theta_{j-1}\\ \theta_i\\ \theta_{j+1}\\ \vdots\\ \theta_{i-1}\\ \theta_j\\ \theta_{i+1}\\ \vdots\\ \theta_n\end{bmatrix}.$$ But clearly

$\rho(\theta_1,\dots,\theta_{j-1},\theta_i,\theta_{j+1},\dots,\theta_{i-1},\theta_j,\theta_{i+1},\dots\theta_n)$ is conjugate to $\rho(\theta_1,\dots,\theta_n)$ by a coordinate change, i.e. by $\gamma_{ij}(z_1,\dots,\overset{j^{th}}{z_j},\dots,\overset{i^{th}}{z_i},\dots z_n) = (z_1,\dots,\overset{j^{th}}{z_i},\dots,\overset{i^{th}}{z_j},\dots,z_n)$. Hence, $C(\mathbb{T}^n)\times_{\rho(\theta_1,\dots,\theta_n)}\mathbb{Z}$ is $*$-isomorphic to $C(\mathbb{T}^n)\times_{\rho(\theta_1,\dots,\theta_{j-1},\theta_i,\theta_{j+1},\dots,\theta_{i-1},\theta_j,\theta_{i+1},\dots\theta_n)}\mathbb{Z}$.

Finally, we consider the case when $0 = j < i$. Clearly,

$$I_{ij} * \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} \theta_{1/\theta_i} \\ \vdots \\ \theta_{i-1/\theta_i} \\ 1/\theta_i \\ \theta_{i+1/\theta_i} \\ \vdots \\ \theta_n \end{bmatrix}$$

we may assume $i = n$. Our task is to show that $C(\mathbb{T}^n) \times_{\rho(\frac{\theta_1}{\theta_n},\ldots,\frac{\theta_{n-1}}{\theta_n},\frac{1}{\theta_n})} \mathbb{Z}$ is strongly Morita equivalent to $C(\mathbb{T}^n) \times_{\rho(\theta_1,\ldots,\theta_n)} \mathbb{Z}$.

By Green's Theorem (4.8), we let $P = \mathbb{T}^{n-1} \times \mathbb{R}^1$ and $H = \mathbb{Z}$ act by α on P by translation of integers on the last coordinate. Let $K = \mathbb{Z}$ act by β on P by $\beta_k(z_1,\ldots,z_{n-1},t)$ $= (z_1 \cdot e^{2\pi i \theta_1 k},\ldots,z_{n-1} \cdot e^{2\pi i \theta_{n-1} k}, t+k\theta_n)$. Then both actions are free and wandering. Hence $C(P/H) \times_\beta K$ and $C(P/K) \times_\alpha H$ are strongly Morita equivalent since H and K act commutatively on P. Now $P/H = \mathbb{T}^n$ is the standard torus, and the induced action of K on P/H is just $\rho(\theta_1,\ldots,\theta_n)$, i.e. $C(P/H) \times_\beta K = A$. We claim P/K is also homeomorphic to $\mathbb{T}^n$. Let $\psi : P/K \to \mathbb{T}^n$ by

$$\psi((z_1,\ldots,z_{n-1},t)_K) = (z_1 \cdot e^{-2\pi i \frac{\theta_1}{\theta_n} t},\ldots,z_{n-1} \cdot e^{-2\pi i \frac{\theta_{n-1}}{\theta_n}}, e^{2\pi i \frac{t}{\theta_n}})$$

then we check that ψ is well-defined. We know $(z_1,\ldots,z_{n-1},t)_K$ $= (z_1 \cdot e^{2\pi i \theta_1},\ldots,z_{n-1} \cdot e^{2\pi i \theta_{n-1}}, t+\theta_n)_K$ in P/K, but

$$\psi((z_1 e^{2\pi i\theta_1},\ldots,z_{n-1}e^{2\pi i\theta_{n-1}},t+\theta_n)_K)$$

$$= (z_1 e^{2\pi i\theta_1}\cdot e^{-2\pi i\frac{\theta_1}{\theta_n}(t+\theta_n)},\ldots,z_{n-1}e^{2\pi i\theta_{n-1}}\cdot e^{-2\pi i\frac{\theta_{n-1}}{\theta_n}(t+\theta_n)},e^{2\pi i\frac{t+\theta_n}{\theta_n}})$$

$$= (z_1 e^{-2\pi i\frac{\theta_1}{\theta_n}t},\ldots,z_{n-1}e^{-2\pi i\frac{\theta_{n-1}}{\theta_n}t},e^{2\pi i\frac{t}{\theta_n}})$$

$$= \psi((z_1,\ldots,z_{n-1},t)_K).$$

The one-to-one and onto as well as continuity of ψ is easy to verify. So we have a homeomorphism ψ from P/K onto $\mathbb{T}^n$. The induced action H on P/K is transferred by ψ to $\tilde{\alpha} : \mathbb{T}^n \to \mathbb{T}^n$,

$$\tilde{\alpha}(z_1,\ldots,z_n) = \psi\alpha_1\psi^{-1}(z_1,\ldots,z_n) = \psi\alpha_1\psi^{-1}(z_1,\ldots,z_{n-1},e^{2\pi i t}),t \text{ in } [0,1)$$

$$= \psi\alpha_1(z_1 e^{2\pi i\theta_1 t},\ldots,z_{n-1}e^{2\pi i\theta_{n-1}t},\theta_n t)_K) = \psi((z_1 e^{2\pi i\theta_1 t},\ldots,z_{n-1}e^{2\pi i\theta_{n-1}t},\theta_n t+1)_K)$$

$$= (z_1 e^{2\pi i\theta_1 t}\cdot e^{-2\pi i\frac{\theta_1}{\theta_n}(\theta_n t+1)},\ldots,z_{n-1}\cdot e^{2\pi i\theta_{n-1}t}\cdot e^{-2\pi i\frac{\theta_{n-1}}{\theta_n}(\theta_n t+1)},e^{2\pi i(\frac{\theta_n t+1}{\theta_n})})$$

$$= (z_1 e^{-2\pi i\frac{\theta_1}{\theta_n}},\ldots,z_{n-1}e^{-2\pi i\frac{\theta_{n-1}}{\theta_n}},z_n e^{-2\pi i\frac{1}{\theta_n}})$$

Let $\overline{\alpha} : \mathbb{T}^n \to \mathbb{T}^n$ be the map

$$\overline{\alpha}(z_1,\ldots,z_n) = (z_1 e^{2\pi i\frac{\theta_1}{\theta_n}},\ldots,z_{n-1}e^{2\pi i\frac{\theta_{n-1}}{\theta_n}},z_n e^{2\pi i\frac{1}{\theta_n}})$$

and $\gamma : \mathbb{T}^n \to \mathbb{T}^n$ be the map

$$\gamma(z_1,\ldots,z_n) = (\overline{z}_1,\ldots,\overline{z}_n)\ .$$

Then it is easy to check that

$$\gamma^{-1}\tilde{\alpha}\gamma(z_1,\ldots,z_n) = \overline{\alpha}(z_1,\ldots,z_n).$$

This tells us that the C*-algebra $C(P/K)\times_\alpha H$ is isomorphic to $C(\mathbb{T}^n)\times_{\rho(\frac{\theta_1}{\theta_n},\ldots,\frac{\theta_{n-1}}{\theta_n},\frac{1}{\theta_n})}\mathbb{Z}$. Hence $C(\mathbb{T}^n)\times_{\rho(\theta_1\ldots\theta_n)}\mathbb{Z}$ is strongly Morita equivalent to $C(\mathbb{T}^n)\times_{\rho(\frac{\theta_1}{\theta_n},\ldots,\frac{\theta_{n-1}}{\theta_n},\frac{1}{\theta_n})}\mathbb{Z}$.

This completes the proof of the theorem.

§5. Affine transformations on tori with quasi-discrete spectrum

In Section 3 we have classified the C*-algebras associated with certain Furstenberg transformations on tori. The method of the classification is a combination of computing the K-groups and the ranges of tracial states on K_0-groups along with some other ad hoc techniques. Since Rieffel in [50], Pimsner and Voiculescu in [44] and Riedel in [48] used the same idea of computing the ranges of tracial states on K_0-groups we find that there is a uniform classification combining the classification of minimal rotation C*-algebras on tori and the classification of the Furstenberg transformation C*-algebras we considered in previous sections. In other words, we will consider a special class of affine transformations on an n-dimensional torus and obtain results on classifications which will include those in [48] and in §3 of this thesis. This may be thought as the first step of solving the problem of classifying all C*-algebras associated with affine transformations on tori. For simplicity, we only consider the following class of affine transformations. The general case of combining Furstenberg transformations and rotations on tori can be carried out in analogous fashion.

(5.1) Let $K = (k_1,\ldots,k_n)$ be a descending sequence of integers as defined in (2.16). Let $\Theta = (\theta_1,\ldots,\theta_n)$ be an n-tuple of numbers in [0,1). We denote by $F_{K,\Theta}$ the transformation on $\mathbb{T}^{n+1}$ defined by

$$F_{K,\Theta}(z_1,\ldots,z_n,z_{n+1},\ldots,z_{n+m}) = (z_1 z_2^{k_1},\ldots,z_n z_{n+1}^{k_n},e^{2\pi i\theta_1} z_{n+1},\ldots,e^{2\pi i\theta_m} z_{n+m}).$$

(5.2) Remark. $F_{K,\Theta}$ is an affine transformation on $\mathbb{T}^{n+m}$ with quasi-discrete spectum (see [23]). The C*-algebras associated with affine transformations with quasi-discrete spectrum on $\mathbb{T}^n$ were first considered by J. Packer in [34]. Hence either by a result of Packer in [34], a result in [18], or by our previous calculation using Pimsner's Theorem (2.21), we can calculate the range of any tracial state on the crossed product C*-algebras. We include a sketch of this which is similar to our proof of Proposition (2.22).

(5.3) Proposition. Let τ be a tracial state on $C(\mathbb{T}^{n+m})\times_{F_{K,\Theta}}\mathbb{Z}$. Then $\tau(K_0(C(\mathbb{T}^{n+m})\times_{F_{K,\Theta}}\mathbb{Z}) = \mathbb{Z} + \theta_1\mathbb{Z} +\ldots+ \theta_m\mathbb{Z}$.

Proof (sketch). Since the C*-algebras $A_\Theta = C(\mathbb{T}^m)\times_\Theta\mathbb{Z}$ is naturally contained in $A_{F_{K,\Theta}} = C(\mathbb{T}^{n+m})\times_{F_{K,\Theta}}\mathbb{Z}$, one can show as in the proof of Theorem (2.22) that

$$\tau(K_0(C(\mathbb{T}^{n+m}))\times_{F_{K,\Theta}}\mathbb{Z}) = \mathbb{Z} + \theta_1\mathbb{Z} +\ldots+ \theta_m\mathbb{Z},$$

by Theorem (4.18).

Let $A_{\theta_i} = C(\mathbb{T}^1)\times_{\theta_i}\mathbb{Z}$ be the rotation C*-algebra on $\mathbb{T}^1$. Then each A_{θ_i} is contained in A_Θ and hence in $A_{F_{K,\Theta}}$ in

$A_{F_{K,\Theta}}$ in a natural way. By the same argument in the proof of Theorem (2.22) and using Theorem (2.21), we obtain that the kernel of the map $(F_{K,\Theta}^{-1})_*-I$ on $K_1(C(\mathbb{T}^{n+m}))$ intersected with the subgroup generated by $[z_1],\ldots,[z_n],\ldots,[z_{n+m}]$ of $K_1(C(\mathbb{T}^{n+m}))$ where z_i's are the coordinate functions of $\mathbb{T}^{n+m}$, is just the subgroup of $K_1(C(\mathbb{T}^{n+m}))$ generated by $[z_{n+1}],\ldots,[z_{n+m}]$. We then determine the contribution of these generators of the range of the trace and find that $\tau(K_0(A_{F_{K,\Theta}})) = \mathbb{Z} + \theta_1\mathbb{Z}+\cdots+\theta_m\mathbb{Z}$, exactly as in the proof of Theorem (2.22).

(5.4) We now assume Θ is a rationally independent set. A theorem due to Hahn in [23] says that $F_{K,\Theta}$ is minimal and uniquely ergodic. Minimality follows from the uniqueness of the invariant probability measure (see Theorem 8 and Theorem 15 of [23]). Since Θ is rationally independent, Corollary (1.13) implies there is a unique tracial state on $A_{F_{K,\Theta}}$ for $n \geq 0$. Note that if $n = 0$, $F_{K,\Theta}$ reduces to the rotation ρ_Θ on $\mathbb{T}^m$. Thus we have the following theorem.

(5.5) <u>Theorem</u>. Let $F_{K,\Theta}$ be as in (5.1). Suppose Θ is rationally independent. Then $A_{F_{K,\Theta}} = C(\mathbb{T}^{n+m})\times_{F_{K,\Theta}}\mathbb{Z}$ is a simple C*-algebra with a unique tracial state.

(5.6) Proposition (2.17) can also be extended to the following form and the proof is just the same as that of (2.17).

<u>Proposition</u>. Let $F_{K,\Theta}$ be as in (5.1). Then the torsion subgroup

of $K_*(A_{F_{K,\Theta}})$ is isomorphic to $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus \cdots \oplus \mathbb{Z}_{k_n}^{(m_n)}$.

(5.7) Lemma. Let $F_{K,\Theta}$ be as in (5.1) and $F_{|K|,\Theta}$ be defined by $F_{|K|,\Theta}(z_1,\ldots,z_n,z_{n+1},\ldots,z_{n+m}) = (z_1 z_2^{|k_1|},\ldots,z_n z_{n+1}^{|k_n|}, e^{2\pi i\theta_1}\cdot z_{n+1}, \ldots, e^{2\pi i\theta_m}\cdot z_{n+m})$. Then $F_{K,\Theta}$ is topologically conjugate to $F_{|K|,\Theta}$.

The proof is similar to that of Lemma (3.8).

We can now prove the following theorem which contains the main result of Riedel for minimal rotations on tori (see [48]) and a classification result in § 3 of this thesis.

(5.8) Theorem. Let $F_{K,\Theta}$ and $F_{K',\Theta'}$ be as in (5.1) where $K = (k_1,\ldots,k_n)$, $K' = (k_1',\ldots,k_n')$, $\Theta = (\theta_1,\ldots,\theta_m)$ and $\Theta' = (\theta_1',\ldots,\theta_m')$. Suppose Θ is rationally independent and if $n > 0$, $m > 1$, we assume also that $\theta_1' = \theta_1$ or $1-\theta_1$. Then $A_{F_{K,\Theta}}$ is $*$-isomorphic to $A_{F_{K',\Theta'}}$ if and only if $|K'| = |K|$ and

$$\mathbb{Z} + \theta_1\mathbb{Z} + \cdots + \theta_m\mathbb{Z} = \mathbb{Z} + \theta_1'\mathbb{Z} + \cdots + \theta_m'\mathbb{Z} \qquad (*)$$

(5.9) Remark. When $n=0$, $F_{K,\Theta}$ is just the rotation ρ_Θ on $\mathbb{T}^m$, and when $n > 0$, $m = 1$, $F_{K,\Theta}$ is just the affine Furstenberg transformation $F_{K,\theta}$ on $\mathbb{T}^{n+1}$.

Proof of Theorem (5.8). We consider the case $n > 0$. By the assumption, $\theta_1' = \theta_1$ or $1-\theta_1$. If we set $\Theta'' = (1-\theta_1',\theta_2',\ldots,\theta_m')$, then clearly $F_{K',\Theta'}$ is conjugate to $F_{K',\Theta''}$. Hence, we assume that $\theta_1' = \theta_1$.

If $A_{F_{K,\Theta}}$ is *-isomorphic to $A_{F_{K',\Theta'}}$, then by Proposition (5.6), $|K| = |K'|$, and by Proposition (5.3), (*) holds.

If $|K| = |K'|$ and (*) holds, $\theta_1' = \theta_1$, we will show that $A_{F_{K,\Theta}}$ is *-isomorphic to $A_{F_{K',\Theta'}}$.

By Lemma (5.7) one can assume $K = K' = |K|$. Since Θ is rationally independent and $\mathbb{Z} + \theta_1\mathbb{Z}+\dots+\theta_m\mathbb{Z} = \mathbb{Z}+\theta_1'\mathbb{Z}+\dots+\theta_m'\mathbb{Z}$, by Lemma (4.6) there is a matrix $A = (a_{ij})_{m+1}$ in $GL(m+1;\mathbb{Z})$, so that
$$A * \begin{pmatrix}\theta_1\\ \vdots\\ \theta_m\end{pmatrix} = \begin{pmatrix}\theta_1'\\ \vdots\\ \theta_m'\end{pmatrix}.$$
Hence Θ' is also rationally independent. By the assumption $\mathbb{Z} + \theta_1\mathbb{Z}+\dots+\theta_m\mathbb{Z} = \mathbb{Z} + \theta_1'\mathbb{Z}+\dots+\theta_m'\mathbb{Z}$ and $\theta_1' = \theta_1$, we see that the matrix $A = (a_{ij})_{m+1}$ is of the form
$$A = \begin{pmatrix}\pm1 & 0 & 0 & \cdots & 0\\ 0 & \pm1 & 0 & \cdots & 0\\ a_{20} & a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mm}\end{pmatrix} = \begin{pmatrix}\pm1 & 0 & \\ 0 & \pm1 & 0\\ a_{20} & a_{21} & \\ \vdots & \vdots & A_{m-1}'\\ a_{m0} & a_{m1} & \end{pmatrix}$$
where A_{m-1}' is in $GL(m-1;\mathbb{Z})$. Without loss of generality, we may assume
$$A = \begin{pmatrix}1 & 0 & \\ 0 & 1 & 0\\ & a & A_{m-1}'\end{pmatrix}, \quad \text{where} \quad a = \begin{pmatrix}a_{20} & a_{21}\\ \vdots & \vdots\\ a_{m0} & a_{m1}\end{pmatrix}_{(m-1)\times 2}.$$
Let G denote all such matrices in $GL(m+1;\mathbb{Z})$. Then G is a subgroup of $GL(m+1;\mathbb{Z})$. The above analysis shows that we need only to prove the converse for those Θ' which are in the same orbit of Θ by the action of G. But clearly, by Lemma (4.21), G is

contained in the group generated by the following elementary matrices in $GL(m+1;\mathbb{Z})$,

$$\{\varepsilon_k \mid k = 2,\ldots,m\}, \text{ where } \varepsilon_k = \begin{array}{c} \begin{array}{cccccc} 0^{th} & & & k^{th} & & \end{array} \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \end{array},$$

and $\{C_{ij} \mid 1 \le j < i \le m\}$, where

$$C_{ij} = \begin{array}{c} \\ \\ j^{th} \\ \\ \\ \\ i^{th} \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & 0 & \ddots & & & & \\ & & \vdots & & \ddots & & & \\ & & 0 & & & \ddots & & \\ & & 1 & 0 & \cdots & 0 & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{bmatrix}$$

and also $\{I_{ij} \mid 2 \le j < i \le m\}$, where

$$I_{ij} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & \cdots & 0 & 1 & & \\ & & & \vdots & 1 & & 0 & & \\ & & & \vdots & & \ddots & \vdots & & \\ & & & 0 & & 1 & \vdots & & \\ & & & 1 & 0 & \cdots & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix}$$

as defined in Lemma (4.21).

Therefore, we need only to check those Θ' which are the images of Θ by the actions of these matrices.

First of all, $\varepsilon_k * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_m\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ -\theta_k\\ \vdots\\ \theta_m\end{bmatrix}$ and therefore,

$F_{K,(\theta_1,\ldots,-\theta_k,\ldots,\theta_m)}$ is conjugate to $F_{K,(\theta_1,\cdots,\theta_n)} = F_{K,\Theta}$.

Next, for $C_{ij}, 1 \le j < i \le m$, we have $C_{ij} * \begin{bmatrix}\theta_1\\ \vdots\\ \theta_m\end{bmatrix} = \begin{bmatrix}\theta_1\\ \vdots\\ \theta_i+\theta_j\\ \vdots\\ \theta_m\end{bmatrix} \leftarrow i^{th}$

Since $i > j \ge 1$ we may assume $i = m$. Now the affine transformations $F_{K,\Theta'}$ is conjugate to $F_{K,\Theta}$, where $\Theta' = (\theta_1,\ldots,\theta_{m-1},\theta_m+\theta_j)$, as follows. Let $\phi : \mathbb{T}^{n+m} \to \mathbb{T}^{n+m}$ be the map defined by

$$\phi(z_1,\ldots,z_{n+m}) = (z_1,\ldots,z_{n+m-1},z_{n+m}\cdot z_{n+j}).$$

Then we have,

$$\begin{aligned}
&\phi\, F_{K,\Theta}(z_1,\ldots,z_{n+m})\\
&= \phi(z_1 z_2^{k_1},\ldots,z_n z_{n+1}^{k_n},e^{2\pi i\theta_1}\cdot z_{n+1},\ldots,e^{2\pi i\theta_m}\cdot z_{n+m})\\
&= (z_1 z_2^{k_1},\ldots,z_n z_{n+1}^{k_n},e^{2\pi i\theta_1}\cdot z_{n+1},\ldots,e^{2\pi i(\theta_j+\theta_m)}\cdot z_{n+m}\cdot z_{n+j}),
\end{aligned}$$

and $F_{K,\Theta'}\phi(z_1,\ldots,z_{n+m})$

$$= F_{K,\Theta'}(z_1,\ldots,z_{n+m-1},z_{n+m}\cdot z_{n+j})$$

$$= (z_1 z_2^{k_1}, \ldots, z_n z_{n+1}^{k_n}, e^{2\pi i\theta_1} \cdot z_{n+1}, \ldots, e^{2\pi i\theta_{m-1}} \cdot z_{n+m-1}, e^{2\pi i(\theta_m+\theta_j)} \cdot z_{n+m} \cdot z_{n+j})$$

and hence, $\phi F_{K,\Theta} = F_{K,\Theta'}\phi$.

Finally, for I_{ij}, $2 \le j < i \le m$ we have,

$$I_{ij} * \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_j \\ \vdots \\ \theta_i \\ \vdots \\ \theta_m \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_i \\ \vdots \\ \theta_j \\ \vdots \\ \theta_m \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow j^{th}, \quad 2 \le j < i \le m. \\ \\ \leftarrow i^{th} \\ \\ \\ \end{array}$$

It is obvious (since $2 \le j < i$) that $F_{K,\Theta'}$ is conjugate to $F_{K,\Theta}$, where $\Theta' = (\theta_1, \ldots, \underset{\substack{\uparrow \\ j^{th}}}{\theta_i}, \ldots, \underset{\substack{\uparrow \\ i^{th}}}{\theta_j}, \ldots, \theta_m)$, by a coordinate change.

Therefore, $A_{F_{K,\Theta}}$ is $*$-isomorphic to $A_{F_{K',\Theta'}}$ if $|K| = |K'|$ and $\mathbb{Z} + \theta_1\mathbb{Z} + \ldots + \theta_m\mathbb{Z} = \mathbb{Z} + \theta_1'\mathbb{Z} + \ldots + \theta_m'\mathbb{Z}$, where $m > 1$ and $n > 0$, $\theta_1' = \theta_1$ or $1 - \theta_1$.

Now if $m = 1$, $n > 0$, we get a descending Furstenberg transformation $F_{K,\theta}$, and the results in Section 3 yields the desired conclusion.

If $m > 0$ and $n = 0$, then we have a rotation ρ_Θ on $\mathbb{T}^n$. The above proof also applies to this situation.

This completes the proof of the theorem.

§6. Concluding remarks

(6.1) In previous sections, by means of K-theory, we have classified the C*-algebras associated with special classes of Furstenberg transformations on tori. We also determined strong Morita equivalence for the C*-algebras associated with Furstenberg transformations which are affine transformations on the two dimensional torus $\mathbb{T}^2$. It is natural to ask if it is possible to classify the C*-algebras associated with all Furstenberg transformations and to determine the strong Morita equivalence for them. The answer is no at present for lacking of new invariants of C*-algebras. For example, if we consider the affine Furstenberg transformation $F_{(a,b,\theta)}$ on the 3-torus $\mathbb{T}^3$, defined by

$$F_{(a,b,\theta)}(z_1,z_2,z_3) = (z_1 \cdot z_2^a, z_2 \cdot z_3^b, e^{2\pi i\theta} \cdot z_3)$$

where a and b are positive integers. Then for i = 0,1,

$$K_i(A_{F_{(a,b,\theta)}}) \cong K_i(A_{F_{(b,a,\theta)}})$$

and a tracial state on $A_{F_{(a,b,\theta)}}$ or on $A_{F_{(b,a,\theta)}}$ has the same range $\mathbb{Z} + \theta\mathbb{Z}$. Moreover, the positive cones of $K_0(A_{F_{(a,b,\theta)}})$ and $K_0(A_{F_{(b,a,\theta)}})$ are also the same. But $F_{(b,a,\theta)}$ is not topologically conjugate to $F_{(a,b,\theta)}$ or $F^{-1}_{(a,b,\theta)}$.

Another example. If $F_{f,\theta} : \mathbb{T}^2 \to \mathbb{T}^2$ is the Furstenberg transformation defined by $F_{f,\theta}(z_1,z_2) = (z_1 \cdot f(z_2), e^{2\pi i\theta} \cdot z_2)$

where the degree of f is k, then $F_{f,\theta}$ is homotopic to $F_{k,\theta}$ as in Proposition (4.9). Unlike the situation for rotations on $\mathbb{T}^1$, the C*-algebras $A_{F_{f,\theta}}$ and $A_{F_{k,\theta}}$ not only have the same K-theory but also have the same trace range on the K_0-groups. But, in general, we do not know if $F_{f,\theta}$ is conjugate to $F_{k,\theta}$ or $F_{k,\theta}^{-1}$. These two examples makes us believe that to determine the isomorphism classes of the C*-algebras is a very subtle matter. A possible way of attacking the classification of the C*-algebras in the second example is to consider $A_{F_{f,\theta}}$ as the crossed product C*-algebra of the C*-dynamical system $(A_\theta, \alpha_f, \mathbb{Z})$, where A_θ is the rotation C*-algebra on $\mathbb{T}^1$ with two canonical generators of unitary elements U, V, s.t. $UV = e^{2\pi i\theta}VU$, and α_f is defined to be $\alpha_f(v) = v$ and $\alpha_f(u) = uf(v)$, where f(v) is the functional calculus of v by the function f. One can easily check that α_f is a *-automorphism of A_θ, by the universal property of A_θ. On this non-commutative C*-algebra A_θ, we might have more room for effecting the <u>exterior</u> <u>equivalence</u> of α_f and α_{z^k} (see [37], [38]).

(6.2) In Section 4 we have shown that for rational θ in [0,1), the C*-algebras $A_{F_{k,\theta}}$ on $\mathbb{T}^2$ are all strongly Morita equivalent to $A_{F_{k,0}}$. Let $\theta = \frac{q}{p}$ and $\theta' = \frac{q'}{p'}$ where (p,q) = 1, (q',p') = 1 and p, q, p', q' are all positive. If $p \neq p'$, then as we indicated in the last paragraph in §3, $A_{F_{k,\frac{q}{p}}}$ is not *-isomorphic to $A_{F_{k,\frac{q'}{p'}}}$. But if $p = p'$ and $q \neq -q'$, we do not know the answer yet. However, the idea for the classification for rational rotation C*-algebras A_θ on $\mathbb{T}^1$ given in [52] could provide a method for classifying the algebras $A_{F_{k,\theta}}$ with rational in [0,1). But in this case, since the primitive ideal space of $A_{F_{k,\theta}}$ is quite complicated, one has to find some new arguments to deal with it. We would like to conjecture that $A_{F_{k,\theta}}$ is *-isomorphic to $A_{F_{k,\theta'}}$ for θ and θ' in [0,1), which are rational, if and only if $\theta' = \theta$ or $1 - \theta$.

(6.3) In Section 4, we also determined the strong Morita equivalence for the minimal rotation C*-algebras on tori considered by Riedel [48]. In general, certain non-minimal rotation C*-algebras on tori have been classified in [14] and [56]. The strong Morita equivalence for these C*-algebras is still open. Using the idea of §4 and the proof of Theorem (4.12), the strong Morita equivalence for non-minimal rotation C*-algebras on tori is at hand. We will give a proof for this in [27]. The idea in §4 can also be applied to give a proof of the strong Morita equivalence for certain non-commutative tori, (see also [27]), although the classification of non-commutative tori is a very difficult matter, see [17], [10] and [14].

(6.4) There is a possible direction to generalize the present work of this thesis which will be of interest. We consider the so-called "affine transformations" on non-commutative tori. Let $u_1,\dots,u_n$ be n-unitaries satisfying the commutation relations $u_i u_j = \lambda_{ij} u_j u_i$ (*), for all $i < j = 1,2,\dots,n$, where λ_{ij} is in $\mathbb{C}$ with $|\lambda_{ij}| = 1$. The universal representation of these n-unitaries satisfying (*), gives a class of C*-algebras which are called the non-commutative tori, (see [17]). Another way to interpret these non-commutative tori is as the twisted group C*-algebras, $C^*(\mathbb{Z}^n;\sigma)$ (see [60], where σ is a two cocycle of $\mathbb{Z}_n$), which is generated by n unitaries $\{u_{I_i}\}_{i=1}^n$ satisfying the relations $u_{I_i} u_{I_j} = \sigma(I_i,I_j) u_{I_j} u_{I_i}$ where $\{I_i\}_{i=1}^n$ are the canonical generators of $\mathbb{Z}^n$.

One would expect to define the affine Furstenberg transformations on $C^*(\mathbb{Z}^n;\sigma)$ by

$$F_{K,\theta} : u_i \to u_i u_{i+1}^{k_{i+1}^i} \cdots u_n^{k_n^i} \qquad i = 1,2,\dots,n-1,$$

and $F_{K,\theta} : u_n \to e^{2\pi i\theta} \cdot u_n$.

But a simple calculation shows that, this could be done for all two cocycles if and only if $n = 2$. Packer in [35] and [36] classified $C^*(\mathbb{Z}^2;\sigma) \times_{F_{1,\theta}} \mathbb{Z}$ where θ is in $[0,1)$ and $F_{1,\theta}(u_1) = u_1 u_2$, $F_{1,\theta}(u_2) = e^{2\pi i\theta} \cdot u_2$, as described above. For $n \geq 3$ this could be also done either for some affine Furstenberg transformations or for all affine Furstenberg transformations but some very special 2-cocycles. Hence we would like to reformulate the question, that is, we consider the twisted group

C^*-algebra $C^*(\mathbb{Z}^n \times_A \mathbb{Z};\sigma)$, where $\mathbb{Z}^n \times_A \mathbb{Z}$ is the semidirect product of the group $\mathbb{Z}^n$ by an automorphism A of $\mathbb{Z}^n$, and σ is a two cocycle of $\mathbb{Z}^n \times_A \mathbb{Z}$. Denote ρ the restriction of σ to $\mathbb{Z}^n$ and denote by u_A the unitary in $C^*(\mathbb{Z}^n \times_A \mathbb{Z};\sigma)$ which represents the element $(0,\ldots,0,1)$ in $\mathbb{Z}^n \times_A \mathbb{Z}$. Then u_A induces the automorphism $\mathrm{ad}u_A$ defined by conjugation on $C^*(\mathbb{Z}^n;\rho)$ which is what we would like to call an "affine transformation" on the non-commutative torus $C^*(\mathbb{Z}^n;\rho)$. If A is of the form

$$\begin{bmatrix} 1 & k_2^1 & \cdots & & k_n^1 \\ & 1 & k_3^2 & \cdots & k_n^2 \\ & & \ddots & & \vdots \\ & & & 1 & k_n^{n-1} \\ & & & & 1 \end{bmatrix}, \quad k_{i+1}^i \neq 0,\ i = 1,2,\ldots,n-1, \text{ then we would}$$

like to call $\mathrm{ad}u_A$ an affine Furstenberg transformation on $C^*(\mathbb{Z}^n;\rho)$. If ρ is not rational in the sense of Rieffel in [53], this is, the range of ρ is not contained in the set $\{\lambda\varepsilon\, \mathbb{T}^1 | e^{2\pi i\theta} = \lambda \text{ and } \theta \text{ is rational}\}$, then $C^*(\mathbb{Z}^n;\rho)$ has cancellation (see [53]), and therefore, the computation of the range of a tracial state on $K_0(C^*(\mathbb{Z}^n;\rho)\times_{\mathrm{ad}u_A}\mathbb{Z}) = K_0(C^*(\mathbb{Z}^n \times_A \mathbb{Z};\sigma))$ is similar to that given in Section 2 without difficulties. To compute the K-theory for $C^*(\mathbb{Z}^n \times_A \mathbb{Z};\sigma) \cong C^*(\mathbb{Z}^n;\rho)\times_{\mathrm{ad}u_A}\mathbb{Z}$, we first have to analyse $K_*(C^*(\mathbb{Z}^n;\sigma))$ which is computed in [17] to be $\Lambda\mathbb{Z}^n$ as an abelian group. Hence, by a computation similar to what we did in §2, we can find the group $K_*(C^*(\mathbb{Z}^n \times_A \mathbb{Z};\sigma))$ or at least know the torsion subgroup of it. Therefore, the classification of these C*-algebras is possible. We will consider this program in a subsequent paper.

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Appendix

In Section 2, we stated Theorem (2.13). In this appendix, we will give a complete proof.

The cohomology appearing in this appendix is the Čech cohomology or sheaf cohomology.

(A.1) <u>Proposition</u>. The cohomology algebra $H^*(\mathbb{T}^n;\mathbb{Z})$ is naturally isomorphic to the exterior algebra $\Lambda^* H^1(\mathbb{T}^n;\mathbb{Z})$.

<u>Proof</u>. For n = 2 it is proved in [22] and [57]. In general, by the Künneth formula, $H^*(\mathbb{T}^n;\mathbb{Z})$ is isomorphic to the graded tensor product algebra $\hat{\otimes}_n H^*(\mathbb{T}^1;\mathbb{Z})$ of n-copies of the graded algebra $H^*(\mathbb{T}^1;\mathbb{Z})$. By the construction of the isomorphism and assuming $\{[z_i]\}_{i=1}^n$ is the canonical basis in $H^1(\mathbb{T}^n;\mathbb{Z})$ given by the coordinate functions $\{z_i\}_{i=1}^n$ in $C^{-1}(\mathbb{T}^n)$ (the invertible complex valued continuous functions over $\mathbb{T}^n$), we know that each $[z_i]$ in $H^1(\mathbb{T}^n;\mathbb{Z})$ corresponds to $1\hat{\otimes}\ldots\hat{\otimes}1\hat{\otimes}\underset{i^{th}}{[z]}\hat{\otimes}1\hat{\otimes}\ldots\hat{\otimes}1$ in $\otimes_n H^*(\mathbb{T}^1;\mathbb{Z})$. Moreover, the cup product in $H^*(\mathbb{T}^n;\mathbb{Z})$ corresponds to the graded cup product in $\hat{\otimes}_n H^*(\mathbb{T}^1;\mathbb{Z})$, that is, if a in $H^i(\mathbb{T}^n;\mathbb{Z})$ and b in $H^j(\mathbb{T}^n;\mathbb{Z})$ have images $a_{i_1}\hat{\otimes}a_{i_2}\hat{\otimes}\ldots\hat{\otimes}a_{i_n}$ and $b_{j_1}\hat{\otimes}\ldots\hat{\otimes}b_{j_n}$ in $\hat{\otimes}_n H^*(\mathbb{T}^1;\mathbb{Z})$, respectively, where a_{i_ℓ} is of degree i_ℓ and b_{j_ℓ} is of degree j_ℓ, $\ell = 1,2,\ldots,n$, and $i_1+\ldots+i_n = i$, $j_1+\cdots+j_n = j$, then the images of $a \cup b$ is $(-1)^{(i_1+\ldots+i_n)j_1+\ldots+i_n\cdot j_{n-1}}(a_{i_1}\cup b_{j_1})\hat{\otimes}\ldots\hat{\otimes}(a_{i_n}\cup b_{j_n})$. In

particular, the image of $[z_i] \cup [z_i]$ is

$1 \hat{\otimes} \dots \hat{\otimes} 1 \hat{\otimes} \underset{i^{\text{th}}}{([z] \cup [z])} \hat{\otimes} 1 \hat{\otimes} \dots \hat{\otimes} 1 = 0$ in $\underset{n}{\otimes} H^*(\mathbb{T}^1;\mathbb{Z})$. Therefore,

$[z_i] \cup [z_i] = 0$ in $H^*(\mathbb{T}^n;\mathbb{Z})$. On the other hand, $[z_i] \cup [z_j] = -[z_j] \cup [z_i]$ by the construction of the cup product. Thus we have shown that $H^*(\mathbb{T}^n;\mathbb{Z})$ is naturally isomorphic to $\Lambda^* H^1(\mathbb{T}^n;\mathbb{Z})$ as rings, where the cup product "$\cup$" in $H^*(\mathbb{T}^n;\mathbb{Z})$ corresponds to the wedge product "Λ" in $\Lambda^* H^1(\mathbb{T}^n;\mathbb{Z})$.

(A.2) Next, we state several theorems from [24], which are crucial in the proof of integrality of the Chern character on $K^*(\mathbb{T}^n)$.

(A.3) <u>Theorem</u> (Theorem 3.2.1 [24]). The isomorphism classes of fibre bundles over X with structure group G and fibre F (with a given effective continuous action of G on F) are in a natural one-one correspondence with the elements of the cohomology set $H^1(X;G_c)$. The trivial bundle $W = X \times F$ corresponds to the distinguished element $\mathbb{1}$ in $H^1(X;G_c)$, where G is a topological group and G_c is the sheaf of germs of continuous functions on X with values in G.

(A.4) <u>Proposition</u> ((3.8) in [24]). The map $\delta^1_* : H^1(X;\mathbb{C}^*_c) \to H^2(X;\mathbb{Z})$ is a natural isomorphism, where δ^1_* is the connecting homomorphism in the long exact sequence,

$$\cdots \to H^1(X;\mathbb{C}_c) \to H^1(X;\mathbb{C}^*_c) \xrightarrow{\delta^1_*} H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C}_c) \to \cdots$$

where $\mathbb{C}$ is the topological group of all complex numbers, $\mathbb{C}^*$ is the group of all non-zero complex numbers and $\mathbb{Z}$ is the group of integers.

(A.5) Theorem (Theorem 4.3.1, [24]). Let ξ be a complex line bundle over X and let $c_1(\xi)$ be the first Chern class of ξ. Then $c_1(\xi) = \delta^1_*(\xi)$, where ξ is viewed as an element of $H^1(X;\mathbb{C}^*_c)$ by Theorem (A.3).

(A.6) Corollary. Let $X = S\mathbb{T}^n$ be the reduced suspension of the n-torus $\mathbb{T}^n$. Then the first Chern class c_1 defines a one-to-one and onto natural isomorphism from the set of all isomorphic classes of complex line bundles over $S\mathbb{T}^n$ to the set of the first Čech cohomology group $H^1(\mathbb{T}^n;\mathbb{Z})$.

Proof. Since $H^2(S\mathbb{T}^n;\mathbb{Z})$ is naturally isomorphic to $H^1(\mathbb{T}^n;\mathbb{Z})$, the corollary follows from Theorem (A.5) and Proposition (4.4).

(A.7) Theorem. The Chern character $ch: K^*(\mathbb{T}^n) \to H^*(\mathbb{T}^n;\mathbb{Q})$ is integral, and it maps $K^*(\mathbb{T}^n)$ naturally and isomorphically onto $H^*(\mathbb{T}^n;\mathbb{Z})$.

Proof. Clearly $K^*(\mathbb{T}^n)$ and $H^*(\mathbb{T}^n;\mathbb{Z})$ both are isomorphic to $\mathbb{Z}^{2^n}$. By the second corollary of (2.5) in [2] ch is injective. It suffices to show that the image of ch contains $H^1(\mathbb{T}^n;\mathbb{Z})$. Because, if so, then $ch(K^*(\mathbb{T}^n))$ contains $H^*(\mathbb{T}^n)$ since ch is a ring isomorphism and $H^1(\mathbb{T}^n)$ generates $H^*(\mathbb{T}^n;\mathbb{Z})$ by Proposition

(A.1). Since ch is injective and $K^*(\mathbb{T}^n)$ and $H^*(\mathbb{T}^n;\mathbb{Z})$ have the same rank as free abelian groups we must have $ch(K^*(\mathbb{T}^n))$ is equal to $H^*(\mathbb{T}^n;\mathbb{Z})$ considered as a subring of $H^*(\mathbb{T}^n;\mathbb{Q})$. The naturality of ch is also known as a theorem in [AH].

Let $[\xi] - [\mathbb{1}]$ be the formal difference of the complex line bundle ξ and the trivial complex line bundle $\mathbb{1}$ over $S\mathbb{T}^n$. By the natural isomorphism $K^1(\mathbb{T}) \overset{\psi}{\cong} \tilde{K}^\circ(S\mathbb{T}^n)$, where $\tilde{K}^\circ$ is the reduced K°-group (see [A]), we can view $[\xi] - [\mathbb{1}]$ an element in $K^1(\mathbb{T}^n)$. Since ch is additive, $ch([\xi]-[\mathbb{1}]) = ch([\xi]) - ch([\mathbb{1}])$, where $ch[\xi] = ch\ \xi$ and $ch[\mathbb{1}] = ch\ \mathbb{1}$ are both in $H^*(S\mathbb{T}^n;\mathbb{Q})$. Since all higher powers of elements in $H^1(\mathbb{T}^n;\mathbb{Z})$ are equal to zero, i.e., if x is in $H^1(\mathbb{T}^n;\mathbb{Z})$, then $x^n = 0$ in $H^*(\mathbb{T}^n;\mathbb{Z})$ for $n \geq 2$, by Theorem (A.1), and we have $ch\ \xi = \text{rank}\ \xi + c_1(\xi) = 1 + c_1(\xi)$ and $ch\ \mathbb{1} = 1$. Here we have used the fact that the Chern classes $c_i(\xi) = 0$ for $i \geq 2$, because ξ is a complex line bundle (see [31]). Hence, $ch([\xi]-[\mathbb{1}]) = c_1(\xi)$ is in $H^2(S\mathbb{T}^n;\mathbb{Z})$. Let φ be the natural isomorphism from $H^2(S\mathbb{T}^n;\mathbb{Z})$ onto $H^1(\mathbb{T}^n;\mathbb{Z})$. We have, $\phi \circ ch \circ \psi : \psi^{-1}([\xi]-[\mathbb{1}]) \to \phi(c_1(\xi))$ is the usual Chern character defined on $\psi^{-1}([\xi]-[\mathbb{1}])$ as an element in $K^1(\mathbb{T}^n)$ (see [2]). According to Corollary (A.6), c_1 is a natural bijection between isomorphism classes of complex line bundles over $S\mathbb{T}^n$ and $H^1(\mathbb{T}^n;\mathbb{Z})$. Letting ξ range over $S\mathbb{T}^n$, we get all of the elements in $H^1(\mathbb{T}^n;\mathbb{Z})$ since ϕ is an isomorphism. In other words, $ch(K^1(\mathbb{T}^n))$ contains all elements in $H^1(\mathbb{T}^n;\mathbb{Z})$, which completes the proof.