Projective Structures on Riemann Surfaces

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Abstract of the Dissertation

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Let Γ be a Fuchsian group acting on the upper half-plane U and having signature $(p,n,0;\nu_1,\nu_2,\cdots\nu_n)$; $2p-2+\sum_{j=1}^{n}(1-1/\nu_j)>0$. $B_2(U,\Gamma)$ is the space of bounded quadratic differentials for Γ. There is a complex vector bundle $\mathfrak{B}(T(\Gamma))$ over the Teichmüller space $T(\Gamma)$ of Γ such that the fiber of $\mathfrak{B}(T(\Gamma))$ over the point representing the group Γ is the space $B_2(U,\Gamma)$.

A given $(t,\varphi)\in\mathfrak{B}(T(\Gamma))$ defines an equivalence class of bounded projective structures with Schwarzian derivative

φ, and there exists a conjugacy class of a homomorphism of Γ into the Moebius group G. Let Hom(Γ,G) denote the set of all homomorphisms of Γ into G. Then we have a well defined map

$$\Phi : \mathcal{B}(T(\Gamma)) \to \mathrm{Hom}(\Gamma,G)/G.$$

We prove that the map Φ is a holomorphic local homeomorphism. Earle, Hejhal and Hubbard proved this result for compact surfaces of genus > 1.

We also formulate a uniqueness theorem for bounded reflectable deformations of a Fuchsian group of the second kind. This gives a generalization of a uniqueness theorem by Kra for deformations of a Fuchsian group of the first kind.

We obtain a splitting property of a well known short exact sequence which associates to a Fuchsian group of the second kind its Schottky-double.

To my parents:

Srimati Shova Rani Dana

and

Sri Manik Chandra Dana

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Chapter 0

Introduction

A projective structure on a Riemann surface M is an equivalence class of a coordinate coverings of M where the transition maps are projective maps (that is, Moebius transformations).

By Koebe's Uniformization Theorem, any compact Riemann surface M of genus > 1 can be expressed as the quotient of its universal covering U by a group of covering transformations Γ; U is the upper half-plane and Γ is a discontinuous subgroup of the group G of Moebius transformations.

Given $\varphi \in Q_2(U,\Gamma)$, the space of quadratic differentials for Γ, let f be a meromorphic solution in U of the Schwarzian differential equation $\mathcal{S}f = \varphi$. Then f is a holomorphic local homeomorphism from U into the extended complex plane $\hat{\mathbb{C}}$, and there exists a homomorphism χ from Γ into G such that

$$(0.1) \qquad f(\gamma(z)) = \chi(\gamma)(f(z)), \quad \text{for all } \gamma \in \Gamma, \quad z \in U.$$

The mapping f can be viewed as describing a projective structure on M. Kra defined the pair (f,χ) as a deformation of Γ.

Modern writers who have investigated projective structures include Earle, Gunning, Hejhal, Hubbard, Kra, Maskit and others.

Most of their attention has focused on the Monodromy map. There are two versions of the Monodromy map in the literature. We define the simpler one first.

Any meromorphic function g on U with $\mathcal{S}g = \varphi$ is given by $g = A \circ f$, $A \in G$, where f is a unique, suitably normalized, function such that $\mathcal{S}f = \varphi$. Replacing f by $A \circ f$ has the effect of conjugating χ by A in (0.1). Let $\mathrm{Hom}(\Gamma, G)$ denote the set of all homomorphisms of Γ into G. Then, we have a well defined map

$$\phi_M : Q_2(U,\Gamma) \to \mathrm{Hom}(\Gamma,G)/G.$$

ϕ_M is called the Monodromy map.

Another version of the Monodromy map involves variation in the quadratic differentials on a varying Riemann surface. Let T_p be the Teichmüller space of compact Riemann surfaces of genus $p > 1$. There exists a complex vector bundle TQ of rank $3p-3$ over T_p whose fiber over a point representing M is the space $Q_2(U,\Gamma)$ [5].

For a given $(t,\varphi) \in TQ$ we get an equivalence class of projective structures and a conjugacy class of $\chi \in \mathrm{Hom}(\Gamma,G)$ as before. Therefore, we have a map

$$\phi : TQ \to \mathrm{Hom}(\Gamma(G)/G.$$

ϕ is also called the Monodromy map. This map restricted to the fiber over the point in T_p representing M is just the map ϕ_M.

ϕ has been shown to be a holomorphic local homeomorphism by Hejhal first [12], then by Earle [7] and Hubbard [14]. In this dissertation, we extend this result for surfaces with punctures and ramification points.

We have also focused on the map ϕ_M. Kra has proven that ϕ_M is injective [15]. He has also proven the injectivity in the case of surfaces with punctures,, of course, restricting the domain to the finite dimensional space of bounded quadratic differentials [17]. One can, naturally, think of generalizing this result for bordered Riemann surfaces. Gallo and Porter have constructed examples in [9] to show that the restriction of ϕ_M to the space of bounded reflectable quadratic differentials $B_2^*(U,\Gamma)$ may not be injective. They have also generalized the injectivity for a some what different Monodromy map for surfaces with one boundary curve [9]. In this direction, we formulate a uniqueness theorem for bounded reflectable projective structures.

In Chapter I, we include some preliminary definitions and discuss some well known interesting properties of

Moebius transformations.

In Chapter II, we find the set of regular points in Hom(Γ,G). This technical result is needed to prove the main theorem in Chapter III.

In Chapter III, we prove the Monodromy theorem. In [7], Earle derived variational formulas from which the result followed easily. A simple extension of his work gives us the desired generalization.

Finally, in Chapter IV, we prove a uniqueness theorem for bounded reflectable deformations for a Fuchsian group of the second kind. We obtain a splitting property of a well known short exact sequence which associates to a Fuchsian group of the second kind its Schottky-double. We also study the monodromy of bounded reflectable deformations.

Chapter I

Preliminaries

§0. We start with some properties of Moebius transformations. We define Fuchsian groups, their regions of discontinuity, limit sets and Fundamental domains.

§1. A Moebius transformation g is a conformal self-mapping of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; hence it is of the form

$$g(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C} \text{ with } ad-bc=1.$$

We denote the group of all Moebius transformations by G. We have a natural isomorphism

$$G \cong {}^{SL(2,\mathbb{C})}/_{\pm I},$$

where $SL(2,\mathbb{C})$ is the group of 2×2 complex matrices with determinant 1 and I is the identity matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

A transformation $g \in G$, $g(z) = \frac{az+b}{cz+d}$, g not being identity, is called parabolic if and only if $tr^2 g = (a+d)^2 = 4$; is called elliptic if and only if $tr^2 g = (a+d)^2 \in [0,4)$; is called loxodromic in all other cases. A loxodromic transformation g for which $tr^2 g > 4$ is called hyperbolic.

A transformation is parabolic if and only if it has one fixed point. All other transformations have two fixed points.

For a loxodromic transformation g, of these two fixed points one, we denote by x, is repelling, the other, we denote by y, is attracting; $g^{-n}(z) \to x$ and $g^{n}(z) \to y$ for $n = 1,2,\ldots$.

Any g with two fixed points x and y may be written in the form

$$(1.1) \qquad \frac{g(z)-x}{g(z)-y} = k^2 \frac{z-x}{z-y};$$

k is a constant. k^2 is called the <u>multiplier</u> of g. If $k^2 = -1$, we can not distinguish between the fixed points of g.

If g is loxodromic, $|k^2| > 1$;

If g is hyperbolic, $k^2 > 1$;

If g is elliptic, $|k^2| = 1$.

§2. $SL(2,\mathbb{C})$ is a complex 3-dimensional Lie group. We denote the Lie Algebra of $SL(2,\mathbb{C})$ by $\mathfrak{G}$. The Lie algebra $\mathfrak{G}$ is identified with the tangent space to $SL(2,\mathbb{C})$ at I as follows. We consider a one parameter family of elements $\varphi(t)$ of $SL(2,\mathbb{C})$ such that $\varphi(0) = I$. Then

$$\lim_{t\to 0} \frac{\varphi(t)-I}{t} = \frac{\partial\varphi}{\partial t}\Big|_{t=0}$$

determines a vector $u \in \mathfrak{G}$. Now it is easy to check that $\mathfrak{G}$

consists of 2×2 complex matrices with trace zero.

The adjoint representation $u \to u^A$ of $SL(2,\mathbb{C})$ in $\mathfrak{g}$ is defined by

$$u^A = \mathrm{Ad}\, A(u), \; u \in \mathfrak{g}, \; A \in SL(2,\mathbb{C}),$$

where

$$\mathrm{Ad}\, A : \mathfrak{g} \to \mathfrak{g}$$

is the differential at I of the map

$$SL(2,\mathbb{C}) \ni x \mapsto A^{-1} \circ x \circ A \in SL(2,\mathbb{C}).$$

Explicitly

$$u^A = \lim_{t \to 0} \frac{A^{-1} \circ e^{tu} \circ A - I}{t} = A^{-1} \circ u \circ A, \; A \in SL(2,\mathbb{C}), \; u \in \mathfrak{g} .$$

At this moment it is necessary to introduce some notations that will be used throughout this section. If $p \in M$ is a point on a complex analytic manifold, then we denote by $T_p(M)$ the tangent space to M at p. If

$$f : M \to N$$

is a complex analytic map between two manifolds M and N, then we denote by

$$(df)(p) : T_p(M) \to T_{f(p)}(N)$$

the differential of f at $p \in M$.

Let us consider a holomorphic one parameter family of elements

$$\psi(t) = Be^{tu},\ B \in SL(2,\mathbb{C}),\ u \in \mathfrak{g},$$

such that $\psi(0) = B$. Then $\frac{\partial \psi}{\partial t}\Big|_{t=0} = Bu$ is a vector in $T_B(SL(2,\mathbb{C}))$. The left translation

$$\mathfrak{g} \ni u \mapsto Bu \in T_B(SL(2,\mathbb{C}))$$

is an isomorphism between $\mathfrak{g}$ and $T_B(SL(2,\mathbb{C}))$. Hence, in practice, we shall identify $T_B(SL(2,\mathbb{C}))$ at an arbitrary point B with $\mathfrak{g}$. We define the differential of f at B as follows. For $u \in \mathfrak{g}$,

$$(df)(B)(u) = \lim_{t\to 0} \frac{f(Be^{tu})-f(B)}{t}.$$

A parabolic transformation with fixed point $x \neq \infty$ can be written as an element of $SL(2,\mathbb{C})$ as $\begin{pmatrix} 1+px & -px^2 \\ p & 1-px \end{pmatrix}$; $p \neq 0$, which is unique up to multiplication by -1 [17]. We consider the natural map

$$\pi : SL(2,\mathbb{C}) \to G$$

which is two-to-one and unramified. Let P be the set of parabolic transformations in G. Each parabolic transformation corresponds to two matrices in $SL(2,\mathbb{C})$, one of which has trace 2 and the other has trace -2. Thus $\pi^{-1}(P)$ consists of two disjoint sets P^+ and P^-, where

P^+ = the set of elements in $SL(2,\mathbb{C})$ with trace $2\setminus\{I\}$,

P^- = the set of elements in $SL(2,\mathbb{C})$ with trace $-2\setminus\{-I\}$.

We prove the following Lemma which has been proven by Gardiner and Kra in [10] in a slightly different manner. We shall adopt the calculations from [10].

Lemma 1.1. Let $f : SL(2,\mathbb{C}) \to \mathbb{C}$ be the mapping defined by

$$f(x) = \operatorname{tr} x.$$

If $u \in \ker(df)(B)$ with $B \in P^+$, then there exists a $v \in \mathfrak{g}$ such that

$$u = v^B - v.$$

Proof. f is holomorphic. Let $B \in P^+$. Then there exists an $A \in SL(2,\mathbb{C})$ such that

$$A^{-1}BA = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}.$$

We consider the function

$$SL(2,\mathbb{C}) \ni B \overset{F}{\mapsto} A^{-1}BA \in SL(2,\mathbb{C}).$$

Since F is a holomorphic isomorphism,

$$u \in \ker d(f\circ F)(B) \Leftrightarrow (dF)(B)u \in \ker(df)(FB).$$

Moreover, for $v \in \mathfrak{G}$, $B \in SL(2,\mathbb{C})$, $A \in SL(2,\mathbb{C})$

$$u = v^B - v \Leftrightarrow u^A = v^{B\circ A} - v^A = v_1^{A^{-1}BA} - v_1;\ v_1 = v^A,$$

and $$(dF)(B)(u) = u^A.$$

Thus it suffices to assume that $B = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$. For $u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{G}$,

$$\begin{aligned}
(df)(B)(u) &= \lim_{t\to 0} \frac{f(Be^{tu}) - f(B)}{t} \\
&= \lim_{t\to 0} \frac{f\{\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+at & bt \\ ct & 1-at \end{pmatrix} + o(t)\} - f\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}}{t} \\
&= \lim_{t\to 0} \frac{f\{\begin{pmatrix} 1+at+pct & bt+p(1-at) \\ ct & 1-at \end{pmatrix} + o(t)\} - 2}{t} \\
&= \lim_{t\to 0} \frac{2+pct-2}{t} \\
&= pc \quad .
\end{aligned}$$

Thus if $u \in \ker(df)(B)$, $c = 0$; that is, $u = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. We check that there exists a $v = \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \in \mathfrak{G}$ such that

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = B^{-1}\begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} B - \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}$$

since

$$B = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad B^{-1}\begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} B - \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} = \begin{pmatrix} -pc' & -c'p^2+2a'p \\ 0 & c'p \end{pmatrix}.$$

We choose $c' = -\frac{a}{p}$, $a' = \frac{b-ap}{2p}$, and b' arbitrarily. This completes the proof of the Lemma.

In the above calculation for $(df)(B)$ with $B = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ we notice that, for $u \in \mathfrak{G}$,

$$(df)(B)(u) = pc.$$

Since $p \neq 0$, $c \neq 0$, $(df)(B)$ is surjective. Again the differential of the map $F : x \to A^{-1}xA$, $x \in SL(2,\mathbb{C})$, $A \in SL(2,\mathbb{C})$ is surjective. Hence $(df)(B)$ is surjective for any $B \in P^+$. Therefore, df has maximal rank at each point of P^+; that is, P^+ is the set of regular points of f in $f^{-1}(2)$ and hence P^+ is a submanifold of $SL(2,\mathbb{C})$ of dimension 2 by the Implicit function theorem. Moreover, for $B \in P^+$,

$$T_B(P^+) = \ker(df)(B).$$

Hence from the above Lemma we conclude that

$$T_B(P^+) = \{u \in \mathfrak{G};\ u = v^B - v \ \text{ for some } v \in \mathfrak{G}\}.$$

Similarly, we can show that P^- is a submanifold of $SL(2,\mathbb{C})$

of dimension 2 and for $B \in P^-$,

$$T_B(P^-) = \{u \in \mathcal{G};\ u = v^B - v \text{ for some } v \in \mathcal{G}\}.$$

Since P^+ and P^- project to P in G, P is a submanifold of G of dimension 2. Thus we prove the following

<u>Corollary 1</u>. P is a submanifold of G of dimension 2. Moreover, for $g \in P$,

$$T_g(P) = \{u \in \mathcal{G};\ u = v^g - v \text{ for some } v \in \mathcal{G}\}.$$

An elliptic transformation g with the fixed points x and y can be written as

$$\frac{g(z)-x}{g(z)-y} = k^2\,\frac{z-x}{z-y},$$

where k^2 is the multiplier of g, $k^2 \neq 1$. Choosing a positive square root of k^2, we write $k^2 = {}^{k}/_{1/k}$. Then solving the above equation we can write in the matrix form

$$g = \frac{1}{x-y}\begin{pmatrix} x/k-yk & xy(k-1/k) \\ 1/k-k & xk-y/k \end{pmatrix}$$

which is unique up to multiplication by -1 [20]. If $k^2 = -1$, the above expression for g is symmetric in x and y.

Let E be the set of all elliptic transformations with the multiplier k^2. Each elliptic transformation in E corresponds to two matrices in $SL(2,\mathbb{C})$, one of which has trace $k+1/k$, and the other has trace $-(k+1/k)$. Hence if $k^2 \neq -1$, $\pi^{-1}(E)$ consists of two disjoint sets E^+ and E^-, where

E^+ = the set of elements in $SL(2,\mathbb{C})$ with trace $k+1/k$,

E^- = the set of elements in $SL(2,\mathbb{C})$ with trace $-(k+1/k)$.

If $k^2 = -1$, $\pi^{-1}(E)$ is just one set; we denote it by E^o, where E^o = the set of elements in $SL(2,\mathbb{C})$ with trace zero. As before, we have the following.

Lemma 1.2. Let $f : SL(2,\mathbb{C}) \to \mathbb{C}$ be the mapping defined by

$$f(x) = tr(x).$$

Then if $u \in \ker(df)(B)$, $B \in E^+$, then there exists a $v \in \mathfrak{g}$ such that

$$u = v^B - v.$$

Proof. The idea of the proof is same as it is in the Lemma 1.1. Without loss of generality we assume that $B = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$. Then for $u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}$,

$$(df)(B)(u) = \lim_{t\to 0} \frac{f\{\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}\begin{pmatrix} 1+at & bt \\ ct & 1-at \end{pmatrix} + o(t)\} - f\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}}{t}$$

$$= \lim_{t\to 0} \frac{f\{\begin{pmatrix} k(1+at) & kbt \\ 1/k\ ct & 1/k(1-at) \end{pmatrix} + o(t)\} - f\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}}{t}$$

$$= \lim_{t\to 0} \frac{(k+1/k) + at(k-1/k) + o(t) - (k+1/k)}{t}$$

$$= a(k-1/k).$$

Hence if $u \in \ker(df)(B)$, $a = 0$; that is, $u = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. We check that there exists a $v = \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \in \mathfrak{G}$ such that

$$B^{-1}vB - v = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Since $B = \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$, $B^{-1}vB - v = \begin{pmatrix} 0 & b'(1/k^2-1) \\ c'(k^2-1) & 0 \end{pmatrix}$.

We choose $b' = \dfrac{b}{1/k^2-1}$, $c' = \dfrac{c}{k^2-1}$ and a' arbitrarily.

This completes the proof of the Lemma. ∎

Once again, we observe that $(df)(B)$ is surjective for $B \in E^+$, since $a \neq 0$ and $k^2 \neq 1$. Hence at each point of E^+ df has maximal rank, and hence $E^+ = f^{-1}(k+1/k)$ is a submanifold of $SL(2,\mathbb{C})$ of dimension 2. Moreover,

$$T_B(E^+) = \ker(df)(B).$$

Hence

$$T_B(E^+) = \{u \in \mathfrak{G};\ u = v^B - v \quad \text{for some } v \in \mathfrak{G}\}.$$

Similarly, we can prove the same results for E^- as well as for E^o. When $k^2 \neq -1$, E^+ and E^- are submanifolds of $SL(2,\mathbb{C})$. Since E^+ and E^- project to E in G, E is a submanifold of G. When $k^2 = -1$, E^o is a submanifold of $SL(2,\mathbb{C})$. Hence $E = E^o/\pm I$ is a submanifold of G. Thus we prove the following.

<u>Corollary 2</u>. E is a submanifold of G of dimension 2. Moreover, for $g \in E$,

$$T_g(E) = \{u \in \mathfrak{G};\ u = v^g - v \quad \text{for some } v \in \mathfrak{G}\}.$$

§3. We shall be studying groups Γ whose elements are Moebius transformations. Let Γ be a subgroup of G and z be a point in $\hat{\mathbb{C}}$. We denote by Γ_z the stabilizer of z in Γ; that is,

$$\Gamma_z = \{\gamma \in \Gamma;\ \gamma(z) = z\}.$$

We say that Γ acts discontinuously at z if

(i) Γ_z is finite, and

(ii) there is a neighborhood V of z such that

$\gamma(V) = V$ for all $\gamma \in \Gamma_z$ and

$\gamma(V) \cap V$ is empty for $\gamma \in \Gamma \setminus \Gamma_z$.

Let $\Omega(\Gamma) = \{z \in \hat{\mathbb{C}};\ \Gamma$ acts discontinuously at $z\}$. We call $\Omega(\Gamma)$ the <u>region of discontinuity</u> of Γ. The group Γ is called <u>discontinuous</u> if $\Omega(\Gamma)$ is not empty. We call $\hat{\mathbb{C}}\setminus\Omega(\Gamma)$ the <u>limit set</u> of Γ and denote it by $\Lambda(\Gamma)$. Ω is an open Γ-invariant subset of $\hat{\mathbb{C}}$. The cardinality of $\Lambda(\Gamma)$ is 0,1,2 or ∞. If the cardinality of $\Lambda(\Gamma)$ is 0,1 or 2, Γ is called elementary; Γ is called <u>non-elementary</u> otherwise.

If Γ is discontinuous and there exists a circle C in $\hat{\mathbb{C}}$ such that Γ fixes the interior of C, then Γ is called <u>Fuchsian</u>. In this case $\Lambda(\Gamma) \subseteq C$. If $\Lambda(\Gamma) = C$, Γ is called the <u>Fuchsian of the first kind</u>, Γ is called <u>Fuchsian of the</u> second kind otherwise. We shall work with finitely generated Fuchsian groups fixing the upper half-plane U. For such a group Γ, the quotient space U/Γ is a Riemann surface of finite genus with a finite number of possible punctures and ramification points and with a finite number of possible analytic boundary curves. Let the genus of U/Γ be p. Let the total number of punctures and ramification points be n and let the number of boundary curves be m. Then we say that the surface U/Γ, equivalently the group Γ, is of type (p,n,m). If Γ is of the first kind, $m = 0$; $m > 0$ otherwise. The surfaces with $m > 0$ may also be called <u>bordered Riemann</u> surfaces. Let $\{x_1, x_2, \ldots, x_n\}$ be the set of points on U/Γ

that are either punctures or ramification points. Let ν_j be the ramification index of $\pi^{-1}(x_j)$, where

$$\pi : U \to U/\Gamma$$

is the natural projection map, and we set $\nu_j = \infty$ for punctures. Then we call the sequence

$$\{p,n,m;\nu_1,\nu_2,\ldots,\nu_n\}$$

the <u>signature</u> of the group Γ.

<u>Definition</u>. By a fundamental domain for a finitely generated Fuchsian group Γ acting on U we mean an open subset D of U such that

(i) whenever $\gamma z = \zeta$ for some $\gamma \in \Gamma$, $z \in D$, $\zeta \in D$, then $\gamma = \mathrm{Id}$,

(ii) for every point $\zeta \in U$, there is a $\gamma \in \Gamma$ and a $z \in C\ell D$ such that $\gamma(z) = \zeta$,

(iii) the boundary of D in U, δD, consists of a finite number of piecewise analytic arcs, and

(iv) for every arc $c \in \delta D$, there is an arc $c' \in \delta D$ and an element $\gamma \in \Gamma$ such that $\gamma(c) = c'$.

In U we introduce the Poincaré metric based on the line element $ds = (\operatorname{Im} z)^{-1} |dz|$; this makes U into a model of the non-Euclidean plane. The geodesics of this metric are arcs of Euclidean circles or straight lines orthogonal to the real line. The above metric is invariant under Γ. Thus it can be projected to U/Γ to obtain a metric which, of course, has singularities at the ramification points and the punctures. But these singularities are not too bad in the sense that the Poincaré metric on U/Γ is locally square integrable at ramification points as well as in a deleted neighborhood of each puncture [8].

Chapter II

The set of regular points in $\mathrm{Hom}(\Gamma,G)$.

§0. In this chapter Γ is a Fuchsian group of signature $\{p,n,0;\nu_1\nu_2,\cdots,\nu_n\}$; $2p-2+\sum_{j=1}^{n}(1-1/\nu_j) > 0$. $\mathrm{Hom}(\Gamma,G)$ is the set of all homomorphisms of Γ into G. In this chapter we prove that a certain subset of $\mathrm{Hom}(\Gamma,G)$ is a manifold. The case $n=0$ has been studied by Gunning [11].

§1. Let $a_1,b_1,a_2,b_2\cdots a_p,b_p,c_1,\cdots c_m,c_{m+1},\cdots,c_n$ be a fixed set of generators of Γ satisfying the following relations.

$$\prod_{i=1}^{p}[a_i,b_i]\prod_{j=1}^{n}c_j = I$$

and

$$c_{m+j}^{\nu_j} = I,\ j = 1,2,\cdots,n-m,$$

where $[a_i,b_i] = a_ib_ia_i^{-1}b_i^{-1}$, and $c_1,c_2,\cdots,c_m$ are the parabolic generators, and $c_{m+1},\cdots,c_n$ are the elliptic generators with periods $\nu_1,\nu_2,\cdots,\nu_{n-m}$, respectively. We also assume that the multiplier of c_{m+j} is k_j^2, $j = 1,2,\cdots,n-m$.

Let $\mathrm{Hom}^*(\Gamma,G)$ be the subset of $\mathrm{Hom}(\Gamma,G)$ consisting of those homomorphisms χ which preserve the parabolic transformations and the multipliers of the elliptic transformations of Γ. A homomorphism $\chi \in \mathrm{Hom}(\Gamma,G)$ is completely determined by $2p+n$ Moebius transformations

$$\chi(a_i) = s_i$$

$$\chi(b_i) = t_i$$

$$\chi(c_j) = w_j \qquad (1 \leq i \leq p,\ 1 \leq j \leq n)$$

satisfying the following relations

$$\prod_{i=1}^{p} [s_i, t_i] \prod_{j=1}^{n} w_j = I$$

and

$$w_{m+j}^{\nu_j} = I, \qquad j = 1,2,\cdots,n-m.$$

P is the set all parabolic transformations and E_j is the set of elliptic transformations with the multiplier k_j^2; $j = 1,2,\cdots,n-m$. If $\chi \in \mathrm{Hom}^*(\Gamma,G)$, $\chi(c_j) = w_j \in P$ for $j = 1,2,\cdots,m$; $\chi(c_{m+j}) = w_{m+j} \in E_j$ for $j = 1,2,\cdots,n-m$. Hence $(s_1,s_2,\cdots s_p,t_1,t_2,\cdots,t_p,w_1,w_2,\cdots w_n)$ is a point in $G^{2p}\times P^m\times E_1\times E_2\times\cdots\times E_{n-m}$. We denote $(s_1,\cdots,s_p,t_1,\cdots,t_p,$ $w_1,\cdots,w_n)$ by (s_i,t_i,w_j) and $G^{2p}\times P^m\times E_1\times\cdots\times E_{n-m}$ by $G_{2p,n}$ for short. We introduce a function F on $G_{2p,n}$ defined by

$$F(s_i,t_i,w_j) = \prod_{i=1}^{p} [s_i,t_i] \prod_{j=1}^{n} w_j.$$

This is a complex analytic function from $G_{2p,n}$ into G. The subset

$$R = \{(s_i,t_i,w_j) \in G_{2p,n};\ F(s_i,t_i,w_j) = I\}$$

is then a complex analytic subvariety of $G_{2p,n}$; the mapping

$$\mathrm{Hom}^*(\Gamma,G) \ni \chi \mapsto (\chi(a_i),\chi(b_i),\chi(c_j)) \in G_{2p,n}$$

identifies $\mathrm{Hom}^*(\Gamma,G)$ with this subvariety and thus establishes a complex structure on $\mathrm{Hom}^*(\Gamma,G)$. Our goal is to find the set of regular points in R.

From the Lemma 1.1 and the Lemma 1.2 we conclude that $G_{2p,n}$ is a complex analytic manifold of dimension $6p+2n$. The set of regular points in R consists of precisely those points where the differential of F has rank 3, and hence it is a submanifold of $G_{2p,n}$ of dimension $6p+2n-3$ by the Implicit function theorem. In the next section we study the differential of F.

§2. Let $d_\chi F$ denote the differential of F at a point $\chi = (s_i,t_i,w_j) \in G_{2p,n}$. Then $d_\chi F$ is the induced linear map from the tangent space of $G_{2p,n}$ at the point χ, $T_\chi(G_{2p,n})$, to the tangent space of G at $F(\chi)$. We know that

$$T_\chi(G_{2p,n}) \cong \mathfrak{G}^{2p} \times \prod_{j=1}^{n} \mathfrak{G}_{w_j},$$

where, for $j = 1,2,\cdots m$, $\mathfrak{G}_{w_j}$ is the subspace of $\mathfrak{G}$ isomorphic to $T_{w_j}(P)$, and for $j = 1,2,\ldots,n-m$, $\mathfrak{G}_{w_{m+j}}$ is the subspace of $\mathfrak{G}$ isomorphic to $T_{w_{m+j}}(E_j)$. We have already discussed $T_{w_j}(P)$ and $T_{w_{m+j}}(E_j)$ in Chapter I.

Let $(x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p, z_1, \cdots z_n)$ be a point in $G^{2p} \times \prod_{j=1}^{n} G_{w_j}$. We denote $(x_1, x_2, \cdots, x_p, y_1, \cdots, y_p, z_1, \cdots z_n)$ by (x_i, y_i, z_j). Then, by definition,

$$d_\chi F(x_i, y_i, z_j) = \lim_{t \to 0} \frac{F(s_i e^{tx_i}, t_i e^{ty_i}, w_j e^{tz_j}) - F(s_i, t_i, w_j)}{t}.$$

In other words, $d_\chi F(x_i, y_i, z_j)$ is the coefficient of t in the Taylor expansion of $F(s_i e^{tx_i}, t_i e^{ty_i}, w_j e^{tz_j})$.

We check the following results

$$e^{ta} e^{tb} = I + t(a+b) + o(t)$$

and

$$s e^{ta} s^{-1} = e^{tsas^{-1}}$$

which we shall use in our calculations.

We begin with

$$[s_1 e^{tx_1}, t_1 e^{ty_1}]$$

$$= s_1 e^{tx_1} t_1 e^{ty_1} e^{-tx_1} s_1^{-1} e^{-ty_1} t_1^{-1}$$

$$= s_1 e^{tx_1} s_1^{-1} s_1 t_1 e^{ty_1} (s_1 t_1)^{-1} s_1 t_1 e^{-tx_1} (s_1 t_1)^{-1} s_1 t_1 s_1^{-1} e^{-ty_1}$$

$$(s_1 t_1 s_1^{-1})^{-1} s_1 t_1 s_1^{-1} t_1^{-1}$$

$$=e^{ts_1x_1s_1^{-1}}e^{ts_1t_1y_1(s_1t_1)^{-1}}e^{-ts_1t_1x_1(s_1t_1)^{-1}}e^{-ts_1t_1s_1^{-1}y_1(s_1t_1s_1^{-1})^{-1}}[s_1,t_1]$$

$$=(I+t(s_1x_1s_1^{-1}+s_1t_1y_1(s_1t_1)^{-1}-s_1t_1x_1(s_1t_1)^{-1}-s_1t_1s_1^{-1}y_1(s_1t_1s_1^{-1})^{-1}) + o(t))[s_1,t_1]$$

$$=[s_1,t_1]+t\mathrm{Ad}(s_1t_1)^{-1}((I-\mathrm{Ad}s_1)y_1-(I-\mathrm{Ad}t_1)x_1)[s_1,t_1]+o(t).$$

It is convenient to let

$$Q_i = \mathrm{Ad}(s_it_i)^{-1}((I-\mathrm{Ad}s_i)y_i-(I-\mathrm{Ad}t_i)x_i).$$

Then we can write

$$[s_1e^{tx_1},t_1e^{ty_1}] = [s_1,t_1]+tQ_1[s_1,t_1]+o(t).$$

Hence

$$\prod_{i=1}^{2}[s_ie^{tx_i},t_ie^{ty_i}]$$

$$= ([s_1,t_1]+tQ_1[s_1,t_1]+o(t))([s_2,t_2]+tQ_2[s_2,t_2]+o(t))$$

$$= [s_1,t_1][s_2,t_2]+t\,Q_1[s_1,t_1][s_2,t_2]+t\,[s_1,t_1]Q_2[s_2,t_2]+o(t)$$

$$= \prod_{i=1}^{2}[s_i,t_i]+t(Q_1+\mathrm{Ad}([s_1,t_1])^{-1}(Q_2))\prod_{i=1}^{2}[s_i,t_i]+o(t).$$

At the next step we have

$$\prod_{i=1}^{3}[s_i e^{tx_i}, t_i e^{ty_i}] = \prod_{i=1}^{3}[s_i,t_i] + t(Q_1 + Ad([s_1,t_1])^{-1}(Q_2)$$
$$+ Ad(\prod_{i=1}^{2}[s_i,t_i]^{-1}(Q_3)) \prod_{i=1}^{3}[s_i,t_i] + o(t).$$

Therefore, generalizing this result we have

$$(2.1) \quad \prod_{i=1}^{p}[s_i e^{tx_i}, t_i e^{ty_i}] = \prod_{i=1}^{p}[s_i,t_i] +$$
$$+ t \sum_{i=1}^{p} Ad(\prod_{k=1}^{i-1}[s_k,t_k])^{-1}(Q_i) \prod_{i=1}^{p}[s_i,t_i] + o(t).$$

Similarly,

$$w_1 e^{tz_1} w_2 e^{tz_2} = w_1 e^{tz_1} w_1^{-1} w_1 w_2 e^{tz_2} (w_1 w_2)^{-1} w_1 w_2$$
$$= e^{tw_1 z_1 w_1^{-1}} e^{tw_1 w_2 z_2 (w_1 w_2)^{-1}} w_1 w_2$$
$$= w_1 w_2 + t(w_1 z_1 w_1^{-1} + w_1 w_2 z_2 (w_1 w_2)^{-1}) w_1 w_2 + o(t)$$
$$= w_1 w_2 + t(Ad(w_1)^{-1}(z_1) + Ad(w_1 w_2)^{-1}(z_2)) w_1 w_2 + o(t).$$

Hence we have

$$(2\cdot2) \quad \prod_{j=1}^{n} w_j e^{tz_j} = \prod_{j=1}^{n} w_j + t \sum_{j=1}^{n} Ad(\prod_{k=1}^{j} w_k)^{-1}(z_j) \prod_{j=1}^{n} w_j + o(t).$$

Mutiplying (2·1) and (2·2) we get

$$\prod_{i=1}^{p}[s_i e^{tx_i}, t_i e^{ty_i}] \prod_{j=1}^{n} w_j e^{tz_j} = \prod_{i=1}^{p}[s_i,t_i] \prod_{j=1}^{n} w_j +$$
$$+ t \sum_{i=1}^{p} Ad(\prod_{k=1}^{i-1}[s_k,t_k])^{-1}(Q_i) \prod_{i=1}^{p}[s_i t_i] \prod_{j=1}^{n} w_j$$
$$+ t \prod_{i=1}^{p}[s_i t_i] \sum_{j=1}^{n} Ad(\prod_{k=1}^{j} w_k)^{-1}(z_j) \prod_{j=1}^{n} w_j + o(t)$$

$$= \prod_{i=1}^{p} [s_i,t_i] \prod_{j=1}^{n} w_j + t \sum_{i=1}^{p} \mathrm{Ad}(\prod_{k=1}^{i-1} s_k,t_k])^{-1}(Q_i) \prod_{i=1}^{p} [s_i,t_i] \prod_{j=1}^{n} w_j$$

$$+ t\,\mathrm{Ad}(\prod_{i=1}^{p} [s_i,t_i])^{-1}(\sum_{j=1}^{n} \mathrm{Ad}(\prod_{k=1}^{j} w_k)^{-1}(z_j)) \prod_{i=1}^{p} [s_i,t_i] \prod_{j=1}^{n} w_j + o(t)$$

$$= I + t \sum_{i=1}^{p} \mathrm{Ad}(\prod_{k=1}^{i-1} [s_k,t_k])^{-1}(Q_i) + t \sum_{j=1}^{n} \mathrm{Ad}(\prod_{i=1}^{p} [s_i,t_i] \prod_{k=1}^{j} w_k)^{-1}(z_j) + o(t),$$

since

$$\prod_{i=1}^{p} [s_i,t_i] \prod_{j=1}^{n} w_j = I.$$

Hence we have

$$d_\chi F(x_i,y_i,z_i) = \sum_{i=1}^{p} \mathrm{Ad}(\prod_{k=1}^{i-1} [s_k,t_k])^{-1}(Q_i) + \sum_{j=1}^{n} \mathrm{Ad}(\prod_{i=1}^{p} [s_i,t_i] \prod_{k=1}^{j} w_k)^{-1}(z_j),$$

where

$$Q_i = \mathrm{Ad}(s_i t_i)^{-1}((I-\mathrm{Ad}\, s_i)y_i - (I-\mathrm{Ad}\, t_i)x_i).$$

Since

$$(\prod_{k=1}^{i-1} [s_k,t_k]s_i t_i)^{-1} = s_i^{-1} t_i^{-1} \prod_{k=i+1}^{p} [s_k,t_k]$$

and

$$(\prod_{k=1}^{p} [s_k,t_k] \prod_{k=1}^{j} w_k)^{-1} = \prod_{k=j+1}^{n} w_k,$$

we have

$$(2\cdot3)\qquad d_\chi F(x_i,y_i,z_j) = \sum_{i=1}^{p} \mathrm{Ad}\, s_i^{-1} t_i^{-1} \prod_{k=i+1} [s_k,t_k] ((I-\mathrm{Ad}\, s_i)y_i - (I-\mathrm{Ad}\, t_i)x_i) + \sum_{j=1}^{n} \mathrm{Ad} \prod_{k=j+1}^{n} (z_j).$$

§3. We define an action of Γ on $\mathfrak{G}$ as follows.

For $u \in \mathfrak{G}$ and $\gamma \in \Gamma$, we define

$$u\cdot\gamma = u\cdot\chi(\gamma) = \mathrm{Ad}\,\chi(\gamma)(u).$$

We rewrite (2·3) in the following way.

$$(2\cdot4)\qquad d_\chi F(x_i,y_i,z_j) = \sum_{i=1}^{p} (x_i\cdot(b_i-I)+y_i\cdot(I-a_i))\cdot a_i^{-1}b_i^{-1} \prod_{k=i+1}^{p} [a_k,b_k] + \sum_{j=1}^{n} z_j\cdot \prod_{k=j+1}^{n} c_k.$$

We want to check when $d_\chi F$ is surjective. To do that we follow Ahlfors' method in [2,§5]. We introduce notations $R_o = I$ and

$$R_i = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_i b_i a_i^{-1} b_i^{-1}$$

$$R_{p+j} = R_p c_1 c_2 \cdots c_j$$

$$\overline{a_i} = R_{i-1} b_i^{-1} R_i^{-1}$$

$$\overline{b_i} = R_i a_i^{-1} R_{i-1}^{-1}$$

$$\overline{c_j} = R_{p+j} c_j R_{p+j}^{-1} \qquad (1 \le i \le p,\ 1 \le j \le n).$$

Then $\bar{a}_i, \bar{b}_i, \bar{c}_j$ are generators of Γ. Moreover, the equation (2·4) becomes

$$d_\chi F(x_i, y_i, z_j) = \sum_{i=1}^{p} x_i \cdot a_i^{-1} R_{i-1}^{-1} (I - \bar{a}_i) + \sum_{i=1}^{p} y_i \cdot b_i^{-1} R_i^{-1} (\bar{b}_i - I)$$
$$+ \sum_{i=1}^{n} z_j \cdot R_{p+j}^{-1}.$$

We suppose that the map

$$d_\chi F : \mathfrak{G}^{2p} \times \prod_{j=1}^{n} \mathfrak{G}_{w_j} \to \mathfrak{G}$$

is not surjective. Then there exists a nonzero linear functional v^* on $\mathfrak{G}$ that vanishes on all the subspaces $\mathfrak{G}\cdot(\bar{a}_i - I)$, $\mathfrak{G}\cdot(\bar{b}_i - I)$ and $\mathfrak{G}\cdot(c_j - I) R_{p+j}^{-1} = \mathfrak{G}\cdot R_{p+j}^{-1}(\bar{c}_j - I) = \mathfrak{G}\cdot(\bar{c}_j - I)$. If v^* annihilates $v\cdot(a-I)$ and $v\cdot(b-I)$ for all $v \in \mathfrak{G}$, it annihilates $v\cdot(ab-I) = v\cdot a(b-I) + v\cdot(a-I)$. Since $\{\bar{a}_i, \bar{b}_i, \bar{c}_j\}$ is a system of generators of Γ, it follows that v^* annihilates $v\cdot(a-I)$ for all $v \in \mathfrak{G}$ and all $a \in \Gamma$.

We assume first that there is a loxodromic element $\chi(a)$, $a \in \Gamma$. We may take

$$\chi(a)(z) = k^2 z; \quad |k^2| \neq 1.$$

For $\quad v = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathfrak{G}, \; v\cdot(a-I) = \begin{pmatrix} 0 & q(\frac{1}{k^2} - 1) \\ r(k^2-1) & 0 \end{pmatrix}$.

Therefore, v^* must be multiple of the linear functional that maps any v on its first entry. It follows that the first entry of $v\cdot(b-I)$ is zero for all $v \in \mathcal{G}$ and all $b \in \Gamma$. We take $\chi(b)(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, and apply the above result on $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we get $\alpha\beta = \gamma\delta = 0$. This is true only when $\chi(b)$ is a multiple of z or $1/z$.

Next, we assume that there is a parabolic element $\chi(a)$, $a \in \Gamma$. We take

$$\chi(a)(z) = z + 1.$$

Then for $v = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathcal{G}$, $v\cdot(a-I) = \begin{pmatrix} -r & 2p-r \\ 0 & r \end{pmatrix}$. Therefore, v^* must be a multiple of the linear functional that maps any v on its third entry. It follows that $v\cdot(b-I)$ has zero third entry for all $v \in \mathcal{G}$, all $b \in \Gamma$. As before, we assume that $\chi(b)(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, and apply the above result on $v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We get $\gamma = 0$, $\alpha^2 = 1$. This is true only when $\chi(b)(z) = z + \beta'$; $\beta' \neq 0$.

Finally, we assume that there is no loxodromic or parabolic element in $\chi(\Gamma)$; that is, all elements of $\chi(\Gamma)$ are elliptic. Hence $\chi(\Gamma)$ is finite.

Combining all these we conclude that $d_\chi F$ is surjective if none of the following statements holds.

(i) $\chi(\Gamma)$ is finite;

(ii) all elements of $\chi(\Gamma)$ are multiples of z or $\frac{1}{z}$;

(iii) all elements of $\chi(\Gamma)$ are of the form $z \to z+\beta$, $\beta \neq 0$.

Now we are in a position to prove the following

<u>Theorem 1</u>. Let R_o be the subset of $\mathrm{Hom}^*(\Gamma,G)$ consisting of those homomorphism χ for which $\chi(\Gamma)$ is non-elementary; that is, $\chi(\Gamma)$ is not a finite extension of an abelian group. Then R_o is a complex manifold of dimension $6p+2n-3$.

<u>Proof</u>. We recall that

$$\mathrm{Hom}^*(\Gamma,G) = R = \{(s_i,t_i,w_j) \in G_{2p,n};\ F(s_i,t_i,w_j) = I\}.$$

We have shown above that $d_\chi F$ is surjective if $\chi \in R_o$; that is, $d_\chi F$ has rank 3 for $\chi \in R_o$. Hence R_o is a submanifold of $G_{2p,n}$ of dimension $6p+2n-3$ by the Implicit function Theorem. This completes the proof of the theorem.

<u>Remark</u>. It follows from the condition (iii) that the above theorem also holds when $\chi(\Gamma)$ is some of the elementary groups.

Chapter III

The Monodromy map.

§0. In this chapter we work with a Fuchsian group Γ acting on the upper half plane and having signature $\{p,n,0;\nu_1,\nu_2,\dots,\nu_n\}$; $2p-2+\sum_{j=1}^{n}(1-1/\nu_j)>0$. We study the variation of projective structures on a varying Riemann surface corresponding to Γ, associated with the bounded quadratic differentials. We prove that the Monodromy map is a holomorphic local homeomorphism. The case $n=0$ gives the previously known result by Hejhal [12], Earle [7] and Hubbard [14]. We shall adopt with almost no change Earle's arguments in [7].

§1. Definition. Let a group Γ act discontinuously on a domain $\Omega\subset\hat{\mathbb{C}}$. We denote by $Q_2(\Omega,\Gamma)$ the complex vector space of quadratic differentials for Γ; $Q_2(\Omega,\Gamma)$ consists of function φ, holomorphic on Ω satisfying

$$(\varphi\circ\gamma)\gamma'^2=\varphi \quad \text{for all } \gamma\in\Gamma.$$

We denote by $B_2(\Omega,\Gamma)$ the subspace of $Q_2(\Omega,\Gamma)$ consisting of bounded quadratic differentials for Γ; $B_2(\Omega,\Gamma)$ consists of $\varphi\in Q_2(\Omega,\Gamma)$ for which

$$\sup_{z\in\Omega}\{\lambda_\Omega^{-2}|\varphi(z)|\}<\infty,$$

where λ_Ω is the Poincaré metric on Ω.

The Schwarzian differential operator $\mathcal{S}$ is defined by

$$\mathcal{S}f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

We shall use the following properties of $\mathcal{S}$ [13].

(i) For meromormorphic functions f and g,

$$\mathcal{S}(f\circ g) = (\mathcal{S}f\circ g)g'^2 + \mathcal{S}g$$

(ii) $\mathcal{S}f = \mathcal{S}g$ if and only if $g = A\circ f$ for some Moebius transformation A;

(iii) $\mathcal{S}f = 0$ if and only if f is a Moebius transformation.

In this chapter, Γ is a finitely generated Fuchsian group acting on the upper half-plane U (and the lower half-plane U*), and Γ has signature $\{p,n,0;\nu_1,\nu_2,\ldots,\nu_n\}$; $2p-2+\sum_{j=1}^{n}(1-1/\nu_j) > 0$.

<u>Definition</u>. A deformation of Γ is a pair (f,χ), where f is a holomorphic local homeomorphism of U into $\hat{\mathbb{C}}$ and χ is a homomorphism of Γ into G, the group of all Moebius transformations, satisfying

(3·1) $f\circ\gamma = \chi(\gamma)\circ f$ for all $\gamma \in \Gamma$.

The above local homeomorphism f also describes a projective structure on the Riemann surface U/Γ (provided Γ is torsion free). So we also call f a projective structure on U/Γ. We call two projective structures f and g equivalent if $g = A \circ f$ for some Moebius transformation A.

Using the properties of the Schwarzian differential operator we can establish a one-to-one correspondence between the set of equivalence classes of projective structures on U/Γ and the space of quadratic differentials $Q_2(U,\Gamma)$ in the following way.

Let $\varphi \in Q_2(U,\Gamma)$. Then a solution of the Schwarzian differential equation

$$(3\cdot 2)\ldots \mathcal{S}f = \varphi$$

is a complex analytic local homeomorphism of U into $\hat{\mathbb{C}}$. Since $\mathcal{S}(f\circ\gamma) = (\mathcal{S}f\circ\gamma)\gamma'^2 = \mathcal{S}f$ for all $\gamma \in \Gamma$, $f\circ\gamma = \hat{\gamma}\circ f$ for some $\hat{\gamma} \in G$. The correspondence $\gamma \to \hat{\gamma}$ defines a homomorphism of Γ into G which is denoted by χ. Hence f satisfies (3·1).

Conversely, if f is a complex analytic local homeomorphism of U into $\hat{\mathbb{C}}$ and f satisfies (3·1), then $\mathcal{S}f \in Q_2(U,\Gamma)$. For $A \in G$, $A\circ f$ and f both satisfy (3·2). Thus we have a one-to-one correspondence between the set of equivalence classes of projective structures and the space of quadratic differentials.

§2. We need to develop some more definitions before we state the main result in this chapter.

We denote the set of all quasiconformal automorphisms of U by Q. Every element $w \in Q$ can be extended, by continuity, to an automorphism of $U \cup \hat{\mathbb{R}}$; $\hat{\mathbb{R}}$ is the extended real line. We call this extension w again. The set of elements $w \in Q$ normalized by the conditions $w(0) = 0$, $w(1) = 1$ and $w(\infty) = \infty$ will be denoted by Q_{norm}. $w \in Q$ is called compatible with the group Γ if $w \circ \gamma \circ w^{-1}$ is conformal for every $\gamma \in \Gamma$. We shall denote the set of elements $w \in Q$ compatible with Γ by $Q(\Gamma)$. Let $Q_{norm}(\Gamma) = Q(\Gamma) \cap Q_{norm}$. Two elements $w_1, w_2 \in Q$ will be called equivalent if $w_1|_{\mathbb{R}} = w_2|_{\mathbb{R}}$. The <u>Teichmüller space</u> $T(\Gamma)$ of Γ is the set of equivalence classes $[w]$ of elements $w \in Q_{norm}(\Gamma)$. Let $L_\infty(U)$ denote the complex Banach space of bounded measurable functions μ on U. Let $L_\infty(U)_1$ be its open unit ball. Let $L_\infty(U,\Gamma)$ be the subspace of $L_\infty(U)$ consisting of μ satisfying

$$\mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z), \text{ for all } \gamma \in \Gamma, \text{ a.e. } z \in U.$$

Let $L_\infty(U,\Gamma)_1 = L_\infty(U)_1 \cap L_\infty(U,\Gamma)$. For every $w \in Q$, its Beltrami coefficient $\mu = \frac{w_{\bar{z}}}{w_z} \in L_\infty(U)_1$. Every $\mu \in L_\infty(U)_1$

determines a unique quasiconformal self-map w of U, fixing 0,1 and ∞, and satisfying $w_{\bar z} = \mu w_z$ [1, Chapter V]. We denote this w by w_μ. We notice that $w_\mu \in Q(\Gamma)$ if and only if $\mu \in L_\infty(U,\Gamma)$ for the following reason.

If $w_\mu \in Q(\Gamma)$, then $\hat{\gamma} = w_\mu \circ \gamma \circ w_\mu^{-1}$ is a Moebius transformation for every $\gamma \in \Gamma$. We write $w_\mu \circ \gamma = \hat{\gamma} \circ w_\mu$. We have

$$\frac{\frac{\partial}{\partial \bar z}(w_\mu \circ \gamma)}{\frac{\partial}{\partial z}(w_\mu \circ \gamma)} = \frac{\frac{\partial}{\partial \bar z}\, w_\mu(\gamma)\overline{\gamma'}}{\frac{\partial}{\partial z}\, w_\mu(\gamma)\gamma'} = \mu(\gamma)\frac{\overline{\gamma'}}{\gamma'} \quad \text{and}$$

$$\frac{\frac{\partial}{\partial \bar z}(\hat{\gamma} \circ w_\mu)}{\frac{\partial}{\partial z}(\hat{\gamma} \circ w_\mu)} = \frac{\hat{\gamma}'(w_\mu)\frac{\partial}{\partial \bar z}(w_\mu)}{\hat{\gamma}'(w_\mu)\frac{\partial}{\partial z}(w_\mu)} = \mu.$$

Thus, $\mu(\gamma)\frac{\overline{\gamma'}}{\gamma'} = \mu$ for all $\gamma \in \Gamma$. Hence $\mu \in L_\infty(U,\Gamma)$.

Conversely, let $\mu \in L_\infty(U,\Gamma)$. $w_\mu \circ \gamma$ has the Beltrami coefficient $\mu(\gamma)\frac{\overline{\gamma'}}{\gamma'}$ which equals μ, since $\mu \in L_\infty(U,\Gamma)$. Therefore, $w_\mu \circ \gamma$ and w_μ have the same Beltrami coefficient. Hence they differ by a conformal map; that is, $w_\mu \in Q(\Gamma)$.

Therefore, there is a canonical bijection

$$L_\infty(U,\Gamma)_1 \to Q_{norm}(\Gamma).$$

We endow $T(\Gamma)$ with the quotient topology associated with the surjective map $\mu \to [w_\mu]$. $T(\Gamma)$, with this topology, can be realized as a bounded open set in $B_2(U^*,\Gamma)$. A proof can be found in [1, Chapter VI]. Since it is an open set in $B_2(U^*,\Gamma)$, $T(\Gamma)$ is a complex manifold modelled on $B_2(U^*,\Gamma)$. The space $B_2(U^*,\Gamma)$, and hence $T(\Gamma)$ has dimension $3p-3+n$ when Γ is of type $(p,n,0)$.

We take $\mu \in L_\infty(U,\Gamma)_1$ and extend it to be zero on the rest of $\hat{\mathbb{C}}$. There exists a unique quasiconformal self-map of $\hat{\mathbb{C}}$, fixing 0,1 and ∞, which has the Beltrami coefficient μ on U and which is conformal on U^*[1, Chapter VI]. We denote this w by w^μ. $w^\mu|_{\mathbb{R}}$, hence $w^\mu|_{U^*}$ depends only on $[w_\mu]$ [1, Chapter VI]. Therefore, $w^\mu(U)$ depends only on $[w_\mu]$. We denote $w^\mu(U)$ by $D(t)$, where $t = [w_\mu] \in T(\Gamma)$. The boundary of $w^\mu(U)$ is $w^\mu(\hat{\mathbb{R}})$. The group $w^\mu\Gamma(w^\mu)^{-1}$ fixes this boundary, which is a Jordan curve. Hence, $w^\mu\Gamma(w^\mu)^{-1}$ is quasi-Fuchsian; a discontinuous group is quasi-Fuchsian if it fixes a directed Jordan curve. We denote $w^\mu\Gamma(w^\mu)^{-1}$ by $\Gamma(t)$.

The <u>Bers fiber space</u> $F(\Gamma)$ over $T(\Gamma)$ is the set of pairs (t,z) with $t \in T(\Gamma)$, $z \in D(t)$.

§3. As we have seen in the last section, for each $t \in T(\Gamma)$, there exist a quasi-Fuchsian group $\Gamma(t)$ and a Jordan domain $D(t)$. To each point t we associate the complex vector space $B_2(D(t),\Gamma(t))$ of bounded quadratic differentials for $\Gamma(t)$.

We form

$$\mathcal{B}(T(\Gamma)) = \bigcup_{t\in T(\Gamma)} B_2(D(t),\Gamma(t))$$

as a fiber space over $T(\Gamma)$. Locally, the space $B_2(D(t),\Gamma(t))$ can be supplied with a basis $\{\varphi_k(z;t)\}_{k=1}^{3p-3+n}$. We can fix t_o such that $\Gamma(t_o) = \Gamma$ and $D(t_o) = U$. We express the basis $\{\varphi_k(z;t_o)\}_{k=1}^{3p-3+n}$ in terms of the Poincaré series of rational functions R_k as follows [19].

$$\varphi_k(z,t_o) = \sum_{\gamma\in\Gamma} R_k(\gamma(z))\gamma'(z)^2, \; z \in U, \; k = 1,2,\cdots,3p-3+n.$$

Then

$$\varphi_k(z;t) = \sum_{\gamma\in\Gamma} R_k(\gamma^t(z))\gamma^{t'}(z)^2, \; z \in D(t), k = 1,2,\cdots,3p-3+n$$

shall yield a basis for $B_2(D(t),\Gamma(t))$. Hence locally, $\mathcal{B}(T(\Gamma))$ is just $T(\Gamma)\times\mathbb{C}^{3p-3+n}$. Since $T(\Gamma)$ is contractible, we actually have [21]

$$\mathcal{B}(T(\Gamma)) \cong T(\Gamma) \times \mathbb{C}^{3p-3+n};$$

$\cong$ is a topological isomorphism. Thus $\mathcal{B}(T(\Gamma))$ forms a complex vector bundle of rank $3p-3+n$ over $T(\Gamma)$. We denote the points of $\mathcal{B}(T(\Gamma))$ by $(t,\varphi(t))$ where $\varphi(t) \in B_2(D(t),\Gamma(t))$.

Each $(t,\varphi(t)) \in B_2(D(t),\Gamma(t))$ determines a holomorphic local homeomorphism

$$f(z;t) : D(t) \to \hat{\mathbb{C}}$$

and a homomorphism $\chi_o : \Gamma(t) \to G$ satisfying

(3·3) $f(\gamma^t(z)) = \chi_o(\gamma^t)(f(z)), \ \gamma^t \in \Gamma(t), \ z \in D(t),$

where $\gamma^t = w^\mu \circ \gamma \circ (w^\mu)^{-1}$ for a fixed $\mu \in L_\infty(U,\Gamma)_1$. Let $\theta^\mu : \gamma \to \gamma^t$ be the isomorphism of Γ onto $\Gamma(t)$ induced by the quasiconformal map w^μ. We take $\chi = \chi_o \circ \theta^\mu$. Then χ is a homomorphism of Γ into G induced by $f \circ w^\mu$ and we have form (3·3).

(3·4) $f \circ w^\mu \circ \gamma = \chi(\gamma) \circ f \circ w^\mu$, for all $\gamma \in \Gamma$.

f and $A \circ f$, $A \in G$, have the same Schwarzian derivative φ. Since replacing f by $A \circ f$ has the effect of replacing χ by $A\chi A^{-1}$, we have a well defined map

$$\phi : \mathfrak{B}(T(\Gamma)) \to \mathrm{Hom}(\Gamma,G)/G.$$

We call ϕ the Monodromy map. Our main result in this chapter is the following.

<u>Theorem 2</u>. The Monodromy map is a holomorphic local homeomorphism.

We want to study the local behaviour of ϕ. For this purpose, we fix the origin $t_o \in T(\Gamma)$ so that $D(t_o) = U$ and $\Gamma(t_o) = \Gamma$; we set $t_o = [id]$. We consider the vector space W of functions $\mu : \mathbb{C} \to \mathbb{C}$ satisfying the following conditions

$$\mu(z) = (\operatorname{Im} z)^2\overline{\varphi(z)}, \; z \in U, \quad \text{for some } \varphi \in B_2(U,\Gamma)$$
$$= 0, \quad \text{outside } U.$$

Let W_1 be the subset of W consisting of μ with $\|\mu\|_\infty < 1$. For each $\mu \in W_1$ there exists a unique quasi-conformal self-map $w = w^\mu$ of $\hat{\mathbb{C}}$, fixing 0, 1 and ∞, and such that w has the Beltrami coefficient μ in U. Moreover, $w^\mu(U)$ is a Jordan domain and $w^\mu\Gamma(w^\mu)^{-1}$ is a quasi-Fuchsian group fixing $w^\mu(U)$. There exists a neighborhood W_o of zero in W_1 which provides a local coordinate at t_o in such a way that for every t in a sufficiently small neighborhood of t_o, $D(t)$ is the Jordan domain $w^\mu(U)$ and $\Gamma(t)$ is the quasi-Fuchsian group $w^\mu\Gamma(w^\mu)^{-1}$ for some $\mu \in W_o$. We choose W_o so small that a point $z_o \in w^\mu(U)$ for all $\mu \in W_o$ whenever $z_o \in U$.

Now for $\mu \in W_o$ and $\varphi \in B_2(w^\mu(U), w^\mu\Gamma(w^\mu)^{-1})$, we consider the ordinary differential equation

$$(3\cdot5) \qquad 2\eta'' + \varphi\eta = 0$$

in $w^\mu(U)$. Let $z_o \in U$. Let η_1 and η_2 be two linearly independent solutions of (3·5) satisfying the initial conditions

$$(3\cdot6) \qquad \begin{cases} \eta_1(z_o) = 0, & \eta_1'(z_o) = 1 \\ \eta_2(z_o) = 1, & \eta_2'(z_o) = 0 \end{cases} .$$

Let
$$g_{\varphi} = \frac{\eta_1}{\eta_2}.$$

Then $g = g_{\varphi}$ is the unique function whose Schwarzian derivative is φ[13], and g satisfies the normalization

(3·7) $\quad g(z_o) = 0,\ g'(z_o) = 1,\ g''(z_o) = 0.$

The derivative of $\eta_2\eta_1' - \eta_1\eta_2'$ with respect to z is zero. Hence $\eta_2\eta_1' - \eta_1\eta_2'$ is constant as a function of z. Since at z_o its value is 1, it is identically 1. Hence

$$g'(z) = \frac{1}{\eta_2^2(z)} \text{ and } g''(z) = -2\,\frac{\eta_2'(z)}{\eta_2^3(z)},$$

and the normalization (3·6) gives the normalization (3·7).

Any function f satisfying $\mathcal{S}f = \varphi$ is given by $f = A\circ g$ for some $A \in G$. Hence for $\mu \in W_o$ and $\varphi \in B_2(w^{\mu}(U), w^{\mu}\Gamma(w^{\mu})^{-1})$, we have from (3·4),

$$A\circ g\circ w^{\mu}(\gamma(z)) = \chi(\gamma)\circ A\circ g\circ w^{\mu}(z), \quad \text{for all } \gamma \in \Gamma,\ z \in U.$$

We take

$$h = A\circ g\circ w^{\mu}.$$

In our setup, w^{μ} is a C^{∞}-function on U; w^{μ} has all higher order continuous partial derivatives in U. Hence h is also a C^{∞}-function on U. h satisfies

$$h\circ\gamma = \chi(\gamma)\circ h, \quad \text{for all } \gamma \in \Gamma.$$

Since g depends on φ and w^{μ} depends on the Beltrami coefficient μ, h is a function of A,μ and φ. Hence so is χ. We denote the map

$$G \times \mathcal{B}(T(\Gamma)) \ni (A,\mu,\varphi) \mapsto \chi \in \mathrm{Hom}(\Gamma,G)$$

by ϕ^*. We shall show that ϕ^* is holomorphic in the next section.

§4. The Lemmas in this section have been taken from [7]. These Lemmas do not need any adjustment for the parabolic and elliptic elements. But, for the sake of completeness, we include the proofs. First, we prove that h depends holomorphically on A,μ and φ; if A,μ,φ are holomorphic functions of a complex variable τ, then h is also a holomorphic function of τ.

<u>Lemma 3.1</u>. Let A,μ and φ be a functions of a complex variable τ such that $A(z,\tau) \in G$, $\mu(z,\tau) \in W_o$, and $\varphi(z,\tau)$ is in $B_2(w^{\mu}(U), w^{\mu}\Gamma(w^{\mu})^{-1})$ for all τ; $|\tau| < \varepsilon$. We assume that

$$(3\cdot 8)\quad \begin{cases} A(z,\tau) = A_o(z) + \tau\dot{A}(z) + o(\tau) \\ \mu(z,\tau) = \tau\dot{\mu}(z) + o(\tau) \\ \varphi(z,\tau) = \varphi_o(z) + \tau\dot{\varphi}(z) + o(\tau) \qquad \text{for } |\tau| < \varepsilon, \end{cases}$$

where $A_o(z) = A(z,0)$, $\varphi_o(z) = \varphi(z,0)$ and the dot denotes the derivative with respect to τ at $\tau = 0$. We set $\mu_o(z) = \mu(z,0) = 0$. Then h has a power series expansion

$$(3\cdot 9) \qquad h(z,\tau) = h_o(z) + \tau\dot{h}(z) + o(\tau), \quad \text{for } |\tau| < \varepsilon,$$

where $h_o(z) = h(z,0)$ and $\dot{h}(z) = \frac{\partial h}{\partial \tau}\Big|_{\tau=0}$.

Proof. From the theory of differential equations we know that the solution $g = g_\varphi$ of the differential equation $\mathcal{S}g = \varphi$ depends holomorphically on φ. It is known that if μ depends holomorphically on τ, then w^μ depends holomorphically on τ [3]. Thus,

$$w(z,\tau) = z + \tau\dot{w}(z) + o(\tau), \quad \text{for } |\tau| < \varepsilon,$$

where $\dot{w}$ is given by the following integral [1, Chapter V].

$$(3\cdot 10) \qquad \dot{w}(z) = \frac{z(z-1)}{2\pi i} \iint_U \frac{\dot{\mu}(\zeta)d\zeta\wedge d\bar{\zeta}}{\zeta(\zeta-1)(\zeta-z)}$$

which satisfies $\dot{w}_{\bar{z}} = \dot{\mu}$.

Finally, $h = A\circ g\circ w^\mu$ is a holomorphic function of τ and hence h has a power series expansion

$$h(z,\tau) = h_o(z) + \tau\dot{h}(z) + o(\tau), \quad \text{for } |\tau| < \varepsilon.$$

This completes the proof of the Lemma.

We notice some results from (3·10) for further use. From (3·10), we get that $\dot{w} = 0$ if $\dot{\mu} = 0$. Moreover, if

$\dot{w} = 0$, $\dot{\mu} = \dot{w}_{\bar{z}} = 0$. Therefore, we conclude that

(3·11) $\dot{w} = 0 \Leftrightarrow \dot{\mu} = 0$.

We take $h^* = \frac{\dot{h}}{h_o'}$. We also observe some useful properties of h^*. We have

$$h = f \circ w^{\mu} ,$$

where $f = A \circ g$; that is,

$$f(z,\tau) = A(g(z,\tau),\tau).$$

Then

$$\dot{f}(z) = \frac{\partial f}{\partial \tau}\Big|_{\tau=0} = A'(g(z,0),0)\dot{g} + \dot{A}(g(z,0));$$

(3·12) $\dot{f}(z) = A_o'(g_o(z))\dot{g} + \dot{A}(g_o(z))$,

where the prime denotes the derivative with respect to z. Again,

$$\dot{h}(z) = \frac{\partial h}{\partial \tau}\Big|_{\tau=0} = f'(w^{\mu}(z,0),0)\dot{w} + \dot{f}(w^{\mu}(z,0));$$

(3·13) $\dot{h}(z) = f_o'(z)\dot{w} + \dot{f}(z)$,

since $w^{\mu}(z,0) = z$. We know that

$$h_o = A_o \circ g_o = f_o.$$

We define $f^* = \frac{\dot{f}}{f_o'}$. Then (3·13) gives

(3·14) $h^* = \dot{w} + f^*$.

From (3·12) we get

$$f^*(z) = \frac{\dot{f}(z)}{f_o'(z)} = \frac{\dot{g}(z)}{g_o'(z)} + \frac{\dot{A}(g_o(z))}{A_o'(g_o(z))g_o'(z)} .$$

We know that

$$g(z,\tau) = \frac{\eta_1(z,\tau)}{\eta_2(z,\tau)} .$$

Thus

$$\dot{g}(z) = \frac{\eta_2(z,0)\dot{\eta}_1(z) - \eta_1(z,0)\dot{\eta}_2(z)}{\eta_2^2(z,0)}$$

and

$$g_o'(z) = \frac{\eta_2(z,0)\eta_1'(z,0) - \eta_1(z,0)\eta_2'(z,0)}{\eta_2^2(z,0)}$$

$$= \frac{1}{\eta_2^2(z,0)} \text{, since } \eta_2\eta_1' - \eta_1\eta_2' = 1 \text{ for all } z \in U$$

We see that $\frac{\dot{g}}{g_o'}$ is holomorphic on U. $\frac{\dot{A}}{A_o'}$ is also holomorphic. Hence f^* is holomorphic on U. From (3·14) we conclude that h^* is a C^∞-function on U. As a consequence of all the above results we have the following.

<u>Lemma 3.2</u>. $h^* = 0 \Leftrightarrow \dot{A} = \dot{u} = \dot{\varphi} = 0$.

<u>Proof</u>. Since g depends holomorphically on φ, $\dot{g} = 0$ if $\dot{\varphi} = 0$. Thus if $\dot{A} = \dot{\mu} = \dot{\varphi} = 0$, (3·11), (3·12) and (3·13) together imply that $h^* = 0$.

Conversely, let $h^* = 0$. Then from (3·14) we have

$$h^*_{\bar{z}} = \dot{w}_{\bar{z}} = 0, \quad \text{and hence}$$

$\underline{\dot{\mu} = 0}$, since $\dot{\mu} = \dot{w}_{\bar{z}}$. But then $\dot{w} = 0$ from (3·11). Hence from (3·13) we have $\dot{f} = 0$, and hence $Sf = Sf_o + o(\tau)$; that is, $\varphi = \varphi_o + o(\tau)$ which implies that $\underline{\dot{\varphi} = 0}$. Consequently $\dot{g} = 0$. Finally, from (3·23) we have $\underline{\dot{A} = 0}$.

Next we shall show with the help of Lemma 3.1 that χ depends holomorphically on A, μ, φ. To show this we prove the following.

<u>Lemma 3.3</u>. Let A, μ and φ satisfy (3·8) and let h satisfy (3·9). Then $\chi(\gamma)$, $\gamma \in \Gamma$, has the following power series expansion

(3·15) $\quad \chi(\gamma) = \chi_o(\gamma) + \tau\dot{\chi}(\gamma) + o(\tau)$ for $|\tau| < \varepsilon$

and for all $\gamma \in \Gamma$, where

(3·16) $\quad \dot{\chi}(\gamma)(h_o(z)) = (h_o \circ \gamma)'(z)[h^*(\gamma)\gamma'(z)^{-1} - h^*(z)]$, $z \in U$,

for all $\gamma \in \Gamma$.

<u>Proof</u>. We fix a $\gamma \in \Gamma$. We choose a compact domain $D_o \subset U$ such that $h(D_o)$ and $h(\gamma(D_o))$ are bounded regions in $\mathbb{C}$ for $|\tau| < \varepsilon$. Using analyticity of $\chi(\gamma)$ on $h(D_o)$ and $h_o(D_o)$ we get, in D_o,

(3·17) $\quad \chi(\gamma)(h(z)) = \chi(\gamma)(h_o(z)) + \tau\dot{h}(z)\chi(\gamma)'(h_o(z)) + o(\tau)$,

since $h(z,\tau) = h_o(z) + \tau\dot{h}(z) + o(\tau)$. We also have

$$(3\cdot 18)\qquad h(\gamma(z)) = h_o(\gamma(z)) + \tau\dot{h}(\gamma(z)) + o(\tau).$$

Since $\qquad h\circ\gamma = \chi(\gamma)\circ h \quad$ for all $\gamma \in \Gamma$,

we have from (3·17) and (3·18)

$$\chi(\gamma)(h_o(z)) = h_o(\gamma(z)) + \tau(\dot{h}(\gamma(z)) - \dot{h}(z)\chi(\gamma)'(h_o(z))) + o(\tau);$$

that is,

$$\chi(\gamma)(h_o(z)) = \chi_o(\gamma)(h_o(z)) + \tau(\dot{h}(\gamma(z)) - \dot{h}(z)\chi(\gamma)'(h_o(z))) + o(\tau),$$

since $h_o\circ\gamma = \chi_o(\gamma)\circ h_o$. Since $h_o(D_o)$ is open, we have

$$\chi(\gamma) = \chi_o(\gamma) + \tau\dot{\chi}(\gamma) + o(\tau) \qquad \text{for } |\tau| < \varepsilon \text{ and}$$

for all $\gamma \in \Gamma$ with

$$\begin{aligned}
\dot{\chi}(\gamma)(h_o(z)) &= \lim_{\tau\to 0} \frac{\chi(\gamma)(h_o(z)) - \chi_o(\gamma)(h_o(z))}{\tau} \\
&= \dot{h}(\gamma(z)) - \dot{h}(z)\chi_o(\gamma)'(h_o(z)) \\
&= h^*(\gamma(z))\ h_o'(\gamma(z)) - \chi_o(\gamma)'(h_o(z))h^*(z)\ h_o'(z) \\
&= h^*(\gamma(z))\ h_o'(\gamma(z)) - (\chi_o(\gamma)\circ h_o)'(z)h^*(z) \\
&= h^*(\gamma(z))\ h_o'(\gamma(z)) - (h_o\circ\gamma)'(z)h^*(z) \\
&= (h_o\circ\gamma)'(z)[h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)],\ z \in D_o.
\end{aligned}$$

Once we show that both sides of the above equation are meromorphic functions, we have (3·16) since D_o is open in U.

We rewrite the above equation in the following form.

$$\frac{\dot{\chi}(\gamma)(h_o(z))}{\chi_o(\gamma)'(h_o(z))} = h_o'(z)[h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)], \tag{3·19}$$

since $h_o \circ \gamma = \chi_o(\gamma) \circ h_o$. $\frac{\dot{\chi}(\gamma)}{\chi_o(\gamma)'}$ is a polynomial. Thus we just need to show that the right hand side of (3·19) is a meromorphic function in U.

We shall show that $h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)$ is meromorphic as follows.

$$\frac{\partial}{\partial \bar{z}}[h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)]$$

$$= h^*_{\bar{z}}(\gamma(z))\overline{\gamma'(z)}\gamma'(z)^{-1} - h^*_{\bar{z}}(z)$$

$$= \dot{\mu}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} - \dot{\mu}(z),$$

since $h^*_{\bar{z}} = \dot{\mu}$. When $\mu \in W_o$, $\dot{\mu} \in W$; that is,

$$\dot{\mu} = (\operatorname{Im} z)^2\overline{\varphi(z)} \qquad \text{for some } \varphi \in B_2(U,\Gamma).$$

Therefore,

$$\dot{\mu}(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \dot{\mu}(z),$$

and hence

$$\frac{\partial}{\partial \bar{z}}[h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)] = 0$$

∎

This completes the proof.

§5. The Lemma 3.3 has the following.

<u>Corollary 1</u>. $\dot{\chi}(\gamma) = 0$ for all $\gamma \in \Gamma$ if and only if $h^* = 0$ in U.

Proof. From (3·19) we have

$$\frac{\dot{\chi}(\gamma)(h_o(z))}{\chi_o(\gamma)'(h_o(z))} = h_o'(z)[h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)].$$

Since $\frac{\dot{\chi}(\gamma)(z)}{\chi_o(\gamma)'(z)}$ is a polynomial, and $h_o(U)$ is open, $\dot{\chi}(\gamma) = 0$ if $h^* = 0$ in U.

Now we assume that $\dot{\chi}(\gamma) = 0$ for all $\gamma \in \Gamma$. Then

$$h^*(\gamma(z))\gamma'(z)^{-1} = h^*(z), \text{ for all } \gamma \in \Gamma, z \in U.$$

Hence h^* is a C^∞ (-1) differential for Γ. We shall show that h^* is actually holomorphic in U under the assumption that $\dot{\chi}(\gamma) = 0$ for all $\gamma \in \Gamma$. We intend to apply Stokes' Theorem on U/Γ. Since U/Γ has punctures, Stokes' Theorem cannot be applied directly. We follow Bers [4] to handle this situation.

Let us recall that Γ is finitely generated and has m parabolic generators. Thus the fundamental domain D of Γ is bounded by finitely many arcs pairwise identified by elements of Γ and D contains m cusped regions. Let ∂D be the positively oriented boundary of D. Let C and C' be two sides of ∂D that are identified by an element $\gamma \in \Gamma$. For convenience, we relabel C and C' and assume that

$$\gamma(C) = -C'.$$

We draw in each cusped region a smooth curve C_s, $s = 1,2,\cdots,m$ so that (i) C_s joins two points ζ_s and ζ_s' on ∂D which are identified by an element of Γ, and (ii) C_s and $C_{s'}$ do not meet, for $s \neq s'$. In this manner we obtain a relatively compact subset D^* of D which is bounded by part of ∂D and the curves $C_1, C_2, \cdots, C_m$.

For any $\varphi \in B_2(U,\Gamma)$, $h^*\varphi$ is a C^∞-differential for Γ. Let φ be arbitrary. By Stokes' Theorem we have

$$\iint_{D^*} d(h^*\varphi dz) = \int_{\partial D^*} h^*\varphi dz = \sum_{s=1}^{m} \int_{C_s} h^*\varphi dz;$$

the integrals along two identified sides C and C' on ∂D cancell each other since $\gamma(C) = -C'$ and $h^*\varphi dz$ is Γ-invariant. The integral $\iint_{D^*} d(h^*\varphi dz)$ is very close to the integral $\iint_D d(h^*\varphi dz)$ whenever $\zeta_s \to a_s$; a_s is the fixed point of the parabolic transformation A_s identifying ζ_s and ζ_s'. Hence we can show that

$$\iint_D d(h^*\varphi dz) = 0$$

by showing that $\lim_{\zeta_s \to a_s} \int_{C_s} h^*\varphi dz = 0$, for $s = 1,2,\cdots,m$.

It suffices to assume that $s = 1$, $A_s(z) = z+1$ and $a_s = \infty$. Then the cusped region belonging to ∞ is the region

$$U_c = \{z \in \mathbb{C};\ 0 \le \operatorname{Re} z < 1,\ \operatorname{Im} z > c\}.$$

Hence

$$(3\cdot 20)\qquad \int_{C_1} h^*\varphi dz = \int_0^1 h^*(x+ib)\varphi(x+ib)dx,$$

where $\zeta_1 = ib$; $b > c$, hence $\zeta_1' = 1+ib$. Since $\varphi \in B_2(U,\Gamma)$, $\varphi(z+1) = \varphi(z)$ which implies that $\varphi(z)$ has a Fourier series expansion

$$\varphi(z) = \sum_{n=-\infty}^{\infty} {}_n e^{2\pi inz},\ z \in U.$$

Since $\sup_{z\in U}\{(\operatorname{Im} z)^2|\varphi(z)|\} < \infty$, $a_n = 0$ for $n \le 0$. Therefore,

$$\varphi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inz},$$

and hence

$$(3\cdot 21)\qquad |\varphi(x+ib)| \le \text{const.}\ e^{-2\pi b}.$$

From (3·15) we have

$$h^* = f^* + \dot{w},$$

where

$$\dot{w}(z) = \frac{z(z-1)}{2\pi i}\iint_U \frac{\dot{u}(\zeta)d\zeta\wedge d\bar{\zeta}}{(\zeta-z)\zeta(\zeta-1)}.$$

It is known [18, Chapter IV] that

$$\dot{w}(z) = 0(|z|\log|z|)\ \text{as}\ z \to \infty,$$

and hence

$$(3\cdot 22)\qquad |\dot{w}(x+ib)| \le \text{const.}(x^2+b^2)\log(x^2+b^2)\quad \text{as}\ b \to \infty.$$

Finally, we shall find a growth condition on f^*. For this purpose we study the behaviour of f^* in the cusped region U_c.

From (3·4) it follows that

$$f \circ w^{\mu} \circ A_1 = \chi(A_1) \circ f \circ w^{\mu};$$

(3·23) $$f \circ w^{\mu} \circ A_1 \circ (w^{\mu})^{-1} = \chi(A_1) \circ f.$$

Let

$$A_\tau = w^{\mu} \circ A_1 \circ (w^{\mu})^{-1}.$$

A_τ is parabolic, since A_1 is parabolic. Since w^{μ} fixes 0, 1 and ∞, A_τ fixes ∞, and takes 0 to 1. Hence $A_\tau(z) = z+1$, for all τ. Moreover, $\chi(A_1)$ is parabolic if A_1 is parabolic [15]. Let $B_\tau(z) = \dfrac{1}{z - P_\tau}$, where P_τ is the fixed point of $\chi(A_1)$. Then

$$B_\tau \circ \chi(A_1) \circ B_\tau^{-1}(z) = z + b_\tau, \; b_\tau \neq 0 .$$

We replace f by $B_\tau \circ f$ so that $\chi(A_1)$ is replaced by $B_\tau \circ \chi(A) \circ B_\tau^{-1}$, and we get from (3·23)

(3·24) $$B_\tau \circ f \circ A_\tau = B_\tau \circ \chi(A_1) \circ B_\tau^{-1} \circ B_\tau \circ f .$$

We take

$$F = B_\tau \circ f$$

and check that

$$\frac{\dot{F}}{F_o'} = \frac{\dot{f}}{f_o'} ,$$

since $\dot{B} = 0$. From (3·24), we have

$$F \circ A_\tau(z) = B_\tau \circ \chi(A_1) \circ B_\tau^{-1} \circ F(z);$$

$$F(z+1) = F(z) + b_\tau, \quad z \in w^\mu(u).$$

Differentiating with respect to z we get

$$F'(z+1) = F(z).$$

Therefore, $F'(z)$ is periodic in z and has a Fourier series expansion

$$(3\cdot25) \qquad F'(z,\tau) = \sum_{n=-\infty}^{\infty} a_n(\tau) e^{2\pi i n z}, \quad z \in w^\mu(U).$$

Now we follow the arguments in [15].

F is locally one-to-one in $w^\mu(U)$. Hence F' is a non-zero meromorphic function of z. F' has only double poles and all its residues vanish. Hence $y = (F')^{-1/2}$ is a well defined holomorphic function in $w^\mu(U)$. An easy calculation shows that y satisfies

$$(3\cdot26) \qquad 2y'' + \varphi(z,\tau)y = 0, \; z \in w^\mu(U), \; \tau \in \Delta_\varepsilon = \{\tau; |\tau| < \varepsilon\}.$$

Since $\varphi \in B_2(w^\mu(U), w^\mu\Gamma(w^\mu)^{-1})$,

$$\varphi(z+1) = \varphi(z),$$

and hence

$$y(z+1) = \pm y(z).$$

We may assume, without loss of generality, that

$$y(z+1) = y(z);$$

otherwise we shall replace A_1 by A_1^2, and A_1^2 will be new A_1. Hence there exist functions $\tilde{\varphi}$ and $\tilde{y}$ in $\Delta \setminus \{0\}$; Δ is a small disc (in a quasi-disc) such that

$$y(z) = \tilde{y}(e^{2\pi iz}) \text{ and } \varphi(z) = \tilde{\varphi}(e^{2\pi iz}).$$

Let $\zeta = e^{2\pi iz}$. Then (3.26) becomes

$$\frac{\partial^2 \tilde{y}}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial \tilde{y}}{\partial \zeta} - \frac{1}{8\pi^2} \frac{\tilde{\varphi}(\zeta,\tau)}{\zeta^2} \tilde{y} = 0.$$

Two independent solutions for the above differential equation are

$$y_1(\zeta,\tau) = 1 + \sum_{n=1}^{\infty} a_n(\tau)\zeta^n$$

and

$$y_2(\zeta,\tau) = y_1(z,\tau)\log \zeta + \sum_{n=1}^{\infty} b_n(\tau)\zeta^n,$$

where $y_1(0,\tau) = 1$ for all τ [16]. Thus the general solution is given by

$$\tilde{y}(\zeta,\tau) = c_1(\tau)y_1(z,\tau) + c_2(\tau)(y_1(z,\tau)\log \zeta + \sum_{n=1}^{\infty} b_n(\tau)\zeta^n);$$

$$\zeta \in \Delta \setminus \{0\}, \text{ and } \tau \in \Delta_\varepsilon.$$

Since $\tilde{y}(\zeta,\tau)$ is a single valued function in ζ, $c_2(\tau) = 0$ for all τ and we have

$$\tilde{y}(\zeta,\tau) = c_1(\tau)y_1(\zeta,\tau);\ y_1(0,\tau) = 1 \text{ and } c_1(\tau) \neq 0.$$

Thus as $z \to \infty$ through a cusped region belonging to ∞ in $w^{\mu}(u)$, F' has a nonzero finite limit. Hence we have

$$a_n(\tau) = 0 \quad \text{for } n < 0 \quad \text{and}$$

$$a_o(\tau) \neq 0.$$

Thus from (3·25) we have

$$(3\cdot27) \qquad F'(z,\tau) = a_o(\tau) + \sum_{k=1}^{\infty} a_k(\tau)e^{2\pi ikz},$$

where $a_o(\tau) = b_\tau \neq 0$. Moreover,

$$a_o(\tau) = \int_{z_o}^{z_o+1} F'(z,\tau)dz \quad \text{and}$$

$$b_k(\tau) = \int_{z_o}^{z_o+1} e^{-2k\pi iz}F'(z,\tau)dz, \quad z_o \in w^{\mu}(u).$$

Integrating (3·27) we get

$$(3\cdot28) \qquad F(z,\tau) = b_\tau z + \sum_{k=1}^{\infty} c_k(\tau)e^{2\pi ikz}, c_k(\tau) = \frac{a_n(\tau)}{2k\pi i}.$$

b_τ and $c_k(\tau)$ are holomorphic in τ, hence they have power series expansions in τ which are uniformly convergent in Δ_ε. Thus from (3·28), taking derivative with respect to τ at $\tau = 0$, we get

$$\dot{F}(z) = \dot{b}z + \sum_{k=1}^{\infty} \dot{c}_k e^{2\pi ikz}, z \in U.$$

We know that

$$B_\tau \circ \chi(A_1) \circ B_\tau^{-1}(z) = z + b_\tau; \text{ that is,}$$

$$B_\tau \circ \chi(A_1)(z) = B_\tau(z) + b_\tau.$$

Differentiating with respect to τ at $\tau = 0$ we get

$$B_o'(\chi_o(A_1))(z)\dot{\chi}(A_1) = \dot{B}(z) + \dot{b} = \dot{b},$$

since $\dot{B} = 0$. Thus $\dot{\chi}(A_1) = 0$ implies that $\dot{b} = 0$, and we have

$$\dot{F}(z) = \sum_{k=1}^{\infty} \dot{c}_k e^{2\pi ikz}, \quad z \in U.$$

From (3·27) we also get

$$F_o'(z) = F'(z,0) = b_o + \sum_{k=1}^{\infty} a_k(0) e^{2\pi ikz}, z \in U.$$

Hence

$$\frac{\dot{F}(z)}{F_o'(z)} = \sum_{k=1}^{\infty} \dot{c}_k e^{2\pi ikz} \left(b_o + \sum_{k=1}^{\infty} a_k(0) e^{2\pi ikz}\right)^{-1}$$

$$= \sum_{k=1}^{\infty} d_k e^{2\pi ikz}.$$

Hence we have

$$(3\cdot29) \qquad f^*(z) = F^*(z) = \sum_{k=1}^{\infty} d_k e^{2\pi ikz}, \quad z \in U.$$

From (3·29) it follows that

$$(3\cdot30) \qquad |f^*(x+ib)| \le \text{const.} e^{-2\pi b}.$$

We recall that in the integral (3·20)

$$h^*\varphi = (f^* + \dot{w})\varphi = f^*\varphi + \dot{w}\varphi.$$

From (3·21), (3·22) and (3·30) we conclude that

$$|h^*(x+ib)\varphi(x+ib)| \le \text{cost.}\left(e^{-4\pi b} + \frac{(x^2+b^2)\log(x^2+b^2)}{e^{2\pi b}}\right) \to 0 \text{ as } b \to \infty$$

and hence

$$\lim_{b\to\infty} \int_{C_1} h^*\varphi dz = \lim_{b\to\infty} \int_0^1 h^*(x+ib)\varphi(x+ib) = 0.$$

Thus we have

$$\iint_D d(h^*\varphi dz) = 0; \text{ that is}$$

$$\iint_D h^*_{\bar{z}}\,\varphi dz \wedge d\bar{z} = 0,$$

since $d(h^*\varphi dz) = \frac{\partial}{\partial \bar{z}}(h^*\varphi)d\bar{z}\wedge dz = h^*_{\bar{z}}\varphi d\bar{z}\wedge dz$. From (3·14) we know that $h^*_{\bar{z}} = \dot{w}_{\bar{z}} = \dot{u}$, hence we have

(3·31) $\iint_D \dot{\mu}\varphi dz \wedge d\bar{z} = 0$ for any $\varphi \in B_2(U,\Gamma)$.

$\dot{\mu} \in W$ if $\mu \in W_o$. Thus

$$\dot{\mu}(z) = (\text{Im } z)^2\,\overline{\varphi_o(z)},\ z \in U, \text{ for some } \varphi_o \in B_2(U,\Gamma).$$

We now take $\varphi = \varphi_o$ in (3·31). Then we have

$$\iint_D (\text{Im } z)^2\,|\varphi_o(z)|^2 dz \wedge d\bar{z} = 0$$

$$\Rightarrow \varphi_o = 0 \Rightarrow \dot{\mu} = 0 \Rightarrow \dot{w} = 0 \Rightarrow h^*_{\bar{z}} = 0.$$

Hence h^* is holomorphic in U. Furthermore, $h^* = f^*$. Thus h^* is a (-1) differential for Γ.

Following Kra [18], we define

$$\text{red ord}_P h^* = \frac{\text{ord}_P h^*}{\Gamma_P}, \text{ if } P \in U,$$

and for each cusp a_s of Γ,

$$\text{red ord}_{a_s} h^* = r$$

if the Fourier series expansion of h^* at ∞ is

$$h^*(z) = \sum_{k=r}^{\infty} a_k e^{2\pi ikz}, \; a_r \neq 0, \; z \in U.$$

Since h^* is holomorphic in U,

$$\text{red ord}_P h^* \geq 0 \quad \text{if } P \in U.$$

From (3-29),

$$\text{red ord}_{a_s} h^* \geq 1 \quad \text{for } s = 1,2,\cdots,m.$$

Thus

$$\sum_{P\in\overline{D}_o} \text{red ord}_P h^* > 0, \quad \text{where } D_o \text{ is a fundamental set in } \overline{U} \text{ for } \Gamma.$$

But

$$\sum_{P\in\overline{D}_o} \text{red ord}_P h^* = -(2p-2 + \sum_{j=1}^{n} (1 - \frac{1}{\nu_j})) \text{ by Kra [18]},$$

and it is negative since $2p - 2 + \sum_{j=1}^{n} (1 - \frac{1}{\nu_j}) > 0$. This contradiction leads to the conclusion that $h^* = 0$. This completes the proof of the Corollary.

<u>Proof of the Theorem</u>. For an arbitrary point $t \in T(\Gamma)$, there exists a map taking t to a given point $t_o \in T(\Gamma)$. This map is a holomorphic homeomorphism [5]. Hence it is sufficient to prove the theorem in a neighborhood of the origin $t_o \in T(\Gamma)$.

In §3 we have seen that, in a neighborhood of t_o, ϕ is induced by ϕ^*. ϕ^* is holomorphic by the Lemma 3.3 . The Lemma 3.2 and the Corollary of the Lemma 3.3 together imply that the differential of ϕ^* is injective. It is known that χ preserves the parabolic elements and the multipliers of the elliptic elements in Γ. Moreover, $\chi(\Gamma)$ is non-elementary by Kra [16]. Hence the image χ of ϕ^* is a manifold point in $\mathrm{Hom}(\Gamma,G)$ by the Theorem 1. Since $G \times \mathfrak{A}(T(\Gamma))$ and $\mathrm{Hom}(\Gamma,G)$ have the same dimension $6p+2n-3$, ϕ^* is a local homeomorphism. Replacing (I,t,φ) by (A,t,φ) in $G \times \mathfrak{A}(T(\Gamma)$ has the effect of conjugating χ by A. Hence we conclude that ϕ is holomorphic and a local homeomorphism in a neighborhood of t_o. This completes the proof.

Chapter IV

Projective structures on bordered Riemann surfaces

§0. Kra [15] defined the concept of a deformation of a Fuchsian group in order to study projective structures on a Riemann surface without using cohomology theory. He proved a uniqueness theorem about deformations of finitely generated Fuchsian groups of the first kind [15], [17]. In this chapter we prove a uniqueness theorem about deformations of a finitely generated Fuchsian group of the second kind. We also study the Monodromy of such deformations.

§1. Let Γ be a Fuchsian group acting on the upper half-plane U and having signature $\{p,n,m;\nu_1,\nu_2,\cdots\nu_n\}$; $2p-2+m+\sum_{j=1}^{n}(1-1/\nu_j) > 0$. Let Ω be the region of discontinuity of Γ; Ω is connected and we assume that $\Omega \cap \hat{\mathbb{R}}$ is not empty. Let $I_1, I_2, \cdots, I_m$ be a maximal set of inequivalent components in $\Omega \cap \hat{\mathbb{R}}$.

Let us recall the definition of a deformation. A pair (f,χ) is a deformation of Γ if

$$f : U \to \hat{\mathbb{C}}$$

is a holomorphic local homeomorphism and

$$\chi : \Gamma \to G$$

is a homomorphism satisfying

$$f \circ \gamma = \chi(\gamma) \circ f \quad \text{for all } \gamma \in \Gamma.$$

Definition. Let $B_2^*(\Gamma)$ denote the space of bounded reflectable quadratic differentials for Γ; $B_2^*(\Gamma)$ consists of functions φ, holomorphic on U, which have holomorphic extensions to $\Omega \cap \hat{\mathbb{R}}$ and which satisfy the following conditions

(i) $\varphi(\gamma)\gamma'^2 = \varphi$ for all $\gamma \in \Gamma$

(ii) $\sup\{(\operatorname{Im} z)^2 |\varphi(z)|\} < \infty$

(iii) $\varphi(z) \in \mathbb{R}$, $z \in \Omega \cap \hat{\mathbb{R}}$.

We shall call the deformations arising from $B_2^*(\Gamma)$ bounded reflectable deformations of Γ.

We observe that a solution of the differential equation

$$(4 \cdot 1) \quad \mathcal{S}f = \varphi, \quad \varphi \in B_2^*(\Gamma)$$

maps each interval in $\Omega \cap \hat{\mathbb{R}}$ into a circle; this will be shown below.

We choose an interval $I_o \subset \Omega \cap \hat{\mathbb{R}}$. Let x_o be an arbitrary point in I_o. Let y_1 and y_2 be two linearly independent solutions of the ordinary differential equation

$$(4 \cdot 2) \quad 2y'' + \varphi y = 0$$

satisfying the initial conditions

$$y_1(x_o) = 0, \quad y_1'(x_o) = 1$$

$$y_2(x_o) = 1, \quad y_2'(x_o) = 0 .$$

Then $g = \frac{y_1}{y_2}$ is the unique solution of (4.1) satisfying

$$g(x_o) = 0, \quad g'(x_o) = 1, \quad g''(x_o) = 0$$

as we have seen earlier in §3 in Chapter III. Hence any solution f of (4·1) is of the form

$$f = A \circ g,$$

for some Moebius transformation A.

Since φ has a holomorphic extension to I_o, in a neighborhood N_{x_o} of x_o; $N_{x_o} \subset \Omega$, φ has the following power series expansion

$$\varphi(z) = \sum_{n=0}^{\infty} a_n (z-x_o)^n, \quad z \in N_{x_o}$$

since φ is real on I_o, every a_n is real.

Let

$$y_1 = \sum_{n=0}^{\infty} b_n (z-x_o)^n \text{ and } y_2 = \sum_{n=0}^{\infty} c_n (z-x_n)^n, \; z \in N_{x_o} .$$

Since y_1 satisfies (4·2), we have

(4·3) $\quad 2y_1'' + \varphi y_1 = 0;$

$$y_1'' = \sum_{n=2}^{\infty} n(n-1) b_n (z-x_o)^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} (z-x_o)^n .$$

Substituting the series expansion of y_1'', y_1 and φ in (4·3), we get

$$2\sum_{n=0}^{\infty}(n+2)(n+1)b_{n+2}(z-x_o)^n + \sum_{n=0}^{\infty}a_n(z-x_o)^n \sum_{n=0}^{\infty}b_n(z-x_o)^n = 0.$$

Equating the coefficients of powers of z to zero we find b_n in terms of a_n. We get the following equations

$$4b_2 + a_o b_o = 0$$

$$12b_3 + a_1 b_o + a_o b_1 = 0$$

$$24b_4 + a_o b_2 + a_1 b_1 + a_1 b_2 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad .$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad .$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad .$$

and so on. With the help of the initial conditions $b_o = 0$ and $b_1 = 1$ we can express b_n; $n = 2,3,\cdots$ in terms of a_n; $n = 0,1,2,\cdots$. Since each a_n is real, each b_n is also real. Similarly, we find c_n and see that each c_n is real. Hence g is real on I_o. Since $f = A\circ g$ for some Moebius transformation A, f maps I_o into a circle.

§2. We prove the following.

<u>Theorem 3</u>. Let (f,χ_1) and (g,χ_2) be two bounded reflectable deformations of a Fuchsian group Γ of signature $\{p,n,m;\nu_1,\nu_2,\ldots,\nu_n\}$; $n > 0$, $m > 0$ and $2p - 2 + m + \sum_{j=1}^{n}(1-1/\nu_j) > 0$. We assume that $f(I_j)$ and $g(I_j)$ lie on the same circle C_j,

$j = 1,2,\cdots,m$. Then $\chi_1 = \chi_2$ implies $f = g$.

Proof. We take

$$F(z) = \frac{(f(z)-g(z))^2}{f'(z)g'(z)}, \quad z \in U.$$

Then $F(z)$ is holomorphic in U. Let

$$\chi_1(\gamma) = \chi_2(\gamma) = A_\gamma, \quad \text{for } \gamma \in \Gamma.$$

Then

$$(4\cdot 3) \qquad \begin{cases} f\circ\gamma = A_\gamma \circ f; \\ g\circ\gamma = A_\gamma \circ g, \end{cases}$$

from which we get

$$(4\cdot 4) \qquad \begin{cases} f'(\gamma) = A'_\gamma(f)f'/\gamma' \\ g'(\gamma) = A'_\gamma(g)g'/\gamma'. \end{cases}$$

It is easy to check that, for a Moebius transformation A,

$$(4\cdot 5) \qquad (A\circ f - A\circ g)^2 = (f-g)^2 A'(f)A'(g).$$

Using (4·3), (4·4) and (4·5) we get, for $z \in U$ and $\gamma \in \Gamma$,

$$\begin{aligned} F(\gamma(z)) &= \frac{(f(\gamma(z))-g(\gamma(z)))^2}{f'(\gamma(z))g'(\gamma(z))} \\ &= \frac{(A_\gamma(f(z))-A_\gamma(g(z)))^2\gamma'(z)^2}{A'_\gamma(f(z))A'_\gamma(g(z))f'(z)g'(z)} \\ &= \frac{(f(z)-g(z))^2 A'_\gamma(f(z))A'_\gamma(g(z))\gamma'(z)^2}{A'_\gamma(f(z))A'_\gamma(g(z))f'(z)g'(z)} \\ &= \frac{(f(z)-g(z))^2}{f'(z)g'(z)}\gamma'(z)^2 = F(z)\gamma'(z)^2. \end{aligned}$$

Therefore, F is a (-2) differential for Γ.

f and g map $I_j \subset \Omega \cap \hat{\mathbb{R}}$ into the circle C_j, $j = 1,2,\cdots,m$. There is a Moebius transformation A_j mapping C_j onto $\hat{\mathbb{R}}$. We have, from (4.5),

$$\frac{(A_j \circ f - A_j \circ g)^2}{(A_j \circ f)'(A \circ g)'} = \frac{(f-g)^2}{f'g'} .$$

$A_j \circ f$ and $A_j \circ g$ are real on I_j. Hence F is real on every I_j. Since

$$F(\gamma(z))\gamma'(z)^2 = F(z), \text{ for all } \gamma \in \Gamma,$$

it is clear that F is real on $\Omega \cap \hat{\mathbb{R}}$. Therefore, F extends to all of Ω.

We study the behaviour of F in a cusped region belonging to each puncture on Ω/Γ; Ω/Γ is a surface of type $(P,2n,0)$, $P = 2p+m-1$. We assume that $\gamma(z) = z+1$ determines a puncture. We use the fact that $\chi_1(\gamma) = \chi_2(\gamma) = A_\gamma$ is also parabolic since $\mathcal{S}f$ and $\mathcal{S}g \in B_2(U,\Gamma)$ [16]. We replace f and g by $C \circ f$ and $C \circ g$, respectively; C is a fixed Moebius transformation. Then A_γ is replaced by $C \circ A_\gamma \circ C^{-1}$ so that

$$C \circ A_\gamma \circ C^{-1}(z) = z+b, \quad b \neq 0 .$$

By the same reasoning as in [16], which we have gone through in Chapter III, we conclude that

$$f'(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$$

and

$$g'(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz}, \quad a_o = b_o = b \neq 0.$$

Hence we have

$$f(z) = bz + \sum_{n=1}^{\infty} A_n e^{2\pi inz}$$

$$g(z) = bz + \sum_{n=1}^{\infty} B_n e^{2\pi inz}.$$

Then we have

$$F(z) = \frac{\left(\sum_{n=1}^{\infty} A_n e^{2\pi inz} - \sum_{n=1}^{\infty} B_n e^{2\pi inz}\right)^2}{\sum_{n=0}^{\infty} a_n e^{2\pi inz} \sum_{n=0}^{\infty} b_n e^{2\pi inz}}$$

$$= \left(\sum_{n=1}^{\infty} A_n e^{2\pi inz} - \sum_{n=1}^{\infty} B_n e^{2\pi inz}\right)^2 \left(a_o + \sum_{n=1}^{\infty} a_n e^{2\pi inz}\right)^{-1} \left(b_o + \sum_{n=1}^{\infty} b_n e^{2\pi inz}\right)^{-1}$$

$$= \sum_{n=2}^{\infty} d_n e^{2\pi inz}.$$

Therefore, the order of $F(z)$ at ∞ is at least 2 [18]. Hence ord $F \geq 2$ at each parabolic fixed point.

We already know that F is holomorphic in Ω. Hence

$$\sum_{Q \in \overline{D}_o} \text{red ord}_Q F > 0;$$

$\overline{D}_o$ is a fundamental set for Γ; that is, $\overline{D}_o$ is in one-to-one correspondence with the points in $\overline{\Omega/\Gamma}$. But, from [18],

$$\sum_{Q\in \overline{D}_0} \text{red ord}_Q F = -2(2P-2+\sum_{j=1}^{2n}(1-1/\nu_j))$$

$$= -4(2p-2+m+\sum_{j=1}^{n}(1-1/\nu_j))$$

$$< 0,$$

since $2p - 2 + m + \sum_{j=1}^{n}(1-1/\nu_j) > 0$. This contradiction leads to the conclusion that $F = 0$. Hence $f = g$. ∎

§3. Let

$$h : U \to \Omega$$

be a holomorphic universal covering map. Let

$$\hat{\Gamma} = \{\hat{\gamma} \in \text{Aut}\, U;\ h\circ\hat{\gamma} = \gamma\circ h \quad \text{for some } \gamma \in \Gamma\}.$$

We call $\hat{\Gamma}$ the <u>Schottky-double</u> of Γ. The correspondence $\mu : \hat{\gamma} \to \gamma$ defines a homomorphism of $\hat{\Gamma}$ onto Γ; (h,μ) is a deformation of $\hat{\Gamma}$. We also say that $(\hat{\Gamma},h)$ uniformizes (Γ,Ω). Let

$$K = \{k \in \text{Aut}\, U;\ h\circ k = h\}.$$

Then K is a subgroup of $\hat{\Gamma}$ which is the covering group of h. We have an exact sequence of groups and group homomorphisms [18].

(4·6) $\quad \{I\} \to K \hookrightarrow \hat{\Gamma} \xrightarrow{\mu} \Gamma \to \{I\}.$

We shall show later that this sequence splits.

We choose a connected component U_o of $h^{-1}(U)$. Let Γ_o be the stabilizer of U_o in $\hat{\Gamma}$. Since h is a covering map, the restriction of h on each connected component of $h^{-1}(U)$ is a homeomorphism onto U. Hence

$$h : U_o \to U$$

is conformal. We call this restriction map h_o. h_o induces an isomorphism

$$\nu : \gamma \to h_o^{-1} \circ \gamma \circ h_o$$

of Γ onto Γ_o. Moreover, U_o/Γ_o is conformally equivalent to U/Γ. We prove the following.

<u>Lemma 4.1</u>. Every element $\hat{\gamma} \in \hat{\Gamma}$ can be represented uniquely as $\hat{\gamma} = k \circ \gamma_o$ with $k \in K$, $\gamma_o \in \Gamma_o$.

<u>Proof</u>. We have the homomorphism

$$\nu \circ \mu : \hat{\Gamma} \to \Gamma_o$$

that satisfies

(4·7) $\quad \nu \circ \mu(\gamma_o) = \gamma_o$, for all $\gamma_o \in \Gamma_o$,

since, for $z \in U_o$, $\gamma_o \in \Gamma_o$,

$$\begin{aligned} \nu \circ \mu(\gamma_o)(z) &= h_o^{-1} \circ \mu(\gamma_o) \circ h_o(z) \\ &= h_o^{-1} \circ h_o \circ \gamma_o(z) \\ &= \gamma_o(z) . \end{aligned}$$

Using (4·7) we can show that $K \cap \Gamma_o = \{I\}$ in the following way. Let $\hat{\gamma} \in K$. Then

$$\nu \circ \mu(\hat{\gamma}) = \nu(I) = I.$$

Let $\hat{\gamma} \in \Gamma_o$, then by (4·7),

$$\nu \circ \mu(\hat{\gamma}) = \hat{\gamma}.$$

Thus if $\hat{\gamma} \in K \cap \Gamma_o$, $\hat{\gamma} = I$.

Now we shall show that every $\hat{\gamma} \in \hat{\Gamma}_o$ has a unique representation

$$\hat{\gamma} = k \circ \gamma_o;\ k \in K,\ \gamma_o \in \Gamma_o.$$

We notice that, for $\gamma_o \in \Gamma_o$, and $z \in U_o$

$$h_o \circ \gamma_o(z) = \mu(\gamma_o) \circ h_o(z).$$

Hence

$$\mu(\gamma_o) = h_o \circ \gamma \circ h_o^{-1} \qquad \text{for } \gamma_o \in \Gamma_o.$$

Since $\Gamma = h_o \Gamma_o h_o^{-1}$, for any $\gamma \in \Gamma$,

$$\gamma = h_o \circ \gamma_o \circ h_o^{-1}, \qquad \text{for some } \gamma_o \in \Gamma_o.$$

Hence

$$\gamma = \mu(\gamma_o), \qquad \text{for some } \gamma_o \in \Gamma_o.$$

Again, for every $\hat{\gamma} \in \hat{\Gamma}$,

$$\mu(\hat{\gamma}) = \gamma \in \Gamma.$$

Hence

$$\mu(\hat{\gamma}) = \mu(\gamma_o); \text{ that is}$$

$$\mu(\hat{\gamma}\circ\gamma_o^{-1}) = I; \text{ that is}$$

$$\hat{\gamma}\circ\gamma_o^{-1} = k \quad \text{for some } k \in K,$$

and hence $\hat{\gamma} = k\circ\gamma_o$ for some $k \in K$ and $\gamma_o \in \Gamma_o$.

Let

$$\hat{\gamma} = k_1\gamma_1, \quad K_1 \in K, \quad \gamma_1 \in \Gamma_o$$

and also

$$\hat{\gamma} = k_2\gamma_2, \quad k_2 \in K, \; \gamma_2 \in \Gamma_o.$$

Then $k_1\gamma_1 = k_2\gamma_2$; that is $k_2^{-1}k_1 = \gamma_2\gamma_1^{-1}$. Since $k_2^{-1}k_1 \in K$, $\gamma_2\gamma_1^{-1} \in \Gamma_o$ and $K \cap \Gamma_o = \{I\}$,

$$k_2 = k_1$$

$$\gamma_2 = \gamma_1.$$

Hence the representation of $\hat{\gamma}$ is unique. This completes the proof of the lemma.

Let $K * \Gamma_o$ denote a product of K and Γ_o. We define the product of (k_1,γ_1) and $(k_2\gamma_2) \in K * \Gamma_o$ by

$$(k_1,\gamma_1)(k_2,\gamma_2) = (k_1k_2^{\gamma_1},\gamma_1\gamma_2); \; k_2^{\gamma_1} = \gamma_1k_2\gamma_1^{-1}.$$

Under this operation, $K * \Gamma_o$ becomes a group. We call $K * \Gamma_o$ the <u>semi-direct product</u> of K and Γ_o. We have the following.

Corollary. $\hat{\Gamma} \cong K * \Gamma_o$; the sequence (4.6) is split exact.

Proof. We define a map

$$\psi : K * \Gamma_o \to \hat{\Gamma}$$

by $\psi(k,\gamma_o) = k\gamma_o$. ψ is well defined. We prove that ψ is an isomorphism.

ψ is a homomorphism

$$\begin{aligned} \psi\{(k_1,\gamma_1)(k_2,\gamma_2)\} &= \psi(k_1 k_2^{\gamma_1}, \gamma_1\gamma_2) \\ &= k_1 k_2^{\gamma_1}\gamma_1\gamma_2 \\ &= k_1\gamma_1 k_2\gamma_1^{-1}\gamma_1\gamma_2 \\ &= k_1\gamma_1 k_2\gamma_2 \\ &= \psi(k_1,\gamma_1)\psi(k_2,\gamma_2). \end{aligned}$$

ψ is one-to-one

Let $\psi(k_1,\gamma_1) = \psi(k_2,\gamma_2) = \hat{\gamma} \in \hat{\Gamma}$; that is,

$$k_1\gamma_1 = k_2\gamma_2 = \hat{\gamma}.$$

By the Lemma (4.1), $\hat{\gamma}$ has a unique representation, hence

$$k_1 = k_2,\ \gamma_1 = \gamma_2.$$

ψ is onto: Again by the Lemma (4.1), any $\hat{\gamma} \in \hat{\Gamma}$ can be represented uniquely as

$$\hat{\gamma} = k \circ \gamma_o; \quad k \in K, \quad \gamma_o \in \Gamma_o.$$

Then for $\hat{\gamma} \in \hat{\Gamma}$, $\hat{\gamma} = \psi(k, \gamma_o)$ for a unique $k \in K$ and a unique $\gamma_o \in \Gamma_o$. ∎

§4. Starting with a bounded reflectable deformation (f, χ) of Γ, we shall construct a deformation $(\hat{f}, \hat{\chi})$ of $\hat{\Gamma}$; $\hat{\Gamma}$ is the Schottky-double of Γ. Our goal is to find a relation between them. We proceed as follows.

We fix a point a_o in U. Let (f, χ) be a bounded reflectable deformation of Γ and f satisfies the normalization

$$f(a_o) = 0, \; f'(a_o) = 1, \; f''(a_o) = 0.$$

We define

$$\hat{\varphi}(z) = \mathcal{S}f \circ h(z) h'(z)^2 + \mathcal{S}h(z), \; z \in U.$$

We claim $\hat{\varphi} \in B_2(U, \hat{\Gamma})$. Since $(\hat{\Gamma}, h)$ uniformizes (Γ, Ω), $\mathcal{S}h \in B_2(U, \hat{\Gamma})$ [16]. Thus we just need to show that $(\mathcal{S}f \circ h) h'^2 \in B_2(U, \hat{\Gamma})$. It is easy to check that

$$(\mathcal{S}f \circ h \circ \hat{\gamma}) h'^2(\hat{\gamma}) \hat{\gamma}'^2 = (\mathcal{S}f \circ h) h'^2, \quad \text{for all } \hat{\gamma} \in \hat{\Gamma}.$$

Let λ and $\hat{\lambda}$ be the Poincaré metrics for Ω and $U = h^{-1}(\Omega)$, respectively. Then we have

$$\lambda(h(z)) |h'(z)| = \hat{\lambda}(z), \quad z \in U.$$

Hence

$$\sup_{z\in U}\{\hat{\lambda}(z)^{-2}|(\mathcal{S}f\circ h)h'(z)^2|\} = \sup_{z\in U}\{\lambda(h(z))^{-2}|\mathcal{S}f\circ h(z)|\}$$

$$= \sup_{w\in\Omega}\{\lambda^{-2}(w)\,|\mathcal{S}f(w)|\}$$

$$< \infty,$$

since $\mathcal{S}f \in B_2(\Omega,\Gamma)$. Hence $(\mathcal{S}f\circ h)h'^2 \in B_2(U,\hat{\Gamma})$. We choose $b_o \in U_o$ such that $h(b_o) = a_o$. We solve the Schwarzian differential equation

$$\mathcal{S}\hat{f}(z) = \hat{\varphi}(z), \quad z \in U$$

with the initial conditions

$$\hat{f}(b_o) = f\circ h(b_o),\ \hat{f}'(b_o) = (f\circ h)'(b_o),\ \hat{f}''(b_o) = (f\circ h)''(b_o).$$

We get a deformation $(\hat{f},\hat{\chi})$ of $\hat{\Gamma}$. Since

$$\mathcal{S}\hat{f}(z) = \mathcal{S}(f\circ h(z)); \quad z \in U_o,$$

we have, for some Moebius transformation A,

$$\hat{f}(z) = A\circ f\circ h(z), \quad z \in U_o.$$

But, the initial conditions for $\hat{f}$ implies that $A(z) = z$ and we have

$$(4\cdot 8) \qquad f(z) = \hat{f}\circ h(z), \quad z \in U_o.$$

We notice that

$$\hat{\chi}(\hat{\gamma})\circ\hat{f}(z) = \hat{f}\circ\hat{\gamma}(z), \quad \text{for all } \hat{\gamma} \in \Gamma, \quad z \in U.$$

Therefore, for $\gamma_o \in \Gamma_o$, $z \in U_o$, we have

$$\hat{\chi}(\gamma_o)\circ\hat{f}(z) = \hat{f}\circ\gamma_o(z)$$

$$= f\circ h\circ\gamma_o(z)$$

$$= f\circ\gamma\circ h(z), \quad \gamma = h_o\circ\gamma_o\circ h_o^{-1}.$$

Hence

$$\chi(\gamma_o)\circ\hat{f}(z) = \chi(\gamma)\circ f\circ h(z)$$

$$= \chi(\gamma)\circ\hat{f}(z)$$

which implies that, for $\gamma_o \in \Gamma_o$,

$$\hat{\chi}(\gamma_o) = \chi(\gamma); \quad \gamma = h_o\circ\gamma_o\circ h_o^{-1} \in \Gamma.$$

By the Lemma (4.1), every $\hat{\gamma} \in \hat{\Gamma}$ can be written as

$$\hat{\gamma} = k\circ\gamma_o, \quad k \in K, \quad \gamma_o \in \Gamma_o.$$

Therefore, for $\hat{\gamma} \in \hat{\Gamma}$,

$$\hat{\chi}(\hat{\gamma}) = \hat{\chi}(k)\circ\hat{\chi}(\gamma_o);$$

$$\hat{\chi}(\hat{\gamma}) = \hat{\chi}(k)\circ\chi(\gamma); \quad \gamma = h\circ\gamma_o\circ h_o^{-1}. \tag{4.9}$$

We want to investigate $\hat{\chi}|_K$.

Let $\pi_1(\Omega, a_o)$ denote the fundamental group of Ω relative to the base point $a_o \in \Omega$. There is a natural isomorphism between K and $\pi_1(\Omega, a_o)$. In the next section we study the analytic continuation of f along the elements of $\pi_1(\Omega, a_o)$ which will help determine $\hat{\chi}|_K$.

§ 5. Every $\varphi \in B_2^*(\Gamma)$ extends holomorphically to all of Ω by reflection in $\mathbb{R}$. Due to this fact, any local solution of

$$(4.10) \qquad \mathcal{S}f(z) = \varphi(z), \quad z \in \Omega$$

can be extended meromorphically along any curve in Ω in the following way.

Let C be any curve in Ω with the initial point $z_o \in \Omega$ and the end point $z_n \in \Omega$. We take intermediate points $z_1, z_2, \cdots, z_{n-1}$ on C and discs $N_{z_o}, N_{z_1}, \cdots, N_{z_n}$ in Ω centered at $z_o, z_1, \cdots, z_n$, respectively, so that each disc intersects the preceding one. We solve the equation (4.10) in N_{z_o}. Let f_{z_o} be a solution. We solve (4.10) again in N_{z_1} with the initial conditions

$$f(w_o) = f_{z_o}(w_o), \; f'(w_o) = f'_{z_o}(w_o) \text{ and } f''(w_o) = f''_{z_o}(w_o);$$

$w_o \in N_{z_o} \cap N_{z_1}$. We call this solution f_{z_1}. We notice that f_{z_1} and f_{z_o} agree in $N_{z_o} \cap N_{z_1}$. Thus f_{z_1} is an extension of f_{z_o} to N_{z_1}. Continuing this process we get f_{z_n} which gives a continuation of f_{z_o} along C. However, if we continue f_{z_o} along a closed curve C; $z_o = z_n$ in this case, function

f_{z_n} does not necessarily agree with f_{z_o} in $N_{z_o} \cap N_{z_n}$. But these two functions differ by a Moebius transformation since they have the same Schwarzian derivative.

Let $\pi_1(\Omega, z_o)$ denote the fundamental group of Ω relative to the base point at z_o. Continuing f_{z_o} along $C \in \pi(\Omega, z_o)$, we get f_{z_n}. Let us denote f_{z_o} by f and f_{z_n} by $\tilde{f}$. This continuation depends only on the homotopy class of C by the Monodromy Theorem. Let [C] denote the homotopy class of C. Then we have

$$\tilde{f}(z) = A_{[C]} \circ f(z), \; z \in N_{z_o} \cap N_{z_n} \text{ and } A_{[C]} \in G.$$

We notice that the Moebius transformation we get depends on the function f in the following sense. We choose another solution g such that $g(z) = B \circ f(z)$, $z \in N_{z_o}$, $B \in G$. Continuing g along C we obtain $\tilde{g}$. Let

$$\tilde{g}(z) = \tilde{A}_{[C]} \circ g(z), \; z \in N_{z_o} \cap N_{z_n}.$$

Then

$$\tilde{A}_{[C]} = B \circ A_{[C]} \circ B^{-1} \text{ as will be shown below.}$$

First we shall show that

$$\tilde{g}(z) = B \circ \tilde{f}(z), \quad z \in N_{z_o} \cap N_{z_n}.$$

In $N_{z_o} \cap N_{z_1}$,

$$g_{z_1} = g_{z_o} = B \circ f; \quad g_{z_o} = g, \; f_{z_o} = f.$$

and

$$f_{z_1} = f_{z_o} = f.$$

Thus

$$g_{z_1} = B \circ f_{z_1} \quad \text{in } N_{z_o} \cap N_{z_1}.$$

Similarly,

$$g_{z_i} = B \circ f_{z_i} \quad \text{for } i = 1, 2, \cdots n-1,$$

and finally,

$$g_{z_n} = B \circ f_{z_n}; \text{ that is, } \tilde{g} = B \circ \tilde{f} \quad \text{in } N_{z_o} \cap N_{z_n}.$$

Then we have, in $N_{z_o} \cap N_{z_n}$,

$$\tilde{g}(z) = B \circ \tilde{f}(z)$$

$$= B \circ A_{[C]} \circ f(z)$$

$$= B \circ A_{[C]}, B^{-1} \circ B \circ f(z)$$

$$= B \circ A_{[C]} B^{-1} \circ g(z).$$

Hence

$$\tilde{A}_{[C]} = B \circ A_{[C]} \circ B^{-1}.$$

Let us recall the group operation $*$ in $\pi_1(\Omega, z_o)$. For $C_1, C_2 \in \pi_1(\Omega, z_o)$, $C_1 * C_2$ denotes the closed curve in $\pi_1(\Omega, z_o)$ tracing C_1 first and then C_2. For a fixed f, $[C] \to A_{[C]}$ defines a homorphism

$$\tilde{\chi} : \pi_1(\Omega, z_o) \to G,$$

as follows.

Let $[C_1]$ and $[C_2] \in \pi_1(\Omega, z_o)$. Continuing f along $[C_1]$ we obtain $\tilde{f}_1$ satisfying

$$\tilde{f}_1(z) = \tilde{\chi}[C_1]\circ f(z);\ z \in N_{z_o} \cap N_{z_n}.$$

Then continuing $\tilde{f}_1(z)$ further along $[C_2]$ we obtain $\tilde{f}_2$ satisfying

$$\begin{aligned}\tilde{f}_2(z) &= \tilde{\chi}[C_1]\circ\tilde{\chi}[C_2]\circ\tilde{\chi}[C_1]^{-1}\circ\tilde{f}_1(z),\ z \in N_{z_o} \cap N_{z_n} \\ &= \tilde{\chi}[C_1]\circ\tilde{\chi}[C_2]\circ\tilde{\chi}[C_1]^{-1}\circ\tilde{\chi}[C_1]\circ f(z) \\ &= \tilde{\chi}[C_1]\circ\tilde{\chi}[C_2]\circ f(z).\end{aligned}$$

We could also obtain $\tilde{f}_2$ continuing f along $[C_1 * C_2]$ satisfying

$$\tilde{f}_2(z) = \tilde{\chi}[C_1 * C_2]\circ f(z) \text{ in } N_{z_o} \cap N_{z_n}.$$

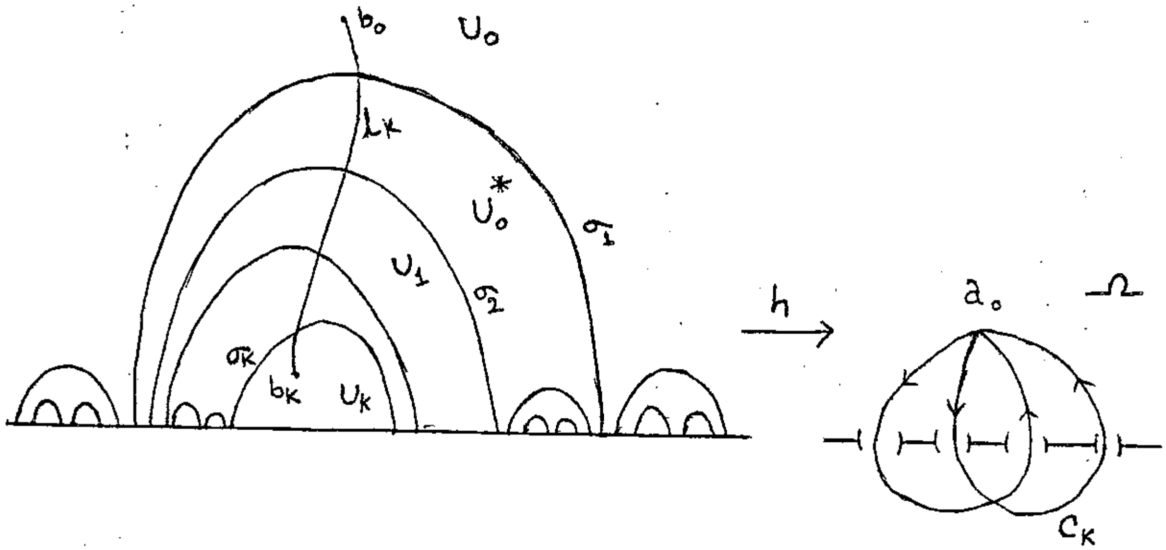
Thus we have

$$\tilde{\chi}[C_1 * C_2] = \tilde{\chi}[C_1]\circ\tilde{\chi}[C_2].$$

§6. We prove the following.

<u>Lemma 4.2</u> $\hat{\chi}|_K = \tilde{\chi}$.

Proof. Each connected component of $h^{-1}(\Omega\cap\hat{\mathbb{R}})$ is a half-circle orthogonal to the real line. We choose U_o to be the connected component of $h^{-1}(U)$ whose boundary contains ∞, as shown in the diagram below.



Let $I_1 \subset \Omega \cap \hat{\mathbb{R}}$ and σ_1 be a connected component of $h^{-1}(I_1)$. Let U_o^* be the component of $h^{-1}(U^*)$ sharing the boundary σ_1 with U_o. U_o^* is bounded by σ_1 and infinitely many half-circles in $h^{-1}(\Omega \cap \hat{\mathbb{R}})$ lying inside σ_1; let σ_2 be one of them. There will be a connected component U_1 of $h^{-1}(U)$ bounded by σ_2 and infinitely many half-circles lying inside σ_2. This process will go on.

We recall from §3 that we have the bounded reflectable deformation (f,χ) of Γ with f satisfying

$$f(a_o) = 0,\ f'(a_o) = 1,\ f''(a_o) = 0;\ a_o \in \Omega.$$

We choose $b_o \in U_o$ such that $h(b_o) = a_o$. We have the deformation $(\hat{f},\hat{\chi})$ of $\hat{\Gamma}$, $\hat{f}$ normalized at b_o in such a way that

$$\hat{f}|_{U_o} = f\circ h\ .$$

Every element $k \in K$, $k \neq I$, maps U_o onto a connected component $U_k \neq U_o$ of $h^{-1}(U)$. For, if $U_k = k(U_o)$ was a component of $h^{-1}(U^*)$, $hk = h$ would imply that $h(U_o) = U^*$, which is a contradiction. $k(b_o) = b_k \in U_k$ such that $h(b_k) = a_o$. We join b_o and b_k by an arc ℓ_k; $h(\ell_k) = C_k$, $C_k \in \pi_1(\Omega, a_o)$.

We extend f along C_k and obtain $\tilde{f}$ satisfying

$$\tilde{f}(z) = \tilde{\chi}[C_k]\circ f(z); \quad z \in U.$$

We define

$$f_k(z) = \tilde{f}\circ h(z), \quad z \in U_k.$$

Then f_k is a continuation of $f\circ h$ to U_k along ℓ_k; the continuation of f across the intervals $h(\sigma_1)$, $h(\sigma_2), \cdots, h(\sigma_k)$ justify the continuation of $f\circ h$ across $\sigma_1, \sigma_2, \cdots, \sigma_k$. But, $\hat{f}$ is the continuation of $f\circ h$ to all of U. Hence

$$\hat{f}(z) = \tilde{f}\circ h(z), \quad z \in U_k.$$

We know that, for $k \in K$,

$$\hat{\chi}(k)\circ \hat{f}(z) = \hat{f}\circ k(z), \quad z \in U_o$$

Therefore, for $z \in U_o$,

$$\hat{\chi}(k)\circ \hat{f}(z) = \tilde{f}\circ h\circ k(z),$$

since $k(z) \in U_k$. Hence we have, for $z \in U_o$,

$$\hat{\chi}(k)\circ\hat{f}(z) = \tilde{\chi}[C_k]\circ f\circ h\circ k(z)$$

$$= \tilde{\chi}[C_k]\circ f\circ h(z)$$

$$= \tilde{\chi}[C_k]\circ\hat{f}(z),$$

which implies

$$\hat{\chi}(k) = \tilde{\chi}[C_k], \quad \text{for } k \in K.$$

We notice that $k \to [C_k]$ is the natural isomorphism of K onto $\pi_1(\Omega, a_o)$. Hence we conclude that

$$\hat{\chi}|_K = \tilde{\chi}.$$

Remark. There exists a relation between the elements of the groups $\chi(\Gamma)$, $\hat{\chi}(\hat{\Gamma})$ and $\tilde{\chi}(\pi_1(\Omega, a_o))$. From (4·9) in §4, it follows that, for $\hat{\gamma} \in \hat{\Gamma}$,

$$\hat{\chi}(\hat{\gamma}) = \hat{\chi}(k)\circ\chi(\gamma), \; k \in K, \; \gamma \in \Gamma.$$

Using the Lemma (4.2), we can write

$$\hat{\chi}(\hat{\gamma}) = \tilde{\chi}[C_k]\circ\chi(\gamma), \text{ for } [C_k] \in \pi_1(\Omega, a_o), \quad \gamma \in \Gamma.$$

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