A RESIDUE THEOREM FOR SECONDARY INVARIANTS OF COLLAPSED RIEMANNIAN MANIFOLDS

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An F-structure is essentially a collection of commutative local Killing vector fields. For a $(4k-1)$-dimensional manifold N with a nonsingular F-structure $\mathfrak{F}$, its secondary invariants are defined and proved to be topological invariants of the pair $(N, \mathfrak{F})$. We generalize Bott's residue theorem to the case of an F-structure. The generalized residue theorem is then applied in the calculations of the secondary invariants of the pair $(N, \mathfrak{F})$.

To My Dearest Father

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0. Introduction

The secondary geometric invariants of Riemannian manifolds were studied in the work of Chern-Simons, Cheeger-Simons, and Atiyah-Patodi-Singer [1], [8], [9], etc. In the presence of some auxiliary topological structure on a closed oriented $(n-1)$-manifold N^{n-1}, the secondary geometric invariants can be made into topological invariants of the manifold N^{n-1} and the auxiliary topological structure on it.

Assume that N^{n-1} is a boundary, say, $N^{n-1} = \partial M^n$ for some compact oriented n-manifold M^n. The orientation on N^{n-1} is chosen so that Stokes' theorem takes the form

$$\int_M d\theta = \int_N \theta \tag{0.1}$$

for all smooth $(n-1)$-form $\theta \in A^{n-1}(M)$.

Let n be even, say, $n = 2k$. Let g be a Riemannian structure on N^{n-1}. For any invariant integral polynomial $P \in I_o^k(O(n))$, set

$$P(M,N,g) = \frac{1}{(2\pi)^k} \int_{(M,\tilde{g})} P(\Omega) \tag{0.2}$$

where $\tilde{g}$ is any extension of the Riemannian structure g on N to M such that the restriction of $\tilde{g}$ to a collar neighborhood of N in M is the product metric $dt^2 + g$.

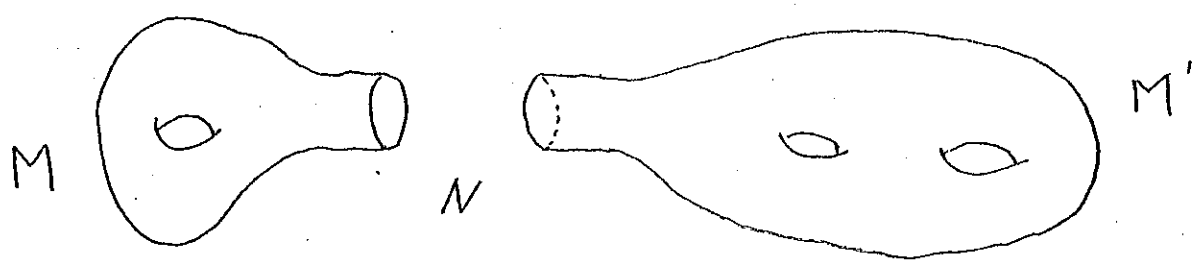
This curvature integral is independent of such extension $\tilde{g}$

since the Pontrjagin number $P(Y)$ of the double $Y = M \cup -M$ is always zero.

The associated secondary geometric invariant $SP(N,g)$ is by definition the residue class modulo $\mathbf{Z}$ of the above curvature integral, that is,

$$(0.3) \qquad SP(N,g) \equiv P(M,N,g) \mod Z$$

which is independent of the filling M.



Let $(M',\tilde{g}')$ be another such filling. Let $Y = M \cup -M'$ be the closed oriented Riemannian manifold obtained from M and M' by identifying along their boundary, thus

$$(0.4) \qquad P(M,N,G) - P(M',N,g) = \frac{1}{(2\pi)^k} \int_Y P(\Omega) \in Z$$

since $P \in I_o^k(O(n))$ is an invariant integral polynomial.

However, these secondary geometrical invariants do depend on the metric g on N.

In 1967, R. Bott developed a residue formula for characteristic numbers of a closed orientable Riemannian manifold M in terms of the data provided by a Killing field.

(See [4], [5]. There was also a complex counterpart for a holomorphic vector field, but we will discuss it elsewhere.)

Proposition 0.1 [Bott, Baum-Cheeger]. Let M^{2k} be a closed oriented even dimensional Riemannian manifold equipped with a nontrivial Killing vector field X. Let $P \in I^k(O(n))$ be any invariant polynomial of degree k. There is a $(2k-1)$-form η defined on $M\backslash Zero(X)$, such that

$$(0.5) \qquad P(\Omega^k)+d\eta = 0$$

$$(0.6) \qquad P(TM) = \frac{-1}{(2\pi)^k} \sum_Z \mathrm{Res}(\eta, Z)$$

where the summation index Z runs over all the connected components of $Zero(X)$, and $\mathrm{Res}(\eta, Z)$ is a topological invariant of the tangent bundle and the normal bundle of $Z \subset M$ and the action induced by L_X --the bracket operation, on the normal bundle.

Bott's idea was to manufacture a canonical locally computable $(2k-1)$-form η out of the Killing vector field X on $M\backslash Zero(X)$ such that

$$(0.5) \qquad P(\Omega) + d\eta = 0 \quad \text{on} \quad M\backslash Zero(X) .$$

The problem was then reduced, by Stokes' theorem, to compute the residues at the singular set $Zero(X)$. In fact, the residues were estimated by Bott when X has only isolated zeros; the estimation for the general singularities was carried out immediately after Bott's work by Baum-Cheeger (see [2]). These three papers ([2], [4], [5]) we just cited are part of the motivation for this thesis.

Bott's idea can be applied immediately in the estimation of the secondary geometric invariants as long as $(M,\tilde{g})$ admit a Killing vector field such that $X|_N$ is nonvanishing and tangent to N. Let X be such a Killing vector field on $(M,\tilde{g})$ then, η is well-defined on $N = \partial M$ and the equation (0.5) remains true since η is constructed by local forms. It follows from the residue theorem that

$$(0.7) \qquad P(M,N,g) = \frac{1}{(2\pi)^k} \int_{(M,\tilde{g})} P(\Omega)$$

$$= \frac{-1}{(2\pi)^k} \sum_Z \operatorname{Res}(\eta,Z) - \frac{1}{(2\pi)^k} \int_{(N,g)} \eta$$

According to Bott and Baum-Cheeger, the residue $\sum_Z \operatorname{Res}(\eta,Z)$ is a topological invariant which is independent of the invariant metrics $\tilde{g}$ and g. Thus, the modified curvature integral

$$(0.8) \qquad P(M,N,g) + \frac{1}{(2\pi)^k} \left\{ \int_{(M,\tilde{g})} p(\Omega) + \int_N \eta \right\} = - \frac{1}{(2\pi)^k} \sum_Z \operatorname{Res}(\eta,Z)$$

is a topological invariant. Formula (0.8) suggests the following definition.

(0.9) $$P(M,N,X) = \frac{1}{(2\pi)^k}\left\{\int_{(M,\tilde{g})} P(\Omega) + \int_N \eta\right\}$$

Note that $P(M,N,X)$ is already well-defined when X is a killing vector field defined only on a neighborhood of $\partial M = N$.

Corollary 0.2. Assume that (N^{n-1},g) is a closed orientable Riemannian manifold which is the boundary of a compact oriented Riemannian manifold $(M,\tilde{g})$, here $\tilde{g}$ is any extension of g to M (but $\tilde{g}$ is not required to be the product metric near the boundary $\partial M = N$ as we did before). Let X be a nonvanishing killing vector field on (N,g). Then, the modified curvature integral $P(M,N,X)$ is a topological invariant of (M,N,X) as long as X can be extended to a global killing vector field on $(M,\tilde{g})$. Moreover,

(0.10) $$P(M,N,X) = -\frac{1}{(2\pi)^k}\sum_Z \operatorname{Res}(\eta,Z).$$

Proof. This is a consequence of Bott's residue theorem and definition (0.9).

However, a transgression argument (see §8) shows that the extendability of X to all of $(M,\tilde{g})$ is superfluous. We have the following Gauss-Bonnet type theorem for the invariant $P(M,N,X)$ and the associated secondary invariant.

Theorem 0.3. Let (N^{2k-1},g) be a closed oriented

Riemannian manifold which is the boundary of a compact oriented manifold M^{2k}. Let X be a nonvanishing Killing vector field for (N^{2k-1},g). Then

$$(0.9)\qquad P(M,N,X) = \frac{1}{(2\pi)^k}\left\{\int_{(M,\tilde{g})} P(\Omega) + \int_{(N,g)} \eta\right\}$$

depends only on (M,N,X) and $P \in I^k(O(2k))$, where $\tilde{g}$ is any extension of the metric g onto M^{2k} such that $\tilde{g}$ is invariant under X in a neighborhood of $N = \partial M$. In particular,

$$(0.11)\qquad SP(N,X) \equiv P(M,N,X) \mod \mathbb{Z}$$

is a topological invariant of the pair (N,X) for $P \in I_o^k(O(2k))$.

This thesis is essentially a generalization of the above three theorems to the case where $(M,\tilde{g})$ admits a collection of commutative local Killing vector fields.

As a global torus action is the underlying topological structure for a collection of commutative global Killing vector fields, there is also an underlying topological structure for a collection of commutative local Killing vector fields, it is the so called F-structure recently introduced by Cheeger and Gromov (see [6], [7]). A collection of commutative local Killing vector fields gives rise to a local integrable distribution. It is called a polarization of the underlying F-structure if the leaves of the integral of the distribution are not all closed submanifolds. Compare with [7] section 1. We will, however, not distinguish an F-structure from a polarization of an F-structure, and call them both an F-structure. But the fact that an F-structure has or has no singularities is essential to us.

The topologicalization of the secondary geometric invariants associated to a nonsingular F-structure is not discovered, nevertheless, by the above modification to the curvature form $P(\Omega)$. It is originally discovered instead by Cheeger and Gromov via their collapsing idea.

Proposition 0.4. (Cheeger,Gromov, see [7], theorem 5.2.)
1. If Y^n admits a nonsingular F-structure $\mathcal{F}$, on the complement of a compact subset, C , then Y^n admits a complete invariant metric g , with $|K_g| \leq 1$ and $\mathrm{Vol}(Y^n,g) < \infty$.
2. If C is empty, Y^n admits a family g_s of such complete metrics, with $|K_{g_s}| \leq 1$ and $\lim_{s \to 0} \mathrm{Vol}(Y^n, g_s) = 0$.

Let $N^{n-1} = \partial M^n$ be a closed manifold with a non-singular F-structure $\mathcal{F}$. Let g be a complete invariant metric for $N \times R$ as in Proposition 0.4. Attach the collapsed tail $N \times \mathbf{R}^+$ to M^n along their boundary N^{n-1} , we obtain a complete manifold

$$Y^n = M^n \cup N \times \mathbf{R}^+ \tag{0.12}$$

with an invariant metric g , $|K_g| \leq 1$, $\mathrm{Vol}(Y^n,g) < \infty$. The integral

$$(0.13) \qquad P(M,N,\mathfrak{F}) \equiv \frac{1}{(2\pi)^k} \int_{(Y^n,g)} P(\Omega)$$

is thus well-defined for $P \in I^k(O(n))$, $n = 2k$. It turns out to be independent of the invariant metric g as long as $|K_g| < 1$, $\mathrm{Vol}(Y,g) < \infty$ which justifies the notation $P(M,N,\mathfrak{F})$ in equation (0.13) (see [6]). Instead of modifying the integral $P(M,N,g) = \frac{1}{(2\pi)^k} \int_{(M,\tilde{g})} P(\Omega)$ by some boundary integrals, as in the above theorem, the collapse kills these boundary integrals. A detailed computation will be carried out in section 10 where $\mathfrak{F}$ is a pure F-structure without singularities. If $P \in I_o^k(O(n))$,

$$(0.14) \qquad SP(N,\mathfrak{F}) \equiv P(M,N,\mathfrak{F}) \mod Z$$

is the secondary topological invariant of the pair $(N,\mathfrak{F})$.

An immediate corollary of the collapsing proposition 0.4 is that if the nonsingular F-structure $\mathfrak{F}$ on N can be extended to a nonsingular F-structure on all of M , then

$$(0.15) \qquad P(M,N,\mathfrak{F}) = 0$$

In particular, if there is a filling M for N such that $\mathfrak{F}$ can be extended to M without singularities, then

$$(0.16) \qquad SP(N,\mathfrak{F}) = 0$$

In general, these invariants may assume any real numbers. (See section 12, example 4.)

In the way of computing the toplogical and secondary topological invariants $P(M,N,\mathfrak{F})$ and $SP(N,\mathfrak{F})$ associated to a nonsingular F-structure $\mathfrak{F}$ which is defined on a neighborhood of $N = \partial M$, we find that there are certain exotic characteristic classes $P(M,\mathfrak{F})$ for manifolds with corners which are equipped with a non-singular F-structure $\mathfrak{F}$ on its boundary. These exotic characteristic classes can be identified as cohomology classes of the De Rham cohomology with compact support for the interior of M. In particular, if $\dim M = 2k$, $P \in I_o^k(0(2k))$, $P(M,N,\mathfrak{F})$ is the functional of the exotic characteristic class $P(M,\mathfrak{F})$ on the fundamental class of $\operatorname{int} M$. The secondary topological invariant $SP(N,\mathfrak{F})$ is then defined by equation (0.14), which is the modulo $\mathbf{Z}$ functional of $P(M,\mathfrak{F})$.

§1. Manifolds with Corners and Stratified Sets

This section is taken, almost without change, from [10].

Let M^n be a Hausdorff topological space. An n-dimension chart with corners is an open subset $U \subset M$ together with a continuous map $\rho: U \to R^n$ which takes U homeomorphically onto a neighborhood of the coordinate origin in $R^k \times (\overline{R}_+)^{n-k} \subset R^n$. Two charts $\rho: U \to R^n$ and $\rho': U' \to R^n$ are said to be compatible if the map $\rho' \cdot \rho^{-1}: \rho(U \quad U') \to \rho'(U \quad U')$ is a diffeomorphism (we note that the values of k for the two charts can be different).

Definition 1.1: The space M is said to be an n-dimension manifold with corners if for some cover $\{U_\alpha\}$ of M there is a set of pairwise compatible n-dimensional charts with corners $\{\rho_\alpha: U_\alpha \to R^n\}$. A charts with corners which is compatible with all the charts of this set is said to be smooth.

Example 1.1. The standard q-simplex $\Delta_q = \{t \varepsilon R^{q+1} | \, t = (t_0, t_1, \ldots, t_q\}, \; t_i > 0, \; \sum_{i=0}^{q} t_i = 1\}$ is a manifold with corners. Manifold with corners is actually modelled after the standard simplexes.

Example 1.2. Let V be a smooth n-dimensional manifold and $\{V_j\}$ a set of n-dimensional submanifolds with boundary in V, where each boundary V_j lies in general position. Then

$M = \bigcap_j V_j$ is a manifold with corners.

Let M be a manifold with corners. Define the tangent bundle TM and the cotangent bundle T^*M as follows: If $U \subset M$, and $\rho: U \to R^k \times (\bar{R}_+)^{n-k} \subset R^n$ is a smooth chart, then we put $TU = \rho^*(TR)^n$, $T^*U = \rho^*(T^*R^n)$. If ρ and ρ' are two smooth charts defined on U and U', respectively, then the restriction of the bundle TU and TU' (or the bundle T^*U and T^*U') are canonically isomorphic on $U \cap U'$. Hence, the bundles TM and T^*M, for which $TM|_U = TU$ and $T^*M|_U = T^*U$ are well-defined. A Riemannian metric on M is a positive definite quadratic form on the tangent bundle TM.

Let $A^q(U)$ be the set of all smooth q-forms on the open subset $U \subset M$. If $V \subset M$ is a subset of M, then $A^q(V)$ is defined to be the set of all q-forms on V which can be extended smoothly to an open neighborhood of V.

Note that if V is open, this definition coincides with the one for open subset.

We shall say that a point $x \in M$ belongs to the k-dimensional skeleton $M(k)$ of the n-dimensional manifold with corners M if there is an open set U containing x and a chart with corners $\rho: U \to R^k \times (\bar{R}_+)^{n-k}$, $k' < k$, such that $\rho(x) = 0$. The closures of the connected components of $M(k) \setminus M(k-1)$ are called the k-dimensional faces of M. The boundary ∂M of M is defined to be the (n-1)-skeleton $M(n-1)$, i.e., $\partial M = M(n-1)$.

Definition 1.2. By a smooth map of manifolds with corners we mean a continuous map $f\colon M \to M'$, satisfying the conditions:

(a) If $\rho\colon U \to R^n$ and $\rho'\colon U \to R^{n'}$ are smooth charts with corners in M and M', respectively, then the map

$$(1.2) \qquad \rho' \circ f \circ \rho^{-1}\colon \rho(U \cap f^{-1}(U')) \to R^{n'}$$

is smooth, i.e., it can be extended to a smooth map $V \to R^{n'}$, where $V \supset \rho(U \cap f^{-1}(U'))$ is an open set in R^n.

(b) If M_i is a face of M and M_i' is a face of M' such that $f(\mathrm{int} M_i) \cap M_i' \neq \phi$, then $f(M_i) \subset M_i'$.

A manifold with corners N is a submanifold of the manifold with corners M if there is a smooth regular inclusion $i\colon N \to M$, which is a homeomorphism from N to $i(N)$ (regularity means that the natural map $i_*\colon TN \to TM$ is an inclusion).

Definition 1.3. A stratified set is a Hausdorff space M together with a locally finite decomposition into nonintersecting connected subsets M_i (the strata) which satisfies the following conditions:

(a) If $\overline{M}_i \cap M_j \neq \phi$, then $\overline{M}_i \supset M_j$.

(b) The closure $\overline{M}_i$ of each stratum M_i has the structure of a manifold with corners, the interior of $\overline{M}_i$ is equal to M_i.

(c) All the strata M_j, contained in $\overline{M}_i$, are the interiors of faces of the manifold with corners $\overline{M}_i$, while the smooth structures on $\overline{M}_i$ and $\overline{M}_j$ coincide.

Manifolds with corners are naturally stratified by the interiors of the faces.

Definition 1.4. A map of stratified manifolds $f: M \to M'$ is said to be smooth if the image of each stratum $M_i \subset M$ is contained in some stratum $M'_j \subset M'$ and $f|_{\overline{M}_i}: \overline{M}_i \to \overline{M}'_j$ is smooth as a map of manifolds with corners.

§2. De Rham Complex for Manifolds with Corners

This section is a refinement of the De Rham complex as introduced by Gabriélov-Gel'fand-Fuchs in [10], section 1.

Definition 2.1. A stratification $\{M_i\}$ of a smooth oriented manifold with corners M^n is said to be subordinate to a locally finite ordered countable open cover $\{U_\alpha\}_{\alpha\in A}$ if the following conditions are satisfied:

(a) Each stratum M_i is orientable.

(b) The n-dimensional strata $\{M_\alpha\}_{\alpha\in A}$ are indexed by the same index set A for the open cover $\{U_\alpha\}_{\alpha\in A}$ such that $\overline{M}_\alpha \subset U_\alpha$ for all $\alpha \in A$.

(c) $\infty \overline{\in} A$. The order on $A \cup \{\infty\}$ is induced from the one on A such that $\alpha < \infty$ for all $\alpha \in A$. For convenience, put $M_\infty = \phi$, $\overline{M}_\infty = \partial M = M \setminus \mathring{M}$. for $\ell = \overline{1, n+1}$, let A^ℓ be the set of all ℓ-tuples $(\alpha_1, \dots, \alpha_\ell) \subset A \cup \{\infty\}$ such that $\alpha_1 < \alpha_2 < \dots < \alpha_\ell$, $\overline{M}_{\alpha_1} \cap \overline{M}_{\alpha_2} \cap \dots \cap \overline{M}_{\alpha_\ell}$ contains at least one $(n+1-\ell)$-dimensional stratum. We put

(2.1) $$M_{(\alpha_1,\dots,\alpha_\ell)} = \cup\{\text{all } (n+1-\ell)\text{-dimensional strata } M_i \text{ which are contained in } \overline{M}_{\alpha_1} \cap \overline{M}_{\alpha_2} \cap \dots \cap \overline{M}_{\alpha_\ell}\}$$

(2.2) $$\overline{M}_{(\alpha_1,\dots,\alpha_\ell)} = \coprod_{M_i \subset M_{(\alpha_1,\dots,\alpha_\ell)}} \overline{M}_i$$

where " $\amalg$ " is the disjoint union.

(d) The set of all connected components of the $M_{(\alpha_1,\dots,\alpha_\ell)}$'s recovers the stratification $\{M_i\}$. We will thus not distinguish the stratification $\{M_i\}$ from $\{M_{(\alpha_1,\dots,\alpha_\ell)}\}$. In fact, by choosing a finer open cover, we may very well assume that $M_{(\alpha_1,\dots,\alpha_\ell)}$ contains exactly one connected components for all $(\alpha_1,\dots,\alpha_\ell) \in A^\ell$, $\ell = \overline{1,n+1}$, and thus $\{M_i\}$ is actually indexed by $\bigcup_{\ell=1}^{n+1} A^\ell$.

Let M^n be an oriented manifold with corners and $\{M_{(\alpha_1,\dots,\alpha_\ell)}\}$ a stratification subordinate to an open cover $\{U_\alpha\}_{\alpha\in A}$ for M^n. The orientation on $\overline{M}_\alpha$ is the one such that the natural embedding into M preserves orientation. Let $(\alpha_1,\dots,\alpha_\ell) \in A^\ell$, each connected component of $\overline{M}_{(\alpha_1,\dots,\alpha_\ell)}$ is an $(n+1-\ell)$-dimensional submanifold of the $(n+2-\ell)$-dimensional manifold $\overline{M}_{(\alpha_1,\dots,\alpha_{\ell-1})}$. The orientation on $\overline{M}_{(\alpha_1,\dots,\alpha_\ell)}$ is so chosen that if V is a representation$ of the orientation in terms of the exterior algebra of the tangent bundle of $\overline{M}_{(\alpha_1,\dots,\alpha_\ell)}$, and v is the outside normal direction of $\overline{M}_{(\alpha_1,\dots,\alpha_\ell)} \subset \overline{M}_{(\alpha_1,\dots,\alpha_{\ell-1})}$, then, $v \wedge V$ is a representative of the chosen orientation on $\overline{M}_{(\alpha_1,\dots,\alpha_{\ell-1})}$. Thus equipped, $\{M_{(\alpha_1,\dots,\alpha_\ell)}\}$ will be called an oriented stratification subordinate to $\{U_\alpha\}_{\alpha\in A}$.

For convenience, we use the following convention. Let $(\alpha_1,\dots,\alpha_\ell) \in A^\ell$, let $\sigma \in s(\ell)$ be a permutation of ℓ

letters, set

$$\overline{M}_{(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(\ell)})} = \mathrm{sgn}(\sigma)\cdot\overline{M}_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\ ,\ \text{i.e.,}$$

$\overline{M}_{(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(\ell)})}$ is identical with $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ except orientation with the same orientation if σ is an even permutation, opposite orientation if σ is odd. Set $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)} \equiv \phi$ if $\{(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(\ell)}) \mid \sigma \in s(\ell)\} \cap A^\ell = \phi$.

Remark. In this convention, Stokes' theorem states that

$$\int_{M_{(\alpha_1,\ldots,\alpha_\ell)}} d\theta = \sum_{\alpha\in A\cup\{\infty\}} \int_{M_{(\alpha_1,\ldots,\alpha_\ell,\alpha)}} \theta \tag{2.3}$$

where θ is an $(n-\ell)$-form on $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$.

We now define the De Rham complex and state some of its basic properties.

Set

$$\Lambda^q(M) = \bigoplus_{\ell=1}^{q+1} \bigoplus_{(\alpha_1,\ldots,\alpha_\ell)\in A^\ell} A^{q+1-\ell}(\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}) \tag{2.4}$$

where $A^q(\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)})$ is the bundle of exterior differential q-forms on $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$. A section θ of the bundle $\Lambda^q(M)$ is given as a set of forms

$\theta_{(\alpha_1,\ldots,\alpha_\ell)} \in A^{q+1-\ell}(\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)})$. Finally, set

$$\Lambda(M) = \bigoplus_{q=0}^{n} \Lambda^q(M) \tag{2.5}$$

Definition 2.2. The De Rham ~~double~~ complex associated to the stratification $\{M_{(\alpha_1,\ldots,\alpha_\ell)}\}$ for M is the complex of bundles $\Lambda\,(M)$ with differential operation $d\colon \Lambda^q(M) \to \Lambda^{q+1}(M)$ defined by the formulae:

$$(2.6) \qquad (d\theta)_{(\alpha)} = d\theta_{(\alpha)}, \ \alpha \in A^1$$

$$(2.7) \qquad (d\theta)_{(\alpha_1,\ldots,\alpha_\ell)} = (-1)^{\ell-1} d\theta_{(\alpha_1,\ldots,\alpha_\ell)} + \sum_{i=1}^{\ell} (-1)^{i+1} \theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)}$$

for $\ell = \overline{2,q+2}$, $\theta \in \Lambda^q(M)$, where the restriction operation on the right hand side to $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ has been suppressed and the caret denotes omission.

Proposition 2.1. $d^2 = 0$.

This proposition verifies Definition 2.2.

Proof. Let $\theta \in \Lambda^q(M)$, by definition,

$$(d^2\theta)_{(\alpha_1,\ldots,\alpha_{\ell+1})} = (-1)^\ell d(d\theta)_{(\alpha_1,\ldots,\alpha_{\ell+1})} + \sum_{i=1}^{\ell+1} (-1)^{i+1} (d\theta)_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}$$

$$= (-1)^\ell \, d\{(-1)^\ell d\theta_{(\alpha_1,\ldots,\alpha_{\ell+1})} + \sum_{j=1}^{\ell+1} (-1)^{j+1} \theta_{(\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_{\ell+1})}\}$$

$$+ \sum_{i=1}^{\ell+1} (-1)^{i+1} \{(-1)^{\ell-1} d\theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}$$

$$+ \sum_{j=1}^{i-1} (-1)^{j+1} \theta_{(\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}$$

$$+ \sum_{j=i+1}^{\ell+1} (-1)^j \theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_{\ell+1})}\} = 0$$

since $\sum_{i=1}^{\ell+1} (-1)^{i+1} \sum_{j=1}^{i-1} (-1)^{j+1} \theta(\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})$

$$+ \sum_{i=1}^{\ell+1} (-1)^{i+1} \sum_{j=i+1}^{\ell+1} (-1)^j \theta(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_{\ell+1}) = 0 .$$

Definition 2.3. Let θ and ω be sections of the bundles $\Lambda^p(M)$ and $\Lambda^q(M)$, respectively. The exterior wedge product $\theta \wedge \omega \in \Lambda^{p+q}(M)$ is defined by the formula

$$(2.8) \qquad (\theta \wedge \omega)(\alpha_1,\ldots,\alpha_\ell) = \sum_{i=1}^{\ell} (-1)^{(p+1-i)(\ell-i)} \theta(\alpha_1,\ldots\alpha_i) \wedge \omega(\alpha_i,\ldots,\alpha_\ell)$$

Proposition 2.2. The wedge product $\wedge$ is associative, i.e.,

$$(2.9) \qquad (\theta \wedge \omega) \wedge \gamma = \theta \wedge (\omega \wedge \gamma)$$

Proof. Straightforward computations.

Let $\theta \in \Lambda^p(M)$, $\omega \in \Lambda^q(M)$, $\gamma \in \Lambda^r(M)$, then

$$((\theta\wedge\omega) \wedge \gamma)(\alpha_1,\ldots,\alpha_\ell) = \sum_{i=1}^{\ell} (-1)^{(p+q+1-i)(\ell-i)} (\theta \wedge \omega)(\alpha_1,\ldots,\alpha_i) \wedge \gamma(\alpha_i,\ldots,\alpha_\ell)$$

$$= \sum_{i=1}^{\ell} \sum_{j=1}^{i} (-1)^{(p+q+1-i)(\ell-i) + (p+1-j)(i-j)} \theta(\alpha_1,\ldots,\alpha_j) \wedge \omega(\alpha_j,\ldots,\alpha_i) \wedge \gamma(\alpha_i,\ldots,\alpha_\ell)$$

$$= \sum_{j=1}^{\ell} \sum_{i=j}^{\ell} (-1)^{(p+q+1-i)(\ell-i) + (p+1-j)(i-j)} \theta_{(\alpha_1,\ldots,\alpha_j)} \wedge \omega_{(\alpha_j,\ldots,\alpha_i)} \wedge \gamma_{(\alpha_i,\ldots,\alpha_\ell)}$$

$$= \sum_{i=1}^{\ell} \sum_{j=i}^{\ell} (-1)^{(p+q+1-j)(\ell-j) + (p+1-i)(j-i)} \theta_{(\alpha_1,\ldots,\alpha_i)} \wedge \omega_{(\alpha_i,\ldots,\alpha_j)} \wedge \gamma_{(\alpha_j,\ldots,\alpha_\ell)}$$

while

$$(\theta \wedge (\omega\wedge\gamma))_{(\alpha_1,\ldots,\alpha_\ell)} = \sum_{i=1}^{\ell} (-1)^{(p+1-i)(\ell-i)} \theta_{(\alpha_1,\ldots,\alpha_i)} \wedge (\omega \wedge \gamma)_{(\alpha_i,\ldots,\alpha_\ell)}$$

$$= \sum_{i=1}^{\ell} \sum_{j=i}^{\ell} (-1)^{(p+1-i)(\ell-i) + (q+i-j)(\ell-j)} \theta_{(\alpha_1,\ldots,\alpha_i)} \wedge \omega_{(\alpha_i,\ldots,\alpha_j)} \wedge \gamma_{(\alpha_j,\ldots,\alpha_\ell)}$$

thus $((\theta \wedge \omega) \wedge \gamma)_{(\alpha_1,\ldots,\alpha_\ell)} = (\theta \wedge (\omega\wedge\gamma))_{(\alpha_1,\ldots,\alpha_\ell)}$

since $(-1)^{(p+q+1-j)(\ell-j) + (p+1-i)(j-i)} = (-1)^{(p+1-i)(\ell-i) + (q+i-j)(\ell-j)}$

Remark. The wedge product thus defined is not skew-commutative.

Proposition 2.3. The following equality holds

$$\mathbf{d}(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^p \theta \wedge \mathbf{d}\omega \tag{2.10}$$

for all $\theta \in \Lambda^p(M)$ and $\omega \in \Lambda^q(M)$.

Proof. By definitions 2.2 and 2.3

$(\mathbf{d}(\theta\wedge\omega))_{(\alpha_1,\ldots,\alpha_\ell)}$

$$= (-1)^{\ell-1} d(\theta\wedge\omega)(\alpha_1,\dots,\alpha_\ell) + \sum_{i=1}^{\ell}(-1)^{i+1}(\theta\wedge\omega)(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_\ell)$$

$$= \sum_{i=1}^{\ell}(-1)^{\ell-1+(p+1-i)(\ell-i)}\{d\theta(\alpha_1,\dots,\alpha_i)\wedge\omega(\alpha_i,\dots,\alpha_\ell)$$

$$+ (-1)^{p+1-i}\theta(\alpha_1,\dots,\alpha_i)\wedge d\omega(\alpha_i,\dots,\alpha_\ell)\}$$

$$+ \sum_{i=1}^{\ell}(-1)^{i+1}\sum_{j=1}^{i-1}(-1)^{(p+1-j)(\ell-1-j)}\theta(\alpha_1,\dots,\alpha_j)\wedge\omega(\alpha_j,\dots,\hat{\alpha}_i,\dots,\alpha_\ell)$$

$$+ \sum_{i=1}^{\ell}(-1)^{i+1}\sum_{j=i+1}^{\ell}(-1)^{(p+2-j)(\ell-j)}\theta(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_j)\wedge\omega(\alpha_j,\dots,\alpha_\ell)$$

meanwhile

$$(\mathbf{d}\theta\wedge\omega)(\alpha_1,\dots,\alpha_\ell) + (-1)^p(\theta\wedge d\omega)(\alpha_1,\dots,\alpha_\ell)$$

$$= \sum_{i=1}^{\ell}(-1)^{(p+2-i)(\ell-i)}(\mathbf{d}\theta)(\alpha_1,\dots,\alpha_i)\wedge\omega(\alpha_i,\dots,\alpha_\ell)$$

$$+ \sum_{i=1}^{\ell}(-1)^{p+(p+1-i)(\ell-i)}\theta(\alpha_1,\dots,\alpha_i)\wedge(\mathbf{d}\omega)(\alpha_i,\dots,\alpha_\ell)$$

$$= \sum_{i=1}^{\ell}(-1)^{(p+2-i)(\ell-i)}\{(-1)^{i-1}d\theta(\alpha_1,\dots,\alpha_i) + \sum_{j=1}^{i}(-1)^{j+1}\theta(\alpha_1,\dots,\hat{\alpha}_j,\dots,\alpha_i)\}$$

$$\wedge\,\omega(\alpha_i,\dots,\alpha_\ell) + \sum_{i=1}^{\ell}(-1)^{p+(p+1-i)(\ell-i)}\theta(\alpha_1,\dots,\alpha_i)\wedge$$

$$\{(-1)^{\ell-i}d\omega(\alpha_i,\dots,\alpha_\ell) + \sum_{j=i}^{\ell}(-1)^{j+i}\theta(\alpha_i,\dots,\hat{\alpha}_j,\dots,\alpha_\ell)\}$$

$$= (\mathbf{d}(\theta\wedge\omega))(\alpha_1,\dots,\alpha_\ell)\,.$$

We introduce the functional

$$\int_{M^n} : \Lambda^n(M^n) \to R$$

(2.11) $$\int_{M^n}\theta = \sum_{\ell=1}^{n+1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}} \theta_{(\alpha_1,\ldots,\alpha_\ell)}$$

for every $\theta \varepsilon \Lambda^n(M)$.

Theorem 2.4. $d(\Lambda^{n-1}(M^n)) \subset \ker \int_{M^n}$

Proof. It is a corollary of Stokes' theorem. Let $\theta \varepsilon \Lambda^{n-1}(M^n)$.

$$\int_{M^n} d\theta = \sum_{\ell=1}^{n+1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}} (d\theta)_{(\alpha_1,\ldots,\alpha_\ell)}$$

$$= \sum_{\ell=1}^{n+1}(-1)^{\ell-1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}} d\theta_{(\alpha_1,\ldots,\alpha_\ell)}$$

$$+ \sum_{\ell=2}^{n+1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \sum_{i=1}^{\ell}(-1)^{i+1} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}} \theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)}$$

Apply Stokes' theorem, i.e., equation (2.3), to the first sum. Note that $\theta_{(\alpha_1,\ldots,\alpha_{n+1})} = 0$ since $\overline{M}_{(\alpha_1,\ldots,\alpha_{n+1})}$ is simply an isolated point set if it is not empty. One has

$$\int_{M^n} d\theta = \sum_{\ell=1}^{n}(-1)^{\ell-1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \sum_{\alpha\varepsilon A \cup \{\infty\}} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell,\alpha)}} \theta_{(\alpha_1,\ldots,\alpha_\ell)}$$

$$+ \sum_{\ell=1}^{n} \sum_{(\alpha_1,\ldots,\alpha_{\ell+1})\varepsilon A^{\ell+1}} \sum_{i=1}^{\ell+1}(-1)^{i+1} \int_{\overline{M}_{(\alpha_1,\ldots,\alpha_{\ell+1})}} \theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}$$

$$= \sum_{\ell=1}^{n} (-1)^{\ell-1} \sum_{(\alpha_1,\dots,\alpha_{\ell+1})\in A^{\ell+1}} \sum_{i=1}^{\ell} \int_{M_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_{\ell+1},\alpha_i)}} \theta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_{\ell+1}}$$

$$+ \sum_{\ell=1}^{n} \sum_{(\alpha_1,\dots,\alpha_{\ell+1})\in A^{\ell+1}} \sum_{i=1}^{\ell+1} (-1)^{i+1} \int_{M_{(\alpha_1,\dots,\alpha_{\ell+1})}} \theta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_{\ell+1})}$$

$$= 0 .$$

Here the summation is rearranged and the orientation convention is used.

Let $A_o^q(M)$ be the set of all smooth q-forms which are supported in the interior of M. There is a homomorphism $i: A_o^q(M) \to \Lambda^q(M)$, where for any $\theta \in A_o^q(M)$,

$$(2.12) \qquad (i\theta)_{(\alpha_1)} = \theta_{(\alpha_1)} = \theta|_{\overline{M}_{(\alpha_1)}}$$

$$(2.13) \qquad (i\theta)_{(\alpha_1,\dots,\alpha_\ell)} = 0 \quad \text{for all } \ell > 1$$

where $\theta_{(\alpha_1)} = \theta|_{\overline{M}_{(\alpha_1)}}$ is an element of $A^q(\overline{M}_{(\alpha_1)})$ such that θ is a representative of $\theta_{(\alpha_1)}$. The diagram

$$(2.14) \qquad \begin{array}{ccc} A_o^{q+1}(M) & \xrightarrow{i} & \Lambda^{q+1}(M) \\ d\uparrow & & \uparrow d \\ A_o^q(M) & \xrightarrow[i]{} & \Lambda^q(M) \end{array}$$

commutes. In fact, for every $\theta \in A_o^q(M)$,

$$(i \circ d(\theta))_{(\alpha_1)} = (d\theta)|_{\bar{M}_{(\alpha_1)}}$$

$$(i \circ d(\theta))_{(\alpha_1,\ldots,\alpha_\ell)} = 0 \quad \text{for all} \quad \ell > 1 \; .$$

while $(\mathbf{d} \circ i(\theta))_{(\alpha_1)} = d\theta_{(\alpha_1)}$

$$(\mathbf{d} \circ i(\theta))_{(\alpha_1,\ldots,\alpha_\ell)} = (-1)^{\ell-1} d(i(\theta))_{(\alpha_1,\ldots,\alpha_\ell)}$$

$$+ \sum_{j=1}^{\ell} (-1)^{j+1} (i(\theta))_{(\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_\ell)}$$

$= 0$ for all $\ell > 1$.

i induces a homomorphism in the cohomology level,

(2.15) $$i_*: \; H^q(A_o^*(M),d) \to H^q(\Lambda(M),\mathbf{d}) \; .$$

§3. Pure F-structure and F-structure

The main reference for F-structure is the paper, "Collapsing Riemannian Manifolds while Keeping Their Curvature Bounded," by Cheeger and Gromov ([see 7]). Also see Appendix 2 of [11]. For the convenience of our exposition, we will take a more geometric point of view and define first a pure F-structure as a sheaf of germs of mutually commutative Killing vector fields on a manifold, an F-structure on a manifold is a number of compatible local pure F-structures.

Let (M,g) be a Riemannian manifold with corners. A local Killing vector field X defined on an open subset $U \subset M$ is an infinitesimal isometry on U, it is equivalent to the condition that the Lie differentiation of the Riemannian metric g with respect to X is zero, i.e.,

$$L_X g = 0 \qquad \text{on } U \tag{3.1}$$

where L_X is the Lie differentiation with respect to X.

Definition 3.1. Let $\mathfrak{H}$ be a sheaf of germs of smooth tangent vector fields on an n-manifold with corners M. $\mathfrak{H}$ will be called a pure polarized F-structure of rank r if it satisfies the following conditions:

(a). Each stalk $\mathfrak{H}_x$ at $x \in M$ is an abelian algebra in the Lie bracket operation. $\mathfrak{H}_x$ is thus a finite dimensional vector space for all $x \in M$, the dimension of $\mathfrak{P}_x$ is equal to $r > 0$.

(b) For all open subset $U(\subset M)$ diffeomorphic to R^n, $\Gamma(\mathfrak{P}(U))$ is a vector space of dimension r. The r-dimensional integrable distribution (might be singular) determined by $\mathfrak{P} \otimes C^\infty(M)$ is complete, where $\Gamma(\mathfrak{P}(U))$ is the set of all smooth sections of $\mathfrak{P}$ on U.

(c). There exists a Riemannian metric g on M such that for all open subset $U \subset M$ and all section $X \in \Gamma(\mathfrak{P}(U))$, the equation (3.1) holds on U, i.e., $L_X g = 0$, on U, the restriction of X to the faces of M are tangent to the faces. Thus, any local section X of the sheaf $\mathfrak{P}$ is a Killing vector field with respect to g. Such a Riemannian metric g will be called an invariant metric for $\mathfrak{P}$.

(d) $\mathfrak{P}$ will be called a pure F-structure of rank r if, in addition to the above three conditions, each leaf of the r-dimensional integrable distribution determined by $\mathfrak{P} \otimes C^\infty(M)$ is compact. Compactness and completeness imply that each leaf is actually a closed submanifold.

Remark. If N is a face of M, the completeness condition implies that $\mathfrak{P}|_N$ is a pure polarized F-structure of rank r.

Definition 3.2. A polarized F-structure $\mathfrak{F}$ on a manifold with corners M is a countable locally finite open cover $\{U_\alpha\}_{\alpha \in A}$ and a pure polarized F-structure $\mathfrak{P}_\alpha$ of rank r_α on U_α for each $\alpha \in A$ such that

(a) There exists an invariant Riemannian metric g on M such that $g|_{U_\alpha}$ is an invariant metric of $\mathfrak{P}_\alpha$ for all $\alpha \in A$.

(b) For all $x \in M$, all germs of $\{\mathfrak{P}_\alpha\}_{\alpha\in A}$ at x commute to each other. Thus for all nonempty intersections $U_{(\alpha_1,\ldots,\alpha_\ell)} = U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_\ell} \neq \phi$, $\{\mathfrak{P}_{\alpha_i}|_{U_{(\alpha_1,\ldots,\alpha_\ell)}}\}_{i=1}^{\ell}$ generates a pure polarized F-structure $\mathfrak{P}(\alpha_1,\ldots,\alpha_\ell)$ on $U_{(\alpha_1,\ldots,\alpha_\ell)}$ of rank $r_{(\alpha_1,\ldots,\alpha_\ell)}$.

(c) $\mathfrak{F}$ will be called an F-structure if all leaves of $\{\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)}\}$ are closed for all nonempty intersection $U_{(\alpha_1,\ldots,\alpha_\ell)}$, $\ell > 1$.

Remark. If $\mathfrak{F}$ is a polarized F-structure on a compact manifold with corners, there exists an F-structure $\overline{\mathfrak{F}}$ such that $\mathfrak{F}$ is a substructure of $\overline{\mathfrak{F}}$ in the sense that if $\overline{\mathfrak{F}} = \{(U_\beta, \mathfrak{P}_\beta)\}_{\beta\in B}$, $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha\in A}$, then, $\{U_\alpha\}_{\alpha\in A}$ is finer than $\{U_\beta\}_{\beta\in B}$, $\mathfrak{P}_\alpha$ is a subsheaf of $\mathfrak{P}_\beta|_{U_\alpha}$ for all $U_\alpha \subset U_\beta$. However, we will not distinguish a polarized F-structure (pure polarized F-structure) from an F-structure (pure F-structure) since they make no essential difference in our context. We will call both of them an F-structure (pure F-structure).

Let $\mathfrak{P}$ be a pure F-structure of rank r on a manifold with corners M^n, let $U \subset M^n$ be an open subset of M which is diffeomorphic to R^n, choose a basis $X_1, X_2, \ldots, X_r$ for the r-dimensional vector space $\Gamma(\mathfrak{P}(U))$, for $x \in U$,

$X_1(x),\ldots,X_r(x) \subset T_xM$ is in general not an independent set of tangent vectors at x. x is said to be a singular point of $\mathfrak{P}$ if the rank $r(x)$ of the set $X_1(x),\ldots,X_r(x)$ is strictly less than r. We denote by $Z(\mathfrak{P}) \subset M$ the set of all singular points of $\mathfrak{P}$ on M. $\mathfrak{P}$ is called a pure F-structure of rank r without singularities if $Z(\mathfrak{P}) = \phi$, i.e., the rank function $r(x) = r$ for all $x \in M$.

Given an F-structure $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha \in A}$ on a manifold with corners M, the singular set $Z(\mathfrak{F})$ is defined to be the disjoint union of the singular sets $Z(\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)})$ of the pure F-structures $\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)}$, i.e.,

$$(3.2) \qquad Z(\mathfrak{F}) = \coprod_{U_{(\alpha_1,\ldots,\alpha_\ell)} \neq \phi} Z(\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)}) .$$

Definition 3.3. Let $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha \in A}$ be an F-structure on an orientable manifold with corners M^n. A stratification $\{M_{(\alpha_1,\ldots,\alpha_\ell)}\}_{(\alpha_1,\ldots,\alpha_\ell) \in A^\ell, \ell=\overline{1,n+1}}$ of M subordinate to $\{U_\alpha\}_{\alpha \in A}$ is said to be compatible with $\mathfrak{F}$ if

$$\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)}\Big|_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}}$$

is a pure F-structure of rank $r_{(\alpha_1,\ldots,\alpha_\ell)}$ on $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ for all $(\alpha_1,\ldots,\alpha_\ell) \in A^\ell$, where $r_{(\alpha_1,\ldots,\alpha_\ell)}$ is the rank of the pure F-structure $\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)}$ on $U_{(\alpha_1,\ldots,\alpha_\ell)}$. Hence, any integral leave of the distribution $\mathfrak{P}_{(\alpha_1,\ldots,\alpha_\ell)} \otimes C^\infty(U_{(\alpha_1,\ldots,\alpha_\ell)})$ which intersects with

$\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ is entirely contained in $\widetilde{M}_{(\alpha_1,\ldots,\alpha_\ell)}$.

Compatible stratifications always exist. Given an F-structure $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha\in A}$, a compatible stratification $\{M_{(\alpha_1,\ldots,\alpha_\ell)}\}_{\substack{(\alpha_1,\ldots,\alpha_\ell)\in A^\ell \\ \ell=\overline{1,n+1}}}$ will be chosen without being specifically mentioned. g will always be a fixed invariant Riemannian metric of $(M,\mathfrak{F})$.

For examples of F-structues on manifolds, we refer again to [7]. However, there are two kinds of typical examples of F-structures which are especially easy to describe.

Example 1. If M^n is a compact, flat manifold, by the Bieberbach theorem, there is a finite normal covering $\widetilde{M}^n$, which is isometric to an n-torus. The isometric action of this torus on itself induces a pure F-structure $\mathfrak{P}$ of rank n without singularities on M^n. $\mathfrak{P} \otimes C^\infty(M^n)$ is the sheaf of germs of all smooth tangent vector fields on M^n.

Example 2. A torus bundle $T^k \to E \to S^1$ on a circle with holonomy $A \in SL(k,\mathbf{Z})$ is obtained as following. Since $A \in SL(k,\mathbf{Z})$ is an automorphism of the torus T^k, we identify the two copies of T^k of the boundary $T^k \times [0,1]$ by A, i.e.,

$$(3.3) \qquad (y,1) \sim (Ay,0)$$

and

$$E = T^k \times [0,1]/\sim \tag{3.4}$$

T^k acts on the fibres by

$$t(x,s) = (x+t,s), \text{ for } t \in T^k = R^k/Z^k \tag{3.5}$$

by equation (3.3), the automorphism A induces a new action of T^k on $T^k \times \{0\}$ by $(x,0) \to (x+At,0)$. The two local actions $(x,s) \to (x+t,s)$, and $(x,s) \to (x+At,s)$ are easily seen to commute. They define a pure F-structure $\mathfrak{F}$ of rank k without singularities on E. Each fibre of the bundle E is a leave of the integrable distribution $\mathfrak{F} \otimes C^\infty(E)$. If $A: R^k \to R^k$ has nontrivial invariant subspaces, then, each invariant subspace of A defines a pure polarization of $\mathfrak{F}$ without singularities in an obvious way. In general, if $T^k \to E \to T^n$ is a torus bundle over a torus, the T^k action on the fibres induces a pure F-structure of rank k without singularities on E.

Example 3. F-structures on the unit 3-sphere $S^3 = \{(z_1,z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subset C^2$. There is a canonical T^2-action induced by this embedding in C^2,

$$(t_1,t_2)(z_1,z_2) = (e^{2\pi i t_1} z_1, e^{2\pi i t_2} z_2) \tag{3.6}$$

Let us denote by T^2 this pure F-structure of rank 2.

The singular set

(3.7) $$Z(T^2) = \{(0,z_2)\varepsilon S^3\} \cup \{(z_1,0)\varepsilon S^3\}$$

is the union of two circles. T^2 has a global basis $\frac{\partial}{\partial\theta_1}$, $\frac{\partial}{\partial\theta_2}$ of Killing vector fields, where θ_1, θ_2 are defined by the multipolar coordinate $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. For any $a, b \in \mathbf{R}$, $a \cdot b \neq 0$,

(3.8) $$X = a\frac{\partial}{\partial\theta_1} + b\frac{\partial}{\partial\theta_2}$$

defines a pure substructure of T^2 of rank 1 without singularities. The leaves of X are not closed if a/b is irrational.

There are also nonpure substructures of T^2 on S^3. Let $a_1, a_2, b_1, b_2 \in \mathbf{R}$, $b_1 \cdot a_2 \neq 0$, set

(3.9) $$X_i = a_i\frac{\partial}{\partial\theta_1} + b_i\frac{\partial}{\partial\theta_2}, \quad i = 1,2$$

X_i is a pure F-structure of rank 1 without singularities on

(3.10) $$U_i = \{(z_1,z_2) \in S^3 \mid |z_i| < 1\}$$

$\mathfrak{F} = \{(U_1,X_1),(U_2,X_2)\}$ defines an F-structure without singularities on S^3, which is not pure. The secondary topological invariants associated to this F-structure on S^3 will be computed explicitly in section 12.

§4. Preliminary

We will work in the real, smooth, and orientable category throughout unless otherwise indicated. Let $\pi: E \to M$ be a vector bundle over the manifold M, and let $T^* = T^*M$ be the cotangent bundle of M. A connection on E is a differential operator.

$$(4.1) \qquad \nabla: \Gamma(E) \to \Gamma(T^* \otimes E)$$

which, relative to smooth functions, satisfies the derivation law

$$(4.2) \qquad \nabla(fs) = df \otimes s + f\nabla s \ , \ s \in \Gamma(E)$$

where Γ denotes smooth sections.

One extends ∇ as an antiderivation to the whole exterior complex of forms on M with values in E. ∇^2 is easily seen to be linear over the module of smooth functions $C^\infty(M)$. ∇^2 can be realized by a 2-form Ω with values in the endomorphism bundle of E, that is, for all $V \in \Gamma(E)$

$$(4.3) \qquad \nabla^2 V = \Omega V$$

where $\Omega \in \Gamma(A^2(T^*M) \otimes \mathrm{End}\, E)$ and A^2 denotes the second exterior power. Moreover, one has the Bianchi Identity

(4.4) $$\nabla\Omega = 0$$

Ω is called the curvature of the connection ∇. If $e = \{e_i\}$ is a frame for E over $U \subset M$, then Ω determines a matrix $\Omega(e) = [\Omega_{ij}(e)]$ of two forms on U, it is given by the formula

(4.5) $$\Omega e_j = \sum_i \Omega_{ij}(e) e_i$$

or

(4.6) $$\Omega e = e\Omega(e)$$

if $e' = eB$ is another such frame, then

(4.7) $$\Omega(e') = B^{-1}\Omega(e)B$$

The connection ∇ determines a matrix $\omega(e) = [\omega_{ij}(e)]$ of one-forms on U, it is defined by the formula

(4.8) $$\nabla e_j = \sum_i \omega_{ij}(e) e_i$$

or

(4.9) $$\nabla e = e\omega(e)$$

(4.10) $$\omega(e') = B^{-1}dB + B^{-1}\omega(e)B$$

We will often suppress the specific local frame e, and write ω, Ω for $\omega(e)$, $\Omega(e)$, respectively. The local connection and curvature forms ω and Ω are related by

$$\Omega = d\omega + \omega \wedge \omega \tag{4.11}$$

Let ∇_0, ∇_1 be two connections on the same bundle $\pi: E \to M$, the difference $\nabla_1 - \nabla_0$ defines a one-form with values in the endomorphism bundle of E. Moreover, the set of all connections on E is a convex set, i.e., for any smooth function $f: M \to R$

$$\nabla_f = (1-f)\nabla_0 + f\nabla_1 \tag{4.12}$$

is a connection on E.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G, let $\mathfrak{g}^k = \underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{k}$. Polynomials of degree k are defined to be symmetric, multilinear maps from $\mathfrak{g}^k \to R$. G acts on $\mathfrak{g}^k$ by inner automorphism. Polynomials invariant under this action are called invariant polynomials of degree k, the set of all these invariant polynomials of degree k is denoted by $I^k(G)$. The polynomial product gives a ring structure on $I(G) \equiv \oplus I^\ell(G)$.

The Weil homomorphism

$$W: \; I^k(G) \to H^{2k}(M,R) \tag{4.12}$$

can be defined by evaluating an invariant polynomial P of degree k on the curvature form Ω of a connection ∇ on the bundle, and resulting in a closed $2k$- form $P(\Omega^k)$ on the base manifold. Set

$$(4.13) \qquad I_0^k(G) = \{P \in I^k(G) \mid W(P) \in H^{2k}(BG,\mathbf{Z})\}$$

where BG is the classifying space of G. An integral invariant polynomial is an invariant polynomial in $I_0^k(G)$. Since we will deal with oriented vector bundles with a Riemannian metric structure, the group G will always be the special orthogonal group $SO(n)$. In fact, $I(SO(n))$ is generated by the Euler polynomial χ and the Pontrjagin polynomial P_i's which in order represent the Euler class and the i^{th} Pontrjagin class under the Weil homomorphism. (See [13].)

Let M be a Riemannian manifold with a Riemannian structure g, there is a unique Riemannian connection D on the tangent bundle TM of M such that

$$(4.14) \qquad dg(V,W) = g(DV,W) + g(V,DW)$$

$$(4.15) \qquad D_V W - D_W V = [V,W]$$

where $[\cdot,\cdot]$ is the Lie bracket operation, $V,W \in \Gamma(TM)$.

A local frame $e = \{e_\alpha\}$ of TM is said to be invariant with respect to $X \in \Gamma(TM)$ if $[X,e_\alpha] = 0$. A connection on

TM is invariant with respect to X if it is invariant in the one parameter transformations generated by X, in terms of local invariant frame e and its associated local connection forms ω,

(4.16) $$L_X\omega = 0 .$$

It is thus clear that the local curvature form Ω associated to e is also invariant, i.e.,

(4.17) $$L_X\Omega = 0 .$$

Theorem 4.1. Let X be a Killing vector field of (M,g), the Riemannian connection associated to (M,g) is invariant.

Proof. The one parameter transformations generated by X are isometries and the Riemannian connection is uniquely determined by the metric g.

A Killing vector field X also defines a skew symmetric endomorphism S_X on TM,

(4.18) $$S_X(V) = -D_V X = [X,V] - D_X V \ , \ V \in \Gamma(TM)$$

S_X is skew-symmetric since

$$0 = (L_X g)(V,W) = Xg(V,W) - g([X,V],W) - g(V,[X,W])$$

$= g(D_X V, W) + g(V, D_X W) - g([X,V], W) - g(V, [X,W])$

$= -g(S_X(V), W) - g(V, S_X(W))$.

Let $\mathfrak{F}$ be an F-structure for M , and g is an invariant metric for $\mathfrak{F}$. A differential geometrical object on M is said to be invariant with respect to $\mathfrak{F}$ if it is locally invariant for all local sections of $\mathfrak{F}$. In addition to invariant Riemannian metric g , the associated Riemannian connection D of g and its curvature transformation are invariant. For each $X \varepsilon \mathfrak{F}(U)$, S_X is invariant on U . The existence of local invariant frames associated to an F-structure guarantees that we can always work locally in the invariant category.

For any $p \varepsilon M$, let U be an open neighborhood of p , $X_1, \ldots X_r \ \varepsilon \mathfrak{F}(U)$. Suppose that $X_1(p), \ldots, X_r(p)$ are linearly independent, then, on a possibly smaller open subset $V \subset U$, one can choose a coordinate system $x = (x_1, \ldots, x_n)$, such that $x(p) = 0$, $\frac{\partial}{\partial x_i} = X_i$, $i = \overline{1,r}$. $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$ is therefore an invariant frame. Furthermore, by use of the Gram-Schmidt orthonormalization process, we can construct an invariant orthonormal set $\{e_{r+1}, \ldots, e_n\}$ on V such that $[X_i, e_j] = 0$, $g(X_i, e_j) = 0$ for all $i < r < j$, and $\{X_i, \ldots, X_r, e_{r+1}, \ldots, e_n\}$ forms a local invariant frame on V .

§5. Singular Basic Connections Associated to a Pure F-structure

In this section, $\mathfrak{P}$ will always be a pure F-structure of rank r on a compact oriented n-dimensional Riemannian manifold with corners (M,g).

Definition 5.1. A singular basic connection ∇^b associated to $\mathfrak{P}$ is an invariant connection of $\mathfrak{P}$, which is defined only on the tangent bundle TM restricted to the complement $M\backslash Z(\mathfrak{P})$ of $Z(\mathfrak{P})$, and singular on $Z(\mathfrak{P})$. Moreover, for all local section $X \in \Gamma(\mathfrak{P}(U))$ of the pure F-structure $\mathfrak{P}$ and all invariant vector field $Y \in \Gamma(TU)$, Y is parallel along X, i.e. $[X,Y] = 0$ implies that

$$\nabla^b_X Y = 0 \tag{5.1}$$

Definition 5.2. A differential form $\theta \in A^*(M)$ is said to be basic (with respect to $\mathfrak{P}$) if θ is invariant and for all local section $X \in \Gamma(\mathfrak{P}(U))$,

$$i(X)\theta = 0 \tag{5.2}$$

where $i(X)$ is the contraction in the direction X.

Remark. By definition, θ is invariant means that for all local section $X \in \Gamma(\mathfrak{P}(U))$,

$$L_X\theta = 0 \tag{5.3}$$

Also note that

$$L_X = i(X) \circ d + d \circ i(X) \tag{5.4}$$

Thus a form θ is basic if and only if the contraction of both θ and $d\theta$ along all local section X of $\mathfrak{P}$ are zero. A d-closed form which satisfies equation (5.2) must be basic.

Let $A^*(M, \mathfrak{P}) = \bigoplus_{p=0}^{n-r} A^p(M, \mathfrak{P})$ be the set of all smooth basic forms on M with respect to $\mathfrak{P}$. It is clear that the exterior differentiation of a basic form is basic. $A^*(M, \mathfrak{P})$ is a graded differential subalgebra of $A^*(M)$.

We introduce the concept of a basic connection because it has the following basicness property.

Theorem 5.1. Suppose that ∇^b is a singular basic connection associated to $\mathfrak{P}$ which is defined on $M \backslash Z(\mathfrak{P})$. Let ω_b and Ω_b be the local connection and curvature forms assoicated to an invariant local frame $e = \{e\alpha\}$ on $U \subset M \backslash Z(\mathfrak{P})$. Then both ω_b and Ω_b are basic.

Proof. First, the discussion at the end of section 4 shows the existence of invariant local frame $e = \{e\alpha\}$.

The local connection forms ω_b is defined by

(5.5) $$\nabla^b e = e\omega_b$$

ω_b is invariant since both ∇^b and e are invariant. Since $e = \{e\alpha\}$ is an invariant frame, equation (5.1) implies that $\nabla^b_X e = 0$ for all $X \in \Gamma(\mathfrak{P}(U))$. It follows that

$$e\omega_b(X) = i(X)\nabla^b e = \nabla^b_X \; e = 0$$

i.e., $\omega_b(X) = 0$ for all $X \in \Gamma(\mathfrak{P}(U))$.

The basicness of Ω_b follows from the structure equation

(5.6) $$\Omega_b = d\omega_b + \omega_b \wedge \omega_b$$

and the fact that $A^*(M, \mathfrak{P})$ is a graded differential subalgebra of $A^*(M)$.

Let g be an invariant Riemannian metric on M for $\mathfrak{P}$. Note $\mathfrak{P}$ defines a singular integrable distribution of dimension $r = \operatorname{rank} \mathfrak{P}$. A tangent vector on $M \backslash Z(\mathfrak{P})$ will be called horizontal if it is g perpendicular to the distribution $\mathfrak{P}$. We now construct a canonical singular basic connection ∇^b for $\mathfrak{P}$ associated to the invariant metric g such that the covariant differentiation ∇^b along all horizontal tangent vector coincide with the Riemannian covariant differentiation D determined by g .

Theorem 5.2. The following formulae define a basic connection ∇^b for $\mathfrak{P}$ on $M \backslash (Z(\mathfrak{P}))$.

(5.7) $$\nabla^b_{v(x)} W = D_{v(x)} W$$

for all horizontal tangent vector $v \in \Gamma(T(M \setminus Z(\mathfrak{p})))$ and all tangent vector field $W \in \Gamma(TM)$.

(5.8) $$\nabla^b_{X(x)} W = [X, W] = D_X W - D_W X$$

for all germs $X \in \mathfrak{p}_x$, $x \in M \setminus Z(\mathfrak{p})$. ∇^b is then extended by linearity to all tangent vector $v \in T_x(M \setminus Z(\mathfrak{p}))$.

Proof. ∇^b is well-defined since for all $x \in M \setminus Z(\mathfrak{p})$, $v \in T_x M$, there is a unique germ $X \in \mathfrak{p}_x$, such that

(5.9) $$v^h \equiv v(x) - X(x)$$

is horizontal, the decomposition $v(x) = X(x) + v^h$ is unique. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{p}(U)$ for some small neighborhood $U \subset M \setminus Z(\mathfrak{p})$ of x, extend it to an invariant basis $X_1, \ldots, X_r, e_{r+1}, \ldots, e_n$ as at the end of section 4, such that

(5.10) $$[X_i, e_j] = 0$$

(5.11) $$g(X_i, e_j) = 0$$

Let $\{\pi_i\}_{i=1}^r$ be the dual 1-forms of $\{X_i\}_{i=1}^r$ such that

$$\pi_i(X_j) = \delta_{ij} \tag{5.12}$$

$$\pi_i(e_j) = 0 \tag{5.13}$$

We will call $\{\pi_i\}_{i=1}^{r}$ the metric dual of $\{X_i\}_{i=1}^{r}$.

It is obvious that the local 1-forms $\{\pi_i\}_{i=1}^{r}$ are invariant since

$$(L_{X_i}\pi_j)(X_s) = X_i(\pi_j(X_s)) - \pi_j([X_i, X_s]) = 0$$

$$(L_{X_i}\pi_j)(e_s) = X_i(\pi_j(e_s)) - \pi_j([X_i, e_s]) = 0$$

by the equations (5.10), (5.12), (5.13), and the commutativity of the sections of $\mathfrak{D}(U)$.

Set

$$S_i = -DX_i \tag{5.14}$$

S_i is invariant since $S_i(Y) = -D_Y X_i = [X_i, Y] - D_{X_i} Y$.

Set

$$S = \nabla^b - D \tag{5.15}$$

S is a skew symmetric endomorphism valued 1-form on $TM|_U$. It is easily checked that

(5.16) $$S = \sum_{i=1}^{r} \pi_i S_i$$

where $S(Y) = \sum_{i=1}^{r} \pi_i S_i(Y)$ is a vector valued 1-form. ∇^b is thus a derivation and is invariant since D, π_i, and S_i are all invariant. For an invariant vector field $w \varepsilon \Gamma(TU)$ and for all $X \varepsilon \mathcal{F}(U)$, $[X,w] = 0$, by equation (5.8), one has

(5.17) $$\nabla^b_X w = [X,w] = 0 .$$

∇^b will be called the cannonical singular basic connection of $\mathcal{F}$ associated to g because of its canonical relations (5.7) and (5.8) with the Riemannian connection D. It is completely determined by the pure F-structure $\mathcal{F}$ and the invariant metric g.

§6. Transgression and the Associated Difference-Differential Formula

The transgression idea was first introduced by S.S. Chern and A. Weil. It serves as a standard argument both for the theory of characteristic classes and late for the theory of secondary characteristic classes in the differential geometrical approach. In these cases, it was used only up to the first stage. We find that the complete transgression has already been used in their computation for a combinatoric Pontrjagin number in 1972 in a series of papers by Fuchs-Gabrielov-Gel'fand [10].

Let $\pi: E \to M$ be an ℓ-plane bundle with a family of connections ∇_t smoothly parametrized on a compact oriented manifold with corner V. Let $f: V \times M \to M$,

$$f(t,x) = x \tag{6.1}$$

be the projection map. The pullback of the family of connections ∇_t defines a connection ∇ on the pullback vector bundle $f^*E = V \times E$ by the formula

$$(\nabla(f^*W))_{(t,x)} = f^*_{(t,x)}(\nabla_t W) \qquad \text{for all } W \in \Gamma(E) \tag{6.2}$$

For each local frame $e = \{e_1, e_2, \ldots, e_\ell\}$ of E on U, the pullback $f^*e = \{f^*e_1, \ldots, f^*e_\ell\}$ is a local frame of f^*E on

$V \times U$. Let ω_t, ω and Ω_t, Ω be the local connection forms and local curvature forms of ∇_t and ∇ for E and f^*E with respect to the local frame e and f^*e , respectively. Let d and d' be the exterior differentiation on M and V , respectively. Then $d' + d$ is the exterior differentiation on $V \times M$. We will identify the differential forms on M with its pullback under f .

Theorem 6.1. These local connection and curvature forms ω_t, ω and Ω_t, Ω are related by the following formulae:

(6.3) $$\omega\big|_{(t,x)} = \omega_t\big|_x$$

(6.4) $$\Omega = \nabla\omega = (d' + d)\omega + \omega\wedge\omega$$

(6.5) $$\Omega_t = \nabla_t\omega_t = d\omega_t + \omega_t \wedge \omega_t$$

(6.6) $$\Omega = d'\omega_t + \Omega_t$$

Proof. Equation (6.3) follows from equation (6.2) and the identification of forms on M with its pullback under f .

Equations (6.4) and (6.5) are the structure equations of (f^*E, ∇) and (E, ∇_t) , respectively.

Equation (6.6) results from equations (6.3), (6.4) and (6.5).

Lemma 6.2. Let $P \in I^k(SO(\ell, R))$ be an invariant polynomial of degree k, then

$$(6.7) \qquad dP(\Omega_t^k) = 0$$

$$(6.8) \qquad (d'+d)P(\Omega^k) = 0$$

Proof. This is the fundamental lemma in the Chern-Weil description of characteristic classes. They are proved via the Bianchi identity,

$$(6.9) \qquad \nabla_t \Omega_t = 0$$

thus
$$dP(\Omega_t^k) = kP(\nabla_t \Omega_t, \Omega_t^{k-1}) = 0 .$$

where
$$P(\nabla_t \Omega_t, \Omega_t^{k-1}) \equiv P(\nabla_t \Omega_t, \overbrace{\Omega_t, \ldots, \Omega_t}^{(k-1)}) .$$

The graded differential algebra $(A^*(V \times M), d'+d)$ is naturally split into a double differential complex $A^*(V \times M) = \bigoplus_{p,q} A^{p,q}(V \times M)$ by the product structure of $V \times M$. The diagram

$$(6.10) \qquad \begin{array}{ccc} A^{p+1,q}(V\times M) & \xrightarrow{d} & A^{p+1,q+1}(V\times M) \\ d' \uparrow & & \uparrow d' \\ A^{p,q}(V\times M) & \xrightarrow{d} & A^{p,q+1}(V\times M) \end{array}$$

commutes.

Equation (6.8) implies that all of its (p,q)-components of $(d'+d)P(\Omega^k)$ in the splitting must be zero.

Corollary 6.3. The $(i,2k-i)$-forms

(6.11) $$\binom{k}{i}P((d'\omega_t)^i,\Omega_t^{k-i}), \quad i = \overline{0,k}.$$

are well-defined on $V \times M$. Moreover,

(6.12) $$d'P((d'\omega_t)^k) = 0$$

(6.13) $$d'\{\binom{k}{i}P((d'\omega_t)^i,\Omega_t^{k-i})\} + d\{\binom{k}{i+1}P((d'\omega_t)^{i+1},\Omega_t^{k-i-1})\} = 0$$

$$0 \leqslant i < k.$$

Let $m = \dim.V. \leqslant k+1$. Then,

(6.14) $$\int_{\partial V}\binom{k}{m-1}P((d'\omega_t)^{m-1},\Omega_t^{k-m+1}) + (-1)^m d\{\int_V \binom{k}{m}P((d'\omega_t)^m,\Omega_t^{k-m})\} = 0$$

Let $\{\nabla_i\}_{i=0}^m$ be $(m+1)$ connections on the bundle $\pi: E \to M$. The linear interpolation

(6.15) $$\nabla_t = \sum_{i=0}^m t_i\nabla_i, \quad t = (t_0,t_1,\dots,t_m) \in \Delta_m$$

defines a family of connections smoothly parametrized on the standard m-simplex Δ_m. Let

(6.16) $$\Delta_m(\hat{i}) = \{t \in \Delta_m | t_i = 0\}$$

be the i^{th} face of Δ_m. Set

$$(6.17) \qquad P_{(0,1,\ldots,m)} = \int_{\Delta_m} \binom{k}{m} P((d'\omega_t)^m, \Omega_t^{k-m})$$

$$(6.18) \qquad P_{(0,1,\ldots,\hat{i},\ldots,m)} = \int_{\Delta_m(\hat{i})} \binom{k}{m-1} P((d'\omega_t)^{m-1}, \Omega_t^{k-m+1})$$

Note that $\partial\Delta_m = \sum_{i=0}^{m} (-1)^i \Delta_m(\hat{i})$.

Corollary 6.4. (The Difference-Differential Formula)

$$(6.19) \qquad \sum_{i=0}^{m} (-1)^i P_{(0,1,\ldots,\hat{i},\ldots,m)} + (-1)^m dP_{(0,1,\ldots,m)} = 0$$

Equation (6.19) is a special case of equation (6.14) with $V = \Delta_m$.

When the $(m+1)$ connections in equation (6.15) are linearly dependent, i.e., there is a nonzero vector $(a_0, a_1, \ldots, a_m) \in R^{m+1}$, such that

$$(6.20) \qquad \sum_{i=0}^{m} a_i \nabla_i = 0$$

then, the $(2k-m)$-form $P_{(0,1,\ldots,m)}$ defined by the equation (6.17) vanishes.

Theorem 6.5. Assume that the $(m+1)$-connections $\{\nabla_i\}_{i=0}^{m}$ are linearly dependent, i.e., equation (6.20) is true for some

$(a_0, a_1, \ldots, a_m) \neq 0$, then

(6.21) $$P_{(0,1,\ldots,m)} \equiv \int_{\Delta m} \binom{k}{m} P((d'\omega t)^m, \Omega_t^{k-m}) = 0$$

(6.22) $$\sum_{i=0}^{m} (-1)^i P_{(0,1,\ldots,\hat{i},\ldots,m)} = 0$$

Proof. Equations (6.19) and (6.21) imply (6.22). To show that $P_{(0,1,\ldots,m)} = 0$, we first note that equation (6.20) implies that $\sum_{i=0}^{m} a_i = 0$. Without loss of generality, we may assume that $a_m \neq 0$. Now consider the nonsingular linear transformation

$$f:\ R^{m+1} \to R^{m+1}$$

(6.23)

$$(s_0, s_1, \ldots, s_m) \to (t_0, t_1, \ldots, t_m) = (s_0 + a_0 s_m, \ldots, s_{m-1} + a_{m-1} s_m, a_m s_m)$$

One has

$$\nabla_t = \sum_{i=0}^{m} t_i \nabla_i = \sum_{i=0}^{m-1} s_i \nabla_i + s_m \sum_{i=0}^{m} a_i \nabla_i = \sum_{i=0}^{m-1} s_i \nabla_i$$

$$= \nabla_0 + \sum_{i=1}^{m-1} s_i B_i$$

where $B_i = \nabla_i - \nabla_0$ is a one-form with values in Hom(E,E), since

$$1 = \sum_{i=0}^{m} t_i = \sum_{i=0}^{m-1} s_i + s_m \sum_{i=0}^{m} a_i = \sum_{i=0}^{m-1} s_i \,.$$

therefore $d'\omega_t = \sum_{i=1}^{m-1} d's_i \wedge B_i$

$$P_{(0,1,\ldots,m)} = \int_{\Delta_m} \binom{k}{m} P((d'\omega_t)^m, \Omega_t^{k-m})$$

$$= \int_{f^{-1}(\Delta_m)} \binom{k}{m} P\left(\left(\sum_{i=1}^{m-1} d's_i \wedge B_i\right)^m, \Omega_s^{k-m}\right)$$

$$= 0$$

since it is an m-form which can be expressed by $d's_1, \ldots, d's_{m-1}$.

§7. The Generalized Bott Forms Associated to a Sequence of Pure F-Structures.

Let $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha\varepsilon A}$ be an F-structure on a compact oriented n-dimensional manifold with corners M^n, $\{M_{(\alpha_1,\ldots,\alpha_\ell)}\}_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell}$, $\ell=\overline{1, n+1}$ a stratification of M which is compatible with $\mathfrak{F}$ as defined in section 3.

Let $(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell$, $\alpha_\ell \varepsilon A^1$, i.e., $\alpha_\ell \neq \infty$, for simplicity, we will often write $(1,2,3,\ldots,\ell)$ for $(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ while there is no confusion, then, $(1,2\ldots,j)\varepsilon A^j$ for $j = \overline{1, \ell}$ by the definition of A^j, $\mathfrak{P}_{(1,\ldots,j)}$ is a pure F-structure of rank $r_{(1,2,\ldots,j)}$ on $U_{(1,2,\ldots,j)}$. The restriction of the sequence $\mathfrak{P}_{(1)}, \mathfrak{P}_{(1,2)},\ldots, \mathfrak{P}_{(1,2,\ldots,\ell)}$ of pure F-structures onto $U_{(1,2,\ldots,\ell)}$ is an increasing sequence, i.e. $\mathfrak{P}_{(1,2,\ldots,j)}|_{U_{(1,2,\ldots,\ell)}}$ is a substructure of $\mathfrak{P}_{(1,2,\ldots,j+1)}|_{U_{(1,2,\ldots,\ell)}}$. Put $\mathfrak{P}^{(j)} = \mathfrak{P}_{(1,2,\ldots,j)}$, $r^{(j)} = r_{(1,2,\ldots j)}$, $j = 1,2,\ldots,\ell$ to save notation. Let ∇_j^b be the canonical singular basic connection of $\mathfrak{P}^{(j)}$ on $U_{(1,2,\ldots,j)}$. Set $\nabla_0 = D$ the Riemannian connection of g, $\nabla_j = \nabla_j^b$, $j = \overline{1,\ell}$. Let $P\varepsilon I^k(O(n))$. Define the $(2k-\ell)$-form $P_{(0,1,\ldots,\ell)}$ by equation (6.17). Similarly, for each rearrangement $(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\ldots,\alpha_{\sigma(\ell)})$, of $(\alpha_1,\ldots,\alpha_\ell)$

where $\sigma \in S(\ell)$, the corresponding sequence

$\mathfrak{P}_{(\sigma(1))}, \mathfrak{P}_{(\sigma(1),\sigma(2))}, \ldots, \mathfrak{P}_{(\sigma(1),\ldots,\sigma(\ell))}$, restricted onto $U_{(1,2,\ldots\ell)}$ is an increasing sequence. Let $\nabla^b_{\sigma(j)}$ be the canonical singular basic connection of $\mathfrak{P}_{(\sigma(1),\ldots,\sigma(j))}$ on $U_{(\sigma(1),\sigma(2),\ldots,\sigma(j))}$. The connections $\nabla_0 = D$, $\nabla_j = \nabla^b_{\sigma(j)}$, define a $(2k-\ell)$-form $P_{(0,\sigma(1),\ldots,\sigma(\ell))}$ by equation (6.17) on $U_{(\sigma(1),\ldots,\sigma(\ell))} = U_{(1,2,\ldots,\ell)}$. Set

$$(7.1) \qquad P_{(\alpha_1,\ldots,\alpha_\ell)} = \sum_{\sigma \in S(\ell)} \operatorname{sgn}(\sigma) P_{(0,\sigma(1),\ldots,\sigma(\ell))}$$

$P_{(\alpha_1,\ldots,\alpha_\ell)}$ will be called the generalized Bott form associated to the pure F-structures $\{\mathfrak{P}_{\alpha_i}\}_{i=1}^{\ell}$. It will serve as the basic building block of the exotic characteristic classes for manifolds with corners which admit a nonsingular F-structure in a neighborhood of its boundary.

Let $P_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)}$ be the generalized Bott form associated to the pure F-structures $\{\mathfrak{P}_{\alpha_i}\}_{\substack{i=1 \\ i\neq s}}^{\ell}$ on $U_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)}$.
Note that

$$(7.2) \quad P_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)} = (-1)^{\ell+s} \sum_{\substack{\sigma \in S(\ell) \\ \sigma(\ell)=s}} \operatorname{sgn}(\sigma) P_{(0,\sigma(1),\ldots,\sigma(\ell-1))}$$

where $P_{(o,\sigma(1),\ldots,\sigma(\ell-1))}$ is the $(2k-\ell+1)$-form on $U_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)}$ defined by the increasing sequence of pure F-structures

$\mathfrak{P}_{(\sigma(1))}, \mathfrak{P}_{(\sigma(1),\sigma(2))}, \ldots, \mathfrak{P}_{(\sigma(1),\ldots,\sigma(\ell-1))}$.

The difference-differential formula (6.19) gives

$$(7.3)\qquad P_{(\sigma(1),\sigma(2),\ldots,\sigma(\ell))} + \sum_{i=1}^{\ell} (-1)^i P_{(0,\sigma(1),\ldots,\widehat{\sigma(i)},\ldots,\sigma(\ell))} + (-1)^{\ell} dP_{(0,\sigma(1),\ldots,\sigma(\ell))} = 0$$

where $P_{(\sigma(1),\sigma(2),\ldots,\sigma(\ell))}$ is the $(2k-\ell+1)$-form defined by the connections $\nabla^b_{\sigma(j)}$, $j = \overline{1,\ell}$ on $U_{(\alpha_1,\ldots,\alpha_\ell)}$, $P_{(0,\sigma(1),\ldots\widehat{\sigma(i)},\ldots,\sigma(\ell))}$ by the connections D, $\nabla^b_{\sigma(j)}$, $j \neq i$ on $U_{(\alpha_1,\ldots,\alpha_\ell)}$, via equation (6.18).

Remark. If $r_\alpha \geq r > 0$ for all $\alpha\varepsilon A$, then, by the basicness of $\nabla^b_{\sigma(j)}$, $P_{(\sigma(1),\ldots,\sigma(\ell))}$ is at least r-dimensional basic. Thus, the restriction of $P_{(\sigma(1),\ldots,\sigma(\ell))}$ onto $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ vanishes if $2k + r > n$.

Take the alternating sum of equation (7.3) over $S(\ell)$,

$$(7.4)\qquad \sum_{\sigma\varepsilon S(\ell)} \operatorname{sgn}(\sigma) P_{(\sigma(1),\sigma(2),\ldots,\sigma(\ell))}$$

$$+ \sum_{i=1}^{\ell} (-1)^i \sum_{\sigma\varepsilon S(\ell)} \operatorname{sgn}(\sigma) P_{(0,\sigma(1),\ldots,\widehat{\sigma(i)},\ldots,\sigma(\ell))}$$

$$+ (-1)^{\ell} dP_{(\alpha_1,\ldots,\alpha_\ell)} = 0$$

Note that $(-1)^{\ell} \sum_{\sigma \varepsilon S(\ell)} \operatorname{sgn}(\sigma) P_{(o,\sigma(1),\ldots,\sigma(\ell-1))}$

$$= (-1)^{\ell} \sum_{s=1}^{\ell} \sum_{\substack{\sigma \varepsilon S(\ell) \\ \sigma(\ell)=s}} \operatorname{sgn}(\sigma) P_{(0,\sigma(1),\ldots,\sigma(\ell-1)}$$

$$= \sum_{s=1}^{\ell} (-1)^{s} P_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)}$$

Set

(7.5) $$B(\sigma,j) = \nabla^{b}_{\sigma(j)} - D$$

(7.6) $$\nabla^{\sigma}_{t} = \sum_{j=1}^{\ell} t_j \nabla^{b}_{\sigma(j)} + t_0 D$$

$$= D + \sum_{j=1}^{\ell} t_j B(\sigma,j)$$

Let ω^{σ}_{t} be the local connection forms of ∇^{σ}_{t} , then

(7.7) $$d' \omega^{\sigma}_{t} = \sum_{j=1}^{\ell} d' t_j \ B(\sigma,j)$$

For $1 \leqslant s < \ell$, let $\tau_s \varepsilon S(\ell)$ be the permutation $\tau_s(s) = s + 1$, $\tau_s(s+1) = s$, $\tau_s(j) = j$ for $j \neq s, s + 1$.

(7.8) $$\tau_s : S(\ell) \longrightarrow S(\ell)$$
$$\sigma \longmapsto \sigma \circ \tau_s$$

defines an automorphism of $S(\ell)$.

(7.9) $$\operatorname{sgn}(\sigma \circ \tau_s) = -\operatorname{sgn}(\sigma)$$

moreover, as a subset of $(1,2,\ldots,\ell)$,

(7.10) $$\sigma(1,2,\ldots,j) = \sigma \circ \tau_s(1,2,\ldots,j) \quad \text{for} \quad j \neq s$$

Note that $\nabla^b_{\sigma(j)} = \nabla^b_{\sigma'(j)}$ if and only if

$$\sigma(1,2,\ldots,j) = \sigma'(1,2,\ldots,j)$$

therefore $B(\sigma,j) = B(\sigma \circ \tau_s,j)$ for $j \neq s$, while on $\Delta_\ell(\hat{s})$, $t_s = 0$, thus

(7.11) $$\nabla^\sigma_t \Big|_{\Delta_\ell(\hat{s})} = \nabla^{\sigma\circ\tau_s} \Big|_{\Delta_\ell(\hat{s})}$$

(7.12) $$d'\omega^\sigma_t \Big|_{\Delta_\ell(\hat{s})} = d'\omega_t^{\sigma\circ\tau_s} \Big|_{\Delta_\ell(\hat{s})}$$

It follows that

(7.13) $$P_{(0,\sigma(1),\ldots,\hat{\sigma}(s),\ldots,\sigma(\ell))}$$

$$= P_{(0,\sigma\circ\tau_s(1),\ldots,\widehat{\sigma\circ\tau_s(s)},\ldots,\sigma\circ\tau_s(\ell))}$$

combine equation (7.9) with (7.13), one has

(7.14) $$\sum_{\sigma\in S(\ell)} \operatorname{sgn}(\sigma) P(0,\sigma(1),\ldots,\widehat{\sigma(s)},\ldots,\sigma(\ell)) = 0$$

for $0 < s < \ell$ on $U_{(\alpha_1,\ldots,\alpha_\ell)}$.

We have thus proved the following

Theorem 7.1

(7.15) $$\sum_{s=1}^{\ell}(-1)^s P_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots\alpha_\ell)} + (-1)^\ell dP_{(\alpha_1,\ldots,\alpha_\ell)}$$

$$= - \sum_{\sigma\varepsilon S(\ell)} \operatorname{sgn}(\sigma) P_{(\sigma(1),\sigma(2),\ldots,\sigma(\ell))}$$

In particular, if $n < 2k + \min_{i=\overline{1,\ell}} r_{\alpha_i}$, then

(7.16) $$\sum_{s=1}^{\ell} (-1)^s P_{(\alpha_1,\ldots,\hat{\alpha}_s,\ldots,\alpha_\ell)}\Big|_{M_{(\alpha_1,\ldots,\alpha_\ell)}} + (-1)^\ell dP_{(\alpha_1,\ldots,\alpha_\ell)}\Big|_{M_{(\alpha_1,\ldots,\alpha_\ell)}} = 0$$

for $\ell = \overline{1,\ 2k+1}$, $P(\hat{\alpha}_1) = P(\Omega)|_{U_{(\alpha_1)}}$ for $\alpha_1 \varepsilon A^1$. For $\ell=1,2$, we have

(7.17) $$P(\Omega^k) + dP_{(\alpha_1)} = 0$$

(7.18) $$-P_{(\alpha_2)} + P_{(\alpha_1)} + dP_{(\alpha_1,\alpha_2)} = 0$$

Theorem 7.2. Let $\mathfrak{P}_{\alpha_1},\ldots,\mathfrak{P}_{\alpha_\ell}$ be ℓ pure F-structures on $U_{(\alpha_1,\ldots,\alpha_\ell)}$, which are commuted to each other. Let $\mathfrak{P}_{(\alpha_{i_1},\ldots,\alpha_{i_j})}$ be the pure F-structure generated by $\mathfrak{P}_{\alpha_{i_1}},\ldots,\mathfrak{P}_{\alpha_{i_j}}$, if for all $i = \overline{1,\ell}$, $\mathfrak{P}_{\alpha_i}$ is a substructure of $\mathfrak{P}_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)}$, then

(7.19) $$P_{(\alpha_1,\dots,\alpha_\ell)} = 0$$

Proof. Under the assumption, $\nabla_o = D$, $\nabla^b_{\sigma(j)}$, $j = \overline{1,\ell}$, are linearly dependent. It follows from the vanishing theorem 6.5 that

(7.20) $$P_{(0,\sigma(1),\dots,\sigma(\ell))} = 0$$

for all $\sigma\varepsilon S(\ell)$. Thus

(7.21) $$P_{(\alpha_1,\dots,\alpha_\ell)} = \sum_{\sigma\varepsilon S(\ell)} \operatorname{sgn}(\sigma) P_{(o,\sigma(1),\dots,\sigma(\ell))} = 0$$

§8. Nonsingular F-structures and Characteristic Classes.

Let $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha\varepsilon A}$ be an F-structure on a compact, oriented manifold with corners M^n, which is either nonsingular in a neighborhood of ∂M or a nonsingular F-structure defined only on a neighborhood of ∂M, in the later case, we put $U_{-\infty} = \mathring{M} =$ the interior of M, and $\{U_\alpha\}_{\alpha\varepsilon A} \cup \{U_{-\infty}\}$ is thus an open cover of M. In both cases, let $\{M_{(\alpha_1,\ldots,\alpha_\ell)}\}$ be a stratification of M compatible with $\mathfrak{F}$. Let

(8.1)
$$r = \min_{x\varepsilon\partial M} r(x)$$

where $r(x)$ is the rank function.

For each invariant polynomial $P\varepsilon I^k(O(n))$ with $2k + r > n$, we now introduce a characteristic class $P(M, \mathfrak{F})$ of $(M, \mathfrak{F})$ which is represented by cocycles in the De Rham complex $\Lambda(M)$, i.e., we will assign to each $(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell$ a $(2k + 1 - \ell)$-form $\tilde{P}_{(\alpha_1,\ldots,\alpha_\ell)}(g)$ on $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ such that the cochain

(8.2)
$$\tilde{P}(g) = \sum_{l=1}^{2k+1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \tilde{P}_{(\alpha_1,\ldots,\alpha_\ell)}(g)$$

is d-closed and the associated cohomology class is independent of the invariant metric g, which depends therefore only on P, M, and $\mathfrak{F}$.

Let g be a Riemannian metric for M which is invariant for $\mathcal{F}$ in a neighborhood of ∂M. Let Ω be the Riemannian curvature forms for (M,g). Set

(8.3) $$\tilde{P}_{(\alpha_1)}(g) = P(\Omega)|_{\overline{M}_{(\alpha_1)}} \quad \text{for all } \alpha_1 \varepsilon A^1$$

(8.4) $$\tilde{P}_{(\alpha_1,\ldots,\alpha_{\ell+1})}(g) = \begin{cases} 0 & \text{for } \alpha_{\ell+1} \neq \infty \\ (-1)^{\ell+1} P_{(\alpha_1,\ldots,\alpha_\ell)}|_{\overline{M}_{(\alpha_1,\ldots,\alpha_{\ell+1})}} & \text{for } \alpha_{\ell+1} = \infty \end{cases}$$

for all $(\alpha_1,\ldots,\alpha_{\ell+1})\varepsilon A^{\ell+1}$, $\ell > 1$, where $P_{(\alpha_1,\ldots,\alpha_\ell)}$ is the $(2k - \ell)$-form defined in the previous section.

Theorem 8.1. The co-Chain

(8.5) $$\tilde{P}(g) = \sum_{\ell=1}^{2k+1} \sum_{(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell} \tilde{P}_{(\alpha_1,\ldots,\alpha_\ell)}(g)\varepsilon\Lambda^{2k}(M)$$

is d-closed. The cohomology class $P(M,\mathcal{F}) = [\tilde{P}(g)]\varepsilon H^{2k}(\Lambda(M),\mathrm{d})$ represented by $\tilde{P}(g)$ is independent of the invariant metric g.

Proof. By definition, for all $(\alpha_1,\ldots,\alpha_{\ell+1})\varepsilon A^{\ell+1}$,

$$\left(\mathrm{d}\tilde{P}(g)\right)_{(\alpha_1,\ldots,\alpha_{\ell+1})} = (-1)^\ell \mathrm{d}\tilde{P}_{(\alpha_1,\ldots,\alpha_{\ell+1})}(g)$$

$$+ \sum_{i=1}^{\ell+1} (-1)^{i+1} \tilde{P}_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}(g)$$

$$
= \begin{cases}
dP(\Omega) & \text{for } \ell = 0 \\[2ex]
0 & \text{for } \alpha_{\ell+1} \neq \infty, \ \ell > 0 \\[2ex]
(-1)^{2\ell+1} dP_{(\alpha_1,\ldots,\alpha_\ell)} + \sum_{i=1}^{\ell} (-1)^{i+1+\ell} P_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)} & \\
\qquad\qquad \text{for } \alpha_{\ell+1} = \infty, \ \ell > 1 \\[2ex]
-dP_{(\alpha_1)} - P(\Omega) & \text{for } \ell = 1, \ \alpha_2 = \infty
\end{cases}
$$

$$= 0$$

Since $dP(\Omega) = 0$

$$(-1)^{2\ell+1} dP_{(\alpha_1,\ldots,\alpha_\ell)} + \sum_{i=1}^{\ell} (-1)^{i+1+\ell} P_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)}$$

$$= (-1)^{\ell} \sum_{\sigma \in S(\ell)} \operatorname{sgn}(\sigma) P_{(\sigma(1),\ldots,\sigma(\ell))}\Big|_{\overline{M}_{(\alpha_1,\ldots,\alpha_{\ell+1})}} = 0$$

by the assumption that $2k + r \geqslant n$.

To prove the second part of the theorem, we construct a transgression cochain $TP(g_o,g_1) \in \Lambda^{2k-1}(M)$ such that

(8.6) $$\tilde{P}(g_1) - \tilde{P}(g_o) = dTP(g_o,g_1)$$

Since both g_o and g_1 are invariant metrics for $\tilde{\mathcal{F}}$ in a neighborhood of ∂M, so is their linear interpolation

(8.7) $$g_s = (1-s)g_o + sg_1$$

Let $(\alpha_1,\ldots,\alpha_\ell)\varepsilon A^\ell$, $\alpha_\ell \varepsilon A^1$. Let $\nabla_{s,o} = D_s$ be the Riemannian connection of the metric g_s, $\nabla_{s,i} = \nabla^b_{s,i}$ the basic connection defined by g_s and $\mathcal{F}_{(1,\ldots,i)}$, $i = \overline{1,\ell}$. Put

(8.8) $$\nabla_{(s,t)} = \sum_{i=0}^{\ell} t_i \nabla_{s,i}$$

for $(s,t)\varepsilon I \times \Delta_\ell = [0,1] \times \Delta_\ell$. We note that

(8.9) $$\partial(I \times \Delta_\ell) = \partial I \times \Delta_\ell - I \times \partial\nabla_\ell$$

Let Ω_o, Ω_1 be the Riemannian curvature forms of the metrics g_o, g_1, respectively. Let $\omega_{s,t}$, $\Omega_{s,t}$ be the local connection and curvature forms of $\nabla_{(s,t)}$, respectively d' the exterior differentiation on $I \times \Delta_\ell$. Put

(8.10) $$\theta_{(0,1,\ldots,\ell)} = \binom{k}{\ell+1} \int_{I \times \Delta_\ell} P((d'\omega_{s,t})^{\ell+1}, \Omega_{s,t}^{k-\ell-1})$$

Equation (6.14) in corollary 6.3 shows that

(8.11) $$(-1)^{\ell+1} d\theta_{(0,1,\ldots,\ell)} + \sum_{i=0}^{\ell} (-1)^{i+1} \theta_{(0,\ldots,\hat{i},\ldots,\ell)}$$

$$+ \binom{k}{\ell} \int_{\Delta_\ell} P((d'\omega_{1,t})^\ell, \Omega_1^{k-\ell}) - \binom{k}{\ell} \int_{\Delta_\ell} P((d'\omega_{o,t})^\ell, \Omega_{o,t}^{k-\ell}) = 0$$

where $\theta_{(0,1,\dots,\hat{i},\dots,\ell)} = \binom{k}{\ell} \int_{I\times\Delta_\ell(\hat{i})} P((d'\omega_{s,t})^\ell, \Omega^{k-\ell}_{s,t})$

Similarly, we define $\theta_{(0,\sigma_{(1)},\dots,\sigma_{(\ell)})}$ by exactly the same procedure with $\nabla_{s,\sigma(i)} = \nabla^b_{s,\sigma(i)}$, the basic connection determined by g_s and $\mathfrak{f}_{(\sigma_{(1)},\dots,\sigma_{(i)})}$, and

(8.12) $$\nabla^\sigma_{(s,t)} = \sum_{i=1}^{\ell} t_i \nabla_{s,\sigma(i)} + t_o \nabla_{s,o}$$

for each $\sigma\varepsilon S(\ell)$. Take the alternating sum over $S(\ell)$, we have

(8.13) $$(-1)^{\ell+1} d \sum_{\sigma\varepsilon S(\ell)} \mathrm{sgn}(\sigma)\theta_{(0,\sigma(1),\dots,\sigma(\ell))}$$
$$+ \sum_{i=0}^{\ell} (-1)^{i+1} \sum_{\sigma\varepsilon S(\ell)} \mathrm{sgn}(\sigma)\theta_{(0,\sigma(1),\dots\hat{\sigma}(i),\dots,\sigma(\ell))}$$
$$+ P_{(\alpha_1,\dots,\alpha_\ell)}(g_1) - P_{(\alpha_1,\dots,\alpha_\ell)}(g_o) = 0$$

Set

(8.14) $$\theta_{(\alpha_1,\dots,\alpha_\ell)} = (-1)^{\ell+1} \sum_{\sigma\varepsilon S(\ell)} \mathrm{sgn}(\sigma)\theta_{(0,\sigma(1),\dots,\sigma(\ell))}$$

Let $\theta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots\alpha_\ell)}$ be the corresponding form constructed from the pure F-structures

$\wp_{\alpha_1}, \dots, \wp_{\alpha_{i-1}}, \wp_{\alpha_{i+1}}, \dots, \wp_{\alpha_\ell}$

Thus

(8.15) $$\sum_{\sigma \varepsilon S(\ell)} \operatorname{sgn}(\sigma)\Theta_{(0,\sigma(1),\dots,\sigma(\ell-1))} = \sum_{i=1}^{\ell} (-1)^i \Theta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_\ell)}$$

Lemma 8.2.

(8.16) $$\sum_{\sigma \varepsilon S(\ell)} \operatorname{sgn}(\sigma)\Theta_{(0,\sigma(1),\dots,\hat{\sigma}(i),\dots,\sigma(\ell))} = 0 \;, \quad 1 \leqslant i < \ell$$

(8.17) $$d\Theta_{(\alpha_1,\dots,\alpha_\ell)} + P_{(\alpha_1,\dots,\alpha_\ell)}(g_1) - P_{(\alpha_1,\dots,\alpha_\ell)}(g_0) + \sum_{i=1}^{\ell} (-1)^{\ell+i+1} \Theta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots\alpha_\ell)} = 0$$

Proof. See the argument in section 7.

We now define the transgression cochain $TP(g_o,g_1)$, set

(8.18) $$TP_{(\alpha)} = \binom{k}{1} \int_I P(d'\omega_s, \Omega_s^{k-1})\Big|_{\overline{M}_{(\alpha)}} = \Theta_{(o)}\Big|_{\overline{M}_{(\alpha)}} \;, \quad \alpha \varepsilon A^1$$

(8.19) $$TP_{(\alpha_1,\dots,\alpha_{\ell+1})} = \begin{cases} 0 & , \quad \alpha_{\ell+1} \neq \infty \\ \Theta_{(\alpha_1,\dots,\alpha_\ell)}\Big|_{\overline{M}_{(\alpha_1,\dots,\alpha_{\ell+1})}} & , \quad \alpha_{\ell+1} = \infty \end{cases}$$

for $\ell \geqslant 1$, $(\alpha_1,\dots,\alpha_{\ell+1}) \varepsilon A^{\ell+1}$.

Note that by the basicness and assumption on the rank function,

$$(8.20) \qquad \sum_{\sigma\in S(\ell)} \operatorname{sgn}(\sigma)\theta_{(\sigma(1),\ldots,\sigma(\ell))}\Big|_{\overline{M}_{(\alpha_1,\ldots,\alpha_\ell,\infty)}} = 0$$

It follows from Lemma 8.2 and the equation (8.20) that

$$\left(dTP(g_o,g_1)\right)_{(\alpha_1,\ldots,\alpha_{\ell+1})}$$

$$= (-1)^\ell dTP_{(\alpha_1,\ldots,\alpha_{\ell+1})} + \sum_{i=1}^{\ell+1}(-1)^{i+1}TP_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_{\ell+1})}$$

$$= \begin{cases} dTP_{(\alpha_1)} & \ell = 0\ ,\quad \alpha_1\varepsilon A^1 \\ -d\theta_{(\alpha_1)} - TP_{(\alpha_1)} & \ell = 1\ ,\quad \alpha_{\ell+1} = \infty \\ 0 & \ell > 1\ ,\quad \alpha_{\ell+1} \neq \infty \\ (-1)^\ell d\theta_{(\alpha_1,\ldots,\alpha_\ell)} + \sum_{i=1}^{\ell}(-1)^{i+1}\theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_\ell)}\ , & \\ \qquad \ell > 1\ ,\ \alpha_{\ell+1} = \infty & \end{cases}$$

$$= \tilde{P}_{(\alpha_1,\ldots,\alpha_{\ell+1})}(g_1) - \tilde{P}_{(\alpha_1,\ldots,\alpha_{\ell+1})}(g_o)$$

for all $(\alpha_1,\ldots,\alpha_{\ell+1})\varepsilon A^{\ell+1}$ since

$$dTP_{(\alpha_1)} = (P(\Omega_1) - P(\Omega_o))\Big|_{\overline{M}_{(\alpha_1)}} = \tilde{P}_{(\alpha_1)}(g_1) - \tilde{P}_{(\alpha_1)}(g_o)$$

$$-d\Theta_{(\alpha_1)} - TP_{(\alpha_1)} = P_{(\alpha_1)}(g_1) - P_{(\alpha_1)}(g_o) = \tilde{P}_{(\alpha_1,\infty)}(g_1) - \tilde{P}_{(\alpha_1,\infty)}(g_o)$$

$$(-1)^{\ell} d\Theta_{(\alpha_1,\ldots,\alpha_\ell)} + \sum_{i=1}^{\ell} (-1)^{i+1} \Theta_{(\alpha_1,\ldots,\hat{\alpha}_i,\ldots\alpha_\ell)}$$

$$= (-1)^{\ell+1} P_{(\alpha_1,\ldots,\alpha_\ell)}(g_1) - (-1)^{\ell+1} P_{(\alpha_1,\ldots,\alpha_\ell)}(g_o)$$

$$= \tilde{P}_{(\alpha_1,\ldots,\alpha_\ell,\infty)}(g_1) - \tilde{P}_{(\alpha_1,\ldots,\alpha_\ell,\infty)}(g_o)$$

Thus $\tilde{P}(g_1) - \tilde{P}(g_o) = dTP(g_o, g_1)$.

This completes the proof of theorem 8.1

Remark: The co chain $\tilde{P}(g)$ is simply the form $P(\Omega)$ modified by some canonical boundary terms which are determined by the metric g , the F-structure $\mathfrak{F}$, and P .

If $\mathfrak{F} = \{(U_\alpha, \mathfrak{F}_\alpha)\}_{\alpha\in A}$ is nonsingular on all of M , then we have the following vanishing theorem.

Corollary 8.3. The cohomology class $P(M, \mathfrak{F})$ is trivial for all $P \in I^k(O(n))$, $2k > n - r$, if $\mathfrak{F}$ is nonsingular on M and $r(x) \geq r > 0$ for all $x \in M$.

Proof. We construct again by the transgression formula and basicness a co chain $\eta \in \Lambda^{2k-1}(M)$ such that

(8.21) $$\tilde{P}(g) + d\eta = 0 \,.$$

For each $(\alpha_1,\dots,\alpha_\ell)\in A^\ell$, set

(8.22) $$\eta_{(\alpha_1,\dots,\alpha_\ell)} = \begin{cases} P_{(\alpha_1,\dots,\alpha_\ell)}\big|_{\overline{M}_{(\alpha_1,\dots,\alpha_\ell)}} & , \quad \alpha_\ell \neq \infty \\ 0 & , \quad \alpha_\ell = \infty \end{cases}$$

(8.23) $$\eta = \bigoplus_{\ell=1}^{2k} \bigoplus_{(\alpha_1,\dots,\alpha_\ell)\in A^\ell} \eta_{(\alpha_1,\dots,\alpha_\ell)}$$

then

$$(\tilde{P}(g) + d\eta)_{(\alpha_1,\dots,\alpha_\ell)}$$

$$= \begin{cases} P(\Omega)\big|_{\overline{M}_{(\alpha_1)}} + dP_{(\alpha_1)} & , \quad \ell = 1 \\ (-1)^{\ell+1} dP_{(\alpha_1,\dots,\alpha_\ell)} + \sum\limits_{i=1}^{\ell} (-1)^{i+1} P_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_\ell)} \end{cases}$$

$$= 0 \qquad \text{for} \quad (\alpha_1,\dots,\alpha_\ell)\in A^\ell \,, \quad \alpha_\ell \neq \infty \,.$$

$$(\tilde{P}(g) + d\eta)_{(\alpha_1,\dots,\alpha_\ell,\infty)}$$

$$= (-1)^{\ell+1} P_{(\alpha_1,\dots,\alpha_\ell)} + (-1)^{\ell} d\eta_{(\alpha_1,\dots,\alpha_\ell,\infty)}$$

$$+ \sum_{i=1}^{\ell} (-1)^{i+1} \eta_{(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_\ell,\infty)} + (-1)^{\ell} \eta_{(\alpha_1,\dots,\alpha_\ell)}$$

$= 0 \qquad$ for $(\alpha_1, \ldots, \alpha_\ell, \infty) \varepsilon A^{\ell+1}$.

It follows that $P(M, \mathfrak{F})$ is trivial for all $P \varepsilon I^k(O(n))$ as long as $2k > n - r$.

When M^n is a closed manifold, the cohomology theory of the De Rham complex $\Lambda(M)$ and the usual De Rham complex are canonically isomorphic.

Corollary 8.4. Let M^n be a closed orientable manifold which admits a nonsingular F-structure of rank $r(x) \geqslant r > 0$ for all $x \varepsilon M^n$, then, the Euler class of M^n vanishes, the real Pontrjagin classes of M^n , $\mathrm{Pont}(M) \varepsilon H^{2k}(M,R)$, vanishes for $2k > n - r$.

However, if $\mathfrak{F}$ is only nonsingular in a neighborhood of the boundary ∂M , η is defined only away from the singularities $Z(\mathfrak{F})$ of $\mathfrak{F}$. If

$$(8.24) \qquad Z(\mathfrak{P}_{(\alpha_1, \ldots, \alpha_\ell)}) \cap \overline{M}_{(\alpha_1, \ldots, \alpha_\ell)}$$

$(\alpha_1, \ldots, \alpha_\ell) \varepsilon A^\ell$, are all closed submanifolds of M, we have the following residue theorem which is a direct conclusion of the residue computations in [2], [4] and corollary 8.3.

Corollary 8.5. Assume that $\dim M^n = n = 2k$,

$P \varepsilon I^k(O(2k))$, $\mathfrak{F} = \{U_\alpha , \phi_\alpha)\}_{\alpha \varepsilon A}$ is an F-structure which is nonsingular in a neighborhood of the boundary ∂M . Assume that $\{Z(\phi_{(\alpha_1,\ldots,\alpha_\ell)}) \cap \overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}\}$ are all closed submanifolds of M . Then, the characteristic number

$$P[M,\mathfrak{F}] = \int \tilde{P}(g) = - \sum_{Z} \text{Res } (\eta, Z) \tag{8.25}$$

where the sum is over all connected components of the set of closed submanifolds $\{Z(\mathfrak{F}) \cap M_{(\alpha_1,\ldots,\alpha_\ell)}\}$.

Remark: The residue theorem remains true even for more complicated structure of singularities. However, we will only carry out the residue computations for isolated second order singularities in section 13.

§9. Characteristic Numbers and Secondary Characteristic Numbers Associated to a Nonsingular F-structure

Let N be a closed oriented manifold of dimension $2k-1$, which is the boundary of a $2k$ dimensional compact oriented manifold M, i.e., $N = \partial M$. Let $\mathfrak{F} = \{(U_\alpha, \mathfrak{P}_\alpha)\}_{\alpha\varepsilon A}$ be a nonsingular F-structure on N. $\mathfrak{F}$ thus defines a nonsingular F-structure in a neighborhood of $N = \partial M$. For each $P \varepsilon I_o^k(O(2k))$, the secondary characteristic number associated to $\mathfrak{F}$ is defined by the following formula

$$(9.1) \qquad SP(N, \mathfrak{F}) \equiv \frac{1}{(2\Pi)^k} P[M, \mathfrak{F}] \qquad \mod Z$$

$$\equiv \frac{1}{(2\Pi)^k} \int \tilde{P}(g) \qquad \mod Z$$

where $P[M, \mathfrak{F}]$ is defined by equation (8.25).

This modulo Z number $SP(N, \mathfrak{F})$ is independent of M and dependent only on N and $\mathfrak{F}$, since for a closed manifold Y, its characteristic number associated to P is an integer, i.e. $P[Y] \varepsilon Z$. In general, $SP(N, \mathfrak{F})$ can arrive at any modulo Z number. However, if $\mathfrak{F}$ is a nonsingular F-structure with all of its orbits closed submanifolds, then for all $P \varepsilon I_o^k(O(2k))$, $SP(N, \mathfrak{F})$ is rational.

Corollary 9.1. Assume that M is a filling of N, i.e., $\partial M = N$, suppose that $\mathfrak{F}$ extends to an F-structure of M

(possibly with singularities). Then

(9.2) $$SP(N, \mathfrak{F}) \equiv \frac{-1}{(2\pi)^k} \operatorname{Res}(M, \mathfrak{F}, \eta) \mod Z$$

where η is the cochain such that

(9.3) $$\tilde{P}(g) + d\eta = 0$$

which lives away from the singular set of $\mathfrak{F}$, $\operatorname{Res}(M, \mathfrak{F}, \eta)$ is the residues of the cochain η at the singular set.

§10. Collapsing Riemannian Manifolds and Secondary Characteristic Numbers.

The fact that the secondary characteristic number $SP(N, \mathfrak{F})$ is a topological invariant of $(N, \mathfrak{F})$ is first revealed by collapsing an invariant riemannian metric along the nonsingular F-structure $\mathfrak{F}$. In [~~CG~~ 7], Cheeger and Gromov proved the following:

Proposition 10.1

a). If M^n admits a nonsingular F-structure $\mathfrak{F}$, on the complement of a compact subset C, then M^n admits a complete invariant metric g, with $|K_g| \leq 1$ and $Vol(M^n, g) < \infty$.

b). If C is empty, M^n admits a family g_s of such complete invariant metrics, with $|K_{g_s}| \leq 1$ and $\lim_{s\to\infty} Vol(M^n, g_s) = 0$.

Proof. see [~~CG~~ 7].

Let M^n be an oriented compact manifold with boundary $N^{n-1} = \partial M^n$. Let $\mathfrak{F}$ be a nonsingular F-structure on N. Put

$$M^n_\infty = M^n \cup (N \times (0, +\infty)) \tag{10.1}$$

$$M^n_s = M^n \cup (N \times (0, s)) \tag{10.2}$$

M^n_∞ satisfies the conditions in Proposition 10.1.a). Let g be a complete invariant metric on M^n_∞ with $|K_g| \leq 1$ and $\mathrm{Vol}(M^n_\infty, g) < +\infty$. For all $P \in I^k_0(0(2k))$, $n = 2k$, the curvature integral

$$(10.3) \qquad \int_{(M^n_\infty, g)} P(\Omega)$$

is well-defined. The secondary characteristic number $SP(N, \mathfrak{F})$ is originally defined by the following formula

$$(10.4) \qquad SP(N, \mathfrak{F}) \equiv \frac{1}{(2\pi)^k} \int_{(M^n_\infty, g)} P(\Omega) \qquad \mathrm{mod}\ Z$$

We claim that the two definitions by equations (9.1) and (10.4) for the secondary characteristic numbers are the same, i.e.,

$$(10.5) \qquad \frac{1}{(2\pi)^k} \left\{ \int_{(M^n_\infty, g)} P(\Omega) - \int_{(M^n_s, g)} \tilde{P}(g) \right\} = 0 \qquad \mathrm{mod}\ Z$$

Fix a stratification $\{N_i\}$ for N which is compatible with $\mathfrak{F}$. Let g_s be the induced invariant metric on $N = \partial M_s$. Choose a stratification $\{M_s, (\alpha_1, \ldots, \alpha_\ell)\}_{(\alpha_1, \ldots, \alpha_\ell) \in A^\ell}$, $\ell = \overline{1, n+1}$ for M_s, which is compatible with $\mathfrak{F}$ and the restriction onto the boundary $N = \partial M_s$ is $\{N_i\}$. By the definition of the co-chain $\tilde{P}(g)$, see equatons (8.3) and (8.4), it is clear that

$$(10.6)\quad \int_{M_s^n} \tilde{P}_s(g) = \sum_{(\alpha_1,\dots,\alpha_\ell)} \int_{\overline{M}_{s,(\alpha_1,\dots,\alpha_\ell)}} \tilde{P}_{s,(\alpha_1,\dots,\alpha_\ell)}(g)$$

$$= \int_{(M_s^n,g)} P(\Omega) + \sum_{\ell=1}^{n} \sum_{(\alpha_1,\dots,\alpha_\ell,\infty)\varepsilon A^{\ell+1}} \int_{\overline{M}_{s,(\alpha_1,\dots,\alpha_\ell,\infty)}} \tilde{P}_{s,(\alpha_1,\dots,\alpha_\ell,\infty)}(g)$$

Note $\overline{M}_{s,(\alpha_1,\dots,\alpha_\ell,\infty)} \subset \partial M_s = N$. Equation (10.5) exactly means that

$$(10.7)\quad \lim_{s\to+\infty} \sum_{\ell=1}^{n} \sum_{(\alpha_1,\dots,\alpha_\ell,\infty)\varepsilon A^{\ell}} \int_{\overline{M}_{s,(\alpha_1,\dots,\alpha_\ell,\infty)}} \tilde{P}_{s,(\alpha_1,\dots,\alpha_\ell,\infty)}(g) = 0$$

i.e., as the metric on the boundary $N = \partial M_S^n$ collapses, the contributions from the boundary terms in $\tilde{P}_s(g)$ become negligible. A precise proof of this fact is complicated due to the complicated construction of a collapsing and is not essential. However, if the F-structure on N is a nonsingular pure F-structure $\mathfrak{F}$ of rank $r > 0$, the collapsing is especially simple. The following is a detailed computation which shows that equation (10.7) is true for a nonsingular pure F-structure.

So let $\mathfrak{F}$ be a nonsingular pure F-structure of rank r on N^{n-1} , let $\overline{g}$ be an invariant metric. There is a natural decomposition

$$(10.8)\quad \overline{g} = h + v$$

where h vanishes on vectors tangent to the orbits of $\mathfrak{P}$ and v vanishes on vectors normal to the orbits. Let $\tilde{g}$ be a reimannian metric on M_∞^n such that the restriction on $N\times(0,+\infty)$ is

$$(10.9) \qquad e^{-2s}v + h + ds^2$$

g is thus invariant on $N\times(0,+\infty)$.

Note that since $\mathfrak{P}$ is pure, the stratification $\{\mathring{M}_s, \partial M_s\}$ is a stratification of M_s compatible with $\mathfrak{P}$. By our convention, we label

$$(10.10) \qquad M_{s,(0)} = \mathring{M}_s$$

$$(10.11) \qquad M_{s,(0,\infty)} = \partial M_s = N$$

thus

$$(10.12) \qquad \tilde{P}_s(g) = \tilde{P}_{s,(0)}(g) + \tilde{P}_{s,(0,\infty)}(g)$$

by definition, one has

$$(10.13) \qquad \tilde{P}_{s,(0)}(g) = P(\Omega)|_{\bar{M}_{s,(0)}} = P(\Omega)|_{M_s}$$

$$(10.14) \qquad \tilde{P}_{s,(0,\infty)}(g) = k\int_0^1 P(d'\omega_t, \Omega_t^{k-1})|_{\partial M_s}$$

where ω_t, Ω_t are the local connection and curvature forms of

∇_t respectively.

(10.15) $$\nabla_t = (1-t)D + t\nabla^b , \qquad t \in [0,1]$$

D is the Riemannian connection of g and ∇^b the canonical basic connection of $\mathfrak{F}$ associated to g. We shall show that

(10.16) $$\lim_{s\to\infty} \int_{\partial M_s} \tilde{P}_{s,(0,\infty)}(g) = 0 .$$

For any point $p \in N$, let $\{X_i\}_{i=1}^{r}$ be a local basis of $\mathfrak{F}$ in a neighborhood $U \subset N$ of P. Choose a local coordinate $(x_1,\ldots,x_{n-1})$ with $(0,\ldots,0)$ as the coordinate of p, such that $\frac{\partial}{\partial x_i} = X_i$, $i = \overline{1,r}$. Choose ε_i small enough, we may assume that $U = B_{\varepsilon_1}(0) \times B_{\varepsilon_2}(0)$, where $B_{\varepsilon_1}(0) = \{x\in R^r \mid \sum x_i^2 < \varepsilon_1\}$, $B_{\varepsilon_2}(0) = \{y \in R^{n-r-1} \mid \sum y_i^2 < \varepsilon_2 , \ y_i = x_{i+r} , i = \overline{1,n-r-1}\}$. The coordinate tangent vector fields $\frac{\partial}{\partial x_i}$ is naturally decomposed as a sum of a horizontal component and a vertical component.

(10.17) $$\frac{\partial}{\partial x_i} = Y_i + X_i , \quad i = \overline{1,n-1}$$

where Y_i is the horizontal component, i.e., orthogonal to the orbits of $\mathfrak{F}$, and X_i is the vertical component which is tangent to the orbits of $\mathfrak{F}$. Thus

(10.18) $$X_i = \frac{\partial}{\partial x_i} , \qquad i = \overline{1,r}$$

(10.19) $$X_i = \sum_{j=1}^{r} a_{ji}(y) \frac{\partial}{\partial x_j} , \qquad i = \overline{r+1,n-1}$$

(10.20) $$Y_i = 0 , \; i = \overline{1,r}$$

(10.21) $$\bar{g}(X_i,Y_j) = 0 , \qquad i,j = \overline{1,n-1}$$

(10.22) $$[X_i,Y_j] = 0 , \qquad i = \overline{1,r} \qquad [X_i,X_j] = 0$$

$X_1,\ldots,X_r,Y_{r+1},\ldots,Y_{n-1}$, forms a local invariant basis on U . The matrices

(10.22) $$A(y) = (A_{ij}(y)) = \left(\bar{g}\left(\frac{\partial}{\partial x_i} , \frac{\partial}{\partial x_j} \right)\right)_{i,j=\overline{1,r}}$$

(10.23) $$B(y) = (B_{ij}(y)) = (\bar{g}(Y_i,Y_j))_{i,j=\overline{r+1,n-1}}$$

are both independent of $(x_1,\ldots,x_r)$.

Let $\pi_1,\ldots,\pi_{n-1}$ be the dual basis of $X_1,\ldots,X_r,Y_{r+1},\ldots,Y_{n-1}$. π_i 's are invariant smooth 1-forms on U . $\pi_i = dy_{i-r}$ for $i > r$. Consider the basis $e^s X_1,\ldots,e^s X_r,Y_{r+1},\ldots,Y_n, \frac{\partial}{\partial s}$ on $U\times(0,+\infty)$, whose dual basis is $e^{-s}\pi_1,\ldots,e^{-s}\pi_r,\pi_{r+1},\ldots,\pi_{n-1},ds$. The matrix of g in this basis is

(10.25) $$(g(y)) = \begin{pmatrix} A(y) & 0 & 0 \\ 0 & B(y) & 0 \\ 0 & 0 & 1 \end{pmatrix} = (g_{ij}(y))$$

which depends only on y .

Theorem 10.1. The matrices $(g(y))$, $(g(y))^{-1}$ and the local connection forms ω, curvature forms Ω in the local basis $e^s X_1, \ldots, e^s X_r, Y_{r+1}, \ldots, Y_{n-1}, \frac{\partial}{\partial s}$ on $U \times (0,\infty)$ are uniformly bounded. Hence, the sectional curvature K_g over M_∞ is uniformly bounded. The volume of M_∞ is finite.

Proof. The volume form is given by

$$\sqrt{\det(g(y))}\,(e^{-s}\pi_1) \wedge (e^{-s}\pi_2) \wedge \cdots \wedge (e^{-s}\pi_r) \wedge \pi_{r+1} \wedge \cdots \wedge \pi_{n-1} \wedge ds$$
$$= e^{-rs}\sqrt{\det(g(y))}\,\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r \wedge \cdots \wedge \pi_{n-1} \wedge ds\,.$$

$(g(y))$ uniformly bounded and N is closed imply $\mathrm{Vol}(M_\infty, g) < +\infty$. Also the uniform boundedness of $(g(y))$, $(g(y))^{-1}$, and Ω imply that the sectional curvature K_g is uniformly bounded. It is obvious that $(g(y))$ and $(g(y))^{-1}$ are uniformly bounded since they depend only on y. In view of the structural equation

$$\Omega = d\omega + \omega \wedge \omega \tag{10.26}$$

we need only to verify that ω and $d\omega$ are both uniformly bounded.

For convenience, set

$$\{e_1, \ldots, e_n\} = \{e^s X_1, \ldots, e^s X_r, Y_{r+1}, \ldots, Y_{n-1}, \frac{\partial}{\partial s}\}$$
$$\{e_1^*, \ldots, e_n^*\} = \{e^{-s}\pi_1, \ldots, e^{-s}\pi_r, \ldots, \pi_{r+1}, \ldots, \pi_{n-1}, ds\}\,.$$

Clearly, $\{e_i^*\}$, $\{de_i^*\}$ are all uniformly bounded on $U \times (0,+\infty)$. Set

$$\omega = (\omega_j^i) \tag{10.27}$$

then

(10.28) $$D\{e_1,\ldots,e_n\} = \{e_1,\ldots,e_n\}\,(\omega^i_j)$$

(10.29) $$D_{e_i} e_j = \sum_{\ell=1}^{n} \omega^{\ell}_j(e_i)e_{\ell}$$

(10.30) $$\omega^{\ell}_j(e_i) = \sum_{m=1}^{n} g^{\ell m} g(D_{e_i} e_j, e_m)$$

where $$(g^{\ell m}) = (g(y))^{-1}$$

(10.31) $$\omega^{\ell}_j = \sum_{i=1}^{n} \omega^{\ell}_j(e_i)e^*_i$$

From equations (10.29) and (10.30), it is enough to verify that both $g(D_{e_i} e_j, e_m)$ and $dg(D_{e_i} e_j, e_m)$ are uniformly bounded since $g^{\ell m}$ depends only on y and e^*_i, de^*_i are uniformly bounded. However, since $(g(y)) = (g(e_i,e_j)) = (g_{ij}(y))$ depends only on y, we have

(10.32) $$e_{\ell} g_{ij}(y) = 0 \qquad \text{for } \ell = \overline{1,r},\ \ell = n .$$

$e_{\ell} g_{ij}(y)$, $\ell = \overline{r+1,n-1}$, depend only on y. Moreover,

(10.33) $$[e_n, e_i] = e_i , \qquad i = \overline{1,r}$$

(10.34) $$[e_n, e_i] = 0 , \qquad i = \overline{r+1,n}$$

(10.35) $$[e_i,e_j] = 0 \ , \qquad i = \overline{1,r},\ j = \overline{1,n}$$

(10.36) $$[e_i,e_j] \text{ depends only on } y \text{ if } i,j > r.$$

(10.37) $$g([e_i,e_j],e_n) = 0 \ .$$

(10.38) $$g(D_{e_i}e_j,e_m) = \frac{1}{2}\{e_i g(e_j,e_m)+e_j g(e_i,e_m)$$

$$- e_m g(e_i,e_j)+g([e_i,e_j],e_m)-g([e_i,e_m],e_j)+g([e_m,e_j],e_i)\}$$

To estimate $g(D_{e_i}e_j,e_m)$, we divide it into five cases.

Case 1. $e_n = \frac{\partial}{\partial r} \in \{e_i,e_j,e_m\}$.
Equation (10.38) reduces to

(10.39) $$2g(D_{e_i}e_j,e_m) = g([e_i,e_j],e_m)$$

$$-g([e_i,e_m],e_j) + g([e_m,e_j],e_i)$$

Case 1.a. $$e_i = e_n \ .$$

if $j,m \leq r$, by equations (10.33), (10.37)
$2g(D_{e_n}e_j,e_m) = g(e_j,e_m) - g(e_m,e_j) + g([e_m,e_j],e_n) = 0$.
If $j,m > r$, by equations (10.34) and (10.37),
$2g(D_{e_n}e_j,e_m) = 0$. If $j \leq r < m$ or $m \leq r < j$, by equations (10.21), (10.33), (10.34), and (10.37), $2g(D_{e_n}e_j,e_m) = 0$.

Case 1.b. $e_j = e_n$.

if $i,m \leqslant r$, by equations (10.33) and (10.37),
$g(D_{e_i} e_n, e_m) = -g_{im}(y)$. If $i,m > r$, by equations (10.34) and (10.37), $g(D_{e_i} e_n, e_m) = 0$. If $i \leqslant r < m$ or $m \leqslant r < i$, by equations (10.33), (10.34), and (10.37),
$g(D_{e_i} e_n, e_m) = -\frac{1}{2} g_{im}(y)$.

Case 2. $i,j,m \leqslant r$.

all three brackets of e_i, e_j, e_m vanish
$g(D_{e_i} e_j, e_m) = \frac{1}{2} \{e_i g_{jm}(y) + e_j g_{im}(y) - e_m g_{ij}(y)\} = 0$

Case 3. $n > i,j,m, > r$

$g(D_{e_i} e_j, e_m)$ depends only on y since everything involved depends only on y .

Case 4. $i \leqslant r < j,m < n$

$g(D_{e_i} e_j, e_m) = g(D_{e_m} e_j, e_i) = -g(D_{e_j} e_m, e_i) = g(D_{e_j} e_i, e_m)$

$= \frac{1}{2} g\ ([e_m, e_j], e_i)$

note $e_m = Y_m = \frac{\partial}{\partial y_{m-r}} - X_m$, $e_j = Y_j = \frac{\partial}{\partial y_{j-r}} - X_j$

$[e_m, e_j] = [\frac{\partial}{\partial y_{j-r}}, X_m] - [\frac{\partial}{\partial y_{m-r}}, X_j]$

$$= \sum_{\ell=1}^{r} \{\frac{\partial}{\partial y_{j-r}} a_{\ell m}(y) - \frac{\partial}{\partial y_{m-r}} a_{\ell j}(y)\} \frac{\partial}{\partial x_\ell}$$

$$= e^{-s} \sum_{\ell=1}^{r} \{ \frac{\partial}{\partial y_{j-r}} \mathfrak{a}_{\ell m}(y) - \frac{\partial}{\partial y_{m-r}} \mathfrak{a}_{\ell i}(y)\} e_\ell$$

thus $g(D_{e_i} e_j, e_m) = g(D_{e_m} e_j, e_i) = -g(D_{e_j} e_m, e_i)$

$$= \frac{1}{2} e^{-s} \sum_{\ell=1}^{r} \{ \frac{\partial}{\partial y_{j-r}} \mathfrak{a}_{\ell m}(y) - \frac{\partial}{\partial y_{m-r}} \mathfrak{a}_{\ell j}(y)\} g_{i\ell}(y)$$

Case 5. $i,j \leqslant r < m < n$.

$$g(D_{e_i} e_j, e_m) = g(D_{e_j} e_i, e_m) = -g(D_{e_m} e_i, e_j) = -g(D_{e_m} e_j, e_i)$$

$$= -g(D_{e_i} e_m, e_j) = -g(D_{e_j} e_m, e_i) = -\frac{1}{2} e_m g_{ij}(y)$$

$$= -\frac{1}{2} \frac{\partial}{\partial y_{m-r}} g_{ij}(y)$$

Thus both $g(D_{e_i} e_j, e_m)$ and $dg(D_{e_i} e_j, e_m)$ are uniformly bounded. It follows that ω and Ω are both uniformly bounded. This completes the proof of theorem 10.1.

Note that by equation (5.15)

$$\nabla^b = D + S \tag{10.40}$$

where S is given by equations (5.14) and (5.16), i.e., (10.41) $$S = \sum_{i=1}^{r} \pi_i S_i = \sum_{i=1}^{r} e^*_i\{e^s S_i\}$$

(10.42) $$S_i = -DX_i$$

thus

(10.43) $$\nabla_t = D + t\mathfrak{S}$$

in terms of local connection forms

(10.44) $$\omega_t = \omega + t\mathfrak{S}$$

(10.45) $$\Omega_t = d\omega_t + \omega_t \wedge \omega_t$$

$$= \Omega + t\{dS + \omega \wedge S + S \wedge \omega\} + t^2 S \wedge S$$

(10.46) $$\tilde{P}_{s,(0,\infty)}(g) = k\int_0^1 d'tP(S,\Omega_t^{k-1})|_{\partial M_s}$$

Let $(S_{i,j}^{\ell})$ be the matrix representation of S_i in the basis $\{e_1,\ldots,e_n\}$, i.e.,

(10.47) $$\mathfrak{S}_i\{e_1,\ldots,e_n\} = \{e_1,\ldots,e_n\}(\mathfrak{S}_{i,j}^{\ell})$$

Equation (10.16) follows if we can verify that

(10.48) $$|\mathfrak{S}_{i,j}^{\ell}| \leq C \cdot e^{-s}$$

(10.49) $$|d\mathfrak{S}_{i,j}^{\ell}| \leq C\cdot e^{-s}$$

where C is a constant independent of s.

Indeed,

(10.50) $$\mathfrak{S}_i(e_j) = -D_{e_j}X_i = [X_i, e_j] - D_{X_i}e_j = -D_{X_i}e_j$$
$$= -e^{-s}D_{e_i}e_j = -e^{-s}\sum_{\ell=1}^{n}\{\sum_{m=1}^{n} g^{\ell m}g(D_{e_i}e_j, e_m)\}e_\ell$$

(10.51) $$\mathfrak{S}_{i,j}^{\ell} = e^{-s}\{-\sum_{m=1}^{n} g^{\ell m}g(D_{e_i}e_j, e_m)\}$$

(10.52) $$d\mathfrak{S}_{i,j}^{\ell} = e^{-s}\{\sum_{m=1}^{n}[g^{\ell m}g(D_{e_i}e_j, e_m) - d(g^{\ell m}g(D_{e_i}e_j, e_m))]\}$$

thus $e^s S_{i,j}^{\ell}$ and $e^s dS_{i,j}^{\ell}$ are unifromly bounded. It follows that there is a constant $C > 0$, such that

(10.53) $$|\tilde{P}_{s,(0,\infty)}(g)|\,|_{\partial M_s} \leq C|e_1^* \wedge e_2^* \wedge \dots \wedge e_{n-1}^*|$$
$$= Ce^{-rs}|\pi_1 \wedge \pi_2 \wedge \dots \wedge \pi_{n-1}|$$

and

(10.54) $$\left|\int_{\partial M_s} \tilde{P}_{s,(0,\infty)}(g)\right| \leq Ce^{-rs}\mathrm{Vol}(N,\bar{g})$$

equation (10.16) follows from (10.54).

Theorem 10.2. Let $\mathfrak{F}$ be a nonsingular F-structure on $N = \partial M^n$. Then

$$(10.55)\qquad \int_{(M^n,g)} \tilde{P}(g) = \int_{(M^n_\infty,g)} P(\Omega)$$

The above argument is a proof for $\mathfrak{F}$ being pure.

When the pure F-structure $\mathfrak{P}$ is of rank 1, a direct computation shows that locally,

$$(10.56)\qquad \tilde{P}_{\mathfrak{s}(o,\infty)}(g) = \pi_1 \sum_{j=1}^{k} \binom{k}{j} P(\mathfrak{S}_1^j, \Omega^{k-j})(d\pi_1)^{j-1} = P_{(0)}|_{\partial M_{\mathfrak{s}}}$$

which is exactly the Bott form constructed in [4].

§11. The Explicit Bott Forms Associated to a Pair of Pure F-Structures of Rank One

Let $\mathfrak{F}_1, \mathfrak{F}_2$ be pure F-structures of rank 1 on U. Let $\mathfrak{F}$ be the rank 2 pure F-structure generated by $\mathfrak{F}_1, \mathfrak{F}_2$. Let g be an invariant Riemannian structure for $(U, \mathfrak{F})$. We compute explicitly the Bott form $P_{(1,2)}$ in local coordinate system.

Let X_1, X_2 be local Killing vector fields which represent $\mathfrak{F}_1, \mathfrak{F}_2$ locally, respectively. $[X_1, X_2] = 0$. For simplicity, we use $\langle\cdot,\cdot\rangle$ for the invariant metric g. Let π_i be the canonical dual of X_i associated to the metric g defined by

(11.1) $$\pi_i(Y) = \frac{\langle X_i, Y\rangle}{\langle X_i, X_i\rangle}, \qquad i = \overline{1,2}.$$

Put

(11.2) $$g_{ij} = \langle X_i, X_j\rangle, \quad i,j \leq 2$$

(11.3) $$\Delta = 1 - \frac{g_{12}^2}{g_{11}g_{22}}$$

(11.4) $$\pi_1^{12} = \Delta^{-1}\{\pi_1 - \frac{g_{12}}{g_{11}}\pi_2\}$$

(11.5) $$\pi_2^{12} = \Delta^{-1}\{\pi_2 - \frac{g_{12}}{g_{22}}\pi_1\}$$

$\{\pi_1^{12}, \pi_2^{12}\}$ is the canonical dual of $\{X_1, X_2\}$ associated to the invariant metric g, i.e., (11.6) $$\pi_i^{12}(X_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}$$

(11.7) $$\pi_i^{12}(Y) = 0 \quad \text{if } \langle X_j, Y\rangle = 0,\ j = \overline{1,2}\ .$$

Solve for π_1, π_2 from equations (11.4) and (11.5)

(11.8) $$\pi_1 = \pi_1^{12} + \frac{g_{12}}{g_{11}} \pi_2^{12}$$

(11.9) $$\pi_2 = \frac{g_{12}}{g_{22}} \pi_1^{12} + \pi_2^{12}$$

(11.10) $$\pi_1 \wedge \pi_2 = \Delta \pi_1^{12} \wedge \pi_2^{12}\ .$$

Let ∇_1^b, ∇_2^b be the canonical basic connection associated to $\mathfrak{P}_1, \mathfrak{P}_2$, respectively, $\nabla_{(1,2)}^b$ the canonical basic connection associated to $\mathfrak{P}$. Set

(11.11) $$A(1) = \nabla_1^b - D$$

(11.12) $$A(2) = \nabla_2^b - D$$

(11.13) $$A(1,2) = \nabla_{(1,2)}^b - D$$

by equation (5.16),

(11.14) $$A(i) = \pi_i S_i\ ,\ i = \overline{1,2}$$

(11.15) $$A(1,2) = \pi_1^{12}S_1 + \pi_2^{12}S_2$$

Note that $\pi_i S_i$ is independent of the local representation X_i for $\mathfrak{P}_i$ and thus depends only on $\mathfrak{P}_i$, $i = 1,2$.

(11.16) $$P_{(i)} = \pi_i \sum_{j=1}^{k} \binom{k}{j} P(S_i^j, \Omega^{k-j})(d\pi_i)^{j-1}$$

is the Bott form associated to $\mathfrak{P}_i$. See equation (10.56).

By equation (7.1),

(11.17) $$P_{(1,2)} = P_{(0,1,2)} - P_{(0,2,1)}$$

To compute $P_{(0,1,2)}$, set

(11.18) $$\nabla_t = t_0 D + t_1 \nabla_1^b + t_2 \nabla_{(1,2)}^b$$

$$= D + t_1 A_{(1)} + t_2 A_{(1,2)} , \quad t = (t_0, t_1, t_2) \in \Delta_2 .$$

the local connection forms of ∇_t are given by

(11.19) $$\omega_t = \omega + t_1 \pi_1 S_1 + t_2(\pi_1^{12}S_1 + \pi_2^{12}S_2)$$

its curvature forms are

(11.20) $$\Omega_t = d\omega_t + \omega_t \wedge \omega_t$$

(11.21) $$\Omega_t = \Omega + (t_1 d\pi_1 + t_2 d\pi_1^{12})S_1 + t_2 d\pi_2^{12} S_2$$

$$= \Omega + t_1\left(d\pi_1^{12} + \frac{g_{12}}{g_{11}}\, d\pi_2^{12}\right)S_1 + t_2\left(d\pi_1^{12}S_1 + d\pi_2^{12}S_2\right)$$

modulo the ideal generated by π_1^{12}, π_2^{12}.

(11.22) $$d'\omega_t = d't_1 \wedge \pi_1 S_1 + d't_2 \wedge (\pi_1^{12}S_1 + \pi_2^{12}S_2)$$

By equation (6.17),

(11.23) $$P_{(0,1,2)} = \int_{\Delta_2} \binom{k}{2} P((d'\omega_t)^2, \Omega_t^{k-2})$$

$$= \int_{\Delta_2} d't_1 \wedge d't_2 \wedge \pi_1^{12} \wedge \pi_2^{12}\; 2\binom{k}{2}\left\{ \frac{g_{12}}{g_{11}}\, P(S_1^2, \Omega_t^{k-2}) - P(S_1, S_2, \Omega_t^{k-2})\right\}$$

(11.24) $$\pi_1^{12} \wedge \pi_2^{12} \wedge P(S_i, S_j, \Omega_t^{k-2})$$

$$= \pi_1^{12} \wedge \pi_2^{12} \wedge P\left(S_i, S_j, \left\{\Omega + t_1\left(d\pi_1^{12} + \frac{g_{12}}{g_{11}}\, d\pi_2^{12}\right)S_1 + t_2\left(d\pi_1^{12}S_1 + d\pi_2^{12}S_2\right)\right\}^{k-2}\right)$$

$$= \pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{|\alpha| = k-2} \binom{k-2}{\alpha} t_1^{\alpha_1} t_2^{\alpha_2} \left(d\pi_1^{12} + \frac{g_{12}}{g_{11}}\, d\pi_2^{12}\right)^{\alpha_1} \wedge$$

$$P\left(S_i, S_j, \Omega^{\alpha_0}, S_1^{\alpha_1}, (d\pi_1^{12}S_1 + d\pi_2^{12}S_2)^{\alpha_2}\right)$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ is a multiple index, $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$,

$$\binom{k-2}{\alpha} = \frac{(k-2)!}{\alpha_0!\,\alpha_1!\,\alpha_2!}\,.$$

Lemma 11.1. Let $\beta = (\beta_1, \ldots, \beta_\ell)$ be a multiple index, $\beta_j \in \mathbf{Z}$, $\beta_j \geq 0$. Let $t = (t_1, \ldots, t_\ell)$, $t^\beta = t_1^{\beta_1} t_2^{\beta_2} \ldots t_\ell^{\beta_\ell}$;

$|\beta| = \sum_{j=1}^{\ell} \beta_j$, $d't = d't_1 \wedge \dots d't_\ell$. Then

(11.25)
$$\int_{\Delta_\ell} t^\beta d't = \frac{\beta!}{(\ell+|\beta|)!}$$

where $\beta! = \beta_1!\beta_2! \dots \beta_\ell!$

Proof. Induction on ℓ .

For $\ell = 1$, $\int_{\Delta_1} t_1^{\beta_1} d't_1 = \frac{1}{\beta_1+1} = \frac{\beta_1!}{(\beta_1+1)!}$

In general,

(11.26)
$$\int_{\Delta_\ell} t^\beta d't = \int_0^1 \int_0^{1-t_\ell} \dots \int_0^{1-\sum_{j=3}^{\ell} t_j} \int_0^{1-\sum_{j=2}^{\ell} t_j} t_1^{\beta_1} \dots t_\ell^{\beta_\ell} d't_1 \dots d't_\ell$$

make a transformation

(11.27)
$$\begin{aligned} t_j &= (1-t_\ell)s_j \ , \ j = \overline{1,\ell-1} \\ t_\ell &= t_\ell \end{aligned}$$

then $d't_1 \dots d't_\ell = (1-t_\ell)^{\ell-1} d's_1 \dots d's_{\ell-1} d't_\ell$, substitute into equation (11.26),

(11.28)
$$\int_{\Delta_\ell} t^\beta d't = \int_0^1 \int_0^1 \int_0^{1-s_{\ell-1}} \dots \int_0^{1-\sum_{j=3}^{\ell-1} s_j} \int_0^{1-\sum_{j=2}^{\ell-1} s_j} (1-t_\ell)^{\beta_1+\dots+\beta_{\ell-1}+\ell-1} t_\ell^{\beta_\ell}$$
$$\cdot \ s_1^{\beta_1} \dots s_{\ell-1}^{\beta_{\ell-1}} d's_1 d's_2 \dots d's_{\ell-1} d't_\ell$$

$$= \int_0^1 (1-t_\ell)^{\beta_1+\dots+\beta_{\ell-1}+\ell-1} t_\ell^{\beta_\ell} d't_\ell \; \frac{\beta_1!\beta_2!\dots\beta_{\ell-1}!}{(\ell-1+\beta_1+\dots+\beta_{\ell-1})!}$$

$$= B(\beta_\ell+1, \beta_1+\dots+\beta_{\ell-1}+\ell) \cdot \frac{\beta_1!\beta_2!\dots\beta_{\ell-1}!}{(\ell-1+\beta_1+\dots+\beta_{\ell-1})!}$$

here we used the induction hypotheses for $\ell-1$. B is the Beta function, recall

(11.29)
$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} .$$

Thus

(11.30)
$$\int_{\Delta_\ell} t^\beta d't = \frac{\Gamma(\beta_\ell+1)\Gamma(\beta_1+\beta_2+\dots+\beta_{\ell-1}+\ell)}{\Gamma(\beta_1+\beta_2+\dots+\beta_\ell+\ell+1)} \cdot \frac{\beta_1!\dots\beta_{\ell-1}!}{(\ell-1+\beta_1+\dots+\beta_{\ell-1})!}$$

$$= \frac{\beta_\ell!(\beta_1+\dots+\beta_{\ell-1}+\ell-1)!}{(\beta_1+\beta_2+\dots+\beta_\ell+\ell)!} \cdot \frac{\beta_1!\dots\beta_{\ell-1}!}{(\ell-1+\beta_1+\dots+\beta_{\ell-1})!}$$

$$= \frac{\beta!}{(\ell+|\beta|)!}$$

where $\Gamma(a)$ is the gamma function, $\Gamma(a+1) = a!$ for $a \in \mathbf{Z}^+$. By induction, this proves the lemma.

By lemma 11.1, we have

(11.31)
$$P_{(0,1,2)} = \pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{|\alpha|=k-2} \binom{k}{\alpha_0} (d\pi_1^{12} + \frac{g_{12}}{g_{11}} d\pi_2^{12})^{\alpha_1} \wedge$$

$$\{ \frac{g_{12}}{g_{11}} P(S_1^2, \Omega^{\alpha_0}, S_1^{\alpha_1}, (d\pi_1^{12}S_1 + d\pi_2^{12}S_2)^{\alpha_2})$$

$$-P(S_1, S_2, \Omega^{\alpha_0}, S_1^{\alpha_1}, (d\pi_1^{12}S_1 + d\pi_2^{12}S_2)^{\alpha_2}) \}$$

$$= \pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{|\alpha|=k-2} \binom{k}{\alpha_0} \left(d\pi_1^{12} + \frac{g_{12}}{g_{11}} d\pi_2^{12}\right)^{\alpha_1} \wedge$$

$$\left\{ \frac{g_{12}}{g_{11}} P(S_1^{\alpha_1+2}, \Omega^{\alpha_0}, (d\pi_1^{12}S_1 + d\pi_2^{12}S_2)^{\alpha_2}) - P(S_1^{\alpha_1+1}, S_2, \Omega^{\alpha_0}, (d\pi_1^{12}S_1 + d\pi_2^{12}S_2)^{\alpha_2}) \right\}$$

Simplify the last expression in equation (11.31),

(11.32) $$P_{(0,1,2)} = \pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{0 \leq i+j \leq k-2} \binom{k}{2+i+j} \binom{1+i+j}{j} (d\pi_1^{12})^j (d\pi_2^{12})^i$$

$$\left\{ \left(\frac{g_{12}}{g_{11}} \right)^{i+1} P(S_1^{2+i+j}, \Omega^{k-2-i-j}) - P(S_1^{j+1}, S_2^{i+1}, \Omega^{k-2-i-j}) \right\}$$

$P_{(0,1,2)}$ is obtained by interchanging the subscripts 1 and 2.

(11.33) $$P_{(0,2,1)} = \pi_2^{12} \wedge \pi_1^{12} \wedge \sum_{0 \leq i+j \leq k-2} \binom{k}{2+i+j} \binom{1+i+j}{j} (d\pi_1^{12})^i (d\pi_2^{12})^j$$

$$\left\{ \left(\frac{g_{12}}{g_{22}} \right)^{i+1} P(S_2^{2+i+j}, \Omega^{k-2-i-j}) - P(S_2^{j+1}, S_1^{i+1}, \Omega^{k-2-i-j}) \right\}$$

$$= -\pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{0 \leq i+j \leq k-2} \binom{k}{2+i+j} \binom{1+i+j}{i} (d\pi_1^{12})^j (d\pi_2^{12})^i$$

$$\left\{ \left(\frac{g_{12}}{g_{22}} \right)^{j+1} P(S_2^{2+i+j}, \Omega^{k-2-i-j}) - P(S_1^{j+1}, S_2^{i+1}, \Omega^{k-2-i-j}) \right\}$$

It follows from equation (11.17) that

(11.34) $$P_{(1,2)} = \pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{0 \leq i+j \leq k-2} \binom{k}{2+i+j} (d\pi_1^{12})^j (d\pi_2^{12})^i$$

$$\left\{ \binom{1+i+j}{j} \left(\frac{g_{12}}{g_{11}} \right)^{i+1} P(S_1^{2+i+j}, \Omega^{k-2-i-j}) \right.$$

$$\left. + \binom{1+i+j}{i} \left(\frac{g_{12}}{g_{22}} \right)^{j+1} P(S_2^{2+i+j}, \Omega^{k-2-i-j}) - \binom{2+i+j}{1+i} P(S_1^{j+1}, S_2^{i+1}, \Omega^{k-2-i-j}) \right\}$$

If it happens that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are perpendicular to each other, i.e., locally, $g_{12} = \langle X_1, X_2 \rangle = 0$, equation (11.34) reduces to

$$(11.35) \qquad P_{(1,2)} = -\pi_1^{12} \wedge \pi_2^{12} \wedge \sum_{0 \leq i+j \leq k-2} \frac{k!}{(1+j)!(1+i)!(k-2-i-j)!} (d\pi_1^{12})^j (d\pi_2^{12})^i \cdot P(S_1^{j+1}, S_2^{i+1}, \Omega^{k-2-i-j})$$

In general, if $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ are r orthogonal pure F-structures of rank 1 locally represented by $X_1, \ldots, X_r$, then the canonical metric dual $\pi_1^{12\ldots r}, \ldots, \pi_r^{12\ldots r}$ of $X_1, \ldots, X_r$ associated to $g = \langle \cdot, \cdot \rangle$ coincides with $\pi_1, \ldots, \pi_r$ defined by $\pi_i(Y) = \dfrac{\langle X_i, Y \rangle}{\langle X_i, X_i \rangle}$, $i = \overline{1,r}$. Set

$$(11.36) \qquad \pi = \pi_1 \wedge \pi_2 \wedge \ldots \wedge \pi_r$$

$$(11.37) \qquad d\pi = (d\pi_1, d\pi_2, \ldots, d\pi_r)$$

$$(11.38) \qquad S = (S_1, S_2, \ldots, S_r)$$

for a multiple index $\alpha = (\alpha_1, \ldots, \alpha_r)$, set

$$(11.39) \qquad (d\pi)^\alpha = (d\pi_1)^{\alpha_1} (d\pi_2)^{\alpha_2} \ldots (d\pi_r)^{\alpha_r}$$

$$(11.40) \qquad S^\alpha = (S_1^{\alpha_1}, S_2^{\alpha_2}, \ldots, S_r^{\alpha_r})$$

Equation (11.35) has the following generalization:

(11.41)
$$P_{(1,2,\ldots,r)}$$

$$= (-1)^{r+1}\pi \sum_{\substack{\alpha\geq 0 \\ |\alpha|\leq k-r}} (d\pi)^{\alpha} \frac{k!}{(\alpha+\beta)!(k-|\alpha|-r)!} P(S^{\alpha+\beta}, \Omega^{k-|\alpha|-r}) \, .$$

where $\beta = (1,\ldots,1) \in R^r$.

However, the orthogonality is a very strong assumption, most set of commutative local Killing vector fields does not admit smooth metric such that under this metric they are orthogonal. For example, consider Killing vector fields in R^4,

(11.42)
$$X_i = a_i \frac{\partial}{\partial\theta_1} + b_i \frac{\partial}{\partial\theta_2} \, , \; i = 1,2 \, . \quad a_i b_i \neq 0 \, .$$

then, $[X_1, X_2] = 0$, for any smooth metric g,

(11.43)
$$g(x_1,x_2) = a_1 a_2 g\left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_1}\right) + (a_1 b_2 + a_2 b_1)\, g\left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\right)$$

$$+ b_1 b_2 \, g\left(\frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_2}\right)$$

$$g(x_1,x_2)\big|_{x_2=y_2=0} = a_1 a_2 \, g\left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_1}\right)\Big|_{x_2=y_2=0} \neq 0 \, .$$

since g is positive definite.

§12. Examples.

Let X_1 be a Killing vector field on a Riemannian manifold (M^n,g) . Let Z be a compact connected component of $\mathrm{Zero}(X_1)$. Define in a deleted neighborhood of Z a one-form π_1 by $\pi_1(Y) = g(X_1,Y)/g(X_1,X_1)$. Let $\nu(Z)$ be the normal bundle of Z and let Ω^ν be the curvature of $\nu(Z)$. The Lie derivative of X_1 induces an automorphism $S^\nu_{X_1}$ on $\nu(Z)$. For $\varepsilon > 0$, let N_ε be the set of all points of M whose distance from Z is $\leqslant \varepsilon$.

Proposition 12.1. Let θ be a differential form on M, degree $\theta = n - 2\ell$. Let $i^*(\theta)$ denote the restriction to Z of θ . Orient N_ε coherently with M and orient ∂N_ε as the boundary of N_ε . Orient Z and $\nu(Z)$ so that the resulting direct sum orientation of $\nu(Z) \oplus T(Z)$ is coherent with the orientation of M . Then:

$$\text{(12.1)} \qquad \lim_{\varepsilon\to 0} \int_{\partial N_\varepsilon} \pi_1 \wedge (d\pi_1)^{\ell-1} \wedge \theta = (2\pi)^\ell \int_Z \frac{i^*\theta}{\chi(S^\nu_{X_1} + \Omega^\nu)}$$

if $i^*\theta$ is a closed form, then

$$\text{(12.2)} \qquad \lim_{\varepsilon\to 0} \int_{\partial N_\varepsilon} \pi_1 \wedge (d\pi_1)^{\ell-1} \wedge \theta = +(2\pi)^\ell \frac{c\ell(i^*\theta)}{\chi(S^\nu_{X_1} + \Omega^\nu)} [Z]$$

where $c\ell(i^*\theta)$ is the cohomology class represented by $i^*\theta$, $[Z]$ is the fundamental class of Z . $\chi(\cdot)$ denotes the Euler characteristic class of ( ) .

Proof. See [2].

Example 1. $(\mathbf{R}^{4n}, X_1 = \sum_{i=1}^{2n} a_i \frac{\partial}{\partial \theta_i})$. $a_1 a_2 \dots a_{2n} \neq 0$. Equip $\mathbf{R}^{4n}$ with an invariant metric g which collapses along X_1 . For example, let g_0 be the standard metric on $\mathbf{R}^{4n}$, g_1 the standard metric on S^{4n-1} . Then, g_1 has a natural decomposition

$$g_1 = h + g_1(X_1, X_1)\, \pi_1 \otimes \pi_1 \tag{12.3}$$

where h vanishes along X_1. Let $\phi_1(s), \phi_2(s)$ be a partition of unity for $\mathbf{R}^+$.
$\phi_1(s) = 1$ for $s \leq 1$, $\phi_2(s) = 1$ for $s \geq 2$. Then

$$g = \phi_1 g_0 + \phi_2(h + e^{-2s}\, g_1(X_1, X_1)\pi_1 \otimes \pi_1 + ds^2) \tag{12.4}$$

will do the job.

Let $P \in I_0^{2n}(SO(4n))$. By equations (8.21), (8.22),

$$P(\Omega) + dP_{(1)} = 0 \qquad \text{on } \mathbf{R}^{4n}\setminus\{0\} . \tag{12.5}$$

where

$$P_{(1)} = \pi_1 \sum_{j=1}^{2n} \binom{2n}{j} P(S_1^j, \Omega^{2n-j})(d\pi_1)^{j-1} \tag{12.6}$$

where $S_1 = -DX_1$. Thus

$$\frac{1}{(2\pi)^{2n}} \int_{R^{4n}} P(\Omega) = \frac{+1}{(2\pi)^{2n}} \lim_{\varepsilon\to 0} \int_{S_\varepsilon^{4n-1}} P(1) \tag{12.7}$$

$$- \frac{1}{(2\pi)^{2n}} \lim_{s\to+\infty} \int_{S_s^{4n-1}} P(1)$$

equations (10.16), (10.56), and (12.2) show that

$$\frac{1}{(2\pi)^{2n}} \int_{R^{4n}} P(\Omega) = \frac{P(L_{X_1})}{\chi(L_{X_1})} \tag{12.8}$$

$$SP(S^{4n-1}, X_1) = \frac{P(L_{X_1})}{\chi(L_{X_1})}, \qquad \text{mod } Z. \tag{12.9}$$

where

$$L_{X_1} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \tag{12.10}$$

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{2n} \end{pmatrix}. \tag{12.11}$$

if $n = 1$, P is the first Pontrjagin class under the Chern-Weil homomorphism. $X_1 = a_1 \frac{\partial}{\partial\theta_1} + a_2 \frac{\partial}{\partial\theta_2}$. Then

$$\chi(L_{X_1}) = a_1 a_2$$

$$SP(S^3, x) = \frac{a_1}{a_2} + \frac{a_2}{a_1} \qquad \text{mod } Z . \tag{12.12}$$

Formula (12.12) was obtained in [6] by a direct computation.

Example 2. Let $C^{\ell} \to E \to M^{4n-2\ell}$ be a complex vector bundle on a real closed manifold $M^{4n-2\ell}$. Let J be the complex structure on E. J induces a canonical S^1 action which perserves fibre, i.e.,

$$e^{i\theta}Y = \cos\theta Y + \sin\theta JY \tag{12.13}$$

Choose a local frame $v_1,\ldots,v_{\ell}$ of type (1,0) such that

$$Jv_j = iv_j \ , \ j = \overline{1,\ell} \ , \ i = \sqrt{-1} \ . \tag{12.14}$$

set

$$X_j = v_j + \bar{v}_j \tag{12.15}$$

$$Y_j = i(v_j - \bar{v}_j) \tag{12.16}$$

then $X_1,\ldots,X_{\ell},Y_1,\ldots,Y_{\ell}$ is a local frame for the underlying real vector bundle E_R. The S^1 action is defined locally by the following formula

$$e^{i\theta}\left(\sum_{j=1}^{\ell}(a_jX_j+b_jY_j)\right) = \sum_{j=1}^{\ell}\{(a_j\cos\theta - b_j\sin\theta)X_j+(a_j\sin\theta+b_j\cos\theta)Y_j\} \tag{12.17}$$

the only fixed point set of this S^1 action is the zero section of E_R. Let g be an invariant metric of the S^1 action, which collapses at infinity. Let X be the Killing field given by this action. Identify $M^{4n-2\ell}$ with the zero section of E_R, we have Zero $X = M^{4n-2\ell}$. Moreover, under the local frame $X_1,\ldots,X_\ell,Y_1,\ldots,Y_\ell$,

$$\text{(12.18)} \qquad L_X^V = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} = J$$

$$L_X = \begin{pmatrix} L_X^V & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$$

thus

$$\text{(12.20)} \qquad \frac{1}{(2\pi)^{2n}} \int_{E_R} P(\Omega) = \frac{P(L_X+K)}{\chi(J+K^V)} \quad [M]$$

where K^V is the curvature of the normal bundle of M, K is the curvature of $v(M) \oplus T(M)$.

If E is the tangent bundle of a complex compact manifold M^{2n}, then

$$\text{(12.21)} \qquad \frac{1}{(2\pi)^{2n}} \int_{TM} P(\Omega) = \frac{P\left(\begin{pmatrix} J+K & 0 \\ 0 & K \end{pmatrix}\right)}{\chi(J+K)} \quad [M]$$

where J is the complex structure of TM and K is the curvature of TM.

If E is a complex line bundle over M^{4n-2}, let e be the Euler class of E. Let $P_1, P_2, \ldots, P_{n-1}$ be the Pontrjagin classes of M^{4n-2}, χ the Euler class of M^{4n-2}. Let σ_{2i} be the i^{th} symmetric polynomial of $\lambda_1^2, \lambda_2^2, \ldots, \lambda_{2n}^2$, $\sigma = \lambda_1\lambda_2\cdots\lambda_{2n}$, then $P = P(\sigma_2, \sigma_4, \ldots, \sigma_{2n-2}, \sigma)$ is a polynomial of $\sigma_2, \sigma_4, \ldots, \sigma_{2n-2}, \sigma$. Since $TE|_M = E \oplus TM$, by the product formula,

$$(12.22) \qquad \sigma_{2i}\begin{pmatrix} \begin{pmatrix} 0 & 1+e \\ -1-e & 0 \end{pmatrix} & 0 \\ 0 & K \end{pmatrix} = P_i + P_{i-1}(1+e)^2 \;, \; i \leqslant n-1$$

$$(12.23) \qquad \sigma\begin{pmatrix} \begin{pmatrix} 0 & 1+e \\ -1-e & 0 \end{pmatrix} & 0 \\ 0 & K \end{pmatrix} = (1+e)\chi$$

where K is the curvature of the tangent bundle of M.

Equation (12.20) gives

$$(12.24) \qquad \frac{1}{(2\pi)^{2n}} \int_{E_R} P(\Omega) =$$

$$\frac{P(P_1+(1+e)^2, P_2+P_1(1+e)^2, \ldots, P_{n-1}+P_{n-2}(1+e)^2, (1+e)\chi)}{1+e} [M^{4n-2}]$$

if $P = \sigma$, then

$$(12.25) \qquad \frac{1}{(2\pi)^{2n}} \int_{E_R} \sigma(\Omega) = \chi[M^{4n-2}] \;.$$

if $P = P(\sigma_2, \sigma_4, \ldots, \sigma_{2n-2})$, then

$$(12.26) \qquad \frac{1}{(2\pi)^{2n}} \int_{E_R} P(\Omega)$$

$$= \frac{P(P_1+(1+e)^2, P_2+P_1(1+e)^2, \ldots, P_{n-1}+P_{n-2}(1+e)^2)-(P_1, \ldots, P_{n-1})}{1+e} [M^{4n-2}]$$

since $P(P_1, \ldots, P_{n-1}) = 0$.

Example 3. Let $C^\ell \to E \to M^{4n-2\ell}$ be a complex vector bundle, $E = \bigoplus_{j=1}^{m} E_j$, where each E_j is a complex vector bundle over $M^{4n-2\ell}$, Let J_j be the complex structure on E_j. Let X_j be the vector field of the S^1 action determined by the complex structure J_j. For $(a_1,\ldots,a_m) \in \mathbf{R}^m$, $a_1 \ldots a_m \neq 0$, set

$$X = \sum_{j=1}^{m} a_j X_j \ . \tag{12.27}$$

Let g be an invariant metric on the total space E, which collapses along X. Then

$$\frac{1}{(2\pi)^n} \int_{E_R} P(\Omega) = \frac{P(L_X+K)}{\prod_{j=1}^{m} \chi(a_j J_j + K_j)} \ [M] \tag{12.28}$$

where K_j is the curvature of E_j, K is the curvature of $(\bigoplus_{j=1}^{m} E_j) \oplus TM$.

Example 4. (Nonpure F-structure on S^3).

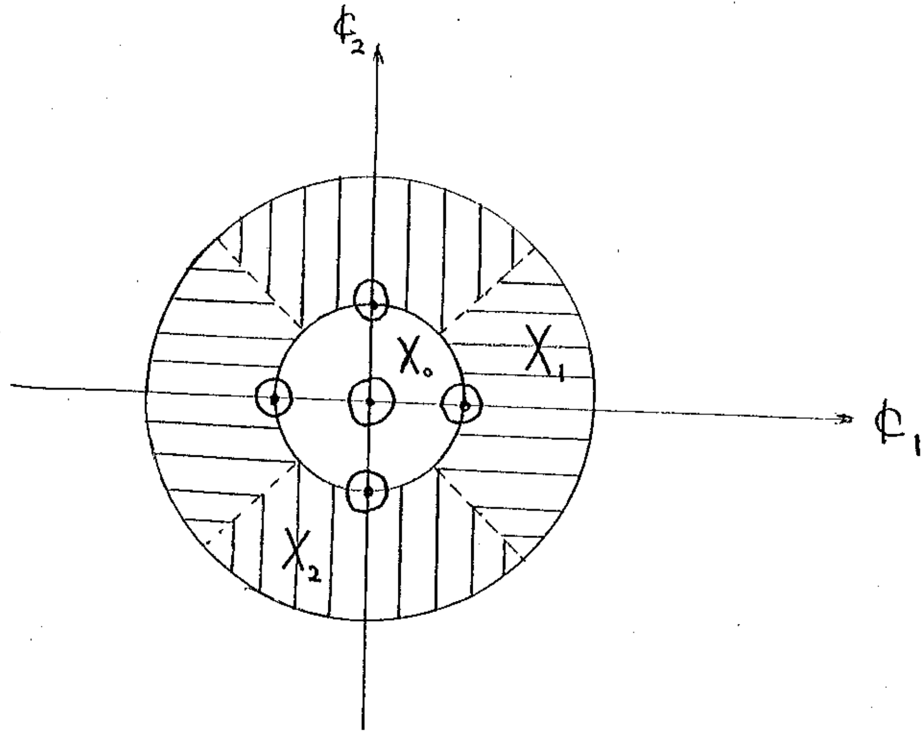
Let (x_1, y_1, x_2, y_2) be the standard coordinate of $\mathbf{R}^4 = C^2$ $(\gamma_1, \theta_1, \gamma_2, \theta_2)$ the multipolar coordinate of $\mathbf{R}^4$, i.e.,

$$\begin{aligned} x_i &= \gamma_i \cos\theta_i \\ y_i &= \gamma_i \sin\theta_i \end{aligned} \qquad i = 1,2 \ . \tag{12.29}$$

Let $D_2^4 \subset \mathbf{R}^4$ be the 4-dimensional disc with the standard metric g of radius 2. $S^3 = \partial D_2^4$. Consider the nonsingular F-structure $\mathfrak{F} = \{(U_i, X_i)\}_{i=1,2}$ on S^3, where

$$X_i = a_i \frac{\partial}{\partial \theta_1} + b_i \frac{\partial}{\partial \theta_2} \ , \ i = 1,2 \qquad a_1 b_2 \neq 0 \ . \tag{12.30}$$

(12.31) $$U_i = \{q \in S^3 \,|\, \gamma_i(q) \neq 0\} \; .$$



Set $\overline{M}_0 = D_1^4$, $\overline{M}_1 = \{q \in D_2^4 \,|\, 1 \leq r_2(q) \leq r_1(q) \leq 2\}$. $\overline{M}_2 = \{q \in D_2^4 \,|\, 1 \leq \gamma_1(q) \leq \gamma_2(q) \leq 2\}$. They determine a stratification of D_2^4 .

$\{M_0, M_1, M_2, M_{(0,1)}, M_{(0,2)}, M_{(1,2)}, M_{(1,\infty)}, M_{(2,\infty)}, M_{(0,1,2)}, M_{(1,2,\infty)}\}$ where $M_{(\alpha_1,\ldots,\alpha_\ell)}$ = interior of $\overline{M}_{\alpha_1} \cap \overline{M}_{\alpha_2} \cap \ldots \cap \overline{M}_{\alpha_\ell}$. Extend X_i to a neighborhood of $\overline{M}_i, i = 1,2$. Let

(12.32) $$X_0 = \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}$$

on a neighborhood of $\overline{M}_0$. The F-structure $\mathfrak{F}$ is thus extended to an F-structure $\tilde{\mathfrak{F}}$ on D_2^4 . The stratification is compatible with the extended F-structure on D_2^4. Let g be the standard metric on $\mathbf{R}^4$ restricted to D_2^4. Let P be the invariant polynomial which represents the first Pontrjagin class under the Chern-Weil homomorphism. Recall equations (8.2), (8.3), and (8.4), we have

(12.33) $$\tilde{P}_{(i)}(g) = P(\Omega)\big|_{\overline{M}_i} = 0 \ , \ i = 0,1,2$$

since $\Omega \equiv 0$.

(12.34) $$\tilde{P}_{(\alpha_1,\alpha_2)}(g) = 0 \quad \text{for} \quad \alpha_2 \neq \infty$$

(12.35) $$\tilde{P}_{(0,1,2)}(g) = 0$$

(12.36) $$\tilde{P}_{(i,\infty)}(g) = P_{(i)}\big|_{\overline{M}_{(i,\infty)}} \ , \ i = 1,2$$

(12.37) $$\tilde{P}_{(1,2,\infty)}(g) = -P_{(1,2)}\big|_{\overline{M}_{(1,2,\infty)}}$$

thus

(12.38) $$\tilde{P}(g) = P_{(1)}\big|_{\overline{M}_{(1,\infty)}} + P_{(2)}\big|_{\overline{M}_{(2,\infty)}} - P_{(1,2)}\big|_{\overline{M}_{(1,2,\infty)}}$$

the singular set of $\tilde{\mathfrak{F}}$ is $Z = Z_0 \cup Z_{(0,1)} \cup Z_{(0,2)}$, where $Z_0 = \{0\} \subset M_0$, $Z_{(0,1)} = \{q \in M_{(0,1)} | \gamma_1(q) = 1\} \subset M_{(0,1)}$, $Z_{(0,2)} = \{q \in M_{(0,2)} | \gamma_2(q) = 1\} \subset M_{(0,2)}$. By equations (8.21), (8.22), and (8.23), there is a co chain $\eta \in \Lambda^3(D_2^4)$, such that away from the singular set Z,

(12.39) $$\tilde{P}(g) + d\eta = 0$$

$$(12.40) \qquad \eta = P_{(0)}\big|\overline{M}_{(0)} + P_{(1)}\big|\overline{M}_{(1)} + P_{(2)}\big|\overline{M}_{(2)} + P_{(0,1)}\big|\overline{M}_{(0,1)}$$

$$+ P_{(0,2)}\big|\overline{M}_{(0,2)} + P_{(1,2)}\big|\overline{M}_{(1,2)} + P_{(0,1,2)}\big|\overline{M}_{(0,1,2)}$$

let $N_\varepsilon(Z_{(\alpha_1,\ldots,\alpha_\ell)})$ be the set of all points in $\overline{M}_{(\alpha_1,\ldots,\alpha_\ell)}$ its distance from $Z_{(\alpha_1,\ldots,\alpha_\ell)} \leq \varepsilon$. Apply theorem 2.4, take care of the orientation, we have

$$(12.41) \qquad \int \tilde{P}(g) = -\int d\eta$$

$$= +\lim_{\varepsilon \to 0} \left\{ \int_{\partial N_\varepsilon(0)} P_{(0)} + \int_{\partial N_\varepsilon(Z_{(0,1)})} -P_{(0,1)} + \int_{\partial N_\varepsilon(Z_{(0,2)})} -P_{(0,2)} \right\}$$

set

$$(12.42) \qquad \Pi_i(Y) = \frac{g(X_i,Y)}{g(X_i,X_i)} \qquad i = 0,1,2$$

$$(12.43) \qquad S_i(Y) = -D_Y X_i \qquad i = 0,1,2$$

Note that $\Omega = 0$

$$(12.44) \qquad P_{(0)} = \Pi_0 d\Pi_0 P(S_0^2)$$

Equation (12.2) shows that

$$(12.45) \qquad \lim_{\varepsilon \to 0} \int_{\partial N_\varepsilon(0)} P_{(0)} = +(2\pi)^2 \frac{P(S_0^2)}{\chi(L_{X_0})} = +(2\pi)^2 2$$

since $\chi(L_{X_0}) = 1$, $P(S_0^2) = 1^2 + 1^2 = 2$.

Note that $k = 2$, use equation (11.34), we have

$$(12.46) \quad P_{(0,1)} = \Pi_0^{01} \wedge \Pi_1^{01} \left\{ \frac{g_{01}}{g_{00}} P(S_0^2) + \frac{g_{01}}{g_{11}} P(S_1^2) - 2P(S_0,S_1) \right\}$$

$$(12.47) \quad P_{(0,2)} = \Pi_0^{02} \wedge \Pi_2^{02} \left\{ \frac{g_{02}}{g_{00}} P(S_0^2) + \frac{g_{02}}{g_{22}} P(S_2^2) - 2P(S_0,S_2) \right\}$$

where $g_{ij} = g(X_i,X_j)$, Π_0^{0i}, Π_i^{0i} is the metric dual of X_0,X_i , i.e., $\Pi_0^{0i}(X_0) = 1$, $\Pi_0^{0i}(X_i) = 0$, $\Pi_i^{0i}(X_0) = 0$, $\Pi_i^{0i}(X_i) = 1$, $\Pi_0^{0i}(Y) = \Pi_i^{0i}(Y) = 0$ for all Y perpendicular to X_0,X_i . Note also

$$(12.48) \quad X_i = a_i \frac{\partial}{\partial \theta_1} + b_i \frac{\partial}{\partial \theta_2} = a_i(-y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}) + b_i(-y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2})$$

$$(12.49) \quad S_i = -DX_i = a_i(dy_1 \frac{\partial}{\partial x_1} - dx_1 \frac{\partial}{\partial y_1}) + b_i(dy_2 \frac{\partial}{\partial x_2} - dx_2 \frac{\partial}{\partial y_2})$$

under the standard basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$

$$(12.50) \quad S_i \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$$

$$= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right) \begin{pmatrix} 0 & a_i & 0 & 0 \\ -a_i & 0 & 0 & 0 \\ 0 & 0 & 0 & b_i \\ 0 & 0 & -b_i & 0 \end{pmatrix}$$

$$(12.51) \quad P(S_i,S_j) = a_i a_j + b_i b_j$$

(12.52)
$$g_{ij} = g(X_i, X_j) = a_i a_j \gamma_1^2 + b_i b_j \gamma_2^2$$

(12.53)
$$\Pi_i = \frac{a_i \gamma_1^2 d\theta_1 + b_i \gamma_2^2 d\theta_2}{a_i^2 \gamma_1^2 + b_i^2 \gamma_2^2}$$

(12.54)
$$\Pi_i \wedge \Pi_j = \frac{(a_i b_j - a_j b_i)\gamma_1^2 \gamma_2^2 d\theta_1 \wedge d\theta_2}{(a_i^2 \gamma_1^2 + b_i^2 \gamma_2^2)(a_j^2 \gamma_1^2 + b_j^2 \gamma_2^2)}$$

set

(12.55)
$$\Delta_{ij} = 1 - \frac{g_{ij}^2}{g_{ii} g_{jj}}, \quad i \neq j$$

then

(12.56)
$$\Pi_i^{ij} = \Delta_{ij}^{-1}\{\Pi_i - \frac{g_{ij}}{g_{ii}} \Pi_j\}$$

(12.57)
$$\Pi_j^{ij} = \Delta_{ij}^{-1}\{\Pi_j - \frac{g_{ij}}{g_{jj}} \Pi_i\}$$

(12.58)
$$\Pi_i^{ij} \wedge \Pi_j^{ij} = \Delta_{ij}^{-1} \Pi_i \wedge \Pi_j$$

$$= \frac{(a_i b_j - a_j b_i)\gamma_1^2 \gamma_2^2 d\theta_1 \wedge d\theta_2}{(a_i^2 \gamma_1^2 + b_i^2 \gamma_2^2)(a_j^2 \gamma_1^2 + b_j^2 \gamma_2^2) - (a_i a_j \gamma_1^2 + b_i b_j \gamma_2^2)^2}$$

$$= \frac{d\theta_1 \wedge d\theta_2}{(a_i b_j - a_j b_i)}$$

(12.59)
$$\lim_{\gamma_1 \to 0} \frac{g_{ij}}{g_{ii}} = \frac{b_j}{b_i}$$

(12.60)
$$\lim_{\gamma_2 \to 0} \frac{g_{ij}}{g_{ii}} = \frac{a_j}{a_i}$$

combine equations (12.46), (12.47), (12.51), (12.58), (12.59), and (12.60), we have

$$(12.61)\qquad \lim_{\gamma_1\to 0} P_{(0,2)} = \frac{d\theta_1 \wedge d\theta_2}{a_0 b_2 - a_2 b_0} \left\{ \frac{b_2}{b_0}(a_0^2+b_0^2) + \frac{b_0}{b_2}(a_2^2+b_2^2) - 2(a_0 a_2 + b_0 b_2) \right\}$$

$$(12.62)\qquad \lim_{\gamma_2\to 0} P_{(0,1)} = \frac{d\theta_1 \wedge d\theta_2}{a_0 b_1 - a_1 b_0} \left\{ \frac{a_1}{a_0}(a_0^2+b_0^2) + \frac{a_0}{a_1}(a_1^2+b_1^2) - 2(a_0 a_1 + b_0 b_1) \right\} .$$

Integrate equation (12.62) on the 2-dimensional torus $\partial N_\varepsilon(Z_{(0,1)})$, note our orientation convention, we have

$$(12.63)\qquad \lim_{\varepsilon\to 0} \int_{\partial N_\varepsilon(Z_{(0,1)})} P_{(0,1)} = -4\pi^2 \frac{\frac{a_1}{a_0}(a_0^2+b_0^2) + \frac{a_0}{a_1}(a_1^2+b_1^2) - 2(a_0 a_1 + b_0 b_1)}{a_0 b_1 - a_1 b_0}$$

$$= -4\pi^2 \frac{a_0 b_1 - a_1 b_0}{a_0 a_1}$$

Similarly,

$$(12.64)\qquad \lim_{\varepsilon\to 0} \int_{\partial N_\varepsilon (Z_{(0,2)})} P_{(0,2)} = 4\pi^2 \, \frac{\frac{b_2}{b_0}(a_0^2+b_0^2) + \frac{b_0}{b_2}(a_2^2+b_2^2) - 2(a_0a_2+b_0b_2)}{a_0b_2 - a_2b_0}$$

$$= +4\pi^2 \, \frac{a_0b_2 - a_2b_0}{b_0b_2} \, .$$

Add up equations (12.45), (12.63), and (12.64),

$$(12.65)\qquad \int \tilde{P}(g) = (2\pi)^2 \left\{ 2 + \frac{a_0b_1 - a_1b_0}{a_0a_1} - \frac{a_0b_2 - a_2b_0}{b_0b_2} \right\}$$

$$= (2\pi)^2 \left\{ \frac{b_1}{a_1} + \frac{a_2}{b_2} \right\}$$

since $X_0 = a_0 \frac{\partial}{\partial\theta_1} + b_0 \frac{\partial}{\partial\theta_2} = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}$.

Therefore

$$(12.66)\qquad SP(S^3, \mathfrak{F}) = \frac{1}{(2\pi)^2} \int \tilde{P}(g) = \frac{b_1}{a_1} + \frac{a_2}{b_2} \qquad \text{mod } Z \, .$$

When $X_1 = X_2$, we obtain formula (12.12) again.

§13. Residue at an Isolated Singular Point of a Pure F-Structure of Rank Two

Let (M^n, g) be a Riemannian manifold with a pure F-structure $\mathfrak{F}$ of rank r. A singular point $q \in M$ is said to be of s dimensional isotropy, $s > 0$, if for some linear independent germs $X_1, \ldots, X_r$ of $\mathfrak{F}$ at q, rank $\{x_1(q), \ldots, x_r(q)\} = r-s$, q is said to be an isolated singular point of s-dimensional isotropy if there exists a neighborhood U of q such that for all $q' \in U \setminus \{q\}$, q' is a point of strictly less than s-dimensional isotropy. In the rest of this section, we will always assume $\mathfrak{F}$ is a pure F-structure of rank 2, $q \in M$ an isolated singular point of 2-dimensional isotropy.

Let v_1, v_2 be two local Killing vector fields on a neighborhood $U \subset M$ of q, which generate $\mathfrak{F}$ on U. According to S. Kobayashi (see [12]), note also that $[v_1, v_2] = 0$, we can choose a normal coordinate

(13.1) $$\phi = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k, z_1, \ldots, z_\ell)$$

such that $\phi(q) = 0$, ϕ preserves orientation, moreover,

(13.2) $$v_1 = \sum_{i=1}^{k} a_i \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) = \sum_{i=1}^{k} a_i \frac{\partial}{\partial \theta_i}$$

(13.3) $$v_2 = \sum_{i=1}^{k} b_i \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) = \sum_{i=1}^{k} b_i \frac{\partial}{\partial \theta_i}$$

(13.4) $$a_i^2 + b_i^2 \neq 0 \quad \text{for all} \quad i = \overline{1,k} \; .$$

There are positive integers $k_1, k_2, \ldots, k_m$, $m \geq 2$, $\sum_{i=1}^{m} k_i = k$, such that

(13.5) $$A_{ij} = a_{\bar{k}_i+1} b_j - a_j b_{\bar{k}_i+1} = 0 \quad \text{for } j \in \{\bar{k}_i+1, \ldots, \bar{k}_{i+1}\}$$

(13.6) $$A_{ij} = a_{\bar{k}_i+1} b_j - a_j b_{\bar{k}_i+1} \neq 0 \quad \text{for all } j \mathrel{\bar{\in}} \{\bar{k}_i+1, \ldots, \bar{k}_i+1\}$$

where

(13.7) $$\bar{k}_i = \sum_{j=1}^{i-1} k_j \; , \quad i = \overline{1,m} \; .$$

Since q is an isolated singular point of 2-dimensional isotropy, we must have $\ell = 0$, i.e., $\phi = (x_1, y_1, \ldots, x_k, y_k)$, and $n = 2k = \dim M$. We may, by a local change of the invariant metric, assume that g is flat on U, thus

(13.8) $$g\left(\frac{\partial}{\partial x_i} , \frac{\partial}{\partial x_j} \right) = g\left(\frac{\partial}{\partial y_i} , \frac{\partial}{\partial y_j} \right) = \delta_{ij}$$

(13.9) $$g\left(\frac{\partial}{\partial x_i} , \frac{\partial}{\partial y_j} \right) = 0$$

Set

(13.10) $$X_i = \sum_{j=1}^{k} A_{ij} \frac{\partial}{\partial \theta_j} = a_{\bar{k}_i+1} v_2 - b_{\bar{k}_i+1} v_1 \; , \quad i = \overline{1,m} \; .$$

Note that the X_i 's are local Killing vector fields uniquely determined by $\mathfrak{H}$ up to a nonzero scalar multiple. Set

$$X_0 = \sum_{j=1}^{k} \left(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}\right) = \sum_{j=1}^{k} \frac{\partial}{\partial \theta_j} \tag{13.11}$$

X_0 is a local Killing vector field (even before the possible local change of the invariant metric.). Moreover, the X_i 's commute to each other, i.e., $[X_i, X_j] = 0$, $i = 0,1,\ldots,m$.

Set

$$\overline{r}_i^{\,2} = \sum_{j=\overline{k}_i+1}^{\overline{k}_{i+1}} r_j^2 \ , \quad r_j^2 = x_j^2 + y_j^2 \ . \tag{13.12}$$

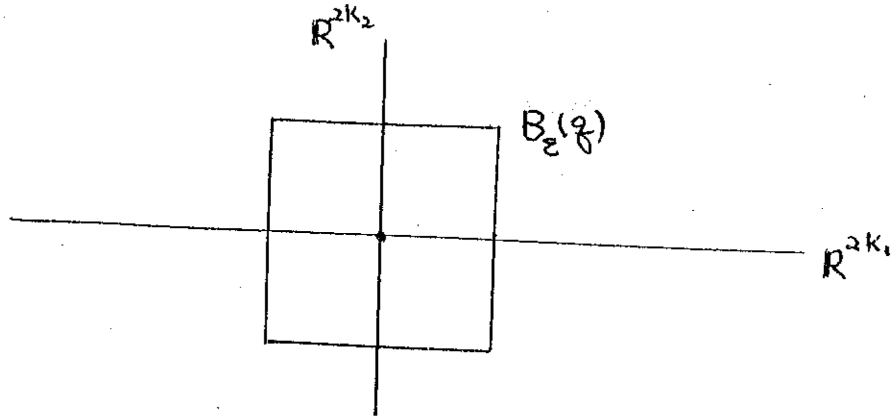
$$Z_i = \{q' \in U \mid \overline{r}_j(q') = 0 \ \text{ for all } j \neq i\} \tag{13.13}$$

then

$$\{q\} = \bigcap_{i=1}^{m} Z_i \ . \tag{13.14}$$

It is clear that X_i vanishes exactly at Z_i .

The residue contribution at q is now computed as follows: Let $B_\varepsilon(q) = B_\varepsilon^{2k_1} \times B_\varepsilon^{2k_2} \times \cdots \times B_\varepsilon^{2k_m} \subset U$, where $B_\varepsilon^{2k_i}$ is a $2k_i$-dimension ball of radius ε centered at q . We use the Killing field X_0 on $B_\varepsilon(q)$, which vanishes only at q .



The residue of X_0 at q is then

$$(13.15) \qquad (2\pi)^k P(S_0^k)$$

since $\chi(S_0) = 1$.

For each pair (X_0, X_1) restricted to

$$(13.16) \qquad \partial_i B_\varepsilon(q) = B_\varepsilon^{2k_1} \times \cdots \times B_\varepsilon^{2k_{i-1}} \times \partial B_\varepsilon^{2k_i} \times B_\varepsilon^{2k_{i+1}} \times \cdots \times B_\varepsilon^{2k_m},$$

the singular set is a sphere of dimension $(2k_i - 1)$,

$$(13.17) \qquad Z_{(0,i)}(\varepsilon) = \{q' \in \partial_i B_\varepsilon(q) \mid \bar{r}_j(q') = 0 \text{ for all } j \neq i\}$$

The idea is first to compute the residue of $P_{(0,i)}$ at $Z_{(0,i)}(\varepsilon)$ and then take the limit as ε tends to 0. The sum of these residues plus the residue of X_0 at q is then the residue contribution of the pure F-structure ϕ at q. Note that from theorem 7.1,

$$(13.18) \qquad P(\Omega^k) = -dP_{(\alpha_1)}$$

(13.19) $$P_{(\alpha_2)} - P_{(\alpha_1)} = dP_{(\alpha_1,\alpha_2)}$$

Let π_0^{0i}, π_i^{0i} be the metric dual of X_0, X_i. Since g is flat on U, $\Omega = 0$. Formula (11.34) simplifies to

(13.20) $$P_{(0,i)} = \pi_0^{0i}\pi_i^{0i} \sum_{j+\ell=k-2} (d\pi_0^{0i})^j (d\pi_i^{0i})^\ell \cdot$$

$$\{\binom{k-1}{j}\left(\frac{g_{0i}}{g_{00}}\right)^{\ell+1} P(S_0^k) + \binom{k-1}{\ell}\left(\frac{g_{0i}}{g_{ii}}\right)^{j+1} P(S_i^k) - \binom{k}{\ell+1} P(S_0^{j+1}, S_i^{\ell+1})\}$$

where $$g_{ij} = g(X_i, X_j) \ .$$

The following computations are carried out on $\partial_i B_\varepsilon(q)$.

(13.21) $$g_{00} = \sum_{j=1}^{k} r_j^2 = \sum_{j\neq i} \bar{r}_j^2 + \bar{r}_i^2 = \varepsilon_i^2 + \varepsilon^2$$

(13.22) $$g_{ii} = \sum_{j=1}^{k} A_{ij}^2 r_j^2$$

(13.23) $$g_{0i} = \sum_{j=1}^{k} A_{ij} r_j^2$$

(13.24) $$g_{0i}^2 \leq g_{ii}\varepsilon_i^2$$

set

(13.25) $$\Delta_{0i} = 1 - \frac{g_{0i}^2}{g_{00}g_{ii}}$$

Fix $\varepsilon > 0$,

(13.26) $$\lim_{\varepsilon_i \to 0} g_{00} = \varepsilon^2 > 0$$

(13.27) $$\lim_{\varepsilon_i \to 0} g_{ii} = 0$$

(13.28) $$\lim_{\varepsilon_i \to 0} g_{0i} = 0$$

(13.29) $$\lim_{\varepsilon_i \to 0} \frac{g_{0i}^2}{g_{00}g_{ii}} = 0$$

(13.30) $$\lim_{\varepsilon_i \to 0} \Delta_{0i} = 1$$

(13.31) $$\lim_{\varepsilon_i \to 0} \frac{g_{0i}}{g_{00}} = 0$$

Note that

(13.32) $$\pi_0^{0i} = \Delta_{0i}^{-1}\{\pi_0 - \frac{g_{0i}}{g_{00}} \pi_1\}$$

(13.33) $$\pi_i^{0i} = \Delta_{0i}^{-1}\{\pi_i - \frac{g_{0i}}{g_{ii}} \pi_0\}$$

(13.34) $$\pi_0^{0i}\pi_i^{0i} = \Delta_{0i}^{-1}\pi_0\pi_i$$

(13.35) $$\lim_{\varepsilon_i \to 0} \pi_0^{0i}\pi_i^{0i} = \lim_{\varepsilon_i \to 0} \pi_0\pi_i$$

(13.36) $$\pi_i = \frac{\sum_{j=1}^{k} A_{ij} r_j^2 d\theta_j}{g_{ii}}$$

(13.37) $$\pi_0 = \frac{\sum_{j=1}^{k} r_j^2 d\theta_j}{\varepsilon_i^2 + \varepsilon^2}$$

(13.38)
$$\lim_{\varepsilon_i \to 0} \pi_0 = \frac{\sum_{j=\overline{k}_i+1}^{\overline{k}_{i+1}} r_j^2 d\theta_j}{\varepsilon^2} = \overset{v}{\pi}_0$$

(13.39)
$$\lim_{\varepsilon_i \to 0} \frac{g_{0i}}{g_{00}} \pi_i$$

$$= \lim_{\varepsilon_i \to 0} \frac{\left[\sum_{j=1}^{k} A_{ij} r_j^2\right] \; \left[\sum_{j=1}^{k} A_{ij} r_j^2 d\theta_j\right]}{[\varepsilon_i^2 + \varepsilon^2] \; \left[\sum_{j=1}^{k} A_{ij}^2 r_j^2\right]} = 0$$

(13.40)
$$\lim_{\varepsilon_i \to 0} \pi_0^{0i} = \lim_{\varepsilon_i \to 0} \pi_0 = \overset{v}{\pi}_0$$

(13.41)
$$\pi_0 \pi_i (d\pi_i^{0i})^{\ell} = \Delta_{0i}^{-\ell} \pi_0 \pi_i \left\{ d\pi_i - \frac{g_{0i}}{g_{ii}} d\pi_0 \right\}^{\ell}$$

(13.42)
$$\lim_{\varepsilon_i \to 0} \pi_0 \pi_i (d\pi_0^{0i})^{\ell} = \lim_{\varepsilon_i \to 0} \pi_0 \pi_i (d\pi_0)^{\ell}$$

Lemma 13.1.

(13.43)
$$\int_{S_{\varepsilon_i}^{2(k-k_i)-1}} \left(\frac{g_{0i}}{g_{ii}} \right)^a \pi_i (d\pi_i)^{k-k_i-1}$$

$$= \frac{(2\pi)^{k-k_i}}{\binom{a+k-k_i-1}{k-k_i-1}} \cdot \frac{\sum_{(i,a)}}{\chi(\overset{v}{S}_i)}$$

where

(13.44)
$$\sum_{(i,a)} = \sum_{\substack{|\alpha| = a \\ \alpha_\ell = 0,\, \ell = \overline{k}_i+1, \overline{k}_{i+1}}} A_{i1}^{-\alpha_1} A_{i2}^{-\alpha_2} \cdots A_{ik}^{-\alpha_k}$$

(13.45) $$\chi(S_i^{\vee}) = A_{i1}A_{i2}\cdots \widehat{A_{i\bar{k}_i}A_{i\bar{k}_i+1}\cdots A_{i\bar{k}_{i+1}}} A_{ik_i+1}\cdots A_{ik}$$

is the Euler polynomial of S_i restricted to the normal bundle of $Z_{(0,i)}(\varepsilon)$, $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a k-tuple index,

$|\alpha| = \sum_{j=1}^{k}\alpha_j$, caret denotes omission. $S_{\varepsilon_i}^{2(k-k_i)-1}$ is the sphere of dimension $2(k-k_i)-1$ with radius ε_i and the standard orientation in the fibre of the normal bundle of $Z_{(0,i)}(\varepsilon)$.

Proof. This is lemma (6.21) in [2].

Now keep track of the orientation convention, we have

(13.46) $$\lim_{\varepsilon_i\to 0}\int_{-S_{\varepsilon_i}^{2(k-k_i)-1}} P_{(0,i)}$$

$$= \sum_{j+\ell=k-2}\lim_{\varepsilon_i\to 0}\int_{S_{\varepsilon_i}^{2(k-k_i)-1}} \pi_i^{0i}(d\pi_i^{0i})^{\ell}\pi_0^{0i}(d\pi_0^{0i})^{j}\left\{\binom{k-1}{j}\left(\frac{g_{0i}}{g_{00}}\right)^{\ell+1}P(S_0^k)\right.$$

$$\left. + \binom{k-1}{\ell}\left(\frac{g_{0i}}{g_{ii}}\right)^{j+1}P(S_i^k) - \binom{k}{\ell+1}P(S_0^{j+1},S_i^{\ell+1})\right\}$$

$$= \sum_{\ell=k-k_i-1}^{k-2}\lim_{\varepsilon_i\to 0}\int_{S_{\varepsilon_i}^{2(k-k_i)-1}} \pi_i(d\pi_i)^{k-k_i-1}\left(\frac{g_{0i}}{g_{ii}}\right)^{\ell-k+k_i+1}(-1)^{\ell-k+k_i+1}$$

$$\binom{\ell}{k-k_i-1}\pi_0(d\pi_0)^{k_i-1}\left\{\binom{k-1}{\ell}\left(\frac{g_{0i}}{g_{ii}}\right)^{k-\ell-1}P(S_i^k) - \binom{k}{\ell+1}P(S_0^{k-\ell-1},S_i^{\ell+1})\right\}$$

$$= \frac{(2\pi)^{k-k_i}\, \overset{v}{\pi}_0 (d\overset{v}{\pi}_0)^{k_i-1}}{\chi(S_i^v)} \sum_{j=0}^{k_i} (-1)^{k_i-j+1} \binom{k}{j} P(S_0^j, S_i^{k-j}) \textstyle\sum_{(i,k_i-j)}$$

Note also

$$(13.47) \qquad \int_{Z_{(0,i)}(\varepsilon)} \overset{v}{\pi}_0 (d\overset{v}{\pi}_0)^{k_i-1} = (2\pi)^{k_i} .$$

It follows that

$$(13.48) \qquad \lim_{\varepsilon\to 0} \int_{Z_{(0,i)}(\varepsilon)} \lim_{\varepsilon_i\to 0} \int_{-S^{2(k-k_i)-1}_{\varepsilon_i}} P_{(0,i)}$$

$$= (2\pi)^k \frac{\sum_{j=0}^{k_i} (-1)^{k_i-j+1} \binom{k}{j} P(S_0^j, S_i^{k-j}) \sum_{(i,k_i-j)}}{\chi(S_i^v)}$$

It is clear that the expression on the right-hand side of formula (13.48) is a homogeneous rational function of degree 0 of $A_{i1}, A_{i2}, \ldots, \hat{A}_{i\overline{k}_i+1}, \ldots, \hat{A}_{i\overline{k}_{i+1}}, \ldots, A_{ik}$.

Apply the formal expansion formula

$$\frac{1}{1-a} = -\frac{a^{-1}}{1-a^{-1}} = -\{a^{-1}+a^{-2}+a^{-3}+ \ldots \}$$

to the product

$$(13.50) \qquad Q_i = (1-A_{i1})^{-1} \ldots (1-A_{i\overline{k}_i})^{-1} (1-A_{i\overline{k}_{i+1}+1})^{-1} \ldots (1-A_{ik})^{-1}$$

we have

$$(13.51) \qquad Q_i = \sum_{s=0}^{\infty} Q_{i,s}(A_{i1}, \ldots, A_{ik})$$

where

(13.52) $$Q_{i,s}(A_{i1},\ldots,A_{ik})$$

$$(-1)^{k-k_i} \sum_{\substack{|\alpha|=s \\ \alpha_\ell=0,\ell=\overline{k}_i+1,\overline{k}_{i+1}}} A_{i1}^{-(\alpha_1+1)} A_{i2}^{-(\alpha_2+1)} \cdots A_{i\overline{k}_i}^{-(\alpha_{\overline{k}_i}+1)} A_{i\overline{k}_{i+1}+1}^{-(\alpha_{\overline{k}_{i+1}+1}+1)} \cdots A_{ik}^{-(\alpha_k+1)}$$

thus

(13.53) $$Q_{i,s}(A_{i1},\ldots,A_{ik}) = \frac{(-1)^{k-k_i} \sum(i,s)}{\chi(S_i^v)}$$

set $Q_{i,s}(A_{i1},\ldots,A_{ik}) = 0$ for all $s < 0$.

Then

(13.54) $$\lim_{\varepsilon\to 0} \int_{Z_{(0,i)}(\varepsilon)} \lim_{\varepsilon_i\to 0} \int_{-S_i} 2(k-k_i)-1 \; P_{(0,i)}$$

$$= -\mathrm{hom}\{P((S_0-S_i)^k)Q_i\}$$

where $\mathrm{hom}\{\quad\}$ is the homogeneous component of degree 0 of $\{\quad\}$.

Theorem 13.2. The residue at an isolated singular point q of 2-dimensional isotropy of a pure F-structure $\mathfrak{P}$ of rank 2 is given by the following formula:

(13.55) $$\mathrm{Res}(P,\mathfrak{P},q)$$

$$= (2\pi)^k \{P(S_0^k) - \text{hom} \sum_{i=1}^{m} P((S_0 - S_i)^k) Q_i \}$$

where $\mathfrak{p}$ is locally generated at q under an appropriate normal coordinate system by

(13.56) $$v_1 = \sum_{i=1}^{k} a_i (-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}) = \sum_{i=1}^{k} a_1 \frac{\partial}{\partial \theta_i}$$

(13.57) $$v_2 = \sum_{i=1}^{k} b_i (-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}) = \sum_{i=1}^{k} b_i \frac{\partial}{\partial \theta_i}$$

(13.58) $$X_i = \sum_{j=1}^{k} A_{ij} \frac{\partial}{\partial \theta_j} = a_{\overline{k}_i+1} v_2 - b_{\overline{k}_i+1} v_1, \quad i = \overline{1,m}$$

(13.59) $$A_{ij} = a_{\overline{k}_i+1} b_j - a_j b_{\overline{k}_i+1}$$

(13.60) $$A_{ij} = 0 \quad \text{for all } j = \overline{\overline{k}_i+1,\ \overline{k}_{i+1}}$$

(13.61) $$A_{ij} \neq 0 \quad \text{for all } j \neq \overline{\overline{k}_i+1,\ \overline{k}_{i+1}}$$

(13.62) $$X_0 = \sum_{j=1}^{k} \frac{\partial}{\partial \theta_j}$$

(13.63) $$S_i = -DX_i$$

Remark. The X_i 's are determined a priori by $\mathfrak{p}$ up to a constant multiple, i.e., up to a constant multiple they are independent of the choice of the local representatives v_1, v_2. It is clear from the expression on the right-hand side of formula (13.55) that $\operatorname{Res}(P, \mathfrak{p}, q)$ is independent of this constant multiple and thus a topological invariant of $\mathfrak{p}$ at q

associated to P, which is a homogeneous rational function of degree zero of the A_{ij}'s.

Corollary 13.3. Let χ be the invariant polynomial which represents the Euler class under the Chern-Weil homomorphism. Then, in the same assumption as in theorem 13.2,

(13.64) $$\operatorname{Res}(\chi,\mathcal{P},q) = (1-m)(2\pi)^k$$

Proof. Note that

(13.65) $$\chi(S_0^j,S_i^{k-j}) = 0 \quad \text{for all} \quad j < k_i$$

(13.66) $$\chi(S_0^{k_i},S_i^{k-k_i}) = \frac{k_i!(k-k_i)!}{k!}\,\chi(S_i^v)$$

Formula (13.64) follows now from formulae (13.48) and (13.55).

Example 13.1. Let v_1,v_2 as defined by equations (13.56) and (13.57) be two vectors in $\mathbf{R}^{2k}$, let X_i be defined by equation (13.58), A_{ij}'s satisfy conditions (13.60) and (13.61). Let g be an invariant metric such that g is the standard metric of $\mathbf{R}^{2k}$ in a neighborhood of the origin and g collapses along, say, $c_1v_1 + c_2v_2$, such that $c_1a_j + c_2b_j \neq 0$ for all j. Such metric always exists since one can average on the torus determined by v_1 and v_2. Decompose $\mathbf{R}^{2k}$,

$$\mathbf{R}^{2k} = \bigoplus_{i=1}^{m} \mathbf{R}_i^{2k_i}$$

identify $\mathbf{R}_i^{2k_i}$ with the canonical embedding as the i^{th} component in the decomposition (13.67). Let

(13.68) $$g_i = g\Big|_{\mathbf{R}_i^{2k_i}}$$

(13.69) $$\Omega_i = \Omega_{g_i}$$

$(\mathbf{R}_i^{2k_i}, g_i)$ is naturally collapsed along $(c_1 v_1 + c_2 v_2)\Big|_{\mathbf{R}_i^{2k_i}}$. Note that

(13.70) $$\chi(S_i+\Omega) = \chi(\Omega_i)\chi(S_i^{\vee}+\Omega^{\vee})$$

The residue theorem and formula (13.55) and (13.64) show that

(13.71)
$$\begin{aligned}
\frac{1}{(2\pi)^k}\int_{\mathbf{R}^{2k}} \chi(\Omega) &= \mathrm{Res}(\chi, \mathfrak{P}) \\
&= \sum_{i=1}^{m} \frac{1}{(2\pi)^{k_i}} \int_{\mathbf{R}^{2k_i}} \frac{\chi(S_i+\Omega)}{\chi(S_i^{\vee}+\Omega^{\vee})} + \mathrm{Res}(\chi, \mathfrak{P}, 0) \\
&= \sum_{i=1}^{m} \frac{1}{(2\pi)^{k_i}} \int_{\mathbf{R}^{2k_i}} \chi(\Omega_i) + (1-m) \\
&= m + 1 - m = 1 .
\end{aligned}$$

Example 13.2. Let P_1 be the invariant polynomial which represents the first Pontrjagin class under the Chern-Weil homomorphism. Let $k = 2$.

(13.72) $$v_1 = a_1 \frac{\partial}{\partial\theta_1} + a_2 \frac{\partial}{\partial\theta_2}$$

(13.73) $$v_2 = b_1 \frac{\partial}{\partial\theta_1} + b_2 \frac{\partial}{\partial\theta_2}$$

(13.74) $$A_{12} = a_1 b_2 - a_2 b_1 \neq 0$$

(13.75) $$\mathrm{Res}(P_1, \mathfrak{F}, q) = (2\pi)^2 \{ P_1(S_0^2) + \frac{P_1(S_1^2)A_{12}^{-1} - 2P_1(S_0, S_1)}{A_{12}} + \frac{P_1(S_2^2)A_{21}^{-1} - 2P_1(S_0, S_2)}{A_{21}} \}$$

$$= (2\pi)^2 \{ 2 + \frac{A_{12} - 2A_{12}}{A_{12}} + \frac{A_{21} - 2A_{21}}{A_{21}} \} = 0 .$$

Example 13.3. Let M^{2k} be a closed oriented manifold, let $\mathfrak{F}$ be a pure F-structure of rank 2 on M with only isolated 2-dimensional isotropy. It is not difficult to see that the singular set $Z(\mathfrak{F})$ is a number of immersed closed totally geodesic submanifolds in M (see [12]). Let $Z_1, \ldots, Z_{m_1}$ be these immersed closed totally geodesic submanifolds, which selfintersect or intersect to each other only at isolated points $q_1, \ldots, q_{m_2}$. $q_1, \ldots, q_{m_2}$ are precicely these points where the isotropy of $\mathfrak{F}$ is of dimension 2. By the residue theorem,

(13.76) $$\frac{1}{(2\pi)^k} \int_{M^{2k}} P(\Omega^k) = \mathrm{Res}(P, \mathfrak{F}, Z(\mathfrak{F}))$$

$$= \sum_{i=1}^{m_1} \frac{P(S_i + \Omega)}{\chi(S_i^\nu + \Omega^\nu)} [Z_i] + \sum_{i=1}^{m_2} \mathrm{Res}(P, \mathfrak{F}, q_i)$$

where S_i is defined by the pure F-structure $\mathfrak{F}_i$ of rank 1 on a neighborhood of z_i, $\mathfrak{F}_i$ is singular precisely at z_i and is canonically determined by $\mathfrak{F}$.

Remark. If M^{2k} is a compact oriented manifold with boundary $N = \partial M$. $\mathfrak{F}$ is a pure F-structure of rank 2 on M with only isolated 2-dimensional isotropy. $\mathfrak{F}|_N$ is nonsingular. We obtain a similar residue formula for the secondary characteristic numbers of $(N, \mathfrak{F}|_N)$.

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