# ON NULL HYPERSURFACES AND SPACELIKE SURFACES IN SPACETIMES

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Abstract of the Dissertation

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Null hypersurfaces in spacetimes are studied. Necessary and sufficient conditions for the separation and foliation of null hypersurfaces by spacelike surfaces are obtained. The deviation of null congruences in null hypersurfaces are discussed. A canonical definition of the second fundamental tensor of a null hypersurface is obtained and its properties are investigated.

The future null cut locus of a spacelike surface is defined and its properties are investigated.

The influence of curvature on the existence of closed trapped surfaces in 4-dimensional spacetimes is

discussed. In cosmological and black hole circumstances, necessary and sufficient conditions for a spacelike surface to be a closed trapped surface diffeomorphic to $S^2$ are obtained. Sufficient conditions for the evolution of closed trapped surfaces from marginally trapped surfaces are obtained and discussed.

To my mother and father

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# INTRODUCTION

It is known from the Hawking-Penrose singularity theorem[1] that the existence of closed trapped surfaces[2] in strongly causal[3] spacetimes yields non-spacelike geodesic incompleteness. However, the crucial ingredient of the theorem is not the existence of closed trapped surfaces but rather the existence of a future trapped set[4] in the presence of the strong causality assumption[5] (see also Figure 1). (Note that a closed trapped surface requires only the weak energy condition[6] (in the case of null geodesic completeness) to be a future trapped set.)[7]

In the literature, the future convergence property[2] of a spacelike surface H is considered to be related with the curvature of spacetime in that H is contained in such a strong gravitational field[8] that the light rays (future directed null geodesics) which are emanating orthogonally from H are dragged back and in fact, are converging. However, the future convergence property of a spacelike surface H may be not a consequence of the curvature but rather of the way H sits inside the

[1]Footnotes to the Introduction are found on pp. 6-7 .

spacetime (see Figure 2). In fact, in any spacetime, one can construct a spacelike surface H with the future convergence property.[9]

We note that a compact spacelike surface H in a spacetime M is a Riemannian surface which is complete in the induced metric. Therefore, each closed trapped surface H is a complete spacelike surface (in the induced metric) with the future convergence property. It seems reasonable to question the influence of curvature on the existence of closed trapped surfaces both from the viewpoint of the relation of curvature to the compactness of future converging complete spacelike surfaces and from the viewpoint of its influence on future convergence.

We shall first investigate the influence of curvature on the compactness of future converging complete spacelike surfaces. We shall obtain in cosmological and black hole circumstances necessary and sufficient conditions for a future converging complete spacelike surface H to be diffeomorphic to $S^2$ (2-sphere). We shall then obtain sufficient conditions for the evolution of closed trapped surfaces from marginally trapped surfaces[10] in the above circumstances. (In cosmological circumstances, necessary and sufficient conditions involve <u>energy</u> <u>density</u> and in black hole circumstances, they involve the <u>surface</u> <u>gravity</u>[11] of H.) These results will appear in Chapter IV.

We shall devote Chapter I to a brief review of <u>Jacobi</u> <u>tensors</u>[12] and the <u>Raychaudhuri</u> <u>equation</u>[12] along null geodesics, which provide a convenient way of studying conjugate and focal points. Furthermore, we shall generalize some theorems of [BE] concerning conjugate points to focal points of spacelike surfaces along null geodesics and we shall discuss the equivalence of Jacobi tensors (along a null geodesic $\gamma$) from the viewpoint of the spaces of Jacobi classes (along $\gamma$) induced by these Jacobi tensors.

In Chapter II, we shall study <u>null</u> <u>hypersurfaces</u>[13] and develop the necessary machinery for Chapter IV. In the literature, null hypersurfaces have been studied either as a part of achronal boundaries[5] or as hypersurfaces constructed from null geodesic congruences orthogonal to spacelike surfaces [cf. [P], pages 21 and 60). However, not every null hypersurface in a spacetime can be constructed by using the above methods.

The major difficulty in the study of null hypersurfaces appears to be that the restriction of the metric of a spacetime M to a null hypersurface S in M is degenerate and therefore there is no well-defined projection of $TM|_S$ onto TS (consequently, it is not possible to define an induced connection on S). We shall partially overcome this problem by observing that the normal bundle $\mathbb{N}(S)$ of a null hypersurface S is a null line subbundle[14] of TS which defines a unique null direction in each $T_pS$ for every $p \in S$. Then, by quotienting out $\mathbb{N}(H)$ from TS, we shall obtain a canonical Riemannian vector bundle G(S) over S which will enable us to study null hypersurfaces using the methods of Riemannian geometry. We shall obtain a canonical definition of the second fundamental tensor $\bar{L}$ of S and show that $\bar{L} = 0$ iff S is a totally geodesic submanifold of M (that is, $\bar{L} = 0$ iff the restriction to S of the connection of spacetime defines a connection on S). Furthermore, we shall investigate the relations between $\bar{L}$ and the deviation of null geodesic congruences in S by relating $\bar{L}$ with Jacobi tensors along null geodesics in S.

In Chapter III, we shall generalize the definition of (future) null cut locus of a point[15] p in a spacetime to (future) null cut locus of a spacelike surface[16] in a spacetime. We shall show that, in a globally hyperbolic[17] spacetime M, the future null cut locus $C_N^+(H)$ of a compact, acausal[5] spacelike surface H is a closed subset of M and each $p \in C_N^+(H)$ is either a focal point of H along some future directed null geodesic which meets H orthogonally or there exist at least two future directed null geodesics from H to p realizing the distance between H and p.

[1]cf. [BE], page 394.

[2]A spacelike surface H is called future converging if the null normals of H are everywhere converging into the future. A compact future converging spacelike surface is called a closed trapped surface.

[3]A spacetime M is called strongly causal if each $p \in M$ has a neighborhood V such that no non-spacelike curve in M intersects V in a disconnected set.

[4]Let H be a subset of a spacetime M. The chronological (respectively, causal) future $I^+(H)$ (respectively $J^+(H)$) is defined to be the set of all points in M which can be joined to H with a past directed timelike (respectively, non-spacelike) curve. H is called a future set iff $I^+(H) = H$. The future horismos $E^+(H)$ of H is defined to be the set $E^+(H) = J^+(H) - I^+(H)$. H is called future trapped if $E^+(H)$ is compact. H is called achronal (respectively acausal) if $I^+(H) \cap H = \emptyset$ (respectively, $J^+(H) \cap H = \emptyset$). Boundary of a future set is called an achronal boundary.

[5]In fact, the essential ingredient of the Hawking-Penrose singularity theorem is the following:

Theorem (Hawking-Penrose): A strongly causal spacetime M cannot be non-spacelike complete if

i) M contains a future trapped set H
ii) every complete non-spacelike geodesic which meet $E^+(H)$ fails to realize distance between its points.

Sketch of Proof: Since M is strongly causal and $E^+(H)$ is compact, there exists an inextendable non-spacelike geodesic $\gamma$ which meets $E^+(H)$ and realizes distance between its points (cf. [BE], pages 208 and 390). But then, since every complete non-spacelike geodesic which meets $E^+(H)$ fails to realize distance between its points, $\gamma$ is necessarily an incomplete geodesic. ∎

Note that, the above theorem predicts non-spacelike geodesic incompleteness in the spacetime which is depicted in Figure 1. (This spacetime is flat and does not contain any closed trapped surface.)

[6]The weak energy condition is defined to be $Ric(u,u) \geq 0$ for every null vector u.

[7]However, a compact spacelike surface need not necessarily be future converging to be a future trapped set (see Figure 1). In fact, the topological structure of the spacetime may play the crucial role in forcing a spacelike surface to be future trapped.

[8]Physically, the curvature of spacetime is manifested in gravitational tidal forces (cf. [SW], page 55).

[9]A spacelike surface with the future convergence property can be constructed in any spacetime as follows: Let V be a geodesically convex, normal neighborhood of a point p of the spacetime. Consider V as a spacetime. Let $\gamma$ and $\beta$ be future directed null geodesics in V emanating from p with non-proportional initial direction. Let $r \neq p$ and $s \neq p$ be points on $\gamma$ and $\beta$ respectively. Then $S_r = \partial J^-(r) - \{r\}$ and $S_s = J^-(s) - \{s\}$ (in V) are smooth null hypersurfaces in V which are transversal to each other since $\gamma$ and $\beta$ have non-proportional initial directions (where $J^-(r)$ and $J^-(s)$ are the causal pasts of r and s in V). Since the null generators of these null hypersurfaces are expanding into the past direction in some neighborhoods of r and s, by choosing r and s sufficiently close to p, we can obtain a future converging spacelike surface H which is contained in $S_r \cap S_s$. (Note that $S_r \cap S_s$ is also a spacelike surface but it may not be future converging everywhere.)

[10]The definition of <u>marginally trapped surface</u> is technical. See Chapter I for the formal definition.

[11]The <u>surface gravity</u> of a spacelike surface H at $p \in H$ is defined as the sectional curvature of the fiber of its normal bundle at p.

[12]cf. [BE], Chapter 11.

[13]A smooth <u>connected</u> hypersurface S in a Lorentzian manifold is called a <u>null hypersurface</u> if $T_pS$ is a null vector space at every point $p \in H$.

[14]A vector bundle with 1-dimensional null fibers is called a <u>null line bundle</u>.

[15]cf. [BE], page 230.

[16]cf. Chapter III.

[17]A strongly causal spacetime M is called globally hyperbolic if $J^-(p) \cap J^+(q)$ is compact for every $p,q \in M$.

# CHAPTER I

## JACOBI TENSORS AND THE RAYCHAUDHURI EQUATION

In this chapter, we shall briefly review some concepts of the Lorentzian geometry which will be frequently referred to in later chapters.

We recall that a spacetime M is a connected, orientable, time oriented n-dimensional Lorentzian manifold M with metric g of signature $(-++\cdots+)$. (We shall also use the notation $\langle , \rangle$ for the metric g.) A non-zero vector v in TM is said to be null (respectively, timelike, non-spacelike, spacelike) if $g(v,v) = 0$ (respectively, $g(v,v) < 0$, $g(v,v) \leq 0$, $g(v,v) > 0$). An embedded, connected, spacelike submanifold of codimension 2 in M is called a spacelike surface.

In this chapter and thereafter, we shall always denote an n-dimensional Lorentzian manifold by M and by I, we shall always denote an interval in the real line.

Definition 1.1: Let $\gamma\colon I \to M$ be a null geodesic and let $N(\gamma)$ and $[\dot{\gamma}]$ be the vector bundles which are defined by $N(\gamma) = \bigcup_{t\in I} N(\gamma(t))$ where $N(\gamma(t)) = \{v\in T_{\gamma(t)}M \mid g(v,\dot{\gamma}(t))=0\}$ and $[\dot{\gamma}] = \bigcup_{t\in I} \{s\dot{\gamma}(t) \mid s\in \mathbb{R}\}$. Then the Riemannian vector

<u>bundle</u> $G(\gamma)$ along $\gamma$ with the metric $\overline{g}$ is defined by $G(\gamma) = N(\gamma)/[\dot{\gamma}]$ and $\overline{g}(\overline{x},\overline{y}) = g(x,y)$ where $x,y \in N(\gamma)$ with $\Pi(x) = \overline{x}$, $\Pi(y) = \overline{y}$ and $\Pi: N(\gamma) \to G(\gamma)$ is the projection. (cf. [BE] page 294).

The covariant derivative of a section of $G(\gamma)$ is defined as follows:

<u>Definition 1.2</u>: Let $\overline{X} \in \Gamma G(\gamma)$. The <u>covariant derivative</u> of $\overline{X}$ along $\gamma$ is defined by $\dot{\overline{X}} = \overline{\nabla}_{\dot{\gamma}}\overline{X} := \Pi(\nabla_{\dot{\gamma}}X)$ where $X \in \Gamma N(\gamma)$ and $\Pi(X) = \overline{X}$. (cf. [BE] page 297)

The generic condition along a null geodesic $\gamma$ can be formulated using the following homomorphism:

<u>Definition 1.3</u>: The bundle homomorphism $\overline{R}(\cdot,\dot{\gamma})\dot{\gamma}: G(\gamma) \to G(\gamma)$ is defined by $\overline{R}(\overline{X},\dot{\gamma})\dot{\gamma} := \Pi(R(X,\dot{\gamma})\dot{\gamma})$ where $\Pi(X) = \overline{X}$ and $R$ is the curvature tensor. It is immediate from the curvature identities that $\overline{R}(\cdot,\dot{\gamma})\dot{\gamma}$ is self-adjoint (cf. [BE] page 298).

<u>Definition 1.4</u>: $\overline{X} \in \Gamma G(\gamma)$ is called a <u>Jacobi class</u> along $\gamma$ if $\ddot{\overline{X}} + R(\overline{X},\dot{\gamma})\dot{\gamma} = \overline{0}$ where $\overline{0}$ is the zero section of $\Gamma G(\gamma)$ (cf. [BE] page 310).

We recall that the normal bundle $N(H)$ of a spacelike surface $H$ in a Lorentzian manifold $M$ has 2-dimensional timelike fibers each of which contains two well-defined null directions (cf. [BE] page 382). Therefore, if $u$ is a null vector which is orthogonal to $H$ at $p \in H$, then $u$ is in the direction of one of the above null directions at $p$. Then $u^{\perp}$ (=orthogonal space to span$\{u\}$) contains $T_pH$ and the restriction of the projection $\Pi: u^{\perp} \to G(u)$ to $T_pH$ is an isomorphism of $T_pH$ onto the quotient space $G(u)$ where $G(u) = u^{\perp}/\text{span}\{u\}$ (cf. [BE] page 382).

Definition 1.5: Let $H$ be a spacelike surface in $M$. The second fundamental tensor of $H$ in the null direction $u \perp H$ at $p \in H$ is defined by $L_u x = -(\nabla_x U)^T$ where $x \in T_pH$, $U$ is an extension of $u$ to a null vector field orthogonal to $H$ and $(\nabla_x U)^T$ is the component of $\nabla_x U$ tangent to $H$. The second fundamental form of $H$ in the null direction $u \perp H$ at $p \in H$ is defined by $II_u(x,y) = \langle L_u x, y\rangle$ where $x,y \in T_pH$.

To show that $L_u$ is well-defined, it suffices to show that $(\nabla_x U)^T$ is independent of the extension $U$ of $u$. Indeed, if $U$ and $U_1$ are two such extensions of $u$ in a neighborhood of $p \in H$ in $H$ then there exists a smooth function $f$ on this neighborhood such that $U = fU_1$ and $f(p) = 1$. Thus

$$\begin{aligned}(\nabla_x U)^T &= (\nabla_x fU_1)^T = (x(f)U_1 + f\nabla_x U_1)^T\Big|_p = f(p)(\nabla_x U_1)^T \\ &= (\nabla_x U_1)^T\end{aligned}$$

since $f(p) = 1$ and $(x(f)U_1)^T = 0$. ∎

Let H be a spaceLlike surface in a spacetime M. Then, at each $p \in H$, there exist two future directed, non-proportional (linearly independent) null vectors u,w orthogonal to H with the normalization $\langle u,w\rangle = -1$ (cf. [BE] page 382). Then, H is said to be future converging (respectively, strongly future converging) if $(\mathrm{tr}L_u)(\mathrm{tr}L_w) > 0$ and $\mathrm{tr}L_u > 0$ (respectively, $II_u(x,x)II_w(y,y) > 0$ and $II_u(x,x) > 0$ for every $x,y \in T_pH$) at every $p \in H$. A future converging spacelike surface H is called a closed trapped surface if H is compact.

Let H be a spacelike surface with trivial normal bundle $\mathfrak{N}(H)$. Let U,W be future directed, non-proportional (linearly independent) null sections of $\mathfrak{N}(H)$. Then H is said to be marginally future converging (respectively, strongly marginally future converging) if $\mathrm{tr}L_W > 0$ and $\mathrm{tr}L_U = 0$ (respectively, if $\mathrm{tr}L_W > 0$ and $L_U = 0$) (or vice versa) along H. A marginally future converging spacelike surface H is called a marginally trapped surface if H is compact.

Definition 1.6: Let H be a spacelike surface in M and u be a null vector orthogonal to H at $p \in H$. Let $G(u) = u^{\perp}/\mathrm{span}\{u\}$. Then the linear map $\overline{L}_u: G(u) \to G(u)$ is defined by $\overline{L}_u\overline{x} = \Pi(L_u x)$ where $\overline{x} \in G(u)$ and $x \in T_pH$ such that $\Pi(x) = \overline{x}$.

Note that $\overline{L}_u$ is well defined since $\Pi|_{T_pH}$ is an isomorphism of $T_pH$ onto $G(u)$ ([BE] page 383).

Definition 1.7. Let $H$ be a spacelike surface in $M$ and let $\gamma\colon [0,a) \to M$ be a null geodesic with $\gamma(0) \in H$ and $\dot\gamma(0) \perp H$. $t \in (0,a)$ is called a focal point of $H$ along $\gamma$ if there exists a Jacobi class $\overline{X} \in \Gamma G(\gamma)$ such that $\overline{X}(0) \neq 0$, $\dot{\overline{X}}(0) = -\overline{L}_{\dot\gamma(0)}\overline{X}(0)$ and $\overline{X}(t) = 0$ (cf. [BE] page 384).

Definition 1.8: Let $\gamma\colon I \to M$ be a null geodesic. A bundle homomorphism $\overline{A}\colon G(\gamma) \to G(\gamma)$ is called a Jacobi tensor along $\gamma$ if it satisfies

(i) $\ddot{\overline{A}} + (\overline{R}(\cdot,\dot\gamma)\dot\gamma)\overline{A} = \overline{0}$

(ii) $\ker \overline{A} \cap \ker \dot{\overline{A}} = \overline{0}$

where $\dot{\overline{A}}(\overline{X}) := \overline{\nabla}_{\dot\gamma}(\overline{A}(\overline{X})) - \overline{A}(\overline{\nabla}_{\dot\gamma}\overline{X})$ for $\overline{X} \in \Gamma G(\gamma)$ (cf. [BE] page 310).

Remark: The condition (i) implies that if $\overline{E} \in \Gamma G(\gamma)$ is a parallel section then $\overline{X} = \overline{A}(\overline{E})$ is a Jacobi class in $G(\gamma)$. The condition (ii) implies that if $\overline{E}_i \in \Gamma G(\gamma)$ are linearly independent parallel sections then $\overline{X}_i = \overline{A}(\overline{E}_i)$ are linearly independent Jacobi classes in $G(\gamma)$.

Proposition 1.9: Let $\gamma\colon I \to M$ be a null geodesic and let $\overline{A}\colon G(\gamma) \to G(\gamma)$ be a bundle homomorphism satisfying $\ddot{\overline{A}} + (\overline{R}(\cdot,\dot\gamma)\dot\gamma)\overline{A} = 0$. Then

$$\ker \overline{A} \cap \ker \dot{\overline{A}} = \overline{0} \text{ iff } \ker \overline{A}(t) \cap \ker \dot{\overline{A}}(t) = 0$$

for some $t \in I$.

Proof: [BE] page 311. ∎

Definition 1.10: A Jacobi tensor $\overline{A}$ (along a null geodesic $\gamma$) is called a Lagrange tensor (along $\gamma$) if $(\dot{\overline{A}})^*\overline{A} = \overline{A}^*\dot{\overline{A}}$ where * denotes the adjoint.

Proposition 1.11: Let $\gamma: I \to M$ be a null geodesic and let $\overline{A}$ be a Jacobi tensor along $\gamma$. Then $\overline{A}$ is a Lagrange tensor along $\gamma$ if $(\dot{\overline{A}}(t))^*\overline{A}(t) = (\overline{A}(t))^*\dot{\overline{A}}(t)$ for some $t \in I$. ∎

Proof: [BE] pages 312 and 313.

Suppose that $\overline{A}$ is a non-singular Jacobi tensor along a null geodesic $\gamma$ and let $\overline{B}: G(\gamma) \to G(\gamma)$ be the bundle homomorphism defined by $\overline{B} = \dot{\overline{A}}(\overline{A})^{-1}$.

Proposition 1.12: The bundle homomorphism $\overline{B}$ is self-adjoint iff $\overline{A}$ is a Lagrange tensor along $\gamma$.

Proof: [BE] page 350. ∎

Remark: It is immediate from Proposition 1.11 that $\overline{B}(t)$ is self-adjoint at some $t$ iff $\overline{B}$ is self-adjoint on $G(\gamma)$.

Now, we shall introduce the concept of the neighbors of a null geodesic $\gamma$ induced by a Jacobi tensor along $\gamma$.

Definition 1.13: Let $\gamma: I \to M$ be a null geodesic and $\overline{A}$ be a Jacobi tensor along $\gamma$. The set of neighbors of $\gamma$ induced by $\overline{A}$ is the set

$$\Gamma_{\overline{A}\text{-nbor}}(\gamma) = \{\overline{X}\in\Gamma G(\gamma) \mid \overline{X}=\overline{A}(\overline{E}) \text{ where } \overline{E} \text{ is a parallel class in } G(\gamma)\}.$$

Remark: Note that $\Gamma_{\overline{A}\text{-nbor}}(\gamma)$ is (n-2)-dimensional vector space of Jacobi classes along $\gamma$.

Proposition 1.14: Let $\gamma: I \to M$ be a null geodesic. Suppose that $\overline{A}$ and $\overline{C}$ are two Jacobi tensors along $\gamma$ with $\overline{A}(t_o)$ and $\overline{C}(t_o)$ non-singular for some $t_o \in I$. Then

$$\Gamma_{\overline{A}\text{-nbor}}(\gamma) = \Gamma_{\overline{C}\text{-nbor}}(\gamma) \text{ iff } \dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1} = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}.$$

Proof:

$\Rightarrow$: Assume $\Gamma_{\overline{A}\text{-nbor}}(\gamma) = \Gamma_{\overline{C}\text{-nbor}}(\gamma)$.

Let $\overline{X} \in \Gamma_{\overline{A}\text{-nbor}}(\gamma)$. Then, by definition 1.13, there exist parallel classes $\overline{E}$ and $\overline{F}$ along $\gamma$ such that $\overline{A}(\overline{E}) = \overline{X} = \overline{C}(\overline{F})$. Thus it follows that

$$\dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1}\overline{X}(t_o) = \dot{\overline{A}}(t_o)\overline{E}(t_o) = \overline{\nabla}_{\dot{\gamma}(0)}(\overline{A}(\overline{E})) - \overline{A}(\overline{\nabla}_{\dot{\gamma}(0)}\overline{E}) = \dot{\overline{X}}(t_o)$$

and

$$\dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}\overline{X}(t_o) = \dot{\overline{C}}(t_o)\overline{F}(t_o) = \overline{\nabla}_{\dot{\gamma}(0)}(\overline{C}(\overline{F})) - \overline{A}(\overline{\nabla}_{\dot{\gamma}(0)}\overline{F}) = \dot{\overline{X}}(t_o)$$

since $\overline{\nabla}_{\dot{\gamma}(t_o)}\overline{E} = 0 = \overline{\nabla}_{\dot{\gamma}(0)}\overline{F}$. Therefore,

$$\dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1} = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}.$$

$\Leftarrow$: Assume $\dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1} = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}$.

Let $\overline{X} \in \Gamma_{\overline{A}\text{-nbor}}(\gamma)$. Then there exists $\overline{Y} \in \Gamma_{\overline{C}\text{-nbor}}(\gamma)$ such that $\overline{X}(t_o) = \overline{Y}(t_o)$. Let $\overline{E}$ and $\overline{F}$ be parallel classes along $\gamma$ such that $\overline{A}(\overline{E}) = \overline{X}$ and $\overline{C}(\overline{F}) = \overline{Y}$. Then

$$\begin{aligned}\dot{\overline{X}}(t_o) &= \dot{\overline{A}}(t_o)\overline{E}(t_o) = \dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1}\overline{X}(t_o) = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}\overline{X}(t_o)\\ &= \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}\overline{Y}(t_o) = \dot{\overline{C}}(t_o)\overline{F}(t_o) = \dot{\overline{Y}}(t_o).\end{aligned}$$

Thus, since $\overline{X},\overline{Y}$ are Jacobi classes along $\gamma$ with $\overline{X}(t_o) = \overline{Y}(t_o)$ and $\dot{\overline{X}}(t_o) = \dot{\overline{Y}}(t_o)$, it follows that $\overline{X} = \overline{Y}$. In other words, $\overline{X} \in \Gamma_{\overline{C}\text{-nbor}}(\gamma)$ and thus $\Gamma_{\overline{A}\text{-nbor}}(\gamma) \subseteq \Gamma_{\overline{C}\text{-nbor}}(\gamma)$. Similarly, $\Gamma_{\overline{C}\text{-nbor}}(\gamma) \subseteq \Gamma_{\overline{A}\text{-nbor}}(\gamma)$ and therefore $\Gamma_{\overline{C}\text{-nbor}}(\gamma) = \Gamma_{\overline{A}\text{-nbor}}(\gamma)$.

As a consequence of the Proposition 1.14, we have the following Corollary:

<u>Corollary 1.15</u>: Let $\gamma$: $I \to M$ be a null geodesic. Suppose that $\overline{A}$ and $\overline{C}$ are Jacobi tensors along $\gamma$ with $\overline{A}(t_o)$ and

$\overline{C}(t_o)$ non-singular and $\dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1} = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}$ for some $t_o \in I$. Then $\overline{A}(t)$ is non-singular for all $t \in I$ iff $\overline{C}(t)$ is non-singular for all $t \in I$. Moreover, $\dim\{\ker \overline{A}(t)\} = \dim\{\ker \overline{C}(t)\}$ for all $t \in I$ and $\dot{\overline{A}}(t)(\overline{A}(t))^{-1} = \dot{\overline{C}}(t)(\overline{C}(t))^{-1}$ for all $t \in I$ whenever $\overline{A}(t)$ and $\overline{C}(t)$ are non-singular.

<u>Proof</u>: From the Proposition 1.14, $\Gamma_{\overline{A}\text{-nbor}}(\gamma) = \Gamma_{\overline{C}\text{-nbor}}(\gamma)$ since $\dot{\overline{A}}(t_o)(\overline{A}(t_o))^{-1} = \dot{\overline{C}}(t_o)(\overline{C}(t_o))^{-1}$. Therefore, if $\overline{A}(t)$ is singular at some $t \in I$ then there exists a Jacobi class $\overline{X} \in \Gamma_{\overline{A}\text{-nbor}}(\gamma)$ with $\overline{X}(t) = 0$. But from Proposition 1.14, $\overline{X}$ is in $\Gamma_{\overline{C}\text{-nbor}}(\gamma)$ and therefore $\overline{C}(t)$ is also singular (and vice versa). That is, $\overline{A}(t)$ is singular iff $\overline{C}(t)$ is singular. On the other hand, since $\dim\{\ker \overline{A}(t)\}$ is equal to the dimension of the space of Jacobi classes in $\Gamma_{\overline{A}\text{-nbor}}(\gamma)$ vanishing at t, it follows that $\dim \ker \overline{A}(t) \leq \dim\{\ker \overline{C}(t)\}$ because $\Gamma_{\overline{A}\text{-nbor}}(\gamma) = \Gamma_{\overline{C}\text{-nbor}}(\gamma)$. Similarly, $\dim \ker \overline{C}(t) \leq \dim \ker \overline{A}(t)$. Finally, it remains to show that $\dot{\overline{A}}(t)(\overline{A}(t))^{-1} = \dot{\overline{C}}(t)(\overline{C}(t))^{-1}$ for all $t \in I$. Let $\overline{x}$ be any vector in $G(\gamma(t))$ at some $t \in I$. Let $\overline{X} \in \Gamma_{\overline{A}\text{-nbor}}(\gamma)$ $(=\Gamma_{\overline{C}\text{-nbor}}(\gamma))$ such that $\overline{X}(t) = \overline{x}$. Then $\dot{\overline{A}}(t)(\overline{A}(t))^{-1}\overline{x} = \dot{\overline{X}}(t) = \dot{\overline{C}}(t)(\overline{C}(t))^{-1}\overline{x}$ as in the proof of Proposition 1.14 since $\overline{X}$ is in both $\Gamma_{\overline{A}\text{-nbor}}(\gamma)$ and $\Gamma_{\overline{C}\text{-nbor}}(\gamma)$. That is, $\dot{\overline{A}}(t)(\overline{A}(t))^{-1}\overline{x} = \dot{\overline{C}}(t)(\overline{C}(t))^{-1}\overline{x}$ for every $\overline{x} \in G\gamma(t))$ and for all $t \in I$. ∎

<u>Remark</u>: We note that the condition $\overline{A}(t_o)$ non-singular is only for definiteness. In fact, $\overline{A}$ is non-singular except on a set of isolated points since zeros of Jacobi classes are isolated.

<u>Theorem 1.16</u>: Let H be a spacelike surface in M and $\gamma\colon [0,a] \to M$ be a null geodesic with $\gamma(0) \in H$ and $\dot{\gamma}(0) \perp H$. Then the following are equivalent:

i) $t \in (0,a)$ is a focal point of H along $\gamma$.

ii) there exists a Lagrange tensor $\overline{A}$ along $\gamma$ such that $\overline{A}(0)$ is non-singular, $\dot{\overline{A}}(0) = -\overline{L}_{\dot{\gamma}(0)}\overline{A}(0)$ and $\ker \overline{A}(t) \neq \{0\}$ where $\overline{L}_{\dot{\gamma}(0)}$ is the second fundamental tensor of H in the direction $\dot{\gamma}(0)$.

<u>Proof</u>: We recall that $t \in (0,a)$ is a focal point of H along $\gamma$ iff there exists a Jacobi class $\overline{X}$ along $\gamma$ with $\overline{X}(0) \neq 0$, $\dot{\overline{X}}(0) = -\overline{L}_{\dot{\gamma}(0)}\overline{X}(0)$ and $\overline{X}(t) = 0$. On the other hand, if $\overline{A}$ is a Jacobi tensor along $\gamma$ and $\overline{X} = \overline{A}(\overline{E})$ where $\overline{E}$ is a parallel section of $G(\gamma)$, then $\dot{\overline{X}} = \overline{\nabla}_{\dot{\gamma}(0)}\overline{X} = \overline{\nabla}_{\dot{\gamma}}(\overline{A}(\overline{E})) = \dot{\overline{A}}(\overline{E}) + \overline{A}(\overline{\nabla}_{\dot{\gamma}}\overline{E}) = \dot{\overline{A}}(\overline{E}) = \dot{\overline{A}}(\overline{A})^{-1}(\overline{X})$ whenever $\overline{A}$ is non-singular. In particular, $\dot{\overline{X}}(0) = \dot{\overline{A}}(0)(\overline{A}(0))^{-1}\overline{X}(0)$. Therefore, by comparing $\dot{\overline{X}}(0) = -\overline{L}_{\dot{\gamma}(0)}\overline{X}(0)$ and $\dot{\overline{X}}(0) = \dot{\overline{A}}(0)(\overline{A}(0))^{-1}\overline{X}(0)$, we see that the set of neighbors $\Gamma_{\overline{A}\text{-nbor}}(\gamma)$, induced by the Jacobi tensor satisfying the initial conditions $\overline{A}(0)$ non-singular and $\dot{\overline{A}}(0) = -\overline{L}_{\dot{\gamma}(0)}\overline{A}(0)$

(that is $\dot{\overline{A}}(0)(\overline{A}(0))^{-1} = -\overline{L}_{\dot{\gamma}(0)}$) are the Jacobi classes along $\gamma$ with $\overline{X}(0) \neq 0$ and $\dot{\overline{X}}(0) = -\overline{L}_{\dot{\gamma}(0)}\overline{X}(0)$. ∎

Now, we shall obtain the Raychaudhuri equation for a Jacobi tensor $\overline{A}$ along a null geodesic $\gamma$.

<u>Lemma 1.17</u>: Let $\gamma\colon I \to M$ be a null geodesic. Then $\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) = \sum_{i=1}^{n-2} \overline{g}(\overline{R}(\overline{X}_i,\dot{\gamma})\dot{\gamma},\overline{X}_i)$ where $\overline{X}_i$, $i=1,2,\ldots,n-2$ are orthonormal sections of $G(\gamma)$.

<u>Proof</u>: [BE] page 352. ∎

<u>Proposition 1.18</u>: Let $\gamma\colon I \to M$ be a null geodesic and $\overline{A}$ be a Jacobi tensor along $\gamma$. Let $\overline{B} = \dot{\overline{A}}(\overline{A})^{-1}$ whenever $\overline{A}$ is non-singular and

$$\overline{\theta} = \mathrm{tr}\,\overline{B} \quad \text{(expansion)}$$

$$\overline{\sigma} = \frac{1}{2}(\overline{B}+\overline{B}^*) - \frac{\overline{\theta}}{n-2}\,\overline{\mathrm{id}} \quad \text{(shear)}$$

$$\overline{\omega} = \frac{1}{2}(\overline{B}-\overline{B}^*) \quad \text{(vorticity)}$$

Then

$$\dot{\overline{\theta}} = -\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) - \mathrm{tr}\,\overline{\omega}^2 - \mathrm{tr}\,\overline{\sigma}^2 - \frac{\overline{\theta}^2}{n-2} \quad \text{(Raychaudhuri equation for } \overline{A}\text{)}.$$

If $\overline{A}$ is a Lagrange tensor then $\overline{\omega} = 0$ since $\overline{B}$ is self-adjoint. Therefore, the Raychaudhuri equation for a

Lagrange tensor $\overline{A}$ is given by

$$\dot{\overline{\theta}} = -\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) - \mathrm{tr}\ \overline{\sigma}^2 - \frac{\overline{\theta}^2}{n-2}\ .$$

Proof: [BE] page 351. ∎

Remark: Since $\overline{\theta} = \mathrm{tr}\ \overline{B} = \mathrm{tr}\ \dot{\overline{A}}(\overline{A})^{-1} = \dfrac{\dot{\overline{\det \overline{A}}}}{\det \overline{A}}$ (cf. [BE] page 351), $\lim\limits_{t\to t_o} |\overline{\theta}(t)| = \infty$ iff $\det \overline{A}(t_o) = 0$ iff $\ker \overline{A}(t_o) \neq \{0\}$ for $t_o \in I$.

Therefore, if $\gamma$: $[0,a) \to M$ is a null geodesic orthogonal to a spacelike surface $H$ at $\gamma(0)$ and $\overline{A}$ is a Jacobi tensor along $\gamma$ with $\overline{A}(0)$ non-singular, $\dot{\overline{A}}(0)(\overline{A}(0))^{-1} = -\overline{L}_{\dot{\gamma}(0)}$ then, $t_o \in (0,a)$ is a focal point of $H$ along $\gamma$ iff $\lim\limits_{t\to t_o} |\theta(t)| = \infty$ where $\overline{\theta} = \mathrm{tr}\ \dot{\overline{A}}(\overline{A})^{-1}$.

Lemma 1.19: Let $\gamma$: $(a,b) \to M$ be a null geodesic, satisfying $\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) \geq 0$ for all $t \in (a,b)$. Let $\overline{A}$ be a Lagrange tensor along $\gamma$ and $\overline{\theta} = \mathrm{tr}\ \dot{\overline{A}}(\overline{A})^{-1}$. Then

i) if $\overline{\theta}(t_1) < 0$ for some $t_1 \in (a,b)$ then $\det \overline{A}(t) = 0$ for some $t \in (t_1, t_1 - \frac{n-2}{\overline{\theta}(t_1)})$, provided that $t_1 - \frac{n-2}{\overline{\theta}(t_1)} < b$

ii) if $\overline{\theta}(t_1) > 0$ for some $t_1 \in (a,b)$ then $\det \overline{A}(t) = 0$ for some $t \in (t_1 - \frac{n-2}{\overline{\theta}(t_1)}, t_1)$, provided that $t_1 - \frac{n-2}{\overline{\theta}(t_1)} > a$.

Proof: [BE] page 355. ■

The above Lemma can be extended to the case where $\overline{\theta}(t_1) = 0$, provided that the generic condition $\overline{R}(\cdot,\dot{\gamma}(t_1))\dot{\gamma}(t_1) \neq 0$ holds at $t_1$.

Lemma 1.20: Let $\gamma$ be a null geodesic and $\overline{A}$ be a non-singular Jacobi tensor along $\gamma$. Then $\dot{\overline{B}} = -\overline{R}(\cdot,\dot{\gamma})\dot{\gamma} - \overline{B}^2$ where $\overline{B} = \dot{\overline{A}}(\overline{A})^{-1}$.

Proof: [BE] page 349. ■

Proposition 1.21: Let $\gamma$: $(a,b) \to M$ be a null geodesic, satisfying $\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) \geq 0$ for all $t \in (a,b)$. Let $\overline{A}$ be a Lagrange tensor along $\gamma$ and $\overline{\theta} = \dot{\overline{A}}(\overline{A})^{-1}$. If $\overline{\theta}(t_1) = 0$ but $\overline{R}(\cdot,\dot{\gamma}(t_1))\dot{\gamma}(t_1) \neq 0$ for some $t_1 \in (a,b)$, then $\det \overline{A}(t) = 0$ for some $t \in (t_2, t_2 - \frac{n-2}{\overline{\theta}(t_2)})$ for each $t_2 > t_1$, provided that $t_2 - \frac{n-2}{\overline{\theta}(t_2)} < b$.

Proof: Since $\overline{\theta}(t_1) = 0$, $\mathrm{Ric}(\dot{\gamma},\dot{\gamma}) \geq 0$ and $\mathrm{tr}\ \overline{\sigma}^2 \geq 0$

$$\dot{\overline{\theta}}(t_1) = -\mathrm{Ric}(\gamma(t_1),\gamma(t_1)) - \mathrm{tr}\ \overline{\sigma}(t_1)^2 - \frac{\overline{\theta}^2(t_1)}{n-2}$$

$$= -\mathrm{Ric}(\gamma(t_1),\gamma(t_1)) - \mathrm{tr}\ \overline{\sigma}(t_1)^2 \leq 0.$$

Thus, we have two cases when $\overline{\theta}(t_1) = 0$.

Case i: If $\dot{\overline{\theta}}(t_1) < 0$ then $\overline{\theta}(t_2) < 0$ for each $t_2 > b_1$. Thus, from the Lemma 1.19-(i), det $\overline{A}(t) = 0$ for some $t \in (t_2, t_2 - \frac{n-2}{\overline{\theta}(t_2)})$, provided that $t_2 - \frac{n-2}{\overline{\theta}(t_2)} < b$ where $t_2 > t_1$.

Case ii: If $\dot{\overline{\theta}}(t_1) = 0$, let $t_1' = \sup\{t \geq t_1 | \dot{\overline{\theta}}([t_1, t_2)) = 0\} \geq t_1$.

If $t_1 = t_1'$ then $\dot{\overline{\theta}}(t_2) < 0$ for each $t_2 > t_1$ and it follows that $\overline{\theta}(t_2) < 0$ for each $t_2 > t_1$. Thus as in the case (i), det $\overline{A}(t) = 0$ for some $t \in (t_2, t_2 - \frac{n-2}{\overline{\theta}(t_2)})$ provided that $t_2 - \frac{n-2}{\overline{\theta}(t_2)} < b$ where $t_2 > t_1$. Therefore it suffices to show that $t_1 = t_1'$. Assume not, that is $t_1' > t_1$. Then $0 = \dot{\overline{\theta}} = -\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \mathrm{tr}\ \overline{\sigma}^2 - \frac{\overline{\theta}^2}{n-2}$ on $[t_1, t_1']$ and therefore $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) = 0$, $\mathrm{tr}\ \overline{\sigma}^2 = 0$, and $\overline{\theta} = 0$ on $[t_1, t_1']$. However, since $\overline{\sigma}$ is self-adjoint, $\mathrm{tr}\ \overline{\sigma}^2 = 0$ implies that $\overline{\sigma} = 0$ and therefore $\overline{B} = 0$ (since $\overline{B} = \frac{\overline{\theta}}{n-2}\ \overline{\mathrm{id}} + \overline{\sigma}$). But this is a contradiction with the assumption $\overline{R}(\cdot, \dot{\gamma}(t_1))\dot{\gamma}(t_1) \neq 0$ since

$$
\begin{aligned}
0 = \dot{\overline{B}}(t_1) &= -\overline{R}(\cdot, \dot{\gamma}(t_1))\dot{\gamma}(t_1) - \overline{B}^2(t_1) \quad \text{(cf. Lemma 1.21)} \\
&= -\overline{R}(\cdot, \dot{\gamma}(t_1))\dot{\gamma}(t_1). \quad \blacksquare
\end{aligned}
$$

<u>Corollary 1.22</u>: Let $\gamma\colon (a,b) \to M$ be a null geodesic, satisfying $\mathrm{Ric}(\dot\gamma,\dot\gamma) \geq 0$ for all $t \in (a,b)$. If $\overline{\theta}(t_1) = 0$ but $\overline{R}(\cdot,\dot\gamma(t_1))\dot\gamma(t_1) \neq 0$ for some $t_1 \in (a,b)$ then $\overline{\theta}(t) < 0$ for all $t > t_1$ and $\overline{\theta}(t) > 0$ for all $t < t_1$.

<u>Proof</u>: Immediate from the proof of Corollary 1.21. ∎

# CHAPTER II

# NULL HYPERSURFACES

We recall that an embedded hypersurface S in a Lorentzian manifold M is called a null hypersurface if $T_pS$ (tangent space of S at p) is a null vector space for each $p \in S$. By definition, the restriction of the metric g of M to S is degenerate with signature $(0{+}{+}\cdots{+})$.

In the literature, null hypersurfaces have been studied either as a part of achronal boundaries or as hypersurfaces constructed from null geodesic congruences orthogonal to spacelike surfaces (cf. [P], pages 21 and 60). However, not every null hypersurface in a spacetime can be constructed by using the above methods.

In this chapter, we shall study null hypersurfaces, in general. In section A, we shall investigate some elementary properties of null hypersurfaces in Lorentzian manifolds. In section B, we shall obtain necessary and sufficient conditions for the separation and foliation of null hypersurfaces by spacelike surfaces. In section C, we shall discuss the deviation of null congruences in a null hypersurface S. Furthermore, we shall obtain a canonical definition of the second fundamental tensor $\overline{L}$ of a null hypersurface which is useful in the analysis of the deviation of null congruences in S. We shall show that $\overline{L} = 0$ iff S is a totally geodesic submanifold (that is, the restriction of the connection of spacetime defines a connection on S). We shall then relate $\overline{L} = 0$ with the deviation of null congruences in S and with the induced metric on S.

*Section A*: Elementary Properties of Null Hypersurfaces

*Definition 2.1*: An embedded, *connected* hypersurface S in an $n(\geq 3)$-dimensional Lorentzian manifold is called a *null hypersurface* if $T_pS$ is a null vector space at each $p \in S$.

We recall that if W is a subspace of a Lorentzian vector space V then $\dim W + \dim W = \dim V$ but $W \cap W^{\perp} \neq \{0\}$ unless W is a non-degenerate subspace of V (where $W^{\perp}$ is the orthogonal space of W) (cf. [0], page 49). If W is a 1-dimensional null subspace of V then $W^{\perp}$ is a null hyperplane which contains W since no non-spacelike vector is orthogonal to a null vector unless they are proportional (cf. [SW], page 21).

Proposition 2.2: Let S be an achronal hypersurface in a Lorentzian manifold M with the property that for every $p \in S$, there passes a null geodesic $\gamma_p\colon (-a,a) \to M$ with $\gamma_p(0) = p$ and $\gamma_p((-a,a)) \subset S$ for some $a > 0$. Then S is an (achronal) null hypersurface.

Proof: Let L be the vector bundle over S defined by

$$L = \bigcup_{p \in S} \{t\dot{\gamma}_p(0) \mid t \in \mathbb{R} \text{ and } \gamma_p\colon (-a,a) \to M \text{ is a null geodesic with } \gamma_p(0) = p \text{ and } \gamma_p((-a,a)) \subset S\}$$

(To show that L is well-defined, it suffices to show that there passes precisely one such $\gamma_p$ (up to parametrization) at each point $p \in S$. In fact, if there passes another null geodesic $\eta_p$ from p with the properties of $\gamma_p$ but not a reparametrization of $\gamma_p$, then $\dot{\gamma}_p(0)$ and $\dot{\eta}_p(0)$ are not proportional and span a timelike vector z tangent to S at p. Thus, if $\alpha$ is a curve in S with $\dot{\gamma}(0) = z$ then $\alpha$ is a timelike curve in some neighborhood of p in S in contradiction with the achronality of S.)

Let $L^\perp$ be the orthogonal bundle to L. Since L has 1-dimensional null fibers, L is a vector bundle along S with (n-1)-dimensional null fibers. We shall show that $L^\perp = TS$. Assume $T_pS \neq L_p^\perp$ for some $p \in S$. By definition,

$L_p^\perp$ contains all vectors which are orthogonal to $\dot{\gamma}_p(0)$ in $T_pM$. Therefore, since $T_pS \neq L_p^\perp$, there exists a vector $v$ in $T_pS$ which is not orthogonal to $\dot{\gamma}_p(0)$. On the other hand, since $L_p^\perp \cap T_pS$ contains $\dot{\gamma}_p(0)$, $v + t\dot{\gamma}_p(0) \in T_pS$ for every $t \in \mathbb{R}$. In fact, we can choose $t$ such that $v + t\dot{\gamma}_p(0)$ is a timelike vector. (Since $v$ is not orthogonal to $\dot{\gamma}_p(0)$, we may assume $\langle v,\dot{\gamma}_p(0)\rangle > 0$. Then, $\langle v+t\dot{\gamma}_p(0), v+t\dot{\gamma}_p(0)\rangle = \langle v,v\rangle + 2t\langle v,\dot{\gamma}_p(0)\rangle$ since $\dot{\gamma}_p(0)$ is a null vector. Therefore, we can choose $t \in \mathbb{R}$ such that $v + t\dot{\gamma}_p(0)$ is a timelike vector.) As before, if $\alpha$ is a curve in $S$ with $\dot{\alpha}(0) = v + t\dot{\gamma}_p(0)$, then $\alpha$ is a timelike curve in some neighborhood of $p$ in $S$ in contradiction with the achronality of $S$. ∎

<u>Remark</u>: The achronality assumption in the above proposition cannot be removed. That is, a hypersurface which admits a tangent null vector field $U$ is not necessarily a null hypersurface (see Figure 2).

We recall that if $W$ is a null hyperplane in a Lorentzian vector space $V$ then $W^\perp$ is a 1-dimensional null subspace of $V$ which is contained in $W$.

Definition 2.3: The normal bundle* $\mathfrak{N}(S)$ of a null hypersurface S in a Lorentzian manifold M is defined by

$$\mathfrak{N}(S) = \bigcup_{p\in S} \{u\in T_pM \mid \langle u,x\rangle = 0 \text{ for every } x \in T_pS\}$$

Note that $\mathfrak{N}(S)$ is a subbundle of TS with 1-dimensional null fibers since $\mathfrak{N}(S) = (TS)^{\perp}$.

Lemma 2.4: Let S be a null hypersurface in a time oriented Lorentzian manifold M. Then $\mathfrak{N}(S)$ is orientable.

Proof: Let Z be a future directed timelike vector field on M. Then, the section U of $\mathfrak{N}(S)$ which is defined by $\langle U_p, Z_p\rangle = -1$ at each $p \in S$ is a (smooth) future directed, nowhere vanishing section of $\mathfrak{N}(S)$. ∎

Remark: Since $\mathfrak{N}(S) \subseteq TS$, if U is a nowhere vanishing section of $\mathfrak{N}(S)$ then U is a null vector field on S. Therefore the above lemma implies that there exists a future directed null vector field U on every null hypersurface in a time oriented Lorentzian manifold. In fact, the vector field U is unique up to rescaling, that is, there exists no other null vector field $\tilde{U}$ on S which is not a scalar multiple of U. Indeed, if $\tilde{U}$ is such a vector field, then some linear combination of $\tilde{U}$ and U is a

* $\mathfrak{N}(S)$ is isomorphic to $TM|_S/TS$, but not canonically.

timelike vector field on S in contradiction with S being a null hypersurface.

We recall that a connected, orientable, time oriented Lorentzian manifold M is called a spacetime.

Corollary 2.5: Every null hypersurface S in a spacetime M is orientable.

Proof: Let Z be a unit future directed vector field on M and $\omega$ be the volume form of M. Since S is a null hypersurface, Z is transversal to S and therefore, the restriction of $i_Z\omega$ to S is a nowhere vanishing (n-1)-form on S. ∎

Corollary 2.6: Let S be a compact null hypersurface in a spacetime M. Then the Euler characteristic $\chi(S) = 0$.

Proof: Immediate from the proof of Lemma 2.4 and the theorem of Hopf (since the vector field U is a nowhere vanishing tangent vector field on S). ∎

Definition 2.7: Let U be a future directed null vector field on a null hypersurface S in a spacetime M. The bundle homomorphism $\nabla U$: $TS \to TS$ is defined by

$$(\nabla U)x := \nabla_x U \text{ where } x \in TS.$$

To show that $\nabla U$ is well-defined, it suffices to show that $\nabla_x U \in TS$ for all $x \in TS$. Indeed, for any $x \in TS$, $\langle\nabla_x U, U\rangle = \frac{1}{2}\, x\langle U,U\rangle = 0$ since $\langle U,U\rangle = 0$. Thus $\nabla_x U \in TS$.

<u>Corollary 2.8</u>: The bundle homomorphism $\nabla U$: $TS \to TS$ is self-adjoint.

<u>Proof</u>: Let X,Y be any two vector fields on S. Then, since $\langle U,X\rangle = \langle U,Y\rangle = \langle U,[X,Y]\rangle = 0$,

$$\begin{aligned}
\langle\nabla_X U, Y\rangle &= X\langle U,Y\rangle - \langle U,\nabla_X Y\rangle \\
&= -\langle U, \nabla_Y X + [X,Y]\rangle \\
&= -\langle U,\nabla_Y X\rangle - \langle U,[X,Y]\rangle \\
&= -Y\langle U,X\rangle + \langle\nabla_Y U, X\rangle \\
&= \langle\nabla_Y U, X\rangle \quad . \qquad \blacksquare
\end{aligned}$$

<u>Proposition 2.9</u>: Let U be a future directed null vector field on a null hypersurface S in a spacetime M. Then $\nabla_U U = fU$ where $f: S \to \mathbb{R}$ is a smooth function.

<u>Proof</u>: It suffices to show that $\nabla_U U$ is orthogonal to every vector on S. Let X be any vector field on S. Then

$$\begin{aligned} \langle \nabla_U U, X\rangle &= U\langle U,X\rangle - \langle U, \nabla_U X\rangle \\ &= -\langle U, \nabla_X U\rangle - \langle U, [U,X]\rangle \\ &= -\tfrac{1}{2}X\langle U,U\rangle = 0 \end{aligned}$$

since $\langle U,X\rangle = \langle U,U\rangle = \langle U,[U,X]\rangle = 0$. Thus, $\nabla_U U$ is a scalar multiple of U at each point. ∎

Remark: Let U be a null vector field on a null hypersurface S in a spacetime M. Then U is a null pregeodesic vector field on S unique up to rescaling from the remark below the Lemma 2.4. Since each integral curve $\gamma$ of U is a null pregeodesic, $\gamma$ can be parametrized to be a null geodesic. Therefore, an integral curve $\gamma$ of a null vector field U on S is called a null generator of S and if $\gamma$ is parametrized as a null geodesic then $\gamma$ is called a null geodesic generator of S.

## Section B: Separation and Foliation of Null Hypersurfaces by Spacelike Surfaces.

We recall that if H is a spacelike surface in a Lorentzian manifold then $\mathbb{N}(H)$ has 2-dimensional timelike fibers, each of which contains two well-defined null directions. If H is contained in a spacetime M then, we can choose two future directed, non-proportional null vectors in each fiber of H. (In fact, using local triviality of $\mathfrak{N}(H)$, locally, we can find two future directed, non-proportional null sections of $\mathfrak{N}(H)$.) Suppose H is a spacelike surface with trivial normal bundle $\mathfrak{N}(H)$. Let U and W be two future directed non-proportional null sections of $\mathfrak{N}(H)$. Define a subbundle of $\mathfrak{N}(H)$ with 1-dimensional null fibers by $\mathfrak{N}_U(H) = \bigcup_{p \in H} \{tU_p \mid t \in \mathbb{R}\}$. Let $\exp^{\perp}$ be the exponential map of $\mathfrak{N}(H)$. Then, there exists an open neighborhood V of H in $\mathfrak{N}_U(H)$ where the restriction of $\exp^{\perp}$ on V is an embedding into M. In fact, $S = \exp^{\perp}(V)$ is a null hypersurface in M with the property that each null geodesic in S intersects H at precisely one parameter value (this is the method of constructing null hypersurfaces that we have mentioned on page 23). This null hypersurface has the property that there exists a <u>geodesic</u> <u>null</u> <u>vector</u> field U on S which is obtained by taking the velocity vectors of the null geodesics emanating from H in the direction $U_p$ at each point p of H.

In general, a null hypersurface S may not have this property, that is, the null vector field U on S may not be rescaled to be a geodesic null vector field. In this section, we shall obtain sufficient conditions for a spacelike surface H in M to have the property that each null geodesic in S intersects H at precisely one parameter value. In this case, we shall show that it is possible to rescale any null vector field U on S to be a geodesic null vector field on S. Furthermore, we shall obtain necessary and sufficient conditions for the foliation of null hypersurfaces by (immersed) spacelike surfaces.

First, we shall show that every spacelike surface in a null hypersurface of a spacetime has trivial normal bundle.

Proposition 2.10: Let H be a spacelike surface in a null hypersurface S of a spacetime M. Then $\mathbf{N}(H)$ is trivial vector bundle and it follows that H is orientable.

Proof: Let U be a future directed null vector field on S. Since $U \perp H$, $U|_H \in \Gamma\mathbf{N}(H)$. Since $\mathbf{N}(H)_p$ is a two-dimensional timelike subspace of $T_pM$ at each $p \in H$, we can define another null section W of $\mathbf{N}(H)$ (which is not a scalar nultiple of $U|_H$) by $\langle W, U|_H \rangle|_p = -1$ at each $p \in H$. Thus, $U|_H$ and W are two future directed, linearly independent null sections of $\mathbf{N}(H)$ which give a trivialization of $\mathbf{N}(H)$. ∎

Corollary 2.11: Let S be a null hypersurface in a 4-dimensional spacetime M. If H is a compact spacelike surface in S then H is classified topologically by its genus.

Proof: Since a spacelike surface H is by definition a connected submanifold of M and H is orientable from the Proposition 2.10, it is classified by its genus. ■

Definition 2.12: Let S be null hypersurface in a spacetime M and let H be a spacelike surface in S. S is said to be causally separated by H if there exists a diffeomorphism $g: S \to H \times \mathbb{R}$ such that, for each $q \in H$, $g^{-1}(\{q\} \times \mathbb{R})$ is a null generator of S.

Theorem 2.13: Let S be a null hypersurface in a spacetime M and let H be a spacelike surface in S with the property that every inextendable null generator of S intersects H at precisely one parameter value. Then H causally separates S.

Proof: Let U be a null vector field on S. Then, U can be rescaled to be a complete null vector field $\tilde{U}$ on S. (That is, there exists a positive function f on S such that $\tilde{U} = fU$ is a complete vector field and therefore, each integral line of $\tilde{U}$ is a reparametrization of an integral curve of U (cf. [BJ], page 85, ex. 2)). Let $\varphi$ be the flow of $\tilde{U}$. Then the map $h: H \times \mathbb{R} \to S$ which is defined by $h(p,t) = \varphi_t(p)$ is a diffeomorphism since every integral curve of $\tilde{U}$ intersects H at precisely one parameter value. ∎

Now, we shall obtain sufficient conditions for the causal separation of null hypersurfaces by spacelike surfaces.

Lemma 2.14: Let S be a simply connected null hypersurface in a spacetime M and let H be a closed (in S) spacelike surface in S. Then, no null generator of S intersects H at more than one parameter value.

Proof: We recall that if H is a closed connected submanifold of codimension 1 in a simply connected manifold S then H separates S (that is S-H is disconnected (cf., [H], page 108). Let U be a future directed null vector field on S. Since H is a spacelike surface in S, H is orthogonal to U. (In particular, U is transversal to H in S.) Therefore U points into the same connected component of S-H everywhere along H. (By this, we mean that the future directed integral curves of U emanating from each point of H enter the same connected component of S-H.) Assume, a future directed integral curve of U emanating from $\gamma(0) = p \in H$ hits H in the future at $\gamma(t) = q \in H$ $(t>0)$. Since $\{\gamma((0,t))\}$ is contained in a connected component of S-H, $\dot{\gamma}(t)$ would point to the other connected component of H-S in contradiction with the fact that U always points to the same connected component of S-H. ∎

Corollary 2.15: Let S be a simply connected null hypersurface in a spacetime M and let H be a closed (in S) spacelike surface in S with the property that every inextendable null generator of S intersects H. Then H causally separates S. In particular, since S is diffeomorphic to $H \times \mathbb{R}$, H is simply connected.

Proof: Let U be a future directed null vector field on S. Then from the assumption that every inextendable null generator of S intersects H and Lemma 2.14, each inextendable null generator of S intersects H at only one parameter value. Then from the Theorem 2.12, H causally separates S. ∎

The above results can be summarized for 4-dimensional spacetimes as follows:

Proposition 2.16: Let S be a simply connected null hypersurface in a 4-dimensional spacetime M. Let H be a closed (in S) spacelike surface in S. Assume each inextendable null generator of S intersects H. Then H causally separates S and is topologically either $S^2$ or $\mathbb{R}^2$ depending on whether it is compact or non-compact.

<u>Proof</u>: From Corollary 2.15, H causally separates S and is simply connected. Thus, it is topologically $S^2$ if it is compact or $\mathbb{R}^2$ if it is non-compact. ∎

In general, one cannot causally separate a null hypersurface by a spacelike surface (for example, the Cauchy horizons in the Taub-NUT spacetime). However, the assumptions of Proposition 2.16 hold on the event horizons of the Kruskal, Reissner-Nordstrom and Kerr black holes which are simply connected and causally separated by spacelike surfaces topologically $S^2$.

In the study of deviation of integral curves of the pregeodesic null vector field U, we shall handle problems which do not involve the whole null hypersurface S, but only a tubular neighborhood of $\gamma$ in S. For such considerations, we shall introduce the concept of elementary neighborhoods of an integral curve $\gamma$ of U (which is localized version of the causal separability of a null hypersurface).

<u>Definition 2.17</u>: Let S be a null hypersurface in a spacetime M and U be a future directed null vector field

on S. Let $\gamma$ be an inextendable integral curve of U. An open subset $V_\gamma$ of S containing $\gamma$ is called an elementary neighborhood of $\gamma$ if there exists a spacelike surface $H_\gamma$ in S and a diffeomorphism $g_\gamma\colon V_\gamma \to H_\gamma \times \mathbb{R}$ such that, for each $q \in H_\gamma$, $g^{-1}(\{q\}\times\mathbb{R})$ is an inextendable integral curve (up to parametrization) of U passing through q.

Remark: By definition, an elementary neighborhood of an inextendable null pregeodesic in a null hypersurface is also a null hypersurface.

Proposition 2.18: Let S be a null hypersurface in a strongly causal spacetime M. Then each inextendable null generator of S has an elementary neighborhood.

Proof: Let U be a future directed null vector field on S and let $\gamma$ be an inextendable integral curve of U. Let p be a point on $\gamma$ and L be a local causality neighborhood of p in M (cf. [P], page 30). Let H be a spacelike surface in $L \cap S$ containing p. Then, no integral curve of U emanating from $H_\gamma$ ever returns to $H_\gamma$. Therefore, the open subset $V_\gamma$ of S which is traced by the inextendable integral curves of U meeting $H_\gamma$ is an elementary neighborhood of $\gamma$ in S. ∎

<u>Theorem 2.19</u>: Let $S$ be a null hypersurface in a spacetime $M$ and $U$ be a future directed null vector field on $S$. If an inextendable integral curve $\gamma$ of $U$ has an elementary neighborhood $V_\gamma$ in $S$ then the vector field $U$ can be rescaled to be a geodesic null vector field along $V_\gamma$.

<u>Proof</u>: Let $H_\gamma$ be a spacelike surface which causally separates $V_\gamma$. Then, by taking the velocity vectors of the inextendable (in $V_\gamma$) null geodesics $\eta$ emanating from each $p \in H_\gamma$ with $\dot{\eta}(0) = U(p)$, we obtain a future directed null vector field on $V_\gamma$. ∎

<u>Remark</u>: By definition, if a null hypersurface $S$ in a spacetime $M$ is causally separated by a spacelike surface $H$ then $S$ is elementary neighborhood of every inextendable null pregeodesic in $S$ and therefore there exists a future directed geodesic null vector field $U$ on $S$.

Now, we shall discuss the foliation of null hypersurfaces by (immersed) spacelike surfaces.

<u>Definition 2.20</u>: Let $S$ be a null hypersurface in an n-dimensional spacetime $M$. Let $\mathcal{D}$ be a subbundle of $TS$ with (n-2)-dimensional spacelike fibers. Then the <u>orthogonal bundle</u> $\mathcal{D}^\perp$ to $\mathcal{D}$ in $TM|_S$ is defined by

$\mathcal{D}^{\perp} = \bigcup_{p \in S} \mathcal{D}_p^{\perp}$ where $\mathcal{D}_p^{\perp}$ is the orthogonal complement of $\mathcal{D}_p$ in $T_pM$.

Note that $\mathcal{D}^{\perp}$ is a vector bundle over S with 2-dimensional timelike fibers and by definition, every future directed null vector field U on S is a section of $\mathcal{D}^{\perp}$. Also, using U, we can determine another future directed null section W of $\mathcal{D}^{\perp}$ by $\langle W,U \rangle = -1$. (Note that W is not orthogonal to S. Thus, U and W are linearly independent and therefore $\mathcal{D}^{\perp}$ is a trivial vector bundle.)

Remark: Let S be a null hypersurface in an n-dimensional spacetime M and let $\mathcal{D}$ be a subbundle of TS with (n-2)-dimensional spacelike fibers. Let U and W be non-proportional (linearly independent) null sections of $\mathcal{D}^{\perp}$. (Thus either U or W is tangent to S.) Then the vector bundle $W^{\perp}$ is defined by $W^{\perp} = \bigcup_p W_p^{\perp}$ where $W_p^{\perp}$ is the orthogonal space to $W_p$. Note that $W^{\perp}$ has null fibers of codimension 1 in $TM|_S$ and since W is a null section of $\mathcal{D}^{\perp}$, $\mathcal{D}$ is a subbundle of $W^{\perp}$. Moreover, every $y \in W^{\perp}$ can be uniquely written as the linear combination of a vector in direction W and a vector in $\mathcal{D}$. That is, if $y \in W_p^{\perp}$ then $y = a_0 W(p)+v$ where $v \in \mathcal{D}_p$. To show that this decomposition is unique, it suffices to show $a_0$ is unique. Indeed,

$a_o = \frac{<y,U>}{<U,W>}$ since $U \in \Gamma\mathcal{D}$ and $<U,W> \neq 0$.

Thus, the projection of a vector $y \in W^{\perp}$ into $\mathcal{D}$ is defined by $\text{proj}_W(y) = v$ where $v$ is the component of $y$ in $\mathcal{D}$.

<u>Definition 2.21</u>: Let S be a null hypersurface in an n-dimensional spacetime M and let $\mathcal{D}$ be a subbundle of TS with (n-2)-dimensional spacelike fibers. Let W be a null section of $\mathcal{D}^{\perp}$. Then the bundle homomorphism $(\nabla W)^T: \mathcal{D} \to \mathcal{D}$ is defined by $(\nabla W)^T x := (\nabla_x W)^T$ where $(\nabla_x W)^T$ is the component of $\nabla_x W$ in $\mathcal{D}$.

To show that $(\nabla W)^T$ is well-defined, it suffices to show that $\nabla_x W \in W^{\perp}$ for all $x \in \mathcal{D}$. Indeed, $<\nabla_x W, W> = \frac{1}{2}x<W,W> = 0$ since $<W,W> = 0$. Therefore $\nabla_x W \in W^{\perp}$.

<u>Theorem 2.22</u>: Let S be a null hypersurface in an n-dimensional spacetime M and let U be a future directed null vector field on S. Let $\mathcal{D}$ be a subbundle of TS with (n-2)-dimensional spacelike fibers and W be a null section of $\mathcal{D}^{\perp}$ with $<U,W> \neq 0$ everywhere. Then, $\mathcal{D}$ is integrable iff $(\nabla W)^T: \mathcal{D} \to \mathcal{D}$ is self-adjoint.

<u>Proof</u>: We shall use the fact that $\mathcal{D}$ is integrable iff, for $X,Y \in \Gamma\mathcal{D}$, $[X,Y] \in \Gamma\mathcal{D}$, that is, iff $[X,Y] \perp \mathcal{D}^{\perp}$. Let $X,Y \perp \Gamma\mathcal{D}$ then

$$\begin{aligned} <[X,Y],U> &= <\nabla_X Y - \nabla_Y X, U> \\ &= <\nabla_X Y, U> - <\nabla_Y X, U> \\ &= X<Y,U> - <Y,\nabla_X U> - Y<X,U> + <X,\nabla_Y U> \\ &= <X,\nabla_Y U> - <Y,\nabla_X U> = 0 \text{ from Corollary 2.8.} \end{aligned}$$

Also,

$$\begin{aligned} <[X,Y],W> &= <\nabla_X Y - \nabla_Y X, W> \\ &= <\nabla_X Y, W> - <\nabla_Y X, W> \\ &= X<Y,W> - <Y,\nabla_X W> + Y<X,W> + <X,\nabla_Y W> \\ &= <X,(\nabla_Y W)^T> - <Y,(\nabla_X W)^T> \\ &= 0 \text{ for all } X,Y \in \Gamma\mathcal{D} \text{ iff } (\nabla W)^T \text{ is self-adjoint.} \end{aligned}$$

(In the third step, we used $<Y,W> = 0 = <X,W>$).
Therefore, $[X,Y] \in \Gamma\mathcal{D}$ iff $(\nabla W)^T$ is self-adjoint. ■

Corollary 2.23: Let S be a null hypersurface in a space-time M. Let U be a future directed tangent null vector field on S and let W be a null vector field along S with $<W,U> \neq 0$ everywhere on S. Then the vector bundle $\mathcal{D} = \text{span}\{U,W\}^\perp$ is integrable iff $(\nabla W)^T: \mathcal{D} \to \mathcal{D}$ is self-adjoint. If $\mathcal{D}$ is integrable then S can be foliated by (immersed) spacelike surfaces.

Proof: Immediate from the Frobenius theorem and Theorem 2.22. ■

Section C: Deviation of Null Generators of Null Hyper-Surfaces

Let U be a nowhere vanishing vector field on a manifold S, $\varphi$ be its flow and $\gamma$: I $\to$ S be an integral curve of U. We recall that a vector field X along $\gamma$ is called Lie parallel (with respect to U) iff $X(s+t) = \varphi_{t_*}X(s)$ whenever $s \in I$ and $s+t \in I$. (cf. [SW], page 39). In fact, it can be shown that a vector field X along $\gamma$ is Lie parallel iff for each $t \in I$, there is a neighborhood J of t in I, a neighborhood V of $\gamma(t)$ in S, and a vector field Y on V such that $\mathfrak{L}_U Y = 0$ and $X = X\circ\gamma$ on J. Thus, it follows that the Lie parallel vector fields along $\gamma$ form an n-dimensional vector space (cf. [SW], page 39). In the context of this study, we shall also indicate the condition that a vector field X along $\gamma$ be Lie parallel by $\mathfrak{L}_U X = 0$.

We recall that there exists a future directed pregeodesic null tangent vector field U on every null hypersurface S in a spacetime M. Let $\varphi$ be the flow of U and $\gamma$: $(-a,a) \to S$ be an integral curve of U. Let $\alpha$: $(-b,b) \to S$ be a spacelike curve with $\alpha(0) = \gamma(0)$. Then, we can define a variation of $\gamma$ to nearby integral curves of U by $\psi(s,t) = \varphi_t(\alpha(s))$. Then the variation vector field $X = \psi_* \left.\frac{\partial}{\partial s}\right|_{o,t} = \varphi_t(\alpha(0))$ is Lie parallel along $\gamma$ since $\psi_* \frac{\partial}{\partial t} = U\big|_{\mathrm{im}\,\psi}$ and $[\psi_*\frac{\partial}{\partial s}, \psi_*\frac{\partial}{\partial t}] = 0$. The vector field X (the variation vector field), by definition, measures the separation of a congruence of integral curves of U. Using the fact that integral curves of U are null pregeodesics, we shall investigate the relations between the topological structure of null hypersurfaces and the curvature by relating Lie parallel vector fields along integral curves of $\gamma$ with Jacobi fields along $\gamma$.

<u>Definition 2.24</u>: Let S be a null hypersurface in a spacetime $(M,g)$. The canonical Riemannian vector bundle $(G(s),\overline{g})$ over S is defined by

$$G(S):=TS/\mathcal{N}(S) \text{ and } \overline{g}(\overline{x},\overline{y}):=g(x,y) \text{ where } \Pi\colon TS \to G(S)$$

is the canonical projection and $\Pi(x) = \overline{x}$ and $\Pi(y) = \overline{y}$. (We shall sometimes denote both metrics $g$ and $\overline{g}$ by $\langle,\rangle$.)

<u>$\overline{g}$ is well-defined</u>: Let $x,x'$ and $y,y'$ be vectors in TS with $\Pi(x) = \overline{x} = \Pi(x')$ and $\Pi(y) = \overline{y} = \Pi(y')$. Then, $x = x_1' + u_1$ and $y = y' + u_2$ for some $u_1,u_2 \in \mathcal{N}(S)$. Thus,

$$\begin{aligned} g(x,y) &= g(x'+u_1,y'+u_2) = g(x',y') + g(x',u_2) + g(u_1,y') \\ &\qquad + g(u_1,u_2) \\ &= g(x',y') \text{ since } g(x',u_2) = g(u_1,y') = g(u_1,u_2) = 0 \end{aligned}$$

which shows that $\overline{g}$ is well-defined. ∎

Definition 2.25: Let S be a null hypersurface in a spacetime M and U be a future directed null vector field on S. The bundle homomorphism $\overline{\nabla}U$: $G(S) \to G(S)$ is defined by $(\overline{\nabla}U)\overline{x} := \overline{\nabla}_{\overline{x}}U := \Pi(\nabla_x U)$ where $\overline{x} \in G(S)$, $x \in TS$ with $\Pi(x) = \overline{x}$.

Note that since U is a pregeodesic null vector field, $\overline{\nabla}U$ is well defined.

Proposition 2.26: $\overline{\nabla}U$: $G(S) \to G(S)$ is self-adjoint.

Proof: Since $\nabla U$: $TS \to TS$ is self-adjoint (see Corollary 2.8), it immediately follows that $\overline{\nabla}U$ is self-adjoint. ∎

Definition 2.27: Let S be a null hypersurface in a spacetime M. The covariant derivative of $\overline{X} \in \Gamma G(S)$ in the direction $u \in N(S)$ is defined by

$$\overline{\nabla}_u\overline{X} := \Pi(\nabla_u X) \text{ where } X \in \Gamma TS \text{ with } \Pi(X) = \overline{X}.$$

To show that $\overline{\nabla}_u\overline{X}$ is well-defined it suffices to show that if $Y \in \Gamma TS$ with $\Pi(Y) = \Pi(X)$ then $\Pi(\nabla_u X) = \Pi(\nabla_u Y)$. Indeed, since $Y = X+U$ for some pregeodesic null vector field U on S, $\Pi(\nabla_u Y) = \Pi(\nabla_u(X+U)) = \Pi(\nabla_u X+\nabla_u U) = \Pi(\nabla_u X)$.

Definition 2.28: The second fundamental tensor $\overline{L}$: $G(S) \times N(S) \to G(S)$ of a null hypersurface S in a spacetime M is defined by

$\bar{L}(\bar{x},u) := -\bar{\nabla}_{\bar{x}}U$ where U is any extension of u to a section of $\mathfrak{N}(S)$.

To show that $\bar{L}$ is well-defined, it suffices to show that $\bar{\nabla}_{\bar{x}}U$ is independent of the extension of u to a section of $\mathfrak{N}(S)$. Let $\tilde{U}$ be another extension of u. Then $\tilde{U} = fU$ for some function f with $f(p) = 1$ where $p \in S$ is such that $u \in \mathfrak{N}(S)p$. Thus $\bar{\nabla}_{\bar{x}}\tilde{U} = \pi(\nabla_x\tilde{U}) = \pi(x(f)\tilde{U}+f(p)\nabla_x U) = \bar{\nabla}_{\bar{x}}U$ since $\pi(\tilde{U}) = 0$ and $f(p) = 1$. ∎

Definition 2.29: Let U be a null vector field on a null hypersurface S of a spacetime M. The divergence of U is defined by

$$\operatorname{div} U := -\operatorname{tr} \bar{L}_U \text{ where } \bar{L}_U = \bar{L}(\cdot,U).$$

Definition 2.30: Let U be a null vector field on a null hypersurface S in a spacetime M. The Lie derivative with respect to U of $\bar{X} \in \Gamma G(S)$ is defined by

$$\bar{\mathfrak{L}}_U\bar{X} = \pi(\mathfrak{L}_U X) \text{ where } X \in \Gamma TS \text{ with } \pi(X) = \bar{X}.$$

A section $\bar{X}$ of G(S) is called Lie parallel with respect to U if $\bar{\mathfrak{L}}_U\bar{X} = 0$.

To show that $\bar{\mathfrak{L}}_U$ is well-defined, it suffices to show that if $X, Y \in \Gamma TS$ with $\pi(X) = \overline{X} = \pi(Y)$ then $\pi(\mathcal{L}_U X) = \pi(\mathcal{L}_U Y)$. Since $X = Y + fU$ for some function f on S,

$$\begin{aligned}\pi(\mathcal{L}_U X) &= \pi(\mathcal{L}_U(Y+fU)) \\ &= \pi(\mathcal{L}_U Y + U(f)U + f\mathcal{L}_U U) \\ &= \pi(\mathcal{L}_U Y) \quad \text{since } \pi(U) = 0 \text{ and } \mathcal{L}_U U = 0.\end{aligned}$$

Remark: Note that $\overline{\mathcal{L}}_{fU}\overline{X} = f\overline{\mathcal{L}}_U\overline{X}$ for any function f on S. For $X \in TS$ with $\pi(X) = \overline{X}$,

$$\overline{\mathcal{L}}_{fU}\overline{X} = \pi(\mathcal{L}_{fU}X) = \pi(f\mathcal{L}_U X - X(f)U) = f\overline{\mathcal{L}}_U\overline{X} \text{ since } \pi(U) = 0.$$

Lemma 2.31: Let $\overline{X} \in \Gamma G(S)$ and $U \in \Gamma N(S)$. Then

$$\overline{\nabla}_U\overline{X} - \overline{\nabla}_{\overline{X}}U = \overline{\mathcal{L}}_U\overline{X}.$$

Proof: Let $X \in \Gamma TS$ with $\pi(X) = \overline{X}$. Since $\nabla_U X - \nabla_X U = \mathcal{L}_U X$,

$\pi(\nabla_U X - \nabla_X U) = \pi(\mathcal{L}_U X)$. Therefore from definitions 2.25, 2.27 and 2.30,

$$\overline{\nabla}_U\overline{X} - \overline{\nabla}_{\overline{X}}U = \overline{\mathcal{L}}_U\overline{X}. \quad \blacksquare$$

Definition 2.32: Let S be a null hypersurface in a space-time M and U be a null vector field on S. The Lie derivative of $\overline{g}$ (of the Riemannian metric of G(S)) with respect to U is defined by

$(\overline{\mathcal{L}}_U\overline{g})(\overline{x},\overline{y}) := \Pi[(\mathcal{L}_U g)(x,y)]$ where $x,y \in TS$ with $\Pi(x) = \overline{x}$ and $\Pi(y) = \overline{y}$.

To show that $\overline{\mathcal{L}}_U g$ is well-defined it suffices to show that the above definition is independent of the choice of $x,y \in TS$ with $\Pi(x) = \overline{x}$ and $\Pi(y) = \overline{y}$. But this is immediate from the facts that $\overline{g}$ and $\overline{\mathcal{L}}_U$ are well-defined (cf. Definitions 2.24 and 2.30).

Proposition 2.33: Let $S$ be a null hypersurface and $U$ be a null vector field on $S$. Then

$$(\overline{\mathcal{L}}_U\overline{g})(\overline{x},\overline{y}) = 2\overline{g}(\overline{\nabla}_{\overline{x}}U,\overline{y}) \text{ for every } \overline{x},\overline{y} \in G(S).$$

Proof: Let $\overline{X} = \Pi(X)$ and $\overline{Y} = \Pi(Y)$ be any local extensions of $\overline{x}$ and $\overline{y}$ on $G(S)$ respectively. Then

$$\begin{aligned}
(\overline{\mathcal{L}}_U\overline{g})(\overline{X},\overline{Y}) &= \Pi[(\mathcal{L}_U g)(X,Y)] \\
&= \Pi[Ug(X,Y) - g(\mathcal{L}_U X,Y) - g(X,\mathcal{L}_U Y)] \\
&= \Pi[g(\nabla_U X,Y) + g(X,\nabla_U Y) - g(\mathcal{L}_U X,Y) - g(X,\mathcal{L}_U Y)] \\
&= \overline{g}(\overline{\nabla}_U\overline{X},\overline{Y}) + \overline{g}(\overline{X},\overline{\nabla}_U\overline{Y}) - \overline{g}(\overline{\mathcal{L}}_U\overline{X},\overline{Y}) - \overline{g}(\overline{X},\overline{\mathcal{L}}_U,\overline{Y}) \\
&= \overline{g}(\overline{\nabla}_{\overline{X}}U,\overline{Y}) + \overline{g}(\overline{X},\overline{\nabla}_{\overline{Y}}U) \\
&= 2\overline{g}(\overline{\nabla}_{\overline{X}}U,\overline{Y}) \text{ since } \overline{\nabla}_U\overline{X} - \overline{\nabla}_{\overline{X}}U = \overline{\mathcal{L}}_U\overline{X},
\end{aligned}$$

$\overline{\nabla}_U\overline{Y} = \overline{\nabla}_{\overline{Y}}U = \overline{\mathcal{L}}_U\overline{Y}$ (cf. Lemma 3.31) and $\overline{\nabla}U$ is self-adjiont (cf. Proposition 2.26). ∎

Definition 2.34. Let S be a null hypersurface in a spacetime M and let $\gamma: I \to S$ be an integral curve of the null vector field U on S. $\overline{X} \in \gamma^*G(S)$ is called a Lie parallel class along $\gamma$ if there exists a Lie parallel (with respect to U) $X \in \Gamma\gamma^*TS$ with $\pi(X) = \overline{X}$.

Note that $\overline{X}$ is Lie parallel along $\gamma$ iff for each $t \in I$, there is a neighborhood J of t in I, a neighborhood V of $\gamma(t)$ in S, and a section $\overline{Y}$ of $G(S)|_V$ such that $\overline{\mathcal{L}}_U\overline{Y} = 0$ and $\overline{X} = \overline{Y}\circ\gamma$ on J (cf. page 42 ). In the context of this study, we shall also interpret the condition that a section $\overline{X}$ of G(S) along $\gamma$ be Lie parallel by $\overline{\mathcal{L}}_U\overline{X} = 0$.

Definition 2.35: Let S be a null hypersurface in a spacetime M and let U be a future directed null vector field on S. The set of U-neighbors of an integral curve $\gamma$ if U in S is defined as the set

$$\Gamma_{U\text{-nbor}}(\gamma) = \{\overline{X}\in\Gamma\gamma^*G(S)\,|\,\overline{\mathcal{L}}_U\overline{X}=0\}.$$

Proposition 2.36: Let S be a null hypersurface in a spacetime M. Let $U_1$ and $U_2$ be two null vector fields on S. Then

$$(\Gamma_{U_1\text{-nbor}}(\gamma_1))\circ h = \Gamma_{U_2\text{-nbor}}(\gamma_2) \text{ where}$$

$\gamma_1$ is an integral curve of $U_1$ and $\gamma_2 = \gamma_1 \circ h$ is reparametrization of $\gamma_1$ as an integral curve of $U_2$.

Proof: Since $U_1 = fU_2$ for some nowhere zero function f on S, $\gamma_1$ can be reparametrized to be an integral curve $\gamma_2 = \gamma_1 \circ h$ of $U_2$. Let $\overline{X} \in \Gamma_{U_1\text{-nbor}}(\gamma_1)$ and $\overline{Y}$ be a local extension of $\overline{X}$ to a Lie parallel section of G(S) (with respect to $U_1$), by so, $\overline{X} = \overline{Y} \circ \gamma_1$. Note that $\overline{\mathcal{L}}_{U_2} \overline{Y} = 0$ since $U_1 = fU_2$ and $\overline{\mathcal{L}}_{U_1} \overline{Y} = 0$ (cf. the remark below the Definition 2.30). Therefore $\overline{X} \circ h = \overline{Y} \circ \gamma_1 \circ h = \overline{Y} \circ \gamma_2$ is a Lie parallel class along $\gamma_2$ (with respect to $U_2$). ∎

Remark: Note that if $\overline{X} \in \Gamma_{U\text{-nbor}}(\gamma)$ and $\overline{X} \neq 0$ then $\overline{X}$ does not vanish along $\gamma$ (since $\overline{X}$ is Lie parallel along $\gamma$) and that $\Gamma_{U\text{-nbor}}(\gamma)$ is an (n-2)-dimensional vector space of Lie parallel classes along $\gamma$ (cf. Definition 2.34).

Proposition 2.37: Let S be a null hypersurface in a spacetime M and let $V_\gamma$ be an elementary neighborhood of an integral curve $\gamma$ of a null geodesic vector field U on $V_\gamma$. (cf. Theorem 2.19). Then every $\overline{X} \in \Gamma_{U\text{-nbor}}(\gamma)$ is a Jacobi class along $\gamma$.

Proof: Let X be a locally defined Lie parallel vector field (with respect to U) with $\Pi(X \circ \gamma) = \overline{X}$ (cf. Definition 2.34). Then

$$\begin{aligned}\bar{R}(\bar{X},U)U &= \pi(R(X,U)U|_\gamma \\ &= \pi(\nabla_X\nabla_U U - \nabla_U\nabla_X U + \nabla_{[U,X]}U)|_\gamma \\ &= -\ (\nabla_U\nabla_X U)|_\gamma + \pi(\nabla_{\mathcal{L}_U X}U)|_\gamma \\ &= -\ \bar{\nabla}_U\bar{\nabla}_{\bar{X}}U,\ \text{since } \nabla_U U = 0 \text{ and } \mathcal{L}_U X = 0.\end{aligned}$$

On the other hand, from $\bar{\nabla}_{\bar{X}}U = \pi(\nabla_X U) = \pi(\nabla_U X) = \bar{\nabla}_U\bar{X}$ (since $[X,U] = 0$), it follows that $\bar{\nabla}_U\bar{\nabla}_U\bar{X} + \bar{R}(\bar{X},U)U = 0$. Thus, $\bar{X}$ is a Jacobi class along $\gamma$ since $U|_\gamma = \dot{\gamma}$. ∎

Now, we shall show that there exist a Jacobi tensor $\bar{A}$ along $\gamma$ such that $\Gamma_{\bar{A}\text{-nbor}}(\gamma) = \Gamma_{U\text{-nbor}}(\gamma)$ (cf. Chapter I).

Proposition 2.38: Let S be a null hypersurface in a spacetime M. Let $V_\gamma$ be an elementary neighborhood of an integral curve $\gamma$ of a null geodesic vector field U on $V_\gamma$. Let $\bar{A}$ be a Jacobi tensor along $\gamma$. Then $\Gamma_{U\text{-nbor}}(\gamma) = \Gamma_{\bar{A}\text{-nbor}}(\gamma)$ iff $\bar{A}$ satisfies the initial conditions $\bar{A}(0)$ non-singular and $\dot{\bar{A}}(0)(\bar{A}(0))^{-1} = \bar{\nabla}U|_{\gamma(0)}$. Moreover, if $\dot{\bar{A}}(0)(\bar{A}(0)^{-1} = \bar{\nabla}U|_{\gamma(0)}$ then $\dot{\bar{A}}(t)(\bar{A}(t))^{-1} = \bar{\nabla}U|_{\gamma(t)}$ along $\gamma$. (Note that since $\bar{\nabla}U$ is self-adjoint, $\bar{A}$ is a Lagrange tensor.)

Proof: We recall that if $\bar{E}$ is a parallel class in $G(\gamma)$ then $\bar{X} = \bar{A}(\bar{E})$ is a Jacobi class along $\gamma$. Then

$$\dot{\bar{A}}(\bar{E}) = \bar{\nabla}_U(\bar{A}(\bar{E})) - \bar{A}(\bar{\nabla}_U\bar{E}) = \bar{\nabla}_U\bar{X} = \dot{\bar{X}} \text{ since } \bar{\nabla}_U\bar{E} = 0.$$

In particular,

$$\dot{\bar{A}}(0)\bar{E}(0) = \dot{\bar{X}}(0).$$

$\Leftarrow$: Assume $\bar{A}$ satisfies the initial conditions $\bar{A}(0)$ non-singular and $\dot{\bar{A}}(0)(\bar{A}(0))^{-1} = \bar{\nabla}U|_{\gamma(0)}$

From Proposition 2.37, $\Gamma_{U\text{-nbor}}$ is a vector space of Jacobi classes which has the same dimension with $\Gamma_{\bar{A}\text{-nbor}}(\gamma)$. Moreover, since every $\bar{Y} \in \Gamma_{U\text{-nbor}}(\gamma)$ is Lie parallel, $\bar{\nabla}_U\bar{Y} = \bar{\nabla}_{\bar{Y}}U$ along $\gamma$ and in particular $\dot{\bar{Y}}(0) = \bar{\nabla}_U\bar{Y}|_{\gamma(0)} = \bar{\nabla}_{\bar{Y}(0)}U$. Thus, it suffices to show that each $\bar{X} \in \Gamma_{\bar{A}\text{-nbor}}(\gamma)$ satisfies $\dot{\bar{X}}(0) = \bar{\nabla}_{\bar{X}(0)}U$. But, if $\bar{X} \in \Gamma_{\bar{A}\text{-nbor}}(\gamma)$ then $\dot{\bar{X}}(0) = \dot{\bar{A}}(0)\ \bar{E}(0) = (\bar{\nabla}U|_{\gamma}\circ\bar{A}(0))\bar{E}(0) = \bar{\nabla}U|_{\gamma(0)}\bar{X}(0) = \bar{\nabla}_{\bar{X}(0)}U$.

$\Rightarrow$: Assume $\Gamma_{U\text{-nbor}}(\gamma) = \Gamma_{\bar{A}\text{-nbor}}(\gamma)$.

Since no non-trivial $\bar{X} \in \Gamma_{U\text{-nbor}}(\gamma)$ vanishes along $\gamma$, $\bar{A}$ is non-singular along $\gamma$. In particular, $\bar{A}(0)$ is non-singular. Let $\bar{X} \in \Gamma_{U\text{-nbor}}(\gamma)(=\Gamma_{\bar{A}\text{-nbor}}(\gamma))$ and $\bar{E}$ be a parallel class in $G(\gamma)$ such that $\bar{A}(\bar{E}) = \bar{X}$. Then $\dot{\bar{A}}(0)(\bar{A}(0))^{-1}\bar{X}(0) = \dot{\bar{A}}(0)\bar{E}(0) = \dot{\bar{X}}(0) = \bar{\nabla}_{\dot{\gamma}(0)}\bar{X} = \bar{\nabla}_{\bar{X}(0)}U$ since $\bar{X} \in \Gamma_{U\text{-nbor}}(\gamma)$. Therefore $\dot{\bar{A}}(0)(\bar{A}(0))^{-1} = \bar{\nabla}U|_{\gamma(0)}$. Moreover, for any $\bar{X} \in \Gamma_{U\text{-nbor}}(\gamma) = \Gamma_{\bar{A}\text{-nbor}}(\gamma)$, since

$\dot{\bar{A}}(t)(\bar{A}(t))^{-1}\bar{X}(t) = \dot{\bar{X}}(t) = \bar{\nabla}_{\bar{X}(t)}U = \bar{\nabla}U|_{\gamma(t)}\bar{X}(t)$, we have

$\dot{\bar{A}}(t)(\bar{A}(t))^{-1} = \bar{\nabla}U|_{\gamma(t)}$ along $\gamma$. ∎

Remark. The above proposition shows that we cannot use arbitrary Jacobi tensors along the (geodesic) integral curves of U to analyze the deviation of a (geodesic) congruence of integral curves of U. In fact, a Jacobi tensor which does not satisfy the above specified initial conditions does not measure the deviation of a geodesic congruence of integral curves of U (see Figure 4).

In general, it may not be possible to find an elementary neighborhood of a null pregeodesic in a null hypersurface. (For example, each closed null geodesic in the Cauchy horizons in Taub-NUT spacetime returns the same point with different velocity and therefore, we cannot find an elementary neighborhood of these geodesics.) Now, we shall extend our discussion to the case when there is no elementary neighborhood of an integral curve of U.

Lemma 2.39: Let $\gamma$ be a smooth curve and X be a vector field along $\gamma$. If $\tilde{\gamma} = \gamma \circ h$ is a reparametrization of $\gamma$ and $\tilde{X} = X \circ h$ then

i) $\nabla_{\dot{\tilde{\gamma}}}\tilde{X} = \dot{h}(\nabla_{\dot{\gamma}}X)\circ h$

ii) $\nabla_{\dot{\tilde{\gamma}}}\nabla_{\dot{\tilde{\gamma}}}X = \ddot{h}(\nabla_{\dot{\gamma}}X)\circ h + \dot{h}^2(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}X)\circ h$

Moreover, if $\gamma$ is a pregeodesic (that is, $\nabla_{\dot\gamma}\dot\gamma = f\dot\gamma$) and $\tilde\gamma = \gamma\circ h$ is a reparametrization of $\gamma$ as a geodesic then h satisfies $\ddot h + (f\circ h)\dot h^2 = 0$.

Proof: The proof can be obtained with a straight forward computation of $\nabla_{\dot{\tilde\gamma}}X$ and $\nabla_{\dot{\tilde\gamma}}\nabla_{\dot{\tilde\gamma}}\tilde X$ (cf. [O], page 93, ex. 3 and page 95, ex. 19). ∎

Proposition 2.40: Let S be a null hypersurface in a spacetime M and let $\gamma$ be an integral curve of a null vector field U on S. If $\bar X \in \Gamma_{U\text{-nbor}}(\gamma)$ then $\tilde{\bar X} = \bar X\circ h$ is a Jacobi class along $\tilde\gamma$ where $\tilde\gamma = \gamma\circ h$ is a reparametrization of $\gamma$ as a null geodesic.

Proof: Since U is a pregeodesic vector field on S, $\nabla_U U = fU$ where f is a smooth function on S. Therefore, if $\gamma$ is an integral curve of U then $\nabla_{\dot\gamma}\dot\gamma = (f\circ\gamma)\dot\gamma$. Let $\tilde\gamma = \gamma\circ h$ be a reparametrization of $\gamma$ as a null geodesic and X be a locally defined Lie parallel vector field (with respect to U) with $\Pi(X\circ\gamma) = \bar X \in \Gamma_{U\text{-nbor}}(\gamma)$. Then, from Lemma 2.39 $\nabla_{\dot{\tilde\gamma}}\nabla_{\dot{\tilde\gamma}}X = \ddot h\,(\nabla_{\dot\gamma}X)\circ h + \dot h^2(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X)\circ h)$ where $\tilde X = X\circ\gamma\circ h$. But since X is Lie parallel (with respect to U),

$$(\nabla_{\dot\gamma}\nabla_{\dot\gamma}X)\circ h = \nabla_U\nabla_U X|_{\gamma\circ h}$$
$$= \nabla_U\nabla_X U|_{\gamma\circ h}$$
$$= (\nabla_X\nabla_U U)\circ h - (R(X,\dot\gamma)\dot\gamma)\circ h$$
$$= (\nabla_X fU)\circ h - (R(X,\dot\gamma)\dot\gamma)\circ h$$
$$= (Xf)\circ hU|_{\gamma\circ h} + (f\circ h)(\nabla_X U)\circ h - (R(X,\dot\gamma)\dot\gamma)\circ h$$

and therefore

$$\nabla_{\dot{\tilde\gamma}}\nabla_{\dot{\tilde\gamma}}X = \ddot h(\nabla_{\dot\gamma}X)\circ h + \dot h^2\ (Xf)\circ hU|_{\gamma\circ h} + \dot h^2(f\circ h)(\nabla_X U)\circ h - \dot h^2(R(X,\dot\gamma)\dot\gamma)\circ h$$

$$= \ddot h(\nabla_{\dot\gamma}X)\circ h + (f\circ h)\dot h^2(\nabla_{\dot\gamma}X)\circ h + \dot h^2(Xf)\circ hU|_{\gamma\circ h} - R(\tilde X,\dot{\tilde\gamma})\dot{\tilde\gamma}.$$

Since $(\ddot h+(f\circ h)\dot h^2)(\nabla_{\dot\gamma}X)\circ h = 0$ (cf. Lemma 2.39),

$$\nabla_{\dot{\tilde\gamma}}\nabla_{\dot{\tilde\gamma}}\tilde X + R(\tilde X,\dot{\tilde\gamma})\dot{\tilde\gamma} = \dot h^2(Xf)\circ hU|_{\gamma\circ h}.$$

Thus

$$\Pi(\nabla_{\dot{\tilde\gamma}}\nabla_{\dot{\tilde\gamma}}\tilde X+R(\tilde X,\dot{\tilde\gamma})\dot{\tilde\gamma}) = \dot h^2(Xf)\circ h\Pi(U)|_{\gamma\circ h} = 0 \text{ since } \Pi(U) = 0.$$

Therefore,

$$\overline\nabla_{\dot{\tilde\gamma}}\overline\nabla_{\dot{\tilde\gamma}}\tilde{\overline X} + \overline R(\tilde{\overline X},\dot{\tilde\gamma})\dot{\tilde\gamma} = 0. \quad \blacksquare$$

Remark: From the above proposition, we conclude that

$$(\Gamma_{U\text{-nbor}}(\gamma))\circ h = \{\overline X\circ h\in G(\gamma\circ h)\,|\,\overline X\in\Gamma_{U\text{-nbor}}(\gamma) \text{ and } \tilde\gamma = \gamma\circ h \text{ is a reparametrization of } \gamma \text{ as a geodesic}\}$$

is an (n-2)-dimensional vector space of Jacobi classes along the null geodesic $\tilde{\gamma}$.

Using the Proposition 2.40, the following generalization of Proposition 2.38 can be similarly proved.

Proposition 2.41: Let S be a null hypersurface in a spacetime M and $\gamma$ be an integral curve of a null vector field U on S. Let $\tilde{\gamma} = \gamma \circ h$ be a reparametrization of $\gamma$ as a null geodesic with $\dot{\tilde{\gamma}}(0) = \dot{\gamma}(0)$ and let $\overline{A}$ be a Jacobi tensor along $\tilde{\gamma}$. Then $(\Gamma_{U\text{-nbor}}(\gamma)) \circ h = \Gamma_{\overline{A}\text{-nbor}}(\tilde{\gamma})$ iff $\overline{A}$ satisfies the initial conditions $\overline{A}(0)$ non-singular and $\dot{\overline{A}}(0)(\overline{A}(0))^{-1} = \overline{\nabla}U|_{\tilde{\gamma}(0)}$. Moreover $\dot{\overline{A}}(t)(\overline{A}(t))^{-1} = \overline{\nabla}U|_{\tilde{\gamma}(t)}$ for all t.

(We note that since $\overline{\nabla}U$ is self-adjoint, $\overline{A}$ is a Lagrange tensor. Also, since $\gamma(0) = \tilde{\gamma}(0)$ and $\dot{\gamma}(0) = \dot{\tilde{\gamma}}(0)$, $h(0) = 0$ and $\dot{h}(0) = 1$.)

Definition 2.42: Let S be a null hypersurface in a spacetime M and let $\gamma$ be an integral curve of a future directed null vector field U on S. Let $\tilde{\gamma} = \gamma \circ h$ be a reparametrization of $\gamma$ to be a future directed null geodesic in S. Then the expansion of U-neighbors of $\gamma$ is defined by $\overline{\theta} \circ h^{-1}$ along $\gamma$ where $\overline{\theta} = \mathrm{tr}\, \dot{\overline{A}}(\overline{A})^{-1}$ and $\overline{A}$ is any Lagrange tensor along $\tilde{\gamma}$ which satisfies the initial conditions $\overline{A}(0)$ non-singular and $\dot{\overline{A}}(0)(\overline{A}(0))^{-1} = \overline{\nabla}U|_{\tilde{\gamma}(0)}$.

(Note that by Corollary 1.15, $\bar{\theta}$ does not depend on the choice of $\bar{A}$ satisfying the specified conditions.)

Remark: Note that since no non-trivial U-neighbor of $\gamma$ vanishes, the neighbors of $\tilde{\gamma}$ induced by $\bar{A}$ do not vanish along $\gamma$ (since $(\Gamma_{U\text{-nbor}}(\gamma)\circ h = \Gamma_{\bar{A}\text{-nbor}}(\gamma))$ and therefore, it follows that $\bar{\theta}$ and $\bar{\theta}\circ h^{-1}$ are finite along $\tilde{\gamma}$ and $\gamma$ respectively (cf. the remark below the Proposition 1.18). Finiteness of $\bar{\theta}$ also can be rephrased as "no null geodesic in S contains a focal point of any spacelike surface H in S which meets $\gamma$" (cf. Chapter I, theorem 1.16).

Definition 2.43: A null hypersurface S in a spacetime M is called a stationary null hypersurface if $\bar{L} = 0$, where $\bar{L}$ is the second fundamental form of S (cf. Definition 2.28).

The well-known examples of stationary null hypersurfaces are the event horizons of time independent black holes (for example, event horizons of the Kruskal and Kerr black holes) and the compact Cauchy horizons (for example, Cauchy horizons in the Taub-NUT spacetime).

Theorem 2.44: Let S be a null hypersurface in a spacetime M. Then the following are equivalent:

i) S is a stationary null hypersurface

ii) $\bar{\nabla}U = 0$ for some null vector field U on S

iii) $\overline{\nabla}U = 0$ for every null vector field $U$ on $S$

iv) For any null vector field $U$ on $S$, $\overline{\nabla}_U\overline{X} = \overline{\mathcal{L}}_U\overline{X}$ for every $\overline{X} \in G(S)$

v) $\overline{\mathcal{L}}_U\overline{g} = 0$ for any null vector field $U$ on $S$.

vi) $S$ is a totally geodesic submanifold in $M$ (that is, the restriction of the connection of $M$ to $S$ defines a connection on $S$).

Proof:

(i) $\Longleftrightarrow$ (ii): $\Rightarrow$:This immediately follows from the definition of $\overline{L}$ (cf. Definition 2.28).

$\Leftarrow$: Let $U$ be a null vector field on $S$ with $\overline{\nabla}U = 0$. Let $\overline{x} \in G(S)$ and $u \in N(S)$. Then, there exists a function $f$ on $S$ such that $u = f(p)\,U(p)$ where $p \in S$ and $u \in N(S)_p$. Then,

$$\overline{L}(\overline{x},u) = -\,\overline{\nabla}_{\overline{x}}(fU) = -\Pi(\nabla_x fU) = -\Pi((xf)U + f\nabla_x U)$$

$$= -f\overline{\nabla}_{\overline{x}}U = 0 \text{ since } \Pi(U) = 0 \text{ and } \overline{\nabla}U = 0.$$

Thus $\overline{L} = 0$.

(ii) $\Longleftrightarrow$ (iii): $\Leftarrow$: obvious

$\Rightarrow$: Let $\tilde{U}$ be any null vector field on $S$. Then there exists a function $f$ on $S$ such that $\tilde{U} = fU$. Thus, for any $\overline{x} \in G(S)$ and $x \in TS$ with $\Pi(x) = \overline{x}$,

$$(\overline{\nabla}\widetilde{U})\overline{x} = \overline{\nabla}_{\overline{x}}\widetilde{U} = \overline{\nabla}_{\overline{x}}(fU) = \Pi(\nabla_x(fU)) = \Pi((xf)U + f\nabla_x U)$$

$$= f\overline{\nabla}_{\overline{x}}U = 0 \text{ since } \Pi(U) = 0 \text{ and } \overline{\nabla}U = 0.$$

(iii) $\Longleftrightarrow$ (iv): This is obvious from $\overline{\nabla}_U\overline{X} - \overline{\nabla}_{\overline{X}}U = \mathcal{L}_U\overline{X}$ (cf. Lemma 2.31).

(iii) $\Longleftrightarrow$ (v): This is obvious from $(\mathcal{L}_U\overline{g})(\overline{x},\overline{y}) = 2\overline{g}(\overline{\nabla}_{\overline{x}}U,\overline{y})$ for every $\overline{x},\overline{y} \in G(S)$ since $\overline{g}$ is a Riemannian metric on $G(S)$ (cf. Proposition 2.33).

(iii) $\Longleftrightarrow$ (vi): Note that, for any null vector field $U$ on $S$, $\overline{\nabla}U = 0$ iff $\nabla_x U \in \mathfrak{N}(S)$ for all $x \in TS$.

Thus, for any $X,Y \in \Gamma TS$,

$$\langle\nabla_X Y,U\rangle = X\langle Y,U\rangle - \langle Y,\nabla_X U\rangle = -\langle Y,\nabla_X U\rangle \text{ since}$$

$$\langle Y,U\rangle = 0.$$

Therefore, $\nabla_X Y \in \Gamma TS$ for every $X,Y \in \Gamma TS$ iff $\nabla_X U \in \Gamma\mathfrak{N}(S)$ for every $X \in \Gamma TS$.

Proposition 2.45: Let $S$ be a stationary null hypersurface in an $n$-dimensional spacetime $M$ and $U$ be a future directed null vector field on $S$. Let $\gamma: I \to S$ be an integral curve of $U$ and $\widetilde{\gamma} = \gamma \circ h$ be a reparametrization of $\gamma$ to be a null geodesic. Let $\overline{A}$ be a Jacobi tensor along $\widetilde{\gamma}$ which satisfies the initial conditions $\overline{A}(t)$

non-singular and $\dot{\overline{A}}(t)(\overline{A}(t))^{-1} = \overline{\nabla}U|_{\tilde{\gamma}(t)}$ for some $t \in I$. Then $\overline{\theta} = \text{tr}\,\overline{B} = 0$, $\overline{\sigma} = \frac{1}{2}(\overline{B} + \overline{B}^*) - \frac{\overline{\theta}}{n-2}\,\overline{\text{id}} = 0$ and $\overline{\omega} = \frac{1}{2}(\overline{B} - \overline{B}^*) = 0$ along $\tilde{\gamma}$ where $\overline{B} = \dot{\overline{A}}(\overline{A})^{-1}$. Furthermore, $\overline{R}(\cdot,U)U = 0$ and it follows that $\text{Ric}(U,U) = 0$.

Proof: Since $\overline{\nabla}U|_{\gamma(t)} = \overline{B}(t)$ for some $t \in I$, $\overline{B}$ is self-adjoint (cf. Proposition 2.41) and it follows that $\overline{\omega} = 0$. Since $\overline{\nabla}U|_{\tilde{\gamma}} = \overline{B} = \overline{\sigma} + \frac{\overline{\theta}}{n-2}\,\overline{\text{id}}$ (cf. [BE], page 351), it follows that $\overline{\sigma} = 0$ and $\overline{\theta} = 0$. Furthermore, since $\overline{\nabla}U = 0$, $\overline{R}(\cdot,U)U = 0$ and it follows that $\text{Ric}(U,U) = 0$. ∎

We recall that an integral curve $\gamma$ of a null vector field $U$ on a null hypersurface $S$ is called a null generator of $S$. Let $\tilde{\gamma} = \gamma \circ h$ be a reparametrization of $\gamma$ to be a null geodesic and $\overline{\theta} \circ h^{-1}$ be the expansion of $U$-neighbors of $\gamma$ (cf. Definition 2.42). Thus, $\overline{\theta} = 0$ iff $\overline{\theta} \circ h^{-1} = 0$.

Theorem 2.46: Let $S$ be a null hypersurface in an $n$-dimensional spacetime $M$ satisfying $\text{Ric}(u,u) \geq 0$ for every $u \in \mathcal{N}(S)$. If the expansion of $U$ is zero for any null vector field $U$ on $S$ then $S$ is a stationary null hypersurface.

Proof: Since $\gamma$ is a null generator of $S$, by definition, $\gamma$ is an integral curve of some null vector field $U$ on $S$. Let $\tilde{\gamma}$ be a reparametrization of $\gamma$ to be a null geodesic and let $\overline{\theta}$ be defined as in the Definition 2.42. Then, $\overline{\nabla}U|_{\tilde{\gamma}(t)} = \frac{\overline{\theta}}{n-2}\,\overline{id} + \overline{\sigma}$ since $\overline{\nabla}U$ is self-adjoint. From the Raychaudhuri equation (cf. Proposition 1.18), it follows that $\dot{\overline{\theta}} = -Ric(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}}) - tr\,\overline{\sigma}^2 = 0$ along every $\tilde{\gamma}$ since $\overline{\theta} = 0$ along every $\tilde{\gamma}$. Thus, since $Ric(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}}) \geq 0$ and $tr\,\overline{\sigma}^2 \geq 0$ (since $\sigma$ is self-adjoint) it follows that $Ric(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}}) = 0$ and $\overline{\sigma} = 0$. Thus $\overline{\nabla}U|_{\tilde{\gamma}} = \frac{\overline{\theta}}{n-2}\,\overline{id} + \overline{\sigma} = 0$ along every $\tilde{\gamma}$ in S and therefore $\overline{\nabla}U = 0$ on S. ∎

# CHAPTER III

## NULL CUT LOCI OF SPACELIKE SURFACES

The future null cut locus $C_N^+(p)$ of a point p in a spacetime M is defined and its properties are investigated in [BE]. In this chapter, we shall generalize this concept to spacelike surfaces. We shall show that the future null cut locus $C_N^+(H)$ of a spacelike surface H in a spacetime M has properties similar to the properties of the future null cut locus of a point in globally hyperbolic spacetimes. Furthermore, by making use of the properties of future null cut locus of a spacelike surface, we shall obtain sufficient conditions for the normal bundle of a spacelike surface to be a trivial vector bundle.

We recall that the <u>future null cut point</u> of a (future directed) null geodesic $\gamma\colon [0,a) \to M$ is defined in [BE] to be the point $q = \gamma(t_0)$ on $\gamma$ where $t_0 = \sup\{t\in[0,a) \mid d(\gamma(0),\gamma(t))=0\}$, provided that $0<t_0<a$ where d is the Lorentzian distance function (cf. [BE], page 230).

Then, the <u>future null cut locus</u> $C_N^+(p)$ of a point p is defined to be the set of future null cut points of all future inextendable null geodesics $\gamma$: $[0,a) \to M$ with $\gamma(0) = p$ (cf. [BE], page 230).

It has been shown in [BE] that $C_N^+(p)$ of a point p in a globally hyperbolic spacetime M is a closed subset of M and each point $x \in C_N^+(p)$ is either a conjugate point to p along some null geodesic $\gamma$ or there exists two null geodesics (which are not reparametrizations of each other), joining p and x.

We recall that the distance between a set H and a point p in a spacetime M is defined by $d(H,P) = \sup\{L(\gamma) | \gamma \in \Omega_{(H,p)}\}$ where $\Omega_{(H,p)}$ is the set of all future-directed (piecewise differentiable) non-spacelike curves $\gamma$ from H to p and $L(\gamma)$ is the length of $\gamma$ (cf. [BE], page 81). (If $\Omega_{(H,p)} = \emptyset$, define $d(H,p)=0$.)

Definition 3.1: Let H be a spacelike surface in spacetime M. Let $\gamma$: $[0,a) \to M$ be a future directed null geodesic with $\gamma(0) \in H$ and $\dot{\gamma}(0) \perp H$. A point $q = \gamma(t_0)$ is said to be the future null cut point of H along $\gamma$ if $0<t_0<a$ and

$$t_0 = \sup\{t \in [0,a) | d(H,\gamma(t))=0\}.$$

Remarks:

i) It is not meaningful to define the future null cut point of a spacelike surface along a future directed null geodesic $\gamma$ which does not meet H orthogonally since such a null geodesic fails to realize distance between H and each of its points (but except possibly the point where it meets H (cf. [O], page 298)).

ii) The assumption $0<t_o$ is technical. We shall make use of this assumption to emphasize the influence of the causal structure of spacetime on the future null cut locus of a spacelike surface.

In fact, it is possible that the future null cut point of a spacelike surface along a null geodesic $\gamma$ may not exist even though the geodesic $\gamma$ fails to realize the distance between H and each of its points (see Figure 6).

It is also possible that the future null cut point q of a spacelike surface along a null geodesic $\gamma$ may fail to be a focal point of H along $\gamma$ and yet there may there exist no other null geodesic from H to q (see Figure 5).

<u>Definition 3.2</u>: Let H be a spacelike surface in a spacetime M. The <u>future null cut locus</u> of H is defined to be the set $C_N^+(H)$ of future null cut points of all future inextendable null geodesics which meet H orthogonally.

We note that even if H is compact and acausal, it is possible that $C_N^+(H)$ is not a closed subset of M (see Figure 6).

We recall that a spacetime M is called <u>causally continuous</u> if each $p \in M$ and any compact set $K \subseteq M - \overline{I^{\pm}(p)}$, there exists a neighborhood U of p such that $K \subseteq M - \overline{I^{\pm}(z)}$ for every $z \in U$ (cf. [HS]) and a spacetime M is called <u>causally simple</u> if $J^{+}(p)$ and $J^{-}(p)$ are closed subsets of M for every $p \in M$ (cf. [HE], page 188). Since, global hyperbolicity implies causal simplicity implies causal continuity (cf. [HS]), it follows that if M is causally simple (or globally hyperbolic) then $\overline{I^{\pm}(p)} = J^{\pm}(p)$ for all $p \in M$.

<u>Lemma 3.3</u>: Let M be a causally simple spacetime and $\{p_n\}, \{q_n\}$ be sequences in M converging to p and q respectively with $p \neq q$ and $p_n < q_n$ for each n. Then $p < q$.

<u>Proof</u>: Assume q is not in $J^{+}(p)$. Then $q \in M - J^{+}(p)$ and since q is compact, there exists an open set U, containing p, such that $q \in M - J^{+}(z)$ for every $z \in U$ from causal continuity.

Let $z_1 \in U \cap I^{-}(p)$. Then there exists $N > 0$ such that $I^{+}(z_1)$ contains all $p_n$ for $n \geq N$ and therefore $I^{+}(z_1)$ contains all $q_n$ for $n \geq N$ since $p_n < q_n$. But then, $M - J^{+}(z_1)$ is an open neighborhood of q which fails to contain infinitely many $q_n$, in contradiction. ∎

Now, by using the Lemma 3.3, we can restate corollary of [BE] (cf. [BE], page 39) without the assumption $p<q$ as follows:

Theorem 3.4: Let M be a globally hyperbolic spacetime and suppose that $\{p_n\}, \{q_n\}$ are sequences in M converging to p and q in M respectively, with $p \neq q$ and $p_n<q_n$ for each n. Let $\gamma_n$ be a future directed non-spacelike curve from $p_n$ to $q_n$ for each n. Then there exists a future directed non-spacelike limit curve $\gamma$ of the sequence $\{\gamma_n\}$ which joins p and q.

Proof: Immediate from the corollary of [BE] (cf. [BE], page 39) and Lemma 3.3 ∎

Let H be a spacelike surface in a spacetime M and let $\mathfrak{N}(H)$ be its normal bundle. We recall from differential topology that there exists an open neighborhood V of H in $\mathfrak{N}(H)$ such that the exponential map $\exp^{\perp}$ of $\mathfrak{N}(H)$ is a diffeomorphism of V onto an open neighborhood W of H in M (namely, $W = \exp^{\perp}(V)$) (cf. [BJ], page 123). Therefore, there exists an open neighborhood W of H in M such that each non-spacelike geodesic emanating orthogonally from H does not contain any focal point of H in W and the non-spacelike geodesics, emanating orthogonally from H do not intersect each other in W.

Theorem 3.5: Let H be a compact, acausal spacelike surface in a globally hyperbolic spacetime M. Let $\gamma$: $[0,a) \to M$ be a future directed null geodesic with $\gamma(0) \in H$ and $\dot{\gamma}(0) \perp H$. Then

$$t_o = \sup\{t \in [0,a) \,|\, d(H,\gamma(t))=0\} > 0.$$

Moreover, if $\gamma(t_o)$ is a future null cut point of H along $\gamma$ then either or both of the following holds:

i) $\gamma(t_o)$ is the first focal point to H along $\gamma$

ii) there exists a least two future dierected null geodesics from H to $\gamma(t_o)$, realizing the distance between H and $\gamma(t_o)$.

Proof: Assume $t_o = 0$. Let $t_n \to 0$ be a strictly decreasing sequence in $(0,a)$. Since M is globally hyperbolic and H is compact, we can find a sequence of timelike geodesics $\{\gamma_n\}$ each of which realizes distance between H and $\gamma(t_n)$ (cf. [HE], page 207 and [P], page 55). Since each $\gamma_n$ realizes distance between H and $\gamma(t_n)$, necessarily, each $\gamma_n$ is orthogonal to H at some point $p_n$. Since H is compact, a subsequence of the sequence $\{p_n\}$ converges to a point $p \in H$. We shall also denote this subsequence by $\{p_n\}$ for brevity. Thus, it remains to show that $p \neq \gamma(0)$. For, then from the theorem 3.4, there exists a non-spacelike limit curve $\alpha$ of $\{\gamma_n\}$ from p to $\gamma(0)$ in contradiction with the acauslity assumption on H. Assume $\gamma(0) = p$. Let V be any local causality neighborhood (cf. [P], page 30) of p. Then V contains all $p_n$ and $\gamma(t_n)$ for large n. Since V is a local causality neighborhood, each $\gamma_n$ is also contained in V for large n. By choosing V small, since $\gamma$ and $\gamma_n$ are non-spacelike geodesics orthogonal to H, we reach a contradiction with the fact that non-spacelike geodesics emanating orthogonally from H cannot intersect each other in some open neighborhood of H in M. That is, $p \neq \gamma(0)$ and $t_o > 0$.

The rest of the claim can be proved following similar lines. Let $\gamma(t_o)$ be the future cut point of H and let $\{t_n\}$ be a strictly decreasing sequence in $(0,a)$, converging to $t_o$. Let $\{\gamma_n\}$ be a sequence of timelike geodesics from some $p_n \in H$ to $\gamma(t_n)$, each of which realizes distance from H to $\gamma(t_n)$. Then from Theorem 3.4, $\{\gamma_n\}$ has a limit curve $\eta$ from a point $p \in H$ to $\gamma(t_o)$. But $\eta$ must be a null geodesic, realizing the distance between H and $\gamma(t_o)$ since $\gamma(t_o)$ is the future null cut point of H along $\gamma$ (in other words $d(H,\gamma(t_o))=0$). Now, we have two cases to consider:

a) if $\underline{\gamma \neq \eta}$, then we obtain the case (ii) of the theorem.

b) if $\gamma = \eta$ (up to parametrization), then $\gamma$ is a limit curve of the sequence $\{\gamma_n\}$ of timelike curves each of which is orthogonal to H. Then, since $\gamma$ is a limit curve of the sequence $\{\gamma_n\}$, there is a subsequence $\{\dot{\gamma}_m(0)\}$ of $\{\dot{\gamma}_n(0)\}$ such that directions of the vectors $\dot{\gamma}_m(0)$ converge to the direction of $\dot{\gamma}(0)$ (cf. [BE], page 231, Lemma 8.14). But then $\gamma(t_o)$ is a focal point of H along $\gamma$ since the exponential map of $\mathcal{N}(H)$ fails to be one to one on every neighborhood of $\dot{\gamma}(0)$ in $\mathcal{N}(H)$. Thus we obtain the case (ii) of the theorem. ∎

<u>Corollary 3.6</u>: Let H be a compact, acausal spacelike surface in a globally hyperbolic spacetime M. Then every future directed null geodesic $\gamma\colon [0,a) \to M$ with $\gamma(0) \in H$ and $\dot{\gamma}(0) \perp H$ realizes distance between its points and H up to and including the future null cut point of H along $\gamma$ (if any). Moreover, either one or both of the following holds for each $x \in C_N^+(H)$:

i) x is the first focal point of H along some geodesic $\gamma$ which meets H orthogonally

ii) there exists at least two future directed null geodesics from H to x realizing the distance between H and x.

Proof: Immediate from the Theorem 3.5.

Theorem 3.7: Let H be a compact, acausal spacelike surface in a globally hyperbolic spacetime M. Then $C_N^+(H)$ is a closed subset of M.

Proof: To show that $C_N^+(H)$ is closed, it suffices to show that if $\{x_n\}$ is a sequence of points in $C_N^+(H)$ with $x_n \to x \in \overline{C_N^+(H)}$ then $x \in C_N^+(H)$.

Let $x_n \in C_N^+(H)$ and $x_n \to x$. First, we shall show that $x \notin H$ and $x \in J^+(H)$.

Let $\gamma_n$ be a null geodesic from some point $y_n \in H$ to $x_n$ realizing the distance between H and $x_n$ for each n. (Such a geodesic always exists since $x_n \in C_N^+(H)$.) Since H is compact, a subsequence of $\{y_n\}$ converges to a point $y \in H$. We shall also denote this subsequence by $\{y_n\}$ for brevity. Thus, to prove the claim, it suffices to show that $x \neq y$ since, then there exists a non-spacelike limit curve $\gamma$ of the sequence $\{\gamma_n\}$ from y to x (cf. Theorem 3.4) and it follows that $x \notin H$ and $x \in J^+(H)$ from the achronality of H.

Assume $x = y$. Let V be a local causality neighborhood of y. Since $x_n \to y$ and $y_n \to y$, there exists $N > 0$ such that V contains all $x_n$ and $y_n$ for $n \geq N$ and therefore

$V$ contains all $\gamma_n$ for $n \geq N$ (since $V$ is a local causality neighborhood). Then, Corollary 3.6 implies that each $x_n$ is either a focal point or there exists two null geodesics (orthogonal to $H$) from $H$ to $x_n$ (or both). Now there are two cases to consider. Either infinitely many $x_n$ are focal points or else no $x_n$ is a focal point for large $n$. In the first case, we can reach a contradiction by choosing $V$ small enough since there exists an open neighborhood of $H$ in $M$ such that no non-spacelike geodesic emanating orthogonally from $H$ contains focal points in this open neighborhood. Therefore, no $x_n$ is a focal point of $H$ for large $n$. But, then Corollary 3.6 implies that there exists a null geodesic $\alpha_n \neq \gamma_n$ from some $z_n \in H$ to $x_n$ realizing distance between the $H$ and $x_n$ for large $n$. Since $H$ is compact, a subsequence of $\{z_n\}$ converges to a point $z \in H$; we shall also denote this subsequence by $z_n$ for brevity. If $z \neq y$ then Theorem 3.4 implies that there exists a non-spacelike limit curve of $\{\alpha_n\}$ from $z$ to $x = y$. But this conflicts with the achronality of $H$. If $z = y$ then all $\alpha_n$ lie in the local causality neighborhood $V$ of $y$ together with all $\gamma_n$ for large $n$. But since the future endpoints of both $\gamma_n$ and $\alpha_n$ is $x_n \in V$ for large $n$ and the past endpoints of $\gamma_n$ and $\alpha_n$ converges to $y$, by choosing $V$

sufficiently small, we again reach a contradiction as before. Thus, this completes the proof that $x \notin H$ and $x \in J^+(H)$.

On the other hand, since the Lorentzian distance function d is continuous in globally hyperbolic space-times (cf. [BE], page 85), $d(H,x) = \lim_{x_n \to x} d(H,x_n) = 0$ and it follows that the non-spacelike limit curve $\gamma$ of $\{\gamma_n\}$ is a null geodesic which realizes the distance between H and x (and therefore $\gamma$ is orthogonal to H).

Now, there are two cases to consider. Either infinitely many points $x_n$ are focal points or else no $x_n$ is a local point for large n. In the first case, x is necessarily a focal point since the set of singular points of the exponential map of $\mathcal{N}(H)$ is closed. If no $x_n$ is a focal point for sufficiently large n then Corollary 3.6 implies that there exists a null geodesic $\beta_n \neq \gamma_n$ from some $\omega_n \in H$ to $x_n$ realizing distance between H and $x_n$ for large n. Let $\beta$ be a non-spacelike limit curve of $\{\beta_n\}$ from H to x (cf. Theorem 3.4).

Case 1: If $\beta \neq \gamma$ then $\gamma$ fails to realize the distance between H and its points to the future of x. Since $\gamma$ realizes distance between H and its points up to x, x is the future null cut point of H along $\gamma$.

Case 2: If $\beta = \gamma$ (up to parametrization) then $\gamma$ is a limit curve of both the sequences $\{\gamma_n\}$ and $\{\beta_n\}$ which are null geodesics emanating orthogonally from H and have the common future and point $x_n$, pairwise. Since $\gamma$ is a limit curve of both sequences, there exists a subsequence $\{\dot{\gamma}_m(0)\}$ of $\{\dot{\gamma}_n(0)\}$ and a subsequence $\{\dot{\beta}_m(0)\}$ of $\{\dot{\beta}_n(0)\}$ such that the directions of the vectors $\dot{\gamma}_m(0)$ and $\dot{\beta}_m(0)$ converge to $\dot{\gamma}(0)$ (cf. [BE], page 231, Lemma 8.14). But then x is a focal point of H along $\gamma$ since the exponential map of $\mathbf{N}(H)$ fails to be one to one on every neighborhood of $\dot{\gamma}(0)$ in $\mathbf{N}(H)$. Therefore, since $\gamma$ realizes distance between the H and its points up to x,x is a future cut point of H along $\gamma$.

Remark: We recall that the future horismos of a set H is defined to be the set $E^+(H) = J^+(H) - I^+(H)$. Thus, if H is a spacelike surface then $C_N^+(H) \subseteq E^+(H)$ which corresponds to those points of $E^+(H)$ where the null geodesics in $E^+(H)$ leave $E^+(H)$ in the future direction. Since every null geodesic $\gamma$ in $E^+(H)$ realizes distance between H and its points, $\gamma$ is orthogonal to H and therefore the set $E^+(H) - \{H \cup C_N^+(H)\}$ is a (smooth) null hypersurface (cf. Chapter II, section B).

Theorem 3.8: Let H be a compact, acausal spacelike surface in a globally hyperbolic spacetime M. Then $E^+(H) - \{H \cup C_N^+(H)\}$ has exactly two connected components each of which is a (smooth) null hypersurface.

Proof: Since $C_N^+(H) \cap H = \emptyset$ (cf. Theorem 3.7 and Corollary 3.6), every null geodesic, emanating orthogonally from H has a non-trivial portion contained in $E^+(H)$. Therefore, since H is smooth, $E^+(H) - \{H \cup C_N^+(H)\}$ is a smooth null (possibly disconnected) hypersurface generated by the null geodesics emanating orthogonally from H. Assume $E^+(H) - \{H \cup C_N^+(H)\} = S$ is connected. Then, since S is a null hypersurface, there exists a null vector field U on S whose integral curves are reparametrizations of the null geodesics in $E^+(H)$ which are orthogonal to H. But this is a contradiction since there exist two linearly independent null directions at each point of H. ∎

Corollary 3.9: Let H be a compact, acausal spacelike surface in a globally hyperbolic spacetime M. Then the normal bundle $\mathbb{N}(H)$ of H is a trivial vector bundle.

Proof: This is immediate from Theorem 3.8 since $E^+(H) - \{H \cup C_N^+(H)\}$ has exactly two connected components. ∎

# CHAPTER IV

# SPACELIKE SURFACES IN 4-DIMENSIONAL SPACETIMES

In this chapter, we shall discuss the influence of curvature on the existence of closed trapped surfaces in 4-dimensional spacetimes which obey the Einstein equation for the stress-energy tensor T (cf. [SW] pages 71 and 111).

In section A, we shall obtain sufficient conditions (which involve curvature in a crucial way) for the compactness of spacelike surfaces. We shall then obtain in section B, some results concerning the evolution of closed trapped surfaces from marginally trapped surfaces.

In achieving the above results, we shall make use of the results of Ambrose and Cohn-Vossen on compactness of complete Riemannian surfaces H by making estimates on the induced curvature $K_H$ of H using the Gauss-Codazzi equation.

We recall that the normal bundle $N(H)$ of a spacelike surface H has two dimensional timelike fibers $N(H)_p$, in each of which, we can find two future directed null vectors $u,w \in N(H)_p$ with $\langle u,w\rangle = -1$. The second fundamental forms of H in the directions u and w at $p \in H$ are defined by $II_u(x,y) = \langle L_u x,y\rangle$ and $II_w(x,y) = \langle L_w x,y\rangle$ where

$x, y \in T_pH$, $L_u$ and $L_w$ are the second fundamental tensors of $H$ in the directions $u$ and $w$ at $p \in H$. Geometrically, $II_u(x,x)$ can be interpreted as follows: Let $\gamma: [0,a) \to M$ be a future directed null geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = u \in N(H)_p$. Let $\overline{X}$ be the Jacobi class along $\gamma$ with $\overline{X}(0) = \Pi(x)$ and $\dot{\overline{X}}(0) = \Pi(L_u x)$. Then, $II_u(x,x)$ is interpreted as the radial velocity $\langle\dot{\overline{X}}(0), \overline{X}(0)\rangle$ of the Jacobi class $\overline{X}$ at $p \in H$, which measures the initial rate of separation of a congruence of null geodesics perpendicular to $H$ (since $\langle\dot{\overline{X}}(0), \overline{X}(0)\rangle = II_u(x,x)$ by definition (cf. [BE], page 382)).

Also, since each fiber $N(H_p)$ of $N(H)$ at $p \in H$ is a timelike plane in $T_pM$, the sectional curvature $K_{H^\perp}(p)$ of $N(H)_p$ is defined by $K_{H^\perp}(p) = -\langle R(n,z)z,n\rangle$ where $\{z,n\}$ is a Lorentzian basis for $N(H)_p$. Geometrically, $K_{H^\perp}(p)$ may be interpreted as follows: Let $\gamma: [0,a) \to M$ be a future directed unit timelike geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) \perp H$. Let $N$ be a Jacobi field along $\gamma$ with $\|N(0)\| = 1$, $N(0) \perp \dot{\gamma}(0)$ and $N(0) \perp H$. Then, $K_{H^\perp}(p)$ can be interpreted as the radial acceleration $\langle\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}N, N\rangle|_p$ of the Jacobi field $N$ at $p \in H$, which measures the separation of a congruence of timelike geodesics (since, $\langle\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}N, N\rangle|_p = \langle -R(N,\dot{\gamma})\dot{\gamma}, N\rangle|_p = K_{H^\perp}(p)$). We shall call a variation of $\gamma$ through timelike geodesics a radial timelike congruence to H at p if the variation vector field is orthogonal to H at $p \in H$.

Now, we shall state the Gauss-Codacci equation for a spacelike surface H in a 4-dimensional spacetime M which obeys the Einstein equation for the stress-energy tensor T. Since its proof is rather long and not available in the recent literature, we shall provide a proof of it at the end of this section.

Gauss-Codazzi equation: Let H be a spacelike surface in a 4-dimensional spacetime M which obeys the Einstein equation for the stress-energy tensor T. Let u,w be future directed null vectors in $\mathcal{N}(H)_p$ with $\langle u,w\rangle = -1$. Then,

$$
\begin{aligned}
K_H(p) &= K_{H^\perp}(p) + \mathrm{Ric}(u,w) + T(u,w) - II_u(e_1,e_1)II_w(e_2,e_2) \\
&\quad - II_u(e_2,e_2)II_w(e_1,e_1) \\
&= K_{H^\perp}(p) + \mathrm{Ric}(u,w) + T(u,w) - (\mathrm{tr}\, L_u)(\mathrm{tr}\, L_w) \\
&\quad + II_u(e_1,e_1)II_w(e_1,e_1) + II_u(e_2,e_2)II_w(e_2,e_2)
\end{aligned}
$$

where $e_1,e_2$ are orthonormal vectors in $T_pH$ for which either $II_u(e_1,e_2) = 0$ or $II_w(e_1,e_2) = 0$.

Proof: see page 87.

We now state Ambrose's lemma and a result of Cohn-Vossen for complete Riemannian surfaces.

Ambrose's lemma: Let H be a connected, complete Riemannian surface. If there is a point $p \in H$ such that along each geodesic $\gamma: [0,\infty) \to H$ with $\gamma(0) = p$, $\int_0^\infty (K_H \circ \gamma)dt = \infty$ then H is compact.

Proof: cf. [A].

Cohn-Vossen's result: Let H be a connected, orientable, complete Riemannian surface. If $\int_H K_H \omega_H > 2\pi$ then H is diffeomorphic to $S^2$ and $\int_H K_H \omega_H = 4\pi$.

Proof: cf. [C].

We shall obtain sufficient conditions for the compactness of spacelike surfaces (to be diffeomorphic to $S^2$) in the following two circumstances:

i) (cosmological case): M is a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $\hat{T} = (\zeta+p)Z \otimes Z + p\hat{g}$ where $\zeta$ (energy density) and p (pressure) are continuous functions on M, Z is a future directed, unit timelike vector field on M and $\hat{g}$ is

(0,2)-tensor field on M which is physically equivalent to the metric $\hat{g}$ of M. (cf. [SW], page 107.)

ii) (Black hole case): M is a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $T = 0$.

We shall need a weak form of spatial isotropy:

Definition 4.1: A spacetime M is said to be weakly spatially isotropic for the instantaneous observer $z \in TM$ ($\langle z,z\rangle = -1$) if $R(x,z)z = kx$ for every $x \in z^{\perp}$ where k is a constant.

Section A: Necessary and Sufficient Conditions for the Spacelike Surfaces to be Diffeomorphic to $S^2$.

Theorem 4.2: Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress energy tensor $\hat{T} = (\zeta+p)Z\otimes Z + p\hat{g}$. Let H be an orientable spacelike surface in M orthogonal to the vector field Z. Suppose that:

a) H is complete in the induced metric.

b) $II_u(x,x)II_w(y,y) \leq 0$ for each $q \in H$ and every $x,y \in T_qH$, where u and w are future directed null vectors in $N(H)_q$ with $\langle u,w\rangle = -1$.

c) M is weakly spatially isotropic for each instantaneous observer in Z along H.

d) At least one of the following holds:

1) there exists a point $q \in H$ such that for every geodesic $\gamma$ of $H$ (in the induced metric) emanating from $q$, $\int_0^\infty (\zeta \circ \gamma)dt = \infty$ and $\int_H \zeta\omega_H \geq 0$.

2) $\int_H \zeta\omega_H > 6\pi$.

Then, $H$ is diffeomorphic to $S^2$ (and if $\zeta > 0$ along $H$ then $\operatorname{diam} H \leq \pi\sqrt{\frac{3}{m}}$ where $m = \min_H(\zeta)$.

Proof: Let $u,w$ be future directed null vectors in $\mathcal{N}(H)_q$ such that $\langle u,Z_q\rangle = -\frac{1}{\sqrt{2}}$ and $\langle u,w\rangle = -1$. Then, necesarily we have $\langle w,Z_q\rangle = -\frac{1}{\sqrt{2}}$ (since $Z_q \in \mathcal{N}(H)_q$, we write $Z_q = au + bw$. Then, since $\langle u,Z_q\rangle = -\frac{1}{\sqrt{2}}$ and $\langle u,w\rangle = -1$, $b = \frac{1}{\sqrt{2}}$. Also, since $\langle Z_q,Z_q\rangle = -1$, $-2ab = -1$. Thus, $a = \frac{1}{\sqrt{2}}$ and therefore $\langle w,Z_q\rangle = -\frac{1}{\sqrt{2}}$). Since $\widehat{\mathrm{Ric}} = (\zeta+p)Z\otimes Z + \frac{1}{2}(\zeta-p)\hat{g}$ from the Einstein equation, using the weak spatial isotropy along $H$, we obtain

$$K_{H^\perp}(p) = -\frac{1}{3}\mathrm{Ric}(Z_p,Z_p) = -\frac{1}{3}(\zeta+p-\frac{1}{2}+\frac{1}{2}p) = -\frac{1}{6}(\zeta+3p).$$

Also, $T(u,w) = \frac{1}{2}(\zeta+p)-p = \frac{1}{2}(\zeta-p)$ and $\mathrm{Ric}(u,w) = \frac{1}{2}(\zeta+p) - \frac{1}{2}(\zeta-p) = p$. Thus, the Gauss-Codazzi equation yields,

$K_H(p) \geq K_{H^\perp}(p) + T(u,w) + Ric(u,w) = -\frac{1}{6}(\zeta+3p) + \frac{1}{2}(\zeta-p)+p = \frac{1}{3}\zeta$. Therefore, if (d-1) holds then $\int_0^\infty (\zeta\circ\gamma)dt = \infty$ implies that H is compact from Ambrose's lemma and $\int_H \zeta\omega_H \geq 0$ implies that H is diffeomorphic to $S^2$ from the Gauss-Bonnet theorem. (Note that $\int_0^\infty (\zeta\circ\gamma)dt = \infty$ implies that $\zeta > 0$ at some point $x \in H$ and therefore $\int_H \zeta\omega_H > 0$.) If (d-2) holds, then H is diffeomorphic to $S^2$ from Cohn-Vossen's result. In either case, if $\zeta > 0$ on H, then from Myer's theorem, diam $H \leq \pi\sqrt{\frac{3}{m}}$ where $m = \min_H \zeta$. ∎

The assumptions of the above theorem are expected to be satisfied by infinitesimally spatially isotropic spacetimes (cf. [K]). In particular, the recollapsing Friedman-Robertson-Walker spacetimes contain strongly marginally future converging spacelike surfaces H with the above properties. (Recall that H is strongly marginally future converging if $\mathcal{N}(H)$ is trivial and tr $L_W = 0$, $L_U = 0$ where U,W are future directed null sections with $\langle U,W\rangle = -1$. (cf. Chapter 1, page 11)).

In the next corollary, we shall obtain necessary and sufficient conditions for a strongly marginally future converging spacelike surface H to be diffeomorphic to $S^2$ (which is then a marginally trapped surface).

Corollary 4.3. Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $\hat{T} = (\zeta+p)Z\otimes Z + p\hat{g}$. Let H be an orientable spacelike surface in M, orthogonal to the vector field Z. Suppose that:

a) H is complete in the induced metric.

b) $II_u(x,x)II_w(y,y) = 0$ at each $q \in H$ and every $x,y \in T_qM$ where u,w are future directed null vectors in $\mathbb{N}(H)_q$ with $\langle u,w\rangle = -1$.

c) M is weakly spatially isotropic for each instantaneous observer in Z along H.

Then, H is diffeomorphic to $S^2$ iff $\int_H \zeta\omega_H > 6\pi$.

Proof: Let u and w be future directed null vectors in $\mathbb{N}(H)_q$ with $\langle u,w\rangle = -1$ at each $q \in H$. Then, as in the proof of the above theorem, $K_H = \frac{1}{3}\zeta$ from (b). Assume H is diffeomorphic to $S^2$. Then, $\int_H K_H\omega_H = \frac{1}{3}\int_H \zeta\omega_H = 4\pi$ from Gauss-Bonnet theorem and it follows that $\int_H \zeta\omega_H = 12\pi > 6\pi$. Assume $\int_H \zeta\omega_H > 6\pi$. Then from the Cohn-Vossen's result, H is diffeomorphic to $S^2$. ∎

Let H be a spacelike surface in a 4-dimensional spacetime M. The shape function $\Phi$: $H \to \mathbb{R}$ of H is defined by $\Phi(q) = II_u(e_1,e_1)II_w(e_2,e_2) + II_u(e_2,e_2)II_w(e_1,e_1)$

where u,w are future directed null vectors in $\mathcal{N}(H)_q$ with $\langle u,w\rangle = -1$ and $e_1, e_2$ are orthonormal vectors in $T_qH$ for which either $II_U(e_1,e_2) = 0$ or $II_w(e_1,e_2) = 0$ (cf. Gauss-Codazzi equation). The shape function $\Phi$ of a spacelike surface H in a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor T is also given by $\Phi(q) = K_{H^\perp}(q) + Ric(u,w) + T(u,w) - K_H(q)$ where u,w are future directed null vectors in $\mathcal{N}(H)_q$ with $\langle u,w\rangle = -1$ (cf. Gauss-Codazzi equation).

Note that if H is a strongly future converging surface then $\Phi > 0$ on H and from the identity

$$(\operatorname{tr} L_u)(\operatorname{tr} L_w) = II_u(e_1,e_1)II_w(e_2,e_2) + II_u(e_2,e_2)II_w(e_1,e_1)$$
$$+ II_u(e_1,e_1)II_w(e_1,e_1) + II_w(e_2,e_2)II_u(e_2,e_2),$$

we see that $(\operatorname{tr} L_u(\operatorname{tr} L_w) > \Phi$ along H.

Theorem 4.4: Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $\hat{T} = (\zeta+p)Z\otimes Z' + p\hat{g}$. Let H be an orientable spacelike surface orthogonal to the vector field Z. Suppose that:

a) H is complete in the induced metric and $\int_{\bar{H}} \omega_H < \infty$.
b) H is strongly future converging.
c) M is weakly spatially isotropic for the instantaneous observers in Z along H.

Then, H is a closed trapped surface diffeomorphic to $S^2$ iff $\int_H \zeta\omega_H > 6\pi + 3\int_H \Phi\omega_H$.

Proof: From the Gauss-Codazzi equation, as in the proof of the Theorem 4.2, we obtain $K_H = \frac{1}{3}\zeta - \Phi$. Then, since $\int_H \zeta\omega_H = 3\int_H K_H\omega_H + 3\int_H \Phi\omega_H$, H is diffeomorphic to $S^2$ iff $\int_H \zeta\omega_H > 6\pi + 3\int_H \Phi\omega_H$ from Cohn-Vossen's result and the Gauss-Bonnet theorem, using the fact that $\int_H \Phi\omega_H > 0$ as in the proof of Theorem 4.3. ■

The conditions of the above theorem may be expected to be satisfied in infinitesimally isotropic spacetimes. In particular, recollapsing Friedman-Robertson-Walker spacetime contains spacelike surfaces that aatisfy the above conditions. The theorem expresses the relation between the energy density on a strongly future converging surface in an isotropic spacetime and the topology of the surface.

Now, we shall discuss the black hole case.

Theorem 4.5: Let H be an orientable spacelike surface in a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor T = 0. Suppose that:

a) H is complete in the induced metric.

b) $II_u(x,x)II_w(y,y) \leq 0$ at each $q \in H$ and every $x,y \in T_qH$ where u,w are future directed null vectors in $N(H)_q$

with $\langle u,w\rangle = -1$.

c) at least one of the following holds:

1) There exists a point $q \in H$ such that every geodesic $\gamma$ of H (in the induced metric) emanating from q satisfies $\int_0 (K_{H^\perp} \circ \gamma)\,dt = \infty$ and $\int_H K_{H^\perp}\omega_H \geq 0$.

2) $\int_H K_{H^\perp}\omega_H > 2\pi$.

Then H is diffeomorphic to $S^2$ (and has diam $H \leq \pi\sqrt{\frac{1}{m}}$ if $K_{H^\perp} > 0$ where $m = \min_H K_{H^\perp}$).

<u>Proof</u>: From the Gauss-Codazzi equation,

$$K_H \geq K_{H^\perp} \text{ since } T = 0 = \mathrm{Ric} \text{ and } II_u(x,x)II_w(y,y) \leq 0.$$

Thus, as in the proof of Theorem 4.2, it is diffeomorphic to $S^2$, either from Ambrose's lemma and the Gauss-Bonnet theorem or from Cohn-Vossen's result. ∎

The following special case of the above theorem may be interesting from the viewpoint of the theory of black holes. We recall that a black hole is said to be time independent if the expansion $\overline{\theta}$ of the generators of its event horizon S vanishes. In Chapter II, Theorem 2.45, we have shown that this condition (together with the weak energy condition) implies that S is a stationary null hypersurface. Furthermore, every spacelike surface H in S has trivial normal bundle $\mathcal{N}(H)$. (cf. Chapter II, Proposition 2.10.) Let U be a future directed null section of $\mathcal{N}(H)$ tangent to S. Then $L_U = 0$ since S is a stationary null hypersurface.

Corollary 4.6: Let H be an orientable spacelike surface in a 4-dimensional spacetime M which obeys the Einstein equation for the stress-energy tensor $T = 0$. Suppose that:

a) H is complete in the induced metric.

b) $II_u(x,x)II_w(y,y) = 0$ at each $q \in H$ and every $x,y \in T_qH$ where u,w are future directed null vectors in $\mathcal{N}(H)_q$ with $\langle u,w\rangle = -1$. Then, H is diffeomorphic to $S^2$ iff $\int_H K_{H^\perp}\omega_H > 2\pi$.

Proof: Since $T = 0$ implies $Ric = 0$, then, from the Gauss-Codazzi equation, $K_H = K_{H^\perp}$. Thus, the claim follows from the Gauss-Bonnet theorem and Cohn-Vossen's result. ∎

Let $\Phi: H \to \mathbb{R}$ be the shape function of H. (Recall that $\Phi > 0$ on each strongly future converging surface H.)

Theorem 4.7: Let H be an orientable spacelike surface in a 4-dimensional spacetime M which obeys the Einstein equation for the stress-energy tensor T = 0. Suppose that:

a) H is complete in the induced metric and $\int_H \omega_H < \infty$.

b) H is strongly future converging.

Then H is a closed trapped surface diffeomorphis to $S^2$ iff $\int_H K_{H^\perp}\omega_H > 2\pi + \int_H \Phi\omega_H$.

Proof: Since T = 0 implies Ric = 0, from Gauss-Codazzi equation, we obtain $K_H = K_{H^\perp} - \Phi$. Then, $\int_H K_{H^\perp}\omega_H = \int_H K_{H^\perp}\omega_H + \int_H \Phi\omega_H$. Thus the claim follows from Gauss-Bonnet and Cohn-Vossen's results since $\Phi > 0$ on H. ∎

Conditions of the above theorem are expected to be satisfied in Ricci flat black hole spacetimes. In particular, the Kruskal black hole contains spacelike surfaces that satisfy the above conditions.

Note that the above theorem expresses the relation between the surface gravity of a strongly future converging spacelike surface in an empty spacetime and the topology of the surface.

Now, we shall verify the Gauss-Codacci equation.

Proof of the Gauss-Codazzi equation:

We shall prove the Gauss-Codazzi equation in two steps.

Step 1. Let H be a spacelike surface in a 4-dimensional spacetime M. Let u,w be future directed null vectors in $N(H)_p$ with $<u,w> = -1$. Then

$$K_H = K(e_1 \wedge e_2) - II_u(e_1,e_1)II_w(e_2,e_2) - II_u(e_2,e_2)II_w(e_1,e_1)$$

where $\{e_1,e_2\}$ is an orthonormal basis for $T_pH$ for which either $II_u(e_1,e_2) = 0$ or $II_w(e_1,e_2) = 0$ and $K(e_1 \wedge e_2)$ is the sectional curvature of the spacelike plane $e_1 \wedge e_2$.

Proof: Let U,W be a local null extensions of u,w in N(H) with $<U,W> = -1$ and X,Y,Z be vector fields on a neighborhood of p in H with $[X,Y] = 0$. Then

(1) $\nabla_Y Z = (\nabla_Y Z)^T + (\nabla_Y Z)^\perp$

$= \tilde{\nabla}_Y Z + (\nabla_Y Z)^\perp$ where $\tilde{\nabla}_Y Z = (\nabla_Y Z)^T$ is the induced connection on H. Since $(\nabla_Y Z)^\perp = -<\nabla_Y Z,W>U - <\nabla_Y Z,U>W$ we can rewrite (1) as

$$\nabla_Y Z = \tilde{\nabla}_Y Z - <\nabla_Y Z,W>U - <\nabla_Y Z,U>W.$$

Thus,

$$\nabla_X\nabla_Y Z = \nabla_X\tilde{\nabla}_Y Z - [X<\nabla_Y Z,W>U + <\nabla_Y Z,W>\nabla_X U]$$
$$- [X<\nabla_Y Z,U>W + <\nabla_Y Z,U>\nabla_X W].$$

Therefore

$$(2)\quad (\nabla_X\nabla_Y Z)^T = (\nabla_X\widetilde{\nabla}_Y Z)^T - \langle\nabla_Y Z,W\rangle(\nabla_X U)^T - \langle\nabla_Y Z,U\rangle(\nabla_X W)^T$$
$$= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z + \langle\nabla_Y Z,W\rangle L_U X + \langle\nabla_Y Z,U\rangle L_W X.$$

On the other hand, from the Ricci identity,

$$\langle\nabla_Y Z,W\rangle = Y\langle Z,W\rangle - \langle Z,\nabla_Y W\rangle = II_W(Z,Y)$$

and

$$\langle\nabla_Y Z,U\rangle = Y\langle Z,U\rangle - \langle Z,\nabla_Y U\rangle = II_U(Z,Y)$$

since $\langle Z,W\rangle = 0 = \langle Z,U\rangle$. Then by substituting in (2), we obtain

$$(\nabla_X\nabla_Y Z)^T = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z + II_W(Z,Y)L_U X + II_U(Z,Y)L_W X.$$

Also, by interchanging X and Y, we obtain

$$(\nabla_Y\nabla_X Z)^T = \widetilde{\nabla}_Y\widetilde{\nabla}_X Z + II_W(Z,X)L_U Y + II_U(Z,X)L_W Y.$$

Therefore,

$$(R(X,Y)Z)^T = \widetilde{R}(X,Y)Z + II_W(Z,Y)L_U X - II_W(Z,X)L_U Y$$
$$+ II_U(Z,Y)L_W X - II_U(Z,X)L_W Y$$

where $\widetilde{R}$ is the curvature of the induced Riemannian structure on H. Thus,

$$(3)\quad \langle R(X,Y)Z,V\rangle = \langle\widetilde{R}(X,Y,Z,V\rangle + II_W(Z,Y)II_U(X,V)$$
$$- II_W(Z,X)II_U(Y,V)$$
$$+ II_U(Z,Y)II_W(X,V)$$
$$- II_U(Z,X)II_W(Y,V)$$

where V is any vector field on H.

Let $\{e_1, e_2\}$ be orthonormal eigenvectors of either $L_W$ or $L_U$ at $p \in H$. Then by making substitutions $X \leftrightarrow e_1$, $Y \leftrightarrow e_2$, $Z \leftrightarrow e_2$, $V \leftrightarrow e_1$ in (3), we obtain

$$\begin{aligned} K(e_1 \wedge e_2) &= \langle R(e_1, e_2)e_2, e_1 \rangle \\ &= \langle \tilde{R}(e_1, e_2)e_2, e_1 \rangle + II_u(e_1, e_1) II_w(e_2, e_2) \\ &\qquad + II_u(e_2, e_2) II_w(e_1, e_1) \end{aligned}$$

since either $II_u(e_1, e_2) = 0$ or $II_w(e_1, e_2) = 0$. Therefore

$$K_H = K(e_1 \wedge e_2) - II_u(e_1, e_1) II_w(e_2, e_2) - II_u(e_2, e_2) II_w(e_1, e_1).$$

<u>Step 2</u>: Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor T. Let $e_1, e_2$ be orthonormal spacelike vectors in $T_pM$ and u,w be future directed null vectors in $(e_1 \wedge e_2)^{\perp}$ with $\langle u, w \rangle = -1$ where $p \in M$ and $(e_1 \wedge e_2)^{\perp}$ is the orthogonal complement of the spacelike plane $e_1 \wedge e_2$ in $T_pM$. Then $K(e_1 \wedge e_2) = K(e_1 \wedge e_2)^{\perp} + Ric(u, w) + T(u, w)$ where $K(e_1 \wedge e_2)^{\perp}$ is the sectional curvature of the timelike plane $(e_1 \wedge e_2)^{\perp}$.

Proof: Let $\{z, n\}$ be the orthonormal Lorentizian basis for $(e_1 \wedge e_2)^{\perp}$ defined by

$$z = \frac{u+w}{\sqrt{2}} \text{ and } n = \frac{u-w}{\sqrt{2}} .$$

Then

(a) $\operatorname{Ric}(e_1,e_1) = -2<R(u,e_1)e_1,w> + K(e_1 \wedge e_2)$

Proof of (a):

$$\begin{aligned} \operatorname{Ric}(e_1,e_1) &= -<R(z,e_1)e_1)e_1,z> + <R(n,e_1)e_1,n> \\ &\quad + <R(e_2,e_1)e_1,e_2> \\ &= -<R(\frac{u+w}{\sqrt{2}},e_1)e_1,\frac{u+w}{\sqrt{2}}> \\ &\quad +<R(\frac{u-w}{\sqrt{2}},e_1)e_1,\frac{u-w}{\sqrt{2}}> \\ &\quad + K(e_1 \wedge e_2) \\ &= -<R(u,e_1)e_1,w> - <R(w,e_1)e_1,u> \\ &\quad + K(e_1 \wedge e_2) \\ &= -2<R(u,e_1)e_1,w> + K(e_1 \wedge e_2) \ \blacksquare \end{aligned}$$

By replacing $e_1$ with $e_2$ in (a) we obtain

(b) $\operatorname{Ric}(e_2,e_2) = -2<R(u,e_2)e_2,w> + K(e_1 \wedge e_2)$

(c) $\operatorname{Ric}(u,w) = -K(e_1 \wedge e_2)^{\perp} + <R(u,e_1)e_1,w> + <R(u,e_2)e_2,w>$

Proof of (c):

Since, $u = \frac{1}{\sqrt{2}}(z+n)$ and $w = \frac{1}{\sqrt{2}}(z-n)$,

$$
\begin{aligned}
\mathrm{Ric}(u,w) &= -\langle R(z,u)w,z\rangle + \langle R(n,u)w,n\rangle \\
&\quad + \langle R(e_1,u)w,e_1\rangle + \langle R(e_1,u)w,e_2\rangle \\
&= -\langle R(z,\tfrac{z+n}{\sqrt{2}})\tfrac{z-n}{\sqrt{2}},z\rangle + \langle R(n,\tfrac{z+n}{\sqrt{2}})\tfrac{z-n}{\sqrt{2}},n\rangle \\
&\quad + \langle R(e_1,u)w,e_1\rangle + \langle R(e_2,u)w,e_2\rangle \\
&= \tfrac{1}{2}\langle R(z,n)n,z\rangle + \tfrac{1}{2}\langle R(n,z)z,n\rangle \\
&\quad + \langle R(u,e_1)e_1,w\rangle + \langle R(u,e_2)e_2,w\rangle \\
&= \langle R(n,z)z,n\rangle + \langle R(u,e_1)e_1,w\rangle + \langle R(u,e_2)e_2,w\rangle \\
&= -K(e_1\wedge e_2)^{\perp} + \langle R(u,e_1)e_1,w\rangle + \langle R(u,e_2)e_2,w\rangle \quad \blacksquare
\end{aligned}
$$

(d) $Sc = -2\mathrm{Ric}(u,w) + \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2)$

<u>Proof of (d)</u>:

$$
\begin{aligned}
Sc &= -\mathrm{Ric}(z,z) + \mathrm{Ric}(n,n) + \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) \\
&= -\mathrm{Ric}(\tfrac{u+w}{\sqrt{2}},\tfrac{u+w}{\sqrt{2}}) + \mathrm{Ric}(\tfrac{u-w}{\sqrt{2}},\tfrac{u-w}{\sqrt{2}}) \\
&\quad + \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) \\
&= -2\mathrm{Ric}(u,w) + \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2). \quad \blacksquare
\end{aligned}
$$

(e) $K(e_1\wedge e_2) = K(e_1\wedge e_2)^{\perp} + 2\mathrm{Ric}(u,w) + \frac{1}{2}(Sc)$

<u>Proof of (e)</u>: from (a) and (b)

$$
\begin{aligned}
\mathrm{Ric}(e_1,e_1)+\mathrm{Ric}(e_2,e_2) &= -2\langle R(u,e_1)e_1,w\rangle - 2\langle R(u,e_2)e_2,w\rangle \\
&\quad + 2K(e_1\wedge e_2) \\
&= -2\,\mathrm{Ric}(u,w) - 2K(e_1\wedge e_2)^{\perp} \\
&\quad + 2K(e_1\wedge e_2) \text{ from (c).}
\end{aligned}
$$

On the other hand, from (d)

$$\mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) = \mathrm{Sc} + 2\,\mathrm{Ric}(u,w).$$

Therefore,

$$K(e_1\wedge e_2) = K(e_1\wedge e_2)^{\perp} + 2\mathrm{Ric}(u,w) + \frac{1}{2}\,\mathrm{Sc}.$$

Finally, from the Einstein equation,

$$\mathrm{Ric}(u,w) + \frac{1}{2}\,\mathrm{Sc} = T(u,w) \text{ since } \langle u,w\rangle = -1.$$

Therefore, from (e)

$$K(e_1\wedge e_2) = K(e_1\wedge e_2)^{\perp} + \mathrm{Ric}(u,w) + T(u,w).$$

Now, by combining Step 1 and Step 2, we obtain

$$\begin{aligned} K_H(p) &= K_{H^{\perp}}(p) + \mathrm{Ric}(u,w) + T(u,w) - II_u(e_1,e_1)II_w(e_2,e_2) \\ &\qquad - II_u(e_2,e_2)II_w(e_1,e_1) \\ &= K_{H^{\perp}}(p) + \mathrm{Ric}(u,w) + T(u,w) - (\mathrm{tr}\,L_u)(\mathrm{tr}L_w) \\ &\qquad + II_u(e_1,e_1)II_w(e_1,e_1) + II_u(e_2,e_2)II_w(e_2,e_2) \end{aligned}$$

since $(\mathrm{tr}\,L_u)(\mathrm{tr}\,L_w) = II_u(e_1,e_1)II_w(e_1,e_1) + II_u(e_2,e_2)II_w(e_2,e_2)$

$$+ II_u(e_1,e_1)II_w(e_2,e_2) + II_u(e_2,e_2)II_w(e_1,e_1). \quad \blacksquare$$

Section B: Evolution of closed trapped surfaces

In this section, we shall be concerned with the evolution of closed trapped surfaces from marginally trapped surfaces in cosmological and black hole circumstances. We shall observe that the evolution of closed trapped surfaces from marginally trapped surfaces is closely related with energy density in cosmological circumstances and is closely related with surface gravity of marginally trapped surfaces in black hole circumstances.

In achieving the above results, we shall make use of the following lemma:

Lemma 4.8: Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor T. Let H be a compact spacelike surface with trivial normal bundle $\mathfrak{N}(H)$ and let U,W be future directed null sections of $\mathfrak{N}(H)$ with $\langle U,W\rangle = -1$. Assume that $L_W = 0$ (where $L_W$ is the second fundamental tensor of H in the direction W). Let $S_U = \exp^{\perp}(Q)$ where Q is an open neighborhood of H in $\mathfrak{N}_U(H)$ on which $\exp^{\perp}$ is an embedding of Q into M. Thus $S_U$ is a null hypersurface in M (cf. Chapter I, section B). Let $\tilde{U}$ be the extension of U to a future directed null geodesic vector field on $S_U$ (cf. Chapter II, section B). Let $\varphi$ be the flow of $\tilde{U}$ and let

$H_t = \varphi_t(H)$. Thus $H_t$ is a compact spacelike surface for sufficiently small t (see Figure 7). Let $\mathcal{D}$ be the sub-bundle to $TS_U$ defined by $\mathcal{D} = \bigcup_t TH_t$ (in some open neighborhood of H in $S_U$) and let $\widetilde{W}$ be the extension of W to a null section of $\mathcal{D}^\perp$ with $\langle\widetilde{U},\widetilde{W}\rangle = -1$ (cf. Chapter II, section B). Then, for any $p \in H$

$$U_p(\mathrm{tr}\nabla\widetilde{W}) = -K_{H^\perp}(p) - \mathrm{Ric}(U,W)|_p + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})|_p + \|\nabla_{\widetilde{U}}\widetilde{W}|_p\|^2$$

$$= T(U,W)|_p - K_H(p) + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})|_p + \|\nabla_{\widetilde{U}}\widetilde{W}|_p\|^2$$

where $\mathrm{div}_H\, \nabla_{\widetilde{U}}\widetilde{W}$ is the divergence of $\nabla_{\widetilde{U}}\widetilde{W}|_H$ on H in the induced metric. (Note that $\nabla_{\widetilde{U}}\widetilde{W}|_H \in \Gamma TH$ (cf. Remark 4.9(a).)

<u>Proof</u>: cf. page 106.

We first note the following consequences of the construction in Lemma 4.8.

<u>Remark 4.9</u>

a) By definition, $\mathcal{D}$ is integrable and therefore $\nabla\widetilde{W}: \mathcal{D}\to\mathcal{D}$ is self-adjoint (cf. Chapter II, Theorem 2.22).

b) Since $\langle\nabla_{\widetilde{U}}\widetilde{W},\widetilde{W}\rangle = \frac{1}{2}\widetilde{U}\langle\widetilde{W},\widetilde{W}\rangle = 0$ and $\langle\nabla_{\widetilde{U}}\widetilde{W},\widetilde{U}\rangle = \widetilde{U}\langle\widetilde{W},\widetilde{U}\rangle - \langle\widetilde{W},\nabla_{\widetilde{U}}\widetilde{U}\rangle = 0$, it follows that $\nabla_{\widetilde{U}}\widetilde{W} \in \Gamma\mathcal{D}$ and therefore $\nabla_{\widetilde{U}}\widetilde{W}|_H \in \Gamma TH$.

c) Let $X$ be a Lie parallel vector field along $\tilde{U}$ with $X|_H \in \Gamma TH$ (thus $X$ is tangent $H_t$ for each $t$, that is $X \in \Gamma \mathcal{D}$). Therefore, $<\nabla_{\tilde{U}}\tilde{W},X> = -<\tilde{W},\nabla_X\tilde{U}>$ since $<\nabla_{\tilde{U}}\tilde{W},X> = \tilde{U}<\tilde{W},X> - <\tilde{W},\nabla_X\tilde{U}> = -<\tilde{W},\nabla_X\tilde{U}>$.

d) If $X \in \Gamma TS_U$ then $X = -<X,\tilde{W}>\tilde{U} + X^T$ where $X^T$ is the component of $X$ in $\mathcal{D}$(cf. Chapter II, section B).

e) If $X \in \Gamma \tilde{W}^{\perp}$ then $X = -<X,\tilde{U}>\tilde{W} + X^T$ where $X^T$ is the component of $X$ in $\mathcal{D}$.

f) Let $X \in TS_U$. Then $\nabla_{\tilde{U}}X \in \Gamma TS_U$ and $\nabla_X\tilde{U} \in \Gamma TS_U$ since $S_U$ is a null hypersurface (cf. Chapter I, section A).

Theorem 4.10: Let $M$ be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $T$. Let $H$ be a spacelike surface diffeomorphic to $S^2$ in $M$. Assume $\mathcal{N}(H)$ is trivial vector bundle and let $U,W$ be future directed null sections of $\mathcal{N}(H)$ with $<U,W> = -1$. Assume also that:

i) $K_{H^{\perp}} + Ric(U,W) > 0$ along $H$

ii) $L_W = -(\nabla W)^T = 0$

iii) $tr\, L_U = -tr(\nabla U)^T \geq 0$ along $H$, and $(R(x,U_p)U_p)^T \neq 0$ for some $x \in T_pH$ whenever $tr\, L_U|_p = 0$

iv) $\langle R(U,W)X,Y\rangle = 0$ for all $X,Y \in \Gamma TH$.

Then M contains closed trapped surfaces diffeomorphic to $S^2$ in $J^+(H)$.

<u>Proof</u>: First, we note that $\langle U,W\rangle = -1$ leaves the freedom $U \to U' = e^f U$ and $W \to W' = e^{-f}W$ where f is a smooth function on H. Now, let's consider the situation in Lemma 4.8 for U' and W'. Then for any $p \in H$,

$$(1)\quad U'_p(\operatorname{tr} \nabla\widetilde{W}') = -K_{H^\perp}(p) - \operatorname{Ric}(U',W')|_p + \operatorname{div}_H(\nabla_{\widetilde{U}'}\widetilde{W}')|_p + \|\nabla_{\widetilde{U}'}\widetilde{W}'\|^2|_p .$$

On the other hand, since $L_{\widetilde{W}'}|_{TH_t}$ and $L_{\widetilde{U}'}|_{TH_t}$ are the second fundamental tensors of the $H_t$ for each t, it suffices to show that $\operatorname{tr} L_{\widetilde{W}'}|_{TH_t} > 0$ and $\operatorname{tr} L_{\widetilde{U}'}|_{TH_t} > 0$ for some $t > 0$. (Thus, $H_t$ is a closed trapped surface diffeomorphic to $S^2$ for some $t > 0$.) But, $\operatorname{tr} L_{\widetilde{U}'}|_{TH_t} > 0$ for all $t > 0$ from condition (iii) and Corollary 1.22 ($\operatorname{Ric}(U',U') \geq 0$ from the weak energy condition). Now, to show that $\operatorname{tr} L_{\widetilde{W}'}|_{TH_t} > 0$ for some $t > 0$, it suffices to show $U'_p(\operatorname{tr} \nabla\widetilde{W}') < 0$ at each $p \in H$. For this, it suffices to show that U' and W' can be chosen so that $\nabla_{\widetilde{U}'}\widetilde{W}'|_H = 0$

Then, $U'_p(\mathrm{tr}\ \nabla\widetilde{W}') < 0$ follows from Lemma 4.8 (since $K_{H^\perp}(p) + \mathrm{Ric}(U',W') > 0$ along H).

Note that, for any $X \in \Gamma TH$,

$$\begin{aligned} \langle \nabla_{\widetilde{U}'}\widetilde{W}', X\rangle &= -\langle W', \nabla_X U'\rangle = -\langle e^{-f}W, \nabla_X e^f U\rangle \\ &= -e^{-f}\langle W, e^f X(f)U + e^f \nabla_X U\rangle \\ &= X(f) - \langle W, \nabla_X U\rangle = df(X) - \langle W, \nabla_X U\rangle \end{aligned}$$

from Remark 4.9(c). Therefore, $\nabla_{\widetilde{U}'}\widetilde{W}' = 0$ along H iff there exists a function $f: H \to \mathbb{R}$ such that $df(X) = \langle W, \nabla_X U\rangle$ for all $X \in \Gamma TH$ iff the 1-form $\alpha = \langle W, \nabla_\cdot U\rangle$ on H is exact. (Note that, from Remark 4.9(b), $\nabla_{\widetilde{U}'}\widetilde{W}' \in \Gamma TH$ and, since H is spacelike, $\nabla_{\widetilde{U}'}\widetilde{W}' = 0$ iff $\langle \nabla_{\widetilde{U}'}\widetilde{W}', X\rangle = 0$ for every $X \in \Gamma TH$). But, since H is diffeomorphic to $S^2$, it suffices to show that $\alpha$ is closed (since H is simply connected).

Let $p \in H$ and X,Y be tangent vector fields to H on some neighborhood of p in H with $[X,Y] = 0$. Then,

$$\begin{aligned} d\alpha(X,Y) &= X\alpha(Y) - Y\alpha(X) = X\langle W, \nabla_Y U\rangle - Y\langle W, \nabla_X U\rangle \\ &= \langle \nabla_X W, \nabla_Y U\rangle + \langle W, \nabla_X\nabla_Y U\rangle - \langle \nabla_Y W, \nabla_X U\rangle \\ &\quad - \langle W, \nabla_Y \nabla_X U\rangle \\ &= \langle \nabla_X W, \nabla_Y U\rangle - \langle \nabla_Y W, \nabla_X U\rangle + \langle W, R(X,Y)U\rangle \end{aligned}$$

since $[X,Y] = 0$.

But, since $\nabla_X W = -\langle\nabla_X W, U\rangle W + (\nabla_X W)^T$ (cf. Remark 4.9(e)),

$$\langle\nabla_X W, \nabla_Y U\rangle = -\langle\nabla_X W, U\rangle\langle W, \nabla_Y U\rangle + \langle(\nabla_X W)^T, \nabla_Y U\rangle$$

$$= \langle\nabla_X W, U\rangle\langle\nabla_Y W, U\rangle + \langle(\nabla_X W)^T, \nabla_Y U\rangle$$

and similarly,

$$\langle\nabla_Y W, \nabla_X U\rangle = \langle\nabla_Y W, U\rangle\langle\nabla_X W, U\rangle + \langle(\nabla_Y W)^T, \nabla_X U\rangle.$$

Thus,

$$d\alpha(X,Y) = \langle W, R(X,Y)U\rangle + \langle(\nabla_X W)^T, \nabla_Y U\rangle - \langle(\nabla_Y W)^T, \nabla_X U\rangle.$$

Therefore, from condition (ii) (that is $L_W = 0$),

$$d\alpha(X,Y) = \langle W, R(X,Y)U\rangle$$

$$= \langle R(U,W)X, Y\rangle$$

$$= 0 \text{ from condition (iv).} \quad \blacksquare$$

Remarks:

i) Note that, $(\nabla_X W)^T = 0$ for every $X \in T_pH$ iff $\nabla_X W = aW_p$ for some $a \in \mathbb{R}$.

ii) Assume that each point $p \in H$ has a neighborhood $V$ in $H$ and on which there exist a function $f\colon V \to \mathbb{R}$ such that $\nabla_X W' = 0$ for every $X \in TH|_V$ where $W' = e^f W$.

Then, $R(X,Y)W = 0$ for every $X, Y \in TH$ and therefore $\langle R(X,Y)W, U\rangle = \langle R(W,U)X, Y\rangle = 0$.

Corollary 4.11: Let H be an orientable spacelike surface in a 4-dimensional spacetime M which obeys the Einstein equation for the stress-energy tensor T = 0. Assume $\mathcal{N}(H)$ is a trivial vector bundle and let U,W be future directed null sections of $\mathcal{N}(H)$ with $\langle U,W\rangle = -1$. Assume also that:

a) H is complete in the induced metric

b) $K_{H^\perp} > 0$ along H

c) $L_W = -(\nabla W)^T = 0$

d) $\operatorname{tr} L_U = -\operatorname{tr}(\nabla U)^T \geq 0$ along H, and $(R(X,U_p)U_p)^T \neq 0$ for some $X \in T_pH$ whenever $\operatorname{tr} L_U|_p = 0$

e) $\langle R(W,U)X,Y\rangle = 0$ for all $X,Y \in TH$

f) $\int_H K_{H^\perp}\omega_H > 2\pi$.

Then there exist closed trapped surfaces diffeomorphic to $S^2$ in $J^+(H)$.

Proof: From the Gauss-Codazzi equation,

$$K_H = K_{H^\perp} \text{ since } T = 0 = \text{Ric and } L_W = 0.$$

Also, since $\int_H K_H\omega_H = \int_H K_{H^\perp}\omega_H > 2\pi$, H is diffeomorphic to $S^2$ from the result of Cohn-Vossen. Thus, there exist closed trapped surfaces in $J^+(H)$ diffeomorphic to $S^2$ from the Theorem 4.10. ∎

<u>Discussion of Corollary 4.11</u>: In fact, the conditions of Corollary 4.11 are more general than needed for the intended application of the theorem to black holes. We shall discuss the conditions and the consequence of the theorem in the Kruskal black hole model. We recall that the Kruskal spacetime is the warped product $\mathbb{R}^2 \times {}_f S^2$ where $\mathbb{R}^2$ is equipped with the Lorentzian metric $g = F(r)$ $(dw \otimes du + du \otimes dw)$ where $F(r) = (\frac{8m^2}{r})\exp(1 - \frac{r}{2m})$ and $r$ is implicitly defined by $(r-2m)\exp(\frac{r}{2m} - 1) = wu$. $S^2$ is equipped with the usual Riemannian metric $h$ of constant curvature 1. The warping function $f$ is given by $f(w,u) = r^2$ (cf. [O], page 389). The null hypersurface $S$ which is given by $r = 2m$ (event horizon) is a stationary null hypersurface diffeomorphic to $\mathbb{R}\times S^2$ and $W = \frac{\partial}{\partial w}\Big|_S$ is a future directed <u>geodesic</u> null vector field on $S$ with $\nabla W = 0$ on $TS$ since $r = 2m$ on $S$. (cf. [O], page 156, exercise 8 and page 206, proposition 35-(2)). Thus, it follows that $R(X,Y)W = 0$ and therefore $\langle R(W,U)X,Y\rangle = \langle R(X,Y)W,U\rangle = 0$ for every spacelike vector $X,Y \in TS$ where $U = \frac{1}{4m}\frac{\partial}{\partial u}\Big|_S$ (cf. the remark below the Theorem 4.10). Since $S$ is a null hypersurface, every spacelike surface $H$ in $S$ has trivial vector bundle and $L_W = 0$ (cf. Chapter II, Theorem 2.45 and Proposition 2.10). Also, it can be

shown that the surface gravity $K_{H^\perp}$ of the spacelike spheres H which are contained in the null hypersurface S is equal to $\frac{1}{8m^2} > 0$ and $\operatorname{tr} L_U > 0$. Furthermore, since they are topologically sphere, $\int_H K_{H^\perp}\omega_H = 4\pi > 2\pi$. Therefore, the corollary implies that there exist closed trapped surfaces in the black hole region which are evolving from the spacelike spheres in the event horizon S.

In the next theorem, we shall discuss the evolution of closed trapped surfaces from marginally trapped surfaces in cosmological circumstances. First we shall state a consequence of the Hodge decomposition theorem for immediate reference in the next theorem.

<u>Lemma 4.12</u>: Let H be a compact, connected, oriented Riemannian manifold and lef $f: H \to \mathbb{R}$ be a smooth function. Then the equation

$$\Delta\psi + f = B \text{ where } B = \frac{\int_H f\omega_H}{\int_H \omega_H}, \text{ has a solution on } H$$

($\Delta$ is the Laplace operator on H).

<u>Proof</u>: From the Hodge decomposition theorem (cf. [W], page 223), $\Delta\psi = B-f$ has a solution on H iff $\int_H (B-f)\omega_H = 0$

(since, the only harmonic functions on a compact, connected, oriented Riemannian manifold are the constant functions). But, by definition of B, $\int_H (B-f)\omega_H = 0$ and therefore there exists a function $\psi: H \to \mathbb{R}$ which satisfies the above equation. ∎

Theorem 4.13: Let M be a 4-dimensional spacetime which obeys the Einstein equation for the stress-energy tensor $\hat{T} = (\zeta+p)Z\otimes Z + p\hat{g}$. Let H be a spacelike surface orthogonal to Z. Assume that $\mathcal{N}(H)$ is trivial and let U,W be future directed null sections of $\mathcal{N}(H)$ with $\langle U,W\rangle = -1$. Assume also that:

a) H is complete in the induced metric

b) $L_W = 0$ and $\operatorname{tr} L_U > 0$ along H.

c) M is weakly spatially isotropic for the instantaneous observers in Z along H.

d) $\int_H \zeta\omega_H > 6\pi$ and $\int_H p\omega_H < 2\pi$

Then M contains closed trapped surfaces in $J^-(H)$ diffeomorphic to $S^2$.

Proof: From the Gauss-Codazzi equation,

$$K_H = K_{H^\perp} + \operatorname{Ric}(U,W) + T(U,W) \quad (\text{since } L_W=0).$$

Again, following the same lines of the proof of Theorem 4.2, we obtain, $K_H = \frac{1}{3}\zeta$. Thus, since $\int_H K_H \omega_H = \frac{1}{3}\int_H \zeta\omega_H > 2\pi$, H is diffeomorphic to $S^2$ (from Cohn-Vossen's result) and $\int_H \zeta\omega_H = 12\pi$.

Note that $\langle U,W\rangle = -1$ leaves the freedom $U \to U' = e^{\psi}U$ and $W' = e^{-\psi}W$ where $\psi: H \to \mathbb{R}$ is a smooth function. Now, let's consider the situation in Lemma 4.8 for U' and W'. Then, for every $q \in H$,

$$U'_q(\mathrm{tr}\ \nabla\widetilde{W}') = T(\widetilde{U}',\widetilde{W}')\big|_q - K_H(q) + \mathrm{div}_H(\nabla_{U'}\widetilde{W}')\big|_q + \|\nabla_{U'}\widetilde{W}'|_q\|^2.$$

Now, we shall show that U' and W' can be chosen so that $U'_q(\mathrm{tr}\ \nabla\widetilde{W}') > 0$ at every $q \in H$.

Now, let's also consider the situation in Lemma 4.8 for U and W. Let $\{X_1, X_2\}$ be an adapted moving frame near q on H in the induced connection. (That is, $\tilde{\nabla}X_i|_q = 0$ and $\tilde{\nabla}_{X_i}X_i = 0$ where $\tilde{\nabla}$ is the induced connection on H (cf. [Po], page 151)). Let $\widetilde{X}_i$ $(i=1,2)$ be the Lie parallel extension of $X_i$ with respect to U' to a neighborhood of H in $S_U$, and $\tilde{\tilde{X}}_i$ $(i=1,2)$ be the Lie parallel extension of $X_i (i=1,2)$ with respect to $\widetilde{U}$ to a neighborhood of H in $S_U$.

Then, since $\nabla_{\tilde{U}'}\tilde{W}'|_H \in \Gamma TH$,

$$\operatorname{div}_H(\nabla_{\tilde{U}'}\tilde{W}')|_q = \sum_{i=1}^{2} \langle \tilde{\nabla}_{X_i} \nabla_{\tilde{U}'}\tilde{W}', X_i\rangle|_q$$

$$= \sum_{i=1}^{2} [X_i \langle \nabla_{\tilde{U}'}\tilde{W}', X_i\rangle|_q - \langle \nabla_{\tilde{U}'}\tilde{W}', \tilde{\nabla}_{X_i} X_i\rangle|_q]$$

(since $\nabla_{\tilde{X}_i} X_i = 0$)

$$= \sum_{i=1}^{2} [\tilde{X}_i(\tilde{U}'\langle \tilde{W}', \tilde{X}_i\rangle)|_q - \tilde{X}_i\langle \tilde{W}', \nabla_{\tilde{U}'}\tilde{X}_i\rangle|_q]$$

$$= \sum_{i=1}^{2} \tilde{X}_i\langle \tilde{W}', \nabla_{\tilde{X}_i}\tilde{U}'\rangle|_q = -\sum_{i=1}^{2} X_i\langle \tilde{W}', \nabla_{X_i}\tilde{U}'\rangle|_q$$

$$= -\sum_{i=1}^{2} X_i\langle e^{-\psi}W, \nabla_{X_i} e^{\psi}U\rangle|_q$$

$$= -\sum_{i=1}^{2} X_i\langle e^{-\psi}W, e^{\psi}X_i(\psi)U + e^{\psi}\nabla_{X_i}U\rangle|_q$$

$$= \sum_{i=1}^{2} [X_i^2(\psi) - X_i\langle W, \nabla_{X_i}U\rangle]|_q$$

$$= \nabla\psi|_q - \sum_{i=1}^{2} [\tilde{\tilde{X}}_i\langle \tilde{W}, \nabla_{\tilde{\tilde{X}}_i}\tilde{U}\rangle|_q]$$

$$= \nabla\psi|_q - \sum_{i=1}^{2} [\tilde{\tilde{X}}_i\langle \tilde{W}, \nabla_{\tilde{U}}\tilde{\tilde{X}}_i\rangle|_q]$$

$$= \nabla\psi|_q + \sum_{i=1}^{2} [\tilde{\tilde{X}}_i\langle \nabla_{\tilde{U}}\tilde{W}, \tilde{\tilde{X}}_i\rangle|_q]$$

$$= \nabla\psi|_q + \sum_{i=1}^{2} [X_i\langle \nabla_{\tilde{U}}\tilde{W}, X_i\rangle|_q]$$

$$= \nabla\psi|_q + \operatorname{div}_H(\nabla_{\tilde{U}}\tilde{W})|_q.$$

Thus

$$U'_p(\mathrm{tr}\ \nabla\widetilde{W}') = T(\widetilde{U}',\widetilde{W}')|_q - K_H(q) + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})|_q + \Delta\psi|_q + \|\nabla_{\widetilde{U}'}\widetilde{W}'|_q\|^2.$$

Since $T(\widetilde{U}',\widetilde{W}')|_H = T(\widetilde{U},\widetilde{W})|_H = T(U,W) = \frac{1}{2}(\zeta - p)$ (cf. proof of the Theorem 4.2),

$$U'_q(\mathrm{tr}\ \nabla\widetilde{W}') = \frac{1}{6}\zeta(q) - \frac{1}{2}p(q) + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})|_q + \Delta\psi|_q + \|\nabla_{\widetilde{U}'}\widetilde{W}'|_q\|^2.$$

Now, let $f = \frac{1}{6}\zeta - \frac{1}{2}p + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})$. Then the equation $\Delta\psi + f = B$ has a solution $\psi$ on $H$ where $B = \dfrac{\int_H f\omega_H}{\int_H \omega_H}$ (cf. Lemma 4.12).

Therefore, for this $\psi$, $U'_q(\mathrm{tr}\ \nabla\widetilde{W}') = B + \|\nabla_{\widetilde{U}'}\widetilde{W}'|_q\|^2$. Thus, to show that $U'_q(\mathrm{tr}\ \nabla\widetilde{W}') > 0$, it suffices to show that $\int_H f\omega_H > 0$. Indeed, $\int_H f\omega_H = \frac{1}{6}\int_H \zeta\omega_H - \frac{1}{2}\int_H p\omega_H + \int_H \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})\omega_H > 0$ since $\frac{1}{6}\int_H \zeta\omega_H = 2\pi$ (since $H \cong S^2$), $\int_H p\omega_H < 2\pi$ and $\int_H \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})\omega_H = 0$.

Now, since $H$ is compact ($\cong S^2$), $\mathrm{tr}\ L_{U'} \geq c > 0$ for some $c \in \mathbb{R}$ from the assumption (b). Also, since $\mathrm{tr}\ L_{W'} = 0$ but $U'(\mathrm{tr}\ L_{W'}) > 0$, it follows that $\varphi_t(H) = H_t$ is a closed trapped surface diffeomorphic to $S^2$ for some $t < 0$ (Corollary 1.22). ∎

<u>Discussion of the Theorem 4.13</u>: We shall discuss the conditions and the consequences of the above theorem in the recollapsing Friedman spacetime. We recall that the recollapsing Friedman spacetime is the warped product $M = \mathbb{R} \times_f S^3$ where $\mathbb{R}$ is equipped with the negative definite

metric $-dt \otimes dt$ and $S^3$ is equipped with the usual Riemannian metric h of constant curvature 1 and f is the warping function which is given by $f' + 1 = A/f$ where A is a positive constant. (cf. [O], page 351.) Thus, stress-energy tensor is $\hat{T} = \zeta Z \otimes Z$ (dust) where $Z = \frac{\partial}{\partial t}$ and this spacetime is spatially isotropic for Z and therefore, weakly spatially isotropic for each instantaneous observer in Z. (cf. [O], page 351.) M contains achronal spacelike surfaces H (diffeomorphic to $S^2$) which are orthogonal to Z with the property that $\operatorname{tr} L_U > 0$ and $L_W = 0$ (cf. [CET], page 186, also see Figure 8). Since these spacelike surfaces are topologically $S^2$, from Gauss-Bonnet theorem $\int_H \zeta \omega_H = 12\pi > 6\pi$. Also, since $p = 0$ (dust), the above theorem implies that there exist closed trapped surfaces diffeomorphic to $S^2$ in the causal pasts of these marginally trapped surfaces.

Now, we shall provide the proof of Lemma 4.8.

Proof of the Lemma 4.8: Let $Y_i$ $i=1,2$ be orthonormal vector fields on H in some neighborhood of $p \in H$.

Let $X_i$ $(i=1,2)$ be vector fields on $S_U$ which are obtained by first Lie translating $Y_i$ $(i=1,2)$ along $\tilde{U}$ and then orthonormalizing them. (Thus, $X_i$ $(i=1,2)$ is an orthonormal basis tangent to $H_t = \varphi_t(H)$). Thus

$$\operatorname{tr}(\nabla\widetilde{W})^T = \sum_{i=1}^{2} <(\nabla_{X_i}\widetilde{W})^T, X_i> = \sum_{i=1}^{2} <\nabla_{X_i}\widetilde{W}, X_i>$$

since $X_i$ $(i=1,2)$ are tangent to $H_t$.

Then

$$\widetilde{U}(\operatorname{tr}\nabla\widetilde{W})^T = \sum_{i=1}^{2} \widetilde{U}<\nabla_{X_i}\widetilde{W}, X_i> = \sum_{i=1}^{2} [<\nabla_{\widetilde{U}}\nabla_{X_i}\widetilde{W}, X_i> + <\nabla_{X_i}\widetilde{W}, \nabla_{\widetilde{U}}X_i>].$$

On the other hand, from

$$\nabla_{\widetilde{U}}\nabla_{X_i}\widetilde{W} = R(\widetilde{U}, X_i)\widetilde{W} + \nabla_{X_i}\nabla_{\widetilde{U}}\widetilde{W} + \nabla_{\nabla_{\widetilde{U}}X_i}\widetilde{W} - \nabla_{\nabla_{X_i}\widetilde{U}}\widetilde{W}$$

we obtain

$$(1)\quad <\nabla_{\widetilde{U}}\nabla_{X_i}\widetilde{W}, X_i> = <R(\widetilde{U}, X_i)\widetilde{W}, X_i> + <\nabla_{X_i}\nabla_{\widetilde{U}}\widetilde{W}, X_i>$$

$$+ <\nabla_{\nabla_{\widetilde{U}}X_i}\widetilde{W}, X_i> - <\nabla_{\nabla_{X_i}\widetilde{U}}\widetilde{W}, X_i>$$

Moreover, since $\nabla_{\widetilde{U}}X_i$ is in $TS_U$ (cf. Remark 4.9-(f)), $\nabla_{\widetilde{U}}X_i = -<\nabla_{\widetilde{U}}X_i, \widetilde{W}>\widetilde{U} + (\nabla_{\widetilde{U}}X_i)^T$ (cf. Remark 4.7-(d)) where $(\nabla_{\widetilde{U}}X_i)^T$ is the component of $\nabla_{\widetilde{U}}X_i$ in $\mathcal{D}$. Therefore

$$(2) <\nabla_{\nabla_{\widetilde{U}}X_i}\widetilde{W}, X_i> = -<\nabla_{\widetilde{U}}X_i, \widetilde{W}><\nabla_{\widetilde{U}}\widetilde{W}, X_i> + <\nabla_{(\nabla_{\widetilde{U}}X_i)^T}W, X_i>$$

$$= <\nabla_{\widetilde{U}}\widetilde{W}, X>^2 + <(\nabla_{\widetilde{U}}X_i)^T, (\nabla_{X_i}\widetilde{W})^T>$$

since $(\nabla\widetilde{W})^T: \mathcal{D} \to \mathcal{D}$ is self-adjoint (cf. Remark 4.9-(a)), and $<\nabla_{\widetilde{U}}\widetilde{W}, X_i> = -<\nabla_{\widetilde{U}}X_i, \widetilde{W}>$ (cf. Remark 4.9-(c)). Similarly since $\nabla_{X_i}\widetilde{U}$ is in $TS_U$, $\nabla_{X_i}\widetilde{U} = -<\nabla_{X_i}\widetilde{U}, \widetilde{W}>\widetilde{U} + (\nabla_{X_i}\widetilde{U})^T$ (cf.

Remark 4.7-(d)) and therefore

$$(3)\quad <\nabla_{\nabla_{X_i}\widetilde{U}}\widetilde{W},X_i> = -<\nabla_{X_i}\widetilde{U},\widetilde{W}><\nabla_{\widetilde{U}}\widetilde{W},X_i>+<\nabla_{(\nabla_{X_i}\widetilde{U})^T}\widetilde{W},X_i>$$

$$= <\widetilde{U},\nabla_{X_i}\widetilde{W}><\nabla_{\widetilde{U}}\widetilde{W},X_i>+<(\nabla_{X_i}\widetilde{U})^T,(\nabla_{X_i}\widetilde{W})^T>$$

from Remarks 4.9-(a) and 4.9-(c).

Finally, since $\nabla_{X_i}\widetilde{W}$ is in $\widetilde{W}^{\perp}$, $\nabla_{X_i}\widetilde{W} = -<\nabla_{X_i}\widetilde{W},\widetilde{U}>W + (\nabla_{X_i}\widetilde{W})^T$ where $(\nabla_{X_i}\widetilde{W})^T$ is the component of $\nabla_{X_i}\widetilde{W}$ in $\mathcal{D}$. Therefore

$$(4)\quad <\nabla_{X_i}\widetilde{W},\nabla_{\widetilde{U}}X_i> = -<\nabla_{X_i}\widetilde{W},\widetilde{U}><\widetilde{W},\nabla_{\widetilde{U}}X_i>+<(\nabla_{X_i}\widetilde{W})^T,\nabla_{\widetilde{U}}X_i>$$

$$= <\widetilde{U},\nabla_{X_i}\widetilde{W}><\nabla_{\widetilde{U}}\widetilde{W},X_i>+<(\nabla_{X_i}\widetilde{W})^T,(\nabla_{\widetilde{U}}X_i)^T>$$

Then, by substituting (2) and (3) in (1), we obtain

$$(5)\quad <\nabla_{\widetilde{U}}\nabla_{X_i}\widetilde{W},X_i> = <R(\widetilde{U},X_i)\widetilde{W},X_i>+<\nabla_{X_i}\nabla_{\widetilde{U}}\widetilde{W},X_i>$$

$$+ <\nabla_{\widetilde{U}}\widetilde{W},X_i>^2 + <(\nabla_{\widetilde{U}}X_i)^T,(\nabla_{X_i}\widetilde{W})^T>$$

$$- <\widetilde{U},\nabla_{X_i}\widetilde{W}><\nabla_{\widetilde{U}}\widetilde{W},X_i>-<(\nabla_{X_i}\widetilde{U})^T,(\nabla_{X_i}\widetilde{W})^T>.$$

Hence, using (4) and (5), we obtain

$$\widetilde{U}(\mathrm{tr}\ \nabla\widetilde{W}) = \sum_{i=1}^{2}[<R(\widetilde{U},X_i)\widetilde{W},X_i>+<\nabla_{X_i}\nabla_{\widetilde{U}}\widetilde{W},X_i>+<\nabla_{\widetilde{U}}\widetilde{W},X_i>^2$$

$$+ <(\nabla_{\widetilde{U}}X_i)^T,(\nabla_{X_i}\widetilde{W})^T>-<(\nabla_{X_i}\widetilde{U})^T,(\nabla_{X_i}\widetilde{W})^T>$$

$$+ <(\nabla_{X_i}\widetilde{W})^T,(\nabla_{\widetilde{U}}X_i)^T>].$$

Thus,

$$(6)\quad U_p(\text{tr } \nabla\widetilde{W}) = \sum_{i=1}^{2} [(<R(U,X_i)W,X_i>|_p + <\nabla_{X_i} \nabla_{\widetilde{U}}\widetilde{W},X_i>|_p + <\nabla_{\widetilde{U}}\widetilde{W},X_i>^2|_p]$$

since $L_{W_p} = -(\nabla\widetilde{W})^T|_p = 0$.

Also, from $\text{Ric}(U,W)|_p + K_{H^\perp}(p) = \sum_{i=1}^{2} <R(U,X_i)X_i,W>$ (cf. Identity c on page 90) and Gauss-Codacci equation, we obtain

$$(7)\quad -\sum_{i=1}^{2} <R(U,X_i)X_i,W>|_p = -K_{H^\perp}(p) - \text{Ric}(U,W)|_p$$

$$= T(U,W)|_p - K_H(p)$$

since $L_W = 0$.

Also, since $\nabla_{\widetilde{U}}\widetilde{W}|_H$ is tangent to H (cf. Remark 4.9-(b)), we have

$$(8)\quad \sum_{i=1}^{2} <\nabla_{X_i} \nabla_{\widetilde{U}}\widetilde{W},X_i>|_p = \sum_{i=1}^{2} <\widetilde{\nabla}_{X_i} \widetilde{\nabla}_{\widetilde{U}}\widetilde{W},X_i>|_p = \text{div}_H(\nabla_{\widetilde{U}}\widetilde{W})|_p$$

and

$$(9)\quad \sum_{i=1}^{2} <\nabla_{\widetilde{U}}\widetilde{W},X_i>^2|_p = ||\nabla_{\widetilde{U}}\widetilde{W}|_p||^2.$$

Finally, by substituting (7), (8) and (9) in (6), we obtain

$$U_p(\mathrm{tr}\ \nabla\widetilde{W}) = -K_{H^{\perp}}(p) - \mathrm{Ric}(U,W)\big|_p + \mathrm{div}_H(\nabla_{\widetilde{U}}\widetilde{W})\big|_p$$

$$+ \|\nabla_{\widetilde{U}}\widetilde{W}|_p\|$$

$$= T(U,W)\big|_p - K_H(p) + \mathrm{div}_H\nabla_{\widetilde{U}}\widetilde{W}\big|_p$$

$$+ \|\nabla_{\widetilde{U}}\widetilde{W}|_p\|^2. \qquad \blacksquare$$

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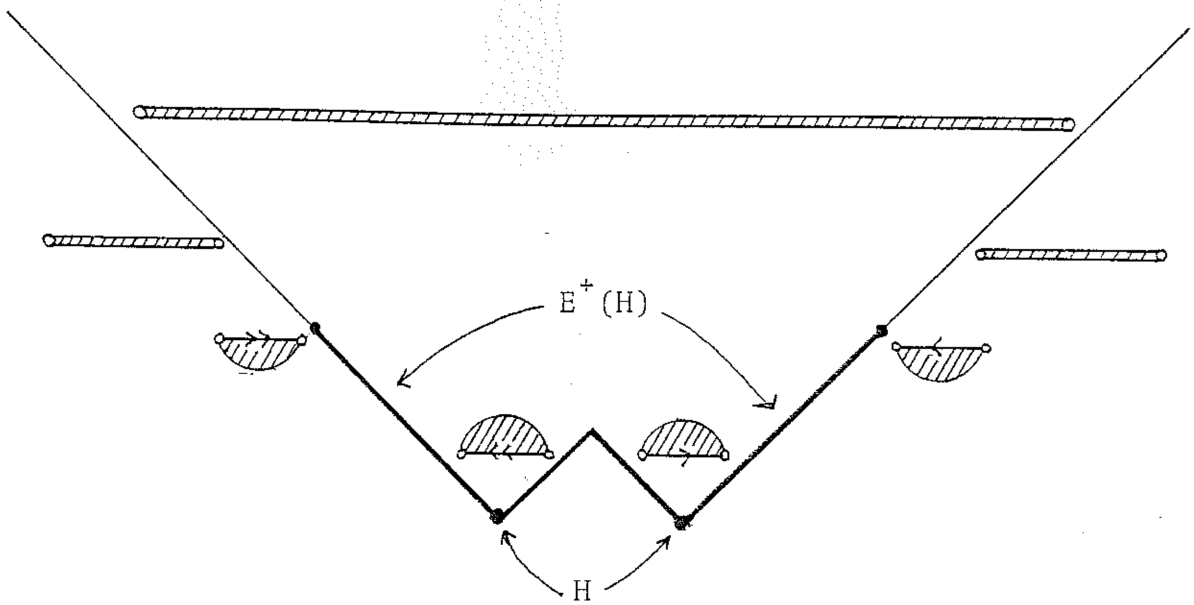



Figure 1. This identified subset of 2-dimensional Minkowski spacetime is strongly causal and contains a future trapped set H. Note that, every complete non-spacelike geodesic which meets $E^+(H)$ fails to realize distance between its points. Therefore, the Hawking-Penrose singularity theorem[5] predicts non-spacelike geodesic incompleteness in this spacetime. (Also, by rotating the figure around the symmetry axis, we obtain a 3-dimensional spacetime, in which H is a compact (non-future converging) spacelike surface with compact $E^+(H)$.

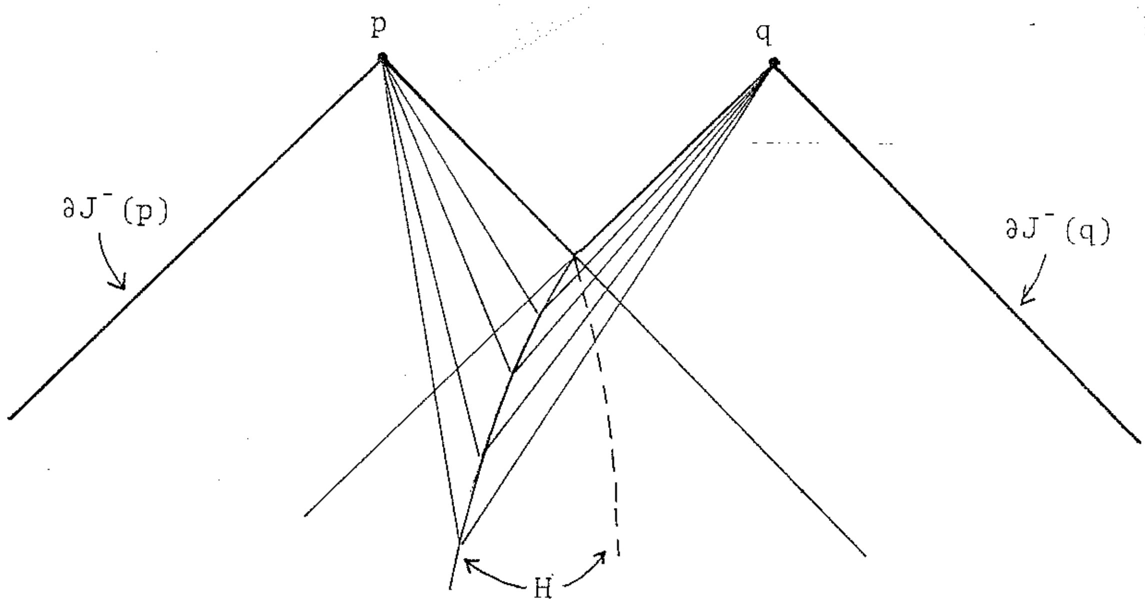



Figure 2. In 3-dimensional Minkowski spacetime, the intersection H $(=\partial J^-(p)\cap\partial J^-(q))$ of the past light cones of the points p and q is a closed (non-compact) strongly future converging spacelike surface. Note that this spacetime does not contain closed trapped surfaces.

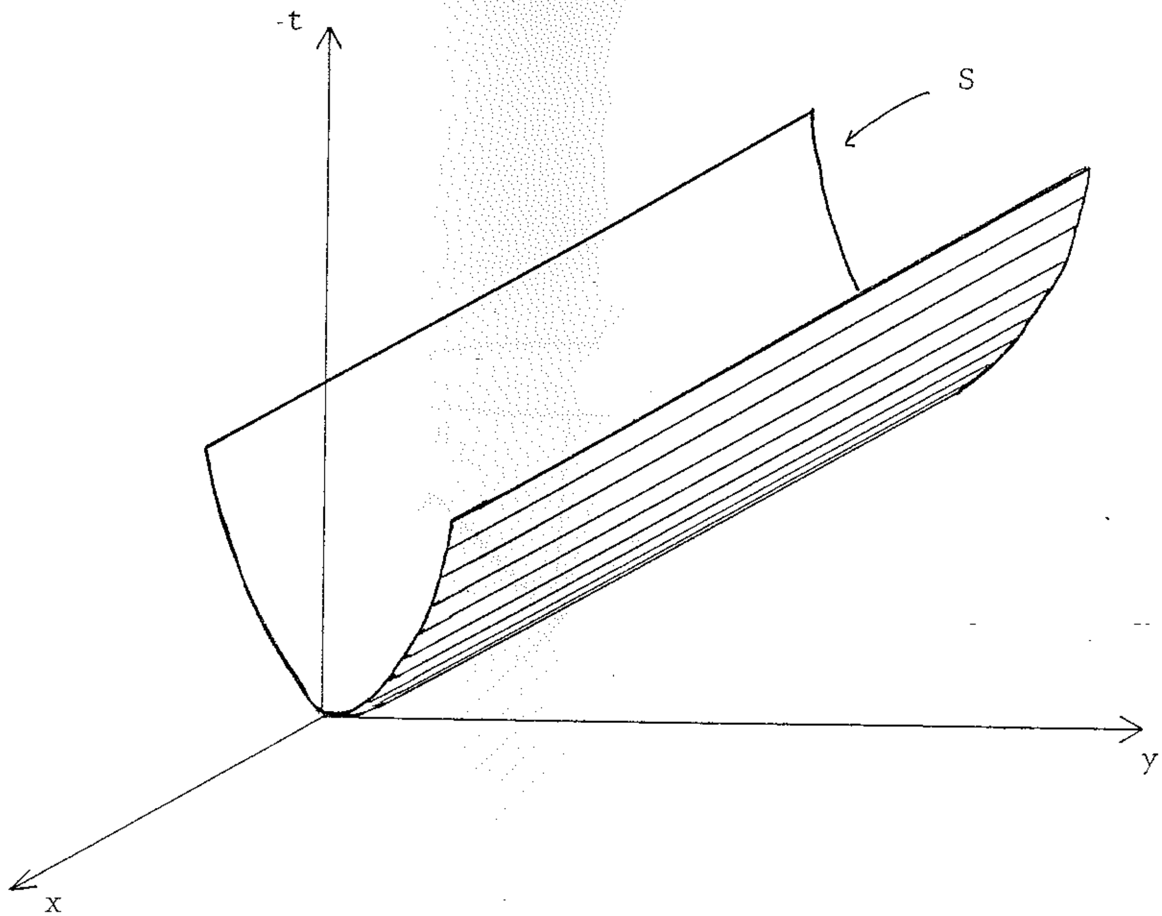



Figure 3. In 3-dimensional Minkowski spacetime, there exists a null vector field U on the smooth hypersurface S, yet S is not a null hypersurface. Note that S is not achronal.

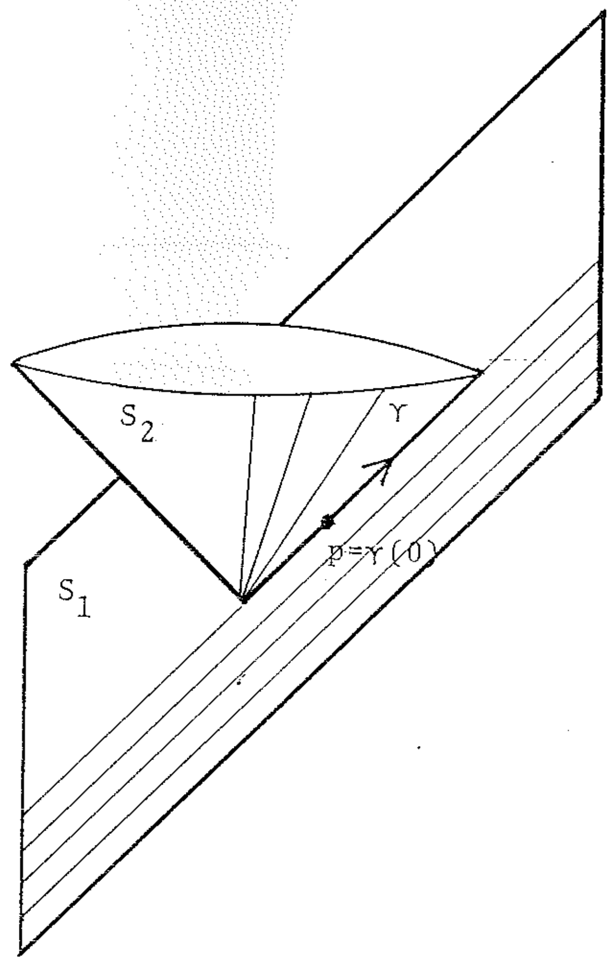



Figure 4. In 3-dimensional Minkowski spacetime, Let $U_1$ and $U_2$ be the null vector fields on the null hypersurfaces $S_1$ and $S_2$ respectively. The null geodesic $\gamma$ is contained in both $S_1$ and $S_2$. However, $\bar{\nabla}U_1|_\gamma = 0$ but $\bar{\nabla}U_2|_\gamma \neq 0$. Therefore, $U_1$-neighbors and $U_2$-neighbors of $\gamma$ we are induced by the different Jacobi tensors along $\gamma$.

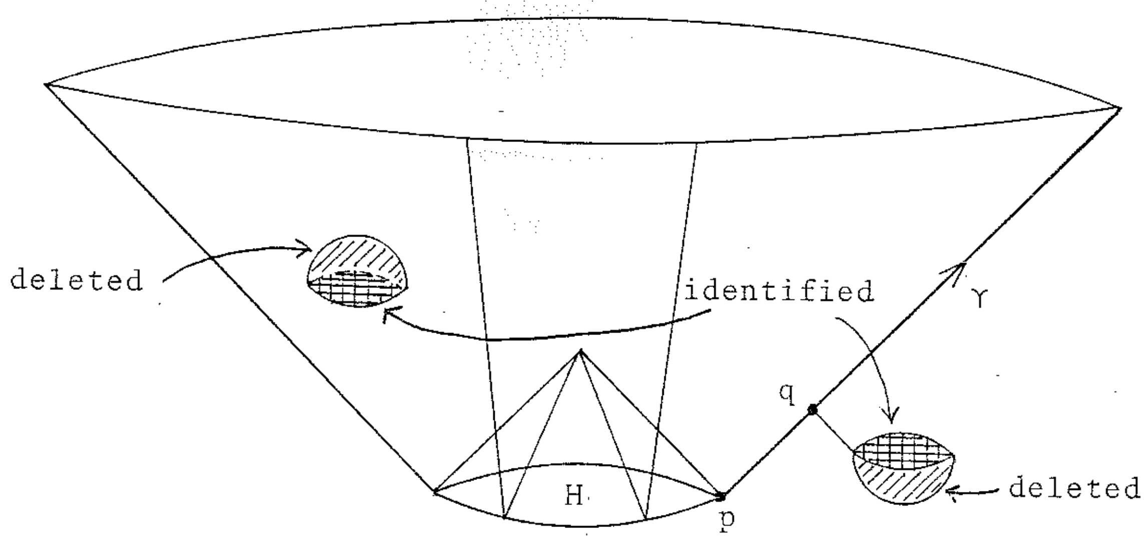



Figure 5. In this strongly causal identified subset of 3-dimensional Minkowski spacetime, q is the future cut point of H along γ. However, q is neither a focal point of H along γ nor there exists another null geodesic, $\eta \neq \gamma$ from H to q.

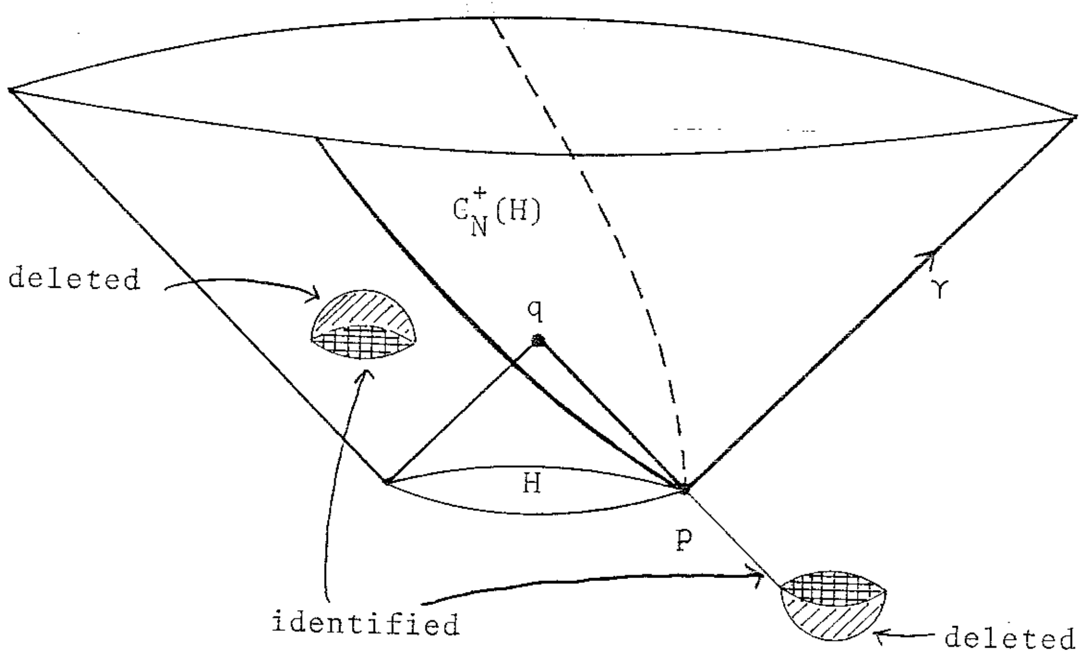



Figure 6. In this strongly causal identified subset of 3-dimensional Minkowski spacetime, the null geodesic $\gamma$ does not contain a future null cut point of H, yet it fails to realize distance between H and its points. Note that, $C_N^+(H)$ is not closed since $p \notin C_N^+(H)$.

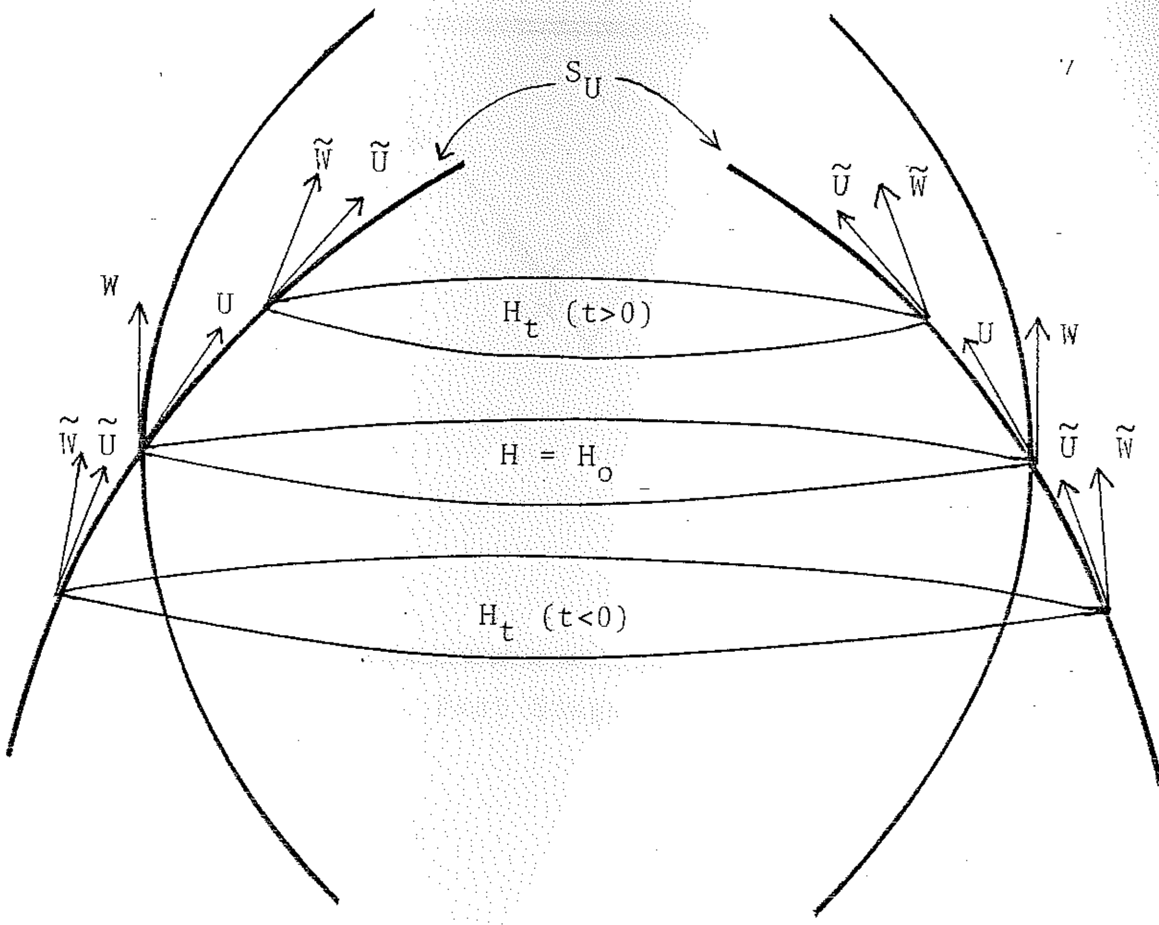
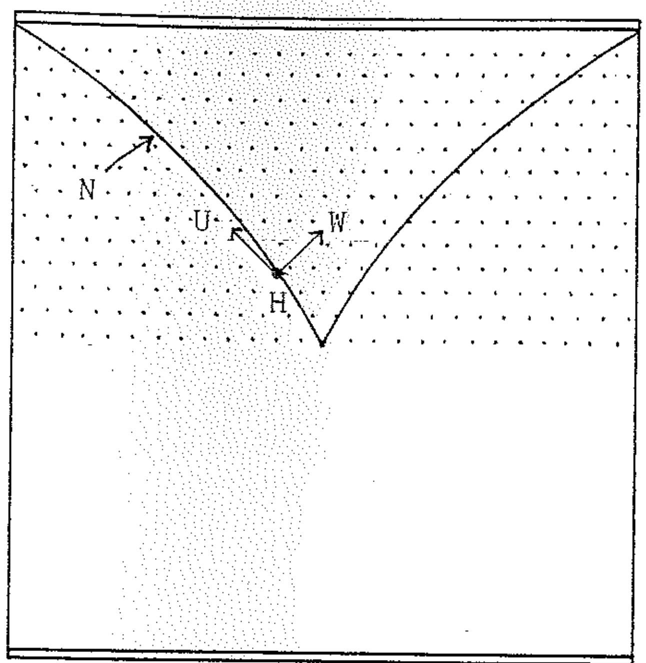



Figure 7. See Lemma 4.8.

Future Singularity




Past Singularity

Figure 8. (See [CET], page 184). Penrose diagram for a recollapsing Friedman Universe. There is a trapped surface through every point in the contracting phase (the dotted region of the figure) and each point on the hypersurface N is a marginally trapped surface diffeomorphic to $S^2$ (in fact, since Friedman universe is globally hyperbolic, it follows from Corollary 3.9, normal bundle $\mathcal{N}(H)$ of H is trivial. Moreover, $L_W = 0$ and $II_U(x,x)>0$ for every $x \in TH$ where U,W are future directed null sections of $\mathcal{N}(H)$).