Applications of Minimal Surface Theory to Topology and Riemannian Geometry, Constructions of Negatively Ricci Curved Manifolds

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by

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Abstract of the Dissertation

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In the first part, we use minimal surface techniques to study circle actions on compact manifolds. The results are applied to the question of the existence of positive scalar curvature metrics. In the second part, we study manifolds of negative Ricci curvature. Finally, we prove the "relative Index Theorem of Gromov-Lawson" for two families of Dirac operators under some technical assumptions.

Part I

We begin by studying a Lie group action on a compact manifold. Let M be a compact, connected, orientable Riemannian

manifold with a connected, compact Lie group G acting as isometries. We assume dim $M \leqslant 7$. Let F be an invariant submanifold of codimension-2, which is homologous to zero in M. By the existence and regularity Theory for minimal surfaces, there exists a C^∞ area-minizing hypersurface Σ with $\partial\Sigma = F$. We prove that for all $g \in G$, either $g(\Sigma) = \Sigma$ or $g(\Sigma) \cap \Sigma = F$. In particular, if G is semisimple, then Σ must be G-invariant. In the case where $G = S^1$, we conclude that either Σ is S^1-invariant or the manifold $M - F$ fibres (equivariantly) over the circle with fibre $(\Sigma - \partial\Sigma)$. The first case must occur if $M - F$ contains a fixed point. The second case must occur if F contains a component of fixed point set.

Using this last result, we prove that a large class of manifolds which admit circle actions must carry metrics of positive scalar curvature. This is true for example if $F = Fix(M, S^1)$ has codimension-2 and is homologous to zero in M^n ($n \leqslant 7$). Some condition on the action is clearly necessary as can be seen in dimension 3. (The torus T^3 or, more generally, circle bundles over surfaces of positive genus, cannot carry positive scalar curvature.) In dimension 3, we have that M (with a circle action) has $\kappa > 0$ iff M has a S^1-action with $Fix(M, S^1) \neq \phi$.

A result similar to the basic one above is proved for a minimizing hypersurface Σ (without boundry) in a homology class $\alpha \in H_{n-1}(M^n;Z)$. That is, we conclude that either Σ is G-invariant or there is an (equivariant) fibre bundle $M \to S^1$ with fibre Σ.

Using these results we can reprove the Raymond-Orlic classification Theorem or orientable 3-manifolds with S^1-action.

Part II

In the second part, we give an affirmative answer to a question of J.P. Bourguignon. We prove that the class of manifolds admitting negatively Ricci curved metrics is stable under connected sums. Moreover, we prove the following. Let M_1, M_2 be two oriented Riemannian manifolds with Ric < 0, and γ_i be a simple closed geodesic in $M_i (i=1,2)$. Without loss of generality, we assume the length of γ_1 equals the length of γ_2. Let N_i be the normal bundle of γ_i, choose tubular neighborhood V_i of γ_i. We identify V_i with $(v \in N_i; \|v\| \leq \epsilon)$. Take any diffeomorphism $\varphi : V_1 \to V_2$, then $\varphi : \partial V_1 \to \partial V_2$ is a diffeomorphism. Let $M_1 \#_\varphi M_2$ be the manifold obtained by gluing $M_1 - V_1$ and $M_2 - V_2$ along the boundaries by φ. We then prove that $M_1 \#_\varphi M_2$ admits a negatively Ricci curved metric.

In dimension 3, we use the above results and some topological constructions to obtain new manifolds with negative Ricci curvature metric from a given manifold with negative Ricci curvature metric. Given M a complete Riemannian three dimensional manifold with negative Ricci curvature metric, then there are complete negative Ricci curvature Riemann metric on $M \# S^2 \times S^1 \# L(p,q)$, and $M \# S^2 \times S^1 \# \Sigma \times S^1$, where Σ is any Riemann surface.

Table of Contents

Page

Acknowledgments

Chapter 1. Applications of minimal surfaces theory to topology and Riemann Geometry

I. Introduction 1

II. Preliminaries 2

III. Uniqueness of minimal submanifolds 4

IV. Group action and minimizing hypersurfaces 16

V. Applications to circle actions on 3-manifolds .. 41

VII. Circle action and scalar curvature 49

Chapter 2. Constructions of negative Ricci curved manifolds

I. Statement of theorems 58

II. Proof of Theorem 1 and Theorem 2 60

III. Proof of Theorem 3 76

IV. Proof of Theorems 4, 5 and 6 93

Reference 96

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Chapter 1

Applications of minimal surface theory to topology and Riemannian geometry

I. Introduction

In this paper, minimal surface techniques are used to study circle action and SO(n) action on compact manifolds. The results are applied to the question of the existence of positive scalar curvature metrics.

First, consider a Lie group action on a compact manifold. Let M be a compact, connected, orientable Riemannian manifold with a connected, compact Lie group G acting as isometries. We assume that dim $M \leq 7$. Let F be an invariant submanifold of codimension 2, which is homologous to zero in M. By the existence and regularity theory for minimal surfaces, there exists a C^∞ area minimizing hypersurface Σ with $\partial\Sigma = F$. We prove that for all $g \in G$, either $g(\Sigma) = \Sigma$ or $g(\Sigma) \cap \Sigma = F$. In particular, if G is semisimple, then Σ must be G-invariant. In the case where $G = S^1$, we conclude that either Σ is S^1-invariant or the manifold $M \sim F$ fibres (equivariantly) over the circle with fibre $(\Sigma \sim \partial\Sigma)$. The first case must occur if $M \sim F$ contains a fixed point. The second case must occur if F contains a component of fixed point set.

Using this last result, we prove that a large class of manifolds, which admit circle actions, must carry metrics of positive scalar curvature. This is true, for example, if $F = Fix(M,S^1)$ has codimension 2 and is homologous to zero in M^n $(n \leq 7)$. Some conditions on the action is clearly necessary as can be seen in dimension 3. (The torus T^3 or, more generally circle bundles over surfaces of positive genus, cannot carry positive scalar curvature metric [GL3].) For other examples and history see [LY]. In dimension 3, M (with a circle action) has $\kappa \geq 0$ if and only if M has a S^1-action with $Fix(M;S^1) \neq \emptyset$.

A result similar to the basic one above is proved for a minimizing hypersurface Σ (without boundary) in a homology class $\alpha \in H_{n-1}(M^n;Z)$. That is, we conclude that either Σ is G-invariant or there is a (equivariant) fibre bundle $M \to S^1$ with fibre Σ.

Using these results we can reprove the Raymond-orlic classification theorem of orientable 3-manifolds with S^1-action, and the classification of SO(n) action on (n+1)-manifolds with Fixed points of W. Y. Hsiang.

II. Preliminaries

This section serves to present some facts needed for this paper. First recall here, in a slightly altered form, Aronszajn's generalization of Carleman's unique continuation theorem [AN].

Theorem 2.1: Let A be a linear elliptic second-order differential operator defined on a domain D in $\mathbb{R}^n$. Let $u = (u^1, \ldots, u^r)$ be a function in D satisfying the differential inequalities

$$|Au^\alpha| \leq \text{const} \cdot \left\{ \sum_{i,\beta} \left| \frac{\partial u^\beta}{\partial x^i} \right| + \sum_\beta |u^\beta| \right\} \quad .$$

If the ∞-jet $J^\infty(u)$ of u vanishes at a single point of D, then $u = 0$ throughout D.

We also collect the maximum principle for a second-order linear elliptic equation [GT].

Theorem 2.2: Let

$$L = \sum a_{ij}(x) D_{ij} + \sum b_i(x) D_i + c(x)$$

be defined in a domain $D \subseteq \mathbb{R}^n$. Suppose that L is uniformly elliptic, then $c \leq 0$, $u \in C^2(D)$ and $Lu \geq 0$ in D. Let $x_0 \in \partial D$ be such that

(i) u is continuous at x_0

(ii) $u(x_0) > U(x)$ for all $x \in D$

(iii) ∂D satisfies an interior sphere condition at x_0.

Then the outer normal derivative u at x_0, if it exists, satisfies

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \quad .$$

Theorem 2.3: Let L be as in Theorem 2.2, and $u \in C^2(D)$, $Lu \geq 0$, then u cannot achieve a non-negative maximum in the interior of D unless it is a constant.

We also like to mention here a result of M. L. Gromov [GM].

Theorem 2.4: On an open n-manifold M $(n > 1)$, there exists a Riemannian metric with everywhere positive sectional curvature and also one with everywhere negative sectional curvature. In both cases the curvature can be chosen to be pinched, i.e., to satisfy $\delta < |K| < 1$ for some $\delta > 0$.

This term "open n-manifold" means a manifold having no compact components.

III. Uniqueness of minimal submanifolds

This section deals with proving some uniqueness theorems for minimal submanifolds in a given Riemannian manifold N.

<u>Theorem 3.1</u>: Let N^{n+m} be a Riemannian manifold and let M_1^n and M_2^n be connected, compact minimal submanifolds with or without boundary of same dimension n. If M_1 and M_2 touch at $x_0 \in \mathrm{Int}(M_1) \cap \mathrm{Int}(M_2)$ to infinitely high order and $\partial M_1 = \partial M_2$, then $M_1 = M_2$.

<u>Proof</u>: Since M_1 and M_2 are tangent to each other at x_0, we can choose a coordinate neighborhood U of x_0 in N such that $M_1 \cap U$ is the slice $x^{n+1} = x^{n+2} = \dots = x^{n+m} = 0$ of U and $M_2 \cap U$ is the graph of $x^{n+\alpha} = u^\alpha(x^1, \dots, x^n)$, $\alpha = 1, \dots, m$; i.e.

$$M_2 \cap U = \left\{(x^1, \dots, x^n; u^1, \dots, u^m) \mid (x^1, \dots, x^n) \in V = M_1 \cap U\right\}.$$

Recall that if $\phi : M \to N$ is a minimal immersion, we have the following (see [EL]). Take the coordinate charts $V \subset M$ and $U \subset N$ such that $\phi(V) \subseteq U$. Writing $x = (x^1, \dots, x^n)$, $\phi(x) = (y^1, \dots, y^{n+m})$ we represent the metric tensors of M and N by $g(x) = \Sigma g_{ij}(x) dx^i dx^j$. Then we have the equation

$$\sum_{i,j} g^{ij} \frac{\partial^2 \phi^\nu}{\partial x^i \partial x^j} - \sum_{i,j} g^{ij} \, {}^M\Gamma^k_{ij} \frac{\partial \phi^\nu}{\partial x^k} + \sum g^{ij} \, {}^N\Gamma^\nu_{\mu\lambda} \frac{\partial \phi^\mu}{\partial x^i} \frac{\partial \phi^\lambda}{\partial x^j} = 0 \qquad (1)$$

where $i,j,k = 1, \dots, n$; $\mu, \lambda, \nu = 1, \dots, n+m$ and $\left\{{}^M\Gamma^k_{ij}\right\}$, $\left\{{}^N\Gamma^\nu_{\mu\lambda}\right\}$ denote the Christoffel symbols of Riemannian connections of M and N.

Consider $f^\alpha \in C^\infty(V)$ and $V \subseteq U \subseteq N$ where $1 \leq \alpha \leq m$ and let V_f be the graph of $\{f^\alpha\}$ over V, i.e. $V_f = \{(x^1, \ldots, x^n; f^1, \ldots, f^m) \mid (x^1, \ldots, x^n) \in V\}$. We can assume that $V_f \subseteq U \subseteq N$. Then V_f is a submanifold of N with the induced metric and connections.

$$g_f = g(x^1, \ldots, x^n; \nabla f^\alpha, f^\alpha)$$

$$^{V_f}\Gamma = \Gamma(x^1, \ldots, x^n; \nabla f^\alpha, f^\alpha)$$

$$^N\Gamma|V_f = {}^N\Gamma(x^1, \ldots, x^n; f^\alpha) .$$

In particular, for the immersion of M_1 and M_2 above, we have

$$g_1 = g(x^1, \ldots, x^n; 0,0) ; \qquad {}_1\Gamma = \Gamma(x^1, \ldots, x^n; 0,0)$$

$$g_2 = g(x^1, \ldots, x^n; \nabla u^\alpha, u^\alpha) ; \qquad {}_2\Gamma = \Gamma(x^1, \ldots, x^n; \nabla u^\alpha, u^\alpha)$$

$$^N_1\Gamma = {}^N\Gamma(x^1, \ldots, x^n; 0) ; \qquad ^N_2\Gamma = {}^N\Gamma(x^1, \ldots, x^n; u^\alpha)$$

where

$$g_1 = g|_{f=u} \Big| , \qquad {}_1\Gamma = {}^{V_f}\Gamma|_{f=u} \Big| , \qquad ^N_1\Gamma = {}^N\Gamma|M_1 \cap U$$

$$g_2 = g|_{f=u} \Big| , \qquad {}_2\Gamma = {}^{V_f}\Gamma|_{f=u} , \qquad ^N_2\Gamma = {}^N\Gamma|M_2 \cap U .$$

Using equation (1), we obtain

$$\sum_{i,j} g_2^{ij} \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \sum_{i,j,k} g_2^{ij}\, {}_2\Gamma^k_{ij} \frac{\partial u^\alpha}{\partial x^k} + \sum_{i,j} g_2^{ij}\, {}^N_2\Gamma^{n+\alpha}_{ij} + 2 \sum_{\substack{i,j \\ 1 \leq \beta \leq m}} g_2^{ij}\, {}^N_2\Gamma^{n+\alpha}_{n+\beta j} \frac{\partial u^\beta}{\partial x^i}$$

$$+ \sum_{\substack{i,j \\ 1 \leq \beta,\gamma \leq m}} g^{ij}\, {}^N_2\Gamma^{n+\alpha}_{n+\beta\; n+\gamma} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} = 0 \qquad (2)$$

and

$$\sum_{i,j} g_1^{ij} \, {}^N_1\Gamma_{ij}^{n+\alpha} = 0 \tag{3}$$

for $\alpha = 1,\ldots,m$. Let

$$A = \sum g_1^{ij} \frac{\partial^2}{\partial x^i \partial x^j} = \sum g^{ij}(x^1,\ldots,x^n;\,0,0) \frac{\partial^2}{\partial x^i \partial x^j} ,$$

then A is an elliptic partial differential operator of second order. If we choose U to be small by the Mean Value Theorem, we have

$$\left| \sum \left(g_2^{ij} - g_1^{ij} \right) \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} \right| \le C \left\{ \sum_{i,\beta} \left| \frac{\partial u^\beta}{\partial x^i} \right| + \sum_\beta |u^\beta| \right\} \tag{4}$$

$$\left| \sum {}^N_2\Gamma_{ij}^{n+\alpha} \, g_2^{ij} - \sum {}^N_1\Gamma_{ij}^{n+\alpha} \, g_1^{ij} \right| =$$

$$\left| \sum {}^N\Gamma_{ij}^{n+\alpha}(x^1,\ldots,x^n,u^s) g^{ij}(x;\nabla u^\beta,u^\beta) - \sum {}^N\Gamma_{ij}^{n+\alpha}(x;0) g^{ij}(x;0,0) \right| \le$$

$$\text{const} \cdot \left\{ \sum_{i,\beta} \left| \frac{\partial u^\beta}{\partial x^i} \right| + \sum_\beta |u^\beta| \right\} . \tag{5}$$

It follows from equations (2), (3), (4) and (5) that

$$|Au^\alpha| \le \text{const} \left\{ \sum_{i,\beta} \left| \frac{\partial u^\beta}{\partial x^i} \right| + \sum |u^\beta| \right\} . \tag{6}$$

But M_1 and M_2 touch at x_0 to infinitely high orders, i.e. $J^{\infty}(u^{\alpha})|_{x=x_0} = 0$ where $\alpha = 1,\dots,m$. Theorem 2.1 implies that $u^{\alpha}|V = 0$ where $\alpha = 1,\dots,m$ that is $M_1 \cap M_2 \supseteq V$ and $M_1 \cap U = M_2 \cap U$. Hence $M_1 \cap U = M_2 \cap U = M_1 \cap M_2 \cap U$. Let $W = \{x \in \mathrm{Int}(M_1) \cap \mathrm{Int}(M_2) | M_1$ and M_2 touch at x to infinitely high orders$\}$. It follows from above that W is open in both $\mathrm{Int}(M_1)$ and $\mathrm{Int}(M_2)$. However, W is clearly closed in both $\mathrm{Int}(M_1)$ and $\mathrm{Int}(M_2)$ and hence $W = \mathrm{Int}(M_1) = \mathrm{Int}(M_2)$. Thus $M_1 = M_2$.

Now turning to the case where the intersection of two minimal submanifolds has a large Haussdorff dimension, we have the following theorem.

Theorem 3.2: Let N be a Riemannian manifold; and let M_1 and M_2 be connected, compact minimal submanifold of the same dimension n and with $\partial M_1 = \partial M_2$ (possibly $= \phi$). For any $s > n-1$, let $\mathcal{H}^s$ be the Haussdorff measure of dimension s on N. If $\mathcal{H}^s(M_1 \cap M_2) > 0$, then $M_1 = M_2$.

In the proof of Theorem 3.2, the following lemmas are needed first.

Lemma 3.3: Suppose Ω is an open chart of a Riemannian manifold M without boundary of dimension n, $F \subseteq M$ is a closed subset and $F \subseteq \Omega$. If $f \in C^{\infty}(\Omega)$ and $f|F = 0$ and the s dimensional Haussdorff measure $\mathcal{H}^s(F) > 0$ for $s > n-1$, then there exists a measurable subset E of F such that

$$J^{\infty}(f)|E = 0$$

and $\mathcal{H}^s(E) = \mathcal{H}^s(F)$.

Sublemma 3.4: Let $F \subseteq \Omega \subseteq M$ as above and let $f \in C^\infty(\Omega)$, $f|F = 0$, $\mathcal{H}^s(F) > 0$, where $s > n-1$. Then there is a closed subset F_1 of F such that $\nabla f|F_1 = 0$ and $\mathcal{H}^s(F_1) = \mathcal{H}^s(F) > 0$.

Proof of Sublemma 3.4: Let

$$U = \left\{x \in \Omega \,\middle|\, |\nabla f|_x > 0\right\}$$

then U is open and $G_1 = U \cap F$ is open in F. For any $x \in G_1$, we have $|\nabla f|_x > 0$. Therefore if we take V to be a very small open neighborhood of x in Ω, then $\{f = 0\} \cap V$ is a (n-1)-dimensional submanifold S_x of $V \subseteq M$. Hence $\mathcal{H}^s(S_x) = 0$. But

$$F \subseteq \left\{x \in \Omega \,\middle|\, f(x) = 0\right\}.$$

Thus we have

$$V \cap G_1 \subseteq V \cap F \subseteq V \cap \left\{x \in \Omega \,\middle|\, f(x) = 0\right\} = S_x$$

so that

$$\mathcal{H}^s(V \cap G_1) \leq \mathcal{H}^s(S_x) = 0 .$$

We can choose a countable covering $\{V_m\}$ of G_1 consisting of such open sets, i.e.

$$\mathcal{H}^s(V_m \cap G_1) = 0 \quad , \quad m = 1,2,3, \ldots .$$

Hence

$$\mathcal{H}^s(G_1) \leq \sum_{m=1}^{\infty} \mathcal{H}^s(V_m \cap G_1) = 0 .$$

Therefore we obtain

$$\mathcal{H}^s(G_1) = 0 .$$

Let $F_1 = F - G_1$, then $F_1 \subseteq F$ is closed and

$$\mathcal{H}^s(F_1) = \mathcal{H}^s(F) > 0 .$$

From the definition, we have $\nabla f | F_1 = 0$.

Proof of Lemma 3.3: Using Sublemma 3.4 to function f and to set F, we have $F_1 \subseteq F$. Using sublemma again to functions $\{\partial f / \partial x^i\}$ and F_1, we have closed subsets $\{E_i^1\}$ of F_1 such that

$$\mathcal{H}^s(E_i^1) = \mathcal{H}^s(F_1) ,$$

and $\qquad i = i,2, \ldots, n$

$$\nabla\left(\frac{\partial f}{\partial x^i}\right) \Big| E_i^1 = 0 .$$

Let $F_2 = \bigcap_{i=1}^{n} E_i^1$, then

$$F_1 \sim F_2 = F_1 \sim \bigcap_{i=1}^{n} E_i^1 = \bigcup_{i=1}^{n} (F_1 \sim E_i^1) .$$

Hence

$$\mathcal{H}^s(F_1 \sim F_2) \leq \sum_{i=1}^{n} \mathcal{H}^s(F_1 \sim E_i^1) = 0 .$$

Therefore, $\mathcal{H}^s(F_2) = \mathcal{H}^s(F_1) = \mathcal{H}^s(F) > 0$ and

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|F_2 = 0 \qquad i,j = 1,2, \ldots, n \quad .$$

By induction and using Sublemma 3.4, we can prove that there is a sequence of closed subsets $\{F_m\}_{m-1}^{\infty}$ of F such that

$$F \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

$$0 < \mathcal{H}^s(F) = \mathcal{H}^s(F_1) = \mathcal{H}^s(F_2) = \mathcal{H}^s(F_3) = \cdots$$

and

$$J^m(f)|F_m = 0 \qquad m = 1,2,3, \ldots \quad .$$

Let $E = \bigcap_{m=1}^{\infty} F_m \subseteq \Omega$, then

$$\mathcal{H}^s(E) = \mathcal{H}^s\left(\bigcap_{m=1}^{\infty}\right)F_m = \lim_{m\to\infty} \mathcal{H}^s(F_m) = \mathcal{H}^s(F) > 0$$

and $J^{\infty}(f)|E = 0$, E is closed in F.

Proof of Theorem 3.2: Since $\mathcal{H}^s(M_1 \cap M_2) > 0$, there is a point $x_0 \in \mathrm{Int}(M_1) \cap \mathrm{Int}(M_2)$ such that for any open $V \subseteq M_1$ and $x_0 \in V$, we have $\mathcal{H}^s(V \cap M_1 \cap M_2) > 0$. We claim that $T_{x_0}(M_1) = T_{x_0}(M_2)$.

Take the coordinate chart of $U_0 \subseteq N$ at x_0 such that $U_0 \cap M_1$ is the slice $x^{n+1} = \cdots = x^{n+m} = 0$, V_0 is a coordinate chart of M_2 at x_0 with coordinate $(v_1, \ldots, v_n)$ and $f(v) = (f^1(v), \ldots, f^{n+m}(v))$ is the embedding from

$V_0 \subseteq M_2$ to N and $f(V_0) \subseteq U_0$. Suppose $T_{x_0}(M_1) \neq T_{x_0}(M_2)$. Since $T_{x_0}(M_1)$ is generated by $\{(\partial/\partial x^1)|_0, \ldots, (\partial/\partial x^n)|_0\}$ hence

$$\left(\frac{\partial f^{n+\alpha}}{\partial v_i}\Big|_{v=0}\right)_{\substack{1 \leq \alpha \leq m \\ 1 \leq i \leq n}} \neq 0 .$$

We may assume that $(\partial f^{n+1}/\partial v_n)|_{v=0} \neq 0$ that is $\nabla f^{n+1}|_{v=0} \neq 0$. If we choose a very small V_0, then $V_0 \cap \{v | f^{n+1}(v) = 0\}$ is a (n-1)-dimensional submanifold of M_2 and $f(V_0 \cap \{v | f^{n+1}(v) = 0\})$ is a (n-1)-dimensional submanifold of U_0, we have

$$\mathcal{H}^s\left(f\{v \in V_0 | f^{n+1}(v) = 0\}\right) = 0$$

But if $U_1 \subseteq U_0$ is very small open set such that $x_0 \subseteq U_1$ and $U_1 \cap M_1 \cap M_2 \subseteq f(V_0) \cap M_1 \subseteq f\{v \in V_0 \; f^{n+1}(v) = 0\}$, then we have $\mathcal{H}^s(U_1 \cap M_1 \cap M_2) \leq \mathcal{H}^s(f\{v \in V_0 | f^{n+1}(v) = 0\}) = 0$. This contradicts the choice of x_0. Therefore $T_{x_0}(M_1) = T_{x_0}(M_2)$.

Choosing a coordinate chart U of N at x_0 such that $U \cap M_1$ is the slice $x^{n+1} = \cdots = x^{n+m} = 0$ of U and $U \cap M_2$ is the graph $\{(x^1, \ldots, x^n; f^1, \ldots, f^m)\}$ (this is possible since $T_{x_0}(M_1) = T_{x_0}(M_2)$). Let

$$G_\alpha = V \cap \left\{x \in V | f^\alpha(x) = 0\right\}$$

where $U \cap M_1 = V = \{(x^1, \ldots, x^n)\}$. Since

$$\mathcal{H}^s(U \cap M_1 \cap M_2) = \mathcal{H}^s\left(\{x \in V | f^1(x) = \cdots = f^m(x) = 0\}\right) > 0 ,$$

we obtain that

$$\mathcal{H}^s(G_\alpha) \geq \mathcal{H}^s(U \cap M_1 \cap M_2) > 0 \quad .$$

Since we take U to be any small neighborhood of x_0 in N, there exists closed subsets $\{F_\alpha\}$ of M_1 such that $F_\alpha \subseteq G_\alpha$ and $\mathcal{H}^s(F_\alpha) > 0$ where $\alpha = 1,2, \ldots, m$. From the definition of G_α, we have $f^\alpha | F_\alpha = 0$ where $(\alpha = 1,2,\ldots,m)$. By Lemma 3.3 there are measurable subsets $E_\alpha \subseteq F_\alpha$ such that $\mathcal{H}^s(E_\alpha) = \mathcal{H}^s(F_\alpha) > 0$ and $J^\infty(f^\alpha)\ E_\alpha = 0$ $(\alpha = 1,2,\ldots,m)$. We can assume also $F_\alpha \supseteq E = U' \cap M_1 \cap M_2$ for some $U' \subseteq U$ open in N and $x_0 \in U'$. Let $\bar{E} = \bigcap_{\alpha=1}^{m} E_\alpha$, then $\bar{E} \cap E = \bigcap_{\alpha=1}^{m} E_\alpha \cap \bar{E}$ and

$$\mathcal{H}^s(E \sim \bar{E} \cap E) = \mathcal{H}^s\left(\bigcap_{\alpha=1}^{m} (E \sim E_\alpha \cap E)\right) \leq \sum \mathcal{H}^s(E \sim E_\alpha \cap E) \quad .$$

But $E_\alpha \subseteq F_\alpha$ and $\mathcal{H}^s(E_\alpha) = \mathcal{H}^s(F_\alpha)$, hence $\mathcal{H}^s(F_\alpha \sim E_\alpha) = 0$ $\mathcal{H}^s(E) = \mathcal{H}^s(F_\alpha \cap E) = \mathcal{H}^s(E_\alpha \cap E) + \mathcal{H}((F_\alpha \sim E_\alpha) \cap E) = \mathcal{H}^s(E_\alpha \cap E)$ and thus $\mathcal{H}^s(E \sim E_\alpha \cap E) = 0$. Therefore $\mathcal{H}^s(E \sim \bar{E} \cap E) = 0$ or equivalently

$$\mathcal{H}^s(\bar{E} \cap E) = \mathcal{H}^s(E) > 0 \ ,$$

since $J^\infty(f^\alpha)|\bar{E} = 0$ where $1 \leq \alpha \leq m$. Let $\bar{x} \in \bar{E}$ then $\bar{x} \in \mathrm{Int}(M_1) \cap \mathrm{Int}(M_2)$ and M_1, M_2 touch at $\bar{x}$ to infinitely high orders. Therefore, Theorem 3.2 follows from Theorem 3.1.

As a consequence of the proof of Theorem 3.2, we have the following.

Corollary 3.5: Let M_1 be the graph of $f : \mathbb{R}^n \to \mathbb{R}^m$ in $\mathbb{R}^{n+m}$. Suppose M_1 is minimal and M_2 is a n-dimensional plane in $\mathbb{R}^{n+m}$ and $\mathcal{H}^s(M_1 \cap M_2) > 0$ for some $s > n-1$, then $M_1 = M_2$.

Turning now to the maximum principle of minimal surfaces, we have the following.

Proposition 3.6: Let S_1 and S_2 be two connected minimal hypersurfaces of a Riemannian manifold N. Suppose that $\partial S_1 = \partial S_2$ and $T_p S_1 = T_p S_2$ for some $p \in \partial S_1 = \partial S_2$ and that in a neighborhood of p one of them lies above the other, then $S_1 = S_2$.

Proof: Choose a coordinate chart U of N at p such that $U \cap S_1 = \{x \in U | x_n \geq 0 ; x_{n+1} = 0\}$ and $U \cap S_2 = \{x \in U | x_n \geq 0 , x_{n+1} = u\}$ for $u = u(x_1, \ldots, x_n) \in C^\infty(U \cap \{x : x_n \geq 0 , x_{n+1} = 0\})$ and $U \cap \partial S_1 = U \cap \partial S_2 = \{x \in U | x_n = x_{n+1} = 0\}$, then

$$u|_{x_n = 0} = 0 , \qquad \left.\frac{\partial u}{\partial x_n}\right|_{x=0} = 0 .$$

We may choose U to be small such that $u \geq 0$ on $\{(x_1, \ldots, x_n) : x_n \geq 0\}$. Since S_1 and S_2 are minimal, we have

$$\sum g_2^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum g_2^{ij} \, {}^{S_2}\Gamma_{ij}^k \frac{\partial u}{\partial x_k} + \sum g_2^{ij} \, {}^{N}\Gamma_{ij}^{n+1} + 2 \sum g_2^{ij} \, {}^{N}\Gamma_{n+1\, j}^{n+1} \frac{\partial u}{\partial x_i}$$
$$+ \sum g_2^{ij} \, {}^{N}\Gamma_{n+1\, n+1}^{n+1} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0 \qquad (6)$$

and

$$\sum g_1^{ij} \, {}^{N}\Gamma_{ij}^{n+1} = 0 \qquad (7)$$

where g_1^{ij}, g_2^{ij} are metric tensors on S_1 and S_2.

Applying the Mean Value Theorem with equations (6) and (7), we obtain

$$\sum_{i,j=1}^{n} g_1^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad (8)$$

where b_i, c are functions of x, u, ∇u. When we consider b_i, c to be functions of x, then equation (8) is a linear elliptic equation of a second order in u. Let $p \in V \subset\subset U$ and

$$V^+ = \{x \in V | x_n \geq 0 , \; x_{n+1} = 0\}$$

$$A = \sum g_1^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \frac{\partial}{\partial x_i} + c$$

then A is strictly, uniformly elliptic on V^+. Let

$$L = \sum g_1^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \frac{\partial}{\partial x_i} + \min\{c,0\}$$

and $v = -u$, then we have

$$\begin{aligned} Lv &= [\min\{c,0\} - c] v + Av \\ &= [\min\{c,0\} - c] v \\ &= -\max\{c,0\} v \geq 0 . \end{aligned}$$

on V^+. Note $\min\{c,0\} \leq 0$.

We claim that $u = 0$ on V^+. First suppose that $u > 0$ on $Int(V^+)$. Applying Theorem 2.2 to L and v on V^+, we have $(\partial v/\partial x_n)(p) > 0$, which is a contradiction since $(\partial v/\partial x_n)(p) = 0$. Therefore $u(x) = 0$ for some $x \in Int(V^+)$, then x is a non-negative maximum of v. By Theorem 2.3 $v \equiv 0$ on V^+. Hence $u = 0$ on V^+ and $V \cap S_1 = V \cap S_2$. Using Theorem 3.1, $S_1 = S_2$.

Similarly we can prove the following.

<u>Proposition 3.7</u>: Let S_1 and S_2 be connected, compact minimal hypersurfaces of N with or without boundary. In the first case, we also assume that $\partial S_1 = \partial S_2$. If there is a neighborhood of some point $x_0 \in Int(S_1) \cap Int(S_2)$ such that in this neighborhood one of them lies above the other, then $S_1 = S_2$.

IV. Group action and minimizing hypersurfaces

In this section we use minimal surface theory to study manifolds with a Group action. Throughout this section N is a compact, connected, oriented C^∞ manifold, G is a compact, connected Lie Group acting on N, and $\dim N \leq 7$. Without the loss of generality, we can assume that N is a Riemannian manifold and G acts on N as a subgroup of isometries of N.

Let F be any submanifold of N without boundary. If F is homologous to zero in N and $\dim F = \dim N - 2$, then, from the existence and regularity theory for minimal surface due to Almgren-Federer-Fleming-Hardt-Simon (see [FH1], [FH2], [AF2], and [HS]), one can solve the Plateau problem with a given boundary F in N i.e. there exists a C^∞ area minimizing hypersurface Σ of N such that $\partial\Sigma = F$. We are now ready for the basic theorem of this section.

<u>Theorem 4.1</u>: Let F_1 be a G-invariant submanifold of N of codimension 2 and let F_1 be homologous to zero in N. Let Σ_1 be any solution of Plateau problem with given boundary $\partial\Sigma_1 = F_1$ and let Σ be any connected component of Σ_1, then for any $g \in G$ where either $g\Sigma = \Sigma$ or $g\Sigma \cap \Sigma = F$ where $F = \partial\Sigma$.

<u>Proof</u>: Let $E = \{g \in G \mid \Sigma \cap g\Sigma \sim F \neq \phi, \Sigma \neq g\Sigma\}$. We need to show $E = \emptyset$.

Since Σ is C^∞ up to the boundary, there is a hypersurface Σ' of N such that $\bar{\Sigma} \subseteq \operatorname{Int}\Sigma'$ and $\bar{\Sigma}'$ is compact. Let $U \cong \Sigma' \times (-\varepsilon,\varepsilon)$ be a tubular neighborhood of Σ' in N and $h : \Sigma' \times (-\varepsilon,\varepsilon) \to U$ given by $h(x,s) = \exp_x(s\nu_x)$ where ν is the unit normal vector field of Σ' (note Σ' is orientable). Note that $(\partial/\partial s)|\Sigma' = \nu$. Let ν_g be the unit normal vector field of $g\Sigma$, then there is an open $W \subseteq G$, $e \in W$ such that $g\Sigma \subseteq U$ and $\langle \nu_g, (\partial/\partial s)\rangle > 1/2$ for $g \in W$. Since G acts on N as isometries, we have $\nu_g = dg(\nu)$. Let $p = U \to \Sigma'$ be the projection map i.e. $p(x,s) = x$. We claim that $\Sigma_g \subseteq \Sigma \times (-\varepsilon,\varepsilon)$ and $p|\Sigma_g : \Sigma_g \to \Sigma$ is a diffeomorphism for $g \in W$ where $\Sigma_g = g\Sigma$.

First we show that $p|\Sigma_g$ is injective and $p|\Sigma_g \subseteq \Sigma$. Suppose (x_1,s_1) and $(x_2,s_2) \in \Sigma_g$ and $p(x_1,s_1) = p(x_2,s_2)$ that is $x_1 = x_2$. Since Σ' is connected, there is an embedded smooth curve $c(\tau) \subseteq \bar{\Sigma}'$ where $-1 \le \tau \le 1$ such that $\|c'(\tau)\| \ne 0$ and $\{c(\tau)\} \cap \partial\Sigma' = \{c(-1);c(1)\}$, $\{c(\tau)\} \cap \partial\Sigma = \{c(\tau_1),c(\tau_2)\}$ where $-1 \le \tau \le 1$ and $-1 < \tau_1 < 0 < \tau_2 < 1$ and $c(\tau_0) = x_1$ for some $-1 < \tau_0 < 1$. Let $X \in T_x\Sigma_g$ for any $x \in \Sigma_g$ and $X \ne 0$. If $dp(X) = 0$, then $X = \alpha(\partial/\partial s)$ where $\alpha \ne 0$, hence $\langle \nu_g, X\rangle = \langle \nu_g, \alpha(\partial/\partial s)\rangle \ne 0$ this contradicts with ν_g being the normal vector field of Σ_g. Thus $dp(X) \ne 0$ $dp(T_x\Sigma_g) = T_{p(x)}\Sigma'$ for any $x \in \Sigma_g$ and $g \in W$. Therefore $p : \Sigma_g \to \Sigma'$ is a local diffeomorphism for $g \in W$.

Consider surface $S = \{(c(\tau),s) \mid -1 < \tau < 1, -\varepsilon < s < \varepsilon\}$ in U. Since $(\partial/\partial s)|S \in TS$ and $\langle \nu_g, (\partial/\partial s)\rangle > 1/2$, it follows that Σ_g is transversed to S in U, and hence $\Sigma_g \cap S$ is one-dimensional compact submanifold of U. We can assume $(x_1,s_1) \in \mathrm{Int}(\Sigma_g)$. Now we first show that $\Sigma_g \cap S$ is connected. Since $\partial(\Sigma_g \cap S) = \partial\Sigma_g \cap S = \partial\Sigma \cap S = \{c(\tau_1),c(\tau_2)\}$, let $\Sigma_g \cap S = \{\gamma_i\}_{i=1}^m$ where γ_i are the connected components of $\Sigma_g \cap S$. Then either γ_i is a segment with $\partial\gamma_i = \{c(\tau_1),c(\tau_2)\}$ or γ_i is a simple curve without boundary and we can only have one γ_i such that $\partial\gamma_i = \{c(\tau_1),c(\tau_2)\}$. We claim that no one of $\{\gamma_i\}$ is a simple closed curve, it will follow from this that $\Sigma_g \cap S$

is connected segment with $\partial(\Sigma_g \cap S) = \{c(\tau_1), c(\tau_2)\}$. Suppose γ_i is a simple closed curve. Since $\gamma_i \subseteq S$ and S is diffeomorphic to $(-1,-) \times (-\varepsilon,\varepsilon)$, we identify S with $(-1,1) \times (-\varepsilon,\varepsilon) = I^2$. Then γ_i is a simple closed curve in I^2. Let $d : I^2 \times I^2 \to \mathbb{R}^+$ be the Euclidean metric on I^2. Since γ_i is simple closed curve, there is a point $(t,s_0) \in \gamma_i$ such that $d((t,s_0), \{(0,s)|-\varepsilon < s < \varepsilon\}) = \max_{y\in\gamma_i} d(y,\{(0,s)|-\varepsilon < s < \varepsilon\})$. It follows that $T_{(t,s_0)}\gamma_i = \{(\partial/\partial s)|_{(t,s_0)}\}$, that is $(\partial/\partial s)|_{(t,s_0)} \in T_{(t,s_0)}\Sigma_g$ and $\langle \nu_g, (\partial/\partial s)|_{(t,s_0)}\rangle = 0$. This contradicts the hypothesis $\langle \nu_g, (\partial/\partial s)\rangle > 1/2$ for $g \in W$.

We know now that $\Sigma_g \cap S$ is connected and $(x_1,s_1), (x_1,s_2) \in \Sigma_g \cap S$. The same argument as above shows that $s_1 = s_2$. That is $(x_1,s_1) = (x_1,s_2)$. It follows that $p : \Sigma_g \to \Sigma'$ is injective. But $\Sigma_g \cap S$ is a segment for any such surface S, it follows that $p(\Sigma_g) \subsetneq \Sigma$. Therefore $p : \Sigma_g \to \Sigma$ is injective and a local diffeomorphism and $p(\text{Int}(\Sigma_g))$ is open in $\text{Int}(\Sigma)$. On the other hand $p(\Sigma_g)$ is closed in Σ hence $p(\text{Int}\,\Sigma_g) = p(\Sigma_g) \sim F$ is closed in $\text{Int}\,\Sigma = \Sigma \sim F$. Since $\text{Int}\Sigma$ is connected, we must have $p(\text{Int}\,\Sigma_g) = \text{Int}\Sigma$ and $p : \Sigma \to \Sigma$ is onto and diffeomorphism for $g \in W$.

We claim now that $W \cap E = \phi$. For any $g \in W$ we can assume by Theorem 3.2 that $\mathcal{H}^s(\Sigma \cap \Sigma) = 0$ for any $s > n-2$, where $n = \dim N$. Let $\Sigma_g \sim \Sigma \cap \Sigma_g = \coprod \Sigma_g^\alpha$; where $\{\Sigma_g^\alpha\}$ are components of $\Sigma_g \sim \Sigma \cap \Sigma_g$. If $g \in E$, choose one Σ_g^α such that $\partial\Sigma_g^\alpha \sim F \neq \phi$. Since $p : \Sigma_p \to \Sigma$ is a diffeomorphism Σ_g is a graph of function $S_g : \Sigma \to (-\varepsilon,\varepsilon)$ i.e.

$$\Sigma_g = \left\{(x, S_g(x) | x \in \Sigma\right\} .$$

By Proposition 3.7, S_g cannot be ≥ 0 (or ≤ 0) unless it is >0 (or <0) on Int Σ i.e. if $g\Sigma \cap \Sigma \neq F$, we can assume S_g changes sign. Let $\Sigma^\alpha = p(\Sigma_g^\alpha)$ then $\Sigma_g^\alpha = \{(x, S_g(x)) | x \in \Sigma^\alpha\}$. We may assume $S_g > 0$ on Σ^α and $\partial\Sigma_g^\alpha = \partial\Sigma^\alpha \subseteq \Sigma \cap \Sigma_g$ and $\{x \in \Sigma | S_g(x) = 0\} = \Sigma \cap \Sigma_g$. Let $C = \{(x,s) | x \in \Sigma^\alpha, 0 < s < S_g$, then C is open connected subset of U and $\partial C = -\Sigma^\alpha + \Sigma_g^\alpha$ (since $\mathcal{H}^{n-1}(\Sigma \cap \Sigma_g) = 0$) where Σ^α, Σ_g^α have the induced orientations. If $M(\Sigma^\alpha) \leq M(\Sigma_g^\alpha)$ let $\Sigma_0 = \Sigma_g - \partial C$, then

$$\partial\Sigma_0 = \partial\Sigma_g = F$$

and

$$\begin{aligned} M(\Sigma_g) \leq M(\Sigma_0) &\leq M(\Sigma_g - \bar{\Sigma}_g^\alpha) + M(\bar{\Sigma}^\alpha) \\ &\leq M(\Sigma_g - \bar{\Sigma}_g^\alpha) + M(\bar{\Sigma}_g^\alpha) = M(\Sigma_g) \quad . \end{aligned}$$

Therefore $M(\Sigma_g) = M(\Sigma_0)$ (same for $M(\Sigma_g^\alpha) \leq M(\Sigma^\alpha)$). Using the regularity theory for minimal surface due to Almgren-Federer-Hardt-Simon, Σ_0 is a C^∞ hypersurface of N. It turns out that if $x_0 \in \partial\Sigma^\alpha \cap \Sigma_g \sim F$, then Σ and Σ_g touch at x_0 to infinitely high order. It follows from Theorem 3.1 that $\Sigma = \Sigma_g$. Therefore if $\Sigma \cap \Sigma_g \sim F \neq \phi$ and $g \in W$, then $\Sigma = \Sigma_g$ that is $W \cap E = \phi$.

We next claim that $G \sim E$ is both open and closed in G or equivalently that $\partial(G \sim E) = \partial E = \phi$. Suppose $\partial(G \sim E) = \partial E \neq \phi$. Let $g_0 \in \partial E$ then there is a sequence $\{g_n\} \subseteq E$ such that the $\lim g_n = g_0$. By the definition of E we have points $x_n \in \Sigma \cap \Sigma_{g_n} \sim F$. Since Σ is compact, we can assume that the $\lim x_n = x_0 \in \Sigma$. It is easy to see that $x_0 \in \Sigma \cap \Sigma_{g_0}$.

We show first that $T_{x_0}\Sigma = T_{x_0}\Sigma_g$. If $x_0 \in \Sigma \sim F$ and $T_{x_0}\Sigma \neq T_{x_0}\Sigma_g$, then there exists A open in Int(Σ) and W' open in G such that $x_0 \in A$

and $g_0 \in W'$ and for any $g \in W'$, $\Sigma \cap \Sigma_g \cap A \neq \phi$, and the intersection of Σ and Σ_g is transversal in A. Hence $W' \subset E$ and so $g_0 \notin \partial E$. If $x_0 \in F$ and $T_{x_0}\Sigma \neq T_{x_0}\Sigma_{g_0}$, there is a neighborhood V of x_0 in N such that $\Sigma \cap V \cap \Sigma_g \cap V = V \cap F$ and hence there exists W'' open in G such that $g_0 \in W''$ and $\Sigma \cap V \cap \Sigma_g \cap V = F \cap V$ for $g \in W''$. But for large n, $g_n \in W''$, $x_n \in V$ and $x_n \in \Sigma \cap V \cap \Sigma_{g_n} \cap V$ and $x_n \notin F$ by the choice of $\{x_n\}$, $\{g_n\}$. Hence $\Sigma \cap V \cap \Sigma_{g_n} \cap V \neq F$ which still leads to contradiction.

Since $T_{x_0}\Sigma = T_{x_0}\Sigma_{g_0}$, we can choose a coordinate chart V of N at x_0 such that $V \cap \Sigma$ is connected and

$$\Sigma \cap V = \begin{cases} \{x \in V | x_n = 0\} & \text{if } x_0 \notin F \\ \{x \in V | x_{n-1} \geq 0 \text{ , } x_n = 0\} & \text{if } x_0 \in F \text{ .} \end{cases}$$

and

$$\Sigma_g \cap V = \{x \in V | x_n = u_g(x_1,\ldots,x_{n-1}) \text{ , } (x_1,\ldots,x_{n-1}) \in \Sigma \cap V\}$$

for $g \in W_1 \subseteq G$ and W_1 is a small neighborhood of g_0. We observe that u_{g_0} does not change sign. Otherwise, there are $x_1, x_2 \in \Sigma \cap V$ such that $u_{g_0}(x_1) < 0 < u_{g_0}(x_2)$. Since $u_{g_0}|\Sigma \cap V \cap F = 0$, we have $x_1, x_2 \in \Sigma \cap V \sim F$. It follows that there is an open subset $W_2 \subseteq G$, $g_0 \in W_2 \subseteq W_1$ such that $u_g(x_1) < 0 < u_g(x_2)$ for $g \in W_2$. It implies that there is $x_g \in \Sigma \cap V \sim F$ such that $u_g(x_g) = 0$ and $\Sigma \cap \Sigma_g \sim F \neq \phi$ and $\neq \Sigma$. Hence $g_0 \in W_1 \cap E$ which contradicts the fact that $g_0 \in \partial E$.

Applying Propositions 3.6 and 3.7 we obtain that $\Sigma = \Sigma_g$ and hence $gg_0\Sigma = g\Sigma$ for $g \in W$ and $Wg_0 \subseteq G \sim E$, so we must have $g_0 \notin \partial E$ which contradicts the choice of g_0. Therefore $\partial(G \sim E) = \partial E = \phi$ and $G = G \sim E$, $E = \phi$.

As a corollary of the proof of above theorem, we can prove similarly (and much more easily than above) the following.

Corollary 4.2: Let N be a complete, connected oriented Riemannian manifold with connected Lie Group G acting on it as isometries. Let γ be a curve with two end points which are fixed by G, or γ is a simple closed curve with one point p on γ which is fixed by G. Suppose γ has minimal length in its homology class. Then, for any $g \in G$ we have

$$\gamma \cap g\gamma = \begin{cases} \gamma & \text{or} \\ \partial\gamma \end{cases} .$$

In the first case and

$$\gamma \cap g\gamma = \begin{cases} \gamma & \text{or} \\ p \end{cases}$$

in the second case.

We now turn to study the global structure of the manifold N. We have the following.

Theorem 4.3: (Fibering theorem). Let F, Σ, N and G be as in Theorem 4.1. Then

$$G(\Sigma) = N \quad \text{or} \quad G(\Sigma) = \Sigma .$$

Moreover, consider the closed subgroup

$$H = \{g \in G \mid g\Sigma = \Sigma\}$$

of G. Then $G(\Sigma) = N$ if and only if $G/H = S^1$ and $N \sim F$ fibers over S^1

with fibre $\mathrm{Int}(\Sigma)$.

Proof: Let $c : S^1 \to G/H$ be any embedding and let $S^1 = I/\{0,1\}$. Considering c as a map of I, we can lift c to a mapping $g : I \to G$ with $\pi \circ g(t) = c(t)$ and $g^{-1}(0)g(1) \in H$.

Consider

$$\phi : \mathrm{Int}(\Sigma) \times (0,1) \longrightarrow N \sim F$$

defined by $\phi(x,t) = g(t)(x)$. Since ϕ is continuous and injective, it follows from the theorem of invariance of domain (see [SM, Vol. I], page 3 or[SE], page 199), ϕ is an open map. Therefore $\mathrm{Im}\,\phi$ is open in $N \sim F$. But $\mathrm{Im}\,\phi = \{g(t)x | x \in \Sigma , t \quad [0,1]\} \sim F$. So $\mathrm{Im}\,\phi$ is closed in $N \sim F$. By the connectedness of N, we know that $N \sim F$ is connected and $\mathrm{Im}\,\phi = N \sim F$. Therefore

$$\{g(t)x | x \in \Sigma , t \in [0,1]\} = N .$$

Now we claim that $\dim G/H \leq 1$. Suppose $\dim G/H \geq 2$, then we can find two disjoint circles $\{c_1(t)\}$, $\{c_2(t)\}$ in G/H. Let $g_1(t)$, $g_2(t)$ be the lifting of c_1, c_2 in G, then we have from the above:

$$\{g_1(t)x | x \in \Sigma , t \in [0,1]\} = N$$

$$\{g_2(t)x | x \in \Sigma , t \in [0,1]\} = N .$$

Hence, there are t_1, t_2 such that

$$g_1(t_1)(\Sigma) \cap g_2(t_2)(\Sigma) \sim F \neq \phi .$$

By Theorem 4.1, $g_1(t_1)(\Sigma) = g_2(t_2)(\Sigma)$ and $g_1^{-1}(t_1)g_2(t_2) \in H$, hence $c_1(t_1) = c_2(t_2)$ which is impossible since $\{c_1\} \cap \{c_2\} = \phi$.

If $\dim G/H = 1$, since G is connected, G/H is also connected and G/H is diffeomorphic to S^1. It is clear that in this case, $N \sim F$ fibres over $S^1 = G/H$ with fibre $\text{Int}\,\Sigma$. If $\dim G/H = 0$, then G/H is a single point and $G = H$, $G(\Sigma) = \Sigma$.

Theorem 4.4: Let F, Σ, N, G be as in Theorem 4.3. In addition, G is semi-simple Lie group. Then Σ is G-invariant i.e. $G(\Sigma) = \Sigma$.

Proof: If $G(\Sigma) \neq \Sigma$, then $G(\Sigma) = N$ and $\dim G/H = 1$, hence H is normal in G.

Theorem 4.5: Let $F, N, G,$ be as in Theorem 4.1. If Σ is not connected, then

$$G(\Sigma) = \Sigma .$$

Proof: Let $\Sigma = \coprod \Sigma_j$ where $\{\Sigma_j\}$ are the connected components of Σ. Note that for every component F_α of F, we have $G(F_\alpha) = F_\alpha$ and hence $G(\partial\Sigma_j) = \partial\Sigma_j$. Since Σ is a manifold with $\partial\Sigma = \coprod \partial\Sigma_j$, we have $\partial\Sigma_j \cap \text{Int}(\Sigma) = \phi$. If Σ is not connected, choose any two of $\{\Sigma_j\}$, say Σ_1, Σ_2. By virtue of Theorem 4.3 we have

$$G(\Sigma_1) = N \quad \text{or} \quad G(\Sigma_1) = \Sigma_1$$

$$G(\Sigma_2) = N \quad \text{or} \quad G(\Sigma_2) = \Sigma_2 .$$

Suppose $G(\Sigma_1) = N$, then there is a $g \in G$ such that $g(\Sigma_1) \cap \partial\Sigma_2 \neq \phi$. Since $\partial\Sigma_1 \cap \partial\Sigma_2 = \phi$, we must have $g(\text{Int}\,\Sigma_1) \cap \partial\Sigma_2 \neq \phi$, and hence

$\mathrm{Int}(\Sigma_1) \cap g(\partial\Sigma_2) = \mathrm{Int}(\Sigma_1) \cap \partial\Sigma_2 \neq \phi$ which is impossible since $\Sigma_1 \cap \Sigma_2 = \phi$. Therefore for any Σ_j, $G(\Sigma_j) = \Sigma_j$ and $G(\Sigma) = \Sigma$.

Remark 4.6: We can prove the similar results for minimal hypersurfaces without boundary of N. (In fact it is easier). In other words, if Σ is a hypersurface of N with $\partial\Sigma = \emptyset$ and if Σ minimizes area in its homology class in N where N and G are as before, then we have the following.

Theorem 4.7: Let Σ, N, G be as above. Then either Σ is G-invariant or $G(\Sigma) = N$. In the latter case, N fibers over a circle $S^1 = G/H$ with fibre Σ where

$$H = \{g \in G \mid g\Sigma = \Sigma\} .$$

Moreover, if either G is semi-simple or Σ is not connected, then Σ is G-invariant.

We now turn to study the special case $G = S^1$. We have the following.

Theorem 4.8: Let M be a compact, connected, oriented manifold with a circle action which acts as isometries and $\dim M \leq 7$. Suppose $F_1 = \mathrm{Fix}(M;S^1) \neq \phi$. If F is an invariant submanifold of M with codimension 2 and $[F] = 0$ in $H_*(M;\mathbf{Z})$ such that $F_1 \subseteq F$. Then

$$S^1(\Sigma) = M .$$

and there is an S^1-equivariant fibration

$$M \sim F \longrightarrow S^1$$

with fibre $\mathrm{Int}\,\Sigma$.

Proof: If $S^1(\Sigma) \neq M$, it follows from Theorem 4.3 that $S^1(\Sigma) = \Sigma$. Let $p \in F_1$ and $\nu(p)$ be the normal vector to $F = \partial\Sigma$ in Σ, which points into Σ. Since S^1 acts on M as isometries, $\nu(p)$ is fixed S^1, and hence $\gamma(t) = \exp_p t\nu(p)$ is also fixed by S^1. Therefore $\gamma \subseteq F_1 \subseteq F$, a contradiction. So we have $S^1(M) = M$.

Corollary 4.9: Let M be a connected, compact, oriented Riemannian manifold with an isometric S^1 action. Suppose $\dim M \leq 7$ and $F = \mathrm{Fix}(M;S^1)$ has codimension 2 and suppose that some non-empty union of oriented components $F_0 \subset F$ is homologous to zero in M. Then $F_0 = F$ and

$$M \sim F \cong \mathrm{Int}\,\Sigma \times S^1$$

where Σ is connected and

$$U_r(F) = \{x \in M | d(x,F) < 2r\} \cong F \times D^2 \qquad (9)$$

for some $r > 0$.

Proof: Let $\varepsilon \in (0,1) \subseteq S^1$ be the smallest element such that $g_\varepsilon(\Sigma) = \Sigma$, then $g_\varepsilon : \Sigma \to \Sigma$ is an isometry whose fixed point set contains $\partial\Sigma$. For any $p \in \partial\Sigma$, let $\nu(p)$ be the normal unit vector to F which points into Σ. Then $\gamma_t(p) = \exp_p t\nu(p)$ is fixed by g_ε. Therefore $\mathrm{Fix}(M,g_\varepsilon)$ is a submanifold of M with dimension $\geq n-1$. Hence $\mathrm{Fix}(M,g_\varepsilon) = M$ and $g_\varepsilon = \mathrm{Id}$. Therefore, we have

$$M \sim F \cong \mathrm{Int}\,\Sigma \times S^1$$

and so $F = F_0$. The diffeomorphism (9) clearly holds for all $r > 0$ sufficiently small.

As a easy consequence, we have the following:

Corollary 4.10: Let M, S^1, Σ, F be as in Corollary 4.9. If $H^2(M;\mathbb{Z}) = 0$ and a component of F has codimension 2. Then F is connected and

$$M \sim F \cong \operatorname{Int} \Sigma \times S^1$$

and

$$U_r(F) = \{x \in M \mid d(x,F) < 2r\} \cong F \times D^2$$

for some $r > 0$.

V. Applications to circle actions on 3-manifolds

The purpose of this section is to give a new geometrical proof of the topological classification of effective actions of the circle group, SO(2), on closed, connected 3-manifolds which was originally proved by P. Orlik and F. Raymond. For technical reasons, we only consider orientable 3-manifolds. In the orientable case, the result of P. Orlik and F. Raymond (see [R] & [OR]) can be stated as follows.

Theorem 5.1: If SO(2) acts effectively on a closed, connected, oriented 3-manifold M, then M is identified as

(1) S^3, $S^2 \times S^1$, L(p,q) admitting actions with or without fixed points,

(2) a connected sum of the above admitting only actions with fixed points,

(3) a quotient of SO(3) or Sp(1) by a finite, non-abelian discrete subgroup, admitting a unique fixed point free action,

(4) a $K(\pi,1)$ whose fundamental group has infinite cyclic center (provided it is not the 3-dimensional torus), admitting a unique action without fixed points and hence not homeomorphic to any other 3-manifold with an SO(2) action.

We are using the results of last section to prove the above theorem. The proof will be in several steps--divided them into propositions which are also interesting in themselves.

Throughout this section, M is a closed, connected, oriented 3-manifold with an effective action of the circle group, $S^1 = SO(2)$. F(M,S) denotes the fixed point set. We assume that M has a metric which is invariant of action of S^1.

Proposition 5.2: If $H_1(M;\mathbf{Z}) = 0$ and $F(M;S^1)$ is non-empty, then $M = S^3$ and the action is the standard linear action.

Proof: By the Corollary 4.10, we must have $F = S^1$. Using the existence and regularity theory of minimal surface of Almgren-Federer-Fleming-Hardt-Simon ([FH1], [FH2], [HS], [AF1], [AF2]), we know that there is an area-minimizing smooth surface Σ such that $\partial\Sigma = F$. Because of Corollary 4.10, we have

$$M - F = \mathrm{Int}(\Sigma) \times S^1 .$$

We claim that Σ is a closed 2-disk D^2. This will prove the theorem. Suppose Σ is oriented surface Σ_g of genus g with a hole on it. Since $H_1(M;\mathbb{Z}) = 0$ using Mayer-Vietoris sequence, we can get that $H_1(\Sigma;\mathbb{Z}) = 0$ which means that $\Sigma = D^2$.

Theorem 5.3: Let M be a oriented 3-manifold with S^1 action, $F \neq \emptyset$ and $H_1(M;\mathbf{Z}) = \mathbf{Z}$. Then $M = S^2 \times S^1$.

Proof: Since $H_1(M,\mathbf{Z}) = \mathbf{Z}$, by the Poincare duality we have $H^2(M,\mathbb{Z}) = \mathbf{Z}$ which implies that $H_2(M;\mathbb{Z}) = \mathbf{Z}$. Using the minimal surface theory of Almgren-Federer-Fleming, there is an area minimizing smooth embedded surface Σ without boundary in M.

Suppose now that $F \neq \phi$. We consider two cases :

(i) F is not connected and

(ii) $F = S^1$ is connected.

In the first case, let S_1, S_2 be any two components of F. Take a minimal geodesic α joining S_1, S_2 such that $\alpha(0) \in S_1$, $\alpha(1) \in S_2$ and the length of α equals the distance of S_1 and S_2 . Then for any $g \in S^1$ and since $\{\alpha(0),\alpha(1)\} \in F$, we must have $g\alpha \cap \alpha = \{\alpha(0),\alpha(1)\}$ and $\{g\alpha | g \in S^1\}$ generates a smooth 2-sphere S^2 in M. Since the intersection number $S_1 \cdot S^2 = \pm 1$; $M \sim S^2$ must be connected. We can cut M along S^2, then cap off equivariantly to get a 3-manifold $\tilde{M}$ with S^1 action such that $F(\tilde{M},S^1) \neq \phi$ and $M = \tilde{M} \# S^2 \times S^1$. Since $H_1(M;\mathbb{Z}) = \mathbb{Z}$, we have $H_2(\tilde{M};\mathbb{Z}) = 0$. By Proposition 5.2, $\tilde{M} = S^3$ and $M \cong S^2 \times S^1$.

In the second case, we claim first F is not homologous to zero in M. Suppose $[F] = [S\ ] = 0$ in $H_1(M;\mathbb{Z})$, then we can solve the plateau problem with the given boundary F to get a smooth surface Σ with boundary $\partial\Sigma = F$. Using Corollary 4.9 we have $M \sim F \cong \mathrm{Int}\,\Sigma \times S^1$. Since $H_1(M;\mathbb{Z}) = \mathbb{Z}$ by using Mayer-Vietoris exact sequence, we have that $H_1(\Sigma;\mathbb{Z}) = 0$ and Σ is a 2-disk. Hence M is the 3-sphere S^3 and $H_1(M;\mathbb{Z}) = 0$ which is impossible.

Now we only have to consider the case $F \cong S^1$ and $[F] \neq 0$. Since $H_1(M;\mathbf{Z}) = \mathbf{Z}$ we use the minimal surface theory to find a surface Σ without boundary in M as we observed in the beginning. By the Remark 4.7 , either Σ is invariant or M fibres over S^1 with fibre Σ. In the first case, Σ is either S^2 or T^2. By Poincaré duality, since $[F] \neq 0$, we must have the intersection number $F \cdot \Sigma \neq 0$. Hence $F \cap \Sigma \neq \phi$. Therefore, Σ has to be the 2-sphere S^2 and $M \sim S^2$ is connected. We cut M along S^2 and cap off equivariantly to get a 3-manifold $\tilde{M}$ such that $M \cong M \# S^2 \times S^1$. Since $H_1(M;Z) = \mathbf{Z}$ and $H_1(\tilde{M} \# S^2 \times S^1,\mathbf{Z}) = H_1(\tilde{M};\mathbf{Z}) \oplus \mathbf{Z}$. Hence $H_1(\tilde{M};\mathbb{Z}) = 0$. By Proposition 5.2, $\tilde{M} = S^3$ and the action is the standard linear action. Therefore

$$M \cong S^2 \times S^1 .$$

In the second case, we must have $\mathrm{Fix}(M;S^1) = \phi$, which is the violation of $F \neq \phi$.

<u>Theorem 5.4</u>: Let M be a connected, compact, oriented 3-manifold with S^1 action and assume that $F(M,S^1)$ to be not empty. Suppose $H_1(M;\mathbf{Z}) = \mathbf{Z}_p$ for some integer $p \geq 2$. Then M is the connected sum of lens spaces $\#_i L(p_i,q_i)$ for some $q_i \geq 1$ where $p = \prod_i p_i$.

<u>Proof</u>: Since $F(M;S\ ) \neq \phi$, every orbit of S^1 in M must be homotopic to zero. Let $\tilde{M} \xrightarrow{p} M$ be the universal covering of M with covering map p. Consider the lifting problem:

$$\begin{array}{ccc} S^1 \times \tilde{M} & \dashrightarrow & \tilde{M} \\ \Big\downarrow \text{Id} \times p & & \Big\downarrow p \\ S^1 \times M & \xrightarrow{\ \ \phi\ \ } & M \end{array} \qquad (*)$$

where $\phi : S^1 \times M \to M$ is the action map.

Since $\phi_*(\text{Id} \times p)_*(\pi_1(S^1 \times \tilde{M})) = 0$, we have a solution $\tilde{\phi} : S^1 \times \tilde{M} \to \tilde{M}$ to the lifting problem (*) with the property that $\tilde{\phi}(1,\tilde{x}_0) = \tilde{x}_0$ for fixed $x_0 \in F$ and $\tilde{x}_0 \in p^{-1}(x_0)$. It is easy to see that $\tilde{\phi} : S^1 \times \tilde{M} \to \tilde{M}$ is a S^1 action on $\tilde{M}$. From the fact that $H_1(M;\mathbf{Z}) = \mathbf{Z}_p$ and Poincaré duality, we have $H_2(M;\mathbf{Z}) = 0$.

We want to show first that $\pi_2(M) = 0$. If not, then by a theorem of Meeks-Yau ([MY], Theorem 7) there is a minimal embedded 2-sphere S^2 in M which is invariant. But since $H_2(M;\mathbf{Z}) = 0$, S^2 is homologous to zero in M, and M is decomposed to the connected sum of two 3-manifolds M_1, M_2 with S^1 action along S^2 and also $F(M_1,S^1) \neq \emptyset$, $F(M_2,S^1) \neq \emptyset$. We note that $H_1(M,\mathbf{Z}) = H_1(M_1,\mathbb{Z}) \oplus H_1(M_2,\mathbf{Z})$, and $H_2(M_1,\mathbf{Z}) = H_2(M_2,\mathbb{Z}) = 0$.
If, for example, $H_1(M_1,\mathbb{Z}) = 0$, then by Proposition 5.2 M_1 is a 3-sphere S^3 and S^2 has to be homotopic to zero in M which is a contradiction.

Therefore

$$H_1(M_1,\mathbf{Z}) \neq 0 \quad \text{and} \quad H_1(M_2,\mathbf{Z}) \neq 0 \quad .$$

We can continue this decomposing process to decompose M. Then we get

$$M = M_1 \# \cdots \# M_\ell$$

and

$$H_1(M_i,\mathbf{Z}) = \mathbf{Z}_{p_i} \neq 0 \quad , \quad p = \pi p_i$$

and also $H_2(M_i,\mathbf{Z}) = 0$. (Note since $H_1(M_i,\mathbb{Z}) \neq 0$ the above process have to stop at finite times.)

Next, we want to show the universal covering M_i of M_i are compact. Let $p_i : \tilde{M}_i \to M_i$ be the covering maps. We know from above that the S^1 action on M_i lift to $\tilde{M}_i$. Suppose $\tilde{M}_i$ is not compact, then $H^3(\tilde{M};\mathbf{Z}) = H_3(\tilde{M};\mathbb{Z}) = 0$. By the Hurewicz theorem, we must have that

$$H_k(\tilde{M}_i;\mathbf{Z}) = 0 \quad , \quad \pi_k(\tilde{M}_i) = 0 \quad , \quad k \geq 1 \quad .$$

Using Smith Homology theory, we have

$$\sum_{k=0}^{1} \text{rank}\ \check{H}^k(F(\tilde{M}_i,S^1);\mathbf{Z}) \leq 1 \quad .$$

Since $\dim F(\tilde{M}_i,S^1) = 1$, $F(\tilde{M}_i,S^1)$ must be connected and $F(\tilde{M}_i,S^1) = \mathbb{R}$. We note that $F(\tilde{M}_i,S^1) = p_i^{-1}(F(M_i,S^1))$ and p_i is a local isometry. Hence we must have $\pi_1(M_i) = \mathbf{Z}$ which implies that $H_1(M_i;\mathbb{Z}) = \mathbb{Z}$. This contradicts the hypothesis that

$$H_1(M_i;\mathbf{Z}) = \mathbf{Z}_{p_i} \quad .$$

Now using Proposition 5.2 $\tilde{M}_i = S^3$ and $F(\tilde{M}_i;S^1) = S^1$. We also have $F(M_i,S^1) = S^1$ and $\pi_1(M_i) = \mathbb{Z}_{p_i}$. Let $G_i = \mathbb{Z}_{p_i}$ be the Deck Transformation group of $\tilde{M}_i$, then it is easy to see that G_i commutes with the S^1 action of $\tilde{M}_i$ and preserves $F(\tilde{M},S^1)$. Let Σ be the solution to the Plateau problem with boundary $\partial\Sigma = F(\tilde{M}_i,S^1)$ in $\tilde{M}_i$. We know from the proof of Proposition 5.2 that Σ must be a 2-disk and $\mathrm{Int}(\Sigma) \times S^1$ is diffeomorphic to $\tilde{M}_i - F(\tilde{M}_i,S^1)$. We use this diffeomorphism to identify $\tilde{M}_i - F(\tilde{M}_i,S^1)$ with $\mathrm{Int}(\Sigma) \times S^1$.

Take any generator of T_i of G_i . We claim that $T_i(\Sigma) = t(\Sigma)$ for some $t \in S^1$. First observe that $T_i(\partial\Sigma) = \partial\Sigma$; consider the projection map f of $\mathrm{Int}(\Sigma) \times S^1$ to the second factor S^1

$$f : \mathrm{Int}(\Sigma) \times S^1 \to S^1$$

since $\mathrm{Int}\, T_i(\Sigma) = T_i(\mathrm{Int}\,\Sigma) \subseteq \tilde{M} - F(\tilde{M},S^1) = \mathrm{Int}(\Sigma) \times S^1$. We may take $g = f \circ j$ where

$$j : \mathrm{Int}\, T_i(\Sigma) \longrightarrow \mathrm{Int}(\Sigma) \times S^1$$

is the inclusion map.

We have two cases to consider. The first case occurs when the image of g is a single point $t_i \in S^1$, then $\mathrm{Int}\, T_i(\Sigma) = \mathrm{Int}\, t_i(\Sigma)$ hence $T_i(\Sigma) = t_i(\Sigma)$.

If the image of g is not a single point in S^1 , then, since $g(\mathrm{Int}\, T_i(\Sigma))$ is connected, $g(\mathrm{Int}\, T_i(\Sigma))$ must contain an open interval. By Sard's theorem, there is a $t \in S^1$ such that t is a regular value of g and t belongs to the image of g . It follows that $\mathrm{Int}\, T_i(\Sigma)$ and $\mathrm{Int}\, t(\Sigma)$ intersect transversally.

On the other hand, let $T \equiv$ the unit tangent field to the foliation of $\tilde{M}_i - \mathrm{Fix}(\tilde{M}_i, S^1)$ by $\{t\overset{\circ}{\Sigma}\}_{t \in S^1}$. Let Θ be the 2-form on $\tilde{M}_i - \mathrm{Fix}(\tilde{M}_i, S^1)$ given by $\Theta(v_1, v_2) \equiv \langle T, v_1 \wedge v_2 \rangle$ for tangent vectors v_1, v_2. Then

(1) $d\Theta = 0$

(2) $|\Theta(\xi_x)| \leq 1$ and $\Theta(\xi_x) = 1$ iff $\xi_x = T_x$

i.e. Θ is a "calibration".

Now

$$\mathrm{Vol}(\Sigma) = \int_\Sigma \Theta = \int_{T_i(\Sigma)} \Theta \leq \mathrm{Vol}(T\Sigma) = \mathrm{Vol}(\Sigma) .$$

The equality implies that if $T_i\Sigma$ intersect $t\Sigma$ for some $t \in S^1$, then $T_i\Sigma$ and $t\Sigma$ have the same tangent space at the intersection points which contradicts the above fact. Hence the second case does not occur.

We have proved that $T_i(\Sigma) = t_i(\Sigma)$ but since $T_i^{p_i} = \mathrm{Id}$ one has that $t_i^{p_i}(\Sigma) = T_i^{p_i}(\Sigma) = \Sigma$ which implies $t_i^{p_i} = \mathrm{Id}$. It is clear that $a_i = t_i^{-1} T_i$ generates a group G_1 which is isomorphic to a quotient of $\mathbb{Z}_{p_i}$ and acts on Σ and $a_i|\partial\Sigma = T_i|\partial\Sigma$. We now claim that the action G_1 on Σ is equivalent to the standard action a 2-disk. Let N be any subgroup of G_1 with prime order $p' > 1$ and let Σ^* be the orbit space of Σ with action N. Using the Smith theory (see[B]), we have

$$\sum_{i \geq 0} \mathrm{rank}\, \check{H}^i(\Sigma^N; \mathbb{Z}_{p^i}) \leq 1 .$$

It follows that Σ^N is a single point in $\mathrm{Int}\,\Sigma$, let us say $\Sigma^N = \{x_0\}$. Take any $S \in N$, we have $Sa_i(x_0) = a_i S(x_0) = a_i(x_0)$, hence $a_i(x_0) = x_0$ and $\Sigma^N = \{x_0\} = \Sigma^{G_1}$. Therefore that action G_1 on Σ is equivalent to a standard orthogonal action on $\Sigma = D^2$.

We have from the above proof that the S^1 action on $\tilde{M}$ is equivalent to the standard S^1 action on S^3 given by

$$e^{2\pi i\theta}(z_1,z_2) = (z_1, e^{2\pi i\theta} z_2)$$

where $(z_1,z_2) \in S^3 \subseteq \mathbf{C}^2$ and

$$T_i(z_1,z_2) = \left(e^{2\pi i(1/p_i)} z_1 , e^{2\pi i(q_i/p_i)} z_2\right)$$

for some $q_i \geq 1$, $(p_i,q_i) = 1$. Thus, M_i is the lens space $L(p_i,q_i)$ with the induced S^1 action from above S^1 action on S^3.

We are now in a position to prove the equivariant classification and topological classification of S^1-actions with fixed point on 3-manifolds. This is the first part of Theorem 5.1.

<u>Theorem 5.5</u> (F. Raymond): Let M be a connected, compact, oriented 3-manifold with an S^1-action such that the fixed point set $F(M,S^1)$ is non-empty. Then M must be a specific equivariant connected sum of $S^2 \times S^1$ and $L(p,q)$. The S^1-action on $S^2 \times S^1$ and $L(p,q)$ are standard.

<u>Proof</u>: Let k = number of components of $F(M;S^1)$, then $F(M;S^1)$ is the the disjoint union of k circles, i.e., $F(M;S^1) = C_1 \amalg C_2 \amalg \ldots \amalg C_k$ and $C_i = S^1$, $i = 1,\ldots,k$. Suppose first $k > 1$. Take any two components C and C' of $F(M;S^1)$ and let γ be a minimizing geodesic segment which connects C and C' such that the length of γ = distance$(C;C')$. Then $\gamma \perp C$ and $\gamma \perp C'$. Since γ is minimizing and two end points are fixed by the group S^1, for any $g \in S^1$, we must have $g(\gamma) \cap \gamma = \{\gamma(0);\gamma(1)\}$. Hence $S^1(\gamma)$ is a

smooth 2-sphere. Since the intersection numbers

$$\#(S^1(\gamma),C) = \pm 1 , \qquad \#(S^1(\gamma),C') = \pm 1$$

$S^1(\gamma)$ is not homologous to zero in M. Thus, $M - S^1(\gamma)$ is connected with two holes. Capping off the two holes with two 3-ball in the equivariant way, we obtain a manifold M_1 with S^1 action and the number of components of $F(M;S^1)$ is $k-1$. Also, it is clear that $M = M_1 \# S^2 \times S^1$ where the connected sum is an equivariant connected sum. We can continue this decomposing process to decompose M. We get

$$M = M_{k-1} \# \underbrace{S^2 \times S^1 \# \cdots \# S^2 \times S^1}_{k-1}$$

and $F(M_{k-1};S^1)$ is connected.

Now we only consider the case where $F(M;S^1) = S^1$ is connected. Suppose

$$H_1(M;\mathbb{Z}) = \mathbb{Z}_{p_1} + \cdots \oplus \mathbb{Z}_{p_m} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n} .$$

If $n+m > 1$, we can choose a class C in $H_1(M;\mathbb{Z})$ such that C is not an Integer multiple of $[F(M;S^1)]$; and has the form

$$C = (0 , \underbrace{\ldots , C_\ell , \ldots}_{m+n} , 0)$$

where C is a generator of the ℓth factor of $H_1(M;\mathbb{Z})$. Let γ be the oriented circle $F(M;S^1)$ and let x_1,x_2 be any two points on γ, then x_1,x_2 cut γ to two parts. Take one of them, call it $\gamma_1 = \gamma|\{x_1;x_2\}$. Let $A = \{\beta | \beta$ is piece-wise C' curve in M with end points on γ such that $\beta \cup \gamma | \{\beta(0);\beta(1)\} \in C\}$. Then we can minimize the length of curves in A as the standard way (see [FH1]) to find a C^∞ curve in A with minimal length denoted $\bar{\alpha}$. We note that $\bar{\alpha}$ is perpendicular to γ at two end points since $\bar{\alpha}$ is minimal.

We claim that the two end points of $\bar{\alpha}$ do not coincide and for any $g \in S^1$, $g(\bar{\alpha}) \cap \bar{\alpha} = \{\bar{\alpha}(0); \bar{\alpha}(1)\}$. If $\bar{\alpha}(0) = \bar{\alpha}(1)$, then let v_0, v_1 be the unit tangent vectors of $\bar{\alpha}$ at $\bar{\alpha}(0)$ and $\bar{\alpha}(1)$. The representation of S^1 on the normal vector space of γ at $\bar{\alpha}(0) = \bar{\alpha}(1)$ is just the standard rotation. Let g_1 be the rotation in S^1 such that $g_1(v_0) = -v_1$. Since $g_1(\bar{\alpha}(t))$ and $\bar{\alpha}(1-t)$ both are geodesics with the same tangent vector at the initial point, we have $g_1(\bar{\alpha}(t)) = \bar{\alpha}(1-t)$ and $g_1(v_1) = -v_0$. It follows that $v_0 = -v_1$ and $\bar{\alpha}$ is a closed geodesic in M. Thus, we proved that either $\bar{\alpha}(0) \neq \bar{\alpha}(1)$ or $\bar{\alpha}$ is a closed geodesic.

Using Corollary 4.2 for any $g_1, g_2 \in S^1$ and $g_1 \neq g_2$, we have

$$g_1(\bar{\alpha}) \cap g_2(\bar{\alpha}) = \{\bar{\alpha}(0);\bar{\alpha}(1)\} . \tag{1}$$

Again, we consider the case that $\bar{\alpha}(0) = \bar{\alpha}(1)$. We are going to show that this case does not occur. Let y_0 be the middle point of $\bar{\alpha}$ and let $R = 1/2$ length of $\bar{\alpha}$. Consider

$$D_1 = \left\{ \exp_{\bar{\alpha}(0)} sg(v_0) \mid 0 \leq s \leq R , g \in S^1 \right\}$$

$$D_2 = \left\{ \exp_{\bar{\alpha}(0)} sg(v_1) \mid 0 \leq s \leq R , g \in S^1 \right\} .$$

By equation (1) D_1 and D_2 are diffeomorphic to a 2-disk and

$$D_1 = D_2 = \left\{ \exp_{\bar{\alpha}(0)} sV \mid V \in N_{\bar{\alpha}(0)}(\gamma) \right\} .$$

Thus $\bar{\alpha} \subseteq D_1 = D_2$ and $\bar{\alpha}$ is homologous to zero in D_1 and hence in M, which is a contradiction.

Now $\{g(\bar{\alpha}) | g \in S^1\}$ is a smooth 2-sphere by equation (1) and the fact that $\bar{\alpha}(0) \neq \bar{\alpha}(1)$. Let us call it $S^1(\bar{\alpha})$. We have two cases to consider. The first case occurs when $S^1(\bar{\alpha})$ is one-sided. In this case $M - S^1(\bar{\alpha})$ is connected. We cap off this manifold as before to obtain a new manifold M_1 with two components of the fixed point set $F(M_1, S^1)$ and

$$M = M_1 \# S^2 \times S^1 .$$

Since $F(M_1;S^1)$ has two components, we have as before that $M_1 = M_2 \# S^2 \times S^1$ where $F(M_2;S^1)$ is connected. Since

$$H_1(M;\mathbf{Z}) = H_1(M_2) \oplus H_1(S^2 \times S^1) \oplus H_1(S^2 \times S^1) ,$$

we must have

$$H_1(M_2;\mathbf{Z}) = \mathbf{Z}_{p_1} \oplus \cdots \oplus \mathbf{Z}_{p_m} \oplus \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n-2} .$$

If $S^1(\bar{\alpha})$ is two-sided, then we can decompose M into the connected sum of two manifolds M_1, M_2 along $S^1(\bar{\alpha})$ where $F(M_i;S^1)$ where $i = 1, 2$ are connected. Thus, we have

$$M = M_1 \# M_2$$

and we note that neither $H_1(M_1;\mathbb{Z})$ nor $H_1(M_2;\mathbb{Z})$ is zero. Otherwise, by Proposition 5.2, one of them would be the 3-sphere and then $C = [\bar{\alpha}] = 0$ in M, which is impossible.

We can continue this process to get a decomposition of M

$$M = M_1 \# \cdots \# M_k \# S^2 \times \underbrace{S^1 \# \cdots \# S^2}_{\mu} \times S^1$$

where $F(M_j;S^1) = S^1$, $j = 1, 2, \ldots, k$ and either $H_1(M_j;\mathbb{Z}) = \mathbb{Z}_p$ for some integer $p \geq 2$ or $H_1(M_j;\mathbb{Z}) = \mathbb{Z}$. By Theorem 5.3 there are no M_j with $H_1(M_j;\mathbb{Z}) = \mathbb{Z}$. By Theorem 5.4 $M_j = \# L(p'_{ji};q_{ji})$. By comparing with the first Homology group of M and reordering, we can assume $\mu = n$, $\prod_i p'_{ji} = p_j$, $j = 1, 2, \ldots, m$ and $\mu = n$. Therefore, we always can decompose the given M in the theorem to

$$M = L(p\ ,q\ ) \# \cdots \# L(p_s,q_s) \# S^2 \times \underbrace{S^1 \# \cdots \# S^2}_{t} \times S^1 \quad .$$

This completes the proof of Theorem 5.5.

We now turn out attention to the case that the fixed point set $F(M;S^1)$ is empty. We are going to prove:

Theorem 5.6 (P. Orlik and F. Raymond): Let M be a connected, compact, orientable 3-manifold with S -action such that $F(M;S^1) = \emptyset$. If M is not $S^2 \times S^1$, S^3. pr a ;ems space, then M is either covered by the 3-sphere with non-abelian fundamental group or it is a $K(\pi,1)$-manifold. Moreover, in both cases, M does not have any S^1-action with fixed points.

In order to prove this theorem, let us first prove some propositions which will be used and will also be interesting in themselves.

Proposition 5.7: Let M be as in Theorem 5.6 with S^1-action such that $F(M;S^1) = 0$ and $\pi_1(M) = 0$. Then M is the 3-sphere.

Proof: Let C be a principle orbit of the S^1 action since $H_1(M;\mathbf{Z}) = 0$. We can solve the plateau problem for the given boundary C to get an area minimizing surface Σ with boundary $\partial\Sigma = C$. We know from the Theorem 4.3 that either Σ is invariant or $M \sim C$ fibres equivariantly over S^1. if Σ is invariant, then Σ must be a 2-disk and there is a fixed point in Σ. This is excluded by the assumption. Hence, $M \sim C$ fibres over S^1. Choose an invariant tubular neighborhood V_1 of C and let $V_2 = M \sim \bar{V}_1$. We have that $M \sim C$ fibres over S^1 with fibre $\operatorname{Int}\Sigma$ and if Σ has genus $g > 0$, then $\pi_1(\Sigma) = \mathbf{Z} * \cdots * \mathbf{Z}$ with $2g$ factors. By the homotopy exact sequence of a fibration, the inclusion

$$\pi_1(\Sigma) \longrightarrow \pi_1(M \sim C) \quad = \quad \pi_1(M \sim V_1)$$

is injective. Using van Kampen's theorem, we have $\pi_1(M) \neq 0$. This is impossible. Therefore, Σ is a 2-disk and $M \sim V_1 \cong D^2 \times S^1$. Since $\bar{V}_1 \cong S^1 \times D^2$ we have a Heegaard splitting of a lens space and so $M \cong L(p,q)$ for some p,q. But since $\pi_1(M) = 0$, M is a 3-sphere.

Remark 5.8: From Proposition 5.2 and Proposition 5.7 we have that 3-sphere S^3 is the only simple connected manifold on which $SO(2)$ operates.

Proof of Theorem 5.6: We first show that $\pi_2(M) = 0$. Suppose $\pi_2(M) \neq 0$, then from a theorem of Meeks-Yau (see [MY], Theorem 7), there is an embedded minimal 2-sphere S^2 in M such that for any $g \in S^1$ either $g(S^2) = S^2$ or $g(S^2) \cap S^2 = 0$. Therefore either S^2 is invariant or M fibres over S^2 with fiber S^2. In the first case, we must have two fixed points on S^2. This is not allowed by hypothesis. For the second case, we must have $M \cong S^2 \times S^1$ which also excluded from the assumption.

Let $\tilde{M}$ be the universal covering manifold of M. If $\tilde{M}$ is not compact, then $H_3(\tilde{M},Z) = 0$ and from the Hurewicz theorem we have $\pi_n(\tilde{M}) = H_n(\tilde{M},Z) = 0$ for $n \geq 1$. Hence $\tilde{M}$ is contractible and M is $K(\pi,1)$ manifold.

Since M is orientable with S^1 action without fixed points, M must be a Seifert fibered manifold (fibered by the orbits). But from the fact that M has a fiber preserving S^1 action, the base surface of the Seifert manifold M must be orientable. Hence M has infinite cyclic center [HJ] (with the exception that M is T^3). It is a immediate consequence of Theorem 5.5. There is no S^1 action on M with fixed points.

Consider now the case that $\tilde{M}$ is compact. Using a theorem of transformation group theory (see [BG], Theorem 9.1, p. 63), we know that there is a covering S^1-action on $\tilde{M}$ of the S^1 action on M. This action commutes with the deck group of the covering. It follows from Proposition 5.7 that $\tilde{M}$ is the 3-sphere S^3. Now if $\pi_1(M)$ is abelian by the proof of Theorem 5.4, M must be a lens space. But M is not a lens space so $\pi_1(M)$ is not abelian. This completes the proof.

VI. The classification of SO(n) actions on (n+1-dimensional manifolds with fixed points

In this section, we generalize the method of the last section to obtain a topological classification of SO(n) actions on (n+1)-manifolds with fixed points.

Throughout this section, we will assume $3 \leq n \leq 6$ and let M be a compact, connected, oriented, (n+1)-manifold with SO(n) action such that the fixed-point set $F(M;SO(n)) \neq 0$.

As an application of minimal surfaces, we are going to prove that the following theorem, which is due to W. Y. Hsiang [HW].

<u>Theorem 6.1</u>: We have

$$H_1(M;\mathbb{Z}) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\ell}$$

and

$$M \cong S^n \times S^1 \# \cdots \# S^n \times S^1 \# S^{n+1}$$

where the actions on $S^n \times S^1$ and S^{n+1} are standard and where the connected sums are equivariant.

We first prove some propositions which are as setups for Theorem 6.1 proof.

<u>Proposition 6.2</u>: If $H_1(M;\mathbb{Z}) = 0$, then $F = F(M;SO(n))$ is connected and

$$M \cong S^{n+1} \quad .$$

Proof: We first show that F is connected. Suppose F is not connected. Since the lowest dimension of the irreducible representations of $SO(n)$ is n, hence each component of F is one dimensional and F is the disjoint union of circles. If F is not connected, let F_1, F_2 be two different components of F. Take a minimal geodesic α which connects F_1 and F_2. Then $\alpha(0) \in F_1$, $\alpha(1) \in F_2$ and α is perpendicular to F_i $(i=1,2)$. For any $g \in SO(n)$, we have

$$g\alpha \cap \alpha = \begin{cases} \alpha & \text{or} \\ \{\alpha(0),\alpha(1)\} \end{cases} .$$

We let $H = \{ g \in SO(n) | g\alpha = \alpha\}$. By looking at the representation of $SO(n)$ on the normal space of F_1 at $\alpha(0)$, we have that $H = SO(n-1)$ and the space $\{g\alpha | g \in SO(n)\}$ is a smooth n-sphere S^n in M and $G/H = S^{n-1}$. Since the intersection number $F_1 \cdot S^n = \pm 1$, $M \sim S^n$ is connected. We can cut M along S^n and cap off equivariantly to get a $(n+1)$ manifold $\tilde{M}$ such that

$$M \cong \tilde{M} \# S^n \times S^1 .$$

Therefore, $H_1(M) = H_1(\tilde{M}) \oplus \mathbf{Z}$ which is impossible.

We now claim that $M \sim F$ fibres over an open surface with fiber S^{n-1}. Let $x \in M \sim F$. Take a minimum geodesic α_x which connects x and F, then α_x is perpendicular to F. From the linear representation of $SO(n)$ on $T_{\alpha_x(1)}M = TF \oplus TF'$ and from uniqueness of geodesics we see that

$$\tilde{H}_x = \{g \in G : g(x) = x\} \supseteq SO(n-1) .$$

Hence, since $SO(n-1)$ is maximal, we have

$$\text{either} \quad \tilde{H}_x = SO(n-1)$$
$$\text{or} \quad \tilde{H}_x = SO(n) \quad .$$

If $\tilde{H}_x = SO(n)$, then $x \in F$ which contradicts that $x \in M \sim F$. Therefore for any $x \in M \sim F$, $\tilde{H}_x = SO(n-1)$ and $G(\alpha_x)$ is a smooth n-disk D^n. Hence, the stability group G_x of x is $H = SO(n-1)$. Since x is arbitrary point of $M \sim F$ which shows that the orbit of x is principle for any $x \in M \sim F$. Therefore the orbit $M \sim F$ has topological type S^{n-1} and $M \sim F$ is fibre bundle over an open surface with fiber $G/H \cong S^{n-1}$. Let V_1 be an invariant tubular neighborhood of F in M and $V_2 = M \sim \bar{V}_1$ and $\bar{V}_2$ fibres over a compact surface Σ with boundary $\partial\Sigma = S^1$ and $V_1 = D^n \times S^1$.

We claim that Σ is a 2-disk. If Σ is a surface with genus $g > 0$, then

$$\pi_1(\Sigma) = \underbrace{\mathbf{Z} * \cdots * \mathbf{Z}}_{2g} \quad .$$

By the homotopy exact sequence of the fiberation $S^{n-1} \to \bar{V}_2 \to \Sigma$, we obtain

$$\pi_1(\bar{V}_2) \cong \pi_1(\Sigma) = \underbrace{\mathbf{Z} * \cdots * \mathbf{Z}}_{2g}$$

Hence

$$H_1(\bar{V}_2) = \underbrace{\mathbf{Z} + \cdots + \mathbf{Z}}_{2g} \quad .$$

By Mayer-Vietorie exact sequence, we must have $H_1(M) \neq 0$ which is the violation of $H_1(M) = 0$. Therefore Σ is a 2-disk.

Hence, $\bar{V}_2 \cong D^2 \times S^{n-1}$ and

$$M = \bar{V}_1 \cup \bar{V}_2 = (D^n \times S^1) \cup (D^2 \times S^{n-1}) \quad .$$

Let $f : S^{n-1} \times S^1 \to S^1 \times S^{n-1}$ be the map $f(x,t) = (t,x)$ and g be the gluing map $\partial\bar{V}_1 \to \partial\bar{V}_2$ of M. Then f,g are equivariant, so $F = f^{-1} \circ g$ is an equivariant diffeomorphism. $F : S^{n-1} \times S^1 \to S^{n-1} \times S^1$ is determined by the image $\{x_0\} \times S^1$. In fact,

$$\{\text{transversal embeddings } S^1 \hookrightarrow S^{n-1} \times S^1 \text{ of degree } 1\}$$

$$\overset{1-1}{\longleftrightarrow} SO(n)\text{-equivariant diffeomorphisms of } S^{n-1} \times S^1 \quad .$$

Thus, F is isotopic to the identity map I and f is isotopic to g. Therefore, M is diffeomorphic to S^{n+1}.

Proposition 6.3: If $H_1(M;\mathbb{Z}) = \mathbb{Z}$, then F has two components and

$$M \cong S^n \times S^1 \quad .$$

Proof: We consider two cases

(i) the number of components of $F > 1$

(ii) F is connected.

In case (i), let F_1, F_2 be two different components. As in the first part of the proof of Proposition 6.2, we have $M \cong M \# S^n \times S^1$. Since $H_1(M;\mathbb{Z}) = \mathbb{Z}$, we must have $H_1(\tilde{M};\mathbb{Z}) = 0$ by Proposition 6.2, $\tilde{M} \cong S^{n+1}$

$$M \cong S^n \times S^1 \quad ,$$

and F has two components.

In case (ii), we consider two subcases:

(a) F is homologous to zero in M

(b) F is not homologous to zero in M.

We claim that case (a) is possible due to $H_1(M;F) = \mathbb{Z}$. Let α be a generator of $H_1(M;F)$. We can minimize the curve in α to obtain a connected smooth geodesic γ with end points on F (see [FH1]) such that $[\gamma] \neq 0$ in $H_1(M;F)$. From the proof of the last section, $\gamma(0) \neq \gamma(1)$ and for any $g \in SO(n)$, either $g\gamma = \gamma$ or $g\gamma \cap \gamma = \{\gamma(0),\gamma(1)\} \subseteq F$. Furthermore, is perpendicular to F at $\gamma(0)$ and $\gamma(1)$. By checking the representation of $SO(n)$ at the normal space of F at $\gamma(0)$, we know that $\{g\gamma | g \in SO(n)\}$ is a smooth n-sphere S^n. If $M \sim S^n$ is connected, then $M \cong \tilde{M} \# S^n \times S^1$ and $F(\tilde{M};SO(n))$ has two components. Hence, we have $\tilde{M} \cong \tilde{\tilde{M}} \# S^n \times S^1$ and $M \cong \tilde{\tilde{M}} \# S^n \times S^1 \# S^n \times S^1$ which is impossible since $H_1(M) = \mathbb{Z}$. If $M \sim S^n$ has two components, then $M \cong M_1 \# M_2$ since $H_1(M) = H_1(M_1) \oplus H_1(M_1) = \mathbb{Z}$, one of $H_1(M_1)$ and $H_1(M_1)$ is zero. Let us say $H_1(M_1) = 0$, then by Proposition 6.2 $M_1 \cong S^{n+1}$ and we must have $[\gamma] = 0$ which contradicts the choice of γ.

In case (b), since $H_1(M) = \mathbb{Z}$ by Poincaré duality, we have $H_n(M) = \mathbb{Z}$. Let Σ be the area minimizing hypersurface in its homology class of M such that $[\Sigma] \neq 0$ in $H_n(M)$. Since $n+1 \leq 7$, Σ is smooth. By Remark 4.7 either Σ is invariant or M fibres over S with fiber Σ. Since $[F] \neq 0$, we have that the intersection number $F \cdot \Sigma \neq 0$, hence $F \cap \Sigma \neq \emptyset$ and Σ must be invariant. We claim $F \cap \Sigma$ contains exactly two points and

$\Sigma \cong S^n$. Suppose that $F \cap \Sigma$ contains a single point. For any $x \in \Sigma \sim F$, take a minimal geodesic α_x from $F \cap \Sigma$ to x, then for any $g \in SO(n)$ either $g\alpha_x \cap \alpha_x = F \cap \Sigma$ or $g\alpha_x = \alpha_x$. Hence, we have that

$$G_x = \{g\alpha_x'(0) = \alpha_x'(0) | g \quad SO(n)\}$$

$$\cong SO(n-1)$$

and all the isotropy groups G_x are conjugate to each other. Therefore, for any $x \in \Sigma \sim F$, the orbit of x in Σ is principal and $\Sigma \sim F$ fibres over a 1-manifold. Since $\Sigma \sim F$ is open and connected, we must have that $\Sigma \sim F/SO(n) \cong \mathbb{R}$ and

$$\Sigma \sim F \cong S^{n-1} \times \mathbb{R} .$$

Thus $H_1(\Sigma \sim F) = 0$ and $H_1(\Sigma) = 0$. Since Σ is a n-dimensional manifold with $SO(n)$ action and the fixed-point set is non-empty. By Proposition 6.2 and Proposition 5.2, we obtain that $\Sigma \cong S^n$. Hence, since $F \cap \Sigma$ is a single point, $\Sigma \sim F \cong S^n \sim F \cap \Sigma = \mathbb{R}^n$ which contradicts $\Sigma \sim F \cong S^{n-1} \times \mathbb{R}$. Therefore, $\Sigma \cap F$ contains more than one point.

Take two different points $p_1, p_2 \in F \cap \Sigma$ and a minimal geodesic α in Σ from p_1 to p_2. We have

$$H = \{g \in SO(n) | g\alpha = \alpha\} = \{g \in SO(n) | g\alpha'(0) = \alpha'(0)\} \cong SO(n-1)$$

and the space $\bigcup_g \{g\alpha | g \in SO(n)\}$ is the n-sphere S^n. Hence $\Sigma = S^n$ and $F \cap \Sigma$ has exactly two points. Since $F \cdot S^n \neq 0$, $M \sim S^n$ is connected and we can cut M along S^n and cap off equivariantly to obtain an (n + 1)-manifold $\tilde{M}$ such that

$$M \cong \tilde{M} \# S^n \times S^1 .$$

Since $H_1(M;\mathbb{Z}) = \mathbf{Z}$, hence $H_1(\tilde{M};\mathbb{Z}) = 0$, and by Proposition 6.2, $\tilde{M} \cong S^{n+1}$ and $M \cong S^n \times S^1$. This completes the proof of Proposition 6.3.

Proposition 6.4: If $H_1(M;\mathbb{Z}) = \mathbb{Z}_p$, then $F = S^1$ and $p = 0$ and

$$M \cong S^{n+1} .$$

Proof: If F has more than one component, then we can decompose M as

$$M \cong \tilde{M} \# S^n \times S^1$$

which is impossible since $H_1(M) = \mathbf{Z}_p$. Hence, $F = S^1$ and as in the proof of Proposition 6.2, we have $M \sim F/SO(n)$ is an open surface. If V_1 is an invariant tubular neighborhood of F with $\bar{V}_1 \cong D^n \times S^1$, then $\bar{V}_2 = M \sim V_1$ fibres over a surface Σ_g of genus g and $\partial\Sigma_g = S^1$. We claim $g = 0$ and Σ_g is a 2-disk. Suppose $g > 0$, then

$$\pi_1(\Sigma_g) = \underbrace{\mathbf{Z} * \cdots * \mathbf{Z}}_{2g} .$$

From the homotology exact sequence of fibration $S^{n-1} = SO(n)/SO(n-1) \to \bar{V}_2 \to \Sigma_g$, we have

$$\pi_1(\bar{V}_2) \cong \pi_1(\Sigma_g) = \underbrace{\mathbf{Z} * \cdots * \mathbf{Z}}_{2g}$$

since $n \geq 3$. Hence

$$H_1(\bar{V}_2) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g} .$$

By using Mayer-Vietoris exact sequence for pair $(\bar{V}_1, \bar{V}_2)$, we obtain

$$H_1(M) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g}$$

Since $H_1(M;\mathbb{Z}) = \mathbb{Z}_p$, we must have $g = 0$ and Σ_g is the 2-disk D^2.

Therefore $\bar{V}_2 \cong S^{n-1} \times D^2$ and

$$M = (D^n \times S^1) \cup (S^{n-1} \times D^2) .$$

Since the gluing map is equivariant, the reason, as before, shows that

$$M \cong S^{n+1} .$$

Using the above propositions, we can apply the same method we used to prove Theorem 5.5 to give a proof of Theorem 6.1. Since they are similar, we will not write out the proof of Theorem 6.1.

VII. Circle action and scalar curvature

It has been known for some years that if a compact spin manifold M admits either a non-trivial S^1 action or a metric of positive scalar curvature, then $\hat{A}(M) = 0$. It has been at times conjectured that these are directly related, in particular that the existence of an S^1-action implies the existence of a metric of positive scalar curvature. This conjecture turns out to be false. In fact, we have many manifolds with S^1-action which does not have any positive scalar curvature metric. This can be seen clearly from recent works of Gromov-Lawson (see [GL3]). We have the following theorem.

Theorem 7.1 (Gromov-Lawson): There is no metric with positive scalar curvature on three-dimensional $K(\pi,1)$ manifold.

Proof: By Theorems 7.1 and 5.1 we have many $K(\pi,1)$ 3-manifolds with S^1-action but no metric with positive scalar curvature. However, all the counterexamples known up to now are have S^1-action without fixed point. In fact for 3-manifold, as a consequence of Theorem 5.1 and the works of Gromov-Lawson (see [GL1]). We have

Corollary 7.2: If M has a S^1-action with fixed points, then there is a metric on M with positive scalar curvature.

Remark 7.3: This corollary was observed by L. B. Bergery (see [BB]).

In this section, we study the relation of the existence of S^1-action with fixed points and the existence of positive scalar curvature in general.

We will prove the following theorem:

Theorem 7.4: Let M be a compact, connected, oriented C manifold of $\dim M \leq 7$ with S^1-action such that the fixed points $F = Fix(M;S^1) \neq 0$. Assume that F has codimension 2 and that F is homologous to zero in M. Then there exists a smooth structure on M such that with this smooth structure there is an invariant Riemannian metric on M with positive scalar curvature.

We first prove a theorem which will be use to prove Theorem 7.3.

Theorem 7.5: Let M be a compact manifold and F a codimension 2 submanifold such that

$$M \sim N_\varepsilon(F) \xrightarrow{\text{diffeomorphic}} \Sigma \times S^1$$

and the tubular neighborhood $N_\varepsilon(F)$ of F in M is trivial, i.e. $N_\varepsilon(F) = F \times D^2$ where Σ is a $(n+1)$-dimensional compact manifold and $\partial\Sigma = F$, $M = N_\varepsilon(F) \cup \Sigma \times S^1$ the gluing map is isotropic to the identity map. Then there exists a metric on M with positive scalar curvature.

Proof: From Theorem 2.4, we known that there is a metric on $\bar{\Sigma}$ with positive sectional curvature, call this metric dS^2. Let $\nu(x)$ be the normal vector field of F in Σ, pointing into Σ. There is a tubular neighborhood U of $F = \partial\bar{\Sigma}$ in $\bar{\Sigma}$ such that $U = \{\exp_x(t\nu(x)) \mid x \in F, 0 \leq t \leq 2r\}$. Hence U is diffeomorphic to $F \times [0,2r)$. Let $V = (U \sim F) \times S^1$. We identify $M \sim F$ with $\Sigma \times S^1$, then $V \subset M$. We are going to construct a metric on M which has the form

$$d\bar{S}^2 = dS_1^2 + f^2(t)d\theta^2$$

where dS_1^2 is a metric which is different from dS^2 only on $U_\varepsilon = \{\exp_x(t\nu(x)) | x \in F, \ 0 \leq t \leq \varepsilon\}$ for some small $\varepsilon < r$ and $f(t)$ is a smooth function which is constant on $\Sigma \sim U$.

First, let us define the function f on $\bar{\Sigma}$. We choose a smooth function $g(t)$ such that

$$g(t) = \begin{cases} \varepsilon^2 \cos\frac{t}{\varepsilon} & \text{on } [0,\varepsilon] \\ 0 & \text{on } [r,2r) \end{cases}$$

and g is decreasing, where $\varepsilon < r$ will be decided later. And let

$$f(t) = \int_0^t g(s)ds = \begin{cases} \varepsilon \sin\frac{t}{\varepsilon} & \text{on } [0,\varepsilon] \\ \int_0^r g(t)dt > 0 & \text{on } [r,2r) \end{cases} .$$

Then we can extend f to $\bar{\Sigma}$ by letting $f \equiv$ constant on $\Sigma \sim U$. Note that $f''(t) = g'(t) \leq 0$.

Let $\{x_1, \ldots, x_n\}$ be a coordinate of F, then $\{x_1, \ldots, x_n, t\}$ is a coordinate of U and we have

$$dS^2 = \sum_{i,j=1}^{n} g_{ij}(t,x)dx_i dx_j + dt^2$$

for $0 \leq t < 2r$.

Let g be any metric on Σ and

$$\bar{g} = g + f^2(x)d\theta^2$$

where f is C^∞ function on $\bar{\Sigma}$. Let $\{\ell_i, \ i = 1,2,\ldots, n+1\}$ be an orthonormal frame of Σ with metric g and $\nu = 1/f(\partial/\partial\theta)$. Let $\bar{R}\langle\ \rangle$, $\overline{\mathrm{Ric}}$, $\bar{R}$ denote the sectional, Ricci, scalar curvatures for $\bar{g}$, and $R\langle\ \rangle$, Ric, R denote the corresponding curvatures for g. Then we have

$$\overline{\mathrm{Ric}}(\ell_i,\ell_i) = \sum_j \langle \bar{R}(\ell_j,\ell_i)\ell_i,\ell_j\rangle + \langle \bar{R}(\nu,\ell_i)\ell_i,\nu\rangle \quad .$$

But

$$\langle \bar{R}(\ell_j,\ell_i)\ell_i,\ell_j\rangle = \langle R(\ell_j,\ell_i)\ell_i,\ell_j\rangle + b_{ii}b_{jj} - b_{ij}^2$$

and for $i \neq j$

$$b_{ij} = \langle \bar{\nabla}_{\ell_i}\nu,\ell_i\rangle = -\langle \nu,\bar{\nabla}_{\ell_i}\ell_j\rangle = -\frac{1}{f}\langle \frac{\partial}{\partial\theta},\bar{\nabla}_{\ell_i}\ell_j\rangle = \frac{1}{f}\langle \bar{\nabla}_{\ell_i}\frac{\partial}{\partial\theta},\ell_j\rangle$$

$$= -\frac{1}{f}\langle \ell_i,\bar{\nabla}_{(\partial/\partial\theta)}\ell_i\rangle = -\frac{1}{f}\langle \ell_i,\bar{\nabla}_{\ell_j}\frac{\partial}{\partial\theta}\rangle = \langle \bar{\nabla}_{\ell_j}\ell_i,\nu\rangle = -\langle \ell_i,\bar{\nabla}_{\ell_j}\nu\rangle$$

$$= -b_{ji} \quad .$$

Hence $b_{ij} = -b_{ji} = -b_{ij} = 0$ for $i \neq j$ and

$$b_{ii} = \langle \bar{\nabla}_{\ell_i}\nu,\ell_i\rangle = -\langle \nu,\bar{\nabla}_{\ell_i}\ell_i\rangle = -\frac{1}{f}\langle \frac{\partial}{\partial\theta},\bar{\nabla}_{\ell_i}\ell_i\rangle = +\frac{1}{f}\langle \bar{\nabla}_{\ell_i}\frac{\partial}{\partial\theta},\ell_i\rangle$$

$$= \frac{1}{f}\langle \bar{\nabla}_{(\partial/\partial\theta)}\ell_i,\ell_i\rangle = \frac{1}{f}\cdot\frac{1}{2}\frac{\partial}{\partial\theta}\langle \ell_i,\ell_i\rangle = 0 \quad .$$

Hence

$$\langle \bar{R}(\ell_i,\ell_j)\ell_j,\ell_i\rangle = \langle R(\ell_i,\ell_j)\ell_j,\ell_i\rangle \quad .$$

By a straightforward computation, we have

$$\bar{R} = \sum \overline{Ric}(\ell_i,\ell_i) + \overline{Ric}(\nu,\nu)$$

$$= \sum Ric(\ell_i,\ell_i) + 2\,\overline{Ric}(\nu,\nu)$$

$$= R + 2\cdot\frac{-\Delta f}{f} = R - 2\cdot\frac{\Delta f}{f}$$

This formula is also in [GL3] where Δ is the Laplace operator of the metric g.

Now consider the metric $g = dS_1^2$ defined by

$$dS_1^2 = \sum g_{ij}(\phi(t),x)dx_i dx_j + dt^2$$

where ϕ is a smooth function such that $\phi(t) = t$ for $t \geq \varepsilon$ and ϕ is constant in a neighborhood of $t = 0$. Let $\gamma_u = g_{ij}(u,x)dx_i dx_j$ be the metric on F and $uR\langle\ \rangle$, $uRic$, uR are the corresponding curvatures of γ_u.

For $\bar{g} = g + f^2(t)d\theta^2 = dS_1^2 + f^2(t)d\theta^2$, we have

$$\Delta f = f''(t)$$

and

$$\bar{R} = R - 2\cdot\frac{f''(t)}{f(t)} \quad . \tag{7.1}$$

By simple calculus, we can obtain

$$R = {}_{\phi(t)}R + \frac{1}{2}\sum\left(g^{jk}\,\frac{\partial^2 g_{jk}}{\partial u^2}\right)(\phi(t))\cdot\phi'^2(t) + \frac{1}{2}\sum\left(g^{jk}\,\frac{\partial g_{jk}}{\partial u}\right)(\phi(t))\cdot\phi''(t) \quad . \tag{7.2}$$

Let $A = F \times [0,r]$. Since A is compact, there are constants $c_1, c_2 > 0$ such that

$$\left| \frac{1}{2} \sum \left(g^{jk} \frac{\partial^2 g_{jk}}{\partial u^2} \right) \right| \leq c_1$$

$$\left| \frac{1}{2} \sum \left(g^{jk} \frac{\partial g_{jk}}{\partial u} \right) \right| \leq c_2 \quad . \tag{7.3}$$

for $0 \leq u \leq r$.

Now we are going to choose the function ϕ . For any $0 < \varepsilon < r$, choose a smooth function $k_\varepsilon(t)$ such that

$$k_\varepsilon(t) = \begin{cases} \frac{1}{4c_2} \cdot \frac{1}{\varepsilon^2} & \text{on } \left[\frac{2}{10}\varepsilon, \frac{9}{10}\varepsilon\right] \\ 0 & \text{on } \left[0, \frac{1}{10}\varepsilon\right] \text{ and } [\varepsilon, 2r) \end{cases}$$

and $0 \leq k_\varepsilon(t) \leq \frac{1}{4c} \cdot \frac{1}{\varepsilon^2}$.

Note that

$$\int_0^\varepsilon k_\varepsilon(t)dt \geq \frac{1}{4c_2} \cdot \frac{1}{\varepsilon^2} \cdot \frac{7}{10}\varepsilon = \frac{7}{40c_2} \cdot \frac{1}{\varepsilon} \longrightarrow \infty$$

when $\varepsilon \to 0$.

Let $c_1' = \sqrt{c_1}$ and we may assume $(1/4c_2) < (1/2c_1')$ and choose ε very small such that

$$\int_0^\varepsilon k_\varepsilon(t)dt = a_\varepsilon >> 1$$

and $\varepsilon < r$, and

$$|{}_u R| < \frac{1}{2} \cdot \frac{1}{\varepsilon^2} \quad \text{for } 0 \leq u \leq r \quad . \tag{7.4}$$

Set $k(t) = (1/a_\varepsilon)k_\varepsilon(t)$ and define

$$\phi(t) = \int_0^t \int_0^s k(x)dx\,ds + \eta$$

where

$$\eta = \varepsilon - \int_0^\varepsilon \int_0^\varepsilon k(x)dx\,ds .$$

Since $\int_0^\infty k(x)dx = 1$ and $\int_0^0 k(x)dx = 0$, we must have

$$\int_0^\varepsilon \int_0^s k(x)dx < \varepsilon$$

and $\eta > 0$. By definition $\phi(t) = \eta$ for $0 \leq t \leq 1/10$ and

$$|\phi'(t)| = \left|\int_0^t \phi''(s)ds\right| \leq \frac{1}{a_\varepsilon}\cdot\frac{1}{4c_2}\cdot\frac{t}{\varepsilon^2} < \frac{1}{4c_2}\cdot\frac{1}{\varepsilon} < \frac{1}{2c_2'}\cdot\frac{1}{\varepsilon} \qquad (7.5)$$

for $0 \leq t \leq \varepsilon$. Furthermore,

$$|\phi''(t)| < \frac{1}{4c}\ \frac{1}{\varepsilon^2} \qquad (7.6)$$

and also $\phi(t) = t$ for $t \geq \varepsilon$. We note ϕ is an increasing function.

We claim now that for the above choice of f and ϕ, the metric defined by

$$d\bar{s}^2 = \sum g_{ij}(\phi(t),x)dx_i dx_j + f^2(t)d\theta^2$$

is well-defined on M with positive scalar curvature. Since $\phi(t) = \eta$ is constant for $t \leq 1/10\varepsilon$ and $d\bar{s}^2$ is a product metric in a neighborhood of F, so $d\bar{s}^2$ is well-defined on M.

For $t \geq \varepsilon$, $\phi(t) = t$ and $f''(t) \leq 0$, hence by equations (7.1), (7.2) and (7.4)

$$\bar{R} = R - 2\,\frac{f''(t)}{f(t)} > 0 .$$

(Note that on Σ, $R > 0$.)

For $t \leq \varepsilon$, $(f''(t)/f(t)) = -(1/\varepsilon^2)$ and by equations (7.3), (7.5), (7.6) and (7.2), we have

$$\bar{R} = \phi(t)^R + \frac{1}{2}\sum\left(g^{jk}\frac{\partial^2 g_{jk}}{\partial u^2}\right)\cdot \phi'(t) + \frac{1}{2}\sum\left(g^{jk}\frac{\partial g_{jk}}{\partial u}\right)\cdot \phi''(t)$$

$$-2\cdot\frac{f''(t)}{f(t)} > -\frac{1}{2}\,\frac{1}{\varepsilon^2} - c_1\cdot\frac{1}{2^2 c_1'^2}\,\frac{1}{\varepsilon^2} - c_2\cdot\frac{1}{4c_2}\cdot\frac{1}{\varepsilon^2} + 2\cdot\frac{1}{\varepsilon^2} = \frac{1}{\varepsilon^2} > 0 .$$

This completes the proof of Theorem 7.4.

Proof of Theorem 7.4: By using Corollary 4.10, we have

$$M \sim F \xrightarrow{\text{homeomorphism}} \mathrm{Int}(\Sigma) \times S^1 .$$

In order to apply Theorem 7.5, we have to define a smooth structure on M which satisfies the condition of that theorem.

We take the product of the smooth structure on $\mathrm{Int}(\Sigma) \times S^1$ and this structure and the homeomorphism (7.7) induce a smooth structure on $M \sim F$. But from the linear representation of S^1 action at the fixed points if the vector field X on M is induced by the S^1 action, then $X(x)$ is transversal to Σ for $x \in \Sigma$ near the boundary $\partial\Sigma$. Therefore, the homeomorphism (7.7) is actually diffeomorphism in a neighborhood of the boundary of $\mathrm{Int}(\Sigma) \times S^1$, hence, the two smooth structures coincide on this neighborhood.

So we can extend this new smooth structure to M and with this smooth structure, M satisfies the condition of Theorem 7.5 so Theorem 7.5 implies Theorem 7.4.

We conclude with the following.

Question: Is Theorem 7.4 true for the given smooth structure?

Chapter 2

The Constructions of Negatively Ricci Curved Manifolds

I. Statement of theorems

Theorem 1: Let M_1 and M_2 be two complete Riemannian manifolds with negative Ricci curvature of same dimension. Then there is a complete Riemannian metric on $M_1 \# M_2$ with negative Ricci curvature.

Let X_1 and X_2 be two oriented C^∞ manifolds, and γ_i is a simple closed curve in $X_i (i=1,2)$. Take a tubular neighborhood V_i of γ_i in X_i, let $\varphi : \partial V_1 \to \partial V_2$ be a diffeomorphism of the boundaries. We use $X_1 \#_\varphi X_2$ to denote the manifold obtained by gluing $X_1 \sim V_1$ and $X_2 \sim V_2$ along the boundaries by φ. We call $X_1 \#_\varphi X_2$ the connected sum along circles of X_1 and X_2.

Theorem 2: Let M_1 and M_2 be two oriented complete Riemannian manifolds with negative Ricci curvature, and let γ_i be a simple closed geodesic of $M_i (i=1,2)$. Without loss of generality, we may assume that γ_1 and γ_2 have the same length. We further assume the holonomy along γ_1 is the same with the holonomy along γ_2. Take an orthonormal parallel frame $\{X_1^i, \ldots, X_{n-1}^i\}$ along γ_i of the normal bundle of γ_i in $M_i (i=1,2)$, let

$$V_i = \{\chi \mid d(\chi, \gamma_i) < \varepsilon\} \quad i = 1,2$$

for small $\varepsilon > 0$, and define $\varphi : \partial V_1 \longrightarrow \partial V_2$ by

$$\varphi\left(\exp \sum_{j=1}^{n-1} a_j X_j^1\right) = \exp \sum_{j=1}^{n-1} a_j X_j^2$$

for $\sum_{j=1}^{n-1} a_j^2 = \varepsilon^2$.

Then there is a complete Riemannian metric on $M_1 \#_\varphi M_2$ with negative Ricci curvature. Here $n = \dim M_1 = \dim M_2$.

For three-dimensional manifold, we have a stronger result.

Theorem 3 : Let M_1 and M_2 be two oriented three-dimensional complete Riemannian manifolds with negative Ricci curvature. Let γ_i be a simple closed geodesic in $M_i (i = 1,2)$. We may assume that γ_1 and γ_2 have the same length. Take $V_i = \{\chi \in M_i \mid d(\chi, \gamma_i) < \varepsilon\}$ $(i = 1,2)$ for small $\varepsilon > 0$. Then, for any $\varphi : \partial V_1 \longrightarrow \partial V_2$, there is a complete Riemannian metric on $M_1 \circledast_\varphi M_2$ with negative Ricci curvature.

Using some topological constructions. We also can obtain a new manifold from an old one in the negative Ricci curvature manifold category.

Theorem 4: Let M be a complete Riemannian n-manifold with negative Ricci curvature. Then there is a complete Riemannian metric on $M \# S^{n-1} \times S^1$ with negative Ricci curvature.

Again, in three-dimensional case, we have more complete results.

Theorem 5: Let M be a complete Riemannian 3-manifold with negative Ricci curvature. Then there is a complete Riemannian metric on $M \# S^2 \times S^1 \# \Sigma \times S^1$ with negative Ricci curvature for any oriented compact surface Σ .

Theorem 6: Let M be as in Theorem 5.

Then there is a complete Riemannian metric on $M \# S^2 \times S^1 \# L(p,q)$ with negative Ricci curvature.

Final Remark: For 3-manifold, the above theorems in conjunction with the work of Thurston conclude that for a given compact 3-manifold M with negative Ricci curvature, there is a Riemannian metric on $M \# S^2 \times S^1 \# P$ with negative Ricci curvature for "almost" all compact, oriented prime manifold P .

II. Proof of Theorem 1 and Theorem 2.

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, and $a \in \mathbb{R}^n$, let

$$U(a,\rho) = \mathbb{R}^n \cap \{x : |x-a| < \rho\} .$$

Let

$$V_\rho = U(0,\rho) \times S^1$$

and S^1 be parametrized by t from 0 to 1 ; i.e., $S^1 = [0,1]/\{0,1\}$. We denote

$$r^2 = \sum_{i=1}^{n} x_i^2 .$$

Throughout this section, we call $x^{n+1} = t$, and the Greek letters always be integers from 1 to n+1 , all sums about Greek letters be taken from 1 to n+1 , and the English letters $i, j, k \ldots$ always be integers from 1 to n , all sums about these letters be taken from 1 to n .

Given a C^∞ manifold X , let

$$\text{Ric}\_(X) = \{\text{all metrics on } X \text{ with negative Ricci curvature}\} .$$

Let ρ_0 be a fixed positive number. The basic fact we need is that we can deform a given metric in a neighborhood to a nice one. We have the following.

Proposition 2.1. Let $g_0, g_1 \in \text{Ric}\_(V_{\rho_0})$ (or $\text{Ric}\_(U(0,\rho_0))$) , and suppose that the 1-jets $J^1(g_0)$, $J^1(g_1)$ are equal on the curve $\gamma \equiv \{r = 0\}$ (or at 0). Then there are a $\bar{g} \in \text{Ric}\_(V_{\rho_0})$ (or $\text{Ric}\_(U(0,\rho_0))$) , and $0 < \rho_2 < \rho_1 < \rho_0$, such that $\bar{g} = g_1$ for $r < \rho_2$, and $\bar{g} = g_0$ for $r > \rho_1$.

Before the proof of Proposition 2.1, let me mention a general lemma of M. Gromov which is pointed out by M. Gromov [GM].

Lemma 2.2. Let V be a manifold, $V_0 \longrightarrow V$ a submanifold, and two sections of a bundle $X \longrightarrow V$, say $f_1, f_2 : V \longrightarrow X$. Suppose that the 1-jets $J^1_{f_1}, J^1_{f_2} : V \longrightarrow J^1(X)$ are equal on V_0 . Let $J^2_t : V \longrightarrow J^2(X)$ be a deformation of the 2-jet $J^2_{f_1}$ to $J^2_{f_2}$ with the following two properties.

(1) The deformation does not move $J^1|V_0$

(2) The deformation keeps the image of V in $J^2(X)$ within a given open subset $\mathcal{W} \subset J^2(X)$.

Then, there is a deformation of <u>sections</u> $f_1 \longrightarrow f_2$, say f_t, which jets $J^2_{f_t}$ deform with the properties (1) and (2).

In fact, we only need a simple corollary of Gromov's lemma. For completeness, we will give the proof of the corollary.

Corollary 2.3. Let g_0, g_1 be as in Theorem 1, then there exists $\bar{\rho} < \rho_0$; and a deformation of $g_0 \longrightarrow g_1$, such that $J^1(g_s)|\gamma = J^1(g_0) = J^1(g_1)$, and $g_s \in Ric\_(V_{\bar{\rho}})$ (or $\in Ric\_(U(0,\bar{\rho}))$) .

Proof of Corollary 2.3. Let

$$g_s = (1-s)g_0 + sg_1 \quad .$$

On γ, we have

$$Ric(g_s) = (1-s)Ric(g_0) + s\ Ric(g_1) < 0 \quad .$$

Therefore, for small $\bar{\rho}$, we have

$$g_s \in Ric\_(V_{\bar{\rho}}) \quad (\text{or} \in Ric\_(U(0,\bar{\rho})))$$

and $$J^1(g_s) = J^1(g_0) = J^1(g_1) \quad . \tag{1}$$

Proof of Proposition 2.1. By the above corollary, we have a deformation g_s. Let

$$g_s = \Sigma\, g_{\alpha\beta}(x\, , t\, ; s)\ dx^\alpha dx^\beta \quad , \quad \text{where} \quad t = x^{n+1} .$$

We denote that $\dot{g}(x\, , t\, ; s) = \frac{\partial}{\partial s} g(x\, , t\, ; s)$, $\ddot{g}(x\, , t\, ; s) = \frac{\partial^2}{\partial s^2} g(x\, , t\, ; s)$.
From (1), we have

$$\dot{g}_{\alpha\beta}(0\,,\,t\,;\,s) = 0\,, \quad \dot{g}^{\alpha\beta}(0\,,\,t\,;\,s) = 0$$

$$\frac{\partial \dot{g}_{\alpha\beta}}{\partial x^\gamma}(0\,,\,t\,;\,s) = 0\,, \quad \frac{\partial \dot{g}^{\alpha\beta}}{\partial x^\gamma}(0\,,\,t\,;\,s) = 0 \tag{2}$$

$$\ddot{g}_{\alpha\beta}(0\,,\,t\,;\,s) = 0\,, \quad \ddot{g}^{\alpha\beta}(0\,,\,t\,;\,s) = 0$$

$$\frac{\partial \ddot{g}_{\alpha\beta}}{\partial x^\gamma}(0\,,\,t\,;\,s) = 0\,.$$

Therefore, there is a $M > 0$, s , t on $V_{\overline{\rho}}$ (or $U(0\,,\overline{\rho})$)

$$|g_{\alpha\beta}(x\,,\,t\,;\,s)| \le M\,, \quad |g^{\alpha\beta}(x\,,\,t\,;\,s)| \le M\,. \tag{3}$$

$$|\dot{g}_{\alpha\beta}(x\,,\,t\,;\,s)| \le M|x|^2\,, \quad |\dot{g}^{\alpha\beta}(x\,,\,t\,;\,s)| \le M|x|^2\,,$$

$$|\Gamma^\gamma_{\alpha\beta}(x\,,\,t\,;\,s)| \le M\,, \quad |\dot{\Gamma}^\gamma_{\alpha\beta}(x\,,\,t\,;\,s)| \le M|x| \tag{4}$$

$$|\ddot{g}_{\alpha\beta}(x\,,\,t\,;\,s)| \le M|x|^2\,, \quad \left|\frac{\partial g^{\alpha\beta}}{\partial x^\gamma}\right| \le M\,, \quad \left|\frac{\partial \dot{g}_{\alpha\beta}}{\partial x^\gamma}\right| \le M|x|\,.$$

where $|x|^2 = r^2 = \sum_{i=1}^{n} (x_i)^2$

We may assume

$$\sum \mathrm{Ric}(g_s)_{\alpha\beta} X^\alpha X^\beta \le -C|X|^2 = -C \sum_{\alpha=1}^{n+1} (X^\alpha)^2\,, \tag{5}$$

for some $C > 0$ on $V_{\overline{\rho}}$ (or $U(0\,,\overline{\rho})$) .

For $\psi \in C^\infty(U(0\,,\rho_0))$, with compact support, $\operatorname{supp}\psi \subset U(0\,,\overline{\rho})$, and $0 \le \psi \le 1$, we let

$$\bar{g} = \Sigma\, g_{\alpha\beta}(x\,,\,t\,,\,\psi(x))dx^{\alpha}dx^{\beta}$$

then $\bar{g}$ is a well defined metric on V_{ρ_0} .

We claim that $\bar{g}$ is the desired metric on V_{ρ_0} for suitable ψ .

Let

$$\bar{g} = \Sigma\, \bar{g}_{\alpha\beta}(x\,,\,t)dx^{\alpha}dx^{\beta}$$

then

$$\bar{g}_{\alpha\beta}(x\,,\,t) = g_{\alpha\beta}(x\,,\,t\,;\,\psi(x))$$

and

$$\mathrm{Ric}(\bar{g})_{\alpha\beta} = \Sigma \frac{\partial\bar{\Gamma}^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial\bar{\Gamma}^{\gamma}_{\gamma\beta}}{\partial x^{\alpha}} + \sum_{\mu,\,\gamma} \bar{\Gamma}^{\mu}_{\alpha\beta}\,\bar{\Gamma}^{\gamma}_{\gamma\mu} - \bar{\Gamma}^{\mu}_{\gamma\beta}\,\bar{\Gamma}^{\gamma}_{\alpha\mu}$$

where

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\,\bar{g}^{\gamma\mu}\left(\frac{\partial\bar{g}_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial\bar{g}_{\beta\mu}}{\partial x^{\alpha}} - \frac{\partial\bar{g}_{\alpha\beta}}{\partial x^{\mu}}\right)$$

$$= \Bigg[\frac{1}{2}\,g^{\gamma\mu}(s)\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right)(s)$$

$$+ \frac{1}{2}\,g^{\gamma\mu}(s)\left(\dot{g}_{\alpha\mu}(s)\,\frac{\partial\psi}{\partial x^{\beta}} + \dot{g}_{\beta\mu}(s)\,\frac{\partial\psi}{\partial x^{\alpha}} - \dot{g}_{\alpha\beta}(s)\,\frac{\partial\psi}{\partial x^{\mu}}\right)\Bigg]_{s\,=\,\psi(x)}$$

$$= \Gamma^{\gamma}_{\alpha\beta}(x\,,\,t\,;\,\psi(x)) + \Delta\Gamma^{\gamma}_{\alpha\beta}$$

and

$$\Delta\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\,g^{\gamma\mu}(x\,,\,t\,;\,\psi(x))\left(\dot{g}_{\alpha\mu}\,\frac{\partial\psi}{\partial x^{\beta}} + \dot{g}_{\beta\mu}\frac{\partial\psi}{\partial x^{\alpha}} - \dot{g}_{\alpha\beta}\,\frac{\partial\psi}{\partial x^{\mu}}\right) \quad . \qquad (6)$$

Therefore, we have

$$\frac{\partial \overline{\Gamma}^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} = \left[\frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} + \dot{\Gamma}^{\gamma}_{\alpha\beta} \cdot \frac{\partial \psi}{\partial x^{\gamma}} \right]_{s = \psi(x)} + \frac{\partial \Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}}$$

$$\begin{aligned}
\mathrm{Ric}(\overline{g})_{\alpha\beta} &= \left[\Sigma \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\gamma\beta}}{\partial x^{\alpha}} + \Sigma\, \Gamma^{\mu}_{\alpha\beta}\, \Gamma^{\gamma}_{\gamma\mu} - \Gamma^{\mu}_{\gamma\beta}\, \Gamma^{\gamma}_{\alpha\mu} \right]_{s = \psi(x)} \\
&+ \left[\Sigma\, \dot{\Gamma}^{\gamma}_{\alpha\beta} \frac{\partial \psi}{\partial x^{\gamma}} - \dot{\Gamma}^{\gamma}_{\gamma\beta} \frac{\partial \psi}{\partial x^{\alpha}} \right]_{s = \psi(x)} + \Sigma \frac{\partial \Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Delta\Gamma^{\gamma}_{\gamma\beta}}{\partial x^{\alpha}} \qquad (7) \\
&+ \Sigma \left(\Gamma^{\mu}_{\alpha\beta}\, \Delta\Gamma^{\gamma}_{\gamma\mu} + \Delta\Gamma^{\mu}_{\alpha\beta} \cdot \Gamma^{\gamma}_{\gamma\mu} - \Gamma^{\mu}_{\gamma\beta}\, \Delta\Gamma^{\gamma}_{\alpha\mu} - \Delta\Gamma^{\mu}_{\gamma} \cdot \Gamma^{\gamma}_{\alpha\mu} \right) \\
&+ \Sigma \left(\Delta\Gamma^{\mu}_{\alpha\beta} \cdot \Delta\Gamma^{\gamma}_{\gamma\mu} - \Delta\Gamma^{\mu}_{\gamma\beta}\, \Delta\Gamma^{\gamma}_{\alpha\mu} \right) \\
&= \mathrm{Ric}(g_s)_{\alpha\beta} \Big|_{s = \psi(x)} + A_{\alpha\beta}
\end{aligned}$$

where

$$\begin{aligned}
A_{\alpha\beta} &= \Sigma\, \dot{\Gamma}^{\gamma}_{\alpha\beta} \frac{\partial \psi}{\partial x^{\gamma}} - \dot{\Gamma}^{\gamma}_{\gamma\beta} \frac{\partial \psi}{\partial x^{\alpha}} + \Sigma \frac{\partial \Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Delta\Gamma^{\gamma}_{\gamma\beta}}{\partial x^{\alpha}} \\
&+ \Sigma \left(\Gamma^{\mu}_{\alpha\beta}\, \Delta\Gamma^{\gamma}_{\gamma\mu} + \Delta\Gamma^{\mu}_{\alpha\beta} \cdot \Gamma^{\gamma}_{\gamma\mu} - \Gamma^{\mu}_{\gamma\beta}\, \Delta\Gamma^{\gamma}_{\alpha\mu} - \Delta\Gamma^{\mu}_{\gamma\beta} \cdot \Gamma^{\gamma}_{\alpha\mu} \right) \qquad (8) \\
&+ \Sigma \left(\Delta\Gamma^{\mu}_{\alpha\beta}\, \Delta\Gamma^{\gamma}_{\gamma\mu} - \Delta\Gamma^{\mu}_{\gamma\beta}\, \Delta\Gamma^{\gamma}_{\alpha\mu} \right) .
\end{aligned}$$

Choose $f : \mathbb{R} \longrightarrow \mathbb{R}$ s.t $f \in C^{\infty}$, and $0 \leq f \leq 1$, $f \equiv 1$ on $(-\infty, 1]$, $f \equiv 0$ on $[2, +\infty)$. Let $N > 0$, such that

$$|f'(s)| \leq N , \quad |f''(s)| \leq N$$

and let $g_\delta(s) = f(\frac{s}{\delta})$, then $0 \leq g_\delta(s) \leq 1$,

and

$$|g_\delta'(s)| \leq \frac{N}{\delta} , \quad |g_\delta''(s)| \leq \frac{N}{\delta^2} \quad . \tag{9}$$

We take

$$\psi(x) = g_\delta(|x|^\lambda) \in C^\infty$$

where $0 < \lambda < 1$, we will determine λ later. Therefore

$$\frac{\partial \psi}{\partial x^i} = g_\delta'(|x|^\lambda) \cdot \lambda |x|^{\lambda-1} \frac{x^i}{|x|}$$

and

$$|\nabla \psi| = \lambda \left| g_\delta'(|x|^\lambda) \right| |x|^{\lambda-1} \leq \lambda \cdot \frac{N}{\delta} \cdot \frac{1}{|x|^{1-\lambda}} \tag{10}$$

$$\frac{\partial^2 \psi}{\partial x^i \partial x^j} = \lambda^2 g_\delta''(|x|^\lambda) \cdot |x|^{2\lambda-2} \frac{x^i x^j}{|x|^2} + \lambda g_\delta'(|x|^\lambda) \frac{\delta_{ij}}{|x|^{2-\lambda}}$$

$$+ \lambda(\lambda-2) g_\delta'(|x|^\lambda) \frac{x^i x^j}{|x|^{4-\lambda}}$$

$$\frac{\partial^2 \psi}{\partial x^{n+1} \partial x^\alpha} \equiv 0$$

hence

$$\left|\frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta}\right| \leq \lambda^2 \cdot \frac{N}{\delta^2} \cdot \frac{1}{|x|^{2-2\lambda}} + \lambda \cdot \frac{N}{\delta} \cdot \frac{1}{|x|^{2-\lambda}}$$

$$+ 2\lambda \cdot \frac{N}{\delta} \cdot \frac{1}{|x|^{2-\lambda}}$$

$$= \lambda^2 \cdot \frac{N}{\delta^2} \cdot \frac{1}{|x|^{2-2\lambda}} + 3 \cdot \lambda \cdot \frac{N}{\delta} \cdot \frac{1}{|x|^{2-\lambda}} \quad . \tag{11}$$

We can assume $2\delta \leq \frac{1}{2}\bar{\rho} \leq \frac{1}{2}$. By (6), we have

$$\frac{\partial \Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\nu}} = \frac{1}{2}\left(\frac{\partial g^{\gamma\mu}}{\partial x^{\nu}} + \dot{g}^{\gamma\mu} \cdot \frac{\partial \psi}{\partial x^{\nu}}\right) \left(\dot{g}_{\alpha\mu} \frac{\partial \psi}{\partial x^{\beta}} + \dot{g}_{\beta\mu} \frac{\partial \psi}{\partial x^{\alpha}} - \dot{g}_{\alpha\beta} \frac{\partial \psi}{\partial x^{\mu}} \right)$$

$$+ \frac{1}{2} g^{\gamma\mu} \left[\left(\frac{\partial \dot{g}_{\alpha\mu}}{\partial x^{\nu}} + \ddot{g}_{\alpha\mu} \cdot \frac{\partial \psi}{\partial x^{\nu}} \right) \frac{\partial \psi}{\partial x^{\beta}} + \dot{g}_{\alpha\mu} \frac{\partial^2 \psi}{\partial x^{\beta} \partial x^{\nu}} + \right.$$

$$+ \left(\frac{\partial \dot{g}_{\beta\mu}}{\partial x^{\nu}} + \ddot{g}_{\beta\mu} \frac{\partial \psi}{\partial x^{\nu}} \right) \frac{\partial \psi}{\partial x^{\alpha}} + \dot{g}_{\beta\mu} \frac{\partial^2 \psi}{\partial x^{\alpha} \partial x^{\nu}}$$

$$\left. - \left(\frac{\partial \dot{g}_{\alpha\beta}}{\partial x^{\nu}} + \ddot{g}_{\alpha\beta} \frac{\partial \psi}{\partial x^{\nu}} \right) \frac{\partial \psi}{\partial x^{\mu}} - \dot{g}_{\alpha\beta} \frac{\partial^2 \psi}{\partial x^{\mu} \partial x^{\nu}} \right] . \qquad (12)$$

Hence, by (6), (12), (4) and (10), (11),

$$|\Delta\Gamma^{\gamma}_{\alpha\beta}| \leq \frac{1}{2}(n+1)M \cdot 3M|x|^2 \cdot \lambda \cdot \frac{N}{\delta} \cdot \frac{1}{|x|^{1-\lambda}}$$

$$= \frac{3}{2}(n+1)M^2 N \cdot \frac{\lambda}{\delta}|x|^{1+\lambda} \qquad (13)$$

$$\left|\frac{\partial \Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\nu}}\right| \leq \frac{1}{2}(n+1)\left(M + M|x|^2 \cdot \lambda \cdot \frac{N}{\delta} \frac{1}{|x|^{1-\lambda}} \right) \cdot 3M|x|^2 \cdot \lambda \frac{N}{\delta} \cdot \frac{1}{|x|^{1-\lambda}}$$

$$+ \frac{1}{2}(n+1)M \cdot 3 \left[\left(M|x| + M|x|^2 \cdot \lambda \cdot \frac{N}{\delta}|x|^{\lambda-1} \right) \cdot \lambda \cdot \frac{N}{\delta}|x|^{\lambda-1} + \right.$$

$$\left. + M|x|^2 \cdot \left(\lambda^2 \cdot \frac{N}{\delta^2} \cdot \frac{1}{|x|^{2-2\lambda}} + 3\lambda \cdot \frac{N}{\delta}|x|^{\lambda-2} \right) \right]$$

$$\leq \frac{3}{2}(n+1)M^2 N \cdot \frac{\lambda}{\delta}\left(1 + \lambda \cdot \frac{N}{\delta}|x|^{1+\lambda} \right) \cdot |x|^{1+\lambda} +$$

$$+ \frac{3}{2}(n+1)M^2 N \cdot \frac{\lambda}{\delta}\left(|x|^{\lambda} + N \cdot \frac{\lambda}{\delta}|x|^{2\lambda} + \frac{\lambda}{\delta}|x|^{2\lambda} + 3|x|^{\lambda} \right)$$

we choose $\lambda^{\frac{1}{2}} = \delta \le \frac{1}{4}\bar{\rho} \le \frac{1}{8} < 1$, then for $|x| \le \bar{\rho}$, $|x|^{1+\lambda} \le |x|^{\frac{1}{2}} \le |x|^{\lambda}$,

and
$$|\Delta\Gamma^{\gamma}_{\alpha\beta}| \le \frac{3}{2}(n+1)M^2N\ \delta\ |x|^{1+\lambda} = C_1\delta|x|^{1+\lambda} \tag{14}$$

$$\left|\frac{\partial\Delta\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\nu}}\right| \le \frac{3}{2}(n+1)M^2N\ \delta(1+N\delta|x|^{1+\lambda})|x|^{\lambda} +$$

$$+\frac{3}{2}(n+1)M^2N\ \delta(|x|^{\lambda} + N\ \delta\ |x|^{2\lambda} + \delta\ |x|^{2\lambda} + 3|x|^{\lambda})$$

$$\le \frac{3}{2}(n+1)M^2N\ \delta(1+N\delta)|x|^{\lambda} + \frac{3}{2}(n+1)M^2N\ \delta(4+N\delta+\delta)|x|^{\lambda}$$

$$= \left[\frac{3}{2}(n+1)M^2N(1+N) + \frac{3}{2}(n+1)M^2N(5+N)\right]\delta\ |x|^{\lambda}$$

$$= C_2\delta|x|^{\lambda}\ . \tag{15}$$

Therefore

$$|A_{\alpha\beta}| \le 2(n+1)M|x| \cdot N\frac{\lambda}{\delta}|x|^{\lambda-1} + 2(n+1)C_2\delta\ |x|^{\lambda}$$

$$+ 4(n+1)^2M\cdot C_1\delta|x|^{1+\lambda} + 2(n+1)^2C_1^2\ \delta^2\ |x|^{2+2\lambda}$$

$$\le \Big[2(n+1)MN|x|^{\lambda} + 2(n+1)C_2|x|^{\lambda} + 4(n+1)^2M\ C_1|x|^{1+\lambda}$$

$$+ 2(n+1)^2C_1^2|x|^{2+2\lambda}\Big]\delta$$

$$\le \Big[2(n+1)MN + 2(n+1)C_2 + 4(n+1)^2M\ C_1 +$$

$$+ 2(n+1)^2\ C_1^2\Big]\delta\ |x|^{\lambda}$$

$$= C_3\ \delta\ |x|^{\lambda} = C_3\lambda^{\frac{1}{2}}|x|^{\lambda}\ . \tag{16}$$

We now take ρ_1 sufficiently small such that

$$2(n+1)C_3 \frac{1}{4} \rho_1 \leq \frac{C}{2} \qquad \text{(see (5))}$$

and $\rho_1 < \bar{\rho}$, then, let $\delta = \frac{1}{4} \rho_1 \leq \frac{1}{4}$,

and hence, for $|x| \leq \rho_1$,

$$\begin{aligned}
|\Sigma A_{\alpha\beta} X^\alpha X^\beta| &\leq C_3 \delta |x|^\lambda \Sigma |X^\alpha X^\beta| \\
&\leq 2\, C_3 |x|^\lambda \delta \sum_{\alpha,\beta} (X^\alpha)^2 + (X^\beta)^2 \\
&= 2(n+1) C_3 |x|^\lambda \delta |X|^2 \\
&\leq 2(n+1) C_3 \delta |X|^2 = 2(n+1) C_3 \frac{1}{4} \rho_1 |X|^2 \\
&\leq \frac{C}{2} |X|^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
&\Sigma \operatorname{Ric}(\bar{g})_{\alpha\beta} X^\alpha X^\beta \\
&\leq \Sigma \operatorname{Ric}(g_s)_{\alpha\beta} X^\alpha X^\beta + |\Sigma A_{\alpha\beta} X^\alpha X^\beta| \\
&\leq -C|X|^2 + \frac{C}{2}|X|^2 \leq -\frac{C}{2}|X|^2 .
\end{aligned}$$

Therefore, for $|x| \leq \rho_1$, we have

$$\operatorname{Ric}(\bar{g}) < 0 , \quad \text{and} \quad \bar{g} \in \operatorname{Ric}\_(V_{\rho_1}) \ (\text{or } \in \operatorname{Ric}\_(\cup(0,\rho_1)))$$

Since $\psi(x) \equiv 1$ for $|x|^\lambda \leq \delta = \frac{1}{4} \rho_1$ so if

$|x| \leq (\frac{1}{4}\rho_1)^{\frac{1}{(\frac{1}{4}\rho_1)^2}} \leq (\frac{1}{4}\rho_1)^{16} \leq \frac{1}{4}\rho_1 < \rho_1 < 1$; then

$\bar{g} = g_1$, and if $(\frac{1}{2}\rho_1)^{\frac{1}{(\frac{1}{4}\rho_1)^2}} \leq |x| < \rho_1$, then $|x|^{\lambda} = |x|^{(\frac{1}{4}\rho_1)^2} \geq \frac{1}{2}\rho_1 = 2\delta$,

and $\psi(x) \equiv 0$, $\bar{g} = g$. Hence $\operatorname{supp}\psi \subset U(0,\rho_0)$, and we can take

$\rho_2 = (\frac{1}{4}\rho_1)^{\frac{1}{(\frac{1}{4}\rho_1)^2}}$ and $\bar{g}$ is the desired metric.

Let X be a manifold without boundary,

$\mathcal{M}(X) = \{$all complete metrics on X with negative Ricci curvature$\}$.

Let h be the standard hyperbolic metric on $U(0,\rho_0)$, i.e.,

$$h = dr^2 + \sinh^2 r \, d\theta^2 \tag{17}$$

where $d\theta^2$ is the standard metric on sphere $S^n = \{r = |x| = 1\}$, and $n+1 = \dim X$.

Proposition 2.4. Let $g \in \mathcal{M}(X)$, and let $p \in X$. Given any normal coordinate neighborhood $U_{\rho_0}(p) = \{|x| < \rho_0\}$ of p , then there is a metric $g_0 \in \mathcal{M}(X)$, and $\eta > 0$; such that g_0 has form (17) on $U_\eta(p) = \{|x| < \eta\}$, i.e., $g_0 = h$ for $|x| < \eta$.

Proof: Since at p , $J^1(g) = J^1(h) = 0$, Proposition 2.4 follows from Proposition 2.1.

Let $\bar{h}$ be the standard hyperbolic metric on $W_{\rho_0} = U(0,\rho_0) \times \mathbb{R}$, i.e.,

$$\bar{h} = \cosh^2 r \, dt^2 + dr^2 + \sinh^2 r \, d\theta^2 \tag{18}$$

and let $A \in O(n)$. Consider the diffeomorphism

$$(x,t) \longrightarrow (Ax\,,\,t+1) : W_{\rho_0} \xrightarrow{A} W_{\rho_0}\ .$$

It is easy to see that $A : (W_{\rho_0}, \bar{h}) \longrightarrow (W_{\rho_0}, \bar{h})$ is an isometry; hence the quotient space $Q_{\rho_0}(A) = (W_{\rho_0}, \bar{h})/\{A\}$ is a disk bundle over S^1 with the induced hyperbolic metric h_A from $\bar{h}$. Note, if $A \in SO(n)$, then $Q_{\rho_0}(A)$ is diffeomorphic to V_{ρ_0}.

Proposition 2.5. Let $g \in \mathcal{M}(X)$, and $\gamma \subseteq X$ be a simple closed geodesic. Then there exists a metric $g_0 \in \mathcal{M}(X)$, such that for some $A \in O(n)$, $g_0 = h_A$ on a tubular neighborhood $Q_\eta(A)$ of γ in X.

Proof: Fix an initial point p on γ, then γ induces an isometry A on $N_p(\gamma) = \{v \in T_pX \mid v \perp \gamma'(p)\} = \mathbb{R}^n$ by parallel translation along γ. Choose small ρ_0, and tubular neighborhood which can be identified to $Q_{\rho_0}(A)$ by a parallel translation along γ. It is clear that $J^1(g) = J^1(h_A)$ along γ. Proposition 2.5 follows from Proposition 2.1.

Now we are ready to prove Theorem 1. By Proposition 2.4, the proof of this theorem is reduced essentially to proving that the connected sum of two hyperbolic manifolds has a metric with negative Ricci curvature. This last fact was known to S. T. Yau, but it has never appeared. For completeness, we give a proof here.

Lemma 2.6. Let $\varphi \in C^\infty([0,\infty))$, and $\eta > 0$. Then there is a C^∞ function g on $(-\infty,\infty)$, such that $g(t) = \varphi(t)$ for $t \geq \eta$, and $g(t) = 0$, for $t \leq 0$. Furthermore, if φ is monotone, increasing, then g is monotone increasing.

Proof: For such $\eta > 0$, there is a C^∞-function $k(t)$ on $\mathbb{R}$, such that $k(t) = 0$, for $t \leq 0$; $k(t) = 1$ for $t \geq \eta$, and $k(t)$ is increasing on $[0,\eta]$. Define $g(t)$ by

$$g(t) = \begin{cases} k(t)\varphi(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then g is C^∞, and has the desired property in Lemma 2.6.

Lemma 2.7. There exists a C^∞-function f on $[0,\infty)$ such that

$$f f'' + Mf'^2 > M \tag{19}$$

on $[0,\infty)$ for any given integer $M \geq 0$, and $f'(0) = 0$, $f''(0) > 0$, $f(0) = \varepsilon > 0$; $f'(r) > 0$, $f''(r) \geq 0$ for $r > 0$; and $f^{(n)}(0) = 0$ for $n \geq 3$. Moreover, for any given $\eta > 0$, we can require $f(r) = \sinh r$ for $r \geq \eta$.

Proof:

Let Δ be the region EACDE. Choose point B in Δ near D. Smooth loop CBAE at corners B and A; by Lemma B, we get a C^∞-function $h(r) \le \cosh r$ and $h(r) = \cosh r$ for $r \ge \eta$ and h is increasing. The shadow region has area ε_1,

$$\varepsilon_1 = \int_{\frac{\eta}{2}}^{\eta} (\cosh r - h(r))dr > 0 \quad .$$

Choose a ray, starting from 0 which intersects with CB at F. Suppose ray OF has slope k such that $k\,\varepsilon_1 > M$ and $\frac{1}{k} < \frac{\eta}{2}$. Smoothing curve OFhE at F, by Lemma 2.6 we get a C^∞ increasing function $g(r)$ such that $g(r) = kr$ for $0 \le r \le \frac{1}{k}$ and $g(r) = h(r)$ for $r \ge \frac{\eta}{2}$; $g(r) \le \cosh r$. Let $f(r) = \int_0^r g(r)dr + \varepsilon$, where

$$\varepsilon = \int_0^{\eta} (\cosh r - g(r))dr \ge \varepsilon_1 \quad .$$

Then $f'(r) = g(r)$ and $g(r) > 1$ for $r > \frac{1}{k}$; $f \ge \varepsilon > 0$, $f''(\frac{1}{k}) = g'(\frac{1}{k}) = k > 0$, $g(\frac{1}{k}) = 1$. Therefore on $[\frac{1}{k}, \infty)$

$$f f'' + Mf'^2 > M \quad .$$

On $[0, \frac{1}{k}]$, $f \ge \varepsilon \ge \varepsilon_1$, $f''(r) = g'(r) = k$, hence $f f'' \ge k\,\varepsilon_1 > M$, and

$$f f'' + Mf'^2 > M \quad .$$

Also we have

$$f(0) = \varepsilon > 0\,, \quad f'(0) = 0\,, \quad f''(0) = k > 0\,,$$

$$f^{(n)}(0) = 0 \quad \text{for} \quad n \ge 3\,.$$

Thus, f is the desired function.

As a corollary, we have:

Corollary 2.8. For any given $\eta > 0$, and integer $M \geq 0$, there is a C^∞ function $f(t)$ on $\mathbb{R}$, such that $f > 0$, $f(-r) = f(r)$, and $f'(r) \geq 0$ for $r \geq 0$; and

$$f f'' + Mf'^2 > M$$

on $\mathbb{R}$, and $f(r) = \sinh r$ for $r \geq \eta$.

Proof of Theorem 1: Suppose X has dimension $n \geq 2$ and a metric with Ricci curvature $\mathrm{Ric} < 0$. Fix $p \in X$ and let $D = \{x \in \mathbb{R}^n : |x| \leq r_0\}$ be a small normal coordinate ball center at p. By Proposition 2.4 we may assume this metric g has the following form:

$$g = dr^2 + \sinh^2 r \, d\theta^2 \quad .$$

We shall change the metric in $D - \{0\}$, while preserving negative Ricci curvature, so that it agrees with the old one near ∂D, and it has the form:

$$g_0 = dr^2 + f^2(r) d\theta^2 \tag{20}$$

where $f(r)$ is the function in Corollary 2.8, and we take $\eta < r_0$. Since $f(r) > 0$, $f'(0) = 0$, $f''(0) = k > 0$, $f^{(\ell)}(0) = 0$ for $\ell \geq 3$, it follows immediately that one can add 1-handles and take connected sums while preserving negative Ricci curvature.

We claim the metric has negative Ricci curvature. Let $\nu = \frac{\partial}{\partial r}$, and $\{\varepsilon_i ; i = 1, \ldots, n-1\}$ be an orthonormal frame on S^{n-1} around a point and let $\ell_i = \frac{1}{f} \varepsilon_i$. Then, by straight calculation, we have

$$\mathrm{Ric}(\nu, \nu) = -(n-1)\frac{f''(r)}{f(r)} < 0$$

$$\mathrm{Ric}(\ell_i, \ell_i) = \frac{1}{f^2}\left[(n-2)-(n-2)f'^2(r) - f(r)f''(r)\right] < 0$$

$$\mathrm{Ric}(\ell_i, \nu) = 0$$

hence, we have, for any unit vector u, we can take ℓ_1 such that $u = \alpha\ell_1 + \beta\nu$, with $\alpha^2 + \beta^2 = 1$ and

$$\mathrm{Ric}(u,u) = \alpha^2\mathrm{Ric}(\ell_1, \ell_1) + \beta^2\mathrm{Ric}(\nu, \nu) < 0$$

which proves the theorem.

Proof of Theorem 2. Let M_1 and M_2 be the given manifolds. Take ε sufficiently small. Using Proposition 2.5, we may assume that the given metric g_i on M_i $(i = 1,2)$ equals h_A on V_i, which is induced by the metric

$$h = \cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\theta^2$$

on $U(0,\varepsilon) \times \mathbb{R}$.

We shall change the metric in $V_i \sim \gamma_i \cong (U(0,\varepsilon) \sim \{0\}) \times S^1$, while preserving negative Ricci curvature, so that it agrees with the given one near ∂V_i, and it is induced from

$$\bar{g}_i = \cosh^2 r\, dt^2 + dr^2 + f^2(r)d\theta^2$$

on $(U(0,\varepsilon) \sim \{0\}) \times \mathbb{R}$, where $f(r)$ is the function in Lemma 2.7, and take $\eta < \varepsilon$. It follows immediately that one can glue these two manifolds together to obtain $M_1 \,\#_\varphi\, M_2$ with negative Ricci curvature metric.

III. The Proof of Theorem 3.

We start with two simple lemmas.

Lemma 3.1. Let g_1, g_2 be two positive C^∞-functions on a neighborhood of r_1 in $\mathbb{R}$ and let $g_1(r_1) = g_2(r_1)$. Suppose also $g_1'(r_1) > g_2'(r_1) \geq 0$; $g_1''(r_1) > g_2''(r_1) \geq 0$. Then, for any $\varepsilon > 0$, there exists a C^∞-function g, such that for some $r_1 - \varepsilon < r_2 < r_1 < r_0 < r_1 + \varepsilon$, we have

$$g(r) = \begin{cases} g_1(r) , & r \geq r_0 \\ g_2(r) , & r \leq r_2 \end{cases}$$

and $g'(r) \geq g_2'(r)$, $g''(r) \geq g_2''(r)$ on $[r_2, r_0]$.

Proof: By hypothesis, we can find $M > 0$, and r_2, ρ such that $r_1 - \varepsilon < r_2 < r_1 < \rho < r_1 + \varepsilon$, and

$$\frac{1}{M} \leq g_1'(r) - g_2'(r) \leq M \tag{21}$$

for $r \in [r_2, \rho]$, and $g_1''(r) - g_2''(r) > 0$.

Since $g_1(r_1) - g_2(r_1) = 0$, there is a $r_0 > r_1$, $r_0 < \rho$, such that

$$\frac{M^2}{r_1 - r_2} \leq \frac{g_1'(r_0) - g_2'(r_0)}{g_1(r_0) - g_2(r_0)} . \tag{22}$$

Let k_δ be a C^∞ function on $\mathbb{R}$, which satisfies:

(a) $0 \leq k_\delta(x) \leq 1$,

(b) $k_\delta(x) \equiv 0$, $x \leq 0$

(c) $k_\delta(x) \equiv 1$, $x \geq 2\delta$

(d) $k_\delta'(x) \geq 0$. on $\mathbb{R}$

Let $\psi_{\varepsilon_1,\delta}(r) = k_\delta(r-(r_0-2\delta-\varepsilon_1))[g_1'(r) - g_2'(r)]$ and

$$\eta = \frac{g_1(r_0) - g_2(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} > 0$$

and let

$$g(r) = g_2(r) + \eta \int_{r_2}^{r} \psi_{\varepsilon_1,\delta}(s)ds \quad .$$

We claim that g is the desired function for small $\delta > 0$ and some $\varepsilon_1 < r_1 - r_2$. Note k_δ is an increasing function and so $g_1' - g_2'$ on $[r_2,r_0]$, hence $\psi_{\varepsilon_1,\delta}$ is increasing on $[r_2,r_0]$. Therefore, we have

$$g'(r) = g_2'(r) + \eta\ \psi_{\varepsilon_1,\eta}(r) \geq g_2'(r)$$

$$g''(r) = g_2''(r) + \eta\ \psi_{\varepsilon_1,\eta}'(r) \geq g_2''(r)$$

on $[r_2,r_1]$.

Since, if $2\eta < r_0 - r_1$, then

$$\frac{\psi_{0,\delta}(r_0)}{\int_{r_2}^{r_0} \psi_{0,\delta}(r)dr} \geq \frac{g_1'(r_0) - g_2'(r_0)}{\int_{r_0-2\delta}^{r_0} [g_1'(r) - g_2'(r)]dr}$$

but

$$\lim_{\delta \to 0^+} \frac{g_1'(r_0) - g_2'(r_0)}{\int_{r_0-2\delta}^{r_0} [g_1'(r) - g_2'(r)]dr} = +\infty$$

We can choose $\delta > 0$, such that $2\delta < r_0 - r_1$, and

$$\frac{\psi_{0,\delta}(r_0)}{\int_{r_2}^{r_0} \psi_{0,\delta}(r)dr} \geq \frac{g_1'(r_0) - g_2'(r_0)}{\int_{r_0-2\delta}^{r_0} [g_1'(r) - g_2'(r)]dr} > \frac{g_1'(r_0) - g_2'(r_0)}{g_1(r_0) - g_2(r_0)} \tag{23}$$

Fix such δ , consider

$$\frac{\psi_{\varepsilon_1,\delta}(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} = \frac{g_1'(r_0) - g_2'(r_0)}{\int_{r_0-2\delta-\varepsilon_1}^{r_0} k_\delta(r-(r_0-2\delta-\varepsilon_1))[g_1'(r)-g_2'(r)]dr}$$

$$\leq \frac{g_1'(r_0)-g_2'(r_0)}{\frac{1}{M}\cdot(\delta+\varepsilon_1)} \leq \frac{M^2}{\delta+\varepsilon_1} \quad . \qquad \text{(by (21))}$$

By (22), if $\varepsilon_1 + \delta > r_1 - r_2$, $\varepsilon < r_1 - r_2$, then

$$\frac{\psi_{\varepsilon_1,\delta}(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} \leq \frac{M^2}{\delta+\varepsilon_1} \leq \frac{M^2}{r_1-r_2} \leq \frac{g_1'(r_0) - g_2'(r_0)}{g_1(r_0) - g_2(r_0)} \quad . \tag{24}$$

Hence, from (23) and (24), there is a $\varepsilon_1 > 0$, $\varepsilon_1 < r_1 - r_2$, such that

$$\frac{\psi_{\varepsilon_1,\delta}(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} = \frac{g_1'(r_0) - g_2'(r_0)}{g_1(r_0) - g_2(r_0)} . \tag{25}$$

By $\psi_{\varepsilon_1,\delta}(r_0) = g_1'(r_0) - g_2'(r_0)$, and (25), we have

$$\eta = \frac{g_1(r_0) - g_2(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} = 1 ,$$

and

$$g'(r) = g_2'(r) + (g_1'(r) - g_2'(r)) = g_1'(r)$$

for $r \geq r_0 - \varepsilon_1$, but

$$g(r_0) = g_2(r_0) + \frac{g_1(r_0)-g_2(r_0)}{\int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr} \cdot \int_{r_2}^{r_0} \psi_{\varepsilon_1,\delta}(r)dr = g_1(r_0)$$

Hence, $g(r) = g_1(r)$ for $r \geq r_0$.

Lemma 3.2. Let g, f be two C^∞ positive functions defined on $(\rho, +\infty)$, such that $g' > 0$, $g'' > 0$; $f' > 0$, $f'' > 0$ on $(\rho, +\infty)$. Then, for any $r_1 > r_2 > \rho$, there are $\delta > 0$ and $s < \rho$, such that, for any $0 < \varepsilon < \delta$, there are $\tilde{g}, \tilde{f} \in C^\infty$ with the following property:

(a) $\tilde{g} > 0$, $\tilde{g}' > 0$, $\tilde{g}'' > 0$; $\tilde{f} > 0$, $\tilde{f}' > 0$, $\tilde{f}'' > 0$ for $r > s$.

(b) $\tilde{g}(s) = \tilde{f}(s) = \varepsilon$, $\tilde{g}'(s) = \tilde{f}'(s) = 0$; $\tilde{g}''(s) = \tilde{f}''(s) > 0$,

$\tilde{g}^{(m)}(s) = \tilde{f}^{(m)}(s) = 0$, for $m \geq 3$.

(c) $\tilde{g}(r) = \tilde{f}(r)$ for r near s .

(d) $\tilde{g}(r) = g(r)$, $\tilde{f}(r) = f(r)$ for $r \geq r_1$.

Proof: Take $\eta < \min\{f(r_2), g(r_2)\}$, and $\eta > 0$.

Consider the functions:

$$h_1 = a_1(r-s_1)^2 + b_1(r-s_1) + \eta$$

$$h_2 = a_2(r-s_1)^2 + b_2(r-s_1) + \eta$$

on $[s_1, +\infty)$, where

$$0 = b_1 = b_2 = \frac{1}{2}\{\min(g(r_2), f(r_2)) - \eta\} \cdot (r_2-s_1)^{-1}$$

$$a_1 = [g(r_2) - b_1(r_2-s_1) - \eta] \cdot (r_2-s_1)^{-2}$$

$$a_2 = [f(r_2) - b_2(r_2-s_1) - \eta] \cdot (r_2-s_1)^{-2} .$$

We choose s_1 such that $s_1 < r_2$ and $2a_1 < g''(r_2)$, $2a_2 < f''(r_2)$, $b_1 + 2a_1(r_2-s_1) < g'(r_2)$; $b_2 + 2a_2(r_2-s_1) < f'(r_2)$. Therefore, we have

$$h_1(r_2) = a_1(r_2-s_1)^2 + b_1(r_2-s_1) + \eta = g(r_2)$$
$$h_1'(r_2) < g'(r_2) , \quad h_1''(r_2) < g''(r_2) .$$
$$h_2(r_2) = a_2(r_2-s_1)^2 + b_2(r_2-s_1) + \eta = f(r_2)$$
$$h_2'(r_2) < f'(r_2) , \quad h_2''(r_2) < f''(r_2) .$$

Using Lemma 1, we can find two C^∞ functions $\tilde{h}_1, \tilde{h}_2$ on $[s_1, +\infty)$, such that

$$\tilde{h}_i > 0 , \quad \tilde{h}_i' > 0 , \quad \tilde{h}_i'' > 0 \quad \text{on} \quad (s_1, +\infty)$$

and

$$\tilde{h}_1(r) = g(r) , \quad \tilde{h}_2(r) = f(r) \quad \text{for} \quad r \geq r$$

and

$$\tilde{h}_i(r) = h_i(r) \quad \text{for } r \text{ near } s_1 .$$

Take $\bar{s} > s_1$ near s_1. Consider $\delta = \eta$,

$$k(r) = a(r-s)^2 + \varepsilon .$$

We choose a and $s < s_1$, such that

$$a = [\eta - \varepsilon](s_1 - s)^{-2}$$

and

$$2a(s_1 - s) < \min \{ \tilde{h}_1'(s_1) , \tilde{h}_2'(s_1) \} =$$

$$2a < \min \{ \tilde{h}_1''(s_1) , \tilde{h}_2''(s_1) \}$$

then

$$k(s_1) = \tilde{h}_1(s_1) = \tilde{h}_2(s_1) = \eta$$

and

$$k'(s_1) < \min \{ \tilde{h}_1'(s_1) ; \tilde{h}_2'(s_1) \}$$

$$k''(s_1) < \min \{ \tilde{h}_1''(s_1) ; \tilde{h}_2''(s_1) \} .$$

Using Lemma 1 to $\{k, \tilde{h}_1\}$ and $\{k, \tilde{h}_2\}$, we obtain two C^∞ functions $\tilde{g}, \tilde{f}$ with the desired properties in Lemma 2.

Proof of Theorem 3: We consider X_1 first. By Proposition 2.5, if we take η small, we can deform the given metric g_1 to a new metric

$\bar{g}_1$ on X_1 with negative Ricci curvature, and $\bar{g}_1 = h_A$ on V_η ; h_A is the metric on V_η as in Proposition 2.5. Since $N_1(\gamma_1)$ is orientable, $A \in SO(2)$. Let

$$A = \begin{pmatrix} \cos\alpha , & \sin\alpha \\ \sin\alpha , & \cos\alpha \end{pmatrix}$$

and let $\{\varepsilon_1(t) , \varepsilon_2(t)\}$ be the orthonormal parallel frame of $N_1(\gamma_1)$, then we have

$$\begin{pmatrix} \varepsilon_1(1) \\ \varepsilon_2(1) \end{pmatrix} = A \begin{pmatrix} \varepsilon_1(0) \\ \varepsilon_2(0) \end{pmatrix}$$

Let

$$\begin{pmatrix} \ell_1(t) \\ \ell_2(t) \end{pmatrix} = \begin{pmatrix} \cos t\alpha , & -\sin t\alpha \\ \sin t\alpha , & \cos t\alpha \end{pmatrix}^{-1} . \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix}$$

then $\{\ell_1(t) , \ell_2(t)\}$ is a smooth orthonormal frame of $N_1(\gamma_1)$.

We have

$$V_\eta = \left\{ \sum_{i=1}^{2} x_i \ell_i(t) \,\middle|\, r = \Sigma (X_i)^2 < \eta \right\}$$

We use exponential map exp to identify V_η with $\{x | d(x , \gamma) < \eta\}$. Let $\{r , \theta\}$ be the polar coordinate on coordinate plane $\{x_1 , x_2\}$, then we have

$$\begin{aligned} h_A &= \cosh^2 r \; dt^2 + dr^2 + \sinh^2 r (d\theta + \alpha dt)^2 \\ &= (\cosh^2 r + \alpha^2 \sinh^2 r) dt^2 + dr^2 + \sinh^2 r \; d\theta^2 + \qquad (26) \\ &\qquad + 2\alpha \sinh^2 r \; d\theta dt \quad . \end{aligned}$$

Consider h_A defined on $V_\eta \sim \gamma_1 \cong T^2 \times (0, \eta)$. We will pull the metric out, such that it defined on $T^2 \times (-L, \eta)$ for large L , which agree with h_A near $T^2 \times \{\eta\}$. Let $\varphi = \theta + t\alpha$, then

$$h_A = \cosh^2 r \; dt^2 + dr^2 + \sinh^2 r \; d\varphi^2 \quad .$$

Consider the metric

$$ds^2 = g^2(r)dt^2 + dr^2 + f^2(r)d\varphi^2 \quad . \tag{27}$$

Note $\{\varphi, t, r\}$ is not global coordinate on $T^2 \times (-\infty, \eta)$, since for $t = 1$, $\varphi = 0 + \alpha$ and $t = 0$, $\varphi = \theta$. Nevertheless, we still can use this coordinate. We have for metric (27),

$$\begin{aligned}
&\mathrm{Ric}\left(\frac{1}{f}\frac{\partial}{\partial\varphi}, \frac{1}{f}\frac{\partial}{\partial\varphi}\right) = -\frac{f''}{f} - \frac{f'}{f}\frac{g'}{g} \\
&\mathrm{Ric}\left(\frac{1}{g}\frac{\partial}{\partial t}, \frac{1}{g}\frac{\partial}{\partial t}\right) = -\frac{f'}{f}\frac{g'}{g} - \frac{g''}{g} \\
&\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -\frac{f''}{f} - \frac{g''}{g} \\
&\mathrm{Ric}\left(\frac{1}{f}\frac{\partial}{\partial\varphi}, \frac{1}{g}\frac{\partial}{\partial t}\right) = 0 \;, \quad \mathrm{Ric}\left(\frac{1}{f}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial r}\right) = 0 \\
&\mathrm{Ric}\left(\frac{1}{g}\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right) = 0
\end{aligned} \tag{28}$$

Using Lemma 3.2, for any $\delta > 0$, $\delta < \eta$, there is a $s < 0$, and two C^∞ functions f, g , such that

(a) $f > 0$, $g > 0$; $f' > 0$, $g' > 0$; $f'' > 0$, $g'' > 0$.

(b) $f(r) = \sinh r$, $g(r) = \cosh r$ for $r \geq \delta$.

(c) $f(r) = g(r)$ for r near s .

Take $s_1 > s$, such that $f(r) = g(r)$ for $s \leq r \leq s_1$. Consider the metric

$$
\begin{aligned}
ds^2 &= f^2(r)dt^2 + dr^2 + f^2(r)d(\theta + t\alpha)^2 \\
&= (1+\alpha^2)f^2(r)dt^2 + dr^2 + f^2(r)d\theta^2 \\
&\quad + \alpha\varphi(r)f^2(r)d\theta dt + \alpha\varphi(r)f^2(r)dtd\theta
\end{aligned} \tag{29}
$$

and let $\varepsilon_1 = \frac{1}{f}\frac{\partial}{\partial\theta}$, $\varepsilon_2 = \frac{1}{\sqrt{1+\alpha^2}\cdot f} \cdot \frac{\partial}{\partial t}$, $\varepsilon_3 = \frac{\partial}{\partial r}$.

Straightforward but rather lengthy calculation shows that

$$\left\langle R\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\right\rangle = -(1+\alpha^2)f^2f'^2 + \frac{1}{4}\alpha^2\varphi'^2f^4 + \alpha^2\varphi\varphi'f^3f' + \alpha^2\varphi^2f^2f'^2$$

$$\left\langle R\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right\rangle = -ff'' + \frac{\frac{1}{4}\alpha^2\varphi'^2f^2}{1+\alpha^2(1-\varphi^2)}$$

$$\left\langle R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right\rangle = -(1+\alpha^2)ff'' + \frac{\frac{1}{4}\alpha^2(1+\alpha^2)\varphi'^2f^2}{1+\alpha^2(1-\varphi^2)}$$

$$\left\langle R\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right\rangle = -\alpha\varphi ff'' - \frac{1}{2}\alpha\varphi''f^2 - \alpha\varphi'ff' - \frac{\frac{1}{4}\alpha^3\varphi\varphi'^2f^2}{1+\alpha^2(1-\varphi^2)}$$

$$\left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right) \frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right\rangle = 0$$

$$\left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\right\rangle = 0$$

and

$$\mathrm{Ric}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = \frac{1}{[1+\alpha^2(1-\varphi^2)]f^2}\left\langle R\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}\right\rangle + \left\langle R\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right\rangle$$

$$= - f'^2 + \frac{\alpha^2\varphi\varphi' f f'}{(1+\alpha^2(1-\varphi^2))} + \frac{\frac{1}{4}\alpha^2\varphi'^2 f^2}{1+\alpha^2(1-\varphi^2)} - f f'' + \frac{\frac{1}{4}\alpha^2\varphi'^2 f^2}{1+\alpha^2(1-\varphi^2)}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{1+\alpha^2}{[1+\alpha^2(1-\varphi^2)]f^2} \left\langle R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right\rangle + \left\langle R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right\rangle$$

$$= -(1+\alpha^2) f'^2 + \frac{\alpha^2(1+\alpha^2)\varphi\varphi' f f'}{1+\alpha^2(1-\varphi^2)} + \frac{\frac{1}{4}\alpha^2(1+\alpha^2)\varphi'^2 f^2}{1+\alpha^2(1-\varphi^2)}$$

$$-(1+\alpha^2 f f'' + \frac{\frac{1}{4}\alpha^2(1+\alpha^2)\varphi'^2}{1+\alpha^2(1-\varphi^2)} f^2$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right) = \left\langle R\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right\rangle$$

$$= -\alpha\varphi f f'' - \frac{1}{2}\alpha\varphi'' f^2 - \alpha\varphi' f f' - \frac{\frac{1}{4}\alpha^3\varphi\varphi'^2 f^2}{1+\alpha^2(1-\varphi^2)}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \frac{1+\alpha^2}{[1+\alpha^2(1-\varphi^2)]f^2} \left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta}, \frac{\partial}{\partial r}\right\rangle$$

$$+ \frac{-2\alpha\varphi}{[1+\alpha^2(1-\varphi^2)]f^2} \left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right\rangle$$

$$+ \frac{1}{[1+\alpha^2(1-\varphi^2)]f^2} \left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial r}\right\rangle$$

$$= -2\frac{f''}{f} + \frac{\frac{1}{2}\alpha^2[(1+\alpha^2)+\alpha^2\varphi^2]\varphi'^2}{[1+\alpha^2(1-\varphi^2)]^2} + \frac{\alpha^2\varphi\varphi''}{1+\alpha^2(1-\varphi^2)}$$

$$+ \frac{2\alpha^2\varphi\varphi'}{1+\alpha^2(1-\varphi^2)} \cdot \frac{f'}{f}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = \left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}\right\rangle \cdot \frac{1}{[1+\alpha^2(1-\varphi^2)]f^2} = 0$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right) = \left\langle R\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right\rangle \cdot \frac{1+\alpha^2}{[1+\alpha^2(1-\varphi^2)]f^2} = 0 \quad .$$

Therefore,

$$\mathrm{Ric}(\varepsilon_1, \varepsilon_1) = -\frac{f'^2}{f^2} - \frac{f''}{f} + \frac{\alpha^2\varphi\varphi'}{1+\alpha^2(1-\varphi^2)} \cdot \frac{f'}{f} + \frac{\frac{1}{2}\alpha^2\varphi'^2}{1+\alpha^2(1-\varphi^2)}$$

$$\mathrm{Ric}(\varepsilon_2, \varepsilon_2) = -\frac{f'^2}{f^2} - \frac{f''}{f} + \frac{\alpha^2(1+\alpha^2)\varphi\varphi'}{1+\alpha^2(1-\varphi^2)} \cdot \frac{f'}{f} + \frac{\frac{1}{2}\alpha^2(1+\alpha^2)\varphi'^2}{1+\alpha^2(1-\varphi^2)}$$

$$\mathrm{Ric}(\varepsilon_1, \varepsilon_2) = -\frac{\alpha\varphi}{\sqrt{1+\alpha^2}} \cdot \frac{f''}{f} - \frac{\frac{1}{2}\alpha\varphi''}{\sqrt{1+\alpha^2}} - \frac{\alpha\varphi'}{\sqrt{1+\alpha^2}} \cdot \frac{f'}{f}$$

$$- \frac{\frac{1}{4}\alpha^3\varphi\varphi'^2}{\sqrt{1+\alpha^2}[1+\alpha^2(1-\varphi^2)]}$$

$$\mathrm{Ric}(\varepsilon_3, \varepsilon_3) = -2\frac{f''}{f} + \frac{\frac{1}{2}\alpha^2[(1+\alpha^2)+\alpha^2\varphi^2]\varphi'^2}{[1+\alpha^2(1-\varphi^2)]^2} + \frac{\alpha^2\varphi\varphi''}{1+\alpha^2(1-\varphi^2}$$

$$+ \frac{2\alpha^2\varphi\varphi'}{1+\alpha^2(1-\varphi^2)} \cdot \frac{f'}{f}$$

$$\mathrm{Ric}(\varepsilon_1, \varepsilon_3) = 0 \quad ; \qquad \mathrm{Ric}(\varepsilon_2, \varepsilon_3) = 0 \quad .$$

We know that metric (29) is defined on $[s, s_1]$. Now we shall extend this metric to $-(N-1) \leq r \leq s_1$ for large N such that for $r < s_1$ near s_1, this metric is the same with (27), and for $r \geq -(N-1)$ near $-(N-1)$, we have $\varphi = 0$ and

$$ds^2 = (1+\alpha^2)f^2(r)dt^2 + dr^2 + f^2(r)d\theta^2 \quad . \tag{30}$$

Take $s_2 \in (s, s_1)$, and $k > 0$, such that

$$2k = \min \left\{ \frac{f'(s_2)}{f(s_2)} \quad ; \quad \sqrt{\frac{f''(s_2)}{f(s_2)}} \right\} \quad . \tag{31}$$

Choose a C^∞ function φ such that $0 \leq \varphi(r) \leq 1$, and $\varphi(r) \equiv 1$ for $r \geq s$; $\varphi(r) \equiv 0$ for large negative γ, and

$$[(1+\alpha^2)^2|\varphi'|\cdot(k+M) + (1+\alpha^2)|\varphi''| +$$
$$+ 2\alpha^2(1+\alpha^2)\varphi'^2] < \frac{1}{2} k^2 \left(1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}}\right) . \tag{32}$$

Let N be a large number, such that if $r \leq -(N-2)$ then $\varphi(r) \equiv 0$. Consider

$$f_1(r) = \delta \cosh(k(r+N))$$

where $\delta = \dfrac{f(s_2)}{\cosh(k(s_2+N))}$.

Take $\varepsilon > 0$ very small, such that

$$\frac{f'(s_2+\varepsilon)}{f_1(s_2-\varepsilon)} < k+M , \quad \frac{f_1''(r)}{f(s_2+\varepsilon)} > \frac{1}{2}k^2 \tag{33}$$

for $r \in [s_2 - \varepsilon , s_2 + \varepsilon]$, where $M = \dfrac{f'(s_2)}{f(s_2)}$.

We note that, from (31), we have

$$f_1(s_2) = f(s_2) , \quad f_1'(s_2) < f'(s_2) , \quad f_1''(s_2) < f''(s_2) \quad .$$

Using Lemma 3.1, we can find a C^∞ function $\tilde{f}$, which satisfies the following:

(a) $\tilde{f}(r) = f_1(r)$ for $r \leq s_2 - \varepsilon$; $\tilde{f}(r) = f(r)$ for $r \geq s_2 + \varepsilon$

(b) $\tilde{f}(r) > 0$; $\tilde{f}'(r) > 0$, $\tilde{f}''(r) > 0$, on $(-N, s_1]$

(c) $\tilde{f}'(r) \geq f_1'(r)$; $\tilde{f}''(r) \geq f_1''(r)$ on $[s_2 - \varepsilon, s_2 + \varepsilon]$.

From (32), we have

$$\frac{\tilde{f}'(r)}{\tilde{f}(r)} \leq \frac{\tilde{f}'(s_2 + \varepsilon)}{\tilde{f}(s_2 - \varepsilon)} = \frac{f'(s_2 + \varepsilon)}{f_1(s_2 - \varepsilon)} < k + M$$

$$\frac{\tilde{f}''(r)}{\tilde{f}(r)} \geq \frac{f_1''(r)}{\tilde{f}(s_2 + \varepsilon)} = \frac{f_1''(r)}{f(s_2 + \varepsilon)} > \frac{1}{2} k^2 \tag{34}$$

for $r \in [s_2 - \varepsilon , s_2 + \varepsilon]$,

and

$$\frac{\tilde{f}'(r)}{\tilde{f}(r)} = k \cdot \frac{\sinh(k(r+N))}{\cosh(k(r+N))} \leq k < k + M$$

$$\frac{\tilde{f}''(r)}{\tilde{f}(r)} = k^2 > \frac{1}{2} k^2 \tag{35}$$

for $r \leq s_2 - \varepsilon$.

In (29), substituting f by $\tilde{f}$, we have that, if $\varepsilon = \xi_1 \varepsilon_1 + \xi_2 \varepsilon_2 + \xi_3 \varepsilon_3$, then

$$\mathrm{Ric}(\varepsilon, \varepsilon) = \xi_1^2 \mathrm{Ric}(\varepsilon_1, \varepsilon_1) + 2\xi_1\xi_2 \mathrm{Ric}(\varepsilon_1, \varepsilon_2) + \xi_2^2 \mathrm{Ric}(\varepsilon_2, \varepsilon_2) + \\ + \xi_3^2 \mathrm{Ric}(\varepsilon_3, \varepsilon_3)$$

$$= -\frac{f''}{f}\left[\xi_1^{\,2}+\xi_2^{\,2}+\frac{\alpha\varphi}{\sqrt{1+\alpha^2}}\cdot 2\xi_1\xi_2+\xi_3^{\,2}\right]-\frac{f'^2}{f^2}(\xi_1^{\,2}+\xi_2^{\,2})$$

$$-\frac{f''}{f}\cdot\xi_3^{\,2}+\left(\frac{\alpha^2\varphi\varphi'}{1+\alpha^2(1-\varphi^2)}\cdot\frac{f'}{f}+\frac{\frac{1}{2}\alpha^2\varphi'^2}{1+\alpha^2(1-\varphi^2)}\right)\xi_1^{\,2}$$

$$+\left[\frac{\alpha^2(1+\alpha^2)\varphi\varphi'}{1+\alpha^2(1-\varphi^2)}\cdot\frac{f'}{f}+\frac{\frac{1}{2}\alpha^2(1+\alpha^2)\varphi'^2}{1+\alpha^2(1-\varphi^2)}\right]\xi_2^{\,2}$$

$$+\left[-\frac{\frac{1}{2}\alpha\varphi''}{\sqrt{1+\alpha^2}}-\frac{\alpha\varphi'}{\sqrt{1+\alpha^2}}\cdot\frac{f'}{f}-\frac{\frac{1}{4}\alpha^3\varphi\varphi'^2}{\sqrt{1+\alpha^2}[1+\alpha^2(1-\varphi^2)]}\right]\cdot 2\xi_1\xi_2$$

$$+\left[\frac{\frac{1}{2}\alpha^2(1+\alpha^2)+\alpha^2\varphi^2)\varphi'^2}{(1+\alpha^2(1-\varphi^2))^2}+\frac{\alpha^2\varphi\varphi''}{1+\alpha^2(1-\varphi^2)}+\frac{2\alpha^2\varphi\varphi'}{1+\alpha^2(1-\varphi^2)}\cdot\frac{f'}{f}\right]\xi_3^{\,2}$$

Hence, for $r \leq s_2 + \varepsilon$, because of (34), (35), and the facts, $0 \leq \varphi \leq 1$, $|2\xi_1\xi_2| \leq \xi_1^{\,2}+\xi_2^{\,2}$, we obtain the following estimates:

$$\mathrm{Ric}(\varepsilon,\varepsilon) \leq -\frac{f''}{f}\left(1-\frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)|\xi|^2-\frac{f'^2}{f^2}(\xi_1^{\,2}+\xi_2^{\,2})$$

$$-\frac{f''}{f}\xi_3^{\,2}+(\alpha^2|\varphi'|\cdot\frac{f'}{f}+\frac{1}{2}\alpha^2\varphi'^2)\xi_1^{\,2}$$

$$+(\alpha^2(1+\alpha^2)|\varphi'|\cdot\frac{f'}{f}+\frac{1}{2}\alpha^2(1+\alpha^2)\varphi'^2)\xi_2^{\,2}$$

$$+(\tfrac{1}{2}|\varphi''|+|\varphi'|\cdot\frac{f'}{f}+\frac{1}{4}\alpha^2\varphi'^2)(\xi_1^{\,2}+\xi_2^{\,2})$$

$$+[\tfrac{1}{2}\alpha^2(1+2\alpha^2)\varphi'^2+\alpha^2|\varphi''|+2\alpha^2|\varphi'|\cdot\frac{f'}{f}]\xi_3^{\,2}$$

$$\leq -\frac{1}{2}k^2\left(1-\frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)|\xi|^2-\frac{f'^2}{f^2}(\xi_1^{\,2}+\xi_2^{\,2})-\frac{f''}{f}\xi_3^{\,2}$$

$$+ (\alpha^2|\varphi'|\cdot(k+M) + \frac{1}{2}\alpha^2\varphi'^2 + \alpha^2(1+\alpha^2)|\varphi'|(k+M) +$$

$$+ \frac{1}{2}\varphi^2(1+\varphi^2)\alpha'^2)(\xi_1^2 + \xi_2^2)$$

$$+ (\frac{1}{2}|\alpha''| + |\alpha'|(k+M) + \frac{1}{4}\alpha^2\varphi'^2)(\xi_1^2 + \xi_2^2)$$

$$+ [\frac{1}{2}\alpha^2(1+2\alpha^2)\varphi'^2 + \alpha^2|\varphi''| + 2\alpha^2|\varphi'|(k+M)]\xi_3^2$$

$$\leq -\frac{1}{2}k^2\left(1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)|\xi|^2 - \frac{f'^2}{f^2}(\xi_1^2 + \xi_2^2) - \frac{f''}{f}\xi_3^2$$

$$+ [(1+\alpha^2)^2|\varphi'|(k+M) + (1+\alpha^2)|\varphi''| + 2\alpha^2(1+\alpha^2)\varphi'^2]|\xi|^2$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

Now applying (32), we have

$$\mathrm{Ric}(\varepsilon, \varepsilon) < -\frac{1}{2}k^2\left(1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)|\xi|^2 - \frac{f'^2}{f^2}(\xi_1^2 + \xi_2^2) - \frac{f''}{f}\xi_3^2$$

$$+ \frac{1}{2}k^2\left(1 - \frac{|\alpha|}{\sqrt{1+\alpha^2}}\right)|\xi|^2 \tag{36}$$

$$= -\frac{f'^2}{f^2}(\xi_1^2 + \xi_2^2) - \frac{f''}{f}\xi_3^2 < 0 .$$

From all of the above, we have found a metric h on $(-N, \eta] \times T^2$, which agrees with $\overline{g}_1 = h_A$ near $\{\eta\} \times T^2$, and equals

$$d\tilde{s} = (1+\alpha^2)\tilde{f}(r)^2dt^2 + dr^2 + \tilde{f}(r)^2d\theta^2 \tag{37}$$

on $(-N, -(N-2)) \times T^2$, and h has negative Ricci curvature.

Since $d\theta^2$ is the metric on S^1 with length 2π, let $d\bar{\theta}^2 = \frac{1}{2\pi} d\theta^2$, then $d\bar{\theta}^2$ is the metric on S^1 with length 1, also dt^2 is the metric on S^1 with length 1, and

$$d\tilde{s} = (1+\alpha^2)\tilde{f}(r)^2 dt + dr^2 + \tilde{f}(r)^2 \cdot \frac{1}{2\pi} d\bar{\theta}^2 \quad .$$

Using Lemma 3.2, we can find C^∞ functions $\tilde{g}_i$ defined on $(-N_1, -(N-2))$ for $N_1 - 2 > N$, such that

$$\tilde{g}_1(r) = \tilde{g}_2(r) \quad \text{for} \quad r \in (-N_1 ; -(N_1 - 2)) \ ,$$

$$\tilde{g}_1(r)^2 = (1+\alpha^2)\tilde{f}(r)^2 \ , \quad \tilde{g}_2(r)^2 = \frac{1}{2\pi}\tilde{f}(r)$$

for $r \in (-(N-1), -(N-2))$, and

$$\tilde{g}_i(r) > 0 \ , \quad \tilde{g}_i'(r) > 0 \ , \quad \tilde{g}_i''(r) > 0$$

on $(-N_1, -(N-2))$, and let $\tilde{g}(r) = \tilde{g}_1(r) = \tilde{g}_2(r)$ for $r \in (-N_1, -(N_1-2))$. For such r, we have the metric

$$d\tilde{s}^2 = \tilde{g}(r)^2 dt^2 + dr^2 + \tilde{g}(r)^2 d\bar{\theta}^2 \quad . \tag{38}$$

Now consider X_2. We can choose polar coordinate $\{\theta_1, t_1 ; r_1\}$ on V_η like we did for V_η above. Using Proposition 2.5, and the above method, we can find a metric on $[-L_1, \eta] \times T^2$ which coincides with the given metric g_2 near $\{\eta\} \times T^2$, and has the form

$$\tilde{g}_2(r)^2 dt_1^2 + dr_1^2 + \tilde{g}_2(r)^2 d\theta_1^2$$

on $[-L_1, -(L_1-2))$ for large $L_1 > 0$, where dt_1^2, $d\theta_1^2$ are metrics on S^1 with length 1.

Let $\varphi : T^2 = S^1 \times S^1 = \{(\bar{\theta}, t)\} \longrightarrow T^2 = \{(\theta_1, t_1)\}$ by any diffeomorphism. Then φ is isotopic to a linear diffeomorphism belonging to $GL(2;\mathbb{Z})$, since the diffeomorphism class of $X_1 \textcircled{\#}_{\varphi} X_2$ only depends on the isotopy class of φ. Hence, we can assume $\varphi \in GL(2;\mathbb{Z})$ and

$$\varphi^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2;\mathbb{Z}) \quad .$$

Let $(\theta_2, t_2) = \varphi \cdot (\bar{\theta}, t)$, then $(\bar{\theta}, t) = \varphi^{-1}(\theta_2, t_2)$ and hence

$$\bar{\theta} = a\theta_2 + bt_2$$

$$t = c\theta_2 + dt_2 \quad .$$

In coordinate $\{\theta_2, t_2, r\}$, metric (18) has the form

$$d\tilde{s}^2 = \tilde{g}(r)^2(b^2+d^2)dt_2^{\,2} + dr^2 + \tilde{g}(r)^2(a^2+c^2)d\theta_2^{\,2} + 2\tilde{g}(r)^2(ad+bc)d\theta_2 dt_2 \tag{39}$$

Using the method above, we can find a metric on $[-L, -(N_1-2)] \times T^2$ which agrees with (39) on $[-(N_1-1), -(N_1-2)]$, and has the form:

$$d\tilde{s}^2 = \tilde{g}_3(r)^2 dt_2^{\,2} + dr^2 + \tilde{g}_3(r)^2 d\theta_2^{\,2}$$

on $[-L, -(L-1)]$.

Note $\varphi : T^2 \longrightarrow T^2$ is the same with $\{\theta_2, t_2\} \longrightarrow \{\theta_1, t_1\}$:

: $\theta_2 \longrightarrow \theta_1$, $t_2 \longrightarrow t_1$.

Using Lemma 3.2, we can assume that

$$g_3(\chi + (L-1)) = g_2(\chi + (L_1-1))$$

for $\chi \in [0,1]$.

Now we can identify $\{-L\} \times \{\theta_2, t_2\}$ with $\{-L_1\} \times \{\theta_1, t_1\}$ to obtain a metric on $X_1 \textcircled{\#}_\varphi X_2$ with negative Ricci curvature, and this completes the proof of Theorem 3.

IV. The Proof of Theorems 4, 5 and 6.

Proof of Theorem 4. In the proof of Theorem 1, since all the procedures are local, instead of two manifolds to start, we start with one given manifold M, and choose two different points p_1, p_2. Take the "connected sum" of M to itself at p_1 and p_2, then we get a manifold which is diffeomorphic to $M \# S^{n-1} \times S^1$, and with negative Ricci curvature metric.

Proof of Theorem 6. Let M be a given 3-manifold with negative Ricci curvature. Take two different points x, y. Applying Proposition 2.4, we can deform the given metric on M near x, y, such that the new metric has the form

$$g = dr^2 + \sinh^2 r \, d\theta^2$$

on geodesic balls $B_x(\eta)$, $B_y(\eta)$ for small $\eta > 0$. Choose unit vectors $V_i (i=1,2)$, $V_1 \in T_x M$, $V_2 \in T_y M$, and $x_\pm = \exp_x(\pm \frac{1}{2} \eta V_1)$, $y_\pm = \exp_y(\pm \frac{1}{2} \eta V_2)$. Then $x_\pm \in B_x(\eta)$, $y_\pm \in B_y(\eta)$.

Apply Theorem 4 to pairs $\{x_-, x_+\}$ and $\{y_-, y_+\}$ to take the connected sums of M to itself at x_- and x_+, and at y_- and y_+. Then, we obtain a manifold which is diffeomorphic to $M \# S^2 \times S^1 \# S^2 \times S^1$, i.e., the given manifold with two handles. By the way are doing, we know that two copies of $S^2 \times S^1 - D^3$ are disjoint, and $\gamma_1(t) = \exp_x(t\,V_1)$, $\gamma_2(t) = \exp_y(t\,V_2)$ are closed geodesics in each of $S^2 \times S^1 - D^3$ (see the proof of Theorem 1). We also can make γ_1 and γ_2 have the same length. Now we use Theorem 3 to small tubular neighborhoods V_1 of γ_1 and V_2 of γ_2, and any $\varphi : \partial V_1 \longrightarrow \partial V_2$, since

$$(S^2 \times S^1 - D^3) \sim V_1 , \qquad (S^2 \times S^1 - D^3) \sim V_2$$

are both solid tori with a ball removed. From the Heegaard splitting of lens space $L(p,q)$ for appropriate choice of φ, we have

$$(S^2 \times S^1 - D^3) \sim V_1 \textcircled{\#}_{\varphi} (S^2 \times S^1 - D^3) \sim V_2$$

$$\cong L(p,q) \sim D^3 \amalg D^3 \quad .$$

Therefore, we obtain a manifold which is diffeomorphic to $M \# S^2 \times S^1 \# L(p,q)$ with a negative Ricci curvature metric.

Proof of Theorem 5. Let M be the given manifold with negative Ricci curvature. As in the proof of Theorem 6, we can touch two handles to M to obtain a manifold $M \# S^2 \times S^1 \# S^2 \times S^1$. We can make the two handles isometric. We have the simple closed geodesic γ_1 and γ_2 inside each handle. We also have the tubular neighborhoods V_1 and V_2. Now we take

$$\varphi : \partial V_1 \longrightarrow \partial V_2$$

to be the canonical diffeomorphism induced by isometry of two handles. Using Theorem 2 to glue $S^2 \times S^1 \sim D^3 \cup V_1$ and $S^2 \times S^1 \sim D^3 \cup V_2$ by φ, we obtain a manifold which is diffeomorphic to

$$M \# S^2 \times S^1 \# S^2 \times S^1 \quad .$$

If we check all the construction, $\partial V_1 \equiv \partial V_2 \subset M \# S^2 \times S^1 \# S^2 \times S^1$ is a total geodesic torus. For any g, we pick up g pairs of parallel closed geodesics in $\partial V_1 \equiv \partial V_2$ in the non-trivial homotopy class of the handle. Apply Theorem 2 g times to each pair to form the "connected sum along circles", then obtain a manifold which is diffeomorphic to

$$M \# S^2 \times S^1 \# \Sigma_g \times S^1$$

where Σ_g is a Riemann surface of genus g. This completes the proof of Theorem 5.

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