On the Smoothness of the Limiting Distribution Functions for Additive Functions

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Abstract of the Dissertation

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ABSTRACT

To a given arithmetic function $f(n)$ we associate a limiting distribution function $F(z)$ which is the limit of the arithmetic mean $F_x(z)$ as x goes to infinity. It is well known that for additive arithmetic functions the existence of such function $F(z)$ is completely characterized by the Erdös-Wintner theorem. Furthermore

this theorem also gives necessary and sufficient conditions on $F(n)$ for $F(z)$ to be a continuous function. In this paper we obtain conditions on the additive arithmetic function $f(n)$ for the limiting distribution function $F(z)$ to be absolutely continuous. The conditions are given on the second moment of $F(z)$, theorem 2.1.

We prove via theorem 2.1 that the functions given by $f(p) = (\log \log p)^{-\alpha}$ with $\alpha > 1$ generate absolutely continuous distribution functions. Also we include a new proof for the absolute continuity of $F(z)$ for the case $f(p) \doteq (\log p)^{-\alpha}$, $0 < \alpha < 1$.

Dedicated to the women in my life: my grandmother, my mother, Teresa, Rubi and Constanza.

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Introduction

We say that $f(n)$, an arithmetic function, is additive if for any pair of positive coprime integers m, n, $f(m.n) = f(m) + f(n)$. An additive function with the property $f(p^\alpha) = f(p)$ for every prime p and positive integer α is called strongly additive.

Clearly, for any positive integer n one has

$$f(n) = \sum_{p^k \| n} f(p^k) = \sum_p f(p^{\alpha_p(n)})$$

where $\alpha_p(n)$ is the positive integer such that $p^{\alpha_p(n)} | n$ and $p^{\alpha_p(n)+1}$ doesn't divide n.

As usual, we define the frequency function $F_x(z)$ as

$$F_x(z) = V_x(n \leq x;\ f(n) \leq z)$$
$$= \frac{1}{x} \sum_{\substack{n \leq x \\ f(n) \leq z}} 1$$

Here we are interested in the smoothness property of the limiting distribution function $F(z)$ defined by

$$F(z) = \lim_{x \to \infty} F_x(z)$$

The existence of such limit is guaranteed by the following Theorem, due to P. Erdős and N. Wintner,

Theorem 0.1 (Erdős - Wintner)

Let $f(n)$ be an additive function.

Define $$A(n) = \sum_{\substack{p \leq n \\ |f(p)| \leq 1}} \frac{f(p)}{p}\ ; \quad \text{then}$$

$V_n(m : f(m) - A(n) \leq z)$ converges to a distribution function $F(z)$ if and only if

$$(0.1) \qquad \sum_{p,\ |f(p)| \leq 1} \frac{f^2(p)}{p} \quad \text{and} \quad \sum_{p,\ |f(p)| > 1} \frac{1}{p} \quad \text{are convergent.}$$

The characteristic function $\psi(t)$ of $F(z)$ has the following form:

$$\psi(t) = \prod_{|f(p)|>1} (1+\omega(p)) \prod_{|f(p)| \leq 1} (1+\omega(p)) \exp\left(\frac{-it\ f(p)}{p}\right).$$

where $\omega(p) = -\frac{1}{p} + (1 - \frac{1}{p}) \sum_{m=1}^{\infty} p^{-m} \exp(it\ f(p^m))$.

The function $F(z)$ is of pure type. Furthermore it is continuous if and only if

(0.2) $$\sum_{f(p)\neq 0} \frac{1}{p} \text{ diverges}$$

This theorem, for strongly additive functions, with the additional hypothesis

(0.3) $$\sum_{p,\,|f(p)|\leq 1} \frac{f(p)}{p} \quad \text{converges}$$

tells us that the characteristic function $\psi(t)$ has the form

$$\psi(t) = \prod_p \left(1 - \frac{1 - \exp(it\, f(p))}{p}\right)$$

The proof of the theorem is based on the strong link between Probability Theory and Number Theory. Namely, the relations between the frequencies generated by additive functions and the sum of independent random variables in some probability space.

P. Erdös [3] proved that if $f(p) = O(p^{-\alpha})$, $\alpha>0$ then $F(z)$ is singular; in other words, $F(z)$ is continuous with $F'(z) = 0$ almost every where. For example, the additive function $f(n) = \log(\phi(n)/n)$ where $\phi(n)$ is the Euler function clearly satisfies

$$|f(p)| \leq \frac{2}{p}, \quad \text{for every prime } p.$$

Similary, $f(n) = \log(\sigma(n)/n)$ where $\sigma(n)$ represents the sum of the divisors of n, satisfies the same condition.

Recently, Babu [1] showed that if

$f(p) = (\log p)^{-a}$, $0 < a < 2$ then $F(z)$ is absolutely continuous.

Therefore all three possible types of $F(z)$: singular, absolutely continuous and discrete are expected to show up in the Erdös-Wintner theorem.

The condition for the continuity of $F(z)$ comes from the Levy's theory of convolutions of infinity many distributions. In Elliot [2] an alternative proof is given using Number Theoretical techniques. But the simplest proof was given by Szüsz [9]. His idea is to apply Chebishev inequality in adequate form.

Here we will give some conditions for the absolute continuity of $F(z)$ and we will compare them with other known results.

Chapter one

I.1 SOME DEFINITIONS AND THEOREMS FROM PROBABILITY THEORY

Let a Probability space $(\Omega, \mathcal{G}, P)$ be given. A measurable functions $f(\omega)$, $\omega \in \Omega$ is called a random variable. If for any set $\alpha_1, \alpha_2, \ldots, \alpha_m$ of m real numbers

$$P[f_1(\omega) < \alpha_1, f_2(\omega) < \alpha_2, \ldots, f_m(\omega) < \alpha_m]$$
$$= \prod_{i=1}^{m} P[f_i(\omega) < \alpha_i]$$

holds, then $f_1(\omega), f_2(\omega), \ldots, f_m(\omega)$ are called independent random variables.

Any real function F which is non-negative, non-decreasing, left continuous with $F(-\infty) = 0$ and $F(+\infty) = 1$ is called a distribution function. It is well known [5] that $F(x)$ can be decomposed in the following way

$$F(x) = a_1 F_d(x) + a_2 F_s(x) + a_3 F_{ac}(x), \quad a_i \geq 0$$
$$\text{with} \quad a_1 + a_2 + a_3 = 1,$$

where F_d is a discrete function, F_s is singular and F_{ac} is absolutely continuous. A function G is singular if and only if G is continuous and $G' = 0$ almost everywhere.

$F(x)$ is called of pure type if $a_i=1$ for some i, $i=1,2,3$.

There is a natural topology for the distribution functions, i.e., $\{F_n\}_n$ a sequence of distribution functions: we say that $\{F_n\}$ converge weakly as $n\to\infty$ if and only if there exists a distribution function G so that

$$\lim_{n\to\infty} F_n(z) = G(z)$$

holds at every point z at which $G(z)$ is continuous.

Given a random variable $f(\omega) = \underline{\overline{X}}(\omega)$, $\omega\in\Omega$, we define

$$F_{\underline{\overline{X}}}(x) = P[\underline{\overline{X}}(\omega) < x].$$

Since $P(\Omega) = 1$, the function $F_{\underline{\overline{X}}}(x)$ is a distribution functions, it is called the distribution function generated by the random variable $\underline{\overline{X}}(\omega)$. We will omit the subscript $\underline{\overline{X}}$ in $F_{\underline{\overline{X}}}(x)$ and we will say that $F(x)$ is the distribution function of $\underline{\overline{X}}(\omega)$.

The convolution of two distribution functions F, G is defined by

$$(F * G)(x) = \int_{-\infty}^{+\infty} F(y-x)d\, G(y)$$

where the integral is in the Lebesgue-Stieltjes sense.

Let $\underline{X}(\omega)$ be a random variable on (Ω, G, P) with distribution function $F(x)$. The $\underline{r^{th}\text{-moment}}$ $E(\underline{X}^r)$ and the $\underline{\text{variance}}$ $D(\underline{X})$ of $\underline{X} = \underline{X}(\omega)$ are defined by

$$E(\underline{X}^r) = \int_\Omega \underline{X}^r(\omega)dP = \int_{-\infty}^{+\infty} x^r \, dF(x) \, , \quad n>0$$

and

$$D^2(\underline{X}) = E((\underline{X} - E(\underline{X}))^2) \text{ as usual.}$$

With these notations the <u>Chebishev inequality</u> becomes

$$P(|\underline{X} - E(\underline{X})| \geq a) \leq a^{-2} D^2(\underline{X})$$

where $a>0$ and $\underline{X}$ is any random variable. This inequality reduces to

$$P(|\underline{X}| \geq a) \leq a^{-2} \int_{-\infty}^{+\infty} x^2 \, dF(x), \quad a>0$$

for $E(\underline{X}) = 0$.

The <u>characteristic function</u> $\psi(t)$ associated with a distribution function $F(x)$ is defined as the Fourier - Stieltjes transform of $F(x)$, namely

$$\psi(t) = \int_{-\infty}^{+\infty} e^{itx} \, dF(x).$$

Consider a sequence $\{\underline{X}_k(\omega)\}_{k=1}^{n}$ of independent random variable and let $\{\psi_k(t)\}_{k=1}^{n}$ be the corresponding sequence of characteristic functions. If we define

$$\varphi_n(t) = \prod_{1 \le k \le n} \psi_k(t)$$

then it is well known that $\varphi_n(t)$ is the characteristic function of $S_n = \sum_{k=1}^{n} \underline{X}_k$. After these introductory concepts and theorems we can state one of the most important theorem about characteristic functions; we will assume that $\{\underline{X}_k(\omega)\}$ is a sequence of independent random variables defined in some probability space $(\Omega, \mathcal{a}, P)$.

Theorem 1.1:

The sequence $F_n(x) = P[S_n \le x]$ of distribution functionns converges weakly to a distribution function $F(x)$ as $n \to \infty$ if and only if there exists a function $\psi(t)$ defined for all real values of t and continuous at zero such that

$$\lim_{n \to \infty} \varphi_n(t) = \psi(t) \qquad \text{for every } t.$$

Furthermore, the characteristic function of $F(x)$ is $\psi(t)$. For the proof of this theorem see Elliot [2], lemma 1.11 and 1.18. Another important theorem about the nature of the limiting distribution function is given by the following,

Theorem 1.2

Let $G(x)$ a distribution function and $\phi(t)$ its characteristic function. If there exists p such that

$$\int_{-\infty}^{+\infty} |\phi(t)|^p \, dt < \infty, \quad 1 \leq p \leq 2$$

Then $G(x)$ is absolutely continuous and its derivative is given by

$$G'(x) = \lim_{T\to\infty} \int_{-T}^{T} e^{-itx} \phi(t)dt \quad \text{for} \quad 1 \leq p < 2,$$

and by

$$G'(x) = \text{L.i.m.} \int_{-T}^{T} e^{-itx} \phi(t)dt \quad \text{for} \quad p = 2$$

Furthermore, $\phi(t) \in L^p(\mathbb{R})$ implies that $G' \in L^q(\mathbb{R})$ where $q = \frac{p}{p-1}$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

For the proof see Kawata [5], Zygmund [11]

I.2 PROBABILISTIC MODEL

Here we will relate the frequency of an additive function with the sum of discrete random variables. For references see Kubilius [7] and Kac [6].

One of the models that we can introduce is the following. Define a sequence of random variables as follows:

Let r be a real number, $r \geq 2$; let N be a positive integer.

Define

$$\underline{X}_p = \begin{cases} f(p^j), & j=1,2,\ldots,N-1, \ n \geq 1 \\ 0 & \end{cases}$$

for every prime p not exceeding r,

where $f(n) = \sum_{\substack{p^j \| n \\ j<N \\ p \leq r}} f(p^j)$ is an additive function

In other words, if we define

$A_p = \{n: p|n\}$, prime, $p \leq r$ and

$A_p(k) = \{n \leq k: n \in A_p\}$, k any positive integer, the density of A_p is given by

$$D\{A_p\} = \lim_{k\to\infty} \frac{1}{k}|A_p(k)|$$

$$= \lim_{k\to\infty}[\frac{k}{p}].\frac{1}{k} = \lim_{k\to\infty}(\frac{k}{p} - \{\frac{k}{p}\})\frac{1}{k} = \frac{1}{p}$$

If we define by $A_{p,q} = \{n/p|n, q|n\}$ p,q primes and $A_{p,q}(k)$ with the obvious meaning then

$$\lim_{k\to\infty}(A_{p,q}(k)) \frac{1}{k} = \frac{1}{pq}$$

therefore:

$$D(A_{pq}) = \frac{1}{p} \cdot \frac{1}{q} = D(A_p) \cdot D(A_q).$$

This is the reason why we can guarantee the independence of the random variables $\{\underline{\overline{X}}_p\}_p$. We also define the law P as follows

$$P[\underline{\overline{X}}_p = f(p^j)] = (1 - \frac{1}{p}) \frac{1}{p^j}, \quad j=1,\ldots,N-1$$

and $$P[X_p = 0] = 1 - \frac{1}{p} + \frac{1}{p^N} .$$

Since $\sum_{j=1}^{N-1}(1 - \frac{1}{p}) \frac{1}{p^j} + (1 - \frac{1}{p}) + \frac{1}{p^N} = 1$ this law is well defined and it is related to the frequency $V_x(n: f(n) \leq z)$ by the following theorem.

Theorem 1.3

Let $f(n)$ be an additive function; then

$$V_x(n: f(n) \leq z) = p(\sum_{p \leq r} \underline{\overline{X}} \leq z) + O(e^{4Nr} x^{-1})$$

holds uniformly for all real numbers $f(p^j)$, z, x, $x>0$ $2 \leq r \leq x$, N positive integer.

For the proof of this theorem and better approximations see Elliot [2], Kubilius [7].

The main fact about this model is that in some sense, $\{\underline{\overline{X}}_p\}_p$, p prime, are independent random variables. We can interpret this by saying: the "events" of being divisible by p and q are independent.

I.3 Some Formulas and Lemmas.

In this section we first state the famous asymptotic formula.

Lemma 1.4:

There are constants A, d such that for any x real, $x \geq 2$

a) $$\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O((\log x)^{-1}).$$

b) $$\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O(\exp(-d\sqrt{\log x})).$$

Lemma 1.5: (Abel's Identity)

For any arithmetical function $a(n)$, let

$$A(x) = \sum_{n \leq x} a(n)$$

with $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then

$$\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)dt$$

For the proofs of these lemmas see any elementary book in Number Theory. For part b) of lemma 1.4 see Babu [1].

Proposition 1.6:

For any real numbers x, α, with $x > e^e$ and $\alpha > 1$ we have

i) $$\sum_{e^e < p} \frac{1}{p(\log\log p)^{\alpha}} \quad \text{converges}$$

ii) $$\sum_{x < p} \frac{1}{p(\log\log p)^{\alpha}} = \frac{1}{\alpha - 1} \frac{1}{(\log\log x)^{\alpha - 1}} + R(x)$$

where $R(x) = O((\log\log x)^{-\alpha})$.

Proof:

First we compute

$$\sum_{e^e \leq p \leq x} \frac{1}{p(\log\log p)^{\alpha}} .$$

Let $b(n)$ be defined as follows

$$b(n) = \begin{cases} 1 & \text{if } n \text{ is prime, } n \geq e^e \\ 0 & \text{otherwise.} \end{cases}$$

We apply lemma 1.5 with $a(n) = \frac{b(n)}{n}$ and $f(n) = (\log\log n)^{-\alpha}$. By lemma 1.4 we obtain

$$\sum_{e^e \leq p \leq x} \frac{1}{p(\log\log p)^{\alpha}} = (\log\log x + R_1(x))\,(\log\log x)^{-\alpha} +$$
$$+ \alpha \int_{e^e}^{x} (\log\log t)^{-\alpha}\,(t \log t)^{-1} dt -$$

$$- \int_{e^e}^{x} R_1(t) \frac{d}{dt}(\log \log t)^{-\alpha} dt$$

where $R_1(x) = O(1)$.

Taking $v = \log \log t$ in the first integral it is easy to see that

$$\int_{e^e}^{x} (\log \log t)^{-\alpha} (t \log t)^{-1} = \frac{1}{1-\alpha} \frac{1}{(\log \log x)^{\alpha-1}} + \frac{1}{\alpha-1}$$

Also, we note that

$$\int_{e^e}^{x} R_1(t) \frac{d}{dt}((\log \log t)^{-\alpha}) dt =$$

$$\int_{e^e}^{x} R_1(t) \frac{d}{dt}((\log \log t)^{-\alpha}) dt - \int_{x}^{\infty} R_1(t) \frac{d}{dt}((\log \log t)^{-\alpha}) dt$$

The first integral in the last identity is finite and we denote it by I_e. Clearly the second integral is $O((\log \log x)^{-\alpha})$.

Therefore we have proved

$$\sum_{e^e \leq p \leq x} \frac{1}{p(\log \log p)^{\alpha}} = \frac{1}{1-\alpha} (\log \log x)^{1-\alpha} + \frac{\alpha}{\alpha-1} +$$

$$- I_e + R(x)$$

where $R(x) = O((\log \log x)^{-\alpha})$. If x goes to infinite we obtain

$$\sum_{e^e \leq p} \frac{1}{p(\log \log p)^\alpha} = \frac{\alpha}{\alpha-1} - I_e - \sum_{e^e \leq p \leq x} \frac{1}{p(\log \log p)^\alpha}$$

the second part of the proposition follows immediately.

The next lemma will be vital for finding out conditions for the square integrability of the characteristic function $\psi(t)$ of the limiting distribution function $F(z)$ generated by the additive function $f(n)$.

Lemma 1.7

Let $\psi(t)$ be the characteristic function of the limiting distribution function $F(z)$ generated by the additive function $f(n)$.
Assume that $f(n)$ satisfies conditions 0.1 and 0.3 of theorem 0.1. Then

$$|\psi(t)|^2 \leq K \exp(-2 \sum_p (1 - \cos(t'f(p)))\frac{1}{p})$$

holds uniformly with respect to N and t.

Proof: we know from theorem 0.1 that

$$|\psi(t)|^2 = \prod_p \left(1 + \frac{e^{it\,f(p)}-1}{p} + \ldots\right).$$

$$.\left(1 + \frac{e^{-it\,f(p)}-1}{p} + \ldots\right).$$

Therefore

$$|\psi(t)|^2 = \prod_p \left(1 + \frac{2(\cos(f(p)t)-1)}{p} + O(\frac{1}{p^2})\right)$$

Taking logarithm one has

$$\log|\psi(t)|^2 = \sum_p \log\left(1 + \frac{2(\cos(f(p)t)-1)}{p} + O(\frac{1}{p^2})\right)$$

$$\leq 2 \sum_p \left(\frac{\cos(f(p)t)-1}{p}\right) + O(1)$$

Applying the exponential function we get the required inequality.

Remark: Notice that each factor in the product is less or equal than one, therefore if we restrict ourselves to a subset of primes we get something which is biger than $|\psi(t)|^2$. For strongly additive functions we can choose such constant K as one.

Chapter Two

Some criteria for absolute continuity of $F(z)$

One of the necessary conditions, theorem 0.1, for the existence of $F(z)$, the limiting distribution function, is the convergence of the series

$$(2.1) \qquad \sum_{|f(p)|\leq 1} \frac{f^2(p)}{p} .$$

Here we study this sum and give conditions on it such that $\psi(t)$, the characteristic function of $F(z)$ is square integrable.

For $\epsilon>0$ define $A_\epsilon(t)$ a subset of the prime numbers, as follows:

$$A_\epsilon(t) = \{p: |f(p)t| < \epsilon\}.$$

It is clear that if the sum 2.1 converges, then

$$\lim_{t\to\infty} \sum_{p\in A_\epsilon(t)} \frac{f^2(p)}{p} = 0$$

where ϵ is a fixed real number.

Also, by lemma 1.7 we know that if $f(n)$ satisfies (0.1), (0.3) of theorem 0.1 then $\psi(t)$, the characteristic function of $F(z)$ satisfies the following inequality

$$(2.2) \qquad |\psi(t)|^2 \leq K \exp(-2 \sum_p \frac{1}{p}(1 - \cos(t\, f(p)))$$

Now we state the following result:

Lemma 2.1

Let $f(n)$ be an additive function which satisfies (0.1), (0.2), (0.3) of theorem 0.1 and the following condition

$$(c) \qquad \sum_{p\in A_\epsilon(t)} \frac{f^2(p)}{p} \geq K_\epsilon \frac{\ln t}{t^2}$$

for t large and ϵ small, with $K_\epsilon > \frac{1}{\cos^2(\epsilon/2)}$. Then $F(z)$ the limiting distribution function is absolutely continuous.

Proof: For each x such that $|x| < \epsilon/2$ we have $|\sin x| \geq (\cos \epsilon/2)|x|$. Hence for $p \in A_\epsilon(t)$ we immediately see that $2(1 - \cos(t\, f(p))) = 4 \sin^2(\frac{t\, f(p)}{2})$ $\geq \cos^2(\epsilon/2)\, t^2 f^2(p)$. Therefore by (2.2)

$$(2.3) \qquad |\psi(t)|^2 \leq \exp(- (\cos(\epsilon/2))^2\, t^2 \sum_{p\in A_\epsilon(t)} \frac{f^2(p)}{p})$$

Then using condition (c) we get that

$$|\psi(t)|^2 \leq t^{-K_\epsilon (\cos(\epsilon/2))^2}$$

which proves that $\psi(t)$ is square integrable. By theorem 1.2 we conclude that $F(z)$ is absolutely continuous.

For the applications of this lemma we need some notation. Let $A_\epsilon(\alpha,\beta)$ the subset of primes defined by

$$A_\epsilon(\alpha,\beta) = A_\epsilon(t^\alpha) - A_\epsilon(t^\beta)$$

where $A_\epsilon(t^\alpha)$ and $A_\epsilon(t^\beta)$ are the set defined as before with $\alpha < \beta$, $\alpha,\beta > 0$.

Consider the strongly additive function defined by

$$f(n) = \sum_{\substack{p \mid n \\ p \text{ prime}}} f(p)$$

where $f(p)$ is given by, $f(p) = (\log p)^{-a}$, where a is a positive real number.

The cardinality of $A_\epsilon(\alpha,\beta)$ is easy to compute, is the set of primes under the condition

$$\exp((t^{\alpha}/_{\epsilon})^{1/a}) < p < \exp((t^{\beta}/_{\epsilon})^{1/a}).$$

Using lemma 1.4 is easy to see that

$$\sum_{p\in A_{\epsilon}(\alpha,\beta)} \frac{1}{p} = \frac{1}{a} \ln|t|^{\beta-\alpha} + R(t)$$

where $R(t)$ is a function that goes to zero as t goes to infinite.

For $p\in A(\alpha,\beta)$ we have

$$f^2(p) \geq \frac{\epsilon^2}{t^{2\beta}}$$

Therefore,

$$\sum_{p\in A_{\epsilon}(\alpha,\beta)} \frac{f^2(p)}{p} \geq \frac{\epsilon^2}{t^{2\beta}} \cdot \frac{1}{a}(\beta-\alpha) \ln|t| + \frac{R(t)}{t^{2\beta}} .$$

Take $\beta = 1$ and choose $\alpha < 1$ such that $1 - \alpha > a$. This α can be chosen for any a in the interval $(0,1)$. Finally we find ϵ such that $\epsilon^2(\frac{1-\alpha}{a}) \cos^2 (\epsilon/2) > 1$. After these choices we easily get that for such a, such that the last inequality holds for some $\alpha_1 < 1$

$$t^2 \sum_{p\in A(\alpha,\beta)} \frac{f^2(p)}{p} \geq K_{\epsilon}\ln t + R(t) ,$$

which is enough for proving the square integrability of

$\psi(t)$. Therefore the distribution function $F(z)$ generated by $f(n)$ is absolutely continuous. This example also give us how fast the second moment (2.1) has to go to zero for getting an absolute continuous distribution function.

We consider $A_1(t)$ as before and $f(n)$ an arbitrary additive function.

Theorem 2.1

Let $f(n)$ be an additive function which satisfies conditions (0.1), (0.2), (0.3) of theorem 0.1. We also assume that there is a constant $M > 0$ such that

$$\text{(d)} \qquad \sum_{p \in A_1(t)} \frac{f^2(p)}{p} = M\, t^{-\beta} + R(t)$$

for t large, where $R(t)$ is a function such that $t^2 R(t)$ is bounded as t goes to infinite and β is a positive constant with $0 \leq \beta < 2$. Then the limiting distribution function generated by $f(n)$ is absolutely continuous.

Proof:

We know from lemma 1.7 that for any t and for $p \in A_1(t)$ we have

$$|\psi(t)|^2 \leq K \exp\Big(-\partial t^2 \sum_{p \in A_1(t)} \frac{f^2(p)}{p}\Big).$$

Applying condition (d) we obtain

$$|\psi(t)|^2 \leq K_1 \exp(-\ dMt^{2-\beta})$$

for some constant $K_1 > 0$. This inequality shows that $|\psi(t)|^2$ is integrable for $\beta < 2$ and hence by theorem 1.2 $F(z)$ is absolutely continuous.

Example:

Consider the strongly additive function given by

$$f(n) = \sum_{p|n} f(p)$$

with $f(p)$ defined by $f(p) = (\log\log p)^{-\gamma}$, $\gamma > 1$. By proposition 1.6 we see that conditions (0.1), (0.2) and (0.3) of theorem 0.1 are satisfied, therefore only remains to check condition (d).

By part ii) of proposition 1.6 we have

$$\sum_{p\in A_1(t)} \frac{f^2(p)}{p} = \sum_{p>h(t)} \frac{1}{p(\log\log p)^{2\gamma}}$$

$$= \frac{1}{2\gamma-1} \frac{1}{(\log\log h(t)^{2\gamma-1}} + R(t)$$

where $h(t) = \exp(\exp(t^{1/\gamma}))$ and $R(t) = O(1/t^2)$.

Therefore

$$\sum_{p\in A_1(t)} \frac{f^2(p)}{p} = \frac{1}{2\gamma-1} \cdot \frac{1}{t^{2-1/\gamma}} + R(t).$$

Since $\gamma \geq 1$ then $0 \leq \beta = 2 - 1/\gamma < 2$.

Appealing theorem 2.1 we conclude that $F(z)$ is absolutely continuous.

II.2 New Proof of the absolute continuity $f(p) = (\log p)^{-\alpha}$.

The method used by Erdös - Katai [4] for proving the absolute continuity of $F(z)$, the limiting distribution function generated by $f(p) = (\log p)^{-1/\alpha}$, $\alpha > 1$ consists in the study of the sum

$$(2.4) \qquad \sum_{\substack{p \\ s(t)<p<S(t)}} \frac{\cos(t\ f(p))}{p}$$

which appears in the inequality (2.2).

They prove that the sum 2.4 is bounded as t goes to infinite if $s(t) = t^{10}$ and $S(t) = e^{t^{\alpha}}$. We notice that the inequality (2.2) holds for p running over the primes number. Since each factor in (2.2) is less than one we obtain that

$$(2.5) \qquad |\psi(t)|^2 \leq \exp(-2 \sum_{p} \frac{1}{p}(1 - \cos(t\ f(p)))$$

$$\leq \exp(-2 \sum_{s(t)<p<S(t)} \frac{1}{p}(1 - \cos(t\ f(p))).$$

Taking $s(t) = t^{10}$ and $S(t) = e^{t^\alpha}$ for $f(p) = (\log p)^{-1/\alpha}$ one gets

$$|\psi(t)|^2 \leq K_2 \exp(- 2 \ln t^\alpha) \exp(\ln \ln t)$$

which proves the square integrability of $|\psi(t)|$.

Here we develop a method for proving the square integrability of $\psi(t)$ via the sum 2.4 with $s(t) = \exp(t^{\alpha/2})$ and $s(t) = \exp(t^\alpha)$.

The main idea is to compare the positive and negative part of the sum (2.4).

Define the following sequence of sets $\{A_k\}_k$.

$$A_k = \{p: f(p)t < (2k + 1)\pi/2\}, \qquad k=0,1,2,\ldots$$

where p runs over the prime numbers.

For each $n=1,2,3,\ldots$ let B_n be the sets given by

$$B_n = A_n - A_{n-1}.$$

Clearly the sets B_n are disjoint. Furthermore the sum (2.4) can be decomposed as follows:

$$(2.6) \qquad \sum_{s(t)<p<S(t)} \frac{\cos\ t\ f(p)}{p} = \sum_{n=1}^{N(t)} \sum_{p\in B_{2n}} \frac{\cos\ (t\ f(p))}{p} + $$

$$+ \sum_{n=1}^{N(t)} \sum_{p\in B_{2n-1}} \frac{\cos(t\ f(p))}{p} +$$

$$+ \sum_{r(t)<p<S(t)} \frac{\cos(t\ f(p))}{p}$$

where $N(t) = \frac{1}{4}(\frac{2}{\pi}\sqrt{t} -1)$, $s(t) = \exp(t^{\alpha/2})$, $S(t) = \exp(t^{\alpha})$ and $r(t) = \exp((\frac{2}{\pi}\ t)^{\alpha})$. It is easy to see that the last sum in (2.6) is bounded because by lemma 1.4

$$(2.7) \qquad \left|\sum_{r(t)<p<S(t)} \frac{\cos(t\ f(p))}{p}\right| \leq \log(\frac{\log\ S(t)}{\log\ r(t)}) = O(1)\ .$$

For estimating the first sum in the right hand side of 2.6 we need to compute the cardinality of B_{2n}. It is clear that $p\in B_{2n}$ if and only if

$$\exp((\frac{2}{\pi}\ \frac{t}{4n+1})^{\alpha}) \leq p \leq \exp((\frac{2}{\pi}\ \frac{t}{4n-1})^{\alpha}).$$

This last inequalities come from the definition of $f(p) = (\log\ p)^{-1/\alpha}$. By lemma 1.4 we obtain

$$\sum_{p\in B_{2n}} \frac{1}{p} = \alpha\ \log((4n+1)\pi/2) - \alpha\ \log((4n-1)\pi/2)$$

$$+ O((\frac{n}{t})^{\alpha}).$$

Since the logarithm is monotonic and by using the Mean Value Theorem we have

$$\sum_{n=1}^{N(t)} \sum_{p \in B_{2n}} \frac{1}{p} \leq \sum_{n=1}^{N(t)} \frac{2\alpha}{4n-1} + O\Big(\sum_{n=1}^{N(t)} \frac{n^{\alpha}}{t^{\alpha}}\Big).$$

Since $\alpha \geq 1$ we also have

$$\text{(2.7a)} \qquad \frac{1}{t^{\alpha}} \sum_{n=1}^{N(t)} n^{\alpha} = O(1) \quad \text{as } t \text{ goes to infinite.}$$

Therefore we have proved

$$\text{(2.8)} \qquad \sum_{n=1}^{N(t)} \sum_{p \in B_{2n}} \frac{1}{p} \leq 2\alpha \sum_{n=1}^{N(t)} \frac{1}{4n-1} + O(1)$$

for $\alpha \geq 1$.

Now we will estimate the second sum in the right hand side of 2.6. Let $B_{2n-1}(\epsilon)$ be a subset of B_{2n-1} defined by

$$B_{2n-1}(\epsilon) = \{p:\ ((4n-3)+\epsilon)\pi/2 < f(p)t < ((4n-1)-\epsilon)\pi/2\}$$

where ϵ is a positive real number, $0 < \epsilon < 1$.

We denote $d(\epsilon)$ the negative number given by $d(\epsilon) = -\sin(\frac{\pi}{2}\epsilon)$. By construction, for any $p \epsilon B_{2n-1}(\epsilon)$ we have

$$\cos(t\, f(p)) \leq d(\epsilon).$$

Using this construction we get the following inequality

$$(2.9) \qquad \sum_{n=1}^{N(t)} \sum_{p \epsilon B_{2n-1}} \frac{\cos(t\, f(p))}{p} \leq d(\epsilon) \sum_{n=1}^{N(t)} \sum_{p \epsilon B_{2n-1}(\epsilon)} \frac{1}{p}$$

The cardinality of $B_{2n-1}(\epsilon)$ is given by the set of prime numbers which are in the interval

$$((4n-3)+\epsilon)\pi/2 \leq p \leq ((4n-1)-\epsilon)\pi/2 .$$

By a similar argument used in 2.8 we get

$$(2.10) \qquad \sum_{n=1}^{N(t)} \sum_{p \epsilon B_{2n-1}(\epsilon)} \frac{1}{p} = \alpha \sum_{n=1}^{N(t)} (\log(((4n-1)-\epsilon)\pi/2) -$$

$$- \log(((4n-3)+\epsilon)\pi/2)) + O(1).$$

Taking the exponential function in (2.6) and applying (2.7), (2.8), (2.9) (2.10) one gets

$$\exp\Big(\sum_{s(t)<p<S(t)} \frac{\cos(t\ f(p))}{p}\Big) \leq K_3 \exp\Big(2\alpha \sum_{n=1}^{N(t)} \frac{1}{4n-1}\Big) \cdot$$

$$\cdot \exp\Big(\alpha d(\epsilon) \sum_{n=1}^{N(t)} \log\Big(\frac{4n-1-\epsilon}{4n-3+\epsilon}\Big)\Big)$$

where K_3 is some positive constant. By the Mean Value Theorem, the monotonicity of logarithm and the fact that $d(\epsilon)$ is a negative number one finally gets

(2.11) $$\exp\Big(\sum_{s(t)<p<S(t)} \frac{\cos(t\ f(p))}{p}\Big) \leq$$

$$\leq K_3 \exp\Big(2\alpha + 2d(\epsilon)\alpha(1-\epsilon)\Big) \sum_{n=1}^{N(t)} \frac{1}{4n-1}\Big)$$

since

$$\Big|\sum_{n=1}^{N(t)} \frac{1}{4n-1} - \sum_{n=1}^{N(t)} \frac{1}{4n}\Big| = O(1) \quad \text{as} \quad t \quad \text{goes to infinite}$$

we have

$$\exp\Big(2 \sum_{s(t)<p<S(t)} \frac{\cos(t\ f(p))}{p}\Big)$$

$$\leq K_4 \exp\Big(\frac{\alpha}{2}(1+d(\epsilon)(1-\epsilon))\ln\ t\Big).$$

Therefore by (2.5) and (2.11) we finally get

II.3 Some Remark about the square integrability of the characteristic function.

Another technique for testing the square integrability of the characteristic function $\psi(t)$ consists in to estimate the set A given by

$$(2.13)\qquad A = \{t\in[0,1]: \ |\Sigma_T \frac{1}{p}\cos(t\ f(p))| \geq \frac{1}{2}\Sigma_T \frac{1}{p}\}$$

for T large, where the sum $\Sigma_T \frac{1}{p}$ denotes the sum over the primes in the interval $[M(T), N(T)]$ with M, N increasing functions which go to infinite as T goes to infinite and $\log(\log \frac{N(T)}{M(T)}) > 0$ for T large.

Following the paper of Szüsz [9] we have that the integral

$$I(T) = \int_0^T |\psi(t)|^2 dt$$

can be written as follows

$$(2.14)\qquad I(T) = T\int_0^1 |\psi(tT)|^2 dt.$$

Denote by Y_T the following expression

$$Y_T = \sum_{M(T)<p<N(T)} \frac{\cos t\ f(p)T)}{p} .$$

By splitting the integral (2.14) with respect to A_T we obtain

$$I(T) = T\int_{A_T} |\psi(tT)|^2 dt + T\int_{A_T^c} |\psi(tT)|^2 dt .$$

Since $|\psi(t)| \leq 1$ for all t and by appealing to lemma 1.7 one gets

$$I(T) \leq T(\exp(-\Sigma_T \frac{1}{p}) + m(A_T^c)).$$

By Chebishev's inequality we have

$$(2.15) \qquad I(T) \leq T(\exp(-\Sigma_T \frac{1}{p}) + \frac{D(Y_T^2)}{(\Sigma_T \frac{1}{p})^2})$$

where $D(Y_T^2)$ is the second moment of Y_T given by

$$(2.16) \qquad D(Y_T^2) = \int_0^1 (\Sigma_T \frac{1}{p} \cos(t\ f(p)T))^2 dt$$

Working out the identity (2.16) we get

$$D(Y_T^2) = O(1) + \sum_{\substack{p,q \\ p \neq q}} \frac{1}{pq} \frac{\sin((f(p) + f(q))T)}{(f(p) + f(q))T} +$$

$$+ \sum_{\substack{p,q \\ p \neq q}} \frac{1}{pq} \frac{\sin((f(p) - f(q))T)}{(f(p) - f(q))T}$$

where p and q run over the primes numbers in the interval $[M(T), N(T)]$. We can assume without loss of generality that $|f(p)| \neq |f(q)|$ for $p \neq q$, see [9] for details.

The first expression in the right hand side of (2.15) tells us that $N(T)$ should be at least $\exp(T^{1+\epsilon})$. The second expression in 2.15 is difficult to handle and conditions like

$$\sum_{\substack{p \\ r(tT)<p<M(tT)}} \frac{\cos(tf(p)T)}{p} = O(1) \quad \text{as } T \text{ goes to infinite}$$

for some functions $r(x)$, $M(x)$ with $r(x) < M(x)$, $r(x) \geq \exp(x^{1+\epsilon})$ assure that $D(Y_T^2)$ is of order $O(\Sigma_T \frac{1}{P})$ and therefore the finiteness of the integral (2.14). But these kind of conditions obviously guaranteed the convergence of (2.14) as $T \to \infty$ as we saw in (2.5). We weren't able to find new conditions on $f(n)$, via the trigonometric series (2.16), such that $I(T)$ is finite.

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