Manifolds of positive scalar curvature, Yang-Mills fields - the Kalusa-Klein model.

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by

Rodney Arthur Carr

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Abstract of the dissertation

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In this dissertation we investigate problems arising from two distinct areas in mathematics. We study manifolds of positive scalar curvature and examine certain problems raised in the study of Yang-Mills fields.

Our study of positive scalar curvature includes both positive and negative results. Our positive results involve giving new examples of manifolds which carry metrics of positive scalar curvature. The basic result here is a surgery-type construction for regular neighbourhoods of

embedded submanifolds of a manifold. Using this we show that any finitely presented group can appear as the fundamental group of a compact 4-manifold of positive scalar curvature. We also examine the space of positive scalar curvature metrics on a fixed manifold M. Here, using the technique of plumbing disc bundles over spheres we find that for $M = S^{4n-1}$, $n > 2$ this space has an infinite number of connected components.

The other main series of results concerning metrics of positive scalar curvature is motivated by the idea that "large" manifolds should not admit such metrics. We develop techniques which involve formalizing a general notion of "largeness" for manifolds by requiring "large" submanifolds. In the first we study manifolds having "large" codimension 2 submanifolds by extending the bad-end results of Gromov and Lawson. In the second we study manifolds having "large" codimension 3 submanifolds by concentrating on "complementary" submanifolds. The latter method gives interesting results in dimension 4 where we are able to use the minimal surface techniques developed by Schoen and Yau and others.

Our results concerning Yang-Mills fields relate to the so-called Kalusa-Klein model which attempts to unify the gravitational field with other interaction fields of physics

by working on the total space of a principal G-bundle $P \to M$. We give a general formulation using a well-known functional - the general idea of the model being that critical points of the functional should give simultaneous solutions of the Einstein field and Yang-Mills equations. We show, however, that this is only the case under certain restrictions and is true in general only if $G = U(1)$. In addition, we obtain a topological rigidity result for Einstein metrics on P. In part this depends on a rigidity result for metrics having harmonic curvature derived from the Bochner-Weitezbock formula.

To Linda - sometimes I may act as if I've forgotten how much I owe to her for her love, support and understanding - I've never really forgotten. I realize just how exceptionally lucky I am to have her for a wife and friend. Certainly, without her none of this work would have been possible.

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Chapter 1. Introduction - Survey of results.

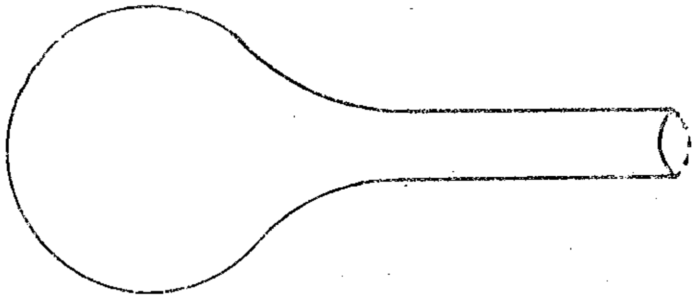
In this chapter we describe our main results and give some background material motivating the studies. There are 4 main chapters in this dissertation. In Chapters 2, 3 and 4 we continue the study of manifolds of positive scalar curvature and in Chapter 5 we consider some problems arising from the study of Yang-Mills fields. These subjects are discussed in Sections 1.1 and 1.2 of this chapter respectively.

§1.1. Positive scalar curvature.

Let M be a complete riemannian manifold. It is a fundamental problem to understand the relationship between the curvature and topology of M. In the first few chapters of this dissertation we continue to investigate this relationship for manifolds carrying metrics of positive scalar curvature.

In general, the results concerning manifolds which admit metrics of any type of positive curvature (sectional, Ricci or scalar) indicate that these manifolds form a rather restrictive class, more specifically, that they are "small" in some sence. To illustrate this we recall the classical theorem of Myers and Bonnet which states that a manifold admitting a complete metric of strictly positive sectional or Ricci curvature has a compact universal cover (and so is itself compact and has a finite fundamental group). With even stronger curvature restrictions the notion of "smallness" becomes more definite - the pinching theorems, for example, conclude that the universal cover of a manifold admitting a sufficiently δ-pinched metric is diffeomorphic to the usual sphere. However, in this dissertation we will be concerned with the weaker restriction that is imposed by requiring positive scalar curvature only. The class of manifolds admitting such metrics is, of course, much wider than the class admitting, say, positive sectional curvature but we still expect them to be "small" in some sence. However, it is difficult to formulate this notion precisely and in fact we have more success in showing that manifolds which are "large" do not admit metrics of positive scalar curvature.

Before attempting to develop a notion of largeness it is useful to study examples of manifolds which do admit metrics of positive scalar curvature. It turns out that we have a great deal of freedom in construction. Firstly, we note that any manifold of the form $M \times S^2$ carries such a metric. This can be generalized in a number of ways. For example, a basic discovery of Gromov and Lawson and Schoen and Yau was that a sphere "factor" could be used to "carry" positive scalar curvature around corners. A simple example of the use of this technique is illustrated in the construction of a complete metric of strictly positive scalar curvature on R^3 where we "pull out" from the standard metric on S^3 - {point} to a complete one which is a product on the end $R \times S^2$.



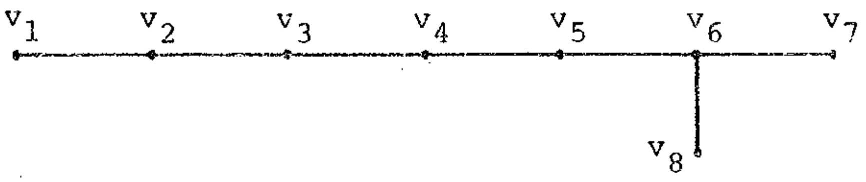
Here the metric on the S^2 factor can be chosen to dominate the negative curvature introduced in going around the corner. Developing this idea Gromov and Lawson [GL_2] and, independently, Schoen and Yau [SY_2] proved that <u>any manifold obtained from a manifold of positive scalar curvature by codimension ≥ 3 surgeries also carries a metric of positive scalar curvature</u>. Chapter 2 of this dissertation is devoted to developing the technique further and studying some natural consequences. We prove the following basic theorem.

Theorem A. Let M be a riemannian manifold with a fixed smooth cell decomposition and K a codimension ≥ 3 subcomplex of M. Then there is a regular neighbourhood U of K in M so that the induced metric on the boundary ∂U has positive scalar curvature.

Using this we are able to construct some new examples of manifolds of positive scalar curvature. In particular, as an interesting consequence we obtain (§2.2)

Theorem B. Let π be any finitely presented group. Then there is a compact 4-manifold M of positive scalar curvature with fundamental group $\pi_1(M) = \pi$.

In Chapter 2 we also show that the space $R^+(M)$ of positive scalar curvature metrics on a fixed manifold M can be quite complicated. The study requires some background material concerning metrics of positive scalar curvature on manifolds with boundary (§2.3), then we indicate one method for detecting non-zero elements of the homotopy groups $\pi_k(R^+(M))$, $k = 0,1$. (§2.4). In §2.5 we use the technique of *plumbing* disc bundles over spheres to study $\pi_0(R^+(S^{4k-1}))$. *By plumbing according to the tree* E_8 *(appropiately weighted)*



we show that the number of connected components of $R^+(S^{4k-1})$ *is infinite*.

Thus, we see that manifolds of positive scalar curvature can be quite complicated and that we have a great deal of freedom to deform such manifolds while retaining positive scalar curvature. Any notion of "largeness" for manifolds which prohibits positive scalar curvature must take this into account. In particular, Theorem B shows that any condition using the fundamental group must involve its interaction with the topology of the manifold. Also, we should avoid manifolds having spheres of dimension $\geqslant 2$ which may be able to carry the positive curvature.

The first successful formulation of a notion of "largeness" was given by Schoen and Yau [$SY_{1,2}$]. They studied compact manifolds having "large" hypersurfaces and proved that such manifolds do not admit metrics of positive scalar curvature. The idea here is that manifolds containing "bad" *hypersurfaces* cannot also contain spheres of dimension $\geqslant 2$ that carry the positive curvature. Their proof used minimal surface techniques and unfortunately broke down in dimensions $\geqslant 8$ due to the failure of internal regularity for solutions of the Plateau problem.

Another natural idea to try in an attempt at formalizing a notion of "largeness" for manifolds is to use the covers - a manifold with "large" covers should not admit a metric positive scalar curvature. For example, recall that a manifold M is called a $K(\pi,1)$ if $\pi_1(M) = \pi$ and $\pi_n(M) = 0$ for $n \geqslant 2$. Then the universal cover of M is contractible and therefore presumably "large" in some sence. Hence we may expect the following to be true.

Conjecture C. *A compact $K(\pi,1)$-manifold does not admit a metric of positive scalar curvature.*

Although this conjecture is plausible no notion of "largeness" which is sufficiently strong to include the general $K(\pi,1)$ case has so far been given. However, a less general but still highly successful "largeness" condition for covers has been given by Gromov and Lawson [GL1,2].

Definition D. A compact manifold X is enlargeable if for any $\varepsilon > 0$ and any riemannian metric on X there is a spin covering manifold $\tilde{X}_\varepsilon \to X$ and a map $f_\varepsilon : \tilde{X}_\varepsilon \to S^n(1)$ onto the euclidean n-sphere of radius 1 which is ε-contracting, constant outside of a compact set and has non-zero degree.

Gromov and Lawson proved that enlargeable manifolds do not admit metrics of positive scalar curvature. Their proof used the Dirac operator and a relative Atiyah-Singer Index Theorem (for non-compact manifolds). Thus it extends earlier work of Lichnerowitz [Li] and Hitchen [H].

The enlargeable result already gives many examples of compact manifolds which cannot carry metrics of positive scalar curvature, the tori T^n, $n \geq 2$, for example (this latter result was also obtained by Schoen and Yau [$SY_{1,2}$] in dimensions ≤ 7). However, the concept of enlargeable is defined in a geometric way and so in cases where we have topological information alone it becomes difficult to work with. For example, it is easy to see that a compact $K(\pi,1)$-manifold which carries a metric having sectional curvatures ≤ 0 is enlargeable (and so does not admit a metric of positive scalar curvature) but the condition does not apply directly to an arbitary compact $K(\pi,1)$.

Thus, in order to exhibit other compact manifolds which do not admit metrics of positive scalar curvature we return to an examination of the covers. If $X \to X_c$ is a cover of a compact manifold of positive scalar curvature then the lifted metric on X has <u>uniformly</u> positive scalar curvature. If X is non-compact this has stronger implications than requiring only positive scalar curvature. This can be seen in the following examples. Let Σ be a (compact) enlargeable manifold. Then

1. $\Sigma \times R$ cannot carry a metric of positive scalar curvature
2. $\Sigma \times R^2$ cannot carry a metric of uniformly positive scalar curvature
3. $\Sigma \times R^3$ can carry a metric of uniformly positive scalar curvature.

These examples illustrate a general phenomenom, namely, that a non-compact manifold's inability to carry a metric of uniformly positive scalar curvature can result from having a high dimensional enlargeable (= "bad") submanifold which is "almost" a factor. Also note that the R and R^2 factors in the Type 1 and 2 cases do not have dimension ≥ 2 sphere "factors" which could carry all the positive curvature. Thus, these types are clearly "large" in some sence.

Of course, the general classes of manifolds enjoying properties akin to those of the above simple products are much wider - all we really require is that X have a "bad" submanifold Σ which is sufficiently non-trivally embedded. For example, the manifolds with "large" hypersurfaces which were studied by Schoen and Yau fall into the general class of Type 1 manifolds. Gromov and Lawson [GL3] studied the class of so-called Λ^2-enlargeable manifolds which are also of this type. In both cases it was shown that the manifolds do not carry metrics of positive scalar curvature.

Manifolds of Type 2 (that is, ones with a "large" codimension 2 submanifold) have been studied by Gromov and Lawson. They introduced the notion of a manifold having a "bad end" and proved that such manifolds do not carry metrics of positive scalar curvature. Various manifolds having "large" codimension 2 submanifolds were then shown to have bad ends.

Manifolds of Type 1 or 2 can be used directly in showing that certain compact manifolds X_c do not carry metrics of positive scalar curvature - we need only show that X_c has an infinite cover X falling into one of these classes. A standard method for attempting this is to require that X_c itself have embedded a "large" submanifold $\Sigma \xrightarrow{i} X_c$ so that when lifted to the cover $X \to X_c$ corresponding to the subgroup $i_*(\pi_1(\Sigma)) \subset \pi_1(X_c)$ the submanifold $\Sigma \xrightarrow{\tilde{i}} X$ is sufficiently "bad". The bad-end results of Gromov and Lawson depend on this construction. For example, a compact 3-dimensional $K(\pi,1)$-manifold has an embedded circle representing an element of infinite order. The corresponding $K(Z,1)$ cover is of Type 2 (it has a bad end) so cannot carry a metric of uniformly positive scalar curvature. Hence we obtain the result that no compact 3-dimensional $K(\pi,1)$-manifold admits a metric of positive scalar curvature (when combined with the Prime Decomposition Theorem for 3-manifolds [M3] and the surgery results for manifolds of positive scalar curvature this gives an almost complete classification of compact 3-manifolds of positive scalar curvature). In Chapter 3 we extent these results and prove a more general "bad end"-type theorem.

<u>Theorem E.</u> <u>Let X_c be a compact manifold containing a compact codimension 2 submanifold $\Sigma \xrightarrow{i} X_c$ so that the following conditions hold</u>

1. $|\pi_1(X_c)/\pi_1(\Sigma)| = \infty$
2. <u>The normal circle about Σ in X_c has infinite order</u>

in $\pi_1(X_c-\Sigma)$

3. *If* $\Gamma = i_*(\pi_1(\Sigma))$ *then there is a surjective map* f_* *of* Γ *onto the fundamental group of the (n-2)-torus* T^{n-2}, $f_*:\Gamma \to Z^{n-2}$.

Let $X \overset{\pi}{\to} X_c$ *be the cover corresponding to the subgroup* $\tilde{i}(\pi_1(\Gamma)) \subset \pi_1(X)$, $\tilde{i}: \Sigma \to X$ *the lift of* $i:\Sigma \to X_c$, *and* Z *the boundary of a tubular neighbourhood of* $\tilde{i}(\Sigma)$ *in* X. *Let* $f:X \to T^{n-2}$ *be the map inducing* f_*. *In addition to the above conditions we require that*

4. *The composition* $f\circ\tilde{i}:\Sigma \to T^{n-2}$ *has non-zero degree*
5. *There is an* S^1*-bundle* $Z_0 \to T^{n-2}$ *so that the bundle* $Z \to \Sigma$ *is the pull-back of* Z_0 *via* $f\circ\tilde{i}$.

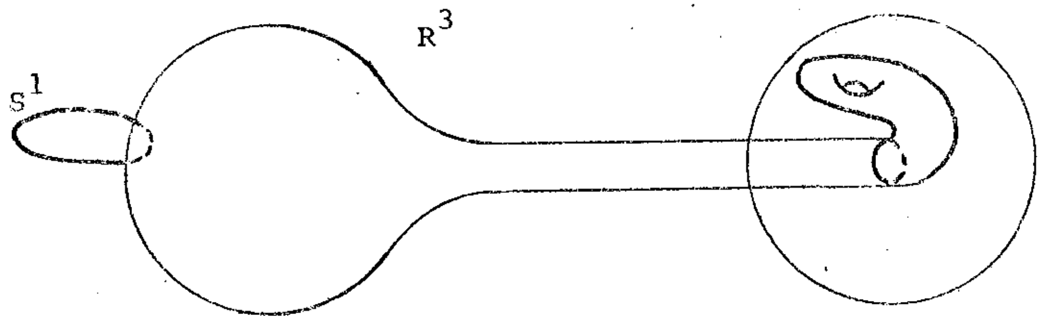
Then $X \to X_c$ *has a cover which does not admit a metric of uniformly positive scalar curvature. In particular,* X_c *cannot carry a metric of positive scalar curvature.*

This theorem can be used to exhibit further examples of compact manifolds which do not admit metrics of positive scalar curvature. Unfortunately, however, we have not been able to extend the $K(\pi,1)$ result to higher dimensions using this method, although this seems to be due to a lack of understanding of the fundamental groups which can appear rather than a fault of method itself.

We now turn our attention to non-compact manifolds of Type 3, that is, ones with codimension 3 enlargeable submanifolds. On the surface it does not appear that these manifolds will be very useful in showing which compact manifolds do not have metrics of positive scalar curvature, since they could, concievably, cover such manifolds. Maybe this is true in general. However, consider the simple 4-dimensional example $X = S^1 \times R^3$ with the product metric of uniformly positive scalar curvature

$$d\theta^2 + (S^3 - \{\text{point}\} \text{ "pulled out"}).$$

Here the R^3 factor (which has 2-dimensional sphere "factors" to carry the positive scalar curvature) has a "long and thin" property so that truncating leaves a small boundary.



If we now assume that X covers a compact manifold we are able to "cap-off" this boundary with a compact manifold and form a 3-cycle which we can arrange to represent a non-zero class in $H_3(X)$ (simply by truncating a long way out and capping off so that the intersection number with the

S^1 doesn't change - we may have to take a finite cover $\overline{X} \to X$ first). This contradiction shows that X with this metric cannot be the riemannian cover of a compact manifold.

Thus, potentially, we have another method for showing that a compact manifold X_c does not admit a metric of positive scalar curvature - we (simply) have to show that it has a non-compact cover $X \to X_c$ having properties similar to those just described for $S^1 \times R^3$. The idea is that if X has a codimension 3 "bad" submanifold Σ then we may be able to find a complementary 3-dimensional submanifold M X which "carries" the positive scalar curvature and is (hopefully) "long and thin" in some sence. We can truncate this submanifold and may be able to cap off the resulting boundary in X to form a non-zero element of $H_3(X)$. This will be a contradiction in some cases.

The most delicate part of this method is in the finding of a suitable complementary 3-dimensional submanifold M and showing that it is sufficiently "long and thin". Here is one possible method for doing this. We notice that the R^3-factor in our $S^1 \times R^3$ example is a stable minimal 3-manifold. Therefore, a natural choice for our 3-manifold M is to pick it to be a stable minimal submanifold in X which intersects the "bad" submanifold Σ non-trivally. Such a choice might work in all dimensions but, for the moment, at least, it is only in the study of 4-manifolds that current knowledge enables progress to be made. The reason is that in this case (alone) the 3-manifold M "carrying" the positive curvature is a hypersurface and the techniques developed by Schoen and Yau [$SY_{1,2,3}$] and Fischer-Colbrie and Schoen [F-CS] can be used effectively in conjunction with results concerning 3-manifolds of positive scalar curvature obtained by Gromov and Lawson [GL_3]. A complete description of this with details is given in Chapter 4. We obtain a theorem which can be used to exhibit new examples of compact 4-manifolds which do not admit metrics of positive scalar curvature. Unfortunately, however, we have been unable to give a complete proof that covers compact $K(\pi,1)$ 4-manifolds. We have recently learned that Schoen and Yau have filled in the missing link and have thus proved the theorem that these manifolds do not admit such metrics.

§1.2. Yang-Mills fields - Kalusa-Klein model.

In Chapter 5 of this dissertation we consider some problems arising out of the study of Yang-Mills fields. Yang-Mills fields are used in physics in an attempt to give a classical description of interaction fields. The subject has a long history begining with the early work of

Weyl [W], Kalusa [Ka] and Klein [Kl] in the 1920's. The work of Kalusa and Klein is particually interesting. They attempted to combine the gravitational and electromagnetic forces into a unified theory by working on the total space P of a principal $U(1)$-bundle over space-time M. The motivation for much of our current study here comes from an attempt to understand this model in a wider setting.

In 1954 Yang and Mills [YM] gave the first useful description of interaction fields other than the electromagnetic (and gravitational) fields. Their work involved using the non-abelian Lie group $SU(2)$ and was couched in a differential-geometric setting. It was recognized that the mathematical model was nothing more than a connection on a principal bundle so the generalization to arbitary Lie groups appeared soon after. The theory developed quickly but an added boost was given in 1967 when Weinberg and Salam gave a model which unified the weak and electromagnetic interactions. Since then thousands of papers have appeared in the physics literature.

In this dissertation we will not be concerned with the physical aspects of Yang-Mills fields. In particular, we do not dicuss how they arise naturally and simply start by giving a mathematical definition. This is done in Section 5.1. We introduce the Yang-Mills functional YM and the combined Einstein-Yang-Mills functional E. These are defined on the space Met(M) Conn(P) where P M is our principal bundle, Met(M) the space of riemannian metrics on M and Conn(P) the space of connections on P. Critical points of YM (for a fixed metric) are Yang-Mills potentials and critical points for E satisfy the combined Einstein field and Yang-Mills equations.

In Section 5.2 we study various aspects of the Kalusa-Klein model. We give the formulation in a general setting using a well-known functional E (the total curvature functional) on the space of metrics of the total space P. Critical points of E restricted to the subspace of bundle metrics on P are shown to give solutions of the combined Einstein-Yang-Mills equations on M. However, we also show that in general such critical points satisfy the vacuum Einstein field equation on P only in the case that the group is the unitary group U(1). This latter result clarifies some confusion prevalent in the mathematical community - it was generally believed that the combined Einstein-Yang-Mills equations on M were equivalent to the simpler vacuum Einstein field equations on P.

We also study other critical points of E. For example, Einstein metrics are critical points of $E|_{\mathrm{Met}_V(P)}$ where $\mathrm{Met}_V(P)$ $\mathrm{Met}(P)$ is the subspace of metrics having fixed volume V. We investigate the conditions under which an Einstein bundle metric actually has constant positive curvature by firstly showing that $P \to M$ *admits a bundle metric having constant positive curvature only if it is one of the standard examples, namely*

$$S^{2n+1}/Z_p \to P^n(C) \quad \underline{\text{or}} \quad S^{4n+1} \to P^n(H).$$

We then use a rigidity result derived from the Bochner-Weitezbock formula and observe that *if* $P \to M$ *admits an Einstein bundle metric which is sufficiently close to having constant positive curvature then the metric actually has constant positive curvature*, and we are back in the situation just described. Thus we have a type of topological quantization for Einstein bundle metrics on P.

Chapter 2. Construction of metrics of positive scalar curvature.

In this chapter we contribute to the study of manifolds of positive scalar curvature by giving some new examples.

In the first section (§2.1) we prove a basic surgery--type theorem (Theorem 2.1.1) for hypersurfaces in an arbitary manifold. The underlying idea here is similar to that used by Gromov and Lawson [GL_2] and Schoen and Yau [SY_1] in obtaining their surgery results for manifolds of positive scalar curvature. We use the fact that spheres of dimension ≥ 2 can "carry" positive scalar curvature around corners. However, we work ambiently.

In subsequent sections we use Theorem 2.1.1 to exhibit new riemannian manifolds of positive scalar curvature. In Section 2.2. we use it to show that any finitely presented group can appear as the fundamental group of a compact 4-manifold of positive scalar curvature. In Sections 2.3 - 2.5 we use it in conjunction with some results concerning metrics of positive scalar curvature on manifolds with boundary to study the space of positive scalar curvature metrics on a fixed manifold M. By using the technique of plumbing disc bundles over spheres we obtain interesting results for $M = S^{4m-1}$.

§2.1. Positive scalar curvature on boundaries of regular neighbourhoods.

In this section we prove the following theorem.

Theorem 2.1.1. *Let M be a riemannian manifold of dimension n with a fixed smooth cell decomposition and K a codimension $q \geq 3$ subcomplex of M. Then there is a regular neighbourhood U of K in M so that the induced metric on the boundary ∂U has positive scalar curvature.*

Idea of proof. We show how to construct the neighbourhood U inductively. We successively construct tubular neighbourhoods U_i about the i-skeletons K^i of K, $i=0,1,\ldots,n-q$, so that at each step the induced metric on the boundary ∂U_i has positive scalar curvature. The inductive step is achieved by taking U_i and constructing U_{i+1} by "pulling out" around the (i+1)-cells of the skeleton K^{i+1}. At the final step we will have constructed $U = U_{n-q}$. We note that at each step we are "pulling out" spheres of dimension $\geq q-1 \geq 2$ which are used to carry the positive curvature. Of course, we are working ambiently but a basic fact which indicates that such an idea will work is that the principal curvatures of a small distance sphere

$$S_p(\varepsilon) = \{\ x \in M \mid \mathrm{dist}_M(x,p) = \varepsilon\ \}, \quad p \in M,$$

are close to those of the usual sphere, that is, of the form $-1/\varepsilon + O(\varepsilon)$. The problem then is similar to that of Gromov and Lawson [GL_2] in that we must show that the negative curvature introduced in "pulling out" can indeed be dominated by the positively curved sphere "factors".

Proof of theorem. As indicated above we will construct U inductively by forming successive regular neighbourhoods U_i of the i-skeletons K^i of K, $i = 1,2,\ldots,n-q$.

To begin the induction we form U_0, a regular neighbourhood of the 0-skeleton $K_0 = \{v_1,\ldots,v_r\}$ of K. For $\varepsilon > 0$ and $p \in M$ let

$$B_p(\varepsilon) = \{\ x \in M \mid \mathrm{dist}_M(p,x) \leq \varepsilon\ \}$$

be a small ball centered at p with boundary $S_p(\varepsilon)$. Then set $U_0 = B_{v_1}(\varepsilon_1) \cup \cdots \cup B_{v_r}(\varepsilon_r)$ where the $\varepsilon_i > 0$ are chosen so that

i) $S_{v_i}(\varepsilon_i) \cap S_{v_j}(\varepsilon_j) = O$ for $i \neq j$

ii) $S_{v_i}(\varepsilon_i)$ has positive scalar curvature for $i = 1,\ldots,r$

iii) each 1-cell of K coming into the vertex v_i intersects $S_{v_i}(\varepsilon_i)$ transversely in one point.

For the inductive step we assume that U_{p-1} is a regular neighbourhood the (p-1)-skeleton K^{p-1} such that

a) with the induced metric the hypersurface $H \equiv \partial U_{p-1}$ has positive scalar curvature,

and

b) each p-cell σ of K which intersects H does so transversely in a $(p-1)$-sphere.

We fix a diffeomophism $\phi_\sigma : \sigma \cap K \to S^{p-1}$ onto the $(p-1)$-sphere.

For each p-cell σ of K we now show how to "pull out" U_{p-1} around σ while retaining positive scalar curvature on the boundary. Let η be the outward unit normal to H. By altering K a little, if necessary, we assume that for each $y \in S^{p-1}$ and sufficiently small $\delta > 0$ we have $\exp_{\phi_\sigma^{-1}(y)}(t) \in \sigma$ for all $t \in (-\delta, \delta)$. We define distance functions $\overline{t}: M \to R$ and $\overline{r}: M \to R^+$ as follows. Set

(2.1.2) $$\overline{t}(x) = \mathrm{dist}_M(x, H),$$

where we measure distance positively if $x \in M$ is in the η direction and negatively if x is in the $-\eta$ direction,

and

(2.1.3) $$\overline{r}(x) = \mathrm{dist}_M(x, \sigma).$$

Let $\{e_i\}_{i=1}^{n-q=p}$ be orthonormal sections of the normal bundle of σ, with $e_q = \mathrm{grad}(\overline{r})$. For $r > 0$ let

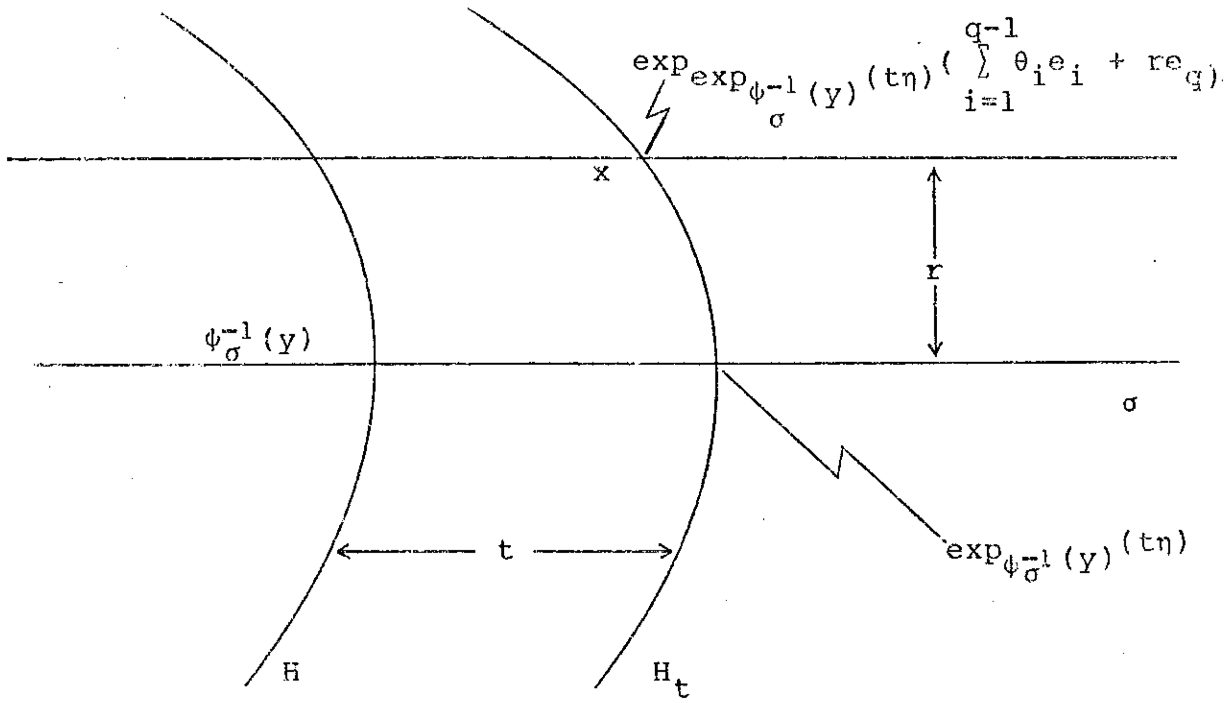
$D^q(R) = \{ x \in R^q \mid \|x\| \leq r \}.$

For sufficiently small $R > 0$ and $0 < T < \delta$ we will identify the set

$$V_{R,T} = \{ x \in M \mid \bar{r}(x) \leq R,\ 0 \leq \bar{t}(x) \leq T \}$$

with $D^q(R) \times [0,T] \times S^{p-1}$ by the map which sends $x \in V_{R,T}$ to $(\theta_1,\ldots,\theta_{q-1},r,t,y)$ where $(\theta_1,\ldots,\theta_{q-1},r) \in D^q(R)$, $t \in [0,T]$ and $y \in S^{p-1}$ are such that $H_t \quad \sigma_{(\theta,r),y} = \{x\}$. Here H_t is the hypersurface a constant distance t from H and

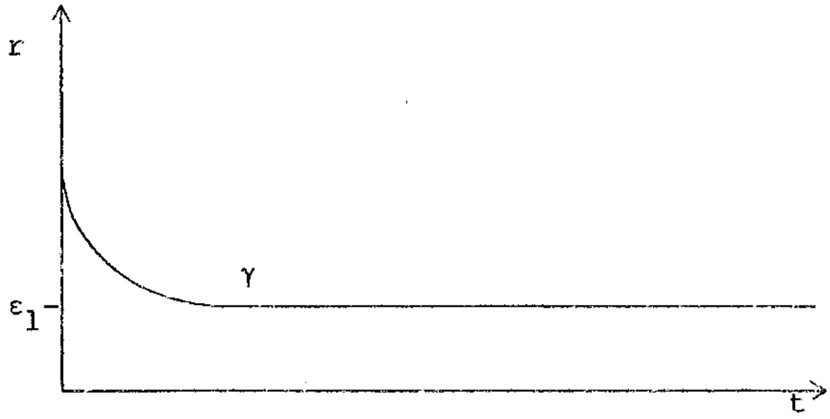
$$\sigma_{(\theta,r),y} = \{ \exp_{\exp_{\psi_\sigma^{-1}(y)}(t\eta)}(\sum_{i=1}^{q-1} \theta_i e_i + r e_q) \mid t \in (-\delta,\delta) \}.$$



We wish to show that in $V_{R,T}$ we are able to choose a hypersurface H' of the form

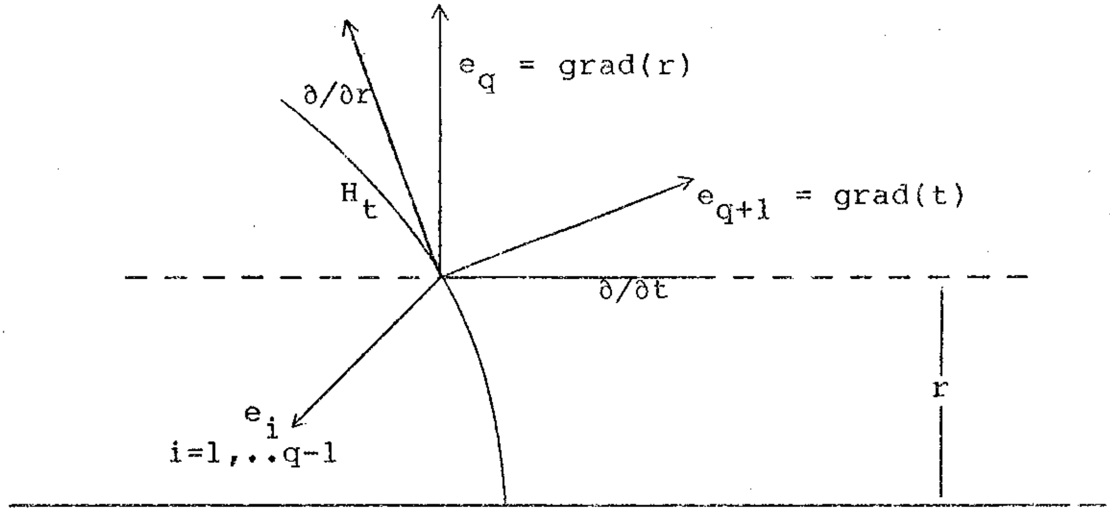
$$H' = \{ (\theta_1,\ldots,\theta_{q-1},r,t) \mid r = \gamma(t) \}$$

where γ is a curve having a graph as follows



and such that the induced metric on H' has positive scalar curvature. The important points about γ are that it starts at $t = 0$ and ends up as a constant $\varepsilon_1 > 0$. The hypersurface H' then has the property that it is pulled out from $H = H_0$ and ends up as a tube a constant distance ε_1 from the cell σ.

For the purposes of calculating the scalar curvature of H' it is convienient to introduce the frame field $\{e_i\}_{i=1}^{n-1}$ on $V_{R,T} = D^q(R) \times [0,T] \times S^{p-1}$ where $e_1,\ldots,e_q$ are, as before, given by polar coordinates about σ, $e_{q+1} = \text{grad}(\bar{t})$ and $\{e_{q+2},\ldots,e_n\}$ forms an orthonormal frame field for the S^{p-1} factor orthogonal to $\{e_1,\ldots,e_{q+1}\}$.



Note that $e_{q+1} = \partial/\partial t + O(r)e_q$.

The metric g on $V_{R,T}$ is given in terms of this basis as

$$(g_{ij}) = \begin{array}{cl} \left(\begin{array}{c|c|c} \delta_{ij} & O(r) & 0 \\ \hline O(r) & & 0 \\ \hline 0 & 0 & \delta_{ij} \end{array}\right) & \begin{array}{l} 1 \leqslant i \leqslant q \\ i=q+1 \\ q+2 \leqslant i \leqslant n \end{array} \end{array}$$

We require only the following crude estimates for the connection ∇ in terms of this basis

$$\nabla_{e_i} e_i = -\frac{1}{r} e_q + \sum_{k=1}^{n} O(1) e_k \qquad 1 \leq i \leq q-1$$

$$(2.1.5) \qquad \nabla_{e_i} e_q = \frac{1}{r} e_q + \sum_{k=1}^{n} O(1) e_k \qquad 1 \leq i \leq q-1$$

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} O(1) e_k \qquad \text{otherwise.}$$

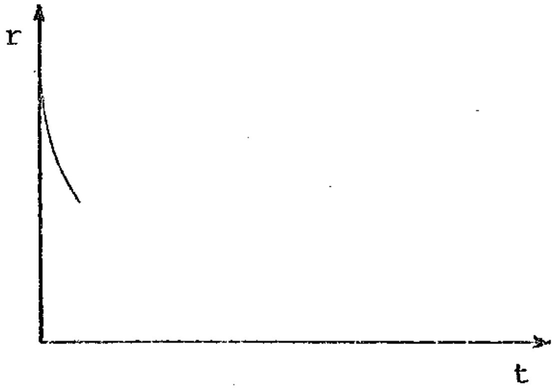
A basis $\{f_i\}_{i=1}^{n}$ for the tangent space of the hypersurface H' in $V_{R,T}$ is given in terms of the basis $\{e_i\}_{i=1}^{n}$ as

$$(2.1.6) \qquad f_i = \begin{cases} \dfrac{e_i + O(1)\gamma(t)\gamma'(t)e_q}{\sqrt{1 + O(1)\gamma(t)^2\gamma'(t)^2}} & i = 1,\ldots,q-1 \\[2ex] \dfrac{e_{q+1} + \gamma'(t)e_q}{\sqrt{1 + \gamma'(t)^2}} & i = q \\[2ex] e_{i+1} & i = q+1,\ldots,n-1. \end{cases}$$

The first expression here can be understood as follows. The angular vector e_i fails to be tangent to the hypersurface H' because the hypersurfaces H_t are not totally geodesic. Hence we have an e_q term the size of which depends to first order on the product of $\gamma(t)$ and $\gamma'(t)$. The unit normal ξ to H' is similarly given as

$$(2.1.7) \qquad \xi = \frac{e_q - \gamma'(t)e_{q+1} + \sum_{k=1}^{q-1} O(\gamma(t)\gamma'(t))e_q}{\sqrt{1 + \gamma'(t)^2 + O(\gamma(t)^2\gamma'(t)^2}}$$

We now proceed to calculate the scalar curvature of the hypersurface H'. Firstly, note that since the condition of having positive scalar curvature is an open condition we can choose an initial segment of our curve γ as shown in the figure below while retaining positive scalar curvature.



Having done this we can assume from now on that γ' is bounded and estimate the scalar curvature of H' in terms of $\gamma(t)$. We need to control $\gamma''(t)$, however.

Firstly, we estimate the sectional curvatures $K_{H'}(f_i \wedge f_j)$, $1 \leq i,j \leq n-1$ of H' using the Gauss curvature equation

$$(2.1.8) \quad K_{H'}(f_i \wedge f_j) = K_M(f_i \wedge f_j) + g(\nabla_{f_i}\xi, f_i)g(\nabla_{f_i}\xi, f_j) \\ -g(\nabla_{f_i}\xi, f_j)^2.$$

Here K_M is the sectional curvature of the ambient manifold M.

Using (2.1.5) and (2.1.7) we have the following

$$
(2.1.9)\quad \begin{cases}
\nabla_{e_i}\xi = \dfrac{1}{\gamma\sqrt{1+\gamma'^2}}\, e_i + \sum\limits_{k=1}^{n} O(1)e_k & i = 1,\ldots,q-1 \\[2ex]
\nabla_{e_q}\xi = \sum\limits_{k=1}^{n} O(1)e_k & \\[2ex]
\nabla_{e_{q+1}}\xi = -\dfrac{\gamma''}{\sqrt{1+\gamma'^2}}(1 + O(\gamma^2))e_{q+1} + \sum\limits_{k=1}^{n} O(1)e_k & \\[2ex]
\nabla_{e_i}\xi = \sum\limits_{k=1}^{n} O(1)e_k & i = q+1,\ldots,n.
\end{cases}
$$

Then from these and (2.1.6)

$$
(2.1.10)\quad \begin{cases}
\nabla_{f_i}\xi = \dfrac{1}{\gamma\sqrt{1+\gamma'^2}}\, e_i + \sum\limits_{k=1}^{n} O(1)e_k & i = 1,\ldots,q-1 \\[2ex]
\nabla_{f_q}\xi = -\dfrac{\gamma''}{\sqrt{1+\gamma'^2}}(1 + O(\gamma^2))e_{q+1} + \sum\limits_{k=1}^{n} O(1)e_k & \\[2ex]
\nabla_{f_i}\xi = \sum\limits_{k=1}^{n} O(1)e_k & i = q,\ldots,n-1.
\end{cases}
$$

From (2.1.8) and the estimates (2.1.10) we have the following estimates for the sectional curvature $K_{H'}$.

$$K_{H'}(f_i\ f_j) = \frac{1}{\gamma^2(1+\gamma'^2)} + O(\tfrac{1}{\gamma}) \qquad 1\leqslant j\leqslant q-1$$

$$K_{H'}(f_i\ f_q) = -\frac{\gamma''}{\gamma(1+\gamma'^2)^2} + O(\gamma'') + O(\tfrac{1}{\gamma}) \qquad 1\leqslant i\leqslant q-1$$

(2.1.11)

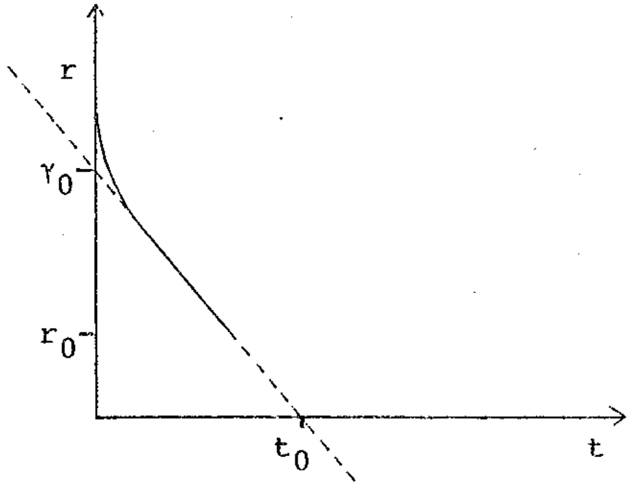
$$K_{H'}(f_i\ f_j) = O(1) \qquad q+1\leqslant j\leqslant n-1$$

$$K_{H'}(f_i\ f_j) = O(1) \qquad q+1\leqslant j\leqslant n-1,\ q\leqslant i\leqslant n-1$$

Finally, the scalar curvature $\kappa_{H'}$ of H' is estimated as

(2.1.12)
$$\begin{aligned}
\kappa_{H'} &= \sum_{i,j=1}^{n-1} K_{H'}(f_i\ f_j) \\
&= \sum_{i,j=1}^{q-1} K_{H'}(f_i\ f_j) + 2\sum_{i=1}^{q-1} K_{H'}(f_i\ f_q) \\
&\quad +2\sum_{i=1}^{q-1}\ \sum_{j=q+1}^{n-1} K_{H'}(f_i\ f_j) \\
&= (q-1)(q-2)\left(\frac{1}{\gamma^2(1+\gamma'^2)} + O(\tfrac{1}{\gamma})\right) \\
&\quad + 2(q-1)\left(-\frac{\gamma''}{\gamma(1+\gamma'^2)^2} + O(\gamma''\gamma) + O(\tfrac{1}{\gamma})\right) \\
&\quad + O(1) \\
&= (q-1)(q-2)\frac{1}{\gamma^2(1+\gamma'^2)} \\
&\quad + 2(q-1)\frac{\gamma''}{\gamma(1+\gamma'^2)^2} + O(\gamma'') + O(\tfrac{1}{\gamma})
\end{aligned}$$

Following a procedure similar to that in [GL_2] we now show how to choose the curve γ so that $\kappa_{H'}$ remains positive. We have already noted that we may choose the initial segment. Extent γ from this as a straight line with equation $\gamma(t) = \gamma_0 - \rho t$ for $t \leqslant t_0 < \gamma_0/\rho$, where $\rho > 0$ is the slope of γ after the initial bend and γ_0 is, as is shown on the graph below, a measure of where we start the bend.

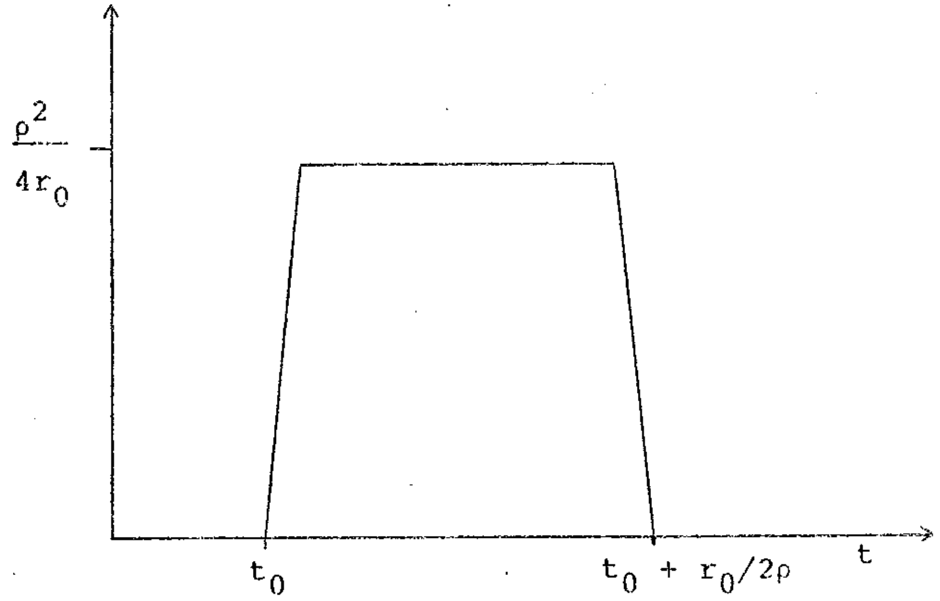


On the straight section

$$\kappa_{H'} = \frac{(q-1)(q-2)}{(\gamma_0 - \rho t)^2(1 + \rho^2)} + O(\frac{1}{\gamma_0 - \rho t})$$

which can be made positive by choosing the parameter γ_0 small enough.

Continue this straight section until we reach a point with $r = r_0$ and then bend γ by choosing γ'' of the form shown below



Note that $\gamma(t) > r_0/2$ throughout this bending. Also, we have that

$$\kappa_{H'} = \frac{q-1}{1+\gamma'^2}\left(\frac{q-2}{\gamma^2} - \frac{2\gamma''}{\gamma(1+\gamma'^2)}\right) + O(\gamma'') + O(\frac{1}{\gamma})$$

$$\geq \frac{q-1}{1+\gamma'^2}\left(\frac{q-2}{r_0^2} - \frac{2(\rho^2/4r_0)}{(r_0/2)^2(1+(\rho-\rho/8)^2)}\right) + O(\frac{1}{r_0})$$

$$= \frac{q-1}{1+\gamma'^2}\cdot\frac{1}{r_0^2}\left((q-2) - \frac{\rho^2}{1+49\rho^2/64}\right) + O(\frac{1}{r_0})$$

$$= c\cdot\frac{1}{r_0^2} + O(\frac{1}{r_0})$$

where the constant $c > 0$ since $q-2 \geq 1$. So choosing the parameter r_0 sufficiently small will assure that $\kappa_{H'} > 0$ around this bend. At the end of the bend the slope of γ has changed by an amount

$$\int_{t_0}^{t_0+r_0/2\rho} \gamma'' \approx \rho/8,$$

so that finally $\gamma' \approx -\rho + \rho/8$. Note that by an appropiate choice of γ'' as above we can actually achieve any change of γ' in the interval $(0,\rho/8)$.

We continue γ as a straight line with the new slope and repeat our bending process as above. After 9 such bends we can achieve $\gamma' = 0$, as required.

This completes the demonstration that we may "pull out" the regular neighbourhood U_{p-1} around a p-cell of K while retaining positive scalar curvature on the boundary. Doing this for each p-cell of K constructs a regular neighbourhood U_p of the p-skeleton. Proceeding inductively we eventually construct $U = U_{n-q}$, a regular neighbourhood of $K^{n-q} = K$, so that the induced metric on the boundary ∂U has positive scalar curvature. This completes the proof of the theorem. QED

§2.2. Positive scalar curvature and the fundamental group.

In this section we use Theorem 2.1.1 to show that manifolds of positive scalar curvature can be topologically complicated, at least on the level of their fundamental group.

Theorem 2.2.1. *Let π be any finitely presented group. Then there is a compact 4-manifold M of positive scalar curvature with $\pi_1(M) = \pi$.*

Proof. Let K be a 2-complex with $\pi_1(K) = \pi$. Such a complex can be constructed by taking a bouquet of circles $S^1 \vee S^1 \vee \ldots \vee S^1$ corresponding to the generators of π then attaching 2-cells according to the relations. By the Whitney Embedding Theorem we can find an embedding ϕ of K into R^5, $\phi : K \to R^5$.

Since K has codimension 3 in R^5, Theorem 2.1.1 shows that K has a regular neighbourhood U so that the induced metric on the boundary $\partial U \equiv M$ has positive scalar curvature. To complete the proof we need only show that $\pi_1(M) = \pi$.

We have that the inclusion $K \hookrightarrow U$ is a homotopy equivalence, so $\pi_1(U) = \pi_1(K) = \pi$. Also, the relative homotopy groups $\pi_1(U,M)$ and $\pi_2(U,M)$ are zero, since any map $f:(D^i,S^{i-1}) \to (U,M)$ representing an element of $\pi_i(U,M)$, $i = 1,2$ can be deformed away from K and then pushed out to $M = \partial U$ radially from K. The long exact sequence for the pair (U,M), namely,

$$\begin{array}{ccccccc} \to \pi_2(U,M) & \to & \pi_1(M) & \to & \pi_1(U) & \to & \pi_1(U,M) \\ \wr\wr & & & & \wr\wr & & \wr\wr \\ 0 & & & & \pi & & 0 \end{array}$$

then shows that $\pi_1(M) = \pi$, as required. QED

§2.3. Metrics of positive scalar curvature on manifolds with boundary.

In this section we proof some results concerning metrics of positive scalar curvature on manifolds with boundary. The class of metrics we study are ones which are a product near the boundary.

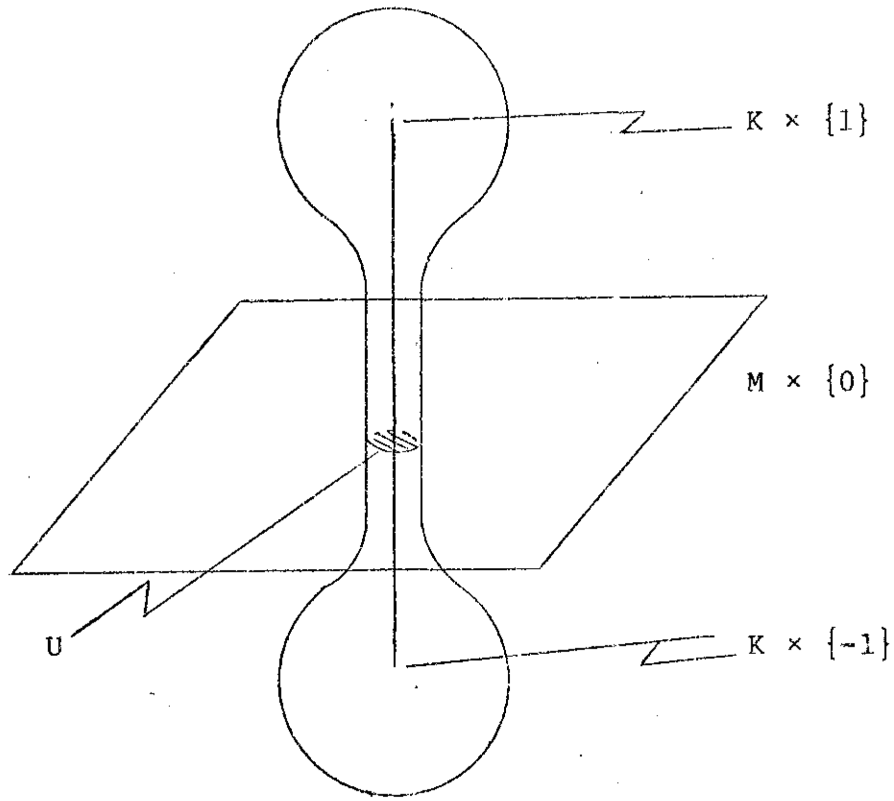
We firstly prove the following extension of Theorem 2.1.1.

Theorem 2.3.1. *Let K be a codimension $q \geq 3$ subcomplex of a riemannian manifold M. Let $U \subset M$ be a regular neighbourhood of K. Then on U there is a metric*

<u>of positive scalar curvature which is a product on a tubular neighbourhood of the boundary.</u>

<u>Proof</u>. We work in the riemannian product $M \times R$ where we take the usual metric on the R factor. In $M \times R$ consider the set $K \times [-1,1]$ which we make into a complex as follows. The zero-skeleton consists of the zero-skeletons of $K \times \{-1\}$ and $K \times \{1\}$, considered as copies of K. Then inductively we obtain the (k+1)-skeleton by taking the (k+1)-skeletons of $K \times \{-1\}$ and $K \times \{1\}$ together with (k+1)-cells of the form $\sigma \times [-1,1]$ where σ is a k-cell of K.

The complex $K \times [-1,1]$ has codimension $q \geqslant 3$ in $M \times R$ so we can apply Theorem 2.1.1 to conclude that it has a regular neighbourhood V with the induced metric on the boundary $H \equiv \partial V$ having positive scalar curvature. The choice of cell structure for $K \times [-1,1]$ and the method of construction of V given in the proof of Theorem 2.1.1 (by "pulling out" to hypersurfaces which are a constant distance from the cells) shows that we may actually choose $V \subset M\times(-\delta,\delta)$ to be of the form $U \times (-\delta,\delta)$ for some $U \subset M$ and positive $\delta < 1$.



We note that U is a regular neighbourhood of K (in $M \times \{0\}$) and is diffeomorphic to $H \quad M\times[0,\infty)$, the top half of the surface of the "dumbell". The induced metric on $H \quad M\times[0,\infty)$ has positive scalar curvature and is a product on a tubular neighbourhood of the boundary, $H \cap M\times[0,\delta)$. We have constructed the metric as required. QED

Now let X_1 and X_2 be n-manifolds having metrics of positive scalar curvature which are products near the boundary. It will be neccessary in later sections to consider their connected sum at the boundary $X_1 \#_\partial X_2$. We wish to know that this also has a metric of positive scalar curvature which is a product near the boundary. This construction is actually an example of the process of adding a solid 1-handle to a manifold X. For $r = 1,\ldots,n-1$ a solid r-handle H is a product $D^r \times D^{n-r}$, and is attached an n-manifold X with boundary ∂X by a map $\phi:\partial D^r \times D^{n-r} \to \partial X$. We have the following theorem.

Theorem 2.3.2. *Let X be an n-manifold carrying a metric of positive scalar curvature which is a product near the boundary. Let H be a solid r-handle where $n-r \geq 4$. Then $X \cup_\phi H$ also carries a metric of positive scalar curvature which is a product near the boundary.*

An immediate corollary of this is the following.

Corollary 2.3.3. *For $n \geq 4$, let $(X_1,\partial X_1)$ and $(X_2,\partial X_2)$ be n-manifolds having metrics of positive scalar curvature which are products near the boundary. Then the connected sum at the boundary $(X_1 \#_\partial X_2, \partial X_1 \# \partial X_2)$ also has a metric of this type.*

Proof of Theorem 2.3.2. We shall give the proof for $r = 1$ - the general result follows by a similar argument.

Since the metric on X is a product near the boundary its double $D(X)$ carries a metric of positive scalar curvature which is a product on a neighbourhood of the join. We also have an involution $\sigma : D(X) \to D(X)$, $\sigma^2 = $ Identity map, which is an isometry and is such that $D(X)/\sigma = X$. Note that the fixed-point set $Fix(\sigma)$ of σ corresponds to ∂X.

Pick $S^0 = \{p_0, p_1\} \subset Fix(\sigma)$. As shown in [GL2] we can deform the original metric in small neighbourhoods about p_0 and p_1 by "pulling out" to a product on the cylinders $R \times S^{n-1}$. This can be done while retaining positive scalar curvature and symmetrically so that

$$\sigma|_{D(X)-S^0} : D(X)-S^0 \to D(X)-S^0$$

will still be an isometry. This shows that we may add a handle $D^1 \times S^{n-1}$ to $D(X)$ while retaining positive scalar curvature and so that we still have an involution

$$\sigma : (D(X)-(S^0 \times D^n)) \cup (D^1 \times S^{n-1}) \to (D(X)-(S^0 \times D^n)) \cup (D^1 \times S^{n-1})$$

which is an isometry.

We consider the induced metric on the manifold with attached handle

$$X \cup (D^1 \times D^{n-1}) = (\ ((D(X)-(S^0 \times D^n)) \cup (D^1 \times S^{n-1}))\ )/\sigma .$$

Outside of a small neighbourhood of the handle this metric is the original one on X and so is a product near the boundary. The handle $D^1 \times D^{n-1}$ has a metric which is very close to being a product of the form (usual metric on D^1) × (hemispherical metric on D^{n-1}). We may deform the hemispherical metric into a "bullet-shape" which is a product near the boundary. Hence $X \cup H$ carries a metric of positive scalar curvature which is a product near the boundary. QED

Note. If X is a manifold carrying a metric of positive scalar curvature with the mean curvature of the boundary equal to zero (not necessarily a product near the boundary) then by a result of Almeida [A] we know that the double $D(X)$ still carries a metric of positive scalar curvature. By using an argument similar to that given in the proof of Theorem 2.3.2 we then have the following.

Theorem 2.3.4. *Let X be an n-manifold carrying a metric of positive scalar curvature with the mean curvature of the boundary equal to zero. Let H be a solid r-handle where $n-r \geqslant 4$. Then $X \cup_\phi H$ also carries a metric of positive scalar curvature with the mean curvature of the boundary equal to zero.*

§2.4. Homotopy groups of the space of positive scalar curvature metrics.

Let M be a compact n-manifold. Let $\mathrm{Met}(M)$ denote the space of riemannian metrics on M and $R^+(M) \subset \mathrm{Met}(M)$ the subset of metrics having positive scalar curvature. In this section we give a procedure for distinguishing non-zero elements of the homotopy groups $\pi_k(R^+(M))$, $k=0,1$. We start with the following.

Lemma 2.4.1. *Assume that $R^+(M)$ is non-empty. For $k = 0,1$ suppose that $[g] \in \pi_k(R^+(M))$ is represented by $g: S^k \to R^+(M)$. Then if $[g] = 0$ there is a metric $\tilde{g}$ of positive scalar curvature on $D^{k+1} \times M$ which is a product of the form*

(2.4.2) $$\tilde{g} = R\cdot g_{S^k} + g(x) + dt^2, \quad x \in S^k$$

on a tubular neighbourhood $S^k \times M \times (1-\delta,1]$ of the boundary $S^k \times M$. Here $R > 0$ is some sufficiently large constant, $R\cdot g_{S^k}$ the usual metric on the sphere of radius R and dt^2 is the usual metric on the interval $(1-\delta,1]$, $\delta>0$.

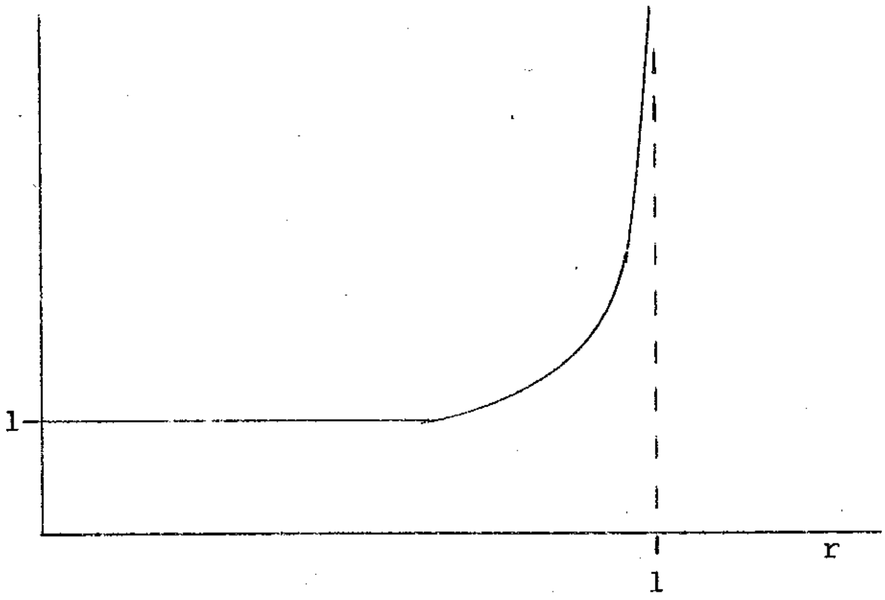
Notes. 1. For $k \geq 2$ this lemma is true without any restriction on $[g]$ being zero in $\pi_k(R^+(M))$ (simply take a "bullet-shaped" metric on the disc factor). Hence our method for distinguishing non-zero elements of $\pi_k(R^+(M))$ will fail for $k \geq 2$.

2. In the case $k = 0$ Lemma 2.4.1 is equivalent to the following statement. If metrics g_0 and g_1 on a manifold M are in the same connected component of $R^+(M)$, that is, are homotopic through metrics of positive scalar curvature, then they are H-cobordant, that is, on the manifold $W = M \times [0,1]$ there is a metric $\tilde{g}$ of positive scalar curvature which is a product near the boundary with $\tilde{g}|_{M\times\{0\}} = g_0$ and $\tilde{g}|_{M\times\{1\}} = g_1$.

Proof of Lemma 2.4.1. Since $[g] = 0$ in $\pi_k(R^+(M))$ we have, by definition, that g extends to a map $\bar{g}: D^{k+1} \to R^+(M)$, $\bar{g}|_{S^k} = g$. Let $g_{D^{k+1}}$ be the usual flat metric on the unit disc D^{k+1}. Let $\{f_\alpha\}_{\alpha=1}^{n}$ be a frame field for M and $\{e_i\}_{i=1}^{k+1}$ an orthonormal frame field for D^{k+1} with $e_{k+1} = \partial/\partial r$, the radial vector field. For $R > 0$ consider the metric h given by

$$h = R\cdot g_{D^{k+1}} + \bar{g}(x), \qquad x \in D^{k+1}.$$

We claim that choosing R large enough will assure that $\kappa_h > 0$, but, unfortunately, h will not in general be a product near the boundary. We may overcome this by deforming it into one that is as follows. Firstly, form a complete metric, call it h', on the open manifold $(D^{k+1}\times M) - \partial(D^{k+1}\times M)$ by multiplying the radial component of h by $f(r)$, where $f:[0,1) \to R^+$ has the graph shown below.



The metric h' has scalar curvature

$$\kappa_{h'} = \sum_{\alpha,\beta=1}^{n} K_{f_\alpha f_\beta} + 2\sum_{\alpha=1}^{n}\sum_{i=1}^{k} K_{f_\alpha e_i} + 2\sum_{\alpha=1}^{n} K_{f_\alpha e_{k+1}}$$

$$+ \sum_{i,j=1}^{k} K_{e_i e_j} + 2\sum_{i=1}^{k} K_{e_i e_{k+1}}$$

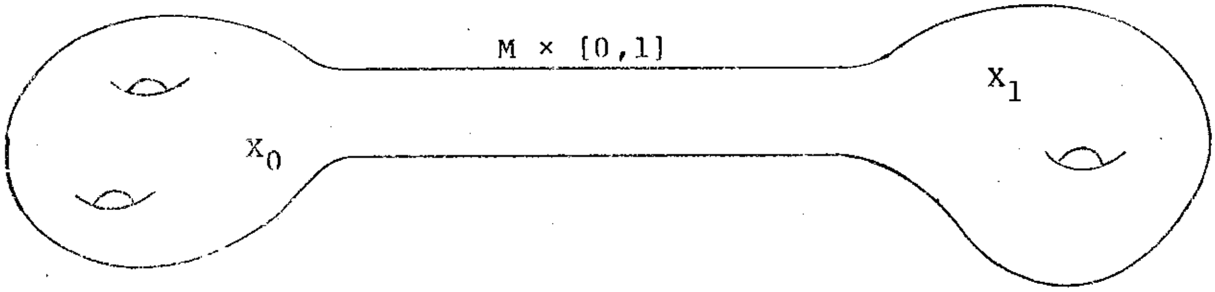
$$= \kappa_{g(x)} + \frac{1}{R} c_1 + \frac{1}{f(r)R} c_2 + 0$$

$$+ \frac{k}{R^2} \left(\frac{-f''(r)^2}{f(r)(1 + f'(r)^2)^2} \right)$$

where $c_1, c_2 : D^{k+1} \times M \to \mathbb{R}$ are independent of R. Note that the term in brackets in the last expression is simply the curvature of the surface given by rotating the graph of f about the r-axis and, by choice of f, is positive. Hence we see that choosing R large enough will assure that $\kappa_{h'} > c_0 > 0$. (Here $c_0 = \min\{ \kappa_{g(x)}(m)/2 \mid (x,m) \in D^{k+1} \times M \}$, say). By a slight deformation we may alter h' so that outside of $D^{k+1}(1-\delta) \times M$, that is, on $(S^k \times M) \times (1-\delta, 1]$, the metric is a product of the form (2.4.2). This completes the proof. QED

Using Lemma 2.4.1 we now show how it is possible in some cases to detect non-zero elements of $\pi_k(R^+(M))$. Suppose that we construct a manifold X with boundary $\partial X = M \times S^k$ together with a metric of positive scalar curvature so that near the boundary the metric is a product of the form (2.4.2) induced by a map $g: S^k \to R^+(M)$. Lemma 2.4.1 shows that if $[g]$ is zero in $\pi_k(R^+(M))$ then we can "cap off" X by attaching a copy of $M \times D^{k+1}$ along their common boundary $M \times S^k$ thus forming a closed manifold $\tilde{X} = X \cup (M \times D^{k+1})$ which still carries a metric of positive scalar curvature. However, we may be able to calculate, for example, the $\hat{A}$-genus of $\tilde{X}$, and if this $\neq 0$ we obtain a contradiction to the theorem of Lichnerowitz [L] and conclude that $[g] \neq 0$ in $\pi_k(R^+(M))$.

In the following section we carry out such a procedure for $k = 0$. We construct manifolds X_0 and X_1 with ∂X_0 and ∂X_1 diffeomorphic, via f_0 and f_1, to M, say, together with metrics $\overline{g}_0$ and $\overline{g}_1$ of positive scalar curvature which are products near the boundary. If $f_{0*}(g_0)$ and $f_{1*}(g_1)$, the induced metrics on M, are in the same connected component of $R^+(M)$ then we know that there is a metric of positive scalar curvature on $\tilde{X} = X_0 \cup M\times[0,1] \cup X_1$.

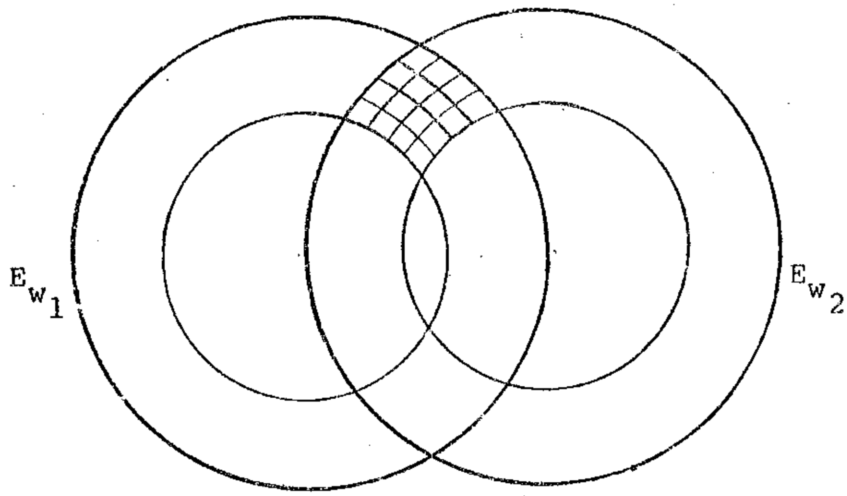


We calculate $\hat{A}(\tilde{X})$ and show that by appropriate choices we can arrange to have it $\neq 0$. This contradiction shows that g_0 and g_1 are not in the same connected component of $R^+(M)$.

§2.5. Plumbing.

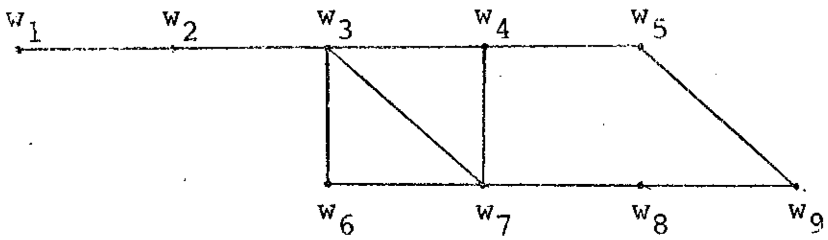
Using the procedure given in the last section we obtain results concerning the connected components of the space of positive scalar curvature metrics on the spheres S^{4m-1}, $m=2,3,\ldots$ We show, in particular, that the number of connected components of the space of positive scalar curvature metrics on S^{4m-1} is infinite. We use a method of constructing manifolds called plumbing, and we wish to emphasize the ideas and results involved in this construction as much as the conclusions we draw.

Plumbing is a method of constructing even-dimensional manifolds with boundaries, given as follows. We recall that D^n-bundles over S^n are classified by $\pi_{n-1}(SO(n))$. Take $w_1, w_2 \in \pi_{n-1}(SO(n))$ and let E_{w_1}, E_{w_2} be the corresponding D^n-bundles over S^n. Let D_i^n S_i^n, $i=1,2$ be discs embedded in the base spheres and let $f_i: D_i^n \times D^n$ $E_{w_i}|_{D_i^n}$ be local trivializations over D_i^n. We plumb E_{w_1} to E_{w_2} by identifying $f_1(x,y)$ with $f_2(y,x)$.



This gives a 2n-manifold with boundary that is smooth everywhere except on $f_1(S^{n-1} \times S^{n-1})$. Along this set we can smooth as shown in [HNK].

Repeated plumbings as above can be conveniently described using a plumbing diagram. This is simply a $\pi_{n-1}(SO(n))$-weighted graph G, that is, a graph with each vertex v_i assigned a weight $w_i \in \pi_{n-1}(SO(n))$, $i = 1,\ldots,N$ We will denote such an object by (G,w_i) or, if there is no confusion, by G.



We do not require that G be connected.

We form a connected manifold with boundary $(X_G, \partial X_G)$ from G as follows. Firstly take the bundles E_{w_i}, $i=1,\ldots,N$, and plumb E_{w_i} to E_{w_j} if v_i is connected to v_j by an edge in G. This will give a connected manifold X_{G_k} for each connected component G_k, $k=1,\ldots,K$ of G. We then form X_G by setting $X_G = X_{G_1} \#_\partial X_{G_2} \#_\partial \ldots\ldots \#_\partial X_{G_K}$, where $\#_\partial$ means connected sum at the boundary.

A basic fact which makes this plumbing technique useful in the study of positive scalar curvature metrics is the following.

<u>Theorem 2.5.1.</u> *If $n \geqslant 3$ and (G, w_i) a $\pi_{n-1}(SO(n))$-weighted graph then the $2n$-dimensional manifold with boundary $(X_G, \partial X_G)$ formed by plumbing according to G has a metric of positive scalar curvature which is a product near the boundary.*

<u>Proof</u>. For each connected component G_k, $k=1,\ldots K$ of G the base spheres of the plumbed bundles form a codimension $n \geqslant 3$ subcomplex of the corresponding manifold

X_{G_i}, and X_{G_i} itself forms a regular neighbourhood of this subcomplex. Theorem 2.3.1 applies directly to show that X_{G_i} has a metric of positive scalar curvature which is a product near the boundary. We then apply Theorem 2.3.2 to conclude that on $X_G = X_{G_1} \#_\partial \ldots \#_\partial X_{G_K}$ there is also a metric of this type. QED

To carry out the program outlined in the previous section we need to answer the following questions.

Q1. For which graphs G_0, G_1 are ∂X_{G_0} and ∂X_{G_1} diffeomorphic?

Q2. If $\partial X_{G_0} \approx \partial X_{G_1} \approx M$ and $\tilde{X} = X_{G_0} \cup M\times[0,1] \cup X_{G_1}$ what is $\hat{A}(\tilde{X})$? In particular, when is $\hat{A}(\tilde{X}) \neq 0$?

To help answer these we gather a number of facts concerning the manifolds X_G. These are based on results given in [HNK, chapter 8].

Lemma 2.5.2. For $n \geq 3$ let G be a $\pi_{n-1}(SO(n))$-weighted graph with components G_k, $k=1,\ldots K$. Suppose that G_k has N_k verticies and is homotopy equivalent to

$S^1 \vee S^1 \vee \ldots \vee S^1$ (p_k times). Set

$$p = \sum_{k=1}^{K} p_k \qquad \text{and} \qquad N = \sum_{k=1}^{K} N_k.$$

Let $(X_G, \partial X_G)$ be the 2n-manifold with boundary given by plumbing according to G. Then

$$H_n(X_G, Z) = \begin{cases} Z & k=0 \\ Z^p & k=1 \\ Z^N & k=N \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, $H_n(X_G, Z)$ is generated by the base spheres of the plumbed bundles.

Proof. We observe that for each connected component G_k of G, the bundles in the corresponding plumbing retract onto their base spheres. So

$$X_{G_k} \underset{h.e.}{\sim} \left(\bigvee_{i=1}^{N_k} S^n\right) \vee \left(\bigvee_{j=1}^{pk} S^1\right).$$

We form X_G by taking the connected sum at the boundary, $X_G = X_{G_1} \#_\partial \ldots \#_\partial X_{G_K}$. The connecting strips retract onto lines. So

$$X_{G_k} \underset{h.e.}{\sim} \bigvee_{k=1}^{K} \left(\bigvee_{i=1}^{N_k} S^n\right) \vee \left(\bigvee_{j=1}^{p_k} S^1\right).$$

This gives the homology of X_G and we note that $H_n(X_G,Z)$ is indeed generated by the base spheres of the plumbed bundles, as claimed. QED

Recall that for any oriented 2n-dimensional manifold X (with possibly non-empty boundary) we have the <u>intersection pairing</u> $\Phi:H_n(X,Z) \times H_n(X,Z) \to Z$. Under the Poincare-Lefschetz duality isomorphism $\zeta\cap:H^{2n-1}(X,\partial X,Z) \to H_k(X,Z)$ this is just the cup product

$$\cup:H^n(X,\partial X,Z) \times H^n(X,\partial X,Z) \to H^{2n}(X,\partial X,Z) \approx Z.$$

The intersection pairing Φ has an associated map, the <u>intersection form of X</u>, called Φ again, defined on the free part of the homology groups

$$\Phi:\mathrm{Free}(H_n(X,Z)) \times \mathrm{Free}(H_n(X,Z)) \to Z$$

Associated to the intersection form Φ is a map

$$\phi:\mathrm{Free}(H_n(X,Z)) \to \mathrm{Free}(H_n(X,Z)))^* \approx \mathrm{Free}(H^n(X,Z))$$

defined by

$$\phi(a)(b) = \Phi(a,b), \quad a,b \in \mathrm{Free}(H_n(X,Z))$$

The map ϕ is called the <u>correlation</u> of Φ. If ϕ is injective we say that Φ is <u>non-degenerate</u>. If ϕ is an isomorphism we say that Φ is <u>non-singular</u>. This latter condition is equivalent to saying that any matrix representation M for Φ (over Z) has $|\det M| = 1$ - we say that such a Φ is <u>unimodular</u>. If n is even, say

$n = 2m$, the intersection form Φ is symmetric and is called the quadratic form of the 4m-dimensional manifold X. In this case Φ can be diagonalized (over R), and if we let α_+ be the number of positive entries and α_- the number of negative entries on the diagonal (they are independent of the choice of representation) we define the signature $\mathrm{sig}(\Phi)$ of Φ by $\mathrm{sig}(\Phi) = \alpha_+ - \alpha_-$. The signature of X, $\mathrm{sig}(X)$, is defined to be the signature of its quadratic form.

In the case we are studying we can easily give the quadratic form. We recall the classical definition of the Euler number of a bundle as the number of "self-intersections" of the zero section (that is, push off the zero section transversely and count the number of intersections of this new surface with the original zero section). With this definition the following is immediate.

Proposition 2.5.3. *For* $m \geq 1$ *let* G *be a* $\pi_{2m-1}(SO(2m))$*-weighted graph with verticies* v_i *of weight* w_i, $i=1,\ldots N$. *Let* $(X_G, \partial X_G)$ *be the 4m-dimensional manifold obtained by plumbing according to* G. *Then in terms of the basis* $\{S_i\}_{i=1}^{N}$ *of* $H_{2m}(X_G)$ *consisting of the base spheres of the plumbed bundles* E_{w_i} *the quadratic form* Φ_G *of* G *has the following matrix representation.*

$$\Phi_G = (\alpha_{ij}), \quad \alpha_{ij} = \begin{cases} e(w_i) & i=j \\ 1 & i \neq j \text{ and } v_i \text{ is connected} \\ & \text{to } v_j \text{ by an edge in } G \\ 0 & i \neq j \text{ otherwise} \end{cases}$$

<u>where</u> $e(w_i)$ <u>is the Euler number of the bundle</u> E_{w_i}.

We should note that in this theorem G need not be connected - if G has components G_k, $k=1,\ldots,K$ with associated quadratic forms Φ_{G_k} then the quadratic form Φ_G of X_G has the representation

$$\Phi_G = \begin{pmatrix} \Phi_{G_1} & & & & 0 \\ & \Phi_{G_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \Phi_{G_K} \end{pmatrix}$$

Hence we have the following.

<u>Corollary 2.5.4.</u> <u>Let</u> G <u>be a</u> $\pi_{2m-1}(SO(2m))$<u>-weighted graph with connected components</u> G_k, $k=1,\ldots,K$. <u>Then the signature of</u> G, $\mathrm{sig}(G)$, <u>is the sum of the signatures</u> $\mathrm{sig}(G_k)$ <u>of the components, that is,</u>

$$\mathrm{sig}(G) = \sum_{k=1}^{K} \mathrm{sig}(G_k).$$

(Here $\mathrm{sig}(G)$ means the signature of the associated $4m$-manifold X_G, that is, the signature of the quadratic form Φ_G).

In particular, we have

Corollary 2.5.5. *Let G be a $\pi_{2m-1}(SO(2m))$-weighted graph. For $k \geq 1$ let kG be the graph consisting of k disjoint copies of G. Then $\mathrm{sig}(kG) = k \cdot \mathrm{sig}(G)$.*

Since we will need the above signature results later we could continue to restrict n to be even without effect. However, the following results (2.5.6) - (2.5.10) are stated for the general case.

We calculate the homology of ∂X_G.

Theorem 2.5.6. *Let G and $(X_G, \partial X_G)$ be as in Lemma 2.5.2. Let $\phi_G : H_n(X_G) \to H_n(X_G)^*$ be the correlation of the intersection form Φ_G of X_G. Then*

$$H_k(\partial X_G, Z) = \begin{cases} Z & k=0, 2n-1 \\ Z^p & k=1, 2n-2 \\ \mathrm{coker}(\phi_G) & k=n-1 \\ \ker(\phi_G) & k=n \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>. The first isomorphism is automatic, since ∂X_G is connected and oriented. For the rest we use the long exact sequence of the pair $(X_G, \partial X_G)$, knowing $H_k(X_G)$ from Lemma 2.5.2.

For $k=1$ we have

$$\begin{tikzcd}
H_2(X_G,\partial X_G) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_1(\partial X_G) \arrow[r] & H_1(X_G) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_1(X_G,\partial X_G) \arrow[d, phantom, sloped, "\cong"] \\
H^{2n-2}(X_G) \arrow[d, phantom, sloped, "\cong"] & & Z^p & H^{2n-1}(X_G) \arrow[d, phantom, sloped, "\cong"] \\
H_{2n-2}(X_G) \arrow[d, phantom, sloped, "\cong"] & & & H_{2n-1}(X_G) \arrow[d, phantom, sloped, "\cong"] \\
0 & & & 0
\end{tikzcd}$$

The two outside vertical isomorphisms are given by Poincare-Lefschetz duality followed by the isomorphism

$$H^k(X_G) \simeq \mathrm{Free}(H_k(X_k)) + \mathrm{Tor}(H_{k-1}(X_G), \quad k=0,\ldots,n$$

and Lemma 2.5.2. So $H_1(\partial X_G) \simeq Z^p$. Also we have

$$H_{2n-2}(\partial X_G) \simeq H^1(\partial X_G) \simeq H_1(\partial X_G) \simeq Z^p.$$

In the middle dimensions, $k = n, n-1$ we have the exact sequence

$$\begin{tikzcd}
H_{n+1}(X_G,\partial X_G) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_n(\partial X_G) \arrow[r] & H_n(X_G) \arrow[r, "i"] \arrow[ddr, "\phi_G"'] & H_n(X_G,\partial X_G) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_{n-1}(\partial X_G) \arrow[r] & H_{n-1}(X_G) \arrow[d, phantom, sloped, "\cong"] \\
0 & & & H^n(X_G) \arrow[d, phantom, sloped, "\cong"] & & 0 \\
& & & H_n(X_G)^* \arrow[uur] & &
\end{tikzcd}$$

The vertical isomorphisms follow from Poincare-Lefchetz duality and Lemma 2.5.2. We note that under the isomorphism

$$H_n(X_G,\partial X_G) \approx H_n(X_G)^*$$

the map i is just the intersection pairing ϕ_G. So we have

$$H_n(\partial X_G, Z) \approx \ker(\phi_G)$$

and

$$H_{n-1}(\partial X_G) \approx H_n(X_G)^*/\mathrm{im}(\Phi_G) \approx \mathrm{coker}(\phi_G).$$

For the other values of k the long exact sequence together with Lemma 2.5.2 shows that $H_k(\partial X_G, Z) \approx 0$. This completes the proof. QED

If we restrict ourselves to $\pi_{n-1}(SO(n))$-weighted graphs G for which the associated intersection form Φ_G is unimodular (so that the correlation ϕ_G is an isomorphism) we have the following corollary of Theorem 2.5.6.

Corollary 2.5.7. *Let G and $(X_G, \partial X_G)$ be as in Lemma 2.5.2, where now we add the assumption that the intersection form Φ_G is unimodular. Then*

$$H_k(\partial X_G, Z) \approx \begin{cases} Z & k=0,2n-1 \\ Z^p & k=1,2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

We now return to the problem of answering Questions Q1 and Q2 on page 47. Firstly we consider Q1. Unfortunately, we cannot answer this question in general but never-the-less we can exhibit an important class of graphs for which an answer is possible. This is the set of trees and for these we have the following well-known theorem.

Theorem 2.5.8. *For $n \geqslant 3$ let T be a (possibly disconnected) $\pi_{n-1}(SO(n))$-weighted tree with unimodular intersection form. Let $(X_T, \partial X_T)$ be the 2n-dimensional manifold with boundary obtained by plumbing according to T. Then ∂X_T ia a homeomorphic (2n-1)-sphere.*

Proof. From Corollary 2.5.6 we have that

$$H_k(\partial X_T, Z) \approx \begin{cases} Z & k=0,2n-1 \\ 0 & \text{otherwise} \end{cases}$$

from which it follows by the Whitehead theorem that ∂X_T is homotopy equivalent to the sphere S^{2n-1}. Then, since $2n-1 \geqslant 5$, the proof of the generalized Poincare conjecture given by Smale [S] shows that it is actually homeomorphic to a (2n-1)-sphere. QED

For $k \geqslant 5$ the set of homeomorphic k-spheres θ_k forms a finite group under the action of the connected sum $\#$ (see [KM]). The identity element is the k-sphere with its usual diffeomorphic structure. So we have the following immediate corollary of Theorem 2.5.8.

Corollary 2.5.9. *For $n \geq 3$ let T be a $\pi_{n-1}(SO(n))$-weighted tree with unimodular intersection form. Let $(X_T, \partial X_T)$ be the associated manifold and let k_T be the order of ∂X_T in the group θ_{2n-1} of homeomorphic $(2n-1)$-spheres. Let $k_T T$ be the tree obtained by taking k_T disjoint copies of T. Then $\partial X_{k_T T}$ is diffeomorphic to the usual $(2n-1)$-sphere. In fact, for each integer $q \geq 1$ we have that $\partial X_{(qk_T)T}$ is diffeomorphic to the usual $(2n-1)$-sphere.*

We will now consider Question Q2 on page 47. In view of Theorem 2.5.8 it would be natural at this point to restrict to $\pi_{n-1}(SO(n))$-weighted *trees*, but we firstly give the following more general answer.

Theorem 2.5.10. *For $n \geq 3$ let G_0 and G_1 be $\pi_{n-1}(SO(n))$-weighted graphs with $(X_{G_0}, \partial X_{G_0})$ and*

$(X_{G_1}, \partial X_{G_1})$ the associated manifolds. Assume that the following conditions hold.

i) If $w \in \pi_{n-1}(SO(n))$ is a weight in G_0 or G_1 then the bundle $E_w \to S^n$ has $p_{n/4}(E_w) = 0$ where $p_{n/4}(E_w) \in H^n(S^n)$ is the $n/4^{th}$ Pontrjagin class.

ii) The manifolds ∂X_{G_0} and ∂X_{G_1} are diffeomorphic (to M, say)

iii) The intersection forms Φ_{G_0} and Φ_{G_1} are unimodular.

Form the compact manifold $\tilde{X} = X_{G_0} \cup M\times[0,1] \cup X_{G_1}$. Then

$$\hat{A}(\tilde{X}) = \begin{cases} 0 & n=2m+1,\ m=1,\ldots \\ c(\mathrm{sig}(G_0)-\mathrm{sig}(G_1)) & n=2m,\ m=2,\ldots \end{cases}$$

where c is some non-zero constant and $\mathrm{sig}(G_i)$ is the signature of the 4m-manifold X_{G_i}, $i=0,1$.

Proof. For $n = 2m+1$ the manifold X has dimension $4m+2$, so $\hat{A}(\tilde{X}) = 0$ automatically.

For $n = 2m$ the manifold $\tilde{X}$ has dimension $4m$ so possibly $\hat{A}(\tilde{X}) \neq 0$. To calculate $\hat{A}(\tilde{X})$ we use its expression in terms of the Pontrjagin classes $p_k \in H^{4k}(\tilde{X})$ of $\tilde{X}$, that is,

$$\hat{A}(\tilde{X}) = \hat{A}(p_1,\ldots,p_m)[\tilde{X}] \tag{2.5.11}$$

where $\hat{A}(p_1,\ldots,p_m) \in H^{4m}(\tilde{X})$ is some universal homogeneous polynomial and $[\tilde{X}]$ is the fundamental class of $\tilde{X}$. There is an explicit formula for the coefficients of $\hat{A}(p_1,\dots,p_m)$ but we only need the fact that

(2.5.12) $$\hat{A}(p_1,\dots,p_m) = c_m p_m + \text{terms involving only } p_{m-1},\dots,p_1.$$

where c_m is some non-zero constant.

Compare this with the similar expression for the signature $\operatorname{sig}(\tilde{X})$ of $\tilde{X}$. We can write

(2.5.13) $$\operatorname{sig}(\tilde{X}) = L(p_1,\dots,p_m)[\tilde{X}]$$

where $L(p_1,\dots,p_m) \in H^{4m}(\tilde{X})$ is the Hirzebruch L-genus of $\tilde{X}$ and, as for $\hat{A}(p_1,\dots,p_m)$, is a homogeneous polynomial. Again, all that we need to know concerning the exact formula for L is that

(2.5.14) $$L(p_1,\dots,p_m) = d_m p_m + \text{terms involving only } p_{m-1},\dots,p_1$$

where d_m is some non-zero constant.

We also have an expression for $\operatorname{sig}(\tilde{X})$ directly in terms of the signatures $\operatorname{sig}(G_0)$ and $\operatorname{sig}(G_1)$ of X_{G_0} and X_{G_1}

(2.5.15) $$\operatorname{sig}(\tilde{X}) = \operatorname{sig}(G_0) - \operatorname{sig}(G_1)$$

We now make the following claim.

Claim. All the lower Pontrjagin classes p_k, $k=1,\dots,m-1$ of $\tilde{X}$ are zero.

With this claim we complete the proof of the theorem since from Equations (2.5.11) to (2.5.15) we have

$$\begin{aligned}\hat{A}(\tilde{X}) &= (c_m p_m)[\tilde{X}], \quad c_m \neq 0\\ &= (c_m/d_m)\tilde{\mathrm{sig}}(X), \quad d_m \neq 0\\ &= (c_m/d_m)(\mathrm{sig}(G_0)-\mathrm{sig}(G_1))\\ &= c(\mathrm{sig}(G_0)-\mathrm{sig}(G_1)), \quad c \neq 0,\end{aligned}$$

as required.

Proof of claim. Consider the following portion of the Mayer-Vietoris sequence for $\tilde{X} = X_{G_0} \cup X_{G_1}$ where $X_{G_0} \cap X_{G_1} = M$.

$$\cdots \to H_{4k}(X_{G_0}) \oplus H_{4k}(X_{G_1}) \to H_{4k}(X) \to H_{4k-1}(M) \to \cdots$$

We have that the first term $= 0$ if $4k \neq n$, that is, $k \neq m/2$, by Lemma 2.5.2, and that the last term $= 0$ by Corollary 2.5.7. Hence we have that $H_{4k}(\tilde{X}) = 0$ for $k \neq m/2$. Since everything is torsion-free this implies that $H^{4k}(\tilde{X}) = 0$ for $k \neq m/2$. Hence the Pontrjagin class p_k is equal to zero except possibly in the case that m is even, say $m = 2q$, and then we have to check $p_q \in H^{4q}(\tilde{X})$.

For this exceptional case we know that $H_{4q}(\tilde{X}) = \mathbf{Z}^N$ where $N = N_0 + N_1$ is the sum of the number of verticies of the graphs G_0 and G_1, with the base spheres S_i, $i=1,\ldots,N$, of the plumbed bundles forming a basis. We wish to show that $p_q = p_q(T\tilde{X})$ is zero, where $T\tilde{X} \to \tilde{X}$ is the tangent bundle, so it suffices to calculate the evaluation $p_q[S_i]$ on each basis element S_i of $H_{4q}(\tilde{X})$. The integer $p_q(T\tilde{X})[S_i]$ is just the qth Pontrjagin number of the bundle $T\tilde{X}|_{S_i} \to S_i$. We have that $T\tilde{X}|_{S_i} = TS_i \oplus NS_i$ where $TS_i \to S_i$ is the tangent bundle to the sphere S_i and $NS_i \to S_i$ is the normal bundle of S_i in $\tilde{X}$. By construction of $\tilde{X}$, the bundle $N_{S_i} \to S_i$ is just $E_{w_i} \to S_i$ where $w_i \in \pi_{4q-1}(SO(4q))$ is the weight assigned to the vertex corresponding to S_i and E_{w_i} is the associated bundle. By Hypothesis i), $p_q(E_{w_i}) = 0$. Hence

$$\begin{aligned} p_q(T\tilde{X})[S_i] &= p_q(T\tilde{X}|_{S_i})[S_i] \\ &= p_q(TS_i \oplus NS_i)[S_i] \\ &= p_q(TS_i \oplus E_{w_i})[S_i] \\ &= (p_q(TS_i) + p_q(E_{w_i}))[S_i] \\ &= 0 \end{aligned}$$

where in the last equality we have used the fact that $p_q(TS^r) = 0$ (since the tangent bundle of a sphere S^r is stably parallizable). Hence $p_q(T\tilde{X}) = p_q(\tilde{X}) = 0$, as claimed. This completes the proof of the theorem. QED

We now restrict n to be even, $n = 2m$.

By combining Corollaries 2.5.5 and 2.5.9 and Theorem 2.5.10 we have a satisfactory answer to Questions Q1 and Q2 on page 47.

The following theorem summarizes the situation.

Theorem 2.5.16. *For* $m \geq 2$ *let* (T, w_i) *be a* $_{2m-1}(SO(2m))$*-weighted tree so that*

a) $p_{m/2}(w_i) = 0$ *for all* i.

b) *The quadratic form* Φ_T *is unimodular and has* $\operatorname{sig}(\Phi_T) \neq 0$.

Let $(X_T, \partial X_T)$ *be the 4m-manifold with boundary given by plumbing according to* T. Then

1. ∂X_T *is a homeomorphic (4m-1)-sphere.*

If $k_T \geq 1$ *is the order of* ∂X_T *in* θ_{4m-1}, *the group of homeomorphic (4m-1)-spheres, then also*

2) *For each integer* $q \geq 1$ *the manifold* $\partial X_{(qk_T)T}$ *is diffeomorphic to the usual (4m-1)-sphere.*

and

3) *For each pair of integers* $q_0, q_1 \geq 1$, $q_0 \neq q_1$, *the manifold*

$$\tilde{X} = X_{(q_0 k_T)T} \cup S^{4m-1} \times [0,1] \cup X_{(q_1 k_T)T}$$

has $\hat{A}(\tilde{X}) \neq 0$.

We now need to show that for $m \geq 2$ there is a $\pi_{2m-1}(SO(2m))$-weighted tree (T, w_i) having the Properties a) and b) in Theorem 2.5.16. A good way of finding such a tree is to try and actually realize a given quadratic form. We firstly give a brief review of quadratic forms (see [MH], for example, for further details)

2.5.17. Quadratic forms. Some of the definitions in this section have already been given in the context of intersection forms of even-dimensional manifolds. We give them again for completeness.

Let V be a free abelian group of rank r. A quadratic form Q over $\mathbf{Z}$ is a symmetric bilinear map $Q: V \times V \to \mathbf{Z}$. The rank $\operatorname{rank}(Q)$ of Q is the integer r. The correlation of Q is the associated map $\phi_Q: V \to V^*$ defined by $\phi_Q(v)(w) = Q(v,w)$, $v, w \in V$. The quadratic form is non-singular or unimodular if ϕ_Q is an isomorphism. This is equivalent to the fact that the determinant of any matrix representation M_Q for Q is equal to ± 1. The matrix M_Q may be diagonalized (over $\mathbf{R}$) and if we then let α_+ equal the number of positive entries and α_- equal the number of negative entries on the diagonal we define the signature $\operatorname{sig}(Q)$ of Q as $\operatorname{sig}(Q) = \alpha_+ - \alpha_-$. If $\operatorname{sig}(Q) = \operatorname{rank}(Q)$ we say that Q is definite. Otherwise it is indefinite. We say that Q is even if $Q(v,v)$ is even for all $v \in V$. This is equivalent to saying that the diagonal entries in any matrix representation M_Q for Q are all even. A quadratic form which is not even is called <u>odd</u>.

We may attempt to classify quadratic forms. If we restrict ourselves to unimodular quadratic forms the known results of interest to us here may be conveniently summarized in the following table.

	even	odd
definite	$\mathrm{sig}(Q) \equiv 0 \pmod 8$ One of rank 8, E_8 Other examples, becoming very complicated as rank increases.	Examples, $\mathrm{diag}(1,1,\ldots)$ $\mathrm{diag}(E_8,1)$ Becomes very complicated as rank increases
indefinite	$Q \simeq \mathrm{diag}(H,\ldots,H,E_8,\ldots E_8)$	$Q \simeq \mathrm{diag}(1,\ldots 1,-1,\ldots -1)$

Here

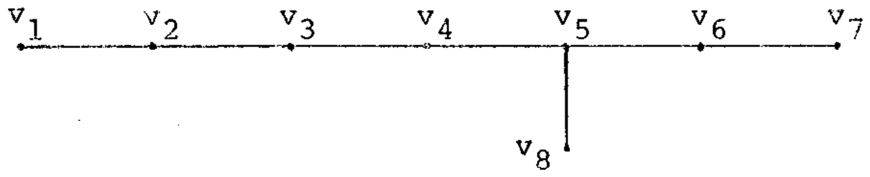
$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.5.18. Plumbing according to the tree E_8.

In this section we show how to find a $\pi_{2m-1}(SO(2m))$-weighted tree (T,w_i) which has the Properties a) and b) given in Theorem 2.16. We concentrate on definite unimodular forms so that Condition b) is automatically satisfied.

From the partial classifaction given in Section 2.5.17 we see that the simplest definite unimodular quadratic form we may try to obtain from a $\pi_{2m-1}(SO(2m))$-weighted tree with $m \geq 2$ is the one represented by the matrix (1). The next simplest is E_8.

The even unimodular definite quadratic form E_8 is actually the easiest to realize by a $\pi_{2m-1}(SO(2m))$-weighted tree (T,w_i), $i=1,\ldots,8$ for which $p_{m/2}(w_i) = 0$ (Condition a) of Theorem 2.5.17). We know (see [MS]) that for each $m \geq 1$ the tangent bundle $TS^{2m} \to S^{2m}$ has Euler number $e(TS^{2m}) = 2$ and $p_{m/2}(TS^{2m}) = 0$. Let $w_{TS} \in \pi_{2m-1}(SO(2m))$ be the classifying element for this bundle. Form the well-known tree T_{E_8}:



and set the weights w_i equal to w_{TS}, $i = 1,\ldots,8$. Then the $\pi_{2m-1}(SO(2m))$-weighted tree (T_{E_8}, w_i) has the properties

a) $p_{m/2}(w_i) = p_{m/2}(w_{TS}) = 0$

and

b) $\Phi_{T_{E_8}} = E_8$, which is unimodular and has non-zero signature.

To simplify our notation we denote this tree by E_8.

Theorem 2.5.16 applies directly and we have

Theorem 2.5.19. For $m \geq 2$ let $(X_{E_8}, \partial X_{E_8})$ be the 4m-manifold-with-boundary obtained by plumbing according to E_8. Then

1. ∂X_{E_8} is a homeomorphic $(4m-1)$-sphere.

If k_{E_8} is the order of ∂X_{E_8} in the group θ_{4m-1} of homeomorphic $(4m-1)$-spheres then also

2. For each integer $q \geq 1$ the manifold $\partial X_{(qk_{E_8})E_8}$ is diffeomorphic to the usual $(4m-1)$-sphere.

and

3. For each pair of distinct integers $q_0, q_1 \geq 1$ the manifold

$$\tilde{X} = X_{(q_0 k_{E_8})E_8} \cup S^{4m-1}\times[0,1] \cup X_{(q_1 k_{E_8})E_8}$$

has $\hat{A}(\tilde{X}) \neq 0$.

We now apply Theorem 2.5.1 and conclude that for $m \geq 2$ and each $n \geq 1$ there is a metric of positive scalar curvature on the 4m-manifold $(X_{nE_8}, \partial X_{nE_8})$ given by plumbing according to the $\pi_{2m-1}(SO(2m))$-weighted tree nE_8 which is a product near the boundary. In particular, for each $q \geq 1$ there is such a metric on $(X_{(qk_{E_8})E_8}, \partial X_{(qk_{E_8})E_8})$. From Theorem 2.5.19 we know that $\partial X_{(qk_{E_8})E_8}$ is diffeomorphic (via $f_q : \partial X_{(qk_{E_8})E_8} \to S^{4m-1}$, say) to the usual (4m-1)-sphere. So letting g_q be the metric induced on $\partial X_{(qk_{E_8})E_8}$ we have a metric $f_q{*}(g_q)$ of positive scalar curvature on the usual sphere S^{4m-1}, $m \geq 2$. Conclusion 3 of Theorem 2.5.19 (together with Lemma 2.4.1) then shows that for distinct integers $q_0, q_1 \geq 1$ the metrics $f_{q_0}{*}(g_{q_0})$ and $f_{q_1}{*}(g_{q_1})$ are not homotopic through metrics of positive scalar curvature. Thus

Theorem 2.5.20. *For $m \geq 2$ the space $R^+(S^{4m-1})$ has an infinite number of connected components.*

Chapter 3. Bad ends and co-dimension 2 enlargeable submanifolds.

In this chapter we prove a generalization of Gromov and Lawson's result concerning manifolds having bad ends. This notion was introduced in [GL_3]:

Definition 3.1. A non-compact manifold X has a bad end X_+ if there exists an enlargeable hypersurface $Z \subset X$ so that X_+ is a non-compact component of $X - Z$ such that

1. the homomorphism $\pi_1(Z) \to \pi_1(\overline{X_+})$ is injective

and

2. there is a map $\overline{X}_+ \to Z_0$ onto an enlargeable manifold Z_0 so that the composition $\overline{Z} \to X_+ \to Z_0$ has non-zero degree.

Gromov and Lawson proved the following.

Theorem 3.2. *Let X_+ be a bad end (of some manifold X) and assume that there is an exhaustion function $F:X \to \mathbb{R}^+$ satisfying*

$$\|\nabla F\| \leqslant C \quad \text{and} \quad \Delta F \leqslant C \tag{3.3}$$

for some constant C. Then there can be no complete metric on X which has uniformly positive scalar curvature on the end X_+.

The proof of this theorem is achieved by using the "inflating balloon" trick. The idea is that if $\overline{X}_+$ carries a metric of uniformly positive scalar curvature then we can use the exhaustion function to construct a metric of positive scalar curvature on $D(\overline{X}_+) \times S^2$ which is a product $D(\overline{X}_+) \times S^2(R)$ outside of a compact set $K \times S^2 \subset D(\overline{X}_+) \times S^2$ (here $D(\overline{X}_+)$ is the double of $\overline{X}_+$). On the other hand, it is shown that this metric is Λ^2-enlargeable and so cannot have positive scalar curvature.

Theorem 3.2 is useful for showing that certain compact manifolds X_c do not admit a metrics of positive scalar curvature since

1) all infinite covers $X \to X_c$ admit an exhaustion function $F:X \to R^+$ having Properties 3.3

and

2) if X_c does carry a metric of positive scalar curvature then the metric lifted to X has uniformly positive scalar curvature.

Thus, if X has a bad end we reach a contradiction and conclude that X_c cannot carry a metric of positive scalar curvature.

To actually give examples of compact manifolds with such covers it is useful to study its codimension 2 submanifolds. The idea is that if X_c has a sufficiently "bad" submanifold $\Sigma \subset X_c$, then in the cover $X \to X_c$ corresponding to the subgroup $i_*(\pi_1(\Sigma)) \subset \pi_1(X_c)$ the set $X - \Sigma$ may be a bad end. Precise theorems using this idea have been given by Gromov and Lawson. The following is a good illustrative example.

Theorem 3.4. *Let X_c be a compact $K(\pi,1)$-manifold and suppose X_c contains an enlargeable $K(\pi,1)$-submanifold $\Sigma \xrightarrow{i} X_c$ of codimension 2 such that the homomorphism $\pi_1(\Sigma) \xrightarrow{i_*} \pi_1(X_c)$ is injective. Suppose that the boundary Z of a tubular neighbourhood of Σ in X is enlargeable. Then X_c carries no metric of positive scalar curvature.*

This theorem implies, in particular, that a compact 3-dimensional $K(\pi,1)$-manifold does not admit a metric of positive scalar curvature since there is an embedded circle $\Sigma \simeq S^1 \subset X_c$ representing an element of infinite order in $\pi_1(X_c)$.

Unfortunately, Theorem 3.4 and others like it are not so easy to apply in high dimensions - the difficulty arises partly from requiring the homomorphism $\pi_1(\Sigma) \to \pi_1(X_c)$ to be injective. We give a strict generalization in which this condition is relaxed.

Theorem 3.5. Let X_c be a compact manifold containing a compact codimension 2 submanifold $\Sigma \xrightarrow{i} X_c$ so that the following conditions hold

1. $|\pi_1(Xc)/\pi_1(\Sigma)| = \infty$
2. The normal circle about Σ in X_c has infinite order in $\pi_1(X_c - \Sigma)$
3. If $\Gamma = i_*(\pi_1(\Sigma))$ then there is a surjective map f_* of Γ onto the fundamental group of the (n-2)-torus T^{n-2}, $f_*:\Gamma \to Z^{n-2}$.

Let $\tilde{X} \xrightarrow{\pi} X_c$ be the cover corresponding to the subgroup $\Gamma \subset \pi_1(X)$, $\tilde{i}: \Sigma \to X^{\sim}$ the lift of $i:\Sigma \to X_c$, and Z the boundary of a tubular neighbourhood of $\tilde{i}(\Sigma)$ in $\tilde{X}$. Let $f:\tilde{X} \to T^{n-2}$ be the map inducing f_*. In addition to the above conditions we require that

4. The composition $f\circ\tilde{i}:\Sigma \to T^{n-2}$ has non-zero degree
5. There is an S^1-bundle $Z_0 \to T^{n-2}$ so that the bundle $Z \to \Sigma$ is the pull-back of Z_0 via $f\circ\tilde{i}$.

Then $\tilde{X} \to \Sigma$ has a cover which does not admit a metric of uniformly positive scalar curvature. In particular, X_c cannot carry a metric of positive scalar curvature.

Proof. We have the following diagram

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "p"'] & \tilde{X} - \Sigma \arrow[dr, "f"] & Z_0 \arrow[d, "p_0"'] \\
\Sigma & & T^{n-2}
\end{tikzcd}$$

We would like to lift the map f to Z_0. Restricted to Z it has a lift of non-zero degree (by Condition 4.), so a lift over all of $\tilde{X} - \Sigma$ would show that $\tilde{X} - \Sigma$ is a bad end. Then Theorem 3.2 would apply to complete the proof.

The obsrtruction to lifting f is a class $[c(f)] \in H^2(\tilde{X} - \Sigma, Z, \pi_1(S^1))$. It is determined by its values on the 2-cells of $\tilde{X} - \Sigma$ - if σ is a 2-cell then $c(f)(\sigma) \equiv [f_1(\delta\sigma)] \in \pi_1(S^1)$, where $f_1 : K^1 \to Z_0$ is any lift of f over the 1-skeleton K^1 of $\tilde{X} - \Sigma$. Unfortunately, since the homomorphism $\pi_1(Z) \to \pi_1(\tilde{X} - \Sigma)$ is not assumed to be surjective, we may not be able to find a cell-decomposition of $\tilde{X} - \Sigma$ in which all 2-cells σ have $\partial\sigma$ Z. In this case we cannot calculate $c(f)(\sigma)$ directly since we do not have a lift f_1 defined.

We can overcome this problem by taking the cover $\widetilde{\tilde{X} - \Sigma} \xrightarrow{\tilde{\pi}} \tilde{X} - \Sigma$ corresponding to the subgroup $i_*(\pi_1(Z))$ in $\pi_1(\widetilde{\tilde{X} - \Sigma})$. Lift Z to $\widetilde{\tilde{X} - \Sigma}$.

$$\begin{tikzcd}
& \widetilde{\tilde{X} - \Sigma} \arrow[d, "\tilde{\pi}"] & \\
Z \arrow[ur, "i"] \arrow[d, "p"'] & \tilde{X} - \Sigma \arrow[dr, "f"] & Z_0 \arrow[d, "p_0"] \\
\Sigma & & T^{n-2}
\end{tikzcd}$$

Now since $\pi_1(Z) \xrightarrow{i_*} \pi_1(\widetilde{\tilde{X} - \Sigma})$ is surjective we can find a cell-decomposition of $\widetilde{\tilde{X} - \Sigma}$ with all 2-cells attached to Z. We can calculate the obstruction, call it $c(f)$ still, to lifting $f \circ \tilde{\pi}$ to Z_0.

For each 2-cell σ of $\widetilde{\tilde{X} - \Sigma}$ we have $c(f)(\sigma) \equiv [f(\partial\sigma)] \in \pi_1(\text{fiber of } Z_0 \to T^{n-2})$. If $[f(\partial\sigma)] \neq 0$ then by the choice of Z_0 (in 5. above) we have that $[\partial\sigma] \neq 0$ in $\pi_1(\text{fiber of } Z \to \Sigma)$. However, this is impossible in view of Condition 2, since no non-zero multiple of the normal circle about Σ in $\tilde{X}$ can bound a 2-disc (2-cell) in $\widetilde{\tilde{X} - \Sigma}$. Hence $c(f)(\sigma) = 0$ for all 2-cells σ. Hence the obstruction cycle $c(f)$ vanishes (and, of course, so does the obstruction class $[c(f)] \in H_2(\widetilde{\tilde{X} - \Sigma}, Z, \pi_1(S^1)))$ and f lifts to Z_0.

We now proceed as in [GL_3]. Under the assumption that X_c has a metric of positive scalar curvature we have a metric of uniformly positive scalar curvature on the bad end $\widetilde{\tilde{X} - \Sigma}$. From this we construct a metric of positive scalar curvature on $D(\widetilde{\tilde{X} - \Sigma}) \times S^2$ which is a product outside of a compact set by slowly inflating the S^2-factor. We see, however, that this metric is Λ^2-enlargeable. This contradiction proves that X_c cannot carry a metric of positive scalar curvature, as required. QED

This theorem generalizes many related theorems of Gromov and Lawson. For example, we retrieve Theorem 3.4 as a special case. Unfortunately, it is still very difficult to apply the theorem to the case of compact $K(\pi,1)$-manifolds of dimensions $\geqslant 4$. The problem is in checking Condition 3, and, in particular, we need to know that the fundamental group of such a manifold has a projection onto $\pi_1(T^{n-2}) = Z^{n-2}$. Whether or not this is true in general appears to be a difficult question.

Chapter 4. 4-manifolds of positive scalar curvature.

In this chapter we study 4-manifolds of positive scalar curvature using the "capping off" ideas outlined in the introduction. The method uses minimal surface techniques which have been developed and used by Schoen and Yau [$SY_{1,2,3}$] and Fischer-Colbrie and Schoen [F-CS] in their study of manifolds of positive scalar curvature. It also relies heavily on some properties of 3-manifolds of positive scalar curvature given by Gromov and Lawson [GL_3].

To understand the procedure it is instructive to work a 3-dimensional example first (§4.1). Then in §4.2 we prove a generalization of a result of Gromov and Lawson concerning 3-manifolds of positive scalar curvature. We give some preliminary covering space lemmas (§4.3) and finally in §4.4 give a "capping off" theorem for 4-manifolds of positive scalar curvature.

§4.1. Outline of procedure.

We prove the following 3-dimensional result.

Theorem 4.1.1. *Let X_c be a compact 3-manifold carrying a metric of positive scalar curvature. Suppose that $u \in \pi_1(X_c)$ has infinite order and let $X \to X_c$ be the* cover corresponding to the cyclic subgroup $\langle u \rangle \subset \pi_1(X_c)$. Then $H_2(X) \neq 0$. In fact there is a class $\tilde{\Sigma} \in H_2(X)$ which has non-zero intersection number with γ, a generator of $H_1(X) \cong \mathbf{Z}$.

To prove this we use the following basic result of Gromov and Lawson [GL_3,§10].

Theorem 4.1.2. Let M be a compact 3-manifold with a possibly empty boundary carrying a metric of positive scalar curvature (assume ≥ 1 for convenience). If $\lambda \subset M$ is a disjoint union of closed curves such that

1) $[\lambda] = 0$ in $H_1(M,\partial M)$

2) $\mathrm{dist}_M(\lambda,\partial M) > 2\pi$

then λ bounds in its 2π-neighbourhood.

Proof of Theorem 4.1.1. Let $\gamma \subset X$ be an embedded circle generating $H_1(X) \cong \mathbf{Z}$. Set $X_\rho = \{x \in X \mid \delta_\gamma(x) \leq \rho\}$ where δ_γ is a smooth approximation to $\mathrm{dist}_X(\cdot,\gamma)$ and where $\rho > 4\pi$ is some regular value of δ_γ. Since $[\gamma]$ has infinite order in $H_1(X_\rho)$ there is a class

$$\omega \in H_2(X_\rho,\partial X_\rho) \cong H^1(X_\rho) \cong \mathrm{Free}(H_1(X_\rho)) + \mathrm{Torsion}$$

so that $(\omega,[\gamma]) \neq 0$. (That is, each 2-cycle representing ω has non-zero intersection number with γ). Let

$(\bar{\Sigma}_\rho, \partial\bar{\Sigma}_\rho) \subset (X_\rho, \partial X_\rho)$ be any surface representing ω.

Consider $\Sigma_\rho = \{x \in \bar{\Sigma}_\rho \mid \delta_\gamma(x) \leq \rho/2\}$ and assume that $\rho/2$ is a regular value of $\delta_\gamma|_{\bar{\Sigma}_\rho}$. Let $\lambda_\rho = \partial\Sigma_\rho$. Clearly we have

i) $[\lambda_\rho] = 0$ in $H_1(X_\rho, \partial X_\rho)$

and

ii) $\mathrm{dist}_{X_\rho}(\lambda_\rho, \partial X_\rho) > 2\pi$.

Hence, by Theorem 4.1.3, λ_ρ bounds a 2-chain Ω in its 2π-neighbourhood.

Form the 2-cycle $\tilde{\Sigma} = \Sigma_\rho \cup \Omega$. This cycle has the same (non-zero) intersection number with γ as Σ_ρ does (since $\Omega \cap \gamma = \phi$) and so, in particular, $[\tilde{\Sigma}] \in H_2(X)$ is non-zero. This completes the proof. QED

If X_c is a compact $K(\pi,1)$ 3-manifold then the cover $X \to X_c$ in Theorem 4.1.1 is a $K(Z,1)$ and we know that $H_n(Z) = 0$ for $n \geq 2$. Hence we conclude:

Corollary 4.1.4. _No compact $K(\pi,1)$ 3-manifold carries a metric of positive scalar curvature._

In the 4-dimensional case we would like to prove a theorem similar to Theorem 4.1.1, namely,

<u>Conjecture 4.1.5</u>. Let X_c be a compact 4-manifold carrying a metric of positive scalar curvature with $\pi_2(X_c) = 0$. Suppose that $u \in \pi_1(X_c)$ has infinite order and let $\overline{X} \to X_c$ be the cover corresponding to the cyclic subgroup $\langle u \rangle$. Then there is finite cover $X \to \overline{X}$ for which $H_3(X) \neq 0$. In fact there is a class $\tilde{\Sigma} \in H_3(X)$ which has non-zero intersection with γ, a generator of $H_1(X) \approx \mathbf{Z}$.

Note that the condition $\pi_2(X_c) = 0$ is equivalent to the condition $H_2(\tilde{X}) = 0$, where $\tilde{X}$ is the universal cover of X_c. This is sufficient to guarantee that all "small" 2-cycles in $\overline{X}$ bound "small" 3-chains. This is, at least, a necessary condition for our "capping off" argument to work. The finite cover $X \to \overline{X}$ appears to be necessary for technical reasons (see §4.3).

If we could prove Conjecture 4.1.5 the 4-dimensional $K(\pi,1)$ result would follow as the 3-dimensional result followed from Theorem 4.1.1. Unfortunately, we have not been able to give a general proof, although our partial results indicate that the theorem is true. Since a general proof will probably rely an these partial results and seeing that they are interesting in their own right we now proceed to develop them.

§4.2. The "long and thin" result for certain 3-manifolds.

In this section we prove a generalization of Theorem 4.2.1 and use it to obtain a long and thin result for simply-connected 3-manifolds.

Theorem 4.2.1. *Let M be a compact 3-manifold with a possibly empty boundary and metric ds^2. Assume that on $M \times S^1$ there is a warped-product metric of the form*

$$(4.2.2) \qquad d\hat{s}^2 = ds^2 + f^2(x)d\theta^2$$

having scalar curvature $\hat{\kappa} \geq 1$. Let σ be a closed curve in M such that

a) $[\sigma] = 0$ in $H_1(M,\partial M)$

and

b) $\text{dist}_M(\sigma,\partial M) > 2\sqrt{3}\pi$.

Then σ bounds in its $2\sqrt{3}\pi$-neighbourhood.

Proof. Pick any surface Σ_0 M with $\partial\Sigma_0 = \sigma$ (mod∂M). Such a surface exists by a). In $(M \times S^1, \partial M \times S^1)$ solve the Plateau problem for the boundary $\sigma \times S^1$ homologous to $\Sigma_0 \times S^1$. Call the solution H. We wish to claim that H is of the form $\Sigma \times S^1$ where Σ is a surface with $\partial\Sigma = \sigma$ (mod∂M). By results of Gao [Gao] we know that either the minimal hypersurface H is S^1-invariant or $\langle\partial/\partial\theta,\nu\rangle > 0$ where the vector field $\partial/\partial\theta$ generates the

S^1-action and ν is the unit normal to H. The second possibility is impossible for the following reason. Set $M_\varepsilon = M - N_\varepsilon(\sigma)$ where $N_\varepsilon(\sigma)$ is a small neighbourhood of σ. The hypersurfsace $\overline{H} \equiv H \cap M_\varepsilon \times S^1$ defines a relative cycle in $M_\varepsilon \times S^1$ homologous to the relative cycle defined by $\overline{\Sigma_0 \times S^1} \equiv \Sigma_0 \times S^1 \cap M_\varepsilon \times S^1$. The orbits of the S^1-action meet $\overline{\Sigma_0 \times S^1}$ with zero intersection number while they meet $\overline{H}$ with positive intersection number if the second alternative holds. This is a contradiction. Hence H is S^1-invariant and can therefore be written as

$$H = \Sigma \times S^1$$

where Σ is a surface with $\partial\Sigma = \sigma \, (\mathrm{mod}\,\partial M)$. The metric on H is a warped-product of the form

$$d\omega_1{}^2 = d\omega^2 + f_1(x)^2 d\theta^2.$$

The minimal surface $\Sigma \times S^1$ is stable in $M \times S^1$ so by results of [GL3], [F-CS] we have that on $(\Sigma \times S^1) \times S^1$ there is a warped-product metric

$$d\omega_2{}^2 = d\omega^2 + f_1(x)^2 d\theta_1{}^2 + f_2(x) d\theta_2{}^2$$

having scalar curvature $\kappa_2 \geq 1$. The idea here is to choose f_2 to be the first eigenvector of the stability operator L for $\Sigma \times S^1$. Since L is S^1-invariant f_2 is independent of θ_1. Actually we must be a little careful because the function f_2 may equal 0 on $\partial\Sigma$. However, we can overcome this by shaving off a small collar of $\partial\Sigma$.

We now show that each point $p_0 \in \Sigma$ is at a distance $\leq \sqrt{3}\pi$ from $\partial\Sigma$ following a construction in [GL3,§12]. In $(\Sigma \times S^1 \times S^1, \partial\Sigma \times S^1 \times S^1)$ solve the Plateau problem for $\{p_0\} \times S^1 \times S^1$. Call the solution H'. Since everything is $S^1 \times S^1$-invariant the same arguments as above show that H' is also $S^1 \times S^1$-invariant and can therefore be written in the form

$$H' = [0,r] \times S^1 \times S^1.$$

The metric induced on H' is a warped-product of the form

$$d\omega_3{}^2 = dt^2 + f_1(t)^2 d\theta_1{}^2 + f_2(t)^2 d\theta_2{}^2.$$

Note that

(4.2.3) $$\mathrm{dist}_M(p_0, \partial\Sigma) \leq r$$

since the curve

$$\begin{aligned} \lambda : [0,r] &\to [0,r] \times S^1 \times S^1 \\ t &\to (t, \text{constant}, \text{constant}) \end{aligned}$$

has length r and goes from p_0 to $\partial\Sigma$.

Again, since $H' = [0,r] \times S^1 \times S^1$ is a stable minimal hypersurface, we have on $([0,r] \times S^1 \times S^1) \times S^1$ a warped-product metric

(4.2.4) $$d\omega_4{}^2 = dt^2 + f_1(t)^2 d\theta_1{}^2 + f_2(t)^2 d\theta_2{}^2 + f_3(t)^2 d\theta_3{}^2$$

having scalar curvature $\kappa_4 \geq 1$. Here f_3 is the first eigenfunction of the $S^1 \times S^1$-invariant stability operator for H' and so is independent of θ_1 and θ_2. Again, we may have to shave the boundary to guarantee that $f_3 > 0$.

Proceeding along the lines of [GL₃,§11] we calculate the scalar curvature of the metric (4.2.4).

$$\kappa_4 = -2\sum_{i=1}^{3}\frac{f_i''}{f_i} - 2\sum_{i<j}\frac{f_i'}{f_i}\cdot\frac{f_j'}{f_j}$$

$$= -2\sum_{i=1}^{3}\frac{f_i''f_i - (f_i')^2}{f^2} - \left(\sum_{i=1}^{3}\frac{f_i'}{f_i}\right)^2 - \sum_{i=1}^{3}\left(\frac{f_i'}{f_i}\right)^2$$

$$\leqslant -2\sum_{i=1}^{3}\frac{f_i''f_i - (f_i')^2}{f^2} - \left(\sum_{i=1}^{3}\frac{f_i'}{f_i}\right)^2 - \frac{1}{3}\left(\sum_{i=1}^{3}\frac{f_i'}{f_i}\right)^2$$

$$= -2\sum_{i=1}^{3}\frac{f_i''f_i - (f_i')^2}{f^2} - \frac{4}{3}\left(\sum_{i=1}^{3}\frac{f_i'}{f_i}\right)^2$$

Setting $F = \log(f_1f_2f_3)$ and recalling that $\kappa_4 \geqslant 1$ we find

$$1 \leqslant \kappa_4 \leqslant -2F'' - \frac{4}{3}(F')^2$$

Setting $u = \frac{2}{\sqrt{3}}F'$ gives

$$\frac{u'}{u^2+1} \leqslant -\frac{1}{\sqrt{3}}$$

which integrates to give

$$\tan^{-1}(u(t)) - \tan^{-1}(u(0)) \leqslant -\frac{1}{\sqrt{3}}t$$

for all t in the interval $[0,r]$.

It follows that

$$t \leqslant \sqrt{3}\pi \quad \text{for all } t \in [0,r].$$

In particular, from (4.2.3) and (4.2.5) we have that

$$\mathrm{dist}_M(p_0,\partial\Sigma) \leqslant r \leqslant \sqrt{3}\pi.$$

Recall by hypothesis b) that $\mathrm{dist}_M(\sigma,\partial M) > 2\sqrt{3}\pi$ so therefore no component of $\partial\Sigma$ can be on $\partial\Sigma$. Hence $\partial\Sigma = \sigma$ and Σ is a surface bounding σ and contained in its $2\sqrt{3}\pi$-neighbouhood. This completes the proof. QED

Theorem 4.2.1 is an extension of Gromov and Lawson's basic theorem (our Theorem 4.1.1) and the same corollaries as given in [GL3,§10] still hold for our more general 3-manifolds. We are particually interested in the "long and thin" result [GL3,Corollary 10.11] but Yau (private communication) has correctly pointed out that there is an error in its proof. As things stand we can only claim the following.

Corollary 4.2.6. *Let M be a complete simply-connected 3-manifold with metric ds^2. Assume that on $M \times S^1$ there is a warped-product metric of the form*

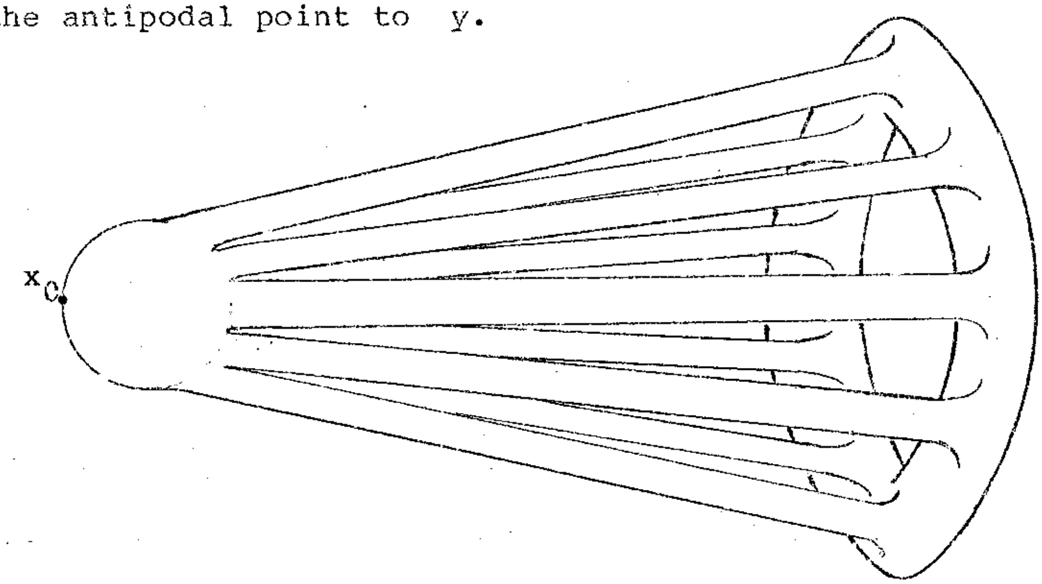
$$d\hat{s}^2 = ds^2 + f(x)^2 d\theta^2$$

having scalar curvature $\hat{\kappa} \geqslant 1$. Fix $x_0 \in M$ and let δ_{x_0} be a smooth approximation to $\mathrm{dist}_M(\cdot,x_0)$. For $\rho > 0$ let Λ_ρ be a connected component of $\delta_{x_0}^{-1}(\rho)$. Then $\mathrm{diam}_M(\Lambda_\rho) \leqslant 12\sqrt{3}\pi$.

Notes. 1. Of course, this theorem is true if M

itself has positive scalar curvature.

2. Without the condition that M be simply-connected the theorem as stated is definitely not true. Here is a counter-example. Take $S^1 \times S^2$ with a large S^1-factor. Remove many points $\{p_i\}_{i=1}^N$ in the set $S^1 \times \{\text{point}\}$ and pull-out the metric around each point to a product $S^2(\varepsilon) \times \mathbf{R}$ while retaining positive scalar curvature. Do the same for a sphere S^3 with points $\{q_i\}_{i=1}^N$ removed from a small ball $B_y(r) \subset S^3$ and join the corresponding ends keeping the connecting "tubes" very long and of approximately the same length. Choose $x_0 \in S^3$ to be the antipodal point to y.

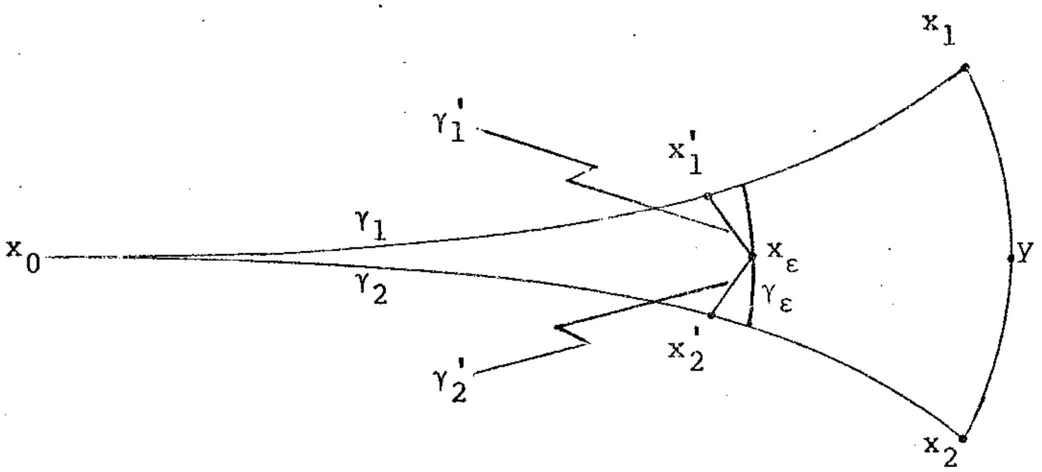


Then the connected components of level sets of the function δ_{x_0} have small diameter for a while but on the $S^1 \times S^2$ part they can become as large as we wish simply by choosing the S^1-factor large and using many points p_i. However, this construction shows only that we cannot use the distance function δ_{x_0}. The "long and thin" result may still hold in general - we need to prove that there is a function $f: M \to \mathbf{R}^+$ which retains some of the properties of the distance function but which avoids the problems. The precise properties we require will become clear in the proof of Theorem 4.2.2. Unfortunately, we have not been able to construct such a function (if, indeed, one exists) and, for the moment, at least, we must be content with the theorem as is.

Proof of Theorem 4.2.2. The proof given by Gromov and Lawson is correct except for one point. We give the details.

We wish to show that if x_1 and x_2 are in the same connected component of $\delta_{x_0}^{-1}(\rho)$ then $\mathrm{dist}_M(x_1,x_2) \leq 12\sqrt{3}\pi$. Construct the triangle $T = \gamma_1\gamma\gamma_2$ where γ is a curve joining x_1 to x_2 on which δ_{x_0} is constant, γ_1 is a minimal geodesic from x_0 to x_1 and γ_2 is a minimal geodesic from x_0 to x_2. Note that T is homotopic to zero since M is assumed to be simply-connected.

We apply Theorem 4.2.1 and conclude that T bounds a surface Σ in its $2\sqrt{3}\pi$-neighbourhood (simply consider the compact sets $M_R = \{ x \in M \mid \delta_{x_0}(x) \leq R \}$). Consider the curves $\gamma_\varepsilon = \{ x \in \Sigma \mid \delta_{x_0}(x) = \delta_{x_0}(x_1) - 2\sqrt{3}\pi - \varepsilon \}$ in Σ. For each regular value $\varepsilon > 0$, the curve γ_ε is a regular compact curve which intersects γ_i in exactly one point, since δ_{x_0} is strictly increasing on γ_i, $i=1,2$. So one component of γ_ε joins γ_1 to γ_2 and there is a point $x_\varepsilon \in \gamma_\varepsilon$ at a distance $< 2\sqrt{3}\pi$ from both γ_1 and γ_2. Join x_ε to γ_i by minimal geodesics γ_i' and let the endpoints on γ_i be x_i', $i=1,2$.



We have

$$\text{(4.2.7)} \qquad \mathrm{dist}_M(x_1,x_2) \leq \mathrm{dist}_M(x_1,x_1') + \mathrm{dist}_M(x_2,x_2') + \mathrm{dist}_M(x_1',x_\varepsilon) + \mathrm{dist}_M(x_2',x_\varepsilon)$$

For $i=1,2$

$$\text{(4.2.8)} \qquad \mathrm{dist}_M(x_i',x_\varepsilon) \leq 2\sqrt{3}\pi$$

and since γ_i is minimal

$$\mathrm{dist}_M(x_i,x_i') = \mathrm{dist}_M(x_0,x_i) - \mathrm{dist}_M(x_0,x_i')$$
$$\leqslant \mathrm{dist}_M(x_1,x_2) - (\mathrm{dist}_M(x_1,x_2) + 2\sqrt{3}\pi)$$
$$(4.2.9) \qquad = (\mathrm{dist}_M(x_0,x_i) - \mathrm{dist}_M(x_0,x_\varepsilon)) + 2\sqrt{3}\pi$$

Let y be the closest point to x_ε on γ. Then

$$\mathrm{dist}_M(x_0,x_i) = \mathrm{dist}_M(x_0,y)$$
$$\leqslant \mathrm{dist}_M(x_0,x_\varepsilon) + \mathrm{dist}_M(x_\varepsilon,y)$$

so

$$\mathrm{dist}_M(x_0,x_i) - \mathrm{dist}_M(x_0,x_\varepsilon) \leqslant 2\sqrt{3}\pi + \varepsilon$$

From (4.2.7) - (4.2.10) we have

$$\mathrm{dist}_M(x_1,x_2) \leqslant 12\sqrt{3}\pi$$

as claimed. QED

§4.3. Covering space lemmas.

In this section we will prove the following theorem.

Theorem 4.3.1. *Let X_c be a compact riemannian manifold with $\pi_2(X_c) = 0$. Suppose that $u \in \pi_1(X_c)$ has infinite order, and let $\bar{X} \to X_c$ be the cover corresponding to the cyclic subgroup $\langle u \rangle$. Let $r_0 > 0$ be any constant. Then there is a finite cover $X \to \bar{X}$ and a constant c_0 so that for all $x \in X$ the inclusion*

$$H_2(B_x(r_0)) \to H_2(B_x(c_0))$$

is zero.

This theorem is intended to partially generalize the following result which holds for regular coverings.

Lemma 4.3.2. Let $X \to X_c$ be a regular cover of a compact manifold with $H_2(X) = 0$. Let $r_0 > 0$ be any constant. Then there is a constant c_0 so that for all $x \in X$ the inclusion

$$H_2(B_x(r_0)) \to H_2(B_x(c_0))$$

is zero.

Proof. Let $K \quad X$ be a (compact) fundamental domain. Since the covering is regular we can find a deck transformation g taking x into K. Let $\{\Sigma_i\}_{i=1}^N$ be a generating set for $H_2(K')$, where $K' = r_0$-neighbourhood of K. Since $H_2(X) = 0$ each Σ_i bounds a 3-chain S_i. Pick $c_0 > 0$ so that $S_1 \cup \ldots \cup S_N \cup K' \subset B_y(c_0)$ for all $y \in K$. Then $H_2(K') \to H_2(B_{g(x)}(c_0))$ is zero, and so the composition $H_2(B_{g(x)}(r_0)) \to H_2(K') \to H_2(B_{g(x)}(c_0))$ is also zero. Apply g^{-1} to conclude that $H_2(B_x(r_0)) \to H_2(B_x(c_0))$ is zero, as required. QED

Before the proof of Theorem 4.3.1 we need

Lemma 4.3.3. *Let* X_c *be a compact riemannian manifold. Suppose that* $u \in \pi_1(X_c)$ *has infinite order. For each integer* n *let* γ_n *be the shortest closed geodesic in the free homotopy class of* $n \cdot u$. *Then*

$$\lim_{n \to \infty} \text{length}(\gamma_n) = \infty.$$

An immediate corollary is

Corollary 4.3.4. *Let* X_c *be a compact riemannian manifold. Suppose* $u \in \pi_1(X_c)$ *has infinite order, and let* $\bar{X} \to X_c$ *be the cover corresponding to* $\langle u \rangle$. *Let* $\tilde{X}$ *be the universal cover of* X_c. *Then given* $c > 0$ *there is a finite cover* $X \to \bar{X}$ *so that balls of radius* $\leq c$ *in* X *are evenly covered by* $\tilde{X} \to X$.

Proof of Lemma 4.3.3. Assume not. Then there is a sequence $\{\gamma_m\}_{m=1}^{\infty}$ of closed geodesics where γ_m is in the free homotopy class of $n_m \cdot u$, $n_m \to \infty$, all having lengths $> L$ (say). By compactness of X_c and properties of ordinary differential equations we know that (a subsequence of) $\{\gamma_m\}$ converges uniformly to a closed geodesic γ. In particular, for m sufficiently large they are homotopic, in contradiction to the initial choice. QED

We can now prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Firstly consider the universal cover $\tilde{X} \to X_c$. This is a regular cover with $H_2(\tilde{X}) = \pi_2(X_c) = 0$, so we can apply Lemma 4.3.2 to deduce that there is a constant c_0 (depending only on r_0) so that for all $y \in \tilde{X}$ the inclusion $H_2(B_y(r_0)) \to H_2(B_y(c_0))$ is zero.

Now apply Corollary 4.3.4 with $c = 2c_0$: there is a finite cover $X \to \overline{X}$ so that balls of radius $\leqslant c_0$ in X are evenly covered by $\tilde{X} \to X$. That the inclusion $H_2(B_x(r_0)) \to H_2(B_x(c_0))$ is zero for any $x \in X$ follows immediately by lifting to the universal cover. This completes the proof. QED

§4.4. The capping off procedure.

In this section we give a partial proof of Conjecture 4.1.5:

Conjecture 4.1.5. Let X_c be a compact 4-manifold carrying a metric of positive scalar curvature with $\pi_2(X_c) = 0$. Suppose that $u \in \pi_1(X_c)$ has infinite order and let $\overline{X} \to X_c$ be the cover corresponding to the cyclic subgroup $\langle u \rangle$. Then there is finite cover $X \to \overline{X}$ for which $H_3(X) \neq 0$. In fact there is a class $\tilde{\Sigma} \in H_3(X)$ which has non-zero intersection with γ, a generator of $H_1(X) \simeq \mathbb{Z}$.

The proof uses the partial "long and thin" result, Corollary 4.2.6, and so is not complete. We point out the extra assumption (4.4.2) we need during the proof.

Set $r_0 = 12\sqrt{3}\pi$ and let $X \to \overline{X}$ be the finite cover given by Theorem 4.3.1: there is a constant $c_0 > 0$ so that for all $x \in X$ the inclusion

$$H_2(B_x(12\sqrt{3}\pi)) \to H_2(B_x(c_0)) \tag{4.4.1}$$

is zero. We now work entirely in X.

Let $\gamma \subset X$ be an embedded circle such that $[\gamma]$

generates $H_1(X) \approx \mathbf{Z}$, and let δ_γ be a smooth approximation to $\mathrm{dist}_X(\cdot,\gamma)$. For each regular value $\rho > 0$ of δ_γ set

$$X_\rho = \{ x \in X \mid \delta_\gamma(x) \leq \rho \}.$$

Since $[\gamma]$ has infinite order there exists an element $\omega \in H_3(X_\rho,\partial X_\rho)$ so that $(\omega,\gamma) \neq 0$. Let $(\overline{\Sigma_\rho,\partial\Sigma_\rho}) \subset (X_\rho,\partial X_\rho)$ be a minimizing 3-manifold with $[\overline{\Sigma}_\rho] = \omega$. By regularity results $\overline{\Sigma}_\rho$ is a regular embedded stable minimal submanifold. By extracting a subsequence we pass to the limit as $\rho \to \infty$ and form Σ. This is a complete regular embedded stable minimal hypersurface.

To carry out the remainder of the proof we

(4.4.2) <u>assume that</u> Σ <u>is simply-connected</u>.

We know, by results of [GL3], [F-CS], that on $\Sigma \times S^1$ there is a warped-product metric of the form

$$d\hat{s}^2 = ds^2 + f(x)^2 d\theta^2$$

having scalar curvature $\hat{\kappa} \geq 1$. Fix $x_0 \in \Sigma \subset X$ and let δ_Σ be a smooth approximation to $\mathrm{dist}_\Sigma(\cdot,x_0)$. For each integer n consider the compact manifold

$$\Sigma_n \equiv \delta_\Sigma^{-1}([0,\rho_n])$$

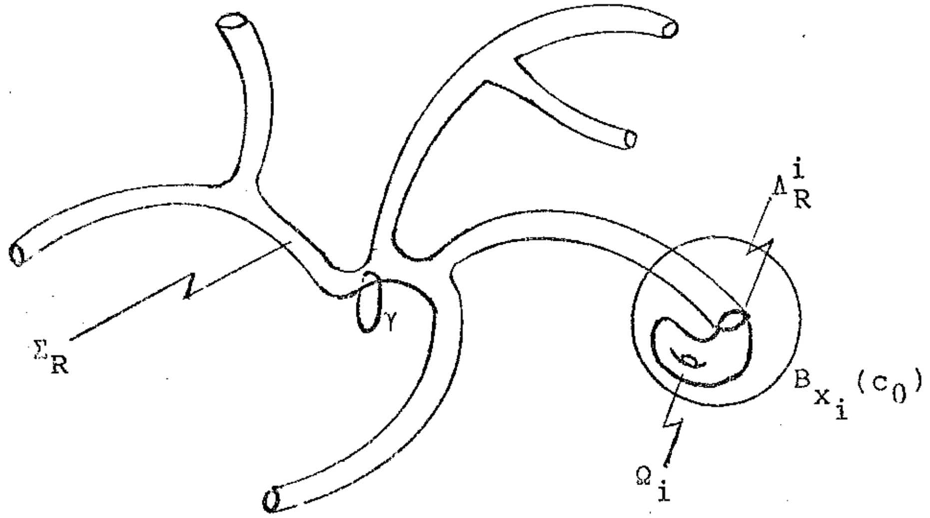
where ρ_n is a regular value of δ_Σ very close to n. Let the connected components of $\partial\Sigma_n = \delta_\Sigma^{-1}(\{\rho_n\})$ be denoted by Λ_n^i, $i=1,\ldots,N_n$. Apply Corollary 4.2.5: since Σ is simply-connected we have that

(4.3.3) $$\mathrm{diam}_{\Sigma}(\Lambda_n^i) \leq 12\sqrt{3}\pi.$$

Fix $\rho > c_0$ and note that for n sufficiently large $\partial\Sigma_n \subset X - X_\rho$. This is evident because both δ_Σ and δ_γ are proper exhaustion functions on Σ. (δ_γ is proper because it is in X and Σ is properly embedded in X). Pick an integer R so that $\Sigma_R \subset X - X_\rho$. Then Σ_R has the property that its boundary components Λ_R^i are at a distance $> c_0$ from γ and from (4.4.3)

$$\mathrm{diam}_X(\Lambda_R^i) \leq \mathrm{diam}_\Sigma(\Lambda_R^i) \leq 12\sqrt{3}\pi.$$

Hence each Λ_R^i is contained in some ball $B_{x_i}(12\sqrt{3}\pi)$ with $B_{x_i}(c_0) \cap \gamma = 0$.

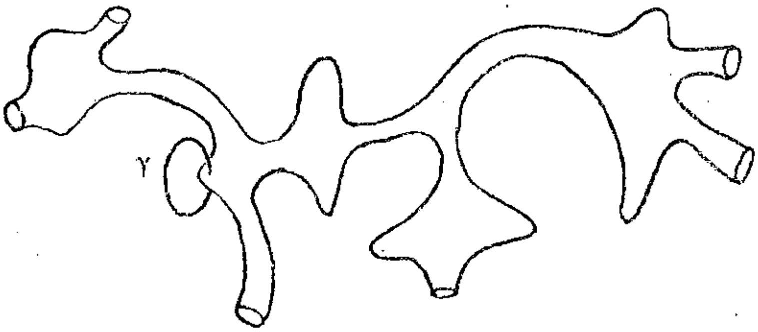


By (4.4.1) we know that each class $\Lambda_R^i \in H_2(B_{x_i}(12\sqrt{3}\pi))$ bounds a 3-chain Ω_i in $B_{x_i}(c_0)$ (with $\Omega_i \cap \gamma = 0$, therefore). The cycle

$$\tilde{\Sigma} = \Sigma_R \cup \Omega_1 \cup \ldots \cup \Omega_{N_R}$$

then has $(\tilde{\Sigma},\gamma) = (\Sigma_R,\gamma) \neq 0$. In particular, the class $[\tilde{\Sigma}] \in H_3(X)$ is non-zero.

Notes. 1. This partial result is not very satisfactory in view of Assumption 4.4.2. However, we do not need to know quite so much about the hypersurface Σ - the proof would still work if Σ is thin "often enough"



or maybe even if it grows at a slow enough rate compared with distance from γ. Of course, if the "long and thin" result holds in general there are no problems.

2. The assumption does hold for $X = S^1 \times R^3$ with the product metric. Here the minimal hypersurface is just $\{\text{point}\} \times R^3$. We conclude that $S^1 \times R^3$ (product metric) cannot be the riemannian cover of any compact manifold.

Chapter 5. Yang-Mills fields - Kalusa-Klein model.

In this chapter we consider problems arising from the study of Yang-Mills fields. It is organized as follows. Section 5.1 contains notation and basic results. We give a precise mathematical definition of a Yang-Mills field as a solution to a certain variational problem and, more generally, show that the combined Einstein-Yang-Mills equations arise from a variational problem. In Section 5.2 we show that this latter problem has a simple formulation if we work on the total space of the principal bundle. This is the Kalusa-Klein model. We study some natural questions concerning this variational problem obtaining in particular a type of topological quantization for certain critical points.

It should be noted that some of the material in this chapter is not new and can be found in greater detail in Bleeker [Bl], for example. However, we clarify some confusing points and develop some of the relevant ideas.

§5.1. Definitions, notation and basic results.

In this section we set-up the mathematical framework in which we work. Our primary aim is to state precisely what a Yang-Mills field is and to give the Yang-Mills and combined Einstein-Yang-Mills equations. This material is quite standard and is given in, for example, [B1] or [BL]. In fact this section is included mainly to fix our notation and for completeness. Kobayashi and Nomizu [KN] is a good source for the basic definitions.

We firstly fix some very general notation. For any manifold X let $TX \to X$ be the tangent bundle for X and if $x \in X$ let T_xX be the tangent space at x. Let T^*X be the cotangent bundle of X. In general, if $E \to X$ is a bundle we denote the sections of E by $\Gamma(E)$. If $E \to X$ is a vector bundle we denote its k^{th}-exterior bundle by $\Lambda^k(E)$. The set of k-forms on X, that is, sections of $\Lambda^k(T^*X)$, will be denoted simply by $\Lambda^k(X)$, and if V is a vector space the set of V-valued k-forms will be denoted by $\Lambda^k(X,V)$. Let $S^2(TX)$ be the set of symmetric (0,2)-tensor fields on X and $Met(X) \subset S^2(TX)$ the space of riemannian metrics. For $g \in Met(X)$ let $Ric_g \in S^2(TX)$ be its Ricci curvature and $\kappa_g : X \to R$ its scalar curvature.

We fix a compact connected orientable manifold M of dimension $n > 2$, a compact connected semi-simple Lie group

G of dimension $r > 1$, and a principal G-bundle $P \xrightarrow{\pi} M$. Let $\mathfrak{g}$ denote the Lie algebra of G and $[\ ,\ ]$ its Lie bracket. We have the adjoint representation $Ad:G \to Gl(\mathfrak{g})$. We fix an Ad-invariant inner product k on $\mathfrak{g}$ or, equivalently, a bi-invariant metric on G. Such a metric is given by the Killing form of $\mathfrak{g}$.

If V is any vector space and $\rho:G \to Gl(V)$ a representation of G on V we can form the associated vector bundle $E \to M$ by setting $E = P \times_\rho V$. If $V = \mathfrak{g}$ and ρ the adjoint representation we denote the associated bundle of Lie algebras by $\hat{\mathfrak{g}}$.

For $g \in G$ let $R_g:P \to P$ be the right action of g on P. Recall that a $\mathfrak{g}$-valued p-form $\phi \in \Lambda^p(P,\mathfrak{g})$ is called tensorial if

a) $R_g^*\phi = Ad_{g^{-1}} \cdot \omega$

and

b) $\phi(X,Y) = 0$ if one of the vector fields $X,Y \in \Gamma(TP)$ is vertical (that is, tangent to fibers of P).

A tensorial $\mathfrak{g}$-valued p-form can be thought of as an element of $\Gamma(\Lambda^p T^*M \otimes \hat{\mathfrak{g}})$.

We now introduce $Conn(P) \subset \Lambda^1(P,\mathfrak{g})$, the set of connection 1-forms on P. A form $\omega \in \Lambda^1(P,\mathfrak{g})$ is a connection form if it satisfies the following conditions

a) If $A \in \mathfrak{g}$ and A^* the corresponding fundamental vector field on P then $\omega(A^*) = A$.

b) If $g \in G$ then $R_g^*\omega = Ad_{g^{-1}} \cdot \omega$.

We note that the difference $\omega_1 - \omega_2$ of two connection 1-forms ω_1 and ω_2 is a tensorial $\mathfrak{g}$-valued 1-form. Hence we can think of $\Gamma(\Lambda^1 T^* M \otimes \hat{\mathfrak{g}})$ as the tangent space at $\omega \in \mathrm{Conn}(P)$. On any associated bundle E a connection $\omega \in \mathrm{Conn}(P)$ induces a covariant derivative $\nabla^\omega : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$.

For $\omega \in \mathrm{Conn}(P)$ we let $d^\omega : \Lambda^p(P,\mathfrak{g}) \to \Lambda^{p+1}(P,\mathfrak{g})$ be the covariant exterior derivative defined by

$$(5.1.1) \qquad d^\omega \phi(X_1,\ldots,X_{p+1}) = d\phi(X_1^H,\ldots,X_{p+1}^H)$$

where $\phi \in \Lambda^p(P,\mathfrak{g})$, $d : \Lambda^p(P,\mathfrak{g}) \to \Lambda^{p+1}(P,\mathfrak{g})$ is the usual exterior derivative, $X_1,\ldots,X_{p+1}$ are vector fields on P and $X_1^H,\ldots,X_{p+1}^H$ their horizontal projections. On a tensorial $\mathfrak{g}$-valued p-form ν, that is, a section of $\Lambda^p T^* M \otimes \hat{\mathfrak{g}} \to M$, the exterior covariant derivative $d^\omega : \Gamma(\Lambda^p T^* M \otimes \hat{\mathfrak{g}}) \to \Gamma(\Lambda^{p+1} T^* M \otimes \hat{\mathfrak{g}})$ has the expression

$$(5.1.2) \qquad d^\omega(\nu) = d\nu + [\omega,\nu]$$

where we define the bracket $[\phi,\psi] \in \Lambda^{k+l}(P,\mathfrak{g})$ of two $\mathfrak{g}$-valued forms $\phi \in \Lambda^k(P,\mathfrak{g})$ and $\psi \in \Lambda^l(P,\mathfrak{g})$ by

$$[\phi,\psi](X_1,\ldots,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma}(-1)^{\mathrm{sgn}(\sigma)}[\phi_{X_{\sigma(1)}\cdots X_{\sigma(p+1)}},\ \psi_{X_{\sigma(p+1)}\cdots X_{\sigma(p+q)}}]$$

Here $X_1,\ldots,X_{p+q} \in \Gamma(TP)$ and σ ranges over all permutations of $1,\ldots,p+q$.

Given $\omega \in \mathrm{Conn}(P)$ we define the associated curvature form $\Omega^\omega \in \Lambda^2(P,\mathfrak{g})$ by $\Omega^\omega = d^\omega\omega$. The curvature form satisfies the following fundamental equations

(5.1.4) $\quad \Omega^\omega = d^\omega + ½[\omega,\omega] \quad$ Structure equation

(5.1.5) $\quad d^\omega\Omega^\omega = 0 \quad$ Bianchi identity

Also Ω^ω is tensorial so that it can be thought of as an element of $\Gamma(\Lambda^2T^*M \otimes \hat{\mathfrak{g}})$ or, equivalently, as a map $\Omega^\omega:\Gamma(\Lambda^2TM) \to \Gamma(\hat{\mathfrak{g}})$.

Using the inner product k on $\mathfrak{g}$ we define, for each $g \in \mathrm{Met}(M)$, a metric $\overline{gk}$ for the bundle $\Lambda^pT^*M \otimes \hat{\mathfrak{g}} \to M$ as follows. For $x \in M$ and $\phi,\psi \in \rho^{-1}(x)$ set

(5.1.6) $$\overline{gk}(\phi,\psi) = \sum_{\alpha_1,\ldots\alpha_p=1}^{n} k(\phi_{E_{\alpha_1}\cdots\cdots E_{\alpha_p}},\psi_{E_{\alpha_1}\cdots\cdots E_{\alpha_p}})$$

where $\{E_1,\ldots,E_n\}$ forms an orthonormal basis for T_xM. As usual this gives an inner product, called $\overline{gk}$ still, on the vector space $\Gamma(\Lambda^pT^*M \otimes \hat{\mathfrak{g}})$ by integration over M

(5.1.7) $$\overline{gk}(\phi,\psi) = \int_M \overline{gk}(\phi_x,\psi_x)\, d\mathrm{vol}_g(x),$$
$$\phi,\psi \in \Gamma(\Lambda p_T{}^*M \otimes \hat{\mathfrak{g}})$$

Finally, for each $(g,\omega) \in \mathrm{Met}(M)\times\mathrm{Conn}(P)$ we define the map $\delta^{\omega,g}:\Gamma(\Lambda^{p+1}T^*M \otimes \hat{\mathfrak{g}}) \to \Gamma(\Lambda^p T^*M \otimes \hat{\mathfrak{g}})$ to be the adjoint of d^ω with respect to the inner product (5.1.7), that is,

$$\overline{gk}(\delta^{\omega,g}\phi,\psi) = \overline{gk}(\phi,d^\omega\psi)$$

for all $\phi \in \Gamma(\Lambda^{p+1}T^*M \otimes \hat{\mathfrak{g}})$ and $\psi \in \Gamma(\Lambda^p T^*M \otimes \hat{\mathfrak{g}})$

Using the riemannian connection D on $\Lambda^p T^*M$ and the connection ∇^ω on $\hat{\mathfrak{g}}$ we have a tensor product connection ∇ on $\Lambda^p T^*M \otimes \hat{\mathfrak{g}}$ defined by $\nabla(\alpha \otimes \xi) = D\alpha \otimes \xi + \alpha\nabla^\omega\xi$. We then have the following handy formulas for d^ω and $\delta^{\omega,g}$.

$$(5.1.8)\qquad (d^\omega\phi)_{X_0,\ldots X_p} = \sum_{i=0}^{p}(-1)^i(\nabla_{X_i}\phi)_{X_0,\ldots\hat{X}_i,\ldots X_p}$$

$$(5.1.9)\qquad (\delta^{\omega,g}\phi)_{X_1,\ldots X_{p-1}} = -\sum_{\alpha=1}^{n}(\nabla_{E_\alpha}\phi)_{E_\alpha,X_1,\ldots X_{p-1}}$$

Here $\phi \in \Gamma(\Lambda^p T^*M \otimes \hat{\mathfrak{g}})$, $\{E_1,\ldots E_n\}$ forms an orthonormal basis for T_xM and $X_0,\ldots X_p$ are arbitary vectors in T_xM.

We now introduce the basic functionals we study in this chapter.

<u>Definition 5.1.10</u>. The functional

$$YM:\mathrm{Met}(M) \times \mathrm{Conn}(P) \to R$$

defined by

$$YM(g,\omega) = \tfrac{1}{4}\int_M \overline{gk}(\Omega^\omega,\Omega^\omega)\ dvol_g$$

is called the Yang-Mills functional.

Definition 5.1.11. Let $c > 0$ be any constant. We define the functional

$$E_c: \mathrm{Met}(M) \times \mathrm{Conn}(P) \to \mathbf{R}$$

by

$$E_c(g,\omega) = \int_M (\kappa_g - \tfrac{1}{4}\overline{gk}(\Omega^\omega,\Omega^\omega) + c)\ dvol_g$$

The constant c is called the cosmological constant. We will call E_c the Einstein-Yang-Mills functional with cosmological constant c.

We usually think of the metric $g \in \mathrm{Met}(M)$ in Definition 5.1.10 as fixed. We then define the g-Yang-Mills functional $YM_g:\mathrm{Conn}(P) \to \mathbf{R}$ by

$$YM_g(\omega) = \tfrac{1}{4}\int_M \overline{gk}(\Omega^\omega,\Omega^\omega)\ dvol_g.$$

A connection ω is called a Yang-Mills potential for the metric g if it is a critical point of YM_g, that is, if $\frac{d}{dt}YM_g(\omega + t\nu)|_{t=0} = 0$ for all variations $\nu \in \Gamma(\Lambda^1T^*M \otimes \hat{\mathfrak{g}})$. The corresponding curvature $\Omega^\omega \in \Gamma(\Lambda^2T^*M \otimes \hat{\mathfrak{g}})$ is called a Yang-Mills field. The first variation formula for YM_g is given in, for example, [BL] and we have the following.

Theorem 5.1.13. *Let* $g \in \mathrm{Met}(M)$. *A connection* $\omega \in \mathrm{Conn}(P)$ *is Yang-Mills for* g *if and only if it satisfies the Yang-Mills equation* -

(5.1.14) $$\delta^{\omega,g}\Omega^{\omega} = 0.$$

There are many interesting problems which arise in the study of the g-Yang-Mills functional for a fixed metric g. However, here we are more interested in allowing g to vary also - the interesting functional in this case is the combined Einstein-Yang-Mills functional E_c. A pair $(g,\omega) \in \mathrm{Met}(M)\times\mathrm{Conn}(P)$ is a critical point of this functional if and only if $\frac{d}{dt}E_c(g + th, \omega + t\nu)|_{t=0} = 0$ for all variations $h \in S^2(TM)$ and $\nu \in \Gamma(\Lambda^1 T^*M \otimes \hat{\mathfrak{g}})$. We have the following theorem.

Theorem 5.1.15. *The pair* $(g,\omega) \in \mathrm{Met}(M) \times \mathrm{Conn}(P)$ *is a critical point of* E_c *if and only if the following equations are simultaneously satisfied*

(5.1.16) $$\mathrm{Ric}_g - \tfrac{1}{2}\kappa_g g - \frac{c}{2}\, g = T_{\omega,g} \quad \text{(\textit{Einstein field equation})}$$

(5.1.17) $$\delta^{\omega,g}\Omega^{\omega} = 0 \quad \text{(\textit{Yang-Mills equation})}$$

Here $T_{\omega,g} \in S^2(TM)$, *the so-called stress-energy tensor, is given by*

$$(5.1.18) \qquad T_{\omega,g}(E_\alpha,E_\beta) = \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma E_\alpha},\Omega_{E_\gamma E_\beta}) - \frac{1}{8}\overline{gk}(\Omega^\omega,\Omega^\omega)g(E_\alpha,E_\beta)$$

where $\{E_1,\ldots,E_n\}$ *is an orthonormal frame field of* TM.

Proof. A proof of this theorem is given in [B1] but we give it again for completeness.

Let $\langle\ ,\ \rangle$ be the metric induced by g on $S^2(TM)$. Then we have the basic results that for variations $h \in S^2(TM)$ and $\nu \in \Gamma(\Lambda^2T^*M \otimes \hat{\mathcal{G}})$

$$(5.1.19) \qquad \frac{d}{dt}\int_M dvol_{g+th}\Big|_{t=0} = \int_M \tfrac{1}{2}\langle g,h\rangle\, dvol_g$$

$$(5.1.20) \qquad \frac{d}{dt}\int_M \kappa_{g+th}\, dvol_{g+th}\Big|_{t=0} = \int_M (\tfrac{1}{2}\kappa_g\langle g,h\rangle - \langle Ric_g,h\rangle)\, dvol_g$$

$$(5.1.21) \qquad \frac{d}{dt}\int_M \overline{(g+th)k}(\Omega^\omega,\Omega^\omega)dvol_{g+th}$$

$$= \int_M \tfrac{1}{2}\overline{gk}(\Omega^\omega,\Omega^\omega)\langle g,h\rangle dvol_g - \int_M 2\langle \sum_{\gamma=1}^{n} k(\Omega_{E_\gamma}\cdot,\Omega_{E_\gamma}\cdot),h\rangle\, dvol_g$$

$$(5.1.22)\quad \frac{d}{dt}\int_M \overline{gk}(\Omega^{\omega+t\nu},\Omega^{\omega+t\nu})\ dvol_g\Big|_{t=0}$$

$$= \int_M 2\overline{gk}(\tfrac{d}{dt}(d(\omega+t\nu) + \tfrac{1}{2}[\omega+t\nu,\omega+t\nu])\big|_{t=0},\Omega^{\omega})\ dvol_g$$

$$= \int_M 2\overline{gk}(d\nu + [\omega,\nu],\Omega^{\omega})\ dvol_g$$

$$= \int_M 2\overline{gk}(d^{\omega}(\nu),\Omega^{\omega})\ dvol_g$$

$$= \int_M 2\overline{gk}(\delta^{\omega,g}\Omega^{\omega},\nu)\ dvol_g.$$

Combining (5.1.19)-(5.1.22) gives

$$\frac{d}{dt}E_c(g+th,\omega+t\nu)\Big|_{t=0} = \int_M \Big\langle \frac{\kappa_g}{2} g - Ric_g - \tfrac{1}{8}\overline{gk}(\Omega^{\omega},\Omega^{\omega})$$

$$+ \tfrac{1}{2}\int_M k(\Omega_{E_\gamma}\cdot,\Omega_{E_\gamma}\cdot) + \tfrac{c}{2}g, h\Big\rangle\ dvol_g$$

$$- \tfrac{1}{2}\int_M \overline{gk}(\delta^{\omega,g}\Omega^{\omega},\nu)\ dvol_g$$

From this it follows that (g,ω) is a critical point if and only if

a) $\quad Ric_g - \frac{\kappa_g}{2}g - \frac{c}{2}g = \frac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma}\cdot,\Omega_{E_\gamma}\cdot) - \frac{1}{8}\,\overline{gk}(\Omega^{\omega},\Omega^{\omega})$

b) $\quad \delta^{\omega,g}\Omega^{\omega} = 0$

which proves the theorem. QED

§5.2. Kalusa-Klein Model.

In this section we show that the Einstein-Yang-Mills functional E_c has a simple formulation if we work on the total space P of our bundle $P \overset{\pi}{\to} M$. This is the Kalusa-Klein model. We then study relationships between the critical points of this new functional. We use the same notation as in the previous section.

To give the Kalusa-Klein formulation we need to embed the space $Met(M)\times Conn(P)$ into $Met(P)$ as follows. For $(g,\omega) \in Met(M)\times Conn(P)$ define the metric $\overline{g\omega} \in Met(P)$ by setting for each $p \in P$ and arbitary $X,Y \in T_pP$

$$(5.2.1) \qquad (\overline{g\omega})_p(X,Y) = g(d\pi_p(X),d\pi_p(Y)) + k(\omega(X),\omega(Y)).$$

This metric is called the <u>bundle metric</u> associated to g and ω. We note that the (right) action of G on P is by isometries of the metric $\overline{g\omega}$. Hence we actually have $Met(M)\times Conn(P) \subset Met_{inv}(P) \subset Met(P)$ where $Met_{inv}(P)$ is the set of G-invariant metrics on P.

We observe that $\pi:P \to M$ is a riemannian submersion for the metrics $\overline{g\omega}$ on P and g on M - in fact an isometry of the horizontal subspaces of T_pP onto $T_{\pi(p)}M$. This observation is important as it shows that we may simply use the O'Neill formulas [O'N] to calculate the curvature of $\overline{g\omega}$. To do this, let $\{E_\alpha\}_{\alpha=1}^n$ be an orthonormal basis for the horizontal subspace of T_pP and $\{F_i\}_{i=1}^{r}$ an orthonormal basis for the vertical subspace tangent to the fibers of $P \xrightarrow{\pi} M$. We have the following.

Theorem 5.2.2. *Let $\overline{g\omega} \in \mathrm{Met}(P)$ be the bundle metric given by the pair $(g,\omega) \in \mathrm{Met}(M)\times\mathrm{Conn}(P)$. Then the Ricci curvature $\mathrm{Ric}_{\overline{g\omega}}$ of $\overline{g\omega}$ is given as follows.*

a) $$\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\beta) = \mathrm{Ric}_g(E_\alpha,E_\beta) - \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma E_\alpha},\Omega_{E_\gamma E_\beta})$$

b) $$\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,F_i) = \tfrac{1}{2}k((\delta^{\omega,g}\Omega)_{E_\alpha},F_i)$$

c) $$\mathrm{Ric}_{\overline{g\omega}}(F_i,F_j) = \mathrm{Ric}_G(F_i,F_j) + \tfrac{1}{2}k(\Omega\circ\Omega^t(F_i),F_j)$$

Here Ric_G is the Ricci curvature of G with its fixed bi-invariant metric k. In the last expression we think of the curvature $\Omega = \Omega^\omega$ as a map $\Omega:\Lambda^2T_xM \to \mathfrak{g}$ and then Ω^t is its adjoint with respect to the riemannian metric on Λ^2T_xM and the metric k on $\mathfrak{g}$. Note that we can write

$$\Omega\circ\Omega^t(F_i) = \Omega\Big(\sum_{\varepsilon<\gamma=1}^{n} k(E_\varepsilon\, E_\gamma,\Omega^t(F_i))(E_\varepsilon\, E_\gamma)\Big)$$
$$= \sum_{\varepsilon<\gamma=1}^{n} k(\Omega_{E_\varepsilon E_\gamma},F_i)\Omega_{E_\varepsilon E_\gamma}$$

so that

$$k(\Omega\circ\Omega^t(F_i),F_j) = \sum_{\varepsilon<\gamma=1}^{n} k(\Omega_{E_\varepsilon E_\gamma},F_i)k(\Omega_{E_\varepsilon E_\gamma},F_j)$$

$$= 1/2 \sum_{\varepsilon<\gamma=1}^{n} k(\Omega_{E_\varepsilon E_\gamma},F_i)k(\Omega_{E_\varepsilon E_\gamma},F_j)$$

Using Theorem 5.2.2 we calulate the scalar curvature

$$\kappa_{\overline{g\omega}} = \sum_{\alpha=1}^{n} \mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\alpha) + \sum_{i=1}^{r} \mathrm{Ric}_{\overline{g\omega}}(F_i,F_i).$$

Theorem 5.2.3. Let $\overline{g\omega}$ be as in Theorem 5.2.2. Then its scalar curvature $\kappa_{\overline{g\omega}}:P \to \mathbf{R}$ is given as

$$\kappa_{\overline{g\omega}} = \kappa_G - 1/4 gk(\Omega,\Omega) + \kappa_G.$$

Here κ_G is the scalar curvature of G with its bi-invariant metric.

Note that, since the action of G on P is by isometries, the scalar curvature $\kappa_{\overline{g\omega}}$ is constant on the fibers. Hence it gives a map $\kappa_{\overline{g\omega}}:M \to \mathbf{R}$ by projection.

We now return to the main point of this section and introduce the total curvature functional $E:\mathrm{Met}(P) \to \mathbf{R}$

$$(5.2.4) \qquad E(h) = \int_P \kappa_h \, dvol_h, \qquad h \in Met(P)$$

The notation here is consistent with that of 5.1.11 since on $Met(M) \times Conn(P) \subset Met(P)$ we have, using Theorem 5.2.3, that

$$E(\overline{g\omega}) = \int_P \kappa_{\overline{g\omega}} \, dvol_{g\omega}$$

$$= \int_P (\kappa_G - \tfrac{1}{4} gk(\Omega,\Omega) + \kappa_G) \, dvol_{g\omega}$$

and since the integrand is constant on the fibers

$$E(\overline{g\omega}) = vol_k(G) \int_M (\kappa_g - \tfrac{1}{4} gk(\Omega,\Omega) + \kappa_G) \, dvol_g$$

$$= vol_k(G) \, E_{\kappa_G}.$$

Here $vol_k(G)$ is the k-volume of G, that is, $vol_k(G) = \int_G dvol_k$. Hence

$$E|_{Met(M) \times Conn(P)} : Met(M) \times Conn(P) \to \mathbf{R}$$

is, up to a constant, the same as $E_{\kappa_G} : Met(M) \times Conn(P) \to \mathbf{R}$. Thus, we may replace E_{κ_G} by the simpler functional $E|_{Met(M) \times Conn(P)}$. This is the Kalusa-Klein formulation.

We are interested in the critical points of E. Of course, on the subspace $Met(M) \times Conn(P) \subset Met(P)$ the critical points agree with those of E_{κ_G} so we have the following re-wording of Theorem 5.1.15.

Theorem 5.2.5. The pair $(g,\omega) \in Met(M)\times Conn(P)$ is a critical point of the functional

$$E|_{Met(M)\times Conn(P)} : Met(M)\times Conn(P) \to R$$

if and only if

a) $$Ric_g - \tfrac{1}{2}\kappa_g g - \tfrac{1}{2}\kappa_G g = \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma}\cdot,\Omega_{E_\gamma}\cdot) - \tfrac{1}{8} g k(\Omega,\Omega) g$$
(Einstein field equation)

b) $\delta^{\omega,g}\Omega = 0$ (Yang-Mills equation)

The critical points of E on other subspaces of $Met(P)$ are more well-known. We have, for example,

Theorem 5.2.6. The metric $\bar{g} \in Met(P)$ is a global crital point of $E:Met(P) \to R$ if and only if it is Ricci-flat, that is, $Ric_{\bar{g}} \equiv 0$.

Let $Met_V(P) \subset Met(P)$ be the subspace of metrics with a fixed volume V. Then, by applying standard Lagrange multiplier techniques (see, for example, [E]), we have

Theorem 5.3.7. The metric $\bar{g} \in Met_V(P)$ is a critical point of $E|_{Met_V(P)} : Met_V(P) \to R$ if and only if it is Einstein, that is, $Ric_{\bar{g}} = \frac{\kappa_{\bar{g}}}{n+r}\,\bar{g}$.

An interesting question to ask is if the critical points of $E|_{Met(M)\times Conn(P)}$ are ever critical points of $E|_{Met_V(P)}$ or of the unrestricted functional E, that is, are they ever Einstein or Ricci flat. Another more general question is to ask for the conditions under which any bundle metric $\overline{g\omega}$ is Einstein. We will consider an aspect of this question later. As far as the first is concerned we have the following.

<u>Theorem 5.2.8.</u> <u>A bundle metric</u>

$$\overline{g\omega} = (g,\omega) \in Met(M)\times Conn(P)$$

<u>that is a critical point of</u> $E|_{Met(M)\times Conn(P)}$ <u>is Einstein if and only if it is Ricci-flat.</u>

<u>Proof.</u> From Theorems 5.2.2, 5.2.3 and 5.2.5 we have

$$\begin{aligned} Ric_{\overline{g\omega}}(E_\alpha,E_\beta) &= Ric_g(E_\alpha,E_\beta) - \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma E_\alpha},\Omega_{E_\gamma E_\beta}) \\ &= \frac{\kappa_g - \tfrac{1}{4}gk(\Omega,\Omega) + \kappa_G}{2}\,\overline{g\omega}(E_\alpha,E_\beta) \\ &= \frac{\kappa_{\overline{g\omega}}}{2}\,\overline{g\omega}(E_\alpha,E_\beta), \qquad \alpha,\beta = 1,\ldots,n \end{aligned}$$

If $\overline{g\omega}$ is Einstein, however, we have

$$Ric_{\overline{g\omega}}(E_\alpha,E_\beta) = \frac{\kappa_{\overline{g\omega}}}{n+r}\,\overline{g\omega}(E_\alpha,E_\beta) \qquad \alpha,\beta = 1,\ldots,n$$

Hence, since $n + r > 2$, we have $\kappa_{\overline{g\omega}} = 0$ and so $\overline{g\omega}$ is Ricci-flat. QED

Thus, we are reduced to asking when a critical point of $E|_{Met(M)\times Conn(P)}$ is Ricci-flat. We actually give the following more general theorem.

Theorem 5.2.9. *Suppose that* $n \geq 3$. *Then a bundle metric* $\overline{g\omega} \in Met(M)\times Conn(P) \quad Met(P)$ *is Ricci-flat if and only if it satisfies the following conditions.*

i) $$Ric_g(E_\alpha, E_\beta) - \frac{\kappa_g}{2} g(E_\alpha, E_\beta) - \frac{\kappa_G}{2} g(E_\alpha, E_\beta) = \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma E_\alpha}, \Omega_{E_\gamma E_\beta}) - \tfrac{1}{4}\, \overline{gk}(\Omega,\Omega) g(E_\alpha, E_\beta)$$

$\alpha, \beta = 1,\ldots,n$ (*Einstein field equation*)

ii) $\delta^{\omega,g}\Omega = 0$ (*Yang-Mills equation*)

iii) $Ric_G(F_i, F_j) = -\tfrac{1}{2}k(\Omega\circ\Omega^t(F_i), F_j)$, $\quad i,j = 1,\ldots,r$

If $n = 2$ *the above is true with the extra condition*

iv) $\kappa_{\overline{g\omega}} = 0$.

Proof. The implication in the forward direction follows immediately from Theorems 5.2.2 and 5.2.3 since then

$$\kappa_{\overline{g\omega}} = \kappa_g - \tfrac{1}{4}\overline{gk}(\Omega,\Omega) + \kappa_G = 0.$$

For the reverse implication we have the following.

a) $\mathrm{Ric}_{\overline{g\omega}}(F_i,F_j) = \mathrm{Ric}_G(F_i,F_j) + \tfrac{1}{2}k(\Omega\circ\Omega^t(F_i),F_j) = 0$

$i,j = 1,\ldots,r$

b) $\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,F_i) = \tfrac{1}{2}k((\delta^{\omega,g}\Omega)_{E_\alpha},F_i) = 0 \qquad i = 1,\ldots,r$

$\alpha = 1,\ldots,n$

c) $$\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\beta) = \mathrm{Ric}_g(E_\alpha,E_\beta) - \tfrac{1}{2}\sum_{\gamma=1}^{n} k(\Omega_{E_\gamma E_\alpha},\Omega_{E_\gamma E_\beta})$$
$$= \frac{\kappa_{\overline{g\omega}}}{2}\,\overline{g\omega}(E_\alpha,E_\beta) \qquad \alpha,\beta = 1,\ldots,n$$

If $n = 2$ condition iv) shows that $\kappa_{\overline{g\omega}} = 0$ and $\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\beta) = 0$. In any case, for $n \geqslant 3$ we have

$$\kappa_{\overline{g\omega}} = \sum \mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\alpha) + \sum \mathrm{Ric}_{\overline{g\omega}}(F_i,F_i)$$
$$= \frac{\kappa_{\overline{g\omega}}}{2}\cdot n + 0$$

and so $\kappa_{\overline{g\omega}} = 0$ still. So $\mathrm{Ric}_{\overline{g\omega}}(E_\alpha,E_\beta) = 0$ for $\alpha,\beta = 1,\ldots,n$. Together a), b) and c) show that $\mathrm{Ric}_{\overline{g\omega}} = 0$.

QED

We note that Conditions i) and ii) in Theorem 5.2.9 characterize a critical point of $E|_{\mathrm{Met}(M)\times\mathrm{Conn}(P)}$ so we have the following corollary.

<u>Corollary 5.2.10</u>. <u>Suppose that</u> $n \geqslant 3$. <u>Then a bundle metric</u> $\overline{g\omega}$ <u>that is a critical point of</u> $E|_{\mathrm{Met}(M)\times\mathrm{Conn}(P)}$ <u>is Ricci-flat (that is, satisfies the vacuum Einstein field</u>

equations) if and only if the following condition holds

i) $\mathrm{Ric}_G(F_i,F_j) = -\frac{1}{2}k(\Omega\circ\Omega^t(F_i),F_j) \qquad i,j = 1,\ldots,n$

If $n = 2$ the above is true with the extra condition

ii) $\kappa_{\overline{g\omega}} = 0.$

An important observation is that if $G = U(1) \approx S^1$, as in the case of electromagnetism, then condition i) in this corollary is unnecessary. This was Kalusa and Klein's observation:

Corollary 5.2.11. Let (M,g) be a compact riemannian 4-manifold (representing space-time), and $P \to M$ a principal $U(1)$-bundle with a connection ω (representing a Maxwell potential). Then the pair (g,ω) satisfies the combined Einstien-Yang-Mills equation if and only if the associated bundle metric $\overline{g\omega}$ on P satisfies the vacuum Einstein field equation, $\mathrm{Ric}_{\overline{g\omega}} = 0$.

For gauge theories with larger groups, however, condition i) in Corollary 5.2.10 does add a restriction. Thus, unfortunately, we cannot claim that in general a pair $(g,\omega) \in \mathrm{Met}(M)\times\mathrm{Conn}(P)$ satisfying the combined Einstein-Yang-Mills equations comes from a bundle metric satisfying the vacuum Einstein equation.

Some more can be said about the theory if we assume that the group G is simple. In that case the Ricci curvature Ric_G, being bi-invariant, must be a multiple of the bi-invariant metric k, that is, $Ric_G = \frac{\kappa_G}{r} k$. Then Condition i) in Corollary 5.2.10 may be written as

i)' $\Omega \circ \Omega^t = -\frac{2\kappa_G}{r} Id$.

Here $Id: \mathfrak{g} \to \mathfrak{g}$ is the identity map. An immediate necessary condition for i)' to hold is that the map $\Omega^t: \mathfrak{g} \to \Lambda^2 T_x M$ be injective for each $x \in M$. Hence, in particular, $r \leqslant n(n-1)/2$. This restricts the choice of group G and, for example, in the physically interesting case where $n = 4$ we have that $r \leqslant 6$. From the classifaction of simple Lie groups (given, for example, in Helgeson [Hel]) we have that the only possibilities in this case are $G = U(1)$ or $SU(2)$.

Turning now to a slightly more general question we ask for conditions under which a bundle metric $\overline{g\omega} \in Met(M) \times Conn(P) \subset Met(P)$ is Einstein. We already know that if $\overline{g\omega}$ is a critical point of $E|_{Met(M) \times Conn(P)}$ then it is Einstein only if it is Ricci-flat. Here we are interested in general metrics contained in $Met(M) \times Conn(P) \subset Met(P)$.

There may be many bundle metrics $\overline{g\omega}$ which are

Einstein. We could write down the equations g and ω must satisfy (using Theorem 5.2.2) but this does not seem particularly enlightening. Instead, we consider some special Einstein metrics - the ones having constant or near constant curvature. We firstly show that P admits a bundle metric or, more generally, any G-invariant metric of constant positive curvature only under very special circumstances.

Before stating our result we point out that there are some examples of principal G-bundles $P \xrightarrow{\pi} M$ which do admit G-invariant metrics of constant positive curvature. We have the S^1-bundle $S^{2n+1} \to P^n(C) = S^{2n+1}/S^1$ which defines the complex projective space $P^n(C)$. The S^1-action on S^{2n+1} is by isometries of the standard metric and the induced metric on $P^n(C)$ is the standard Fubini-Study metric. If we take a finite cyclic group $Z_p \subset S^1$ we can also form the S^1-bundle $S^{2n+1}/Z_p \to P^n(C)$. The "generalized lens space" S^{2n+1}/Z_p still has constant positive curvature. Another related example is the $SU(2)$-bundle $S^{4n+1} \to P^n(H)$ which defines the quaternionic projective space $P^n(H)$. Here, again, the $SU(2)$-action is by isometries of the standard constant curvature metric on S^{4n+1}. The theorem we prove states that these are the only possible examples.

Theorem 5.2.12. Let $P \xrightarrow{\pi} M$ *be a principal G-bundle over an orientable manifold* M. *If* P *admits a G-invariant metric* g *of constant positive curvature then either*

i) $G = U(1)$ *and* $P \xrightarrow{\pi} M$ *is one of the bundles*

$$S^{2n+1}/Z_p \to P^n(C)$$

or

ii) $G = SU(2)$ *and* $P \xrightarrow{\pi} M$ *is the bundle* $S^{4n+1} \to P^n(H)$.

Proof. In the first part we consider the case where M is simply-connected, so let $\tilde{M} \xrightarrow{\rho} M$ be the universal cover and $\tilde{P}$ the corresponding pull-back principal G-bundle over $\tilde{M}$ –

$$\begin{tikzcd} \tilde{P} \arrow[r, "\rho^*"] \arrow[d, "\tilde{\pi}"'] & P \arrow[d, "\pi"] \\ \tilde{M} \arrow[r, "\rho"] & M \end{tikzcd}$$

We determine the bundle $\tilde{P} \xrightarrow{\tilde{\pi}} \tilde{M}$.

Since P has constant curvature so does its cover $\tilde{P}$ and hence its universal cover is the sphere of constant curvature $S^N \xrightarrow{p} \tilde{P}$. Let $\tilde{G} \xrightarrow{p_G} G$ be the universal cover of our group G. We have a free action $\Phi: G \times \tilde{P} \to \tilde{P}$ which lifts to an action $\tilde{\Phi}: \tilde{G} \times S^N \to S^N$.

$$\begin{tikzcd} \tilde{G} \times S^N \arrow[r, "\tilde{\Phi}"] \arrow[d, "p_G \times \rho"'] & S^N \arrow[d, "p"] \\ G \times \tilde{P} \arrow[r, "\Phi"] & \tilde{P} \end{tikzcd}$$

Unfortunately, the action of $\tilde{G}$ on S^N may not be free but we may replace it by one that is by following a standard procedure. We let $H = \{\tilde{g} \in \tilde{G} \mid \tilde{\Phi}(\tilde{g},\cdot) = Id_{S^N}\}$ and consider the quotient $G' = \tilde{G}/H$. Let $\phi':G'\times S^N \to S^N$ be the quotient action. By construction Φ' is effective and by the unique path-lifting property we actually have that since Φ is free so is Φ' (see [Bre] for details). Hence we have the following commutative diagram.

$$\begin{tikzcd} G' \times S^N \arrow[r, "\Phi'"] \arrow[d, "p_{G'} \times \rho"'] & S^N \arrow[d, "p"] \\ G \times \tilde{P} \arrow[r, "\Phi"] & \tilde{P} \end{tikzcd}$$

We now determine G'. Firstly, we note that it is compact. This follows from the fact that $\ker(p_{G'}) \equiv$ the kernal of the homomorphism $p_{G'}:G' \to G$ is contained in

$\Gamma \equiv$ the finite of deck transformations of $S^N \overset{p}{\to} \tilde{P}$ (that is, $\Phi'(g',\cdot) \in \Gamma$ if $g' \in \ker(p_{G'})$). So G' is a compact connected group acting freely on S^N by isometries of the usual metric. There are not many choices - either

i) $G' = U(1)$ acting standardly on S^{2n+1} with quotient

or

ii) $G' = SU(2)$ acting standardly on S^{4n+1} with quotient $P^n(H)$.

We now wish to conclude that $\tilde{M} = P^n(C)$ or $P^n(H)$. To do this we need to show that $\ker(p_{G'}) = \Gamma$ since then

$$\tilde{M} = \tilde{P}/G = {}^{S^N/\Gamma}\!\Big/_{G'/\Gamma} = S^N/G'$$

and S^N/G' is known from i) and ii) above. We have already observed that $\ker(p_{G'}) \subset \Gamma$. The reverse inclusion follows from the fact that $\tilde{M}$ is simply-connected. Pick $\gamma \in \Gamma \simeq \pi_1(\tilde{P})$ - we wish to show that the action of γ on S^N is given by an element $g'_\gamma \in G'$, that is, for all $x \in S^N$, $\gamma(x) = \Phi'(g'_\gamma, x)$. Fix $p \in \tilde{P}$ and let $e \in G$ be the identity. Since $\tilde{M}$ is simply-connected we have from the exact sequence for the fibration $G \to \tilde{P} \overset{\pi}{\to} \tilde{M}$ that the inclusion $\pi_1(G,e) \to \pi_1(\tilde{P},p)$ is onto. Hence γ is represented by a loop in $\tilde{P}$, called γ still, based at p

and contained in the fiber $\tilde{\pi}^{-1}(\tilde{\pi}(p))$. It can be written in the form $t \to \Phi(g(t),p)$, $t \in [0,1]$, where here $g:[0,1] \to G$ is a loop in G based at the identity e. We lift g to a path $g':[0,1] \to G'$ beginning at the identity $e' \in G'$.

$$\begin{tikzcd} & G' \arrow[d] \\ {[0,1]} \arrow[ur, "g'"] \arrow[r, "g"'] & G \end{tikzcd}$$

Then the path $\tilde{\gamma}:[0,1] \to S^N$ given by $\tilde{\gamma}(t) = \Phi'(g'(t),x)$ is the unique lift of the path γ in $\tilde{P}$. Hence the action of γ on S^N, namely $x \to \gamma(x) = \tilde{\gamma}(1)$, is given by the action of $g'_\gamma = g'(1)$. This completes the proof that $\ker(p_{G'}) = \Gamma$ and we have determined that $\tilde{M} = P^n(C)$ or $P^n(H)$ according as to whether $G' = U(1)$ or $SU(2)$.

We also determine the original group $G = G'/\Gamma$. In the case that $G' = U(1)$ we have $\Gamma = Z_p$, a finite cyclic group, and $G = U(1)$ still. In the case that $G' = SU(2)$ we have Γ = trival group and so $G = G' = SU(2)$. Also, in the first case we have that $\tilde{P} = S^{2n+1}/Z_p$ (standard action) and in the second $\tilde{P} = S^{4n+1}$.

Thus, we have determined the G-bundle $\tilde{P} \overset{\tilde{\pi}}{\to} \tilde{M}$. To summarize we have either

i) $G = U(1)$ acting standardly on $\tilde{P} = S^{2n+1}/Z_p$ with quotient $\tilde{M} = P^n(C)$

or

ii) $G = SU(2)$ acting standardly on $\tilde{P} = S^{4n+1}$ with quotient $\tilde{M} = P^n(H)$.

All that is left to do is determine M, that is, we need to decide which manifolds are covered by $P^n(C)$ and $P^n(H)$. We use the Lefschetz fixed point theorem.

Case 1. $P^n(C)$. Let $f: P^n(C) \to P^n(C)$ be a non-trivial deck transformation so that, in particular, it is a fixed-point free diffeomorphism. We show that such a map exists only in the case that n is odd and then it is orientation reversing.

Let $\omega \in H^2(P^n(C), Z)$ be a generator of the cohomology ring $H^*(P^n(C), Z)$ (so that ω^k generates $H^{2k}(P^n(C), Z) = Z$ for $k \leq n$). Since f is a diffeomorphism the induced map on cohomology, $H^*(f)$, is an isomorphism. Hence $H^2(f)(\omega) = \pm\omega$.

If $H^2(f)(\omega) = \omega$ then $H^{2k}(f)(\omega^k) = \omega^k$ for $k=1,2,\ldots.$ and the Lefschetz number $L(f)$ of f defined by

$$L(f) = \sum_{i=1}^{2n} (-1)^i tr(H^i(f))$$

is equal to $\sum_{k=0}^{n} (-1)^{2k} \neq 0$. By the Lefschetz fixed point theorem such a map f has a fixed point. This is a contradiction to f being a deck transformation. Hence $H^2(f) = -\omega$. In that case $H^{2k}(f)(\omega^k) = (-1)^k\omega^k$ and we then have

$$L(f) = \sum_{k=0}^{n} (-1)^{2k}(-1)^k$$
$$= \sum_{k=0}^{n} (-1)^k$$
$$= \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

If n is even we again have a contradiction to f being a deck transformation. Hence the only possibility is that n is odd and then $H^{2n}(f)(\omega^n) = -\omega^n$ so f is orientation reversing. Such a map $P^n(C) \to P^n(C)$, n odd, exists - it is given by complex conjugation. However, the resulting manifold $P^n(C)/\Gamma$ is non-orientable.

Case 2. $P^n(H)$. Following an argument similar to the one just given for $P^n(C)$ (with $\upsilon \in H^4(P^n(H))$ a generator of the cohomology ring) we find that $P^n(H)$ does not cover any manifold if n is even and in the case that n is odd it only covers a non-orientable manifold.

So we have that both $P^n(C)$ and $P^n(H)$ are not non-trivial covers of any orientable manifold. Hence $M = \tilde{M}$ in both cases and the theorem is proved. QED

We now return to the problem of finding G-invariant Einstein metrics on P. Theorem 5.2.12 shows that if such a metric has constant positive curvature then the whole bundle $P \to M$ must be one of the very special ones described. Then, since Einstein metrics are, in particular, harmonic the usual "gap" phenomena may be expected. Here we have in mind results similar to those found by Bourguignon and Lawson [BL] for self-dual connections. They depend on a Bochner-Weitzenboch formula. Indeed, by examining this formula in our case such a "gap" theorem can be proved. The following theorem has been proved by Bourguignon [B] and was discovered independently by the author. We refer the reader to [B] for the proof.

Theorem 5.2.13. Let (X,g) be a compact n-dimensional riemannian manifold having harmonic curvature. If the sectional curvature K_X of X satisfies

$$\frac{5n}{9n-4}K \leq K_X \leq K$$

for some $K > 0$, then (X,g) has constant curvature K.

Thus, if a G-bundle $P \to M$ has a bundle metric $g\omega$ (or any G-invariant metric) which Einstein and close to having constant positive curvature then in fact the metric

<u>is</u> of constant curvature and we are back in the situation described by Theorem 5.2.12. In particular, then, we have a type the of topological quantization since the bundle itself must be one of the special types described.

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