ON THE COHOMOLOGY OF LIE ALGEBRA EXTENSIONS

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William Evan Rosenthal

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Abstract of the Dissertation

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In this thesis we define a Hochschild-Serre-type spectral sequence for an extension $0 \to \mathfrak{I} \to \mathfrak{g} \to \Phi \to 0$ of Lie algebras with abelian kernel $\mathfrak{I}$. We use techniques similar to those employed by Charlap and Vasquez for the group extension case to show that if $\mathfrak{g}$ is the semidirect product of $\mathfrak{I}$ and Φ, the second differential d_2 is identically 0. In general, we show that d_2 is given by cup product with a cohomology class that is the image of the extension under a map determined by a homology multiplication.

To the memory of my father. I am forever grateful that the apple did not fall far from the tree.

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CHAPTER I

INTRODUCTION

In [7], Hochschild and Serre defined two spectral sequences associated with a group extension $0 \to K \to G \to \Phi \to 1$ with abelian kernel K. Charlap and Vasquez [3] computed the second differential d_2 of one of these sequences, the Cartan-Leray sequence. They showed that for $x \in E_2^{p,N}$, $d_2(x)$ is given by cup product involving a canonically determined element $\theta(x)$ with a second factor that decomposes into two summands. One of these (the characteristic class of the Φ-module K) is independent of the extension and the other is determined by applying a map induced by Pontrjagin multiplication to the cohomology class $\alpha \in H^2(\Phi;K)$ classifying the extension.

In [8], Hochschild and Serre introduced a spectral sequence for Lie algebra extensions, analogous to the Lyndon sequence in the group case. However, they did not treat the analogue of the Cartan-Leray sequence. This we do in Chapter III, and subsequently a cup product formula is obtained. Briefly, this formula is described as follows. For $0 \to \mathfrak{I} \to \mathfrak{g} \to \Phi \to 0$ a Lie algebra extension with abelian kernel and A an appropriate coefficient space, $H_N(\mathfrak{I};A)$ and $H^N(\mathfrak{I};A)$ are Φ-modules. Denote by E, $\bar{E}$, and $\hat{E}$ the spectral sequences for the extension with coefficients in A, $H^N(\mathfrak{I};A)$, and $H_N(\mathfrak{I};A)$ respectively. A pairing of spectral sequences with $\bar{E}_2^{p,0} \otimes \hat{E}_2^{0,N} \to E_2^{p,N}$ is obtained, and there is a canonical isomorphism θ: $E_2^{p,N} \approx \bar{E}_2^{p,0}$. The cup product formula (Chapter VI) expresses $d_2(x)$ $(x \in E_2^{p,N})$ as the product of $\theta(x)$ with an element $\alpha^N \in \hat{E}_2^{2,N-1}$. This formula is precisely that of Charlap and Vasquez. However, in our case, there is no "constant term" independent of the extension; more precisely, this term vanishes identically. The main consequence is that if $\mathfrak{g}$ is the semidirect product of $\mathfrak{I}$ and Φ, then the differential d_2 also vanishes identically.

CHAPTER II

PRELIMINARIES

In this chapter the basic facts concerning Lie algebra cohomology are summarized.

1. $\mathfrak{g}$-modules

Let $\mathfrak{g}$ be a Lie algebra over a field k (no restriction is assumed on either the characteristic of k or the dimension of $\mathfrak{g}$). A (left) $\mathfrak{g}$-module is a k-vector space together with an action of $\mathfrak{g}$ on M; i.e., a Lie algebra homomorphism $\rho: \mathfrak{g} \to L(\mathrm{End}_k M)$ where $L(\mathrm{End}_k M)$ denotes the associative algebra $\mathrm{End}_k M$ made into a Lie algebra via $[\alpha,\beta] = \alpha\beta-\beta\alpha$. We think of the elements of $\mathfrak{g}$ acting on M according to the rule $z_1\cdot(z_2\cdot m) - z_2\cdot(z_1\cdot m) = [z_1,z_2]\cdot m$. Right $\mathfrak{g}$-modules are defined similarly. Any right $\mathfrak{g}$-module may be made into a left $\mathfrak{g}$-module by defining $z\cdot m = -m\cdot z$ (and vice versa). The subspace of invariant elements of M consists of all elements of M that are annihilated by each element of $\mathfrak{g}$; i.e., $M^{\mathfrak{g}} = \{m\in M \mid z\cdot m = 0$ for all $z\in\mathfrak{g}\}$. M is called $\mathfrak{g}$-trivial if $M^{\mathfrak{g}} = M$; any k-vector space may be regarded as a trivial $\mathfrak{g}$-module.

As in the category of groups, where for a group G, G-modules correspond to modules over the ring $\mathbb{Z}G$, modules over a Lie algebra $\mathfrak{g}$ correspond to modules over a universal enveloping algebra $U\mathfrak{g}$. This is the quotient of the tensor algebra $T\mathfrak{g}$ by the ideal generated by $z_1 \otimes z_2 - z_2 \otimes z_1 - [z_1, z_2]$. $U\mathfrak{g}$ thus assumes the role for Lie algebra cohomology that $\mathbb{Z}G$ does for group cohomology. There is a map $i: \mathfrak{g} \to U\mathfrak{g}$ given by $\mathfrak{g} \to T\mathfrak{g} \to U\mathfrak{g}$; i is a monomorphism and we consider $\mathfrak{g}$ as imbedded in $U\mathfrak{g}$. It is worthwhile to observe that if $\mathfrak{g}$ is abelian with k-basis $\{z_\gamma\}$, then $U\mathfrak{g}$ is commutative and is the polynomial algebra $k[z_\gamma]$.

The Poincaré-Birkhoff-Witt theorem is fundamental to the theory. It states that if $\mathfrak{g}$ is k-free on an ordered basis $\{z_\gamma\}_{\gamma \in C}$, then $U\mathfrak{g}$ is k-free on $\{z_I\}$, where I is a finite increasing sequence in C and z_I denotes the product $z_{\gamma_1} \cdots z_{\gamma_n}$ in $U\mathfrak{g}$ where $I = (\gamma_1, \ldots, \gamma_n)$. The empty sequence corresponds to $1 \in U\mathfrak{g}$. In a sense $U\mathfrak{g}$ has a more complicated structure as a k-module than does $\mathbb{Z}G$ as a $\mathbb{Z}$-module; $\mathbb{Z}G$ has a $\mathbb{Z}$-basis consisting solely of the elements of G, while a k-basis of $U\mathfrak{g}$ consists of (ordered) products of the elements of $\mathfrak{g}$. It is a consequence of the Poincaré-Birkhoff-Witt theorem that a $\mathfrak{g}$-module M is $U\mathfrak{g}$-free on $\{m_i\}$ if and only if it is k-free on $\{z_I \cdot m_i\}$.

Remark on notation: We will follow the usual convention and write "$\mathfrak{g}$-free" for "$U\mathfrak{g}$-free", "$Hom_{\mathfrak{g}}$" for "$Hom_{U\mathfrak{g}}$", etc. Hom and tensor over the base field k will be denoted simply by "Hom" and "$\otimes$" respectively.

2. Diagonal actions

We shall make frequent use of diagonal actions on Hom and tensor. Let B and A be left $\mathfrak{g}$-modules. For $f \in Hom(B,A)$ and $z \in \mathfrak{g}$, define

$$(2.1) \qquad (z\cdot f)(b) = z\cdot f(b) - f(z\cdot b).$$

This makes Hom(B,A) into a $\mathfrak{g}$-module; note we have $\{Hom(B,A)\}^{\mathfrak{g}} = Hom_{\mathfrak{g}}(B,A)$. Similarly, define

$$(2.2) \qquad z\cdot(b\otimes a) = z\cdot b \otimes a + b \otimes z\cdot a;$$

this gives diagonal action on the tensor product. If B is a right $\mathfrak{g}$-module, (2.2) must be modified to

$$(2.3) \qquad z.(b\otimes a) = -b\cdot z \otimes a + b \otimes z\cdot a .$$

Furthermore, if $\mathfrak{I}$ is an ideal of $\mathfrak{g}$, we have diagonal action of the quotient on $Hom_{\mathfrak{I}}$ and $\otimes_{\mathfrak{I}}$. For B and A left $\mathfrak{g}$-modules and $f \in Hom_{\mathfrak{I}}(B,A)$, define $z\cdot f$ by formula (2.1). Since f is an $\mathfrak{I}$-map, the ideal $\mathfrak{I}$ annihilates $Hom_{\mathfrak{I}}(B,A)$ and the action is well-defined on the quotient. Similarly, for B a right $\mathfrak{g}$-module, A a left $\mathfrak{g}$-module, and $b \otimes a \in B \otimes A$, define $z\cdot(b \otimes a)$ by formula (2.3). Again, the action passes to one of the quotient.

3. Cohomology and homology of Lie algebras

Given a left $\mathfrak{g}$-module A and a right $\mathfrak{g}$-module B, define

$$H^n(\mathfrak{g};A) = \mathrm{Ext}^n_{\mathfrak{g}}(k,A)$$

and

$$H_n(\mathfrak{g};B) = \mathrm{Tor}^{\mathfrak{g}}_n(B,k) \text{ for } n = 0,1,\ldots .$$

As with groups, we have $H^0(\mathfrak{g};A) = A^{\mathfrak{g}}$. $H^n(\mathfrak{g};A)$ may be computed via any $\mathfrak{g}$-projective resolution of the trivial $\mathfrak{g}$-module k. The following resolution, called (by an abuse of terminology) the standard resolution, will be used throughout this paper. Denote the exterior algebra of $\mathfrak{g}$ by $E_*(\mathfrak{g})$ and define $C_*(\mathfrak{g}) = U\mathfrak{g} \otimes E_*(\mathfrak{g})$. For each $n \geq 0$, $C_n(\mathfrak{g}) = U\mathfrak{g} \otimes E_n(\mathfrak{g})$ is a $\mathfrak{g}$-module via the action of $\mathfrak{g}$ on the first factor, which is multiplication in $U\mathfrak{g}$; i.e., $z\cdot(u \otimes X) = zu \otimes X$ for $z \in \mathfrak{g}$, $u \in U\mathfrak{g}$, $X \in E_n(\mathfrak{g})$. It follows that if $\mathfrak{g}$ is k-free on $\{z_\gamma\}$, then $C_n(\mathfrak{g})$ is $\mathfrak{g}$-free on $\{1 \otimes \langle z_{\gamma_1},\ldots,z_{\gamma_n}\rangle \mid \gamma_1<\cdots<\gamma_n\}$; here we follow standard notation and write $\langle z_{\gamma_1},\ldots,z_{\gamma_n}\rangle$ for $z_{\gamma_1}\wedge\cdots\wedge z_{\gamma_n}$.

The differential in $C_*(\mathfrak{g})$ is given by

$$(2.4)\quad d_n(1\otimes\langle z_1,\ldots,z_n\rangle) = \sum_{i=1}^{n}(-1)^{i+1} z_i\otimes\langle z_1,\ldots,\hat{z}_i,\ldots,z_n\rangle$$

$$+ \sum_{1\le i<j\le n}(-1)^{i+j} 1\otimes\langle [z_i,z_j],z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots z_n\rangle$$

and the augmentation $\varepsilon\colon C_o(\mathfrak{g}) \to k$ is the structure map of k considered as a trivial $\mathfrak{g}$-module. It is very important to observe that (2.4) is stated and holds for arbitrary elements $z_1,\ldots,z_n$ of $\mathfrak{g}$; this is because the formula giving the differential is alternating and multilinear and $d^2 = 0$ for arbitrary elements.

Remark 1: A Lie algebra $\mathfrak{g}$ may be regarded as a module over itself via the adjoint action $z\cdot z_1 = [z,z_1]$. This extends to an action of $\mathfrak{g}$ on $E_n(\mathfrak{g})$ by derivations: $z\cdot\langle z_1,\ldots,z_n\rangle = \sum_{i=1}^{n}\langle z_1,\ldots,[z,z_i],\ldots,z_n\rangle$. Hence diagonal action gives another action of $\mathfrak{g}$ on $C_n(\mathfrak{g})$; denote this new $\mathfrak{g}$-module by $C_n'(\mathfrak{g})$ and note that the action of $\mathfrak{g}$ on $C_n(\mathfrak{g})$ is itself diagonal action, with trivial action on the second factor. It can be shown that if M is any $\mathfrak{g}$-module and M_o is the underlying vector space of M; i.e., M made $\mathfrak{g}$-trivial, there is a $\mathfrak{g}$-isomorphism $U\mathfrak{g}\otimes M \approx U\mathfrak{g}\otimes M_o$ that carries $1\otimes m$ to $1\otimes m$. Hence $C_n'(\mathfrak{g})$ is a free $\mathfrak{g}$-module and $C_*'(\mathfrak{g})$ a $\mathfrak{g}$-free resolution of k. We may use either resolution for cohomological purposes, but nothing is gained by using the "more complicated" resolution $C'_*(\mathfrak{g})$. It should be noted that the differential in $C'_*(\mathfrak{g})$ is induced by

$$\begin{array}{ccc} C_n(\mathfrak{g}) & \xrightarrow{d_n} & C_{n-1}(\mathfrak{g}) \\ \wr\wr & & \wr\wr \\ C'_n(\mathfrak{g}) & \xrightarrow[d'_n]{} & C'_{n-1}(\mathfrak{g}) \end{array}$$

and takes a slightly different form than (2.4):

$$d'_n(1\otimes\langle z_1,\ldots,z_n\rangle) = \sum_{i=1}^{n}(-1)^{i+1} z_i \otimes \langle z_i,\ldots,\hat{z}_i,\ldots,z_n\rangle$$

$$+ \sum_{1\le i<j\le n}(-1)^{i+j+1} 1\otimes\langle [z_i,z_j],z_1,\cdots,\hat{z}_i,\cdots\hat{z}_j,\cdots z_n\rangle.$$

Note the different sign in the second summand above.

Remark 2: $H^n(\mathfrak{g};A)$ is the $n^{\underline{th}}$ cohomology space of the cochain complex $C^*(\mathfrak{g},A) = \mathrm{Hom}_{\mathfrak{g}}(C_*(\mathfrak{g}),A)$. Adjoint associativity yields $C^*(\mathfrak{g},A) = \mathrm{Hom}(E_*(\mathfrak{g}),A)$; we will go back and forth between these complexes as we please. In the second complex, the n-cochains of $\mathfrak{g}$ in A are the n-multilinear alternating functions with arguments in $\mathfrak{g}$ and values in A; the coboundary takes the form

$$(2.5)\quad (\delta f)(x_1,\ldots,z_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} z_i\cdot f(z_1,\ldots,\hat{z}_i,\ldots,z_{n+1})$$

$$+ \sum_{1\leq i<j\leq n+1} (-1)^{i+j} f([z_i,z_j],z_1,\ldots\hat{z}_i,\ldots,\hat{z}_j,\ldots z_{n+1}).$$

Observe that if $\mathfrak{g}$ is abelian, the second summand disappears, and if A is $\mathfrak{g}$-trivial, so does the first summand. Hence in this situation we have that $\delta \equiv 0$ and $H^n(\mathfrak{g};A) = C^n(\mathfrak{g},A)$. In this setting, we will not distinguish between cohomology classes and their representative cochains (= cocycles); however, we will continue to write H^n rather than C^n. Similar remarks apply to homology with coefficients in a right $\mathfrak{g}$-module B; here we have $C_*(\mathfrak{g},B) = B \otimes_{\mathfrak{g}} C_*(\mathfrak{g}) = B \otimes E_*(\mathfrak{g})$ and the boundary in the latter complex is

$$(2.6)\quad \partial(b\otimes\langle z_1,\ldots,z_n\rangle) = \sum_{i=1}^{n} (-1)^{i+1}(b\cdot z_i)\otimes\langle z_1,\ldots\hat{z}_i,\ldots,z_n\rangle$$

$$+ \sum_{1\leq i<j\leq n} (-1)^{i+j} b\otimes\langle [z_i,z_j],z_1\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots z_n\rangle.$$

4. Extensions and the semidirect product

The theory of Lie algebra extensions is analogous to that of group extensions. An extension of Lie algebras with abelian kernel $\mathfrak{I}$ is an exact sequence

$$(2.7) \qquad 0 \to \mathfrak{I} \xrightarrow{j} \mathfrak{g} \xrightarrow{\rho} \Phi \to 0 \quad .$$

Regard $\mathfrak{I}$ as an ideal of $\mathfrak{g}$ and Φ as the quotient. $\mathfrak{I}$ is a $\mathfrak{g}$-module via $z \cdot x = [z,x]$; since it is abelian, it becomes a Φ-module via $\sigma \cdot x = [z,x]$, $z \in \rho^{-1}(\sigma)$.

Now start with a Φ-module $\mathfrak{I}$. An extension of Φ by $\mathfrak{I}$ is an extension (2.7) of Lie algebras such that the Φ-structure induced by the extension agrees with the given structure. Equivalence of extensions is defined precisely as with groups, and the equivalence classes of extensions of Φ by $\mathfrak{I}$ are in 1-1 correspondence with the elements of $H^2(\Phi;\mathfrak{I})$. If $\alpha \in H^2(\Phi;\mathfrak{I})$ classifies the extension (2.7), $\mathfrak{g}$ may be described as follows. As a vector space, $\mathfrak{g} = \mathfrak{I} \oplus \Phi$ and the bracket in $\mathfrak{g}$ is given by

$$(2.8) \qquad [(x_1,\sigma),(x_2,\tau)] = (\sigma \cdot x_2 - \tau \cdot x_1 + a(\sigma,\tau),[\sigma,\tau]),$$

where $a \in C^2(\Phi,\mathfrak{I})$ is a 2-cocycle representing α (see Cartan-Eilenberg [2], p. 307). The maps $j\colon \mathfrak{I} \to \mathfrak{g}$ and $\rho\colon \mathfrak{g} \to \Phi$ are then described by $j(x) = (x,0)$ and $\rho(x,\sigma) = \sigma$. The semidirect product $\mathfrak{I} \cdot \Phi$ of $\mathfrak{I}$ and Φ is the extension whose equivalence class corresponds to $\alpha = 0$ in $H^2(\Phi;\mathfrak{I})$; in this case we may take $a = 0$ and the bracket in $\mathfrak{I} \cdot \Phi$ is

(2.9) $$[(x_1,\sigma),(x_2,\tau)] = (\sigma\cdot x_2-\tau\cdot x_1,[\sigma,\tau]).$$

Remark: In general, the complementary subspace Φ (of $\mathfrak{I}$ in $\mathfrak{g}$) is not a Lie subalgebra of $\mathfrak{g}$. However, if $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, (2.9) shows that $[(0,\tau),(0,\tau)] = (0,[\sigma,\tau])$ and here the complementary subspace is a Lie subalgebra. This fact will be crucial in our future calculations.

CHAPTER III

THE SPECTRAL SEQUENCE

Let M be a left $\mathfrak{g}$-module and let $\mathfrak{I}$ be an ideal of $\mathfrak{g}$, with $\Phi = \mathfrak{g}/\mathfrak{I}$. In this chapter we define a double complex and construct a Hochschild-Serre type spectral sequence; i.e., $E_2^{p,q}(M) \approx H^p(\Phi;H^q(\mathfrak{I};M)) \underset{p}{\Rightarrow} H^{p+q}(\mathfrak{g};M)$. This is essentially an adaptation to Lie algebras of the Cartan-Leray spectral sequence for group extensions; our techniques are in essence a rewording of those in MacLane [9] and Rotman [10]. First we need a technical lemma which is the analogue of a standard result for groups (see Hilton-Stammbach [6], p. 213).

Lemma 3.1: Let M be a left $\mathfrak{g}$-module and let M_o denote the underlying vector space of M (i.e., M made $\mathfrak{g}$-trivial). Then $\mathrm{Hom}(U\mathfrak{g},M) \approx \mathrm{Hom}(U\mathfrak{g},M_o)$ as $\mathfrak{g}$-modules with diagonal action.

Proof: An element of $m \in M$ will be written as $m_o \in M_o$. The idea of the proof is to define maps $\mathfrak{c}$: $\mathrm{Hom}(U\mathfrak{g},M) \to \mathrm{Hom}(U\mathfrak{g},M_o)$ and ψ: $\mathrm{Hom}(U\mathfrak{g},M_o) \to \mathrm{Hom}(U\mathfrak{g},M)$ recursively by requiring them to be $\mathfrak{g}$-maps. Let $\alpha \in \mathrm{Hom}(U\mathfrak{g},M)$.

Define $\mathfrak{C}(\alpha)$ by starting with $\{\mathfrak{C}(\alpha)(1)\}_o = \{\alpha(1)\}_o$. Then for $z \in \mathfrak{g}$, force $\mathfrak{C}(\alpha)(z)$ as follows:

$$\begin{aligned}\{(z\cdot\mathfrak{C}(\alpha))(1)\}_o &= z\cdot\{\mathfrak{C}(\alpha)(1)\}_o - \{\mathfrak{C}(\alpha)(z)\}_o \\ &= -\{\mathfrak{C}(\alpha)(z)\}_o\end{aligned}$$

so we must have

$$\begin{aligned}\{\mathfrak{C}(\alpha)(z)\}_o &= -\{\mathfrak{C}(z\cdot\alpha)(1)\}_o \\ &= -\{(z\cdot\alpha)(1)\}_o \\ &= -\{z\cdot\alpha(1)-\alpha(z)\}_o\end{aligned}$$

Continuing in this manner, $\{\mathfrak{C}(\alpha)(zz_1,\ldots,z_n)\}_o$ is constructed from $\{\mathfrak{C}(\alpha)(z_1,\ldots,z_n)\}_o$ and the requirement that $\mathfrak{C}(\alpha)$ be a $\mathfrak{g}$-map; the next step yields

$$\{\mathfrak{C}(\alpha)(z_1z_2)\}_o = \{\alpha(z_1z_2)-z_1\cdot\alpha(z_2)-z_2\cdot\alpha(z_1)+z_2z_1\cdot\alpha(1)\}_o.$$

$\mathfrak{C}(\alpha)$ is well defined on $U\mathfrak{g}$ since α itself is.

Going in the other direction, for $\beta \in \mathrm{Hom}(U\mathfrak{g},M_o)$, start with $\psi(\beta)(1) = \beta(1)$. Then for $z \in \mathfrak{g}$,

$$\begin{aligned}(z\cdot\psi(\beta))(1) &= z\cdot(\psi(\beta)(1)) - \psi(\beta)(z) \\ &= z\cdot\beta(1) - \psi(\beta)(z).\end{aligned}$$

But

$$\begin{aligned}(\psi(z\cdot\beta))(1) &= (z\cdot\psi)(1) = z\cdot\{\beta(1)\}_o-\{\beta(z)\}_o \\ &= -\beta(z)\end{aligned}$$

(regarded in M) so comparing the two expressions, we must have $\psi(\beta)(z) = \beta(z) + z\cdot\beta(1)$. As with $\mathfrak{C}$, we continue in this manner; the next step yields

$$\psi(\beta)(z_1 z_2) = \beta(z_1 z_2) + z_1\cdot\beta(z_2) + z_2\cdot\beta(z_1) + z_1 z_2\cdot\beta(1).$$

One checks at each stage that $\mathfrak{C}$ and ψ are mutual inverses; e.g.,

$$\{(\mathfrak{C}\circ\psi)(\beta)(1)\}_o = \{\psi(\beta)(1)\}_o = \{\beta(1)\}_o,$$

$$(\psi\circ\mathfrak{C})(\alpha)(1) = \mathfrak{C}(\alpha)(1) = \alpha(1),$$

$$\begin{aligned}\{(\mathfrak{C}\circ\psi)(\beta)(z)\}_o &= \{\psi(\beta)(z) - z\cdot\psi(\beta)(1)\}_o \\ &= \{\beta(z) + z\cdot\beta(1) - z\cdot\beta(1)\}_o \\ &= \{\beta(z)\}_o.\end{aligned}$$ ∎

<u>Corollary 3.2</u>: $\mathrm{Hom}_{\mathfrak{I}}(U\mathfrak{g},M) \approx \mathrm{Hom}_{\mathfrak{I}}(U\mathfrak{g},M_o)$ as Φ-modules.

<u>Proof</u>: In $\mathrm{Hom}(U\mathfrak{g},M) \overset{\mathfrak{g}}{\approx} \mathrm{Hom}(U\mathfrak{g},M_o)$, take $\mathfrak{I}$-invariant elements. ∎

There is a result analogous to Lemma 3.1 for the tensor product with diagonal action.

Now consider the cochain complex $\mathfrak{U}^*$ with $\mathfrak{U}^j = \mathrm{Hom}_{\mathfrak{I}}(C_j(\mathfrak{g}),M)$, and coboundary induced by the boundary in $C_*(\mathfrak{g})$. It is a consequence of the Poincare-Birkhoff-Witt theorem that the enveloping algebra $U\mathfrak{g}$ is $\mathfrak{I}$-free. Since the tensor product (over k) of an $\mathfrak{I}$-free module and a vector space is again $\mathfrak{I}$-free; $C_j(\mathfrak{g})$ is $\mathfrak{I}$-free for each $j \geq 0$ and $C_*(\mathfrak{g})$ is an $\mathfrak{I}$-free resolution of the trivial $\mathfrak{I}$-module k. Hence it may be used to compute the cohomology of $\mathfrak{I}$; i.e., $H^*(\mathfrak{I};M) = H^*(\mathfrak{U}^*)$. The diagonal action of Φ on $\mathrm{Hom}_{\mathfrak{I}}(C_*(\mathfrak{g}),M)$ commutes with the coboundary, yielding the action of Φ on $H^*(\mathfrak{I};M)$.

<u>Lemma 3.3</u>: For $p \geq 0$, $q > 0$, $H^q(\Phi;\mathrm{Hom}_{\mathfrak{I}}(C_p(\mathfrak{g}),M)) = 0$.

<u>Proof</u>: It suffices to prove this for $p = 0$; i.e., for $U\mathfrak{g}$, the free $\mathfrak{g}$-module on one generator. For any Lie algebra $\mathfrak{L}$ with Lie subalgebra S, consider the (coinduced) S-module $\mathrm{Hom}_S(UL,A)$, for A an S-module. Shapiro's Lemma for Lie algebras (see Cartan-Eilenberg [2], p. 275) says that $H^q(S;A) \approx H^q(L;\mathrm{Hom}_S(UL,A))$. Take $L = \mathfrak{g}/\mathfrak{I}$ and $S = \mathfrak{I}/\mathfrak{I} = 0$. Then $\mathrm{Hom}_S = \mathrm{Hom}_{U(0)} = \mathrm{Hom}$, and the left side is 0 for $q > 0$, yielding $H^q(\Phi;\mathrm{Hom}(U(\mathfrak{g}/\mathfrak{I}),A)) = 0$ for $q > 0$. But $U(\mathfrak{g}/\mathfrak{I}) \approx k \otimes_{\mathfrak{I}} U\mathfrak{g}$ (as Φ-modules) and adjoint associativity yields $\mathrm{Hom}(U(\mathfrak{g}/\mathfrak{I}),A) \approx \mathrm{Hom}_{\mathfrak{I}}(U\mathfrak{g},A)$. This proves the lemma modulo a subtlety. In Shapiro's lemma, A is taken to be an S-module and here $S = 0$; hence A

is just a vector space. But the action of Φ on $\mathrm{Hom}_{\mathfrak{l}}(U\mathfrak{g},M)$ uses the action of $\mathfrak{g}$ on M. However, corollary 3.2 takes care of this by saying that the Φ-modules in question are isomorphic. ■

Now we construct the spectral sequence. For $p,q \geq 0$, define $\mathcal{L}^{p,q} = C^p(\Phi,\mathfrak{U}^q) = \mathrm{Hom}(E_p(\Phi),\mathfrak{U}^q)$. The bigraded module $\mathcal{L}$ has two coboundary operators: $d_\Phi\colon \mathcal{L}^{p,q} \to \mathcal{L}^{p+1,q}$ is defined by $(d_\Phi f)(c) = (d'f)(c)$ for $c \in E_{p+1}(\Phi)$, where d' is the coboundary in $C^*(\Phi,\mathfrak{U}^q)$, and $d_\mathfrak{U}\colon \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1}$ is defined by $(d_\mathfrak{U} f)(c) = d''\{f(c)\}$ for $c \in E_p(\Phi)$, where d" is the coboundary in $\mathfrak{U}^*$.

In accordance with the "standard sign convention", set $d = d_\Phi + (-1)^p d_\mathfrak{U}$. This makes $\mathcal{L}$ into a double complex.

The (cochain) complex associated with $\mathcal{L}$ is as usual denoted by $\mathrm{Tot}(\mathcal{L})$ and is defined by $\{\mathrm{Tot}(\mathcal{L})\}_n = \bigoplus_{p+q=n} \mathcal{L}^{p,q}$.

Our spectral sequence is the one arising from the first filtration ${}^{I}F(\mathrm{Tot}(\mathcal{L}))$: $\{{}^{I}F(\mathrm{Tot}(\mathcal{L}))\}_i = \bigoplus_{q=0}^{\infty} \bigoplus_{p\geq i} \mathcal{L}^{p,q}$.

Denote this spectral sequence by E(M). It is standard that

$$(3.1) \qquad E_2^{p,q}(M) \underset{p}{\Rightarrow} H^{p+q}(\mathrm{Tot}(\mathcal{L})).$$

<u>Proposition 3.4</u>: $E_2^{p,q}(M) \approx H^p(\Phi;H^q(\mathfrak{I};M)) \underset{p}{\Rightarrow} H^{p+q}(\mathfrak{g};M)$.

<u>Proof</u>: Consider the first iterated cohomology of $\mathfrak{L}$; i.e., $H''^{p,q}(\mathfrak{L}) = \ker(d_{\mathfrak{U}}\colon \mathfrak{L}^{p,q} \to \mathfrak{L}^{p,q+1})/d_{\mathfrak{U}}(\mathfrak{L}^{p,q-1})$. For each fixed p, this is the cohomology of the $p^{\underline{th}}$ column, and $H''^{p,q}(\mathfrak{L})$ is a complex with coboundary $\partial''\colon H''^{p,q}(\mathfrak{L}) \to H''^{p+1,q}(\mathfrak{L})$ induced by d_Φ. Now take cohomology with respect to ∂'' to yield a bigraded module $H'^pH''^{p,q}(\mathfrak{L})$. It is a standard result that $E_2^{p,q}(M) \approx H'^pH''^{p,q}(\mathfrak{L})$. Now $H''^{p,q}(\mathfrak{L})$ is the $q^{\underline{th}}$ cohomology space of the complex $\mathrm{Hom}(E_p(\Phi),\mathfrak{U}^*) = \mathrm{Hom}_\Phi(C_p(\Phi),\mathfrak{U}^*)$. Since $C_p(\Phi)$ is Φ-free, it is Φ-projective and $\mathrm{Hom}\,(C_p(\Phi),-)$ is an exact functor. Hence

$$H''^{p,q}(\mathfrak{L}) = H^q\{\mathrm{Hom}_\Phi(C_p(\Phi),\mathfrak{U}^*)\}$$

$$\mathrm{Hom}_\Phi(C_p(\Phi),H^q(\mathfrak{U}^*)),$$

since cohomology commutes with exact functors. So $H''^{p,q}(\mathfrak{L}) \approx \mathrm{Hom}_\Phi(C_p(\Phi),H^q(\mathfrak{I};M))$. Pass to cohomology: $H'^pH''^{p,q}(\mathfrak{L}) \approx H^p\{\mathrm{Hom}_\Phi(C_*(\Phi),H^q(\mathfrak{I};M))\}$ or $E_2^{p,q}(M) \approx H^p(\Phi;H^q(\mathfrak{I};M))$.

We now show convergence. Consider the spectral sequence arising from the second filtration ${}^{II}F(\mathrm{Tot}(\mathfrak{L}))$ of $\mathrm{Tot}(\mathfrak{L})$: $\{{}^{II}F(\mathrm{Tot}(\mathfrak{L}))\}_i = \bigoplus_{q=0}^{\infty} \bigoplus_{p\geq i} \mathfrak{L}^{p,q}$. Denote this sequence by E'(M). The second iterated cohomology $H''^p H'^{q,p}(\mathcal{L})$ of $\mathcal{L}$ is computed by first taking cohomology of the $p^{\underline{th}}$ row of $\mathcal{L}$, and then of the columns. Again, it is standard that $E_2'^{p,q}(M) \approx H''^p H'^{q,p}(M)$. The $p^{\underline{th}}$ row of $\mathcal{L}$ is $C^*(\Phi, \mathrm{Hom}_{\mathfrak{T}}(C_p(\mathfrak{g}),M))$, so $H'^{q,p}(\mathcal{L})$ is the $q^{\underline{th}}$ cohomology space of Φ with coefficients in $\mathrm{Hom}_{\mathfrak{T}}(C_p(\mathfrak{g}),M)$. Lemma 3.3 says this is 0 if $q > 0$. If $q = 0$, $H^0(\Phi;\mathrm{Hom}_{\mathfrak{T}}(C_p(\mathfrak{g}),M)) \approx \{\mathrm{Hom}_{\mathfrak{T}}(C_p(\mathfrak{g}),M)\}^{\Phi} \approx \mathrm{Hom}_{\mathfrak{g}}(C_p(\mathfrak{g}),M)$, hence

$$H'^{q,p}(\mathcal{L}) = \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(C_p(\mathfrak{g}),M) & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

Passing to cohomology yields

$$E_2'^{p,q}(M) = \begin{cases} H^p(\mathfrak{g};M) & \text{if } q = 0 \\ 0 & \text{if } q > 0\ . \end{cases}$$

Hence we get nonzero terms only on the base $q = 0$; i.e., the spectral sequence collapses. Hence $H^n(\mathfrak{g};M) \approx E_2'^{n,0}(M)$. Now if $\mathcal{E}$ is either spectral sequence derived from a double complex $\mathfrak{C}$ and if $\mathcal{E}$ collapses, then $\mathcal{E}_2^{n,0} \approx H^n(\mathrm{Tot}(\mathfrak{C}))$ (since there is only one nonzero factor in the filtration). In our case, this yields $H^n(\mathfrak{g};M) \approx H^n(\mathrm{Tot}(\mathcal{L}))$, and combining this with (3.1) yields $E_2^{p,q}(M) \underset{p}{\Rightarrow} H^{p+q}(\mathfrak{g};M)$ ∎

CHAPTER IV

THE PRINCIPAL MACHINERY

In ch. III, we have described $H^*(\mathfrak{I};M)$ as $H^*(\mathfrak{U}^*)$. However, if $\mathfrak{V}^*$ is the cochain complex $\mathfrak{V}^* = \mathrm{Hom}_{\mathfrak{I}}(C_*(\mathfrak{I}),M)$, $H^*(\mathfrak{I};M)$ is of course also $H^*(\mathfrak{V}^*)$. Since both $C_*(\mathfrak{I})$ and $C_*(\mathfrak{g})$ are $\mathfrak{I}$-free resolutions of k, there exist $\mathfrak{I}$-module chain maps

$$j_*\colon C_*(\mathfrak{I}) \to C_*(\mathfrak{g}) \quad \text{and} \quad \psi_*\colon C_*(\mathfrak{g}) \to C_*(\mathfrak{I})$$

over the identity map of k. The main focus of this chapter is to describe j_* and ψ_* (and a certain chain homotopy s_*) explicitly, for later use in computations.

Let $\{x_\alpha\}_{\alpha\in A}$ be an ordered k-basis of the abelian ideal $\mathfrak{I}$. Since elements of the form $\langle x_{\alpha_1},\ldots,x_{\alpha_n}\rangle$ with $\alpha_1<\cdots<\alpha_n$ form a k-basis of $E_n(\mathfrak{I})$, elements of the form $1\otimes\langle x_{\alpha_1},\ldots,x_{\alpha_n}\rangle$ form an $\mathfrak{I}$-basis for $C_n(\mathfrak{I})$. Hence we may define an $\mathfrak{I}$-map on $C_n(\mathfrak{I})$ by describing it on such elements; equivalently we may define it on $\langle x_1,\ldots,x_n\rangle$ as long as the given recipe is alternating and multilinear. Indeed, this is what is done for the differential (2.4). Note that since $\mathfrak{I}$ is abelian, the second summand in the differential disappears and we have $d_n^{\mathfrak{I}}(1\otimes\langle x_1,\dots,x_n\rangle)$ $= \sum_{i=1}^{n}(-1)^{i+1}x_i \otimes\langle x_1,\dots,\hat{x}_i,\dots,x_n\rangle$ for $x_i \in \mathfrak{I}$.

The chain map j_* is constructed as follows: for $n = 0$, $C_o(\mathfrak{I}) = U\mathfrak{I}$ and we define $j_o(1) = 1$. For $x \in \mathfrak{I}$, this includes $\mathfrak{I}$ in $\mathfrak{g}$ as $j_o(x) = j(x) = (x,0)$. For $n \geq 1$, $j_n\colon C_n(\mathfrak{I}) \to C_n(\mathfrak{g})$ is defined by

(4.1) $$j_n(1\otimes\langle x_1,\dots,x_n\rangle) = 1\otimes\langle (x_1,0),\dots,(x_n,0)\rangle \quad .$$

Note that j_n is functorial in the sense that $j_n = Uj \otimes E_n j$.

We must show that j_* is a chain map. Firstly, we verify commutativity of

$$\begin{array}{ccccc} U\mathfrak{I} & \xrightarrow{\varepsilon^{\mathfrak{I}}} & k & \longrightarrow & 0 \\ {\scriptstyle j_o}\downarrow & & \| & & \\ U\mathfrak{g} & \xrightarrow[\varepsilon^{\mathfrak{g}}]{} & k & \longrightarrow & 0. \end{array}$$

We have $\varepsilon^{\mathfrak{g}}j_o(1) = \varepsilon^{\mathfrak{g}}(1) = \varepsilon^{\mathfrak{g}}(1)$ so $\varepsilon^{\mathfrak{g}}j_o = \varepsilon^{\mathfrak{I}}$. For $n \geq 1$, commutativity of

$$\begin{array}{ccc} C_n(\mathfrak{I}) & \xrightarrow{d_n^{\mathfrak{I}}} & C_{n-1}(\mathfrak{I}) \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle j_{n-1}} \\ C_n(\mathfrak{g}) & \xrightarrow[d_n^{\mathfrak{g}}]{} & C_{n-1}(\mathfrak{g}) \end{array}$$

must be checked. If n = 1,

$$d_1^{\mathfrak{g}} j_1(1 \otimes x) = d_1^{\mathfrak{g}}(1 \otimes (x,0)) = (x,0) \text{ while}$$

$$j_o d_1^{\mathfrak{I}}(1 \otimes x) = j_o(x) = (x,0).$$

Now take $n \geq 2$. Then

$$j_{n-1} d_n^{\mathfrak{I}}(1 \otimes \langle x_1,\ldots,x_n \rangle) = \sum_{i=1}^{n} (-1)^{i+1} j_{n-1}(x_i \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n \rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} (x_i,0) \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0) \rangle$$

while

$$d_n^{\mathfrak{g}} j_n(1 \otimes x_1,\ldots,x_n) = d_n^{\mathfrak{g}}(1 \otimes \langle (x_1,0),\ldots,(x_n,0) \rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} (x_i,0) \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0) \rangle$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} 1 \otimes \langle [(x_i,0),(x_j,0)],(x_1,0),\ldots,(\hat{x_i},0),\ldots,(\hat{x_j},0),\ldots,(x_n,0) \rangle.$$

Since $\mathfrak{I}$ is abelian, the second summand disappears and $j_{n-1} d_n^{\mathfrak{I}} = d_n^{\mathfrak{g}} j_n$.

The construction of the chain map $\psi_*: C_*(\mathfrak{g}) \to C_*(\mathfrak{I})$ turns out to be surprisingly difficult. Essentially, this is because each $C_n(\mathfrak{g})$ has a far more complicated structure as an $\mathfrak{I}$-module than does each $C_n(\mathfrak{I})$. We now explore this structure. Expand $\{x_\alpha\}_{\alpha\in A}$ by $\{y_\beta\}_{\beta\in B}$ (a complementary basis) to an ordered k-basis of $\mathfrak{g}$, where $A \cap B = \emptyset$ and $A \cup B$ is ordered with each $\alpha \in A$ preceding each $\beta \in B$. Since $U\mathfrak{g}$ is k-free on $\{x_I y_J\}$, I a finite increasing sequence in A and J one in B, it follows that $U\mathfrak{g}$ is $\mathfrak{I}$-free on $\{y_J\}$. Furthermore, $C_n(\mathfrak{g})$ is k-free on $\{x_I y_J \otimes \langle (x_{\alpha_1},0),\ldots,(x_{\alpha_i},0),y_{\beta_1},\ldots,y_{\beta_j}\rangle\}$ where $\alpha_1<\cdots<\alpha_i < \beta_1<\cdots<\beta_j, i,j\geq 0$, and $i+j = n$, so is $\mathfrak{I}$-free on $\{y_J \otimes \langle (x_{\alpha_1},0),\ldots,(x_{\alpha_i},0),y_{\beta_1},\ldots,y_{\beta_j}\rangle\}$.

The existence of ψ_* is of course guaranteed by general theory. However, ψ_* will be required to satisfy

(4.2) $$\psi_n j_n = \mathrm{Id} \quad (\text{for } n \geq 0)$$

(4.3) $$\psi_n(y\otimes\langle (x_1,0),\ldots,(x_n,0)\rangle) = \sum_{i=1}^{n} 1\otimes\langle x_1,\ldots,[y,(x_i,0)],\ldots x_n\rangle$$

for $n \geq 1$, $x_i \in \mathfrak{I}$, and y in the complementary subspace. ψ_* will also be required to satisfy other conditions; these will be dealt with in chapter VII. The method, both here and in the future, will be to show that ψ_* can be inductively defined so that the required conditions are satisfied.

<u>Remark</u>: As with j_*, we will write ψ_* in terms of <u>arbitrary</u> elements x_i in $\mathfrak{I}$ rather than basis elements x_{α_i}. We shall also do this with expressions involving the elements y of the complementary subspace, when permitted (i.e., when the given formula takes the same form on arbitrary elements as on basis elements). This is for typographical convenience only (and to follow convention, as with the differential); it should always be kept in mind that ψ_n is being defined on an $\mathfrak{I}$-basis of $C_n(\mathfrak{g})$.

We first define ψ_o. Define

(4.4) $\qquad \psi_o(1) = 1$

(4.5) $\qquad \psi_o(y_I) = 0$ if $I \neq \emptyset$

Since $\varepsilon^{\mathfrak{I}}\psi_o(1) = \varepsilon^{\mathfrak{I}}(1) = \varepsilon^{\mathfrak{g}}(1) = 1$ and
$\varepsilon^{\mathfrak{I}}\psi_o(y_I) = \varepsilon^{\mathfrak{I}}(0) = 0 = \varepsilon^{\mathfrak{g}}(y_I)$, the diagram

$$\begin{array}{ccccc} U\mathfrak{g} & \xrightarrow{\varepsilon^{\mathfrak{g}}} & k & \longrightarrow & 0 \\ {\scriptstyle\psi_o}\downarrow & & \| & & \\ U\mathfrak{I} & \xrightarrow[\varepsilon^{\mathfrak{I}}]{} & k & \longrightarrow & 0 \end{array}$$

commutes. Note that $\psi_o j_o(1) = \psi_o(1) = 1$ so (4.2) is satisfied; also note that since ψ_o is an $\mathfrak{I}$-map,

$\psi_o((x,0)) = x$ for $x \in \mathcal{I}$.

For $n \geq 1$, to satisfy (4.2) it is necessary and sufficient that $\psi_n j_n(1 \otimes \langle x_1, \ldots, x_n \rangle) = 1 \otimes \langle x_1, \ldots, x_n \rangle$, or

$$\psi_n(1 \otimes \langle (x_1,0), \ldots, (x_n,0) \rangle) = 1 \otimes \langle x_1, \ldots, x_n \rangle \quad (4.6)$$

We need only check that this is compatible with commutativity of

$$\begin{array}{ccc} C_n(\mathfrak{g}) & \xrightarrow{d_n^{\mathfrak{g}}} & C_{n-1}(\mathfrak{g}) \\ \psi_n \downarrow & & \downarrow \psi_{n-1} \\ C_n(\mathcal{I}) & \xrightarrow[d_n^{\mathcal{I}}]{} & C_{n-1}(\mathcal{I}) \end{array} \quad (4.7)$$

If $n = 1$, $d_1^{\mathcal{I}}\psi_1(1 \otimes (x,0)) = d_1^{\mathcal{I}}(x) = x$ while

$\psi_o d_1^{\mathfrak{g}}(1 \otimes (x,0)) = \psi_o((x,0)) = x$. If $n > 1$,

$$d_n^{\mathcal{I}}\psi_n(1 \otimes \langle (x_1,0), \ldots, (x_n,0) \rangle) = d_n^{\mathcal{I}}(1 \otimes \langle x_1, \ldots, x_n \rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} x_i \otimes \langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle$$

while

$$\psi_{n-1} d_n^{\mathfrak{g}}(1 \otimes \langle (x_1,0), \ldots, (x_n,0) \rangle) =$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \psi_{n-1}((x_i,0) \otimes \langle (x_1,0), \ldots, (\hat{x_i},0), \ldots, (x_n,0) \rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot (1 \otimes \langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle) \text{ since}$$

ψ_{n-1} is an $\mathfrak{I}$-map.

$$= \sum_{i=1}^{n} (-1)^{i+1} x_i \otimes \langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle .$$

We now turn to (4.3). The diagram (4.7) indicates that once ψ_{n-1} is known, defining ψ_n involves solving a "differential equation"; i.e., for $\mu \in C_n(\mathfrak{g})$, $\psi_n(\mu)$ is a solution δ of

(4.8) $$d_n^{\mathfrak{I}}(\delta) = \psi_{n-1} d_n^{\mathfrak{g}}(\mu).$$

First take n = 1. We must solve

$d_1^{\mathfrak{I}}(\delta) = \psi_o d_1^{\mathfrak{g}}(y \otimes (x,0))$ for δ. The right side is

$$\psi_o(y \otimes (x,0)) = \psi_o((x,0)y) + \psi_o([y,(x,0)])$$

$$= x \cdot \psi_o(y) + [y,(x,0)] \text{ since } \mathfrak{I} \text{ is an ideal}.$$

$$= [y,(x,0)] \text{ by (4.5).}$$

Since $[y,(x,0)] = d_1^{\mathfrak{I}}(1 \otimes [y,(x,0)])$, define

(4.9) $$\psi_1(y \otimes (x,0)) = 1 \otimes [y,(x,0)].$$

This is (4.3) for n = 1.

Remark: Note the writing of $y(x,0)$ as $(x,0)y + [y,(x,0)]$. This will be repeated almost incessantly in the future.

Now assume (4.3) for n-1. We must show that the right-hand side of (4.3) is a solution of (4.8):

$$\psi_{n-1}d_n^{\mathfrak{g}}(y \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle) =$$

$$= \sum_{i=1}^{n} (-1)^{i+1}\psi_{n-1}(y(x_i,0) \otimes \langle (x_1,0),\ldots,\widehat{(x_i,0)},\ldots,(x_n,0)\rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1}\psi_{n-1}((x_i,0)y \otimes \langle (x_1,0),\ldots,\widehat{(x_i,0)},\ldots,(x_n,0)\rangle)$$

$$+ \sum_{i=1}^{n} (-1)^{i+1}\psi_{n-1}([y,(x_i,0)] \otimes \langle (x_1,0),\ldots,\widehat{(x_i,0)},\ldots,(x_n,0)\rangle)$$

$$= \sum_{i=1}^{n} (-1)^{i+1}x_i \otimes \psi_{n-1}(y \otimes \langle (x_1,0),\ldots,\widehat{(x_i,0)},\ldots,(x_n,0)\rangle)$$

$$+ \sum_{i=1}^{n} (-1)^{i+1}[y,(x_i,0)] \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle.$$

Now apply the induction hypothesis to the first summand. For each fixed i,

$$\psi_{n-1}(y \otimes \langle (x_1,0),\ldots,\widehat{(x_i,0)},\ldots,(x_n,0)\rangle) =$$

$$= \sum_{j<i} 1 \otimes \langle x_1,\ldots,[y,(x_j,0)],\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$+ \sum_{j>i} 1 \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,[y,(x_j,0)],\ldots,x_n\rangle,$$

so summing, the whole expression becomes

$$\sum_{i=1}^{n}(-1)^{i+1}\Big(\sum_{j<i} x_i \otimes \langle x_1,\ldots,[y,(x_j,0)],\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$(4.10)\quad + \sum_{j>i} x_i \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,[y,(x_j,0)],\ldots,x_n\rangle\Big)$$

$$+ \sum_{i=1}^{n}(-1)^{i+1}[y,(x_i,0)] \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle .$$

On the other side, for each fixed i,

$$d_n^{I}(1 \otimes \langle x_1,\ldots,[y,(x_i,0)],\ldots,x_n\rangle) =$$

$$= \sum_{j<i}(-1)^{j+1}x_j \otimes \langle x_1,\ldots,\hat{x}_j,\ldots,[y,(x_i,0)],\ldots,x_n\rangle$$

$$+ (-1)^{i+1}[y,(x_i,0)] \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$+ \sum_{j>i}(-1)^{j+1}x_j \otimes \langle x_1,\ldots,[y,(x_i,0)],\ldots,\hat{x}_j,\ldots,x_n\rangle;$$

summing over i,

$$d_n^{I}\Big(\sum_{i=1}^{n} 1 \otimes \langle x_1,\ldots,[y,(x_i,0)],\ldots,x_n\rangle\Big) =$$

$$(4.11)\quad = \sum_{i=1}^{n}\Big(\sum_{j<i}(-1)^{j+1}x_j \otimes \langle x_1,\ldots,\hat{x}_j,\ldots,[y,(x_i,0)],\ldots,x_n\rangle$$

$$+ \sum_{j>i}(-1)^{j+1}x_j \otimes \langle x_1,\ldots,[y,(x_i,0)],\ldots,\hat{x}_j,\ldots,x_n\rangle\Big)$$

$$+ \sum_{i=1}^{n} (-1)^{i+1} [y,(x_i,0)] \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n \rangle \quad .$$

The third summands of (4.10) and (4.11) agree. As for the others, note that there is a term for each pair of integers $i \neq j$ between 1 and n in each of the expressions; call this $\alpha_{i,j}$ in (4.10) and $\beta_{i,j}$ in (4.11). Since $|\alpha_{i,j}| = |\beta_{j,i}|$ and $\alpha_{i,j}$ occurs with sign $(-1)^{i+1}$ while $\beta_{j,i}$ occurs with sign $(-1)^{j+1}$, the expressions agree.

Remark: When one tries to define ψ_n explicitly on other $\mathfrak{I}$-generators of $C_n(\mathfrak{g})$, it is seen that this is somewhat easier to do in the case $\mathfrak{g} = \mathfrak{I} \cdot \Phi$ in the sense that certain expressions can be taken to be zero that cannot be made to vanish in the general case. However, (4.3) holds for any extension, and it is perhaps instructive to see why. Write $y = (0,\sigma)$; then $[y,(x_i,0)] = \sigma \cdot x_i$ and (4.3) becomes $\psi_n(y \otimes \langle (x_1,0),\ldots,(x_n,0) \rangle) =$ $\sum_{i=1}^{n} 1 \otimes \langle x_1,\ldots,\sigma \cdot x_i,\ldots,x_n \rangle$, which is given solely in terms of the action of Φ on $\mathfrak{I}$. However, expressions such as $\psi_n(y_\beta \otimes \langle y_{\beta_1},\ldots,y_{\beta_n} \rangle)$ are not, and we shall see later that in a sense it is the dependence of ψ_* on the extension that controls the second differential.

By fiat, we have $\psi_n j_n$ = identity of $C_n(\mathfrak{l})$. Although $j_n\psi_n$ is not the identity of $C_n(\mathfrak{g})$, both $j_*\psi_*$ and the identity complete

$$\begin{array}{ccccccccc}
\cdots \to & C_n(\mathfrak{g}) & \xrightarrow{d_n^{\mathfrak{g}}} & C_{n-1}(\mathfrak{g}) \to \cdots \to & C_0(\mathfrak{g}) & \xrightarrow{\varepsilon^{\mathfrak{g}}} & k & \to 0 \\
 & \downarrow & & \downarrow & \downarrow & & \| & \\
\cdots \to & C_n(\mathfrak{g}) & \xrightarrow[d_n^{\mathfrak{g}}]{} & C_{n-1}(\mathfrak{g}) \to \cdots \to & C_0(\mathfrak{g}) & \xrightarrow[\varepsilon^{\mathfrak{g}}]{} & k & \to 0 .
\end{array}$$

Hence they are chain-homotopic via an $\mathfrak{l}$-module homotopy s_*. For future use, we shall desire a homotopy with the following property

(4.12) $\qquad s_n j_n = 0.$

To construct s_*, we follow the same procedure as with ψ_* above. First s_0 is constructed; we then define s_* inductively so that (4.12) is satisfied. For each n, s_n must satisfy

(4.13) $\qquad d_{n+1}^{\mathfrak{g}} s_n + s_{n-1} d_n^{\mathfrak{g}} = j_n\psi_n - \mathrm{Id}$

If $n = 0$, this becomes $d_1^{\mathfrak{g}} s_0 = j_0\psi_0 - \mathrm{Id}$. Since $j_0\psi_0(1) = 1$, we start with $s_0(1) = 0$ (hence $s_0((x,0))=0$); and since $\psi_0(y_I) = 0$ if $I \neq \emptyset$, we must solve $d_1^{\mathfrak{g}}\{s_0(y_I)\} = -y_I$. Note that $d_1^{\mathfrak{g}}(-1 \otimes y) = -y$ so that

(4.14) $s_o(y) = -1 \otimes y$

is appropriate. And $d_1^{\mathfrak{g}}(-y_{\beta_1} \cdots y_{\beta_{p-1}} \otimes y_{\beta_p}) =$
$-y_{\beta_1} \cdots y_{\beta_{p-1}} y_{\beta_p}$ so that for $p \geq 2$, $s_o(y_{\beta_1} \cdots y_{\beta_p}) =$
$-y_{\beta_1} \cdots y_{\beta_{p-1}} \otimes y_{\beta_p}$ works. Since $s_o j_o(1) = s_o(1) = 0$,
(4.12) is satisfied.

Next assume that s_* has been defined so that (4.12) is satisfied for n-1; i.e., $s_{n-1}(1 \otimes \langle (x_1,0),\ldots,(x_{n-1},0)\rangle) = 0$. Denote $1 \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle$ by μ. Since $j_n \psi_n(\mu) = \mu = Id(\mu)$, the right side of (4.13) vanishes. But

$$s_{n-1}d_n^{\mathfrak{g}}(\mu) = \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot s_{n-1}(1 \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0)\rangle)$$

$$= 0$$

by the induction hypothesis; hence $s_n(\mu) = 0$ is suitable.

Our use of these chain maps will be in a comparison of the two cochain complexes that are used to define $H^*(\mathfrak{l};M)$. We have already treated $H^*(\mathfrak{l};M)$ as $H^*(\mathfrak{u}^*)$, $\mathfrak{u}^* = Hom_{\mathfrak{l}}(C_*(\mathfrak{g}),M)$, in chapter III. Now we consider $\mathfrak{v}^*$ where $\mathfrak{v}^n = Hom_{\mathfrak{l}}(C_n(\mathfrak{l}),M)$. For the time being, the assumption that $\mathfrak{l}$ is abelian is dropped. As a vector space, $\mathfrak{v}^n$ is Hom($E_n(\mathfrak{I})$,M). Since $\mathfrak{I}$ is an ideal of $\mathfrak{g}$, the adjoint action of $\mathfrak{g}$ on $\mathfrak{I}$ extends to $E_n(\mathfrak{I})$ by derivations and this turns Hom($E_n(\mathfrak{I})$,M) into a $\mathfrak{g}$-module by diagonal action:

$$(z\cdot f')(\langle x_1,\ldots,x_n\rangle) = z\cdot f'(\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f'(\langle x_1,\ldots,[z,x_i],\ldots,x_n\rangle)$$

or as is more common,

$$(4.15)\quad (z\cdot f')(x_1,\ldots,x_n) = z\cdot f'(x_1,\ldots,x_n) - \sum_{i=1}^{n} f'(x_1,\ldots,[z,x_i],\ldots,x_n).$$

This action has been described by Hochschild and Serre in [8] and it induces the action of Φ on $H^n(\mathfrak{I};M)$. However, (4.15) is not in general an action of Φ on the cochain level and such an action is required for our techniques. Under the vector space isomorphism $\mathrm{Hom}(E_n(\mathfrak{I}),M) \approx \mathrm{Hom}_{\mathfrak{I}}(C_n(\mathfrak{I}),M)$, $f \in \mathrm{Hom}_{\mathfrak{I}}(C_n(\mathfrak{I}),M)$ and $f' \in \mathrm{Hom}(E_n(\mathfrak{I}),M)$ correspond where $f(1 \otimes \langle x_1,\ldots,x_n\rangle) = f'(x_1,\ldots,x_n)$. This endows $\mathfrak{v}^n$ with a $\mathfrak{g}$-structure:

$$(z\cdot f)(1\otimes\langle x_1,\ldots,x_n\rangle) = z\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1,\ldots,[z,x_i],\ldots,x_n\rangle).$$

We once again restrict to $\mathfrak{I}$ abelian and M $\mathfrak{I}$-trivial; then both terms on the right side above vanish if $z \in \mathfrak{I}$. Hence Φ acts on $\mathfrak{v}^n$ via

$$(4.16)\quad (\sigma\cdot f)(1\otimes\langle x_1,\ldots,x_n\rangle) = z\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1 \ldots,[z,x_i],\ldots,x_n\rangle)$$

where $z \in p^{-1}(\sigma)$; we may of course take $z = (0,\sigma)$. Note that if $n = 0$, we identify $f \in \mathfrak{D}^o$ with $f(1) \in M$ and we then have $(\sigma\cdot f)(1) = z\cdot f(1)$.

We return to j_* and ψ_*. Since $j^n = j_n^*: \mathfrak{U}^n \to \mathfrak{D}^n$ and $\psi^n = \psi_n^*: \mathfrak{D}^n \to \mathfrak{U}^n$, if $f \in \mathfrak{D}^n$, we may move f to $\mathfrak{U}^n$ via ψ_n^*, compute the action of Φ on this element, and then move back to $\mathfrak{D}^n$ via j_n^*. The following proposition tells us where we end up.

Proposition 4.1: For $f \in \mathfrak{D}^n$ and $\sigma \in \Phi$, $j_n^*\{\sigma\cdot(\psi_n^*(f))\} = \sigma\cdot f$.

Proof: If $n = 0$, $j_o^*\{\sigma\cdot(\psi_o^*(f))\}(1) =$

$$\{\sigma\cdot(f\circ\psi_o)\}(j_o(1)) =$$

$$\{\sigma\cdot(f\circ\psi_o)\}(1) =$$

$$(0,\sigma)\cdot\{(f\circ\psi_o)(1)\} - (f\circ\psi_o)((0,\sigma)) =$$

$$(0,\sigma)\cdot f(1) \text{ since } \psi_o(y) = 0$$

for all y in the complementary subspace by (4.5). Since $(\sigma\cdot f)(1) = (0,\sigma)\cdot f(1)$, this gives the result if $n = 0$. Now suppose $n \geq 1$. Then

$$j_n^*\{\sigma\cdot(\psi_n^*(f))\}(1\otimes\langle x_1,\ldots,x_n\rangle) =$$

$$\{\sigma\cdot(f\circ\psi_n)\}(j_n(1\otimes\langle x_1,\ldots,x_n\rangle)) =$$

$$\{\sigma\cdot(f\circ\psi_n)\}(1\otimes\langle(x_1,0),\ldots,(x_n,0)\rangle) =$$

$$(0,\sigma)\cdot f\{\psi_n(1\otimes\langle(x_1,0),\ldots,(x_n,0)\rangle)\}-f\{\psi_n((0,\sigma)\otimes\langle(x_1,0),\ldots,(x_n,0)\rangle)\}$$

$$= (0,\sigma)\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1,\ldots,[(0,\sigma),(x_i,0)],\ldots,(x_n,0)\rangle)$$

by (4.3).

$$= (0,\sigma)\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_n\rangle).$$

The right side is

$$(\sigma.f)(1\otimes\langle x_1,\ldots,x_n\rangle) = (0,\sigma)\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1,\ldots,[(0,\sigma),x_i],\ldots,x_n\rangle)$$

$$= (0,\sigma)\cdot f(1\otimes\langle x_1,\ldots,x_n\rangle) - \sum_{i=1}^{n} f(1\otimes\langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_n\rangle) \quad \blacksquare$$

CHAPTER V

HOMOLOGY

The techniques in the next chapter will require the consideration of the homology spaces $H_n(\mathfrak{I};M)$ as Φ-modules. In the case at hand, where $\mathfrak{I}$ is abelian and the $\mathfrak{g}$-module M is $\mathfrak{I}$-trivial, this is quite easy since chains and homology coincide. However, a complete treatment, analogous to that of Hochschild-Serre used in Chapter IV for cohomology, is given below. Again, M is any $\mathfrak{g}$-module and $\mathfrak{I}$ any ideal of $\mathfrak{g}$, until further notice.

We think of the chains $C_n(\mathfrak{I},M)$ as $M \otimes E_n(\mathfrak{I})$ with boundary (2.6). Diagonal action makes $M \otimes E_n(\mathfrak{I})$ into a $\mathfrak{g}$-module:

$$(5.1) \quad z\cdot(m\otimes\langle x_1,\ldots,x_n\rangle) = -m.z \otimes\langle x_1,\ldots,x_n\rangle + m\otimes \sum_{i=1}^{n}\langle x_1,\ldots,[z,x_i],\ldots,x_n\rangle$$

The negative sign is required here since we are starting with a right $\mathfrak{g}$-module M. If M had been taken to be a left $\mathfrak{g}$-module, no negative sign is needed in (5.1), but one would then be required in the first term of (2.6). (5.1) does not make the chains themselves into a Φ-module since the diagonal action is not $\mathfrak{I}$-trivial.

However, a short computation shows that the action commutes with the boundary, so an action of $\mathfrak{g}$ on the homology groups $H_n(\mathfrak{I};M)$ is defined. For this to induce an action of the quotient Φ, the ideal $\mathfrak{I}$ must annihilate $H_n(\mathfrak{I};M)$. This will be shown by establishing identity (5.2) below. The above considerations apply in particular to $\mathfrak{I} = \mathfrak{g}$ itself, and (5.2) will concern this situation. Let $\alpha \in C_n(\mathfrak{g},M)$ be of the form $m \otimes \langle z_1,\ldots,z_n\rangle$ and let $z \in \mathfrak{g}$. Define an element $\alpha_z \in C_{n+1}(\mathfrak{g},M)$ by $\alpha_z = m \otimes \langle z,z_1,\ldots,z_n\rangle$. More precisely, α_z is determined from α as follows: the mapping $h_z\colon \underbrace{\mathfrak{g}\times\ldots\times\mathfrak{g}}_{n \text{ times}} \to E_{n+1}(\mathfrak{g})$ defined by $h_z(z_1,\ldots,z_n) = \langle z,z_1,\ldots,z_n\rangle$ is multilinear and alternating, hence determines $\tilde{h}_z\colon E_n(\mathfrak{g}) \to E_{n+1}(\mathfrak{g})$ with $\tilde{h}_z(\langle z_1,\ldots,z_n\rangle) = \langle z,z_1,\ldots,z_n\rangle$. Then $\alpha_z = f_z(\alpha)$ where $f_z = \mathrm{Id}_M \otimes \tilde{h}_z$.

We now establish an identity relating α_z to the action of $\mathfrak{g}$ on $C_n(\mathfrak{g},M)$.

<u>Proposition 5.1</u>: For $\alpha \in C_n(\mathfrak{g},M)$ and $z \in \mathfrak{g}$,

$$z\cdot\alpha + d(\alpha_z) = -(d\alpha)_z. \tag{5.2}$$

<u>Proof</u>: Firstly, let $n = 0$. Then $\alpha \in C_0(\mathfrak{g},M) = M$ so $\alpha = m \in M$ and $z\cdot\alpha = z\cdot m = -m\cdot z$. Since $\alpha_z = m \otimes z$,

$$d(\alpha_z) = m\cdot z \text{ and } z\cdot\alpha + d(\alpha_z) = 0$$
$$= -(d\alpha)_z.$$

Now suppose $n > 0$. By (2.6),

$$(d\alpha)_z = \sum_{i=1}^{n} (-1)^{i+1}(m\cdot z_i) \otimes \langle z, z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+j} m \otimes \langle z, [z_i, z_j], z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n\rangle.$$

Next the left side of (5.2) is computed.

$$d(\alpha_z) = (m\cdot z) \otimes \langle z_1, \ldots, z_n\rangle + \sum_{i=1}^{n} (-1)^{i+2}(m\cdot z_i) \otimes \langle z, z_1, \ldots, \hat{z}_i, \ldots z_n\rangle$$

$$+ \sum_{i=1}^{n} (-1)^{1+i+1} m \otimes \langle [z, z_i], z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+1+j+1} m \otimes \langle [z_i, z_j], z, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n\rangle;$$

the signs are due to the fact that in α_z, z_i occupies the $(i+1)^{st}$ position.

$$= (m\cdot z) \otimes \langle z_1, \ldots, z_n\rangle + \sum_{i=1}^{n} (-1)^{i}(m\cdot z_i) \otimes \langle z, z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle$$

$$+ \sum_{i=1}^{n} (-1)^{i} m \otimes \langle [z, z_i], z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+j} m \otimes \langle [z_i, z_j], z, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_n\rangle.$$

Add this to

$$z\cdot\alpha = -m\cdot z \otimes \langle z_1,\ldots,z_n\rangle + m \otimes \sum_{i=1}^{n}\langle z_1,\ldots,[z,z_i],\ldots,z_n\rangle$$

$$= -m\cdot z \otimes \langle z_1,\ldots z_n\rangle + \sum_{i=1}^{n}(-1)^{i-1}m \otimes \langle [z,z_i],\ldots,z_1,\ldots,\hat{z}_i,\ldots,z_n\rangle.$$

After cancellation, we obtain

$$\sum_{i=1}^{n}(-1)^{i}(m\cdot z_i) \otimes \langle z,z_1,\ldots,\hat{z}_i,\ldots,z_n\rangle + \sum_{1\le i<j\le n}(-1)^{i+j}m \otimes \langle [z_i,z_j],z,z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots,z_n\rangle$$

$$= \sum_{i=1}^{n}(-1)^{i}(m\cdot z_i) \otimes \langle z,z_1,\ldots,\hat{z}_i,\ldots,z_n\rangle$$

$$+ \sum_{1\le i<j\le n}(-1)^{i+j+1}m \otimes \langle z,[z_i,z_j],z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots,z_n\rangle$$

$$= -(d\alpha)_z.$$

By linearity, (5.2) holds for all $\alpha \in C_n(\mathfrak{g},M)$. ■

<u>Proposition 5.2</u>: Equation (5.1) induces an action of Φ on $H_n(\mathcal{I};M)$.

<u>Proof</u>: We need only show that $\mathcal{I}$ annihilates $H_n(\mathcal{I};M)$. Since α_z is the image of α under the linear map f_z,

$\alpha_z = 0$ if $\alpha = 0$. Let $\alpha \in C_n(\mathcal{I},M)$ be a cycle and let $x \in \mathcal{I}$. The above observation yields $(d\alpha)_x = 0$ and then by proposition 5.1,

$$\begin{aligned} x\cdot\alpha &= -d(\alpha_x) \\ &= d(-\alpha_x), \end{aligned}$$

so $x\cdot\alpha$ is a boundary. ∎

Once again, specialize to the situation where $\mathcal{I}$ is an abelian ideal and M is $\mathcal{I}$-trivial. In this case, the chains and homology coincide, so Φ does act on the chains themselves. This can be seen directly from (5.1): if $x \in \mathcal{I}$, the assumption that $\mathcal{I}$ is abelian forces the second summand to vanish and the assumption that M is $\mathcal{I}$-trivial forces the first summand to vanish.

CHAPTER VI

THE CUP PRODUCT FORMULA

We choose our final coefficients A to be an associative k-algebra with identity for which the hypotheses of the universal coefficient theorem are satisfied (e.g., an extension field of k or an associative division algebra over k). Regard A as $\mathfrak{g}$-trivial and consider A, $H^N(\mathfrak{x};A)$, and $H_N(\mathfrak{x};A)$ as coefficients for the spectral sequence in Chapter III. For a fixed integer $N \geq 1$, set

$$E_r^{p,q} = E_r^{p,q}(A)$$

$$\bar{E}_r^{p,q} = E_r^{p,q}(H^N(\mathfrak{x};A))$$

$$E_r^{p,q} = E_r^{p,q}(H_N(\mathfrak{x};A)), \text{ for } r \geq 2 \text{ and } p,q \geq 0.$$

$H_N(\mathfrak{x};A)$ is an A-module via $a\cdot(a' \otimes X) = aa' \otimes X$. By the universal coefficient theorem, there is a split exact sequence

$$0 \to \mathrm{Ext}_A(H_{N-1}(\mathfrak{x};A),A) \to H^N(\mathfrak{x};A) \xrightarrow{p_N} \mathrm{Hom}_A(H_N(\mathfrak{x};A),A) \to 0.$$

Since A is of course A-free, the Ext term vanishes and p_N is an isomorphism; explicitly, $\{p_N(u)\}(a \otimes C_N) = a\cdot u(C_N)$ for $u \in H^N(\mathfrak{x};A)$, $a \in A$, $C_N \in E_N(\mathfrak{x})$. The inverse

$\bar{p}_N^{-1}$ will be needed later; it is $\{p_N^{-1}(v)\}(C_N) = v(1_A \otimes C_N)$. Note that we are again identifying chains with homology and cochains with cohomology p_N yields a homomorphism

$$q_N\colon H^N(\mathfrak{I};A) \otimes H_N(\mathfrak{I};A) \to A \text{ defined by}$$

$$q_N(u \otimes \beta) = \{p_N(u)\}(\beta); \text{ hence}$$

$$q_N(u \otimes (a \otimes C_N)) = a\cdot u(C_N).$$

It is standard that for $\mathfrak{g}$-modules P,Q,R, a pairing $P \otimes Q \to R$ gives a spectral sequence pairing $E_r^{p,q}(P) \otimes E_r^{p',q'}(Q) \to E_r^{p+p',q+q'}(R)$, where there is diagonal action on $P \otimes Q$. Hence for q_N to yield a spectral sequence pairing, the following proposition is needed.

<u>Proposition 6.1</u>: q_N is a $\mathfrak{g}$-pairing.

<u>Proof</u>: We must show that $q_N(z\cdot(u \otimes (a \otimes C_N))) = z\cdot q_N(u \otimes (a \otimes C_N))$. Since q_N maps to the $\mathfrak{g}$-trivial module A, the right side is 0 and since $A \otimes E_N(\mathfrak{I})$ has trivial action on the first factor, we must show $q_N((z\cdot u) \otimes (a \otimes C_N)) = -q_N(u \otimes (a \otimes (z\cdot C_N)))$. The left side is

$$\{p_N(z\cdot u)\}(a \otimes C_N) = a\cdot\{(z.u)(C_N)\}$$
$$= a\cdot\{z\cdot u(C_N) - u(z\cdot C_N)\}$$
$$= -a\cdot\{u(z\cdot C_N)\} \text{ since } u(C_N) \in A.$$

The right side is

$$-\{p_N(u)\}(a \otimes (z\cdot C_N)) = - a\cdot\{u(z\cdot C_N)\}. \qquad \blacksquare$$

Remark: It is not absolutely necessary that A be assumed to be $\mathfrak{g}$-trivial. If A is allowed to possess a Φ-structure and the above proof is chased through, it is seen to hold so long as the $\mathfrak{g}$-action satisfies the following compatibility condition with the multiplication of A:

$$z\cdot(a_1a_2) = (z\cdot a_1)a_2 + a_1(z\cdot a_2).$$

Hence q_N induces a spectral sequence pairing $\psi\colon \bar{E}_2^{p,q} \otimes \hat{E}_2^{p',q'} \to E_2^{p+p',q+q'}$, and if we denote $\psi(\bar{\beta} \otimes \hat{\beta})$ by $\bar{\beta} \cup \hat{\beta}$, it is standard that

(6.1) $$d_2(\bar{\beta}\cup\hat{\beta}) = \bar{d}_2(\bar{\beta}) \cup \hat{\beta} + (-1)^{p+q}\bar{\beta} \cup \hat{d}_2(\hat{\beta})$$

Also, the isomorphism $\phi^{p,q}\colon E_2^{p,q}(M) \approx H^p(\Phi;H^q(I;M))$ is compatible with this pairing in that

(6.2) $$\phi^{p+p',q+q'}(\bar{\beta}\cup\hat{\beta}) = (-1)^{p'q}\phi^{p,q}(\bar{\beta}) \cup \phi^{p',q'}(\hat{\beta})$$

where on the right side, we take the cup product in $H^*(\Phi;-)$.

As mentioned in the introduction, our cup product formula expresses $x \in E_2^{p,N}$ as the cup product of elements, of $\bar{E}_2^{p,0}$ and $\hat{E}_2^{0,N}$; i.e., we use $\bar{E}_2^{p,0} \otimes \hat{E}_2^{0,N} \to E_2^{p,N}$. The corresponding pairing via ϕ is

$$(6.3) \quad \xi\colon H^p(\Phi;H^0(\mathcal{I};H^N(\mathcal{I};A))) \otimes H^0(\Phi;H^N(\mathcal{I};H_N(\mathcal{I};A))) \to H^p(\Phi;H^N(\mathcal{I};A))$$

Since $p' = q = 0$ in (6.2), there is no sign change. This pairing is explored in the proof of proposition 6.3; now we cook up the second factor in our formula.

The action of $\mathcal{I}$ on $H^N(\mathcal{I};A)$ is trivial, so there is an isomorphism $\theta\colon H^N(\mathcal{I};A) \approx H^0(\mathcal{I};H^N(\mathcal{I};A))$. This induces, for each p, an isomorphism

$\bar{\theta}\colon H^p(\Phi;H^N(\mathcal{I};A)) \approx H^p(\Phi;H^0(\mathcal{I};H^N(\mathcal{I};A)))$, and since

$E_2^{p,N} \approx H^p(\Phi;H^N(\mathcal{I};A))$ and

$\bar{E}_2^{p,0} \approx H^p(\Phi;H^0(\mathcal{I};H^N(\mathcal{I};A)))$, we obtain an isomorphism

$\theta\colon E_2^{p,N} \approx \bar{E}_2^{p,0}$ via

$$(6.4) \qquad \begin{array}{ccc} H^p(\Phi;H^N(\mathcal{I};A)) & \overset{\bar{\theta}}{\approx} & H^p(\Phi;H^0(\mathcal{I};H_N(\mathcal{I};A))) \\ \phi \wr\wr & & \wr\wr \phi \\ E_2^{p,N} & \xrightarrow[\theta]{\quad - - - \quad} & \bar{E}_2^{p,0} \end{array}$$

Again by the universal coefficient theorem, we get a split exact sequence

$$0 \to \mathrm{Ext}_A(H_{N-1}(\mathcal{I};A), H_N(\mathcal{I};A)) \to H^N(\mathcal{I};H_N(\mathcal{I};A)) \overset{p_N}{\to} \mathrm{Hom}_A(H_N(\mathcal{I};A), H_N(\mathcal{I};A)) \to 0.$$

We make the assumption that $H_i(\mathcal{I};A)$ is A-free for all i. Once more, the Ext term disappears yielding p_N: $H^N(\mathcal{I};H_N(\mathcal{I};A)) \approx \mathrm{Hom}_A(H_N(\mathcal{I};A), H_N(\mathcal{I};A))$. Let $g^N \in H^N(\mathcal{I};H_N(\mathcal{I};A))$ correspond to the identity map; i.e., $p_N(g^N) = \mathrm{Id}_{H_N(\mathcal{I};A)}$.

<u>Proposition 6.2</u>: g^N is annihilated by Φ.

<u>Proof</u>: Let $\sigma \in \Phi$ and let $z \in p^{-1}(\sigma)$. Denote $\langle x_1,\ldots,x_N\rangle \in E_N(\mathcal{I})$ by C_N and $\langle x_1,\ldots,[z,x_i],\ldots,x_N\rangle$ by C^i; with this notation, the action of Φ on $g^N \in H^N(\mathcal{I};H_N(\mathcal{I};A))$ becomes

$$(6.5) \qquad (\sigma\cdot g^N)(C_N) = z\cdot g^N(C_N) - \sum_{i=1}^{N} g^N(C^i)$$

For $a \in A$, on the one hand

$\{p_N(g^N)\}(a \otimes C_N) = a\cdot g^N(C_N)$ while on the other,
$\{p_N(g^N)\}(a \otimes C_N) = a \otimes C_N$.

Hence

$$(6.6) \qquad a \cdot g^N(C_N) = a \otimes C_N.$$

Take $a = 1_A$; then $g^N(C_N) = 1_A \otimes C_N$ and also $g^N(C^i) = 1_A \otimes C^i$. So

$$\begin{aligned} z \cdot g^N(C_N) &= z \cdot (1_A \otimes C_N) \\ &= 1_A \otimes (z \cdot C_N). \end{aligned}$$

Now $z \cdot C_N = \sum_{i=1}^{N} C^i$ so

$$\begin{aligned} 1_A \otimes (z.C_N) &= \sum_{i=1}^{N} (1_A \otimes C^i) \\ &= \sum_{i=1}^{N} g^N(C^i) \end{aligned}$$

and by (6.5), $\sigma \cdot g^N = 0$. ∎

So $g^N \in \{H^N(I;H_N(I;A))\}^{\Phi} \approx H^0(\Phi;H^N(I;H_N(I;A))) \approx \hat{E}_2^{0,N}$. Let $h^N \in H^0(\phi;H^N(I;H_N(I,A)))$ and $f^N \in \hat{E}_2^{0,N}$ correspond to g^N.

<u>Proposition 6.3</u>: Let $x \in E_2^{p,N}$. Then $x = \theta(x) \cup f^N$.

<u>Proof</u>: The statement is almost a tautology since x and $\theta(x)$ correspond under a canonical isomorphism and f^N corresponds to an identity. We use the pairing (6.3) and show that if $\alpha \in H^p(\Phi;H^N(I;A))$, $\overline{\alpha} \in H^p(\Phi;H^0(I;H^N(I;A)))$, and $\beta \in H^0(\Phi;H^N(I;H_N(I;A)))$ correspond respectively to

x, $\theta(x)$, and f^N under ϕ, then $\xi(\overline{\alpha} \otimes \beta) = \alpha$. Note that $\beta = h^N$ as defined above and that ξ is constructed as follows: the $\mathfrak{g}$-pairing q_N induces a cup product pairing δ: $H^0(\mathcal{I};H^N(\mathcal{I};A)) \otimes H^N(\mathcal{I};H_N(\mathcal{I};A)) \to H^N(\mathcal{I};A)$ which in turn induces ξ.

Let $u \in H^N(\mathcal{I};A)$ and $\overline{u} \in H^0(\mathcal{I};H^N(\mathcal{I};A))$ correspond to u under $\tilde{\theta}$. We first show that

$$(6.7) \qquad \delta(\overline{u} \otimes g^N) = u.$$

Just compute:

$$\begin{aligned}\{p_N(\delta(\overline{u} \otimes g^N))\}(a \otimes C_N) &= a\cdot\{\delta(\overline{u} \otimes g^N)(C_N)\} \\ &= a\cdot\{q_N(u \otimes g^N(C_N))\} \\ &= a\cdot\{p_N(u)(g^N(C_N))\} \\ &= p_N(u)(a\cdot g^N(C_N))\end{aligned}$$

since $p_N(u)$ is an A-map.

$$= p_N(u)(a \otimes C_N) \text{ by } (6.6).$$

Since p_N is an isomorphism, $\delta(\overline{u} \otimes g^N) = u$.

Now $\xi(\overline{\alpha} \otimes h^N)$ is computed. Let $\overline{a} \in C^p(\Phi,H^0(\mathcal{I};H^N(\mathcal{I};A)))$ be a cocycle representing $\overline{\alpha}$ and observe that $g^N \in H^N(\mathcal{I};H_N(\mathcal{I};A)) = C^0(\Phi,H^N(\mathcal{I};H_N(\mathcal{I};A)))$ represents h^N. Then $\xi(\overline{\alpha} \otimes h^N)$ is represented by $\overline{a} \cup g^N \in C^p(\Phi,H^N(\mathcal{I};A))$ where

$$(\bar{a} \cup g^N)(c_p) = \delta(\bar{a}(c_p) \otimes g^N)$$
$$= a(c_p)$$

by (6.7), where $a(c_p)$ is the element of $H^N(I;A)$ corresponding to $\bar{a}(c_p)$ under $\tilde{\theta}$. This defines an element $a \in C^p(\Phi, H^N(I;A))$; since $a(c_p)$ and $\bar{a}(c_p)$ correspond under $\tilde{\theta}$ for each c_p, a and $\bar{a}$ correspond under θ_1: $C^p(\Phi, H^N(I;A)) \approx C^p(\Phi, H^0(I; H^N(I;A)))$; since $\bar{a}$ is a cocycle, so is a, and it defines an element of $H^p(\Phi; H^N(I;A))$. Since $\bar{\theta}$ is induced by θ_1, $\bar{\alpha}$ and this class correspond under $\bar{\theta}$, but since $\bar{\alpha}$ and $\theta(x)$ correspond under ϕ, so do this class and x by (6.4); i.e., α = class of a. Finally $\bar{a} \cup g^N = a$ gives $\xi(\bar{\alpha} \otimes h^N) = \alpha$. ∎

<u>Proposition 6.4</u>: For $x \in E_2^{p,N}$, $d_2(x) = (-1)^p \theta(x) \cup \hat{d}_2(f^N)$.

<u>Proof</u>: By proposition 6.3,

$$d_2(x) = d_2(\theta(x) \cup f^N)$$
$$= \bar{d}_2(\theta(x)) \cup f^N + (-1)^{p+0}\theta(x) \cup \hat{d}_2(f^N)$$

by (6.1),

$$= (-1)^p \theta(x) \cup \hat{d}_2(f^N)$$

since $\quad \bar{d}_2(\theta(x)) \in \bar{E}_2^{p+2,-1} = 0.$ ∎

Now consider the differential $\hat{d}_2$: $\hat{E}_2^{0,N} \to \hat{E}_2^{2,N-1}$. $\hat{d}_2$ induces a homomorphism

$$H^0(\Phi;H^N(I;H_N(I;A))) \to H^2(\Phi;H^{N-1}(I;H_N(I;A)))$$

by completing the diagram

$$\begin{array}{ccc} \hat{E}_2^{0,N} & \xrightarrow{\hat{d}_2} & \hat{E}_2^{2,N-1} \\ \phi \wr & & \wr \phi \\ H^0(\Phi;H^N(I;H_N(I;A))) & \dashrightarrow & H^2(\Phi;H^{N-1}(I;H_N(I;A))). \end{array}$$

We shall call this map D_α, indicating its dependence on the extension α; we have $\phi^{2,N-1}\hat{d}_2(f^N) = D_\alpha(h^N)$. Note, however, that the domain and range of D_α do not depend on the extension, but only on I, Φ, and the structure of I as a Φ-module.

The remainder of this paper will be spent computing D_α.

CHAPTER VII

COMPUTATION OF THE REPRESENTATIVE

The problem has now been reduced to the computation of the factor $\hat{d}_2(f^N)$, or equivalently, $\phi^{2,N-1}(\hat{d}_2(f^N))$. f^N is an element of $\hat{E}_2^{0,N} = E_2^{0,N}(H_N(\mathfrak{I};A))$, where of course $H_N(\mathfrak{I};A)$ is $\mathfrak{I}$-trivial, so we work in the following setting.

Take M to be any $\mathfrak{I}$-trivial $\mathfrak{g}$-module and let $u \in E_2^{0,N}(M)$. We will first compute $\phi^{2,N-1}(d_2(u))$ and later specialize to $M = H_N(\mathfrak{I};A)$, $u = f^N$ (so $d_2(u) = \hat{d}_2(f^N)$).

For $u \in E_2^{0,N}(M)$, $\phi^{0,N}(u) \in H^0(\Phi;H^N(\mathfrak{I};M)) = \{H^N(\mathfrak{I};M)\}^{\Phi}$. Hence (once again identifying cochains and cohomology), $\phi^{0,N}(u)$ is annihilated by Φ: $\sigma\cdot\{\phi^{0,N}(u)\} = 0$ $\forall\ \sigma \in \Phi$. For notational convenience, write f for $\phi^{0,N}(u)$; think of f as an element of $\mathrm{Hom}_{\mathfrak{I}}(C_N(\mathfrak{I}),M)$. We then have $\sigma\cdot f = 0\ \forall\ \sigma \in \Phi$.

We seek a representative 2-cocycle (in $C^2(\Phi,H^N(\mathfrak{I};M))$) for $\phi^{2,N-1}(d_2(u))$. The following technique (used by Charlap-Vasquez for the group extension case), coming from Godement [6], will be used.

Let $\mathfrak{C}$ be a double complex and E the spectral sequence derived from the first filtration of $\mathrm{Tot}(\mathfrak{C})$. Then $x \in E_2^{p,q}(M)$ can be represented by $x^{p,q} + x^{p+1,q-1} \in \mathfrak{C}^{p,q} \oplus \mathfrak{C}^{p+1,q-1}$ where $d''(x^{p,q}) = 0$ $(d'' \colon \mathfrak{C}^{p,q} \to \mathfrak{C}^{p,q+1})$ and $d'(x^{p,q}) = (-1)^p d''(x^{p+1,q-1})$ $(d' \colon \mathfrak{C}^{p,q} \to \mathfrak{C}^{p+1,q})$. Furthermore, $d_2(x) \in E_2^{p+2,q-1}(M)$ is then represented by $d'(x^{p+1,q-1}) + 0 \in \mathfrak{C}^{p+2,q-1} \oplus \mathfrak{C}^{p+3,q-2}$. In our case, x is u, $\mathfrak{C}^{p,q}$ is $\mathfrak{L}^{p,q} = C^p(\Phi,\mathfrak{U}^q)$, d'' is $d_{\mathfrak{U}}$, and d' is d_Φ (all as defined in Chapter III).

Now $\mathfrak{C}^{0,N} = C^0(\Phi,\mathfrak{U}^N) = \mathfrak{U}^N = \mathrm{Hom}_{\mathfrak{I}}(C_N(\mathfrak{g}),M)$. Define $F = \psi^N(f) \in \mathrm{Hom}_{\mathfrak{I}}(C_N(\mathfrak{g}),M)$. Since ψ_* is a chain map, the diagram

$$\begin{array}{ccc} \mathfrak{V}^{N+1} & \xrightarrow{\ \psi^{N+1}\ } & \mathfrak{U}^{N+1} \\ \delta_N^{\mathfrak{I}} \uparrow & & \uparrow \delta_N^{\mathfrak{g}} \\ \mathfrak{V}^N & \xrightarrow[\ \psi^N\ ]{} & \mathfrak{U}^N \end{array}$$

is commutative. Hence

$$\begin{aligned} d_{\mathfrak{U}}(F) &= \delta_N^{\mathfrak{g}}(F) \\ &= \delta_N^{\mathfrak{g}}(\psi^N(f)) \\ &= \psi^{N+1}(\delta_N^{\mathfrak{I}}(f)) \text{ by } (7.1) \\ &= 0 \end{aligned}$$

Hence F will play the role of $x^{p,q} (= x^{0,N})$ in the representative for u.

Now $d_\Phi : C^0(\Phi, \mathfrak{U}^N) \to C^1(\Phi, \mathfrak{U}^N)$ so $d_\Phi(F) \in C^1(\Phi, \mathfrak{U}^N)$ and for $\sigma \in \Phi$, $\{d_\Phi(F)\}(\sigma) = \sigma \cdot F \in \mathfrak{U}^N$. $\sigma \cdot F$ is computed below:

$$\begin{aligned}
\sigma \cdot F &= \sigma \cdot (\psi^N(f)) \\
&= (Id)^*\{\sigma \cdot (\psi^N(f))\} \\
&= (j_N\psi_N - s_{N-1}d_N^{\mathfrak{g}} - d_{N+1}^{\mathfrak{g}} s_N)^*\{\sigma \cdot (\psi^N(f))\} \\
&= (\psi^N j^N)\{\sigma \cdot (\psi^N(f))\} - (\delta_{N-1}^{\mathfrak{g}} s_{N-1}^*)\{\sigma \cdot (\psi^N(f))\} \\
&\qquad - (s_N^* \delta_N^{\mathfrak{g}})\{\sigma \cdot (\psi^N(f))\} \\
&= \psi^N\{j_N^*(\sigma \cdot (\psi_N^*(f)))\} - \delta_{N-1}^{\mathfrak{g}}\{(\sigma \cdot (\psi^N(f))) \circ s_{N-1}\} \\
&\qquad - s_N^*\{\delta_N^{\mathfrak{g}}(\sigma \cdot (\psi^N(f)))\}
\end{aligned}$$

The last term disappears:

$\delta_N^{\mathfrak{g}}(\sigma \cdot (\psi^N(f))) = \sigma \cdot \delta_N^{\mathfrak{g}}(\psi^N(f))$ (the action is compatible with the coboundary)

and

$$\begin{aligned}
\delta_N^{\mathfrak{g}}(\psi^N(f)) &= \delta_N^{\mathfrak{g}}(f \circ \psi_N) \\
&= (\delta_N^{\mathfrak{I}} f) \circ \psi_{N+1} \quad \text{since } \psi_* \text{ is a chain map} \\
&= 0.
\end{aligned}$$

By proposition 4.1, $j_N^*(\sigma \cdot (\psi_N^*(f))) = \sigma \cdot f$

$= 0$; hence the first term also vanishes. So

$$\sigma \cdot F = -\delta^{\mathfrak{U}}_{N-1}\{(\sigma\cdot(\psi^N(f)))\circ s_{N-1}\} \text{ and}$$

$$\{d_\Phi(F)\}(\sigma) = -\delta^{\mathfrak{U}}_{N-1}\{(\sigma\cdot(\psi^N(f)))\circ s_{N-1}\}.$$

Define $Y \in L^{1,N-1} = C^1(\Phi,\mathfrak{U}^{N-1})$ by $Y(\sigma) = -(\sigma\cdot(\psi^N(f)))\circ s_{N-1}$ so that

$$\begin{aligned}(d_{\mathfrak{U}}(Y))(\sigma) &= \delta^{\mathfrak{U}}_{N-1}(Y(\sigma)) \\ &= \sigma\cdot F \\ &= (d_\Phi(F))(\sigma) \text{ and since } p = 0,\end{aligned}$$

$d_\Phi(F) = (-1)^0 d_{\mathfrak{U}}(Y)$. Hence, Y is an appropriate $x^{p+1,q-1}$ $(=x^{1,N-1})$ for a representative of u, and u is represented by $F + Y \in L^{0,N} \oplus L^{1,N-1}$. Therefore, $d_2(u)$ is represented by $d_\Phi(Y) + 0 = d_\Phi(Y) \in L^{2,N-1} = C^2(\Phi,\mathfrak{U}^{N-1})$. Now we need a representative for $\phi^{2,N-1}(d_2(u))$, where here think of $H^*(\mathfrak{I};M)$ as $H^*(\mathfrak{V}^*)$. Since $j^*: \mathfrak{U}^* \to \mathfrak{V}^*$ induces the isomorphism $H^*(\mathfrak{U}^*) \approx H^*(\mathfrak{V}^*)$, if $x \in E_2^{p,q}(M)$ is represented by $\alpha \in C^p(\Phi,\mathfrak{U}^q)$, $\phi^{p,q}(x) \in H^p(\Phi;H^q(\mathfrak{I};M))$ is represented by $\beta \in C^p(\Phi,H^q(\mathfrak{V}^*))$ where $\beta(\sigma_1,\ldots,\sigma_p) =$ class in $H^q(\mathfrak{V}^*)$ determined by $j^q\{\alpha(\sigma_1,\ldots,\sigma_p)\}$. Since we identify cochains and cohomology using $H^*(\mathfrak{V}^*)$, this says that $\phi^{2,N-1}(d_2(u))$ is represented by $X \in C^2(\Phi,H^{N-1}(\mathfrak{I};M))$ where $X(\sigma,\tau) = j^{N-1}\{(d_\Phi(Y))(\sigma,\tau)\}$.

Now $(d_\Phi(Y))(\sigma,\tau) = \sigma\cdot Y(\tau) - \tau\cdot Y(\sigma) - Y([\sigma,\tau])$ so that

$$X(\sigma,\tau) = j^{N-1}(\sigma\cdot Y(\tau)) - j^{N-1}(\tau\cdot Y(\sigma)) - j^{N-1}(Y([\sigma,\tau]))$$

$$= - j^{N-1}(\sigma\cdot\{(\tau\cdot(\psi^N(f)))\circ s_{N-1}\})$$
$$+ j^{N-1}(\tau\cdot\{(\sigma\cdot(\psi^N(f)))\circ s_{N-1}\})$$
$$+ j^{N-1}(\{[\sigma,\tau]\cdot(\psi^N(f))\}\circ s_{N-1}).$$

But the last term is $\{[\sigma,\tau]\cdot(\psi^N(f))\}\circ s_{N-1}\circ j_{N-1}$

$= 0$ by equation (4.12).

We have proved

<u>Proposition 7.1</u>: Let $u \in E_2^{0,N}(M)$ and denote $\phi^{0,N}(u)$ by f. Then $\phi^{2,N-1}(d_2(u))$ is represented by $X \in C^2(\Phi,H^{N-1}(\mathcal{I};M))$ where

$$X(\sigma,\tau) = j^{N-1}(\tau\cdot\{(\sigma\cdot(\psi^N(f)))\circ s_{N-1}\}) - j^{N-1}(\sigma\cdot\{(\tau\cdot(\psi^N(f)))\circ s_{N-1}\}).$$

The remainder of this chapter will involve a more explicit calculation of this representative. We shall first require more formulae involving the chain map ψ_* and the homotopy s_*. As mentioned in Chapter VI, matters are somewhat simplified when $\mathfrak{g}$ is the semidirect product of $\mathcal{I}$ and Φ.

Proposition 7.2: ψ_1 may be chosen with

(a):

(7.2) $$\psi_1(1 \otimes y) = 0$$

(b):

(7.3) $$\psi_1(y_{\beta_1} \otimes y_{\beta_2}) = 0$$

if $\beta_1 \le \beta_2$. If $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, the same formula holds if $\beta_1 > \beta_2$.

(c):

(7.4) $$\psi_1(y_{\beta_1} y_{\beta_2} \otimes (x,0)) = 1 \otimes [y_{\beta_1},[y_{\beta_2},(x,0)]]$$

if $\beta_1 \le \beta_2$.

If $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, the same formula holds if $\beta_1 > \beta_2$.

Proof: (a) $\psi_o d_1^{\mathfrak{g}}(1 \otimes y) = \psi_o(y) = 0$.

(b) $\psi_o d_1^{\mathfrak{g}}(y_{\beta_1} \otimes y_{\beta_2}) = \psi_o(y_{\beta_1} y_{\beta_2})$

$= 0$

if $\beta_1 \le \beta_2$ since $y_{\beta_1} y_{\beta_2}$ is an $\mathfrak{I}$-basis element of $U\mathfrak{g}$. If $\beta_1 > \beta_2$, we have $y_{\beta_1} y_{\beta_2} = y_{\beta_2} y_{\beta_1} + [y_{\beta_1}, y_{\beta_2}]$. The first term is an $\mathfrak{I}$-basis element, so $\psi_o(y_{\beta_2} y_{\beta_1}) = 0$. If $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, the complementary subspace Φ is a Lie subalgebra of $\mathfrak{g}$; since $\psi_o(y) = 0$ for all y in this subspace, $\psi_o([y_{\beta_1}, y_{\beta_2}]) = 0$ and we have $\psi_o(y_{\beta_1} y_{\beta_2}) = 0$ again.

(c) $\psi_o d_1^{\mathfrak{g}}(y_{\beta_1} y_{\beta_2} \otimes (x,0)) = \psi_o(y_{\beta_1} y_{\beta_2}(x,0))$

$$= \psi_o(y_{\beta_1}(x,0)y_{\beta_2}) + \psi_o(y_{\beta_1}[y_{\beta_2},(x,0)])$$

$$= x\cdot\psi_o(y_{\beta_1}y_{\beta_2}) + [y_{\beta_1},(x,0)]\cdot\psi_o(y_{\beta_2})$$

$$+ [y_{\beta_2},(x,0)]\cdot\psi_o(y_{\beta_1}) + [y_{\beta_1},[y_{\beta_2},(x,0)]].$$

The middle two terms vanish; so does the first if $\beta_1 \leq \beta_2$. We are left with $[y_{\beta_1},[y_{\beta_2},(x,0)]] = d_1^{\mathfrak{I}}(1 \otimes [y_{\beta_1},[y_{\beta_2},(x,0)]])$. If $\beta_1 > \beta_2$ and $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, the argument in (b) shows that the first term again vanishes, hence the formula remains valid. ∎

<u>Remark</u>: It is important to observe that (7.3) does <u>not</u> hold for <u>all</u> y_1, y_2 in the complementary subspace, since it does not hold for all y_{β_1} and y_{β_2}. However, since (7.3) holds for all basis elements if $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, it does hold for all y_1 and y_2 in this case: if $\mathfrak{g} = \mathfrak{I}\cdot\Phi$,

(7.5) $$\psi_1(y_1 \otimes y_2) = 0.$$

Now, we consider ψ_n for $n > 1$.

<u>Proposition 7.3</u>: For $n > 1$, ψ_n may be chosen with

(a):

(7.6) $$\psi_n(1 \otimes \langle (x_1,0),\ldots,(x_{n-1},0),y\rangle) = 0$$

(b):

(7.7) $$\psi_n(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(x_{n-1},0),y_{\beta_2}\rangle) = 0$$

if $\beta_1 \leq \beta_2$. If $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, the same formula holds if $\beta_1 > \beta_2$.

Proof: (a) First let $n = 2$. Then

$$\begin{aligned}\psi_1 d_2^{\mathfrak{g}}(1 \otimes \langle (x,0),y\rangle) &= \psi_1((x,0)\otimes y) - \psi_1(y\otimes(x,0)) - \psi_1(1\otimes[(x,0),y]) \\ &= x\cdot\psi_1(1\otimes y) - 1\otimes[y,(x,0)] - 1\otimes[(x,0),y] \\ &= 0 - 1\otimes[y,(x,0)] + 1\otimes[y,(x,0)] \text{ by (7.2)} \\ &= 0.\end{aligned}$$

Now assume (7.6) holds for n and compute

$$\begin{aligned}&\psi_n d_{n+1}^{\mathfrak{g}}(1\otimes\langle (x_1,0),\ldots,(x_n,0),y\rangle) = \\ &\sum_{i=1}^{n}(-1)^{i+1}x_i\cdot\psi_n(1\otimes\langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y\rangle) \\ &\quad + (-1)^{n+2}\psi_n(y\otimes\langle (x_1,0),\ldots,(x_n,0)\rangle) \\ &\quad + \sum_{i=1}^{n}(-1)^{i+n+1}\psi_n(1\otimes\langle [(x_i,0),y],(x_1,0),\ldots, \\ &\qquad (\hat{x_i},0),\ldots,(x_n,0)\rangle),\end{aligned}$$

where the last summand comes from the fact that the only nonzero brackets are those of an $(x_i,0)$ with y, and y is in the $(n+1)^{\underline{st}}$ slot. The first summand vanishes by the induction hypothesis, the second is computed by (4.3), and the third by (4.2) (since $[(x_i,0),y] \in \mathcal{I}$). We get

$$\sum_{i=1}^{n} (-1)^n 1 \otimes \left\langle x_1,\ldots,[y,(x_i,0)],\ldots,x_n \right\rangle + \sum_{i=1}^{n} (-1)^{i+n+1} 1 \otimes \left\langle [(x_i,0),y],x_1,\ldots,\hat{x}_i,\ldots,x_n \right\rangle .$$

Since $(-1)^{i+n+1} 1 \otimes \left\langle [(x_i,0),y],x_1,\ldots,\hat{x}_i,\ldots,x_n \right\rangle =$

$$(-1)^{i+n+1+i} 1 \otimes \left\langle x_1,\ldots,[y,(x_i,0)],\ldots,x_n \right\rangle =$$

$$(-1)^{n+1} 1 \otimes \left\langle x_1,\ldots,[y,(x_i,0)],\ldots,x_n \right\rangle$$

(one sign interchanges within the bracket and $i-1$ to push the bracketed factor to the $i^{\underline{th}}$ slot), everything cancels.

(b) First let $n = 2$. Then

$$\psi_1 d_2^{\mathfrak{g}}(y_{\beta_1} \otimes \langle (x,0),y_{\beta_2} \rangle) = \psi_1(y_{\beta_1}(x,0) \otimes y_{\beta_2}) - \psi_1(y_{\beta_1} y_{\beta_2} \otimes (x,0)) - \psi_1(y_{\beta_1} \otimes [(x,0),y_{\beta_2}])$$

$$= x \cdot \psi_1(y_{\beta_1} \otimes y_{\beta_2}) + [y_{\beta_1},(x,0)] \cdot \psi_1(1 \otimes y_{\beta_2}) - 1 \otimes [y_{\beta_1},[y_{\beta_2},(x,0)]] - 1 \otimes [y_{\beta_1},[(x,0),y_{\beta_2}]],$$

where the third and fourth terms are by (7.4) and (4.3) respectively. The first two terms vanish by (7.3) and (7.2) and the last two cancel by a sign interchange, so $\psi_2(y_{\beta_1} \otimes \langle (x,0), y_{\beta_2} \rangle) = 0$ is suitable.

Now assume (7.7) holds for n. Then

$$\psi_n d^{g}_{n+1}(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(x_n,0), y_{\beta_2}\rangle) =$$

$$\sum_{i=1}^{n} (-1)^{i+1}\psi_n(y_{\beta_1}(x_i,0) \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y_{\beta_2}\rangle)$$

$$+ (-1)^{n+2}\psi_n(y_{\beta_1}y_{\beta_2} \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle)$$

$$+ \sum_{i=1}^{n} (-1)^{i+n+1}\psi_n(y_{\beta_1} \otimes \langle [(x_i,0),y_{\beta_2}],(x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0)\rangle)$$

$$(7.8) \quad = \sum_{i=1}^{n} (-1)^{i+1} x_i \cdot \psi_n(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y_{\beta_2}\rangle)$$

$$+ \sum_{i=1}^{n} (-1)^{i+1} [y_{\beta_1},(x_i,0)] \cdot \psi_n(1 \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y_{\beta_2}\rangle)$$

$$+ (-1)^{n}\psi_n(y_{\beta_1}y_{\beta_2} \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle)$$

$$+ \sum_{i=1}^{n} (-1)^{i+n}\psi_n(y_{\beta_1} \otimes \langle [y_{\beta_2},(x_i,0)],(x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0)\rangle).$$

The first summand vanishes by the induction hypotheses; the second vanishes by (7.6). The following formula may be seen to hold for the third summand:

$$\psi_n(y_{\beta_1}y_{\beta_2} \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) =$$

(7.9)
$$\sum_{i=1}^{n} 1 \otimes \langle x_1,\ldots,[y_{\beta_1},[y_{\beta_2},(x_i,0)]],\ldots,x_n\rangle$$
$$+ \sum_{1\le i<j\le n} 1 \otimes \langle x_1,\ldots,[y_{\beta_1},(x_i,0)],\ldots,[y_{\beta_2},(x_j,0)],\ldots,x_n\rangle$$
$$+ \sum_{1\le i<j\le n} 1 \otimes \langle x_1,\ldots,[y_{\beta_2},(x_i,0)],\ldots,[y_{\beta_1},(x_j,0)],\ldots,x_n\rangle$$

for $n \geq 1$, if $\beta_1 \leq \beta_2$.

In the first summand of (7.9), the bracketed factor appears in the $i^{\underline{th}}$ slot; in the last two summands, in the $i^{\underline{th}}$ and $j^{\underline{th}}$ respectively. The proof of (7.9) is an extremely tedious induction in the spirit of the proof of (4.3) and will be omitted. Note that for $n = 1$, (7.9) reduces to (7.4).

Since $[y_{\beta_2},(x_i,0)] \in \mathfrak{I}$, the $i^{\underline{th}}$ term in the fourth summand of (7.8) is

$$(-1)^{i+n} 1 \otimes \langle [y_{\beta_1},[y_{\beta_2},(x_i,0)]],x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle$$
$$+ \sum_{j<i} (-1)^{i+n} 1 \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots,\underset{\substack{\downarrow \\ j+1^{\underline{st}} \text{ slot}}}{[y_{\beta_1},(x_j,0)]},\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$+ \sum_{i<j} (-1)^{i+n_1} \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots,\hat{x}_i,\ldots,\underset{\substack{\downarrow \\ j^{\underline{th}} \text{ slot}}}{[y_{\beta_1},(x_j,0)]},\ldots,x_n \rangle .$$

Now sum over i to yield

$$\sum_{i=1}^{n} (-1)^{i+n_1} \otimes \langle [y_{\beta_1},[y_{\beta_2},(x_i,0)]],x_1,\ldots,\hat{x}_i,\ldots,x_n \rangle$$

$$(7.10) \qquad + \sum_{1\le j<i\le n} (-1)^{i+n_1} \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots, \underset{\substack{\downarrow \\ j+1^{\underline{st}} \text{ slot}}}{[y_{\beta_1},(x_j,0)]},\ldots,\hat{x}_i,\ldots,x_n \rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+n_1} \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots,\hat{x}_i,\ldots,\underset{\substack{\downarrow \\ j^{\underline{th}} \text{ slot}}}{[y_{\beta_1},(x_j,0)]},\ldots,x_n \rangle .$$

Note that the third summand of (7.8) is (7.9) with a sign of $(-1)^n$ in each term; call this (7.11). In the first summand of (7.10), push the bracketed factor to the $i^{\underline{th}}$ slot; the sign becomes $(-1)^{i+n+i-1} = (-1)^{n-1}$ and hence this summand cancels with the first summand of (7.11). In the second summand of (7.10), interchange i and j to get

$$\sum_{1\le i<j\le n} (-1)^{j+n_1} \otimes \langle [y_{\beta_2},(x_j,0)],x_1,\ldots, \underset{\substack{\downarrow \\ i+1^{\underline{st}} \text{ slot.}}}{[y_{\beta_1},(x_i,0)]},\ldots,\hat{x}_j,\ldots,x_n \rangle$$

Move $[y_{\beta_2},(x_j,0)]$ to the $j^{\underline{th}}$ slot via j-1 interchanges; this moves $[y_{\beta_1},(x_i,0)]$ to the $i^{\underline{th}}$ slot and makes the sign $(-1)^{j+n+j-1} = (-1)^{n-1}$, so this summand cancels with the second of (7.11). Finally, in the third summand of (7.10), push $[y_{\beta_2},(x_i,0)]$ to the $i^{\underline{th}}$ slot; the other bracketed factor remains in the $j^{\underline{th}}$ slot and the sign is $(-1)^{i+n+i-1} = (-1)^{n-1}$ so this summand cancels with the third of (7.11).

This proves (7.7) for any extension, as long as $\beta_1 \leq \beta_2$. If $\beta_1 > \beta_2$, all the summands in (7.8) will be the same except perhaps for $\psi_n(y_{\beta_1}y_{\beta_2} \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle)$. Thus to show that (7.7) holds for $\beta_1 > \beta_2$ if $\mathfrak{g} = \mathfrak{I}\cdot\Phi$, it suffices to show that (7.9) does:

$$\psi_n(y_{\beta_1}y_{\beta_2} \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) =$$

$$\psi_n(y_{\beta_2}y_{\beta_1} \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) + \psi_n([y_{\beta_1},y_{\beta_2}] \otimes \langle(x_1,0),$$
$$\ldots,(x_n,0)\rangle).$$

Since $y_{\beta_2}y_{\beta_1}$ is now a basis element, the first term is given by (7.9) with β_1 and β_2 reversed:

$$\sum_{i=1}^{n} 1 \otimes \langle x_1,\ldots,[y_{\beta_2},[y_{\beta_1},(x_i,0)]],\ldots,x_n\rangle$$

$$(7.12) \qquad + \sum_{1\leq i<j\leq n} 1 \otimes \langle x_1,\ldots,[y_{\beta_2},(x_i,0)],\ldots,[y_{\beta_1},(x_j,0)],\ldots,x_n\rangle$$

$$+ \sum_{1\leq i<j\leq n} 1 \otimes \left\langle x_1,\ldots,[y_{\beta_1},(x_i,0)],\ldots,[y_{\beta_2},(x_j,0)],\ldots,x_n\right\rangle .$$

Since $[y_{\beta_1},y_{\beta_2}]$ lies in the complementary subspace,

$$\psi_n([y_{\beta_1},y_{\beta_2}] \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) =$$

(7.13) $$\sum_{i=1}^{n} 1 \otimes \left\langle x_1,\ldots,[[y_{\beta_1},y_{\beta_2}],(x_i,0)],\ldots,x_n\right\rangle.$$

Now add (7.12) and (7.13) to get

$$\psi_n(y_{\beta_1} y_{\beta_2} \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) =$$

$$\sum_{i=1}^{n} 1 \otimes \left\langle x_1,\ldots,[y_{\beta_2},[y_{\beta_1},(x_i,0)]] + [[y_{\beta_1},y_{\beta_2}], (x_i,0)],\ldots,x_n\right\rangle$$

(7.14) $$+ \sum_{1\leq i<j\leq n} 1 \otimes \left\langle x_1,\ldots,[y_{\beta_2},(x_i,0)],\ldots,[y_{\beta_1},(x_j,0)],\ldots,x_n\right\rangle$$

$$+ \sum_{1\leq i<j\leq n} 1 \otimes \left\langle x_1,\ldots,[y_{\beta_1},(x_i,0)],\ldots,[y_{\beta_2},(x_j,0)],\ldots,x_n\right\rangle.$$

The second and third summands of (7.14) agree with the third and second respectively of (7.9). By the Jacobi identity,

$$[y_{\beta_2},[y_{\beta_1},(x_i,0)]] + [[y_{\beta_1},y_{\beta_2}],(x_i,0)] = [y_{\beta_1},[y_{\beta_2},(x_i,0)]]$$

hence the first summands also agree. ∎

Again, (7.7) holds for all y_1, y_2 when $\mathfrak{g} = \mathfrak{I}\cdot\Phi$:

(7.15) $$\psi_n(y_1 \otimes \langle(x_1,0),\ldots,(x_{n-1},0),y_2\rangle) = 0$$

if $\mathfrak{g} = \mathfrak{I}\cdot\Phi$.

One more technical result involving the homotopy s_* shall be required before we can return to the main argument.

<u>Proposition 7.4</u>. For $n \geq 1$, s_n may be chosen with

(7.16) $$s_n(y \otimes \langle(x_1,0),\ldots,(x_n,0)\rangle) = (-1)^{n+1} 1 \otimes \langle(x_1,0),\ldots,(x_n,0),y\rangle .$$

<u>Proof</u>. First let $n = 1$ and recall that $s_1 \colon C_1(\mathfrak{g}) \to C_2(\mathfrak{g})$ must satisfy $d_2^{\mathfrak{g}} s_1 = j_1\psi_1 - \mathrm{Id} - s_o d_1^{\mathfrak{g}}$. The right side applied to $y \otimes (x,0)$ yields

$$j_1\psi_1(y \otimes (x,0)) - y \otimes (x,0) - s_o(y(x,0))$$

$$= j_1(1\otimes[y,(x,0)]) - y \otimes (x,0) - x\cdot s_o(y) - [y,(x,0)]\cdot s_o(1)$$

$$= 1 \otimes [y,(x,0)] - y \otimes (x,0) + (x,0) \otimes y$$

since $s_o(1) = 0$ and $s_o(y) = -1 \otimes y$.

$$= d_2^{\mathfrak{g}}(1 \otimes \langle(x,0),y\rangle) \text{ so}$$

$$s_1(y \otimes (x,0)) = 1 \otimes \langle(x,0),y\rangle$$

$$= (-1)^2 1 \otimes \langle(x,0),y\rangle$$ is appropriate.

Now assume that (7.16) holds for n-1 and write δ for $y \otimes \langle (x_1,0),\dots,(x_n,0)\rangle$. Then it must be shown that

$$(7.17) \qquad d^{\mathfrak{g}}_{n+1}((-1)^{n+1} 1 \otimes \langle (x_1,0),\dots,(x_n,0),y\rangle) = j_n\psi_n(\delta) - \delta - s_{n-1} d^{\mathfrak{g}}_n(\delta)\,.$$

Work out the right hand side: by (4.3), we get

$$\sum_{i=1}^{n} j_n(1 \otimes \langle x_1,\dots,[y,(x_i,0)],\dots,x_n\rangle) - y \otimes \langle (x_1,0),\dots,(x_n,0)\rangle$$
$$- \sum_{i=1}^{n} (-1)^{i+1} s_{n-1}(y(x,0) \otimes \langle (x_1,0),\dots,\widehat{(x_i,0)},\dots,(x_n,0)\rangle)$$

$$= \sum_{i=1}^{n} (1 \otimes \langle (x_1,0),\dots,[y,(x_i,0)],\dots,(x_n,0)\rangle) - y \otimes \langle (x_1,0),\dots,(x_n,0)\rangle$$
$$+ \sum_{i=1}^{n} (-1)^{i} x_i \cdot s_{n-1}(y \otimes \langle (x_1,0),\dots,\widehat{(x_i,0)},\dots,(x_n,0)\rangle)$$
$$+ \sum_{i=1}^{n} (-1)^{i} [y,(x_i,0)] \cdot s_{n-1}(1 \otimes \langle (x_1,0),\dots,\widehat{(x_i,0)},\dots,(x_n,0)\rangle).$$

By (4.12), the last summand vanishes. Apply the induction hypothesis to the third summand; the i^{th} term has sign $(-1)^{i+n}$ so the right side becomes

$$\sum_{i=1}^{n} (1 \otimes \langle (x_1,0),\dots,[y,(x_i,0)],\dots,(x_n,0)\rangle) - y \otimes \langle (x_1,0),\dots,(x_n,0)\rangle$$
$$(7.18) \qquad + \sum_{i=1}^{n} (-1)^{i+n} (x_i,0) \otimes \langle (x_1,0),\dots,\widehat{(x_i,0)},\dots,(x_n,0),y\rangle.$$

For the left side,

$$
\begin{aligned}
d^{\mathfrak{g}}_{n+1}((-1)^{n+1}1 \otimes \langle (x_1,0),\ldots,(x_n,0),y\rangle) = \\
\sum_{i=1}^{n} (-1)^{n+1+i+1}(x_i,0) \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y\rangle \\
+ (-1)^{n+1+n+1+1} y \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle \\
+ \sum_{i=1}^{n} (-1)^{n+1+i+n+1} 1 \otimes \langle [(x_i,0),y],(x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0)\rangle .
\end{aligned}
\tag{7.19}
$$

The second summands of (7.18) and (7.19) agree; the first of (7.19) is the third of (7.18). As usual, i interchanges are necessary to convert the $i^{\underline{th}}$ term of the third summand of (7.19) into the form of the first summand of (7.18); the sign becomes $(-1)^{n+1+i+n+1+i} = 1$, as desired. ∎

At long last, we return to the calculation of the representative X in proposition 7.1. Recall that

$$X(\sigma,\tau) = j^{N-1}(\tau\cdot\{(\sigma\cdot(\psi^N(f)))\circ s_{N-1}\}) - j^{N-1}(\sigma\cdot\{(\tau\cdot(\psi^N(f)))\circ s_{N-1}\}).$$

To eliminate all the grouping, we first observe that $\psi^N(f) = f\circ\psi_N \in \mathfrak{M}^N$ and that each expression inside the curly brackets above is in $\mathfrak{M}^{N-1}$. Hence all the actions of the elements of Φ are on elements of $\mathfrak{M}^n = \mathrm{Hom}_{\mathfrak{I}}(C_n(\mathfrak{g}),M)$. Since the Φ-structure is given by diagonal action and $\rho((0,\sigma)) = \sigma$, we have $(\sigma\cdot F)(c_n) = (0,\sigma)\cdot F(c_n) - F((0,\sigma)\cdot c_n)$

for $F \in \mathfrak{U}^n$. Since M is $\mathfrak{I}$-trivial, we may write $\sigma \cdot F(c_n)$ to give

$$(7.20) \qquad (\sigma \cdot F)(c_n) = \sigma \cdot F(c_n) - F((0,\sigma)\cdot c_n).$$

To simplify the notation, let $M_\sigma: M \to M$ be the map $M_\sigma(m) = \sigma \cdot m$. The first term of (7.20) then becomes $(M_\sigma \circ F)(c_n)$. As for the second term, let $A_\sigma: C_n(\mathfrak{g}) \to C_n(\mathfrak{g})$ be the map $A_\sigma(c_n) = (0,\sigma)\cdot c_n$; the second term then becomes $(F \circ A_\sigma)(c_n)$ so that (7.20) becomes

$$(7.21) \qquad \sigma \cdot F = M_\sigma \circ F - F \circ A_\sigma \ .$$

This is now used to decompose $X(\sigma,\tau)$. Observe that it suffices to decompose the first term and then switch the roles of σ and τ; i.e.,

$$X(\sigma,\tau) = X_1(\sigma,\tau) - X_2(\sigma,\tau) \text{ where } X_2(\sigma,\tau) = X_1(\tau,\sigma).$$

(Note that X_1 and X_2 individually are not alternating!) So we write

$$X(\sigma,\tau) = X_1(\sigma,\tau) - X_1(\tau,\sigma); \; X_1(\sigma,\tau) \in \mathrm{Hom}_{\mathfrak{I}}(C_{N-1}(\mathfrak{I}),M).$$

Write a_σ for $\sigma \cdot (\psi^N(f)) = \sigma \cdot (f \circ \psi_N)$. Then we have $X_1(\sigma,\tau) = (\tau \cdot \{a_\sigma \circ s_{N-1}\}) \circ j_{N-1}$. Since $a_\sigma \in \mathfrak{U}^{N-1}$, (7.21) gives

$$X_1(\sigma,\tau) = (M_\tau \circ (a_\sigma \circ s_{N-1}) - (a_\sigma \circ s_{N-1}) \circ A_\tau) \circ j_{N-1}$$

$$= -a_\sigma \circ s_{N-1} \circ A_\tau \circ j_{N-1}$$

since $s_{N-1} \circ j_{N-1} = 0$. But $f \circ \psi_N \in \mathfrak{U}^N$ so $a_\sigma = M_\sigma \circ (f \circ \psi_N) - (f \circ \psi_N) \circ A_\sigma$ hence

(7.22) $X_1(\sigma,\tau) = -M_\sigma \circ f \circ \psi_N \circ s_{N-1} \circ A_\tau \circ j_{N-1} + f \circ \psi_N \circ A_\sigma \circ s_{N-1} \circ A_\tau \circ j_{N-1}$.

The case N = 1 is handled separately; we evaluate

$$\{X_1(\sigma,\tau)\}(1) = (f \circ \psi_1 \circ A_\sigma \circ s_o \circ A_\tau \circ j_o)(1) - (M_\sigma \circ f \circ \psi_1 \circ s_o \circ A_\tau \circ j_o)(1)$$

$$= (f \circ \psi_1 \circ A_\sigma \circ s_o \circ A_\tau)(1) - (M_\sigma \circ f \circ \psi_1 \circ s_o \circ A_\tau)(1)$$

$$= (f \circ \psi_1 \circ A_\sigma \circ s_o)((0,\tau)) - (M_\sigma \circ f \circ \psi_1 \circ s_o)((0,\tau))$$

$$= -(f \circ \psi_1 \circ A_\sigma)(1 \otimes (0,\tau)) + (M_\sigma \circ f \circ \psi_1)(1 \otimes (0,\tau))$$

by (4.14),

$$= -(f \circ \psi_1)((0,\sigma) \otimes (0,\tau));$$

the second term is 0 by (7.2). Hence the second term vanishes identically and we have $\{X_1(\sigma,\tau)\}(1) = -f(\psi_1((0,\sigma) \otimes (0,\tau)))$. Hence

(7.23) $\{X(\sigma,\tau)\}(1) = f(\psi_1((0,\tau) \otimes (0,\sigma))) - f(\psi_1((0,\sigma) \otimes (0,\tau)))$.

If N > 1, we have a similar calculation:

$$\{X_1(\sigma,\tau)\}(1 \otimes \langle x_1,\ldots,x_{N-1}\rangle) =$$

$$(f\circ\psi_N\circ A_\sigma\circ s_{N-1}\circ A_\tau\circ j_{N-1})(1 \otimes \langle x_1,\ldots,x_{N-1}\rangle)$$

$$- (M_\sigma\circ f\circ\psi_N\circ s_{N-1}\circ A_\tau\circ j_{N-1})(1 \otimes \langle x_1,\ldots,x_{N-1}\rangle)$$

$$= (f\circ\psi_N\circ A_\sigma\circ s_{N-1}\circ A_\tau)(1 \otimes \langle (x_1,0),\ldots,(x_{N-1},0)\rangle)$$

$$- (M_\sigma\circ f\circ\psi_N\circ s_{N-1}\circ A_\tau)(1 \otimes \langle (x_1,0),\ldots,(x_{N-1},0)\rangle)$$

$$= (f\circ\psi_N\circ A_\sigma\circ s_{N-1})((0,\tau) \otimes \langle (x_1,0),\ldots,(x_{N-1},0)\rangle)$$

$$- (M_\sigma\circ f\circ\psi_N\circ s_{N-1})((0,\tau) \otimes \langle (x_1,0),\ldots,(x_{N-1},0)\rangle)$$

$$= (-1)^N(f\circ\psi_N\circ A_\sigma)(1 \otimes \langle (x_1,0),\ldots,(x_{N-1},0),(0,\tau)\rangle)$$

$$+ (-1)^{N+1}(M_\sigma\circ f\circ\psi_N)(1\otimes\langle (x_1,0),\ldots,(x_{N-1},0),(0,\tau)\rangle)$$

by (7.18)

$$= (-1)^N(f\circ\psi_N)((0,\sigma) \otimes \langle (x_1,0),\ldots,(x_{N-1},0),(0,\tau)\rangle);$$

the second term vanishes by (7.15). Hence

$$\{X(\sigma,\tau)\}(1 \otimes \langle x_1,\ldots,x_{N-1}\rangle) =$$

$$(-1)^N\{f(\psi_N((0,\sigma) \otimes \langle (x_1,0),\ldots,(x_{N-1},0),(0,\tau)\rangle))$$

$$- f(\psi_N((0,\tau) \otimes \langle (x_1,0),\ldots,(x_{N-1},0),(0,\sigma)\rangle))\} \tag{7.24}$$

Now suppose that $\mathfrak{g}$ is the semidirect product of $\mathfrak{I}$ and Φ. By (7.5), the right side of (7.23) vanishes; by (7.15), the right side of (7.24) vanishes. So the cocycle X representing $\phi^{2,N-1}(d_2(u))$ vanishes identically. We have proved

Theorem 7.5: Let $\mathfrak{g}$ be the semidirect product of $\mathfrak{I}$ and Φ. Then $d_2 = 0$.

We now turn to a computation of (7.23) and (7.24) in the general case. It is necessary to compute $\psi_1(y_{\beta_1} \otimes y_{\beta_2})$ and $\psi_N(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(x_{N-1},0),y_{\beta_2}\rangle)$ when $\beta_1 > \beta_2$. First we handle N = 1.

Proposition 7.6: (a) If $\beta_1 > \beta_2$, we may take

$$\psi_1(y_{\beta_1} \otimes y_{\beta_2}) = 1 \otimes x_{\beta_1}^{\beta_2} \text{ where } [y_{\beta_1},y_{\beta_2}] = (x_{\beta_1}^{\beta_2},\sigma_{\beta_1}^{\beta_2})$$

(b):

$$(7.25)\quad \psi_1(y_{\beta_1} \otimes y_{\beta_2} - y_{\beta_2} \otimes y_{\beta_1}) = 1 \otimes x_{\beta_1}^{\beta_2}$$

for all β_1,β_2.

Proof: (a) $\psi_0 d_1^{\mathfrak{g}}(y_{\beta_1} \otimes y_{\beta_2}) = \psi_0(y_{\beta_1}y_{\beta_2})$

$$= \psi_0(y_{\beta_2}y_{\beta_1}) + \psi_0([y_{\beta_1},y_{\beta_2}])$$

$$= \psi_0([y_{\beta_1},y_{\beta_2}]) \text{ since}$$

$y_{\beta_2} y_{\beta_1}$ is a basis element.

$$= \psi_o((x_{\beta_1}^{\beta_2}, \sigma_{\beta_1}^{\beta_2}))$$

$$= x_{\beta_1}^{\beta_2} = d_1^I(1 \otimes x_{\beta_1}^{\beta_2}).$$

(b) Note that by the anticommutativity of the bracket, $x_{\beta_2}^{\beta_1} = -x_{\beta_1}^{\beta_2}$. If $\beta_1 > \beta_2$, $\psi_1(y_{\beta_1} \otimes y_{\beta_2}) = 1 \otimes x_{\beta_1}^{\beta_2}$ and $\psi_1(y_{\beta_2} \otimes y_{\beta_1}) = 0$ so $\psi_1(y_{\beta_1} \otimes y_{\beta_2} \otimes -y_{\beta_2} \otimes y_{\beta_1}) = 1 \otimes x_{\beta_1}^{\beta_2}$. If $\beta_1 \leq \beta_2$, $\psi_1(y_{\beta_1} \otimes y_{\beta_2}) = 0$ and $\psi_1(y_{\beta_2} \otimes y_{\beta_1}) = 1 \otimes x_{\beta_2}^{\beta_1} = -1 \otimes x_{\beta_1}^{\beta_2}$, so (7.25) again holds. ■

By virtue of the expression in (7.23), it is really (7.25) that we are interested in. Observe (by linearity) that $\psi_1(y_1 \otimes y_2 - y_2 \otimes y_1) = 1 \otimes x_1^2$ for all y_1, y_2, where $[y_1, y_2] = (x_1^2, \sigma_1^2)$. But we can compute both x_1^2 and σ_1^2 in terms of the extension:

Write $y_1 = (0, \sigma_1)$ and $y_2 = (0, \sigma_2)$; then

$$[y_1, y_2] = [(0, \sigma_1), (0, \sigma_2)]$$

$$= (a(\sigma_1, \sigma_2), [\sigma_1, \sigma_2])$$

(recall that a is a 2-cocycle representing the class of the extension) so we get

(7.26) $$[(0,\sigma_1),(0,\sigma_2)] = (a(\sigma_1,\sigma_2),[\sigma_1,\sigma_2])$$

and $x_1^2 = a(\sigma_1,\sigma_2)$ (also $\sigma_1^2 = [\sigma_1,\sigma_2]$). Applying this to (7.25), $\psi_1((0,\tau)\otimes(0,\sigma) - (0,\sigma)\otimes(0,\tau)) = 1\otimes a(\tau,\sigma)$ and this gives

Corollary 7.7: If $N = 1$, $\{X(\sigma,\tau)\}(1) = f(1\otimes a(\tau,\sigma))$.

For $N > 1$, the idea is similar, but the computations are of course more tedious.

Proposition 7.8: (a) If $\beta_1 > \beta_2$, we may take

(7.27) $$\psi_n(y_{\beta_1}\otimes\langle(x_1,0),\ldots,(x_{n-1},0),y_{\beta_2}\rangle = 1\otimes\langle x_1,\ldots,x_{n-1},x_{\beta_1}^{\beta_2}\rangle.$$

(b):

$$\psi_n(y_{\beta_1}\otimes\langle(x_1,0),\ldots,(x_{n-1},0),y_{\beta_2}\rangle - y_{\beta_2}\otimes\langle(x_1,0),\ldots,(x_{n-1},0),y_{\beta_1}\rangle)$$

(7.28) $$= 1\otimes\langle x_1,\ldots,x_{n-1},x_{\beta_2}^{\beta_2}\rangle \text{ for any } \beta_1,\beta_2.$$

Proof: (a) Unfortunately, another tedious induction is required. We first observe that since for $\sigma\in\Phi$, $[(0,\sigma),(x,0)] = \sigma\cdot x$, the identity

$$(7.29)\quad [y_{\beta_1},[y_{\beta_2},(x,0)]] - [y_{\beta_2},[y_{\beta_1},(x,0)]] - [(0,\sigma_{\beta_1}^{\beta_2}),(x,0)] = 0$$

holds since $\sigma_{\beta_1}^{\beta_2} = [\sigma_{\beta_1},\sigma_{\beta_2}]$ and $\sigma_{\beta_1}\cdot(\sigma_{\beta_2}\cdot x)-\sigma_{\beta_2}\cdot(\sigma_{\beta_1}\cdot x)=[\sigma_{\beta_1},\sigma_{\beta_2}]\cdot x$.

Let $n = 2$. Then

$$\psi_1 d_2^{\mathfrak{g}}(y_{\beta_1} \otimes \langle (x,0),y_{\beta_2}\rangle) =$$

$$\psi_1(y_{\beta_1}(x,0)\otimes y_{\beta_2}) - \psi_1(y_{\beta_1}y_{\beta_2}\otimes(x,0)) - \psi_1(y_{\beta_1}\otimes[(x,0),y_{\beta_2}])$$

$$= x\cdot\psi_1(y_{\beta_1}\otimes y_{\beta_2}) + [y_{\beta_1},(x,0)]\cdot\psi_1(1\otimes y_{\beta_2}) - \psi_1(y_{\beta_2}y_{\beta_1}\otimes(x,0))$$

$$- \psi_1([y_{\beta_1},y_{\beta_2}]\otimes(x,0)) + \psi_1(y_{\beta_1}\otimes[y_{\beta_2},(x,0)])$$

$$= x\otimes x_{\beta_1}^{\beta_2} - 1\otimes[y_{\beta_2},[y_{\beta_1},(x,0)]] - \psi_1((x_{\beta_1}^{\beta_2},0)\otimes(x,0))$$

$$-\psi_1((0,\sigma_{\beta_1}^{\beta_2})\otimes(x,0)) + 1\otimes[y_{\beta_1},[y_{\beta_2},(x,0)]]$$

$$= x\otimes x_{\beta_1}^{\beta_2} - 1\otimes[y_{\beta_2},[y_{\beta_1},(x,0)]] - x_{\beta_1}^{\beta_2}\otimes x$$

$$- 1\otimes[(0,\sigma_{\beta_1}^{\beta_2}),(x,0)] + 1\otimes[y_{\beta_1},[y_{\beta_2},(x,0)]].$$

The first and third terms comprise $d_1^{I}(1\otimes\langle x,x_{\beta_1}^{\beta_2}\rangle)$ and the others cancel by (7.29). Hence $\psi_1(y_{\beta_1}\otimes\langle (x,0),y_{\beta_2}\rangle) = 1\otimes\langle x,x_{\beta_1}^{\beta_2}\rangle$ is suitable. Now assume (7.27) for n.

Compute

$$\psi_n d^{\mathfrak{g}}_{n+1}(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(x_n,0),y_{\beta_2}\rangle) =$$

$$\sum_{i=1}^{n}(-1)^{i+1}x_i\cdot\psi_n(y_{\beta_1} \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y_{\beta_2}\rangle)$$

$$+ \sum_{i=1}^{n}(-1)^{i+1}[y_{\beta_1},(x_i,0)]\cdot\psi_n(1 \otimes \langle (x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0),y_{\beta_2}\rangle)$$

$$+ (-1)^n\psi_n(y_{\beta_2}y_{\beta_1} \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle)$$

$$+ (-1)^{n+2}\psi_n((x^{\beta_2}_{\beta_1},0) \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle)$$

$$+ (-1)^n\psi_n((0,\sigma^{\beta_2}_{\beta_1}) \otimes \langle (x_1,0),\ldots,(x_n,0)\rangle)$$

$$+ \sum_{i=1}^{n}(-1)^{i+n}\psi_n(y_{\beta_1} \otimes \langle [y_{\beta_2},(x_i,0)],(x_1,0),\ldots,(\hat{x_i},0),\ldots,(x_n,0)\rangle)$$

(omitting some steps that are the same as in previous calculations). The first term is given by the induction hypothesis, and the second vanishes by (7.6). The third is computed by (7.9), the fourth by (4.4), and the fifth and sixth by (4.3). We get

$$\sum_{i=1}^{n}(-1)^{i+1}x_i \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n,x^{\beta_2}_{\beta_1}\rangle$$

$$+ \sum_{i=1}^{n}(-1)^{n}1 \otimes \langle x_1,\ldots,[y_{\beta_2},[y_{\beta_1},(x_i,0)]],\ldots,x_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{n_1} \otimes \langle x_1,\ldots,[y_{\beta_2},(x_i,0)],\ldots,[y_{\beta_1},(x_j,0)],\ldots,x_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{n_1} \otimes \langle x_1,\ldots,[y_{\beta_1},(x_i,0)],\ldots,[y_{\beta_2},(x_j,0)],\ldots,x_n\rangle$$

$$+ (-1)^{n+2} x_{\beta_1}^{\beta_2} \otimes \langle x_1,\ldots,x_n\rangle + \sum_{i=1}^{n} (-1)^{n_1} \otimes \langle x_1,\ldots,[(0,\sigma_{\beta_1}^{\beta_2}),(x_i,0)],\ldots,x_n\rangle$$

$$+ \sum_{i=1}^{n} (-1)^{i+n_1} \otimes \langle [y_{\beta_1},[y_{\beta_2},(x_i,0)]],x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$+ \sum_{1\le j<i\le n} (-1)^{i+n_1} \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots,\underset{\downarrow\ j+1^{\text{st}}\text{ slot}}{[y_{\beta_1},(x_j,0)]},\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+n_1} \otimes \langle [y_{\beta_2},(x_i,0)],x_1,\ldots,\hat{x}_i,\ldots,\underset{\downarrow\ j^{\text{th}}\text{ slot.}}{[y_{\beta_1},(x_j,0)]},\ldots x_n\rangle .$$

There are now nine terms. The first and fifth together comprise $d^I_{n+1}(1 \otimes \langle x_1,\ldots,x_n,x_{\beta_1}^{\beta_2}\rangle)$. Exactly as in the proof of (7.8), the fourth and eighth cancel, as do the third and ninth. In the seventh, move $[y_{\beta_1},[y_{\beta_2},(x_i,0)]]$ to the i^{th} slot via $i-1$ interchanges; it then occurs with sign $(-1)^{n-1}$ and then the remaining terms cancel by virtue of (7.29).

(b) This follows precisely as in the proof of proposition 7.6(b). ∎

Finally, we note that (7.28) holds for all y_1, y_2:

$$\psi_n(y_1 \otimes \langle (x_1,0),\ldots,(x_{n-1},0),y_2\rangle - y_2 \otimes \langle (x_1,0),\ldots,(x_{n-1},0),y_1\rangle\ ) =$$

(7.30)

$$1 \otimes \langle x_1,\ldots,x_{n-1},x_1^2\rangle$$

for all y_1, y_2, where $[y_1, y_2] = (x_1^2, \sigma_1^2)$.

Finally, applying (7.30) to (7.24) yields our final form for the representative X.

<u>Corollary 7.9</u>: If $N > 1$,

$$\{X(\sigma,\tau)\}(1 \otimes \langle x_1,\ldots,x_{N-1}\rangle) = (-1)^N f(1 \otimes \langle x_1,\ldots,x_{N-1},a(\sigma,\tau)\rangle).$$

CHAPTER VIII

PONTRJAGIN MULTIPLICATION

The main formula of the next chapter will express $\hat{d}_2(f^N)$ in terms of a Pontrjagin multiplication; i.e., a homology product. This multiplication will be described in general terms in this chapter, following techniques in [1]. Due to the specialized nature of our assumptions, however, the main formula could be derived without the general Pontrjagin product.

Let $\mathfrak{A}$ be any abelian Lie algebra and define $\mathfrak{C}: \mathfrak{A} \oplus \mathfrak{A} \to \mathfrak{A}$ by $\mathfrak{C}(x_1, x_2) = x_1 + x_2$. Since both $\mathfrak{A}$ and $\mathfrak{A} \oplus \mathfrak{A}$ are abelian, $\mathfrak{C}$ is a Lie algebra homomorphism (it is not if $\mathfrak{A}$ is not abelian).

Now we recall some general theory. If $\mathfrak{g}$ and $\mathfrak{g}'$ are any two Lie algebras and $\eta: \mathfrak{g} \to \mathfrak{g}'$ a Lie algebra homomorphism, every $\mathfrak{g}'$-module A acquires a $\mathfrak{g}$-structure by change of rings via the induced map $U\eta: U\mathfrak{g} \to U\mathfrak{g}'$: $x \cdot a \equiv \eta(x) \cdot a$ for $x \in \mathfrak{g}$, $a \in A$. Now if R and R' are any two algebras over a commutative ring Λ and if M,M' are R,R'-modules respectively, then the tensor product $M \otimes_\Lambda M'$ is an $R \otimes_\Lambda R'$-module in the natural fashion: $(r \otimes r') \cdot (m \otimes m') = (r \cdot m) \otimes (r' . m')$. Applying this to the situation where M is a $(U)\mathfrak{g}$-module and M' is a $(U)\mathfrak{g}'$-module, $M \otimes M'$ becomes a $U\mathfrak{g} \otimes U\mathfrak{g}'$-module as described. But there is an isomorphism $U\mathfrak{g} \otimes U\mathfrak{g}' \approx U(\mathfrak{g} \oplus \mathfrak{g}')$ where $(z,z') \in \mathfrak{g} \oplus \mathfrak{g}' \subseteq U(\mathfrak{g} \oplus \mathfrak{g}')$ corresponds to $z \otimes 1 + 1 \otimes z'$. So $M \otimes M'$ becomes a $(\mathfrak{g} \oplus \mathfrak{g}')$-module via $(z,z')\cdot(m \otimes m') = (z \otimes 1 + 1 \otimes z')\cdot(m \otimes m')$ or $(z,z')\cdot(m \otimes m') = z\cdot m \otimes m' + m \otimes z'\cdot m'$. Furthermore, if M is $\mathfrak{g}$-free on $\{m_i\}$ and M' is $\mathfrak{g}'$-free on $\{m_j'\}$, then $M \otimes M'$ is $(\mathfrak{g} \oplus \mathfrak{g}')$-free on $\{m_i \otimes m_j'\}$.

Next, suppose that C is a free, acyclic A-complex. By what has just been said, the tensor product complex $C \otimes C$, where $(C \otimes C)_n = \bigoplus_{r+s=n} (C_r \otimes C_s)$ and $\partial(c_r \otimes c_s) = \partial_r(c_r) \otimes c_s + (-1)^r c_r \otimes \partial_s(c_s)$, becomes a free, acyclic $(A \oplus A)$-complex. Since $\mathfrak{C}$ makes C into an acyclic $(A \oplus A)$-complex (although C may not be free as an $(A \oplus A)$-complex), the comparison lemma gives a chain map $h\colon C \otimes C \to C$ of $(A \oplus A)$-modules. Now examine the condition that h is an $(A \oplus A)$-map. On the one hand $h((x_1,x_2)\cdot(c_1 \otimes c_2)) = h(x_1c_1 \otimes c_2 + c_1 \otimes x_2c_2)$; while on the other,

$$\begin{aligned}(x_1,x_2)\cdot h(c_1 \otimes c_2) &= \mathfrak{C}(x_1,x_2)\cdot h(c_1 \otimes c_2)\\ &= (x_1+x_2)\cdot h(c_1 \otimes c_2).\end{aligned}$$

Hence h satisfies

(8.1) $$h(x_1c_1 \otimes c_2 + c_1 \otimes x_2c_2) = (x_1+x_2)\cdot h(c_1 \otimes c_2) .$$

Any h satisfying (8.1) may be used to define the homology product.

Since A is abelian, its enveloping algebra UA is commutative. Hence if M and N are A-modules, $M \otimes_A N$ becomes one via $x\cdot(m \otimes n) = (x\cdot m) \otimes n = m \otimes (x.n)$. Let P be a third A-module. We have seen that $M \otimes N$ is an $(A \oplus A)$-module and $\mathfrak{C}$ turns P into one. Let $\beta : M \otimes N \to P$ be an $(A \oplus A)$-map. We claim that if β is regarded as a map of $M \otimes_A N$ to P, then it is an A-homomorphism. The fact that β is an $(A \oplus A)$-map ensures that β is well-defined (and therefore a A-map): since $\beta(x_1\cdot m \otimes n + m \otimes x_2\cdot n) = (x_1+x_2)\cdot\beta(m \otimes n)$, $x_1 = x$ and $x_2 = 0$ yields $\beta(x\cdot m \otimes n) = x\cdot\beta(m \otimes n)$; $x_1 = 0$ and $x_2 = x$ yields $\beta(m \otimes x\cdot n) = x\cdot\beta(m \otimes n)$.

Returning to $h: C \otimes C \to C$, this shows that when h is regarded as a map from $C \otimes_A C$ to C, it is an A-map. Now $(M \otimes_A C) \otimes_A (N \otimes_A C) \xrightarrow{\text{canonical}} (M \otimes_A N) \otimes_A (C \otimes_A C)$ together with $C \otimes_A C \xrightarrow{h} C$ gives $(M \otimes_A C) \otimes_A (N \otimes_A C)$ $(M \otimes_A N) \otimes_A C$. The passage to homology is well-defined and yields the Pontrjagin product $H_m(A;M) \otimes_A H_n(A;N) \to H_{m+n}(A;M \otimes_A N)$.

One nice feature of the standard resolution $C_*(A)$ is that a suitable h can be explicitly described. Denote

$C_*(A) \otimes C_*(A)$ by $\mathcal{E}_*(A)$ and bear in mind throughout that UA is commutative. We have a map $h_{n,m}: C_n(A) \otimes C_m(A) \to C_{n+m}(A)$ defined by $h_{n,m}((u \otimes X) \otimes (v \otimes Y)) = uv \otimes (X \wedge Y)$ and this defines a map $h_*: \mathcal{E}_*(A) \to C_*(A)$ with $h_p: \mathcal{E}_p(A) \to C_p(A)$.

Proposition 8.1: The map h_* may be used to compute the Pontryagin product.

Proof: We first show that h_* satisfies (8.1);

$$h_{n,m}((x_1(u \otimes X) \otimes (v \otimes Y)) + ((u \otimes X) \otimes x_2 \cdot (v \otimes Y))) =$$

$$h_{n,m}(((x_1 u \otimes X) \otimes (v \otimes Y)) + ((u \otimes X) \otimes (x_2 v \otimes Y))) =$$

$$x_1 uv \otimes (X \wedge Y) + ux_2 v \otimes (X \wedge Y) =$$

$$(x_1+x_2)uv \otimes (X \wedge Y) \quad \text{(since UA is commutative)}$$

$$= (x_1+x_2) \cdot (uv \otimes (X \wedge Y))$$

$$= (x_1+x_2) \cdot h_{n,m}((u \otimes X) \otimes (v \otimes Y)).$$

We next see that h_* is indeed a chain map. The boundary δ_* in $\mathcal{E}_*(A)$ is defined by

$$\delta_n((u \otimes X)_p \otimes (v \otimes Y)_q) = (d_p(u \otimes X)) \otimes (v \otimes Y) + (-1)^p (u \otimes X) \otimes (d_q(v \otimes Y)).$$

The augmentation ε' is defined via

$$C_o(A) \otimes C_o(A) \xrightarrow{\varepsilon\otimes\varepsilon} k \otimes k \underset{\approx}{\xrightarrow{\text{mult}}} k;$$

we check commutativity of

$$\begin{array}{ccccc} E_o(A) & \xrightarrow{\varepsilon'} & k & \longrightarrow & 0 \\ h_o \downarrow & & \| & & \\ C_o(A) & \xrightarrow[\varepsilon]{} & k & \longrightarrow & 0 \end{array} :$$

$\varepsilon h_o(1 \otimes 1) = \varepsilon(1) = 1$ and $\varepsilon'(1 \otimes 1) = \varepsilon(1) \cdot \varepsilon(1) = 1$; on all other generators, both compositions are 0 since $\varepsilon(x) = 0$ and h_o is multiplication in UA.

Now look at

$$\begin{array}{ccc} E_n(A) & \xrightarrow{\delta_n} & E_{n-1}(A) \\ h_n \downarrow & & \downarrow h_{n-1} \\ C_n(A) & \xrightarrow[d_n]{} & C_{n-1}(A) \end{array} .$$

If $p+q = n$, $h_{n-1}\delta_n((u \otimes \langle x_1, \ldots, x_p\rangle) \otimes (v \otimes \langle x_{p+1}, \ldots, x_n\rangle)) =$

$$h_{n-1}(d_p(u \otimes \langle x_1, \ldots, x_p\rangle) \otimes (v \otimes \langle x_{p+1}, \ldots, x_n\rangle)$$

$$+ (-1)^p (u \otimes \langle x_1, \ldots, x_p\rangle) \otimes d_q(v \otimes \langle x_{p+1}, \ldots, x_n\rangle))$$

$$= \sum_{i=1}^{p} (-1)^{i+1} h_{n-1}((ux_i \otimes \langle x_1, \ldots, \hat{x}_i, \ldots, x_p\rangle) \otimes (v \otimes \langle x_{p+1}, \ldots, x_n\rangle))$$

$$+ \sum_{i=p+1}^{n} (-1)^{i+p+1} h_{n-1}((u \otimes \langle x_1, \ldots, x_p\rangle) \otimes (vx_i \otimes \langle x_{p+1}, \ldots, \hat{x}_i, \ldots, x_n\rangle))$$

$$= \sum_{i=1}^{n} (-1)^{i+1} uvx_i \otimes \langle x_1,\ldots,\hat{x}_i,\ldots,x_n\rangle$$

$$= \delta_n(uv \otimes \langle x_1,\ldots,x_n\rangle) = \delta_n h_n((u\otimes\langle x_1,\ldots,x_n\rangle) \otimes (v\otimes\langle x_{p+1},\ldots,x_n\rangle)).$$

Hence h_* is indeed a chain map. ■

We will use h_* in Chapter IX as a tool in the derivation of the main formula.

CHAPTER IX

THE MAIN FORMULA

We apply the results of Chapter VII to the case $M = H_N(I;A)$, $u = f^N$, in order to derive our main formula. Here, $\phi^{2,N-1}(d_2(u)) = \phi^{2,N-1}(\hat{d}_2(f^N))$. Since $\phi^{0,N}(f^N)$ is h^N in $H^0(\Phi;H^N(I;H_N(I;A)))$ and h^N corresponds to $g^N \in H^N(I;H_N(I;A))$, think of $\phi^{0,N}(f^N)$ as g^N, where of course $p_N(g^N)$ = identity map of $H_N(I;A)$. Hence in the notation of Chapter VII, we take the element f to be g^N. Once again, we are identifying cochains with cohomology, and chains with homology.

<u>Lemma 9.1:</u> $g^N(1_{UI} \otimes \langle x_1,\ldots,x_N\rangle) = 1_A \otimes \langle x_1,\ldots,x_N\rangle$

<u>Proof:</u> $1_A \otimes \langle x_1,\ldots,x_N\rangle = \{p_N(g^N)\}(1_A \otimes \langle x_1,\ldots,x_N\rangle)$

$$= 1_A\cdot\{g^N(1_{UI} \otimes \langle x_1,\ldots,x_N\rangle)\}$$

$$= g^N(1_{UI} \otimes \langle x_1,\ldots,x_N\rangle)$$

where the second equality identifies $A \otimes E_N(I)$ with $A \otimes_I C_N(I)$. ∎

From Chapter VIII, we have a Pontrjagin product $\mathbb{P}: H_n(\mathcal{I};N) \otimes_{\mathcal{I}} H_m(\mathcal{I};M) \to H_{n+m}(\mathcal{I};N \otimes_{\mathcal{I}} M)$. Since we consider N and M $\mathcal{I}$-trivial, so is $N \otimes_{\mathcal{I}} M$ and we consider $\mathbb{P}: H_n(\mathcal{I};N) \otimes H_m(\mathcal{I};M) \to H_{n+m}(\mathcal{I};N \otimes M)$. $\mathbb{P}$ is induced by the chain map h* in Chapter VIII and is given by

$$\mathbb{P}((n \otimes_{\mathcal{I}} (u \otimes X)) \otimes (m \otimes_{\mathcal{I}} (v \otimes Y))) = (n \otimes m) \otimes_{\mathcal{I}} h_{n,m}((u \otimes X) \otimes (x \otimes Y))$$

$$= (n \otimes m) \otimes_{\mathcal{I}} (uv \otimes (X \wedge Y)).$$

We now compute

$$\mathbb{P}: H_1(\mathcal{I};k) \otimes H_{N-1}(\mathcal{I};A) \to H_N(\mathcal{I};k \otimes A):$$

$$\mathbb{P}((1_k \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes x)) \otimes (a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1, \ldots, x_{N-1} \rangle))) =$$

$$(1_k \otimes a) \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x, x_1, \ldots, x_{N-1} \rangle).$$

Identify $k \otimes A$ with A as usual and identify $\mathcal{I}$ with $H_1(\mathcal{I};k) = k \otimes_{\mathcal{I}} C_1(\mathcal{I})$ via $x \leftrightarrow 1_k \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes x))$; this gives $\mathbb{P}: \mathcal{I} \otimes H_{N-1}(\mathcal{I};A) \to H_N(\mathcal{I};A)$ as

$$\mathbb{P}(x \otimes (a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1, \ldots, x_{N-1} \rangle))) = a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x, x_1, \ldots, x_{N-1} \rangle),$$

and a map $\mathbb{P}': \mathcal{I} \to \mathrm{Hom}_A(H_{N-1}(\mathcal{I};A), H_N(\mathcal{I};A))$ given by

$$(9.1)\quad \{\mathbb{P}'(x)\}(a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1, \ldots, x_{N-1} \rangle)) = a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x, x_1, \ldots, x_{N-1} \rangle)$$
$$= (-1)^{N-1} a \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1, \ldots, x_{N-1}, x \rangle).$$

Recall that $p_{N-1}^{-1}: \mathrm{Hom}_A(H_{N-1}(\mathcal{I};A),H_N(\mathcal{I};A)) \to H^{N-1}(\mathcal{I};H_N(\mathcal{I};A))$ is given by $\{p_{N-1}^{-1}(\beta)\}(c_{N-1}) = \beta(1_A \otimes c_{N-1})$. Thinking of $\beta \in \mathrm{Hom}_{\mathcal{I}}(C_{N-1}(\mathcal{I}),H_N(\mathcal{I};A))$ instead of in $\mathrm{Hom}(E_{N-1}(\mathcal{I}),H_N(\mathcal{I};A))$, $\{p_{N-1}^{-1}(\beta)\}(1_{U\mathcal{I}}\otimes \langle x_1,\ldots,x_{N-1}\rangle) = \beta(1_A \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle))$. Then

$$\{p_{N-1}^{-1}(\mathbb{P}'(x))\}(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle) = (-1)^{N-1} 1_A \otimes_{\mathcal{I}} (1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1},x\rangle)$$

$$= (-1)^{N-1} 1_A \otimes \langle x_1,\ldots,x_{N-1},x\rangle$$

in $A \otimes E_N(\mathcal{I})$. Denote $p_{N-1}^{-1}(\mathbb{P}'(x))$ by $\mathbb{P}''$:

(9.2) $\mathbb{P}''(x)(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle) = (-1)^{N-1} 1_A \otimes \langle x_1,\ldots,x_{N-1},x\rangle$

Now $\mathbb{P}'': \mathcal{I} \to H^{N-1}(\mathcal{I};H_N(\mathcal{I};A))$ is a Φ-module homomorphism :

$$\{\sigma\cdot\mathbb{P}''(x)\}(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle) = \sigma\cdot\{\mathbb{P}''(x)(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle$$

$$- \sum_{i=1}^{N-1} \mathbb{P}''(x)(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_{N-1}\rangle)$$

$$= (-1)^{N-1}\sigma\cdot(1_A \otimes \langle x_1,\ldots,x_{N-1},x\rangle)$$

$$- (-1)^{N-1} \sum_{i=1}^{N-1} 1_A \otimes \langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_{N-1},x\rangle$$

$$= (-1)^{N-1} \sum_{i=1}^{N-1} 1_A \otimes \langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_{N-1},x\rangle$$

$$+ (-1)^{N-1} 1_A \otimes \langle x_1,\ldots,x_{N-1},\sigma\cdot x\rangle$$

$$- (-1)^{N-1} \sum_{i=1}^{N-1} 1_A \otimes \langle x_1,\ldots,\sigma\cdot x_i,\ldots,x_{N-1},x\rangle$$

$$= (-1)^{N-1} 1_A \otimes \langle x_1,\ldots,x_{N-1},\sigma\cdot x\rangle$$

$$= \{\mathbb{P}''(\sigma.x)\}(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle).$$

If $N = 1$, we have

(9.3) $$\mathbb{P}''(x)(1_{U\mathcal{I}}) = 1_A \otimes x.$$

So $\mathbb{P}''$ induces $\mathbb{P}_*: H^2(\Phi;\mathcal{I}) \to H^2(\Phi;H^{N-1}(\mathcal{I};H_N(\mathcal{I};A)))$: if $\beta \in H^2(\Phi;\mathcal{I})$ is represented by b, $\mathbb{P}_*(\beta)$ is represented by b' where $b'(\sigma,\tau) = \mathbb{P}''(b(\sigma,\tau))$.

<u>Proposition 9.2</u>: $D_\alpha(h^N) = -\mathbb{P}_*(\alpha)$.

<u>Proof</u>: Recall that $h^N \in H^0(\mathcal{I};H^{N-1}(\mathcal{I};H_N(\mathcal{I};A)))$ corresponds to $f^N \in \hat{E}_2^{0,N}$, so $\phi^{2,N-1}(\hat{d}_2(f^N)) = D_\alpha(h^N)$. By corollaries 7.7 and 7.9, taking f to be g^N, a representative of $D_\alpha(h^N)$ is X where

$$\{X(\sigma,\tau)\}(1_{U\mathcal{I}}) = g^N(1_{U\mathcal{I}} \otimes a(\tau,\sigma)) \text{ if } N = 1$$

$$\{X(\sigma,\tau)\}(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1}\rangle) = (-1)^N g^N(1_{U\mathcal{I}} \otimes \langle x_1,\ldots,x_{N-1},a(\sigma,\tau)\rangle)$$

if $N > 1$. By lemma 9.1, this becomes

$$\{X(\sigma,\tau)\}(1_{UI}) = 1_A \otimes a(\tau,\sigma) \quad \text{if } N = 1$$

(9.4) $$\{X(\sigma,\tau)\}(1_{UI} \otimes \langle x_1,\ldots,x_{N-1}\rangle)$$

$$= (-1)^N 1_A \otimes \langle x_1,\ldots,x_{N-1},a(\sigma,\tau)\rangle .$$

But $\mathbb{P}_*(\alpha)$ is represented by a' where $a'(\sigma,\tau) = \mathbb{P}''(a(\sigma,\tau))$. So by (9.2) and (9.3),

$$\{a'(\sigma,\tau)\}(1_{UI}) = 1_A \otimes a(\sigma,\tau) \text{ if } N = 1$$

(9.5) $$\{a'(\sigma,\tau)\}(1_{UI} \otimes \langle x_1,\ldots,x_{N-1}\rangle)$$

$$= (-1)^{N-1} 1_A \otimes \langle x_1,\ldots,x_{N-1},a(\sigma,\tau)\rangle .$$

Comparing (9.4) and (9.5) yields $X = -a'$. ∎

If we identify $D_\alpha(h^N)$ and $\hat{d}_2(f^N)$ via ϕ, combining propositions 9.2 and 6.4 yield the main formula.

Theorem 9.3: For $x \in E_2^{p,N}$, $d_2(x) = (-1)^{p+1}\theta(x) \cup \mathbb{P}_*(\alpha)$.

CHAPTER X

CONCLUDING REMARKS

First and foremost, it is hoped that this spectral sequence can be put to some practical use in cohomology computations.

The vanishing of d_2 for the semidirect product is an interesting phenomenon. Charlap and Vasquez have shown in [4] that this does not always happen for free abelian groups, although it is quite difficult to find examples with $d_2 \neq 0$.

Looking at the problem from a different perspective, both group cohomology and Lie algebra cohomology are special cases of the theory of cohomology of supplemented algebras. We would hope that the problem could be considered in this more general setting and that the answer would include both the results in this thesis and of Charlap and Vasquez. Going one step further, one could consider the problem in the setting of associative algebra cohomology, which itself includes the cohomology of supplemental algebras.

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