Invariant Subspaces of Shift Operators for the Quarter Plane

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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In this paper we decide when two shift operators on $H^2(\mathbb{D}^2)$, the Hardy space, restricted to some invariant subspace, of finite co-dimension, are unitarily equivalent. To such pair of shift operators, there is a naturally associated hermitian holomorphic vector bundle. We use techniques of complex geometry introduced by Cowen and Douglas. Our associated hermitian holomorphic line bundle is holomorphically trivial. In finding a global holomorphic cross-section of the line bundle, we made critical use of a basis for $H^2(\mathbb{D}^2)$, other than the usual one. Using this cross-section, the curvature of the associated line bundle was

computed. We use a theorem of Cowen and Douglas to prove our result.

To my mother, Harbai. To my wife, Michele.

Table of Contents

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I. INTRODUCTION.

In a beautiful paper [3], Beurling studied the invariant subspaces for the unilateral shift operator. He proved that a closed subspace M of $H^2(\mathbb{D})$, the Hardy space, is invariant for T_z, multiplication by the co-ordinate function z on the unit disc $\mathbb{D}$ in $\mathbb{C}$, if and only if $M = \varphi H^2(\mathbb{D})$ where $|\varphi| = 1$ a.e. on $\mathbb{T} = \partial\mathbb{D}$, that is, φ is an inner function. R.G. Douglas (c.f. [6]) has observed that the collection of operators obtained by restricting T_z to its non-zero invariant subspaces are all unitarily equivalent to T_z and has given a proof of Beurling's result based on his observation.

What are the invariant subspaces of $H^2(\mathbb{D}^2)$? Here invariant subspace means invariant under each T_{z_i}, multiplication by the co-ordinate function z_i on the bi-disc $\mathbb{D}^2$ in $\mathbb{C}^2$, for $i = 1,2$. The obvious generalization of Beurling's theorem for $H^2(\mathbb{D}^2)$ fails, that is, it is known (c.f. [8]) that there is an invariant subspace which is not of the form $\varphi H^2(\mathbb{D}^2)$ for any inner function φ. An explicit description of or determining the invariant subspaces of $H^2(\mathbb{D}^2)$ is, it seems, a difficult problem. However, seeking a model for the operators T_{z_i} on $H^2(\mathbb{D}^2)$ restricted to its non-zero invariant subspaces may help to understand the nature of the invariant subspace. To be explicit, let $\mathfrak{a}$ be a subalgebra of $\mathfrak{L}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and let Lat($\mathfrak{a}$) be the lattice of invariant subspaces for $\mathfrak{a}$. One is interested in determining $\mathfrak{m}(\text{Lat}(\mathfrak{a}))$, the space of equivalent representations, that is, algebra homomorphisms from $\mathfrak{a}$ to $\mathfrak{a}_{|M}$ which maps T in $\mathfrak{a}$ to $T_{|M}$, the restriction of T to M, for M in Lat($\mathfrak{a}$). In this generality, it is unlikely to get a usable model for $\mathfrak{m}(\text{Lat}(\mathfrak{a}))$. However, for natural classes of operators, it is not unreasonable to expect a good model for restriction operators. This is evidenced by Douglas' observation of Beurling's theorems in this case for $\mathfrak{a} = \mathfrak{a}(T_z)$, the subalgebra generated by T_z in $\mathfrak{L}(H^2(\mathbb{D}))$, the space $\mathfrak{m}(\text{Lat}(\mathfrak{a}))$ is given by a point.

In seeking models for the operators T_{z_i}, on $H^2(\mathbb{D}^2)$, restricted to its invariant subspace, one possibility is to consider ideals I in $\mathbb{C}[z_1,z_2]$, the algebra of polynomials in two complex variables. If [I] denotes its closures in the Hardy space $H^2(\mathbb{D}^2)$, then [I] is invariant for multiplication by $\mathcal{P}(\mathbb{D}^2)$, the algebra of polynomials in $\mathbb{D}^2$, then $\mathcal{P}(\mathbb{D}^2)_{|[I]}$ is a restriction representation of $\mathcal{P}(\mathbb{D}^2)$. In this case ideals in $\mathbb{C}[z_1,z_2]$ provide a model for the restriction of $\mathcal{P}(\mathbb{D}^2)$ to some invariant subspace. However, not all restriction representation of $\mathcal{P}(\mathbb{D}^2)$ arise from ideals. This follows from the fact that invariant subspaces arising from ideals are all finitely generated and $H^2(\mathbb{D}^2)$ has an invariant subspace which is not finitely generated (c.f. [8]). Which invariant subspaces arise from ideals? In this direction, Ahern and Clark [1] proved: If M is an invariant subspace of $H^2(\mathbb{D}^2)$, of finite co-dimension, then there is an ideal I in $\mathbb{C}[z_1,z_2]$ such that $M = [I]$. Hence for invariant subspaces of $H^2(\mathbb{D}^2)$, of finite co-dimension, the model for the restriction representation of $\mathcal{P}(\mathbb{D}^2)$ is given by ideals in $\mathbb{C}[z_1,z_2]$.

It is not known when different ideals give rise to inequivalent restriction representation of $\mathcal{P}(\mathbb{D}^2)$. However, in a few cases this is known. For example, let $0 \le p_1 < p_2 < \ldots < p_r$ and $0 \le q_r < q_{r-1} < \ldots < q_1$ be integers, and let A be a finite subset of $\mathbb{D}^2$, and let

$$I^A_{p,q} = \{f \in \mathbb{C}[z_1,z_2] : \frac{\partial^{i+j} f}{\partial z_1^i \partial z_2^j}(\lambda) = 0 \text{ for each } \lambda \text{ in } A; \quad i \le p_k,\ j \le q_k,\ 1 \le k \le n\}.$$

Note that $V(I^A_{p,q})$, the set of common zeros of polynomials in $I^A_{p,q}$, is equal to the set A. In the case when the set A consists of just the origin; Berger, Coburn and Lebow [2] showed that all the restriction representations are inequivalent, that is, the representation $\mathcal{P}(\mathbb{D}^2) \to \mathcal{P}(\mathbb{D}^2)|_{[I^{\{0\}}_{p,q}]}$ is unitarily equivalent to the representation $\mathcal{P}(\mathbb{D}^2)|_{[I^{\{0\}}_{\tilde{p},\tilde{q}}]}$ if and only if $p_i = \tilde{p}_i$ and $q_i = \tilde{q}_i$. In [4] arising from ideals are all finitely generated and $H^2(\mathbb{D}^2)$ has an invariant subspace which is not finitely generated (c.f. [8]). Which invariant subspaces arise from ideals? In this direction, Ahern and Clark [1] proved: If M is an invariant subspace of $H^2(\mathbb{D}^2)$, of finite co-dimension, then there is an ideal I in $\mathbb{C}[z_1,z_2]$ such that $M = [I]$. Hence for invariant subspaces of $H^2(\mathbb{D}^2)$, of finite co-dimension, the model for the restriction representation of $\mathcal{P}(\mathbb{D}^2)$ is given by ideals in $\mathbb{C}[z_1,z_2]$.

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Cowen and Douglas gave an alternate proof of this result based on their techniques of complex geometry. In this thesis we generalize this result to the case where the set A consists of one non-zero point. We prove that the representation $\mathcal{P}(\mathbb{D}^2) \to \mathcal{P}(\mathbb{D}^2)\big|_{[I_{p,q}^{\{\lambda\}}]}$ is unitarily equivalent to the representation $\mathcal{P}(\mathbb{D}^2) \to \mathcal{P}(\mathbb{D}^2)\big|_{[I_{\tilde{p},\tilde{q}}^{\{\beta\}}]}$ if and only if $\lambda = \beta$, $p_i = \tilde{p}_i$, $q_i = \tilde{q}_i$ $i = 1\ldots r$ where β is in $\mathbb{D}^2$, that is, all restriction representation of $\mathcal{P}(\mathbb{D}^2)$ are inequivalent. Some of our results generalize to polydisc in $\mathbb{C}^n$. We were unable to prove that the restriction representation of $\mathcal{P}(\mathbb{D}^2)$ are inequivalent if the set A contains more than one point.

CHAPTER I.

In this section we state some of the known facts we need for our purposes. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$.

Definition 1.1: Let Ω be an open connected set in $\mathbb{C}^m$, and let $T_1,\ldots,T_m$ be operators in $\mathcal{L}(\mathcal{H})$. Given an integer $n \geq 1$, we say that $T = (T_1,\ldots,T_m)$ is in $\mathcal{B}_n(\Omega)$ if the following conditions are satisfied:

(1) $\{T_i\}_{i=1}^m$ are pairwise commuting.

(2) ran $D_{T-\lambda}$ is closed for λ in Ω where $D_T : \mathcal{H} \to \underbrace{\mathcal{H}\oplus\ldots\oplus\mathcal{H}}_{m\text{-times}}$ defined by

$D_T x = T_1 x\oplus\ldots\oplus T_m x.$

(3) span$\{\text{Ker } D_{T-\lambda} : \lambda \text{ is in } \Omega\}$ is dense in $\mathcal{H}$.

(4) dim Ker $D_{T-\lambda} = n$ for all λ in Ω.

The class $\mathcal{B}_n(\Omega)$ for $m = 1$ was introduced and studied by Cowen and Douglas in [4] and for $m \geq 2$ by the same authors in a subsequent paper [5], and more recently by Curto and Salinas in [7].

Definition 1.2: Let Ω be a complex manifold and let n be integer ≥ 1. A holomorphic vector bundle of rank n consists of a complex manifold E with a holomorphic map π from E onto Ω such that each fibre $E_\lambda = \pi^{-1}(\lambda)$ is isomorphic to $\mathbb{C}^n$ and such that for each λ_0 in Ω there is an open set U containing λ_0 and holomorphic functions $s_1,\ldots,s_n$ from U to E such that $\{s_1(\lambda),\ldots,s_n(\lambda)\}$ forms a basis for E_λ for all λ in U. A holomorphic cross-section of E is a holomorphic map $s : \Omega \to E$ such that $s(\lambda)$ is in E_λ for each λ in Ω. For $T = (T_1,\ldots,T_m)$ in $\mathcal{B}_n(\Omega)$, let (E_T,π) denote the subbundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by $E_T = \{(\lambda,x) \in \Omega \times \mathcal{H} : x \in \operatorname{Ker} D_{T-\lambda}\}$, $\pi(\lambda,x) = \lambda$. That E_T is a holomorphic vector bundle of rank n follows from the following:

<u>Lemma 1.3</u>: Let $\Omega \subset \mathbb{C}^m$ be an open connected set and let $\mathcal{H}_1,\mathcal{H}_2$ be Hilbert spaces. Let $X : \Omega \to \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ be holomorphic, that is, it can be defined locally by a power series, with coefficients in $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$, which converges in norm. Let $\lambda_0 \in \Omega$ be such that ran $X(\lambda_0)$ is closed and $\dim \operatorname{Ker} X(\lambda) = n$ for λ near λ_0. Then there exist holomorphic $\mathcal{H}_1$-valued functions $s_1,\ldots,s_n$ defined in some neighborhood Ω_0 of λ_0 such that $\{s_1(\lambda),\ldots,s_n(\lambda)\}$ forms a basis for $\operatorname{Ker} X(\lambda)$ for each λ in Ω_0.

<u>Proof</u>: See Cowen and Douglas [5], page 16 or Curto and Salinas [7], page 8.

In order to study simultaneous unitary equivalence we need some more notions from complex geometry.

Definition 1.4: A hermitian holomorphic vector bundle E over Ω is a holomorphic vector bundle such that each fibre E_λ is an inner product space. The bundle is said to have smooth (real analytic) metric if $\lambda \to \|s(\lambda)\|^2$ is smooth (real analytic) for each holomorphic cross-section of E.

1.5: Let E be a hermitian holomorphic vector bundle over Ω. A connection on E is a first order differential operator $D : \mathcal{E}(\Omega,E) \to \mathcal{E}^1(\Omega,E)$ such that $D(f\sigma) = df \otimes \sigma + fD\sigma$ for f in $\mathcal{E}(\Omega)$ and σ in $\mathcal{E}(\Omega,E)$, where $\mathcal{E}(\Omega)$ denotes the algebra of complex valued C^∞-functions on Ω and $\mathcal{E}^p(\Omega,E)$ denotes the spaces of smooth differential p-forms with coefficients in E, that is, $\mathcal{E}^p(\Omega,E) = \mathcal{E}(\Omega,\wedge^p T^*(\Omega)\otimes_{\mathbb{C}} E)$. Now given a connection D on a hermitian holomorphic vector bundle E over Ω, we define an operator $D : \mathcal{E}^p(\Omega,E) \to \mathcal{E}^{p+1}(\Omega,E)$ by using Leibnitz's rule

$$D(f\otimes\sigma) = df \otimes \sigma + (-1)^p f \wedge D\sigma$$

$$\text{for } f \text{ in } \mathcal{E}^p(\Omega) = \mathcal{E}(\Omega,\wedge^p T^*(\Omega)),$$

a p-form on Ω and σ in $\mathcal{E}(\Omega,E)$. An easy calculation shows that $D^2(f\sigma) = f(D^2\sigma)$ for f in $\mathcal{E}(\Omega)$ and σ in $\mathcal{E}(\Omega,E)$.

Thus D^2 is a bundle map from E to $\wedge^2 T^*(\Omega) \otimes E$ and we define the curvature $K(E,D) = K$ as the C^∞-section of

$$\mathrm{Hom}(E,\wedge^2 T^*(\Omega)\otimes E) \text{ by } K = K(E,D) = D^2.$$

For more complete treatment see Wells [9].

How is simultaneous unitary equivalence between two m-tuples of operators $T = (T_1,\ldots,T_m)$ and $\tilde{T} = (\tilde{T}_1,\ldots,\tilde{T}_m)$ in $\beta_n(\Omega)$ related to the associated hermitian holomorphic vector bundle E_T and $E_{\tilde{T}}$? The relation is given by the following:

<u>Proposition 1.6</u>: Let $T = (T_1,\ldots,T_m)$ and $\tilde{T} = (\tilde{T}_1,\ldots,\tilde{T}_m)$ be in $\beta_n(\Omega)$. Then T and $\tilde{T}$ are similtaneously unitarily equivalent if and only if E_T and $E_{\tilde{T}}$ are holomorphically and isometrically equivalent, that is, there exists an isometric holomorphic bundle map from E_T onto $E_{\tilde{T}}$.

<u>Proof</u>: See Cowen and Douglas [5], page 16.

For operators in $\beta_1(\Omega)$, the simultaneous unitary equivalence is related to the curvature of the associated line bundles as the following proposition shows.

<u>Proposition 1.7</u>: Let $T = (T_1,\ldots,T_m)$ and $\tilde{T} = (\tilde{T}_1,\ldots,\tilde{T}_m)$ be in $\beta_1(\Omega)$. Then T and $\tilde{T}$ are simultaneously unitarily equivalent if and only if the curvatures of the associated line bundles are equal.

<u>Proof</u>: See Cowen and Douglas [5], page 16-17.

CHAPTER II.

In this section we state and prove our main result.

2.1. Let $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| < 1 \quad i = 1,2\}$ be the bi-disc in $\mathbb{C}^2$. We let $H^2(\mathbb{D}^2)$ denote the class of holomorphic functions on $\mathbb{D}^2$ which satisfy the following condition:

$$\sup_{0 \le r < 1} \int_{\mathbb{T}^2} |f_r|^2 dm_2 < \infty$$ where $\mathbb{T}^2$ is the distinguished boundary

of $\mathbb{D}^2$, dm_2 is the normalized Lebesque measure on $\mathbb{T}^2$ and $f_r(z) = f(rz_1, rz_2)$ for $z = (z_1, z_2)$ in $\mathbb{T}^2$.

Proposition 2.2: For f in $H^2(\mathbb{D}^2)$, $f^*(z) = \lim_{r \to 1} f_r(z)$ exists a.e. on $\mathbb{T}^2$ and the following are true:

(a) f^* is in $L^2(\mathbb{T}^2)$ and $f_r \to f$ in $L^2(\mathbb{T}^2)$

(b) If $f(z) = \sum_{m,n \ge 0} c_{mn} z_1^m z_2^n$ is the Taylor expansion of f in $H^2(\mathbb{D}^2)$ and

$f^*(e^{i\theta_1}, e^{i\theta_2}) = \sum_{m,n \in \mathbb{Z}^2} a_{mn} e^{im\theta_1} e^{in\theta_2}$ is the Fourier expansion of f^* in $L^2(\mathbb{T}^2)$ then

$c_{mn} = a_{mn}$ for $m,n \ge 0$ and $a_{m,n} = 0$ otherwise.

Proof: See Rudin [8].

Definition 2.3: Let

$$H^2(\mathbb{T}^2) = \{f \in L^2(\mathbb{T}^2)$$

$$: a_{m,n} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2})e^{-im\theta_1}e^{-in\theta_2}dm_2$$

$$= 0 \text{ for } m < 0 \text{ or } n < 0\}.$$

Note that $H^2(\mathbb{T}^2)$ is a closed subspace of $L^2(\mathbb{T}^2)$ and hence a Hilbert space.

Proposition 2.4: The map from $H^2(\mathbb{D}^2) \to H^2(\mathbb{T}^2)$ given by $f \to f^*$ is an isometrical onto isomorphism.

Proof: See Rudin [8].

Under this identification we treat $H^2(\mathbb{D}^2)$ as a closed subspace of $L^2(\mathbb{T}^2)$. For more detailed study of these concepts see Rudin [8].

Let $0 \le p_1 < p_2 < \ldots < p_r$ and $0 \le q_r < q_{r-1} < \ldots < q_1$ be integers and let $\lambda = (\lambda_1, \lambda_2)$ be a point in $\mathbb{D}^2$.

Definition 2.5: We denote $\mathfrak{m}_\lambda^{(p,q)} = \{f \in H^2(\mathbb{D}^2) : \frac{\partial^{i+j} f}{\partial z_1^i \partial z_2^j}(\lambda) = 0$

for $i \le p_k$, $j \le q_k$

all $k, 1 \le k \le r\}$

Observe that $\mathfrak{m}_\lambda^{(p,q)}$ is a closed subspace of $H^2(\mathbb{D}^2)$.

<u>Definition 2.6</u>: We define S_i on $\mathfrak{m}_\lambda^{(p,q)}$ by $S_i f = P_{\mathfrak{m}_\lambda^{(p,q)}}(\bar{z}_i f)$, $i = 1,2$ for f in $\mathfrak{m}_\lambda^{(p,q)}$, where $P_{\mathfrak{m}_\lambda^{(p,q)}}$ is the orthogonal projection on $H^2(\mathbb{D}^2)$ onto $\mathfrak{m}_\lambda^{(p,q)}$ and z_1, z_2 are independent variables.

Note that S_1 and S_2 are bounded linear operators on $\mathfrak{m}_\lambda^{(p,q)}$ and depend not only on the point λ but also on p's and q's.

2.7: Let $\mathcal{H}$ be a functional Hilbert space, that is, $\mathcal{H}$ is Hilbert space of complex valued functions on a non-empty set X such that the evaluation map $f \to f(y)$ is a bounded linear functional for each y in X. Consequently, by the Riesz Representation Theorem, there exists, for each y in X, an element K_y in $\mathcal{H}$ such that $f(y) = \langle f, K_y \rangle$, where $\langle \ \rangle$ denotes the inner product in $\mathcal{H}$. The function K on $X \times X$ defined by $K(x,y) = K_y(x)$ is called the kernel function for $\mathcal{H}$. Observe that $H^2(\mathbb{D}^2)$ is a functional Hilbert space. It's kernel function is given by $K_\omega(z) = \dfrac{1}{(1-\bar{\omega}_1 z_1)(1-\bar{\omega}_2 z_2)}$ for each $\omega \in \mathbb{D}^2$. $P_{\mathfrak{m}_\lambda^{(p,q)}} K_\omega$ is the kernel function for $\mathfrak{m}_\lambda^{(p,q)}$ as can be seen quite easily.

In order to study the pair (S_1, S_2), we require a basis for $H^2(\mathbb{D}^2)$ other than the usual one.

Proposition 2.8: For λ in $\mathbb{D}$, the unit disc in $\mathbb{C}$, the functions defined by $e_m(z) = \dfrac{\sqrt{1-|\lambda|^2}(z-\lambda)^m}{(1-\bar{\lambda}z)^{m+1}}$ form a complete orthonormal basis for $H^2(\mathbb{D})$.

Proof: Suppose $m > n$. Then

$$\langle e_m, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right)^m \frac{\sqrt{1-|\lambda|^2}}{(1-\bar{\lambda}e^{i\theta}}\ \overline{\left(\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right)^n}\frac{\sqrt{1-|\lambda|^2}}{1-\lambda\bar{e}^{i\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right)^{m-n}\frac{(1-|\lambda|^2)}{1-\bar{\lambda}e^{i\theta}}\ \frac{1}{1-\lambda\bar{e}^{i\theta}}\,d\theta$$

$$= \langle f, K_\lambda\rangle = f(\lambda) = 0 \text{ where}$$

$f(z) = \left(\dfrac{z-\lambda}{1-\bar{\lambda}z}\right)^{m-n}\dfrac{(1-|\lambda|^2)}{1-\bar{\lambda}z}$ is in $H^2(\mathbb{D})$. This shows that $\{e_m\}$ is an orthogonal family. Now

$$\|e_m\|^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right|^{2m}\frac{(1-|\lambda|^2)}{|1-\bar{\lambda}e^{i\theta}|^2}\,d\theta = 1 \text{ since } \left|\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right| = 1$$

and $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\frac{1}{|1-\bar{\lambda}e^{i\theta}|^2}\,d\theta = \dfrac{1}{1-|\lambda|^2}$ where K_λ is the kernel function for $H^2(\mathbb{D})$ defined by $K_\lambda(z) = \dfrac{1}{1-\bar{\lambda}z}$. This shows that $\{e_m\}$ is an orthonormal family. It remains to show that this family is complete, that is, if $\langle f, e_m\rangle = 0$ for all $m \geq 0$ then $f \equiv 0$. To show this we prove that such an f has a zero of infinite order at λ; and since f is holomorphic on the open connected set $\mathbb{D}$, f is identically equal to zero. Now we claim that if $\langle f,e_j\rangle = 0$ for $j = 0,1,\ldots,n$ then f has a zero, of order at least $n + 1$, at λ. We use induction. This is obviously true for $n = 0$ since $\langle f,e_0\rangle = 0$, then $0 = \langle f,e_0\rangle = \sqrt{1-|\lambda|^2}\, f(\lambda)$ and hence f has a zero, of order ≥ 1, at λ. Assume $\langle f,e_j\rangle = 0$ for $j = 0,\ldots,n$, then f has a zero, of order $\geq n + 1$, at λ. Suppose $\langle f,e_j\rangle = 0$ for $j = 0,1,\ldots,n+1$.

Then $$0 = \langle f,e_j\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{\left(\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right)^j}\,\overline{\frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}e^{i\theta}}}\,d\theta$$

$j = 0,1,\ldots,n+1$

$$= \frac{(1-|\lambda|^2)}{2\pi i}\int_{\mathbb{T}} \frac{f(z)(1-\bar{\lambda}z)^j}{(z-\lambda)^{j+1}}\,dz = \frac{\sqrt{1-|\lambda|^2}}{j!}g^{(j)}(\lambda)$$

by the Cauchy integral formula, for $j = 0,1,\ldots,n+1$, where $g(z) = f(z)(1-\bar{\lambda}z)^j$

$$= \frac{\sqrt{1-|\lambda|^2}}{j!}\sum_{k=0}^{j}\binom{j}{k}f^{(k)}(\lambda)h_j^{(j-k)}(\lambda)$$

by Leibnitz's rule, where $h_j(z) = (1-\bar{\lambda}z)^j$ for $j=0,1,\ldots,n+1$.

But by the induction hypothesis, f has a zero, of order $\geq n + 1$, at λ, that is, $f^{(k)}(\lambda) = 0$ for $k = 0,1,\ldots,n$.

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Then $$0 = \langle f,e_j\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{\left(\frac{e^{i\theta}-\lambda}{1-\bar{\lambda}e^{i\theta}}\right)^j}\,\frac{\sqrt{1-|\lambda|^2}}{1-\lambda e^{-i\theta}}\,d\theta$$

$j = 0,1,\ldots,n+1$

$$= \frac{(1-|\lambda|^2)}{2\pi i}\int_{\mathbb{T}} \frac{f(z)(1-\bar{\lambda}z)^j}{(z-\lambda)^{j+1}}\,dz = \frac{\sqrt{1-|\lambda|^2}}{j!}g^{(j)}(\lambda)$$

by the Cauchy integral formula, for $j = 0,1,\ldots,n+1$, where $g(z) = f(z)(1-\bar{\lambda}z)^j$

$$= \frac{\sqrt{1-|\lambda|^2}}{j!}\sum_{k=0}^{j}\binom{j}{k}f^{(k)}(\lambda)h_j^{(j-k)}(\lambda)$$

by Leibnitz's rule, where $h_j(z) = (1-\bar{\lambda}z)^j$ for $j=0,1,\ldots,n+1$.

But by the induction hypothesis, f has a zero, of order $\geq n + 1$, at λ, that is, $f^{(k)}(\lambda) = 0$ for $k = 0,1,\ldots,n$.

Hence $\quad 0 = \langle f, e_{n+1}\rangle = \frac{\sqrt{1-|\lambda|^2}}{(n+1)!} \sum_{k=0}^{n+1} f^{(k)}(\lambda) h_{n+1}^{(n+1-k)}(\lambda)$

$$= \frac{\sqrt{1-|\lambda|^2}}{(n+1)!} f^{(n+1)}(\lambda) h_{n+1}(\lambda)$$

which implies $f^{(n+1)}(\lambda) = 0$

since $h_{n+1}(\lambda) = (1-|\lambda|^2)^{n+1} \neq 0$, proving what was required.

Corollary 2.9: For $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$, the family $\{e_{mn}\}_{m,n\geq 0}$ is a basis for $H^2(\mathbb{D}^2)$ where

$$e_{mn}(z) = \frac{\sqrt{(1-|\lambda_1|^2)(1-|\lambda_2|^2)}\ \bar{\lambda}_1^m \bar{\lambda}_2^n (z_1-\lambda_1)^m (z_2-\lambda_2)^n}{\lambda_1^m \lambda_2^n (1-\bar{\lambda}_1 z_1)^{m+1} (1-\bar{\lambda}_2 z_2)^{n+1}} .$$

Proof: Proposition 2.8. shows that the family $\{e_m\}$ where

$e_m(z) = \dfrac{\sqrt{1-|\lambda_1|^2}(z-\lambda_1)^m}{(1-\bar{\lambda}_1 z)^{m+1}}$ is a basis for $H^2(\mathbb{D})$ and hence

$\{f_m\}$ is also a basis for $H^2(\mathbb{D})$ where

$f_m(z) = \sqrt{(1-|\lambda_1|^2)}\ \dfrac{\bar{\lambda}_1^m (z-\lambda_1)^m}{\lambda_1^m (1-\bar{\lambda}_1 z)^{m+1}} = \dfrac{\bar{\lambda}_1^m}{\lambda_1^m} e_m(z)$. It follows

that $e_{mn}(z) = f_m(z_1) f_n(z_2)$ is a basis for $H^2(\mathbb{D}^2)$.

Corollary 2.10: $\{e_{m,n}\}_{\substack{m\geq p_k+1 \\ n\geq q_k+1}}$ $k = 1,\ldots,r$ is an orthonormal basis for $\mathfrak{M}_\lambda^{(p,q)}$ and $\{e_{m,n}\}_{\substack{m\leq p_k \\ n\leq q_k}}$ for all k, $1 \leq k \leq r$ is an orthonormal basis for $\mathfrak{m}_\lambda^{(p,q)^\perp}$ and dim $\mathfrak{m}_\lambda^{(p,q)^\perp}$

$$= \sum_{k=1}^{r} (q_k+1)(p_k - p_{k-1}) \text{ where } p_0 = -1.$$

Proof: This follows from the definition of $\mathfrak{m}_\lambda^{(p,q)}$ and corollary 2.9.

Proposition 2.11: The pair (S_1, S_2) is in $\mathcal{B}_1(\mathbb{D}^2 \setminus \{\lambda\})$.

Proof: The map $\eta(z) = \left(\frac{z_1-\lambda_1}{1-\bar{\lambda}_1 z_1}, \frac{z_2-\lambda_2}{1-\bar{\lambda}_2 z_2}\right)$ is a biholomorphic map from $\mathbb{D}^2$ onto itself. This map η induces a unitary operator $U : \mathfrak{m}_0^{(p,q)} \to \mathfrak{m}_\lambda^{(p,q)}$ defined by

$(Uf)(z) = \eta'(z)^{\frac{1}{2}} f(\eta(z))$ where

$$\eta'(z) = \det\left(\frac{\partial \tau_i}{\partial z_j}\right) i,j = 1,2 \qquad \tau_i(z) = \frac{z_i - \lambda_i}{1-\bar{\lambda}_i z_i} \quad i = 1,2.$$

We get the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{m}_0^{(p,q)} & \xrightarrow{D_S} & \mathfrak{m}_0^{(p,q)} \oplus \mathfrak{m}_0^{(p,q)} \\
U \Big\downarrow & & \Big\downarrow U \oplus U \\
\mathfrak{m}_\lambda^{(p,q)} & \xrightarrow{D_S} & \mathfrak{m}_\lambda^{(p,q)} \oplus \mathfrak{m}_\lambda^{(p,q)}
\end{array}$$

where $D_S f = S_1 f \oplus S_2 f$ acting on $\mathfrak{m}_0^{(p,q)}$ and $D_S f = S_1 f \oplus S_2 f$ acting on $\mathfrak{m}_\lambda^{(p,q)}$. Hence D_S acting on $\mathfrak{m}_\lambda^{(p,q)}$ is unitarily equivalent to D_S acting on $\mathfrak{M}_0^{(p,q)}$. But the pair (S_1,S_2) acting on $\mathfrak{M}_0^{(p,q)}$ is in $\mathcal{B}_1(\mathbb{D}^2\backslash\{0\})$ (see Cowen and Douglas [5] page 20). Hence the pair (S_1,S_2) acting on $\mathfrak{M}_\lambda^{(p,q)}$ is in $\mathcal{B}_1(\mathbb{D}^2\backslash\{\lambda\})$.

<u>Proposition 2.12</u>: Let $\Omega_0 \subset \Omega \subset \mathbb{C}^m$, Ω_0 connected bounded, then $\mathcal{B}_n(\Omega) \subset \mathcal{B}_n(\Omega_0)$.

<u>Proof</u>: See Cowen and Douglas [4], page 193.

We want to calculate the curvature of the associated line bundle E_S, for $S = (S_1,S_2)$ in $\mathcal{B}_1(\mathbb{D}^2\backslash\{\lambda\})$.

<u>Proposition 2.13</u>: $K_S(\omega)$, the curvature of the associated bundle E_S, for $S = (S_1,S_2)$ in $\mathcal{B}_1(\mathbb{D}^2\backslash\{\lambda\})$, is given by

$$K_S(\omega) = \bar{\partial}\partial \log\|K_\omega\|^2 + \bar{\partial}\partial \log F_{p,q,\lambda}(\omega) \text{ where}$$

$$F_{p,q,\lambda}(\omega) = \sum_{k=1}^{r+1} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

$$- \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

and $q_{r+1} = -1$

<u>Proof</u>: A holomorphic cross-section for the line bundle E_S is given by $P_{\mathfrak{M}_\lambda^{(p,q)}}K_\omega$. Hence the curvature for the

bundle E_S is given by $K_S(\omega) = -\partial\bar{\partial} \log\| P_{\mathcal{M}_\lambda(p,q)} K_\omega\|^2$

$= \bar{\partial}\partial \log\| P_{\mathcal{M}_\lambda(p,q)} K_\omega\|^2$. We want to compute the norm

$\| P_{\mathcal{M}_\lambda(p,q)} K_\omega\|^2$. Now by Corollary 2.10 a basis for $\mathcal{M}_\lambda^{(p,q)^\perp}$

is given by $\{e_{ij}\}$ $i \le p_k$, $j \le q_k$, $1 \le k \le r$ where e_{ij} is as in Corollary 2.9.

Hence
$$\| P_{\mathcal{M}_\lambda(p,q)^\perp} K_\omega\|^2 = \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} |\langle P_{\mathcal{M}_\lambda(p,q)^\perp} K_\omega, e_{ij}\rangle|^2$$

$$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} |e_{ij}(\omega)|^2 \text{ since } K_\omega$$

is the kernel function and e_{ij} are in $\mathcal{M}_\lambda^{(p,q)^\perp}$.

$$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \frac{1}{\|K_\lambda\|^2} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2i} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2j} \frac{1}{|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2}$$

$$= \frac{1}{\|K_\lambda\|^2|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2} \sum_{k=1}^{r} \{(\frac{\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}}{1-\left|\frac{\omega_1\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^2})$$

$$\times (\frac{1-\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{1-\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^2})\}$$

bundle E_S is given by $K_S(\omega) = -\partial\bar{\partial} \log\| P_{\mathfrak{m}_\lambda(p,q)} K_\omega \|^2$ $= \bar{\partial}\partial \log\| P_{\mathfrak{m}_\lambda(p,q)} K_\omega \|^2$. We want to compute the norm $\| P_{\mathfrak{m}_\lambda(p,q)} K_\omega \|^2$. Now by Corollary 2.10 a basis for $\mathfrak{m}_\lambda(p,q)^\perp$ is given by $\{e_{ij}\}$ $i \le p_k$, $j \le q_k$, $1 \le k \le r$ where e_{ij} is as in Corollary 2.9.

Hence
$$\| P_{\mathfrak{m}_\lambda(p,q)^\perp} K_\omega \|^2 = \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} |\langle P_{\mathfrak{m}_\lambda(p,q)^\perp} K_\omega, e_{ij} \rangle|^2$$
$$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} |e_{ij}(\omega)|^2 \text{ since } K_\omega$$

is the kernel function and e_{ij} are in $\mathfrak{m}_\lambda(p,q)^\perp$.

$$= \sum_{k=1}^{r} \sum_{i=p_{k-1}+1}^{p_k} \sum_{j=0}^{q_k} \frac{1}{\|K_\lambda\|^2} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2i} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2j} \frac{1}{|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2}$$

$$= \frac{1}{\|K_\lambda\|^2|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2} \sum_{k=1}^{r} \{ (\frac{\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}}{1 - \left|\frac{\omega_1\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^2})$$

$$\times \ (\frac{1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^2}) \}$$

by summing the geometric sequence

$$= \frac{\sum_{k=1}^{r} \{ (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2})(1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}) \}}{\|K_\lambda\|^2 |1-\bar{\lambda}_1\omega_1|^2 |1-\bar{\lambda}_2\omega_2|^2 (1 - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^2)(1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^2)}$$

$$= \frac{\sum_{k=1}^{r} (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}) - \sum_{k=1}^{r} (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}) \times \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{\|K_\lambda\|^2 (|1-\bar{\lambda}_1\omega_1|^2 - |\omega_1-\lambda_1|^2)(|1-\bar{\lambda}_2\omega_2|^2 - |\omega_2-\lambda_2|^2)} .$$

Simplifying both the numerator and the denominator we get

$$= \frac{1 - \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} + \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{(1-|\omega_1|^2)(1-|\omega_2|^2)}$$

$(q_{r+1} = -1)$

by summing the geometric sequence

$$= \frac{\sum_{k=1}^{r} \{ (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2})(1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}) \}}{\|K_\lambda\|^2 |1-\bar{\lambda}_1\omega_1|^2 |1-\bar{\lambda}_2\omega_2|^2 (1 - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^2)(1 - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^2)}$$

$$= \frac{\sum_{k=1}^{r} (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}) - \sum_{k=1}^{r} (\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} - \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}) \times \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{\|K_\lambda\|^2 (|1-\bar{\lambda}_1\omega_1|^2 - |\omega_1-\lambda_1|^2)(|1-\bar{\lambda}_2\omega_2|^2 - |\omega_2-\lambda_2|^2)} .$$

Simplifying both the numerator and the denominator we get

$$= \frac{1 - \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} + \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}}{(1-|\omega_1|^2)(1-|\omega_2|^2)}$$

$(q_{r+1} = -1)$

$= \|K_\omega\|^2(1-F_{p,q,\lambda}(\omega))$

$$\text{where}\quad F_{p,q,\lambda}(\omega) = \sum_{k=1}^{r+1}\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2}\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

$$- \sum_{k=1}\left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2}\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

and $q_{r+1} = -1$.

Now $\|P_{\mathfrak{m}_\lambda(p,q)}K_\omega\|^2 = \|K_\omega\|^2 - \|P_{\mathfrak{m}_\lambda(p,q)^\perp}K_\omega\|^2$

$$= \|K_\omega\|^2 - \|K_\omega\|^2(1-F_{p,q,\lambda}(\omega)) = \|K_\omega\|^2 F_{p,q,\lambda}(\omega)$$

which implies, after taking logarithms of both sides

$$\log\|P_{\mathfrak{m}_\lambda(p,q)}K_\omega\|^2 = \log\|K_\omega\|^2 + \log F_{p,q,\lambda}(\omega)$$

Hence $\quad K_S(\omega) = \bar{\partial}\partial \log\|K_\omega\|^2 + \bar{\partial}\partial \log F_{p,q,\lambda}(\omega).$

2.14: Let $S^{(p,q,\lambda)} = (S_1,S_2)$ and $S^{(\tilde{p},\tilde{q},\beta)} = (\tilde{S}_1,\tilde{S}_2)$ be two pairs of operators on $\mathfrak{m}_\lambda^{(p,q)}$ and $\mathfrak{m}_\beta^{(\tilde{p},\tilde{q})}$, respectively for $\lambda \neq \beta$ in $\mathbb{D}^2$ and let $0 \le \tilde{p}_1 < \ldots < \tilde{p}_s$, $0 \le \tilde{q}_s < \tilde{q}_{s-1} < \ldots < \tilde{q}_1$ be integers. Then by Proposition 2.11 $S^{(p,q,\lambda)}$ is in $B_1(\mathbb{D}^2\setminus\{\lambda\})$ and $S^{(\tilde{p},\tilde{q},\beta)}$ is in $B_1(\mathbb{D}^2\setminus\{\beta\})$. By Proposition 2.12 we obtain $S^{(p,q,\lambda)}$ and $S^{(\tilde{p},\tilde{q},\beta)}$ both are in $B_1(\mathbb{D}^2\setminus\{\lambda,\beta\})$. Now we

$= \|K_\omega\|^2(1-F_{p,q,\lambda}(\omega))$

$$\text{where } F_{p,q,\lambda}(\omega) = \sum_{k=1}^{r+1} \left|\frac{\omega_1-\lambda_1}{1-\overline{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\overline{\lambda}_2\omega_2}\right|^{2q_k+2}$$

$$- \sum_{k=1} \left|\frac{\omega_1-\lambda_1}{1-\overline{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\overline{\lambda}_2\omega_2}\right|^{2q_k+2}$$

and $q_{r+1} = -1$.

Now $\|P_{\mathfrak{m}_\lambda^{(p,q)}}K_\omega\|^2 = \|K_\omega\|^2 - \|P_{\mathfrak{m}_\lambda^{(p,q)\perp}}K_\omega\|^2$

$$= \|K_\omega\|^2 - \|K_\omega\|^2(1-F_{p,q,\lambda}(\omega)) = \|K_\omega\|^2 F_{p,q,\lambda}(\omega)$$

which implies, after taking logarithms of both sides

$$\log\|P_{\mathfrak{m}_\lambda^{(p,q)}}K_\omega\|^2 = \log\|K_\omega\|^2 + \log F_{p,q,\lambda}(\omega)$$

Hence $\quad K_S(\omega) = \overline{\partial}\partial \log\|K_\omega\|^2 + \overline{\partial}\partial \log F_{p,q,\lambda}(\omega)$.

2.14: Let $S^{(p,q,\lambda)} = (S_1,S_2)$ and $S^{(\tilde{p},\tilde{q},\beta)} = (\tilde{S}_1,\tilde{S}_2)$ be two pairs of operators on $\mathfrak{m}_\lambda^{(p,q)}$ and $\mathfrak{m}_\beta^{(\tilde{p},\tilde{q})}$, respectively for $\lambda \neq \beta$ in $\mathbb{D}^2$ and let $0 \le \tilde{p}_1 < \ldots < \tilde{p}_s$, $0 \le \tilde{q}_s < \tilde{q}_{s-1} < \ldots < \tilde{q}_1$ be integers. Then by Proposition 2.11 $S^{(p,q,\lambda)}$ is in $\mathcal{B}_1(\mathbb{D}^2\setminus\{\lambda\})$ and $S^{(\tilde{p},\tilde{q},\beta)}$ is in $\mathcal{B}_1(\mathbb{D}^2\setminus\{\beta\})$. By Proposition 2.12 we obtain $S^{(p,q,\lambda)}$ and $S^{(\tilde{p},\tilde{q},\beta)}$ both are in $\mathcal{B}_1(\mathbb{D}^2\setminus\{\lambda,\beta\})$. Now we state and prove our main result:

Theorem 2.15: Let (p,q) and $(\tilde{p},\tilde{q})$ and λ,β be as before. If the pair $S^{(p,q,\lambda)} = (S_1,S_2)$ is simultaneously unitarily equivalent to the pair $S^{(\tilde{p},\tilde{q},\beta)} = (S_1,S_2)$, then $\lambda = \beta$.

Proof: By the discussion preceding the theorem we see that both $S^{(p,q,\lambda)}$ and $S^{(\tilde{p},\tilde{q},\beta)}$ are in $\beta_1(\mathbb{D}^2\setminus\{\lambda,\beta\})$ and by Proposition 1.7 the curvatures of the associated line bundles are the same. But by Proposition 2.13 $K_{S^{(p,q,\lambda)}}(\omega)$, the curvature, is given by $K_{S^{(p,q,\lambda)}}(\omega) = \bar{\partial}\partial \log\|K_\omega\|^2 + \bar{\partial}\partial \log F_{p,q,\lambda}(\omega)$ and the corresponding curvature for $S^{(\tilde{p},\tilde{q},\beta)}$ has a similar expression. Now the equality of $K_{S^{(p,q,\lambda)}}(\omega)$ with $K_{S^{(\tilde{p},\tilde{q},\beta)}}(\omega)$ on $\mathbb{D}^2\setminus\{\lambda,\beta\}$ implies $\bar{\partial}\partial \log F_{p,q,\lambda}(\omega) = \bar{\partial}\partial \log F_{\tilde{p},\tilde{q},\beta}(\omega)$ on $\mathbb{D}^2\setminus\{\lambda,\beta\}$ and hence equality holds on $\mathbb{T}^2$ since $F_{p,q,\lambda}$ and $F_{\tilde{p},\tilde{q},\beta}$ are both real analytic in a neighborhood of $\mathrm{CL}\,\mathbb{D}^2\setminus\{\lambda,\beta\}$. Now $\bar{\partial}\partial \log F_{p,q,\lambda}(\omega) = \sum_{i,j=1}^{2} \frac{\partial^2 \log F_{p,q,\lambda}(\omega)}{\partial\bar{\omega}_i\partial\omega_j} d\bar{\omega}_i\wedge d\omega_j$ and hence we have

$$\frac{\partial^2 \log F_{p,q,\lambda}(\omega)}{\partial\bar{\omega}_i\partial\omega_j} = \frac{\partial^2 \log F_{\tilde{p},\tilde{q},\beta}}{\partial\bar{\omega}_i\partial\omega_j} \quad \text{on } \mathbb{T}^2 .$$

Recall that

$$F_{p,q,\lambda}(\omega) = \sum_{k=1}^{r+1} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

$$- \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} .$$

Rewriting we get

$$F_{p,q,\lambda}(\omega) = \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} + \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right).$$

Differentiating $F_{p,q,\lambda}$ with respect to $\bar{\omega}_1$, we get

$$\frac{\partial F_{p,q,\lambda}}{\partial \bar{\omega}_1} = \sum_{k=1}^{r} \frac{\partial}{\partial \bar{\omega}_1} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right)$$

$$= \frac{(1-|\lambda_1|^2)(\omega_1-\lambda_1)}{(1-\lambda_1\bar{\omega}_1)^2(1-\bar{\lambda}_1\omega_1)} \sum_{k=1}^{r} (p_k+1) \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k}$$

$$\times \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right)$$

Differentiating, once more, $\frac{\partial F_{p,q,\lambda}}{\partial \bar{\omega}_1}$ with respect to ω_2

$$F_{p,q,\lambda}(\omega) = \sum_{k=1}^{r+1} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_{k-1}+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2}$$

$$- \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} .$$

Rewriting we get

$$F_{p,q,\lambda}(\omega) = \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} + \sum_{k=1}^{r} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right).$$

Differentiating $F_{p,q,\lambda}$ with respect to $\bar{\omega}_1$, we get

$$\frac{\partial F_{p,q,\lambda}}{\partial \bar{\omega}_1} = \sum_{k=1}^{r} \frac{\partial}{\partial \bar{\omega}_1} \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k+2} \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right)$$

$$= \frac{(1-|\lambda_1|^2)(\omega_1-\lambda_1)}{(1-\lambda_1\bar{\omega}_1)^2(1-\bar{\lambda}_1\omega_1)} \sum_{k=1}^{r} (p_k+1) \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k}$$

$$\times \left(\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}+2} - \left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k+2} \right)$$

Differentiating, once more, $\dfrac{\partial F_{p,q,\lambda}}{\partial \bar{\omega}_1}$ with respect to ω_2

we get

$$\frac{\partial^2 F_{p,q,\lambda}}{\partial\omega_2 \partial\bar{\omega}_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(\omega_1-\lambda_1)(\bar{\omega}_2-\bar{\lambda}_2)}{(1-\lambda_1\bar{\omega}_1)^2(1-\lambda_2\bar{\omega}_2)(1-\bar{\lambda}_1\omega_1)(1-\bar{\lambda}_2\omega_2)^2}$$

$$\sum_{k=1}^{r} (p_k+1) \left|\frac{\omega_1-\lambda_1}{1-\bar{\lambda}_1\omega_1}\right|^{2p_k} \{(q_{k+1}+1)\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_{k+1}}$$

$$- (q_k+1)\left|\frac{\omega_2-\lambda_2}{1-\bar{\lambda}_2\omega_2}\right|^{2q_k} \}$$

Now $\dfrac{\partial F_{p,q,\lambda}}{\partial\bar{\omega}_1} = \dfrac{\partial F_{p,q,\lambda}}{\partial\omega_2} = 0$ on $\mathbb{T}^2$

as $\left|\dfrac{\omega_i-\lambda_i}{1-\bar{\lambda}_i\omega_i}\right| = 1$ when $|\omega_i| = 1$, $i = 1,2$

and $F_{p,q,\lambda} = 1$ on $\mathbb{T}^2$ for the same reason.

Differentiating $\log F_{p,q,\lambda}$ first with respect to $\bar{\omega}_1$ and then with respect to ω_2 we get

$$\frac{\partial^2 \log F_{p,q,\lambda}}{\partial\omega_2\partial\bar{\omega}_1} = \frac{1}{(F_{p,q,\lambda})^2}\{F_{p,q,\lambda} \frac{\partial^2 F_{p,q,\lambda}}{\partial\omega_2\partial\bar{\omega}_1} - \frac{\partial F_{p,q,\lambda}}{\partial\omega_2} \cdot \frac{\partial F_{p,q,\lambda}}{\partial\bar{\omega}_1}\}$$

$$= \frac{\partial^2 F_{p,q,\lambda}}{\partial\omega_2\partial\bar{\omega}_1} \quad \text{on } \mathbb{T}^2 \text{ since } F_{p,q,\lambda} = 1,$$

$$\frac{\partial F_{p,q,\lambda}}{\partial \bar{\omega}_1} = \frac{\partial F_{p,q,\lambda}}{\partial \omega_2} = 0 \qquad \text{on } \mathbb{T}^2.$$

Since

$$\frac{\partial^2 \log F_{p,q,\lambda}}{\partial \omega_i \partial \bar{\omega}_j} = \frac{\partial^2 \log F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_i \partial \bar{\omega}_j} \quad \text{for } i,j = 1,2 \qquad \text{on } \mathbb{T}^2$$

we have

$$\frac{\partial^2 F_{p,q,\lambda}}{\partial \omega_2 \partial \bar{\omega}_1} = \frac{\partial^2 F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_2 \partial \bar{\omega}_1} \qquad \text{on } \mathbb{T}^2 \qquad (1)$$

But

$$\frac{\partial^2 F_{p,q,\lambda}}{\partial \omega_2 \partial \bar{\omega}_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(\omega_1-\lambda_1)(\bar{\omega}_2-\bar{\lambda}_2)}{(1-\lambda_1\bar{\omega}_1)^2(1-\bar{\lambda}_2\omega_2)^2(1-\bar{\lambda}_1\omega_1)(1-\lambda_2\bar{\omega}_2)} \sum_{k=1}^{r} (p_k+1)(q_{k+1}-q_k)$$

and

$$\frac{\partial^2 F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_2 \partial \bar{\omega}_1} = \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)(\omega_1-\beta_1)(\bar{\omega}_2-\bar{\beta}_2)}{(1-\beta_1\bar{\omega}_1)^2(1-\bar{\beta}_2\omega_2)(1-\bar{\beta}_1\omega_1)(1-\beta_2\bar{\omega}_2)} \sum_{k=1}^{s} (\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k)$$

$$\text{on } \mathbb{T}^2.$$

Hence from (1) and using $|\omega_i| = 1$ for $i = 1,2$ we get

$$\frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)\omega_1\bar{\omega}_2 \sum_{k=1}^{r} (p_k+1)(q_{k+1}-q_k)}{|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2}$$

$$= \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)\omega_1\bar{\omega}_2 \sum_{k+1}^{s} (\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k)}{|1-\bar{\beta}_1\omega_1|^2|1-\bar{\beta}_2\omega_2|^2} \qquad \text{on } \mathbb{T}^2$$

$$\frac{\partial F_{p,q,\lambda}}{\partial \overline{\omega}_1} = \frac{\partial F_{p,q,\lambda}}{\partial \omega_2} = 0 \qquad \text{on } \mathbb{T}^2.$$

Since

$$\frac{\partial^2 \log F_{p,q,\lambda}}{\partial \omega_i \partial \overline{\omega}_j} = \frac{\partial^2 \log F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_i \partial \overline{\omega}_j} \quad \text{for } i,j = 1,2 \qquad \text{on } \mathbb{T}^2$$

we have

$$\frac{\partial^2 F_{p,q,\lambda}}{\partial \omega_2 \partial \overline{\omega}_1} = \frac{\partial^2 F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_2 \partial \overline{\omega}_1} \qquad \text{on } \mathbb{T}^2 \qquad (1)$$

But

$$\frac{\partial^2 F_{p,q,\lambda}}{\partial \omega_2 \partial \overline{\omega}_1} = \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)(\omega_1-\lambda_1)(\overline{\omega}_2-\overline{\lambda}_2)}{(1-\lambda_1\overline{\omega}_1)^2(1-\overline{\lambda}_2\omega_2)^2(1-\overline{\lambda}_1\omega_1)(1-\lambda_2\overline{\omega}_2)} \sum_{k=1}^{r} (p_k+1)(q_{k+1}-q_k)$$

and

$$\frac{\partial^2 F_{\tilde{p},\tilde{q},\beta}}{\partial \omega_2 \partial \overline{\omega}_1} = \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)(\omega_1-\beta_1)(\overline{\omega}_2-\overline{\beta}_2)}{(1-\beta_1\overline{\omega}_1)^2(1-\overline{\beta}_2\omega_2)(1-\overline{\beta}_1\omega_1)(1-\beta_2\overline{\omega}_2)} \sum_{k=1}^{s} (\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k)$$

$$\text{on } \mathbb{T}^2.$$

Hence from (1) and using $|\omega_i| = 1$ for $i = 1,2$ we get

$$\frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)\omega_1\overline{\omega}_2 \sum_{k=1}^{r} (p_k+1)(q_{k+1}-q_k)}{|1-\overline{\lambda}_1\omega_1|^2|1-\overline{\lambda}_2\omega_2|^2}$$

$$= \frac{(1-|\beta_1|^2)(1-|\beta_2|^2)\omega_1\overline{\omega}_2 \sum_{k+1}^{s} (\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k)}{|1-\overline{\beta}_1\omega_1|^2|1-\overline{\beta}_2\omega_2|^2} \qquad \text{on } \mathbb{T}^2$$

from which it follows that

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)|1-\bar{\beta}_1\omega_1|^2|1-\beta_2\omega_2|^2$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)|1-\lambda_1\omega_1|^2|1-\lambda_2\omega_2|^2 \quad \text{on } \mathbb{T}^2 \quad \ldots.(2)$$

where

$$c = c(p,q) = \sum_{k=1}^{r}(p_k+1)(q_{k+1}-q_k) \neq 0$$

and $$\tilde{c} = \tilde{c}(\tilde{p},\tilde{q}) = \sum_{k=1}^{s}(\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k) \neq 0 .$$

Now

$$|1-\bar{\lambda}_1\omega_1|^2|1-\bar{\lambda}_2\omega_2|^2 = \{(1+|\lambda_1|^2)-\bar{\lambda}_1\omega_1-\lambda_1\bar{\omega}_1\}\{(1+|\bar{\lambda}_2|^2)-\bar{\lambda}_2\omega_2-\lambda_2\bar{\omega}_2\}$$

$$= (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\bar{\lambda}_1(1+|\lambda_2|^2)\bar{\omega}_1$$

$$- \bar{\lambda}_1(1+|\lambda_2|^2)\omega_1-\lambda_2(1+|\lambda_1|^2)\omega_2-\lambda_2(1+|\lambda_1|^2)\bar{\omega}_2$$

$$+ \bar{\lambda}_1\bar{\lambda}_2\omega_1\omega_2+\lambda_1\lambda_2\bar{\omega}_1\bar{\omega}_2+\bar{\lambda}_1\lambda_2\omega_1\bar{\omega}_2+\lambda_1\bar{\lambda}_2\bar{\omega}_1\omega_2$$

(using $|\omega_i| = 1$).

Hence from (2) we get

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\{(1+|\beta_1|^2)(1+|\beta_2|^2)-\bar{\beta}_1(1+|\beta_2|^2)\bar{\omega}_1-\bar{\beta}_1(1+|\beta_2|^2)\bar{\omega}_2$$

$$- \bar{\beta}_2(1+|\beta_1|^2)\omega_2-\beta_2(1+|\beta_1|^2)\bar{\omega}_2+\bar{\beta}_1\bar{\beta}_2\omega_1\omega_2+\beta_1\beta_2\bar{\omega}_1\bar{\omega}_2+\bar{\beta}_1\beta_2\bar{\omega}_1\omega_2$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\ (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\bar{\lambda}_1(1+|\lambda_2|^2)\omega_1$$

from which it follows that

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\,|1-\overline{\beta}_1\omega_1|^2|1-\beta_2\omega_2|^2$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\,|1-\lambda_1\omega_1|^2|1-\lambda_2\omega_2|^2 \qquad \text{on } \mathbb{T}^2 \quad \ldots.(2)$$

where

$$c = c(p,q) = \sum_{k=1}^{r}(p_k+1)(q_{k+1}-q_k) \neq 0$$

and
$$\tilde{c} = \tilde{c}(\tilde{p},\tilde{q}) = \sum_{k=1}^{s}(\tilde{p}_k+1)(\tilde{q}_{k+1}-\tilde{q}_k) \neq 0\ .$$

Now

$$|1-\overline{\lambda}_1\omega_1|^2|1-\overline{\lambda}_2\omega_2|^2 = \{(1+|\lambda_1|^2)-\overline{\lambda}_1\omega_1-\lambda_1\overline{\omega}_1\}\{(1+|\overline{\lambda}_2|^2)-\overline{\lambda}_2\omega_2-\lambda_2\overline{\omega}_2\}$$

$$= (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\overline{\lambda}_1(1+|\lambda_2|^2)\overline{\omega}_1$$

$$- \overline{\lambda}_1(1+|\lambda_2|^2)\omega_1-\lambda_2(1+|\lambda_1|^2)\omega_2-\lambda_2(1+|\lambda_1|^2)\overline{\omega}_2$$

$$+ \overline{\lambda}_1\overline{\lambda}_2\omega_1\omega_2+\lambda_1\lambda_2\overline{\omega}_1\overline{\omega}_2+\overline{\lambda}_1\lambda_2\omega_1\overline{\omega}_2+\lambda_1\overline{\lambda}_2\overline{\omega}_1\omega_2$$

(using $|\omega_i| = 1$).

Hence from (2) we get

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\{(1+|\beta_1|^2)(1+|\beta_2|^2)-\overline{\beta}_1(1+|\beta_2|^2)\overline{\omega}_1-\overline{\beta}_1(1+|\beta_2|^2)\overline{\omega}_2$$

$$- \overline{\beta}_2(1+|\beta_1|^2)\omega_2-\beta_2(1+|\beta_1|^2)\overline{\omega}_2+\overline{\beta}_1\overline{\beta}_2\omega_1\omega_2+\beta_1\beta_2\overline{\omega}_1\overline{\omega}_2+\overline{\beta}_1\beta_2\overline{\omega}_1\omega_2$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\ (1+|\lambda_1|^2)(1+|\lambda_2|^2)-\overline{\lambda}_1(1+|\lambda_2|^2)\omega_1$$

$$- \lambda_1(1+|\lambda_2|^2)\overline{\omega}_1-\overline{\lambda}_2(1+|\lambda_1|^2)\omega_2-\lambda_2(1+|\lambda_1|^2)\omega_2+\overline{\lambda}_1\overline{\lambda}_2\omega_1\omega_2$$

$$+ \lambda_1\lambda_2\overline{\omega}_1\overline{\omega}_1+\overline{\lambda}_1\lambda_2\omega_1\overline{\omega}_2+\lambda_1\overline{\lambda}_2\overline{\omega}_1\omega_2\}.$$

Since these polynomials in ω_1, $\overline{\omega}_1$, ω_2 and $\overline{\omega}_2$ are equal on $\mathbb{T}^2$, the coefficients of these polynomials must be equal. So we have constant term:

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)(1+|\beta_1|^2)(1+|\beta_2|^2)$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)(1+|\lambda_1|^2)(1+|\lambda_2|^2 \quad \ldots. \quad (3)$$

coefficient of $\overline{\omega}_1$: $c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\beta_1(1+|\beta_2|^2)$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\lambda_1(1+|\lambda_2|^2)\ldots(4)$$

Dividing (4) by (3) we get $\dfrac{\beta_1}{1+|\beta_1|^2} = \dfrac{\lambda_1}{1+|\lambda_1|^2}$ .... (5)

Taking the absolute value and cross-multiplying we obtain

$$|\beta_1|(1+|\lambda_1|^2) = |\lambda_1|(1+|\beta_1|^2)$$

$$\Rightarrow |\beta_1| + |\beta_1||\lambda_1|^2 = |\lambda_1|-|\lambda_1||\beta_1|^2 = 0$$

$$\Rightarrow (|\beta_1|-|\lambda_1|)(1-|\lambda_1||\beta_1|) = 0 \Rightarrow |\lambda_1| = |\beta_1|$$

since $|\lambda_1| < 1$ and $|\beta_1| < 1$.

Hence from (5) we get $\lambda_1 = \beta_1$. Similarly equating the

$$- \lambda_1(1+|\lambda_2|^2)\overline{\omega}_1 - \overline{\lambda}_2(1+|\lambda_1|^2)\omega_2 - \lambda_2(1+|\lambda_1|^2)\omega_2 + \overline{\lambda}_1\overline{\lambda}_2\omega_1\omega_2$$

$$+ \lambda_1\lambda_2\overline{\omega}_1\overline{\omega}_1 + \overline{\lambda}_1\lambda_2\omega_1\overline{\omega}_2 + \lambda_1\overline{\lambda}_2\overline{\omega}_1\omega_2\}.$$

Since these polynomials in ω_1, $\overline{\omega}_1$, ω_2 and $\overline{\omega}_2$ are equal on $\mathbb{T}^2$, the coefficients of these polynomials must be equal. So we have constant term:

$$c(1-|\lambda_1|^2)(1-|\lambda_2|^2)(1+|\beta_1|^2)(1+|\beta_2|^2)$$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)(1+|\lambda_1|^2)(1+|\lambda_2|^2 \quad \ldots. \quad (3)$$

coefficient of $\overline{\omega}_1$: $c(1-|\lambda_1|^2)(1-|\lambda_2|^2)\beta_1(1+|\beta_2|^2)$

$$= \tilde{c}(1-|\beta_1|^2)(1-|\beta_2|^2)\lambda_1(1+|\lambda_2|^2) \ldots (4)$$

Dividing (4) by (3) we get $\dfrac{\beta_1}{1+|\beta_1|^2} = \dfrac{\lambda_1}{1+|\lambda_1|^2}$ .... (5)

Taking the absolute value and cross-multiplying we obtain

$$|\beta_1|(1+|\lambda_1|^2) = |\lambda_1|(1+|\beta_1|^2)$$

$$\Rightarrow |\beta_1| + |\beta_1||\lambda_1|^2 = |\lambda_1| - |\lambda_1||\beta_1|^2 = 0$$

$$\Rightarrow (|\beta_1|-|\lambda_1|)(1-|\lambda_1||\beta_1|) = 0 \Rightarrow |\lambda_1| = |\beta_1|$$

since $|\lambda_1| < 1$ and $|\beta_1| < 1$.

Hence from (5) we get $\lambda_1 = \beta_1$. Similarly equating the coefficients of $\overline{\omega}_2$ and dividing by the constant term we get $\frac{\beta_2}{1+|\beta_2|^2} = \frac{\lambda_2}{1+|\lambda_2|^2}$ which implies $\lambda_2 = \beta_2$. Hence $\lambda = \beta$ what we are required to show.

Theorem 2.16: If the pair $S^{(p,q,\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{p,q}$ is simultaneously unitarily equivalent to the pair $S^{(\tilde{p},\tilde{q},\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(\tilde{p},\tilde{q})}$, then $r = s$, $p_i = \tilde{p}_i$ and $q_i = \tilde{q}_i$ for $i = 1,\ldots,r=s$.

Proof: The complete unitary invariants for the pair $S^{(p,q,\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(p,q)}$ are

$$\frac{\partial^2 \log\| P_{\mathfrak{m}_\lambda^{(p,q)}} K_\omega\|^2}{\partial\omega_i \partial\overline{\omega}_j} \quad \text{for } i,j = 1,2.$$

Let $$W_{ij}^{(p,q)}(\omega_1,\omega_2) = \frac{\partial^2 \log F_{p,q,\lambda}(\omega_1,\omega_2)}{\partial\omega_i \partial\overline{\omega}_j} \quad \text{where } F_{p,q,\lambda}$$

is given by Proposition 2.13. Thus the complete unitary invariants are $W_{ii}^{(p,q)} + (1-|\omega_i|^2)^{-2}$ for $i = 1,2$ and $W_{12}^{(p,q)}, W_{21}^{(p,q)}$. Let $\tau_i(\omega_1,\omega_2) = \frac{\omega_i - \lambda_i}{1-\overline{\lambda}_i\omega_i}$ $i = 1,2$ and

$$\psi_{p,q}(z_1,z_2) = \sum_{k=1}^{r+1} |z_1|^{2p_{k-1}+2} |z_2|^{2q_k+2} - \sum_{k=1}^{r} |z_1|^{2p_k+2} |z_2|^{2q_k+2}.$$

Observe that $\psi_{p,q}$ is bi-circularly symmetric. Now

$$\log F_{p,q,\lambda}(\omega_1,\omega_2) = \log \psi_{p,q}(\tau_1(\omega_1,\omega_2),\tau_2(\omega_1,\omega_2)).$$

Hence, by chain rule, we get

$$W_{jj}^{(p,q)}(\omega_1,\omega_2) = \tilde{W}_{jj}^{(p,q)}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 \qquad j = 1,2$$

where $$\tilde{W}_{jj}^{(p,q)}(z_1,z_2) = \frac{\partial^2 \log \psi_{p,q}(z_1,z_2)}{\partial z_j \partial \bar{z}_j}$$

Let $z = re^{i\theta}$, $r > 0$. Then $\frac{\partial}{\partial z} = \frac{1}{2}\, e^{-i\theta}\{\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\}$. Thus

$$\tilde{W}_{jj}^{(p,q)}(r_1,r_2) = \frac{1}{4}\{\frac{\partial^2}{\partial r_j^2} + \frac{1}{r_j}\frac{\partial}{\partial r_j}\}\log \psi_{p,q}(r_1,r_2).$$

Fix $r_2 \neq 0$ and let $G(r_1,r_2) = \log \psi_{p,q}(r_1,r_2)$. Then we have $4r_1\tilde{W}_{11}^{p,q} = \frac{\partial}{\partial r_1}(r_1\frac{\partial G}{\partial r_1})$, so

(*) $$4\int_0^{r_1} s\tilde{W}_{11}^{p,q}(s,r_2)ds = r_1\frac{\partial G}{\partial r_1}$$ and hence

(**) $$G(r_1,r_2) = 4\int_0^{r_1}\frac{1}{t}\int_0^t s\tilde{W}_{11}^{(p,q)}(s,r_2)ds\,dt + G(0,r_2).$$

Using the formula similar to (*) for $\tilde{W}_{22}^{(p,q)}$ when $r_1 \neq 0$ and taking the limit as $r_1 \to 0$ we get, again with $r_2 \neq 0$,

$r_2\frac{\partial G}{\partial r_2}(0,r_2) = 4\lim\limits_{r_1\to 0}\int_0^{r_2}\tilde{W}_{22}^{p,q}(r_1,t)dt$. Hence using the fact that $G(0,1) = 0$ we have

$$G(0,r_2) = 4\int_1^{r_2}\frac{1}{s}\lim_{r_1\to 0}\int_0^s t\tilde{W}_{22}^{(p,q)}(r_1,t)dt\,ds.$$

$$\log F_{p,q,\lambda}(\omega_1,\omega_2) = \log \psi_{p,q}(\tau_1(\omega_1,\omega_2),\tau_2(\omega_1,\omega_2)).$$

Hence, by chain rule, we get

$$W_{jj}^{(p,q)}(\omega_1,\omega_2) = \tilde{W}_{jj}^{(p,q)}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 \qquad j = 1,2$$

where $$\tilde{W}_{jj}^{(p,q)}(z_1,z_2) = \frac{\partial^2 \log \psi_{p,q}(z_1,z_2)}{\partial z_j \partial \bar{z}_j}$$

Let $z = re^{i\theta}$, $r > 0$. Then $\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta}\{\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\}$. Thus

$$\tilde{W}_{jj}^{(p,q)}(r_1,r_2) = \frac{1}{4}\{\frac{\partial^2}{\partial r_j^2} + \frac{1}{r_j}\frac{\partial}{\partial r_j}\}\log \psi_{p,q}(r_1,r_2).$$

Fix $r_2 \neq 0$ and let $G(r_1,r_2) = \log \psi_{p,q}(r_1,r_2)$. Then we have $4r_1\tilde{W}_{11}^{p,q} = \frac{\partial}{\partial r_1}(r_1 \frac{\partial G}{\partial r_1})$, so

(*) $$4\int_0^{r_1} s\tilde{W}_{11}^{p,q}(s,r_2)ds = r_1 \frac{\partial G}{\partial r_1} \text{ and hence}$$

(**) $$G(r_1,r_2) = 4\int_0^{r_1} \frac{1}{t}\int_0^t s\tilde{W}_{11}^{(p,q)}(s,r_2)ds\,dt + G(0,r_2).$$

Using the formula similar to (*) for $\tilde{W}_{22}^{(p,q)}$ when $r_1 \neq 0$ and taking the limit as $r_1 \to 0$ we get, again with $r_2 \neq 0$,

$r_2 \frac{\partial G}{\partial r_2}(0,r_2) = 4 \lim_{r_1\to 0} \int_0^{r_2} \tilde{W}_{22}^{p,q}(r_1,t)dt$. Hence using the fact that $G(0,1) = 0$ we have

$$G(0,r_2) = 4\int_1^{r_2} \frac{1}{s} \lim_{r_1\to 0}\int_0^s t\tilde{W}_{22}^{(p,q)}(r_1,t)dt\,ds.$$

Combining with (**) we get

$$(***) \qquad G(r_1,r_2) = 4\int_0^{r_1}\int_0^t s\widetilde{W}_{11}^{(p,q)}(s,r_2)\,ds\,dt$$

$$+ \int_0^s \frac{1}{s} \lim_{r_1\to 0}\int_0^s t\widetilde{W}_{22}^{(p,q)}(r_1,t)\,dt\,ds\,.$$

Now the pair $S^{(p,q,\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(p,q)}$ is unitarily equivalent to the pair $S^{(\tilde{p},\tilde{q},\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(\tilde{p},\tilde{q})}$ implies $W_{jj}^{(p,q)}(\omega_1,\omega_2) = W_{jj}^{(\tilde{p},\tilde{q})}(\omega_1,\omega_2)$ which in turn implies

$$\widetilde{W}_{jj}^{(p,q)}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 = W_{jj}^{(p,q)}(\omega_1,\omega_2) = \widetilde{W}_{jj}^{(\tilde{p},\tilde{q})}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2$$

and hence $\qquad \widetilde{W}_{jj}^{(p,q)}(z_1,z_2) = \widetilde{W}_{jj}^{(\tilde{p},\tilde{q})}(z_1,z_2)$

since $\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 > 0$. From this and (***) we obtain

$$\log\psi_{p,q}(r_1,r_2) = \log\psi_{\tilde{p},\tilde{q}}(r_1,r_2)$$

which implies $\psi_{p,q}(r_1,r_2) = \psi_{\tilde{p},\tilde{q}}(r_1,r_2)$.

and hence $r = s$, $p_i = \tilde{p}_i$, $q_i = \tilde{q}_i$ for $i = 1,2\ldots r=s$ as $\psi_{p,q}$ and $\psi_{\tilde{p},\tilde{q}}$ are real analytic, in fact polynomials in r_1 and r_2

Combining with (**) we get

$$(***)\qquad G(r_1,r_2) = 4\int_0^{r_1}\int_0^{t} s\widetilde{W}_{11}^{(p,q)}(s,r_2)\,ds\,dt$$

$$+\int_0^{s}\frac{1}{s}\lim_{r_1\to 0}\int_0^{s} t\widetilde{W}_{22}^{(p,q)}(r_1,t)\,dt\,ds\ .$$

Now the pair $S^{(p,q,\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(p,q)}$ is unitarily equivalent to the pair $S^{(\tilde{p},\tilde{q},\lambda)} = (S_1,S_2)$ on $\mathfrak{m}_\lambda^{(\tilde{p},\tilde{q})}$ implies $W_{ij}^{(p,q)}(\omega_1,\omega_2) = W_{ij}^{(\tilde{p},\tilde{q})}(\omega_1,\omega_2)$ which in turn implies $\widetilde{W}_{jj}^{(p,q)}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 = W_{jj}^{(p,q)}(\omega_1,\omega_2) = \widetilde{W}_{jj}^{(\tilde{p},\tilde{q})}(z_1,z_2)\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2$

and hence $\qquad \widetilde{W}_{jj}^{(p,q)}(z_1,z_2) = \widetilde{W}_{jj}^{(\tilde{p},\tilde{q})}(z_1,z_2)$

since $\left|\frac{\partial\tau_j}{\partial\omega_j}\right|^2 > 0$. From this

and (***) we obtain

$$\log\psi_{p,q}(r_1,r_2) = \log\psi_{\tilde{p},\tilde{q}}(r_1,r_2)$$

which implies $\psi_{p,q}(r_1,r_2) = \psi_{\tilde{p},\tilde{q}}(r_1,r_2)$.

and hence $r = s$, $p_i = \tilde{p}_i$, $q_i = \tilde{q}_i$ for $i = 1,2\ldots r{=}s$ as $\psi_{p,q}$ and $\psi_{\tilde{p},\tilde{q}}$ are real analytic, in fact polynomials in r_1 and r_2.

Corollary 2.17: If the pair $S^{(p,q,\lambda)} = (S_1, S_2)$ on $\mathfrak{m}_\lambda^{(p,q)}$ is simultaneously unitarily equivalent to the pair $S^{(\tilde{p},\tilde{q},\beta)} = (S_1, S_2)$ on $\mathfrak{m}_\beta^{(\tilde{p},\tilde{q})}$ then

$$\lambda = \beta,\ r = s,\ p_i = \tilde{p}_i,\ q_i = \tilde{q}_i \qquad i = 1,\dots,r=s.$$

Proof: Combine Theorems 2.15 and 2.16

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