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THE SCHUR MULTIPLIER OF THE EXCEPTIONAL LIE GROUP G_2

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Abstract of the Dissertation

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This dissertation gives a proof that the second homology of the compact exceptional Lie group G_2 considered as a discrete group is isomorphic to the second homology of the special unitary group SU(2), also considered as a discrete group.

The above homology groups are Eilenberg-MacLane homology groups, with G-trivial Z coefficients, and $H_2(G)$ is known as the Schur multiplier of G.

The above result is then used to provide further information on a conjecture of Milnor's. If G is a

compact Lie group, we use G^{δ} to denote G with the discrete topology. Then we show that the natural map $n\colon BG^{\delta} \longrightarrow BG$ between classifying spaces induces an isomorphism

$$n_*\colon H_i(BG^{\delta},\mathbf{F}_p) \dashrightarrow H_i(BG,\mathbf{F}_p)$$

of their homology groups with mod p coefficients for $0 \leqslant i \leqslant 2$, where $G = G_2$. Here $H_*(BG^{\delta})$ is equivalent to the Eilenberg-MacLane homology of the group G, and $H_*(BG)$ is the singular homology of the topological space BG.

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Introduction For any group G, we use $H_i(G)$ to denote the i-th Eilenberg-MacLane homology group with **Z**G-trivial integral coefficients. These homology groups have been used to study the homology of $K(\pi,1)$ spaces, in the solution of scissors congruence problems (see Sah[14]), and algebraic K-theory (Berrick[2], Quillen[12]). The second homology group, $H_2(G)$, is called the Schur multiplier of G. This group was used by Schur[17,18] to study certain representations of finite groups G, and calculation of the Schur multipliers of various finite simple groups was important in the classification of the finite simple groups.

For any Lie group G, one may forget the topology of G and consider it as a discrete group, denoted G^{δ}. We use BG to denote the classifying space of G. Then the following conjecture has been made by Milnor[11] for any Lie group G:

> The map $H_*(BG^{\delta},\mathbf{F}_p) \dashrightarrow H_*(BG,\mathbf{F}_p)$ is an isomorphism for any prime p, where the action on $\mathbf{F}_p$ is trivial.

Milnor has shown that the conjecture holds for G a solvable Lie group, and that it depends only on the Lie algebra of G. The general case is reduced to the case where G is connected, simple, and simply-connected (if convenient). In this case, Milnor has shown that the map is surjective. The conjecture is trivial for H_0 and H_1. Sah[15] has shown that the conjecture is true on the H_2 level for the classical Lie groups, and gives an extensive outline of the known results on the conjecture. Most of the results on H_i for $i > 1$ are on the H_2 level, either as "K_2 calculations", or direct H_2 computations. The higher homology is extremely complicated, and analysis is involved in addition to the geometry and algebra used to calculate H_2; in Parry-Sah [13], Rogers' L-function is used to study $H_3(SL(2,\mathbf{R}))$, and several authors have made use of the dilogarithm to study scissors congruence problems and $H_3(SL(2,\mathbf{C}))$. For further references see Sah[15], Parry-Sah[13] and Dupont-Sah[5].

An important result at the H_2 level is the following theorem of Sah[15] :

> Let $G \neq 1$ be a simple, simply-connected, compact Lie group of classical type. Then $H_2(G) \cong H_2(SU(2))$ under a natural inclusion map.

In this dissertation, we show that this theorem is also true in the case where $G = G_2$, the compact exceptional Lie group of rank 2. We use the representation of G_2 as the group of automorphisms of the octonion algebra to study its action on a special chain complex. The second homology of G^δ is then computed via a spectral sequence. Many of the techniques useful in the case of classical Lie groups are also of use in the present case. In group homology calculations, success usually depends on the proper choice of resolution ; the independent chain complex used by Sah in the classical case is useful also in the present case. In addition to determining $H_2(G_2)$, we show that the conjecture of Milnor is true on the level of H_2 for the group G_2.

It seems reasonable to make the following conjecture: $H_2(G) \cong H_2(SU(2))$ if G is any of the exceptional Lie groups F_4, E_6, E_7, or E_8, and that the conjecture of Milnor also holds for these groups. We hope to consider this in a future work.

I. Preliminaries

In order to study the group G_2, we will need a convenient way to describe it. Just as the orthogonal, unitary, and symplectic groups are groups of transformations of the vector spaces over $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$, we may describe G_2 as the group of automorphisms of a certain algebra, the octonions, denoted by $\mathbf{O}$. In this section, we examine the properties of this algebra, also known as the Cayley division algebra. References for this and generalizations of this algebra are Jacobson[6,7] and Zorn[23].

Definition 1.1 A composition algebra is a (non-associative) algebra A with unit 1, together with a non-degenerate quadratic form N such that N commutes with the product in A, i.e. $N(xy) = N(x)N(y)$ for $x,y \in A$.

The algebra A is a vector space over a field $\mathbf{F}$, together with a bilinear product xy on A. We will only consider the case where $F = R$, although many of the results hold for arbitrary fields. In the case of the octonions, the underlying vector space of the algebra is eight dimensional over the field $F = \mathbf{R}$. We have a symmetric bilinear form $\langle\ ,\ \rangle$ associated to the quadratic form N, given by

$$\langle x,y\rangle = N(x+y) - N(x) - N(y).$$

Using this, we can supply A with an involution $*: A \longrightarrow A$, given by $x^* = \langle x,1\rangle 1 - x$. The factor $\langle x,1\rangle$ is called the trace of x, and is denoted $T(x)$.

Before proving further general properties of composition algebras, we observe that the original Cayley algebra arose as a generalization of the complex numbers and the quaternions. It is frequently convenient to use the following basis for the octonions, and also it gives some indication of the origin of the formulas obtained in the more general setting of composition algebras. We set

$$x = \sum x_i e_i,\ 0 \leqslant i \leqslant 7, \text{ for } x \in \mathbf{O},$$

where e_i is the standard basis for the vector space $\mathbf{R}^8$. With the appropriate definition of multiplication of these basis vectors, the e_i form a basis of the octonion algebra $\mathbf{O}$. For $1 \leqslant i \leqslant 7$, $e_i^2 = -1$ and $e_ie_j = -e_je_i$ for $i \neq j$. The quaternions are contained in $\mathbf{O}$ by including $\mathbf{H}$ in $\mathbf{R}^8$ so that $1 = (1,0,..,0)$, $i = (0,1,0,...,0)$, etc.. The other rules necessary to make this an algebra are obtained from the construction of **O** as the (-1)-double of **H**. (See Jacobson [7],p 425-426). Note that the complex numbers are contained in **O** as $R[e_1]$ where $e_1{}^2 = -1$ as usual. Also, note that the description in Zorn uses the cross-product defined in Definition 1.11 instead of the usual one from [7].

In terms of this basis, we can define the standard norm for the octonions by $N(x) = \sum x_i{}^2$, where the x_i's are the coordinates relative to the basis. The involution becomes

$$x^* = x_0 e_0 - \sum x_i e_i \ , \ 1 \leqslant i \leqslant 7 \ ,$$

analogous to complex conjugation. By writing the norm in terms of the basis, we see that $\langle x,y \rangle$ is positive definite. We record the following properties of the norm and the involution from Jacobson[6] :

(1.2) $\quad xx^* = N(x)1 = x^*x,$

(1.3) $\quad (xy)^* = y^*x^*$

(1.4) $\quad (x^*)^* = x$

(1.5) $\quad x + x^* = T(x)1 \ ,$

(1.6) $\quad x^2y = x(xy) \ , \ (xy)x = x(yx) \ , \ yx^2 = (yx)x \ ,$

(1.7) $\quad x^*y + y^*x = \langle x,y \rangle 1 \ ,$

These relations are true for all composition algebras, and are readily proved. The relations of (1.6) are the alternative laws. In general, composition algebras need not be either commutative or associative: the quaternion algebras are not commutative, and the Cayley algebras are not associative.

Next, we show how composition algebras can be built up from lower dimensional algebras. This construction is needed for the classification of composition algebras, and resembles the construction of the complex numbers from the reals. For proofs, see Jacobson[7] .

If A is a non-associative algebra with unit and involution *, we can construct a new algebra B, called the <u>c-double</u> of A, that is of dimension $2 \cdot \dim A$, where $c \neq 0$ is an element of the base field **F**. The underlying set of B is just A x A, the vector space of ordered pairs of elements of A, with the usual coordinate-wise sum. The product is given by the following:

$$(1.8) \qquad (u,v)(x,y) = (ux + cy^*v, yu + vx^*) .$$

It is easy to see that this makes B an algebra over the base field **F**, with unit $1 = (1,0)$ and involution given by $(x,y)^* = (x^*,-y)$. The symmetric bilinear form is

$$(1.9) \qquad \langle (u,v),(x,y)\rangle = \langle u,x\rangle - c\langle v,y\rangle ,$$

and this is non-degenerate. We note that by dimension counting and Lemma 3 of Jacobson [7], every composition algebra $\neq 1$ is isomorphic to a c-double of a proper subalgebra.

Having indicated the basic construction of the octonion algebra, we now wish to examine their structure more closely. We recall several definitions:

<u>Definition 1.10</u> : We say that elements x,y of A are orthogonal, denoted by $x \perp y$, if $\langle x,y\rangle = 0$. For a subalgebra H of A, we denote by $H^{\perp}$ the set of all $y \in A$ such that $\langle x,y\rangle = 0$ for every x in H.

The elements of **O** orthogonal to $(F\cdot 1)$, i.e. the set of $x \in A$ such that $\langle x,1\rangle = 0 = T(x)$, analogous to "pure" quaternions, are important in the sequel, and the set of these is denoted by $\mathbf{O}_0$. If x,y are **R**-linearly independent vectors in $\mathbf{O}_0$, we note that together with 1, they generate a quaternionic subalgebra of $\mathbf{O}$. We will use H to denote quaternionic subalgebras of O, and K for quadratic subalgebras of $\mathbf{O}$.

The reason for our interest in the Cayley division algebra is that the exceptional Lie group G_2 has been shown to be the group of automorphisms of the composition algebra $\mathbf{O}$. The definition of an automorphism of composition algebras is clear; note in particular that an automorphism must preserve products of elements from O. In addition, it is a subgroup of the orthogonal group of O, O(O,N) ,i.e. the group of isometries of $\mathbf{O}$, relative to the norm N. To help describe this group, we record the following definition:

Definition 1.11 We define the cross-product $a \times b$ by

$$a \times b = 1/2\ [a,b] = 1/2\ (ab-ba), \text{ for } a,b \in \mathbf{O}.$$

Note that this product is zero if a or $b \in F\cdot 1$. Clearly $a \times a = 0$, and the product is $\mathbf{R}$-bilinear. It is easily seen that if $a,b \in O_0$, then $a \times b \in \mathbf{O}_0$. The relation with the ordinary 3-dimensional cross product is clear ; for an intuitive description of the octonions, see Zorn[23].

Definition 1.12 The scalar triple product is defined by

$$[a,b,c] = -\langle a,b \times c\rangle, \text{ for } a,b,c \in \mathbf{O}.$$

This is an **R**-trilinear map from **O** to **R**. As shown in Jacobson[6], $[a,b,c] = [c,a,b]$ for $a,b,c \in O_o$, and this together with $a \times a = 0$ implies that it is an alternating map. From this we obtain the following, giving a insight into the nature of the cross product:

Lemma 1.13 If $a \perp b$, then a, b, and $a \times b$ are pairwise orthogonal.

A useful characterization of $g \in G_2$ is that it is a 1-1 linear map of the subspace O_o into itself that preserves the forms $[a,b,c]$ and $\langle a,b \rangle$. We also note that G_2 is transitive on quaternionic subalgebras, i.e. given H_1 and H_2, we can find $g \in G_2$ so that $g(H_1) = H_2$. The subgroup of the group of automorphisms G of a composition algebra **O** that fixes the elements of a subalgebra H of O is denoted G_H, and is called the Galois group of **O** over H. We record here the following results of Jacobson[6] on the Galois groups of various subalgebras of Cayley algebras:

Theorem 1.14 (Theorem 3, [6]) Let C be a Cayley algebra, k a quadratic subfield. Then the Galois group G_k of C over k is isomorphic to the unimodular unitary group $U^+(k^\perp, \{xy\})$ of the three dimensional space $k^\perp$ over k relative to the norm $\{x,y\}$ defined by

$$\{x,y\} \equiv \langle x,y\rangle + u^{-1}j\langle jx,y\rangle ,$$

where $k = F[j]$, $j^2 = u1$.

Theorem 1.15 (Theorem 5, [6]) Let C be Cayley, B a quaternionic subalgebra. Then the Galois group G_B of C over B is isomorphic to the multiplicative group of elements of norm 1 of B.

Several remarks should be made on the special case of the standard Cayley division algebra. We note that the norm of Theorem 1.14 is just the usual one, given by $N(x) = \sum x_i{}^2$, obtained by setting $u = -1$, and $j = e_4$. We see also that $N(x) \neq 0$ if $x \neq 0$, and so $\mathbf{O}$ is a division algebra, i.e. if $x \neq 0$, we can find x^{-1} such that $xx^{-1} = 1 = x^{-1}x$. By setting $x^{-1} = (N(x))^{-1}x^*$, and applying (1.2), we see that x^{-1} is an inverse for x.

In Theorem 1.15, we see that $G_B = Sp(1)$ by considering $Sp(1)$ as $SU(1,\mathbf{H})$, i.e. 1-dimensional matrices of unit quaternions. Also, we may identify $Sp(1)$ with $SU(2)$ through the standard isomorphism induced from the identification of $\mathbf{H}$ with $\mathbf{C}^2$.

I.2. The independent chain complex

A reference for the results of this and the following section is Sah[15] . We will construct a chain complex over the coset space $G_2/SU(3)$. By Jacobson [6], this is equivalent to the sphere S^6. The identification is given as follows: let S^6 be the set of unit vectors of $\mathbf{O}_o$. From the proof of Theorem 2 of [6], if $g \in G_2$, it has a fixed point $x_0 \neq 0$ in $\mathbf{O}_o$. Since g is $\mathbf{R}$-linear, we may normalize so that $N(x_0) = 1$, which is therefore a point on S^6. (Alternately, one may apply the Lefschetz fixed point theorem to the G_2 action on S^6). If K is the quadratic subalgebra generated by 1 and x_o, then the Galois group G_K of K is isomorphic to SU(3), i.e. a point on the sphere is determined by an element of G_2, up to an element of SU(3), and thus the structure as a coset space is as desired.

In order to apply the spectral sequence, we will need an acyclic chain complex. Following Sah[15], we use the independent chain complex C_* with vertices from S^6. By definition, C_i is the free abelian group on the set of all independent i-cells $(v_0,\ldots,v_i)$. These cells are ordered (i+1)-tuples of elements from S^6 (from the above, a point on S^6 is identified with a unit vector in $\mathbf{O}_o$) such that any ordered subset of the v_i with at most 7 elements are $\mathbf{R}$-linearly independent. The boundary $\partial : C_i \longrightarrow C_{i-1}$ is the usual one, familiar from topology or group homology:

$$\partial_C(v_0,\ldots,v_i) = \sum_j (-1)^j (v_0,\ldots,\hat{v}_j,\ldots,v_i) ,$$

where $\hat{v}_i$ means to omit that vertex.

A complex that is more geometric in nature is the semi-simplicial complex whose nondegenerate cells are the independent i-cells. If we form the standard geometric realization W^{ind} of this complex, then C_* is the complex of cellular chains on W^{ind} with the independent i-cells as a basis. For the construction of the geometric realization, see Milnor[10] or May[9] .

The action of G_2 on S^6 is similar in many ways to the action of $U(n,F)$ on S^m in Sah[15] , essentially since G_2 is a subgroup of $O(O_o) = O(7)$, where we have taken $F = R$. We record here some of the properties which hold for the two chain complexes.

(1.16) C_* is acyclic with augmentation $\mathbf{Z}$. Equivalently, W^{ind} is acyclic.

(1.17) G acts cellularly on W^{ind} and the complex W^{ind}/G is also a CW-complex.

(1.18) $C_* \otimes_G Z = C_*(W^{ind}/G)$, the complex of cellular chains on W^{ind}/G.

(1.19) The action of G is determined by its action on the vertices. W^{ind}/G has only one vertex, or, equivalently, G acts transitively on the vertices of W^{ind}, or $C_0 \otimes_G Z = Z$.

(1.20) The stability subgroup of any i-cell acts trivially on the i-cell.

<u>Proof of 1.16</u> : Given any finite collection of i-cells c_j , $j \in J$, the formal join $x \vee c_j$ is an independent (i+1)-cell if and only if x is a unit vector not lying on the various faces of c_j with less than 7 vertices. Since the index set J is finite, we have a Zariski open dense subset of such unit vectors x in S^6, and thus we can always find a vertex for the cone. Then by the formal cone construction, c_j is homologous to 0 . See Vick[21], Appendix I for more details.

I.3. The spectral sequence and group homology

In this section, we set down notation and state basic results on the homology theory of groups and the spectral sequence that we use to compute $H_2(G_2)$.

For an abstract group G, one may define the group homology as follows:

Let B_n be the free $\mathbf{Z}G$-module with generators $[x_1,\ldots,x_n]$ all n-tuples of elements x_i in G, and $x_i \neq 1$, for all i. We may normalize this by setting $[x_1,\ldots,x_n] = 0$ if $x_i = 1$ for some i. We denote this complex by $B(\mathbf{Z}G)$, or simply B_* for a fixed group G. If A is a G-module, we may define the homology groups of G with coeffiecients in A by

$$H_i(G,A) = H_i(B_* \otimes_G A) .$$

The homology on the right is the homology of the chain complex $B_* \otimes_G A$. By the comparison theorem for resolutions, the homology groups may be computed by using any projective resolution in place of B_* .

For any group G, if the action of the group ring

ZG on the coefficient group A is assumed to be trivial, the Eilenberg-MacLane homology $H_0(G,A) \cong A$. If we use trivial integral coefficients, the first homology $H_1(G,\mathbf{Z}) = G/[G,G]$, where $[G,G]$ is the commutator subgroup of G. The group G is said to be perfect if $G = [G,G]$; note that in the case of trival G-action on $\mathbf{Z}$, this implies that $H_1(G) = 0$.

A standard technique in the homology theory of fibrations and of groups described as extensions of groups is the use of spectral sequences. From knowledge of the homology groups of the base and fibre (resp. kernel and cokernel), plus the action of the base on the homology of the fibre, the homology of the total space can be found in certain special cases. In general, computation of either the E^2 terms or the differentials is too complicated. The spectral sequence we use is the homology spectral sequence with second index filtration described in Sah[15], which we record here for convenience.

The spectral sequence is derived in the usual way from a double complex of ZG-modules. The individual complexes are as follows:

We define an acyclic G-chain complex with augmentation by

$$\cdots \longrightarrow M_j \longrightarrow M_{j-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow A \longrightarrow 0 \ ,$$

which is an exact sequence of G-modules M_j, together with G-homomorphisms which we denote by ∂_M, and an augmentation $\epsilon : M_0 \longrightarrow A$. In the present case, M will be the complex of cellular chains associated to a cell complex defined over a space on which G_2 acts cellularly.

Next, we consider a **Z**G-free or **Z**G-projective resolution of the G-trivial module **Z** :

$$\cdots \longrightarrow C_i \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbf{Z} \longrightarrow 0 \ ,$$

for example the standard bar resolution. Recall that $H_*(G,A) = H_*(C_* \otimes_G A)$ if A is a G-module. We may then form the double complex $C_*(G) \otimes_G M_*$ with (i,j)-th term $C_i(G) \otimes_G M_j$ and total boundary $\partial = \partial_G \otimes 1 + \epsilon \otimes \partial_M$, where the sign ϵ is $(-1)^i$ on $C_i(G)$.

The first index filtration with $'d = \partial_G$ leads to:

$$'E^2_{i,0} = H_i(G,A) \text{ and } 'E^2_{i,j} = 0 \text{ for } j > 0.$$

From this we see that the second index filtration "E converges to $H_*(G)$,

$$H_{i+j}(G,A) \Longleftarrow {}''E^1_{i,j} = H_i(G,M_j), \text{ and } {}''d^1 = -\partial_M.$$

In the present case, we ignore the sign, as it has no effect on our results.

The following lemma, due to Shapiro, allows us to calculate the "E^1 terms of the spectral sequence by reducing the calculation to a subgroup.

Lemma 1.21 (Shapiro's Lemma) If K is a subgroup of the group G and N is any (left) ZK-module, then

$$H_*(G,\mathrm{ind}_K{}^G N) = H_*(K,N)$$

where $\mathrm{ind}_K{}^G N$ denotes the module N considered as a Z G-module, with the action induced by the inclusion of K in G. (see Cartan-Eilenberg [4], X.7.4) .

II. The Stability Subgroups G(j) .

The stability subgroup of G fixing an i-cell $(v_0,...,v_i)$ is denoted by G(i). The knowledge of these subgroups, together with the Shapiro lemma, allow us to calculate the terms of the spectral sequence.

We state the following definition for convenience in describing certain special types of cells:

Definition 2.1: We define an independent 2-cell (u,v,x) to be of Type I if (1,u,v,x) generate a quaternionic subalgebra of **O**. It is said to be of Type II if (1,u,v,x) generate all of O.

Note that because of dimension restrictions on composition algebras and their subalgebras, (they only exist for n = 1, 2, 4, and 8) every independent 2-cell is either Type I, or Type II. We can now state the main result of this section:

Proposition 2.2 : The stability subgroups G(j) of an independent j-cell $v = (v_0,...,v_j)$ are given by:

(a) G(0) = SU(3)

(b) G(1) = SU(2)

(c) G(2) = SU(2) , if v is of Type I

$$\text{(d) } G(j) = \begin{cases} \{1\}, & \text{if } v \text{ is of Type II} \\ \{1\}, & \text{for } j > 2 \end{cases}$$

Furthermore, $G(j)$ fixes pointwise the vertices of v.

<u>Proof</u>: We observe that if a $g \in G$ fixes an independent i-cell v, it fixes (pointwise) the subalgebra of O generated by 1 and the vertices of v. Part (a) is obvious from the representation of S^6 as a homogenous space. The second assertion and the Type I part of (c) both follow from Theorem 1.15 and the isomorphism of Sp(1) with SU(2). The second half of (c) and (d) follow since $(1, v_0, \ldots, v_i)$ generate all of **O** and G acts effectively.

Note that part (c) shows that the stability subgroups of i-cells are not necessarily conjugate to each other in G. Next, we examine 2-simplices and their homology classes, and show that we need only consider Type II simplices in homology calculations. Furthermore, we prove a result on circumcenters for 2-simplices, showing that we need only consider independent Type II 2-simplices that are of the form $x * (u,v)$ in homology calculations. Here * denotes orthogonal join.

Lemma 2.3: The type (I or II) of an independent 2-cell (u,v,x) is preserved under G-action.

Proof: Since elements of G are vector space isomorphisms, the dimension of the subalgebra generated by u,v,x is invariant under G-action, and therefore so is the type.

We recall the following definitions from Sah[15] :

Definition 2.4 We call an independent i-cell $v = (v_0,\dots v_i)$ an iso-i-cell with lateral invariant s if $\langle v_0,v_i\rangle = s$ holds for $i > 0$, and denote it by $v_0 \lhd (v_1,\dots,v_i)$. By convention, 0-cells and 1-cells are iso-i-cells.

In Euclidean space, for $i = 2$, an iso-i-cell is just an isosceles triangle.

Defintion 2.5 A unit vector z is said to be a circumcenter of v with lateral invariant s for an independent i-cell $v = (v_0,\dots,v_i)$ if $\langle v_j,z\rangle = s$ for $0 \leqslant j \leqslant i$.

We use the circumcenter to change problems involving arbitrary i-cells to ones involving iso-i-cells through the following observation:

$$\partial_C(v_0 \triangleleft (v_1,\ldots,v_i)) = (v_1,\ldots,v_i) - v_0 \triangleleft \partial_C(v_1,\ldots,v_i).$$

The product $\triangleleft$ is extended additively to chains. We will say that $(v_1,\ldots,v_i)$ is circum-subdivided into $v_0 \triangleleft \partial_C(v_1,\ldots,v_i)$. The cell $v_0 \triangleleft (v_1,\ldots,v_i)$ must be an independent cell.

<u>Lemma</u> <u>2.6</u> Given an independent 2-simplex (u,v,x) of C_*, there is a circumcenter z. Furthermore, z may be chosen so that the boundary of the resulting independent 3-simplex (u,v,x,z) consists entirely of independent 2-simplices of Type II. In addition, the 2-faces of this 3-simplex are $z * (v_1,v_2)$, where the v_i are taken from u,v,x and * denotes orthogonal join.

<u>Proof</u>: Let H be the quaternionic subalgebra generated by 1,u,v. Choose a unit vector z in $H^\perp$ that is independent from x so that

$$z \in H^\perp \cap (v-x)^\perp \cap (u-x)^\perp$$

(which is at least 2-dimensional over $\mathbb{R}$). Since $z \in H^\perp$, we have $z \perp u$ and $z \perp v$, and these together with

$z \perp (v-x)$ imply that $z \perp x$. A computation of the various inner products (e.g. $\langle u-x, u-z \rangle$) shows that z is a circumcenter of (u,v,x). To prove the second assertion, we note that for any term in the boundary of (u,v,x,z), either two vertices are in H and the remaining one is in $H^{\perp}$; or vice-versa.

Lemma 2.7 Every Type I simplex c is homologous to a sum $\sum v_i$ of Type II simplices. Furthermore, the v_i can be chosen so that $c - \sum v_i$ is the boundary of an iso-3-cell.

Proof: If $c = (u_1, u_2, u_3)$ is of Type I, let H be the quaternionic subspace generated by 1 and the u_i. Choose a circumcenter z for (u_1, u_2, u_3) by lemma 2.6. The simplices (u_i, u_j, z), $i \neq j$, are then of Type II. A calculation of the boundary of (u_1, u_2, u_3, z) gives the result.

Note that in this case also, we may choose z so that the 2-faces of the resulting iso-3-cell are orthogonal joins.

III. Acyclicty results for W^{ind}/G:

We must show that the reduced chain complex C_* is 2-acyclic to make use of the spectral sequence described in section I.3. It may be possible to show n-acylicity, for some $n > 2$; however, we use other arguments to prove results involving higher degree terms.

Theorem 3.1. W^{ind}/G is 2-acyclic .

To show 1-acyclicity of W^{ind} , we proceed as follows: let (u,v) be a independent 1-cell, and let B be the quaternionic subalgebra generated by $1,u,v$. Let x be any unit vector in $B^{\perp}$. Now consider $K = F[x]$. Then the Galois group $G_K = SU(3)$, and acts transitively on S^5 = unit vectors in $O_o \cap F[x]^{\perp}$. Thus we can find $g \in G_K$ which fixes x, and such that $gu = v$, and therefore $(x,u) \sim_G (x,v)$, i.e. (x,u) is congruent under the G-action to (x,v). Note that $\langle x,u\rangle = \langle x,v\rangle$. Then any independent i-cell (u,v) is homologous to 0 in W^{ind}/G, i.e. W^{ind}/G is 1-acyclic.

We now show the 2-acyclicity of W^{ind} . Let c be any 2-cycle of W^{ind}/G_2. By 2.6, we may assume that

$c = \sum_j n_j c_j$ is made up of Type II 2-cells, and each c_j may be chosen so that $x \triangleleft (u,v) = x * (u,v)$. By the transitivity of G on isomorphic non-isotropic subalgebras, we may assume that u_i, v_i lie in a single quaternionic subalgebra, for example H, and therefore, $x_i \in H^{\perp}$. By using elements of G_H, which fix x_i, we see that (u_i,v_i) is G-equivalent to (u_j,v_j) if $\langle u_i,v_i\rangle = \langle u_j,v_j\rangle$. For such (u_i,v_i) we can move x_i by an element of G_H so that $(x_i,x_j) \triangleleft (u_i,v_i)$ is an independent 3-cell. We now have a sum (with possibly different n_i) :

$$c = \sum n_i x_i \triangleleft (u_i,v_i)$$

with $n \in Z$ and $\langle u_i,v_j\rangle$ pairwise distinct. Then

$$\begin{aligned} \partial c &= \sum n_i \; ((u_i,v_i)-(x_i,v_i)+(x_i,u_i)). \\ &= \sum n_i \; (u_i,v_i) \end{aligned}$$

Since the remaining $\langle u_i,v_i\rangle$ are distinct, the n_i must all be zero, i.e. c is a boundary.

IV Main Theorem

Theorem 4.1 $H_2(G_2) \cong H_2(SU(2))$.

Proof: From Sah [15], $H_2(SU(2)) = H_2(SU(3))$, so we will show the isomorphism for SU(3). We first show the surjectivity of the map:

$$i_* : H_2\ (SU(3)) \longrightarrow H_2(G_2)$$

induced by the inclusion i: $SU(3) \longrightarrow G_2$. We use the independent chain complex C_* of section I.2, whose vertices are ordered i-tuples of independent unit vectors in $\mathbf{O}_o$. We then have a spectral sequence converging to the Eilenberg-MacLane homology of G_2 with trivial Z coefficients:

$$(4.1) \qquad H_{i+j}(G_2,\ Z) \Longleftarrow\!\!=\!\!=\!\!=\!\!= {}''E_{i,j} = H_i(G_2,C_j),$$

based on the second index filtration on the complex $C_*(G_2) \otimes_G C_*(S^6)$ with boundary $\partial = \partial_G \otimes 1 + \epsilon \otimes \partial_C$. The sign $\epsilon = (-1)^i$, although it is not important for our purposes, and so formulas may be off by a sign. The differential is

$$d^r: {}''E^r{}_{p,q} \longrightarrow {}''E^r{}_{p+r-1,q-r}\ ,$$

and the graded structure in E^{∞} is given by inclusion of $''E^{\infty}_{p,q}$ in $''E^{\infty}_{p-1,q-1}$ as indicated by the direction of the double arrow in (4.1). We will compute the terms $''E^{1}_{i,j}$ by using the results of section 3.1 and the following consequence of Shapiro's lemma:

$$(4.2) \qquad H_i(G,C_j) = \coprod_{\sigma} H_i(G(j),\mathbf{Z}\cdot\sigma),$$

where σ ranges over distince G-orbits of j-cells on W^{ind}. Since the stability subgroups G(j) of a j-cell act trivially on the j-cell, by using the universal coefficient theorem we obtain:

$$(4.3) \qquad H_i(G,C_j) = \coprod H_i(G(j)) \otimes C_j(W^{ind}/G) ,$$

where the sum ranges over equivalence classes of stability subgroups G(j) under conjugation. For $j \neq 2$, all of the stability subgroups G(j) are conjugate, and so the sum in (4.3) has only one term. For j = 2, the sum contains two terms, reflecting the different stability subgroups for Type I and Type II simplices. Since G(2) = {1} for Type II simplices, there is again only one term in the sum, and so

$$(4.4) \qquad H_i(G,C_j) = H_i(G(j)) \otimes C_j(W^{ind}/G),$$

where we use G(j) = SU(2) for j= 2.

We will now fill in the terms of the spectral sequence by using (4.4). We note that the boundary $"d^1$ is just ∂_C. Since $H_0(G(j)) = Z$, $"E^2_{0,j}$ is just the j-th homology of the reduced chain complex W^{ind}/G. From III.1 , this is 2-acyclic and so $"E^2_{0,1} = "E^2_{0,2} = 0$. For $"E^2_{2,0}$, we have $"E^2_{2,0} = H_2(SU(3))/d^1("E^1_{2,1})$; the image of $"d^1\colon "E^1_{i,1} \longrightarrow "E^1_{i,0}$ is not a priori zero. This will not cause us any difficulty, as we are only showing surjectivity with this argument. Next, since the isotropy subgroups SU(3), SU(2), and {1} are all perfect, $H_1(G(j)) = 0$, for all $j \geqslant 0$; and therefore $"E^1_{1,j} = 0$ for $j \geqslant 0$.

Since $C_*(W^{ind}/G)$ is reduced, $C_0(W^{ind}/G) = Z$, and so $"E^1_{2,0}$ is just $H_2(SU(3))$. We summarize the results in the following diagram for $"E^2$:

$$
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots \\
H_3(W^{ind}/G) & 0 & * & \cdots \\
 & & & \\
0 & 0 & * & \cdots \\
0 & 0 & * & * \\
Z & 0 & "E^2_{2,0} & "E^2_{3,0}
\end{array}
$$

Finally, since "$E^1_{p,q} \longrightarrow$ "$E^\infty_{p,q}$ is surjective, we see that $d\colon H_2(SU(3)) \longrightarrow H_2(G_2)$ is surjective. Note that $E^3_{2,0}$ need not be equal to $E^2_{2,0}$ since the image of d^3:"$E^3_{0,3} \longrightarrow$ "$E^3_{2,0}$ is not necessarily zero.

Rather than show the isomorphism by showing that d^3 is zero or by the 3-acyclicity of W^{ind}/G, we show that the map $H_2(SU(3)) \longrightarrow H_2(G_2)$ is injective by considering the following commutative diagram:

$$
\begin{tikzcd}
G_2 \arrow[r, "i'"] & O(7) \arrow[r, "f_1"] & O(8) \\
SU(3) \arrow[u, "i"] \arrow[rr, "f_3"'] & & SU(4) \arrow[u, "f_2"']
\end{tikzcd}
$$

Here we view the orthogonal groups as transformation groups of $\mathbf{O}$, which is just $\mathbf{R}^8$ if we forget the algebra structure. Then $O(7)$ is included in $O(8)$ as the subgroup leaving $F\cdot 1$ fixed, and G_2 is included in $O(7)$ as the subgroup of orthogonal transformations of $\mathbf{O}_o = R^7$

that also leaves fixed the triple scalar product [a,b,c].

The map from $SU(3)$ to $SU(4)$ is the natural inclusion, and the map from $SU(4)$ to $O(8)$ is given by forgetting the complex structure and taking the real part of the hermitian norm. By theorem 3.1 of Sah[15], the induced maps ${f^*}_j$ are isomorphisms on the level of H_2 for $j = 1,2,3$. Then i_* is injective.

We note that the preceeding techniques would have to be expanded to get results on the H_3 level.

V. The isoorphism conjecture of Milnor

If G is any Lie group, we let G^{δ} denote the same group with the discrete topology. If we use BG to denote the classifying space of G, we can consider the following conjecture of Milnor's :

> The map $H_*(BG^{\delta},\mathbf{F}_p) \dashrightarrow H_*(BG,\mathbf{F}_p)$ induced by the natural map $G^{\delta} \dashrightarrow G$ is an isomorphism, for any prime p.

The homology group $H_*(BG^{\delta},F)$ is the Eilenberg-MacLane homology of the group G, as we see by the following. In the case where G^{δ} is discrete, the space BG^{δ} has fundamental group G^{δ} , and EG^{δ} is the universal covering space of BG^{δ} . By the arguments in MacLane (Theorem 11.5) relating the homology of a space modulo its fundamental group with the group homology of the fundamental group, we see that $H_*(BG^{\delta},A)$ is the Eilenberg-MacLane homology of G. The homology group $H_*(BG,F)$ is the (simplicial) homology of the topological space BG. In both cases the action on the coefficient group F is trivial, i.e. untwisted.

To relate the two homology groups, we will need the following definition: For any Lie group G, the homotopy fibre $\bar{G}$ of the map $G^{\delta} \longrightarrow G$ is a topological group consisting of all pairs (g,f), where $g \in G^{\delta}$, and f is a path from the identity element to the image of g in G. The multiplication is pointwise. (See Berrick[2] or Thurston[24]).

The aim of this section is to prove the following piece of Milnor's conjecture:

Theorem 5.1 For the compact Lie group G_2 and $* \leqslant 2$, the map $H_*(BG^{\delta},\mathbf{F}_p) \longrightarrow H_*(BG,\mathbf{F}_p)$ induced by the natural map from $G^{\delta} \longrightarrow G$ is an isomorphism, for any prime p.

We shall actually prove an equivalent condition shown by Sah[15], namely:

(5.2) For $i > 0$, $H_i(B\bar{G},\mathbf{Z}) = H_i(B\bar{G})$ is a Q-vector space.

Another equivalent formulation is the following:

Lemma 5.3 (Lemma 1,[11]) The isomorphism conjecture is true for a connected Lie group G if and only if the associated space BG has the mod p homology of a point, for every prime p.

We will also need the following:

Lemma 5.4 (Lemma 6,[11]) : If G is a connected, semi-simple Lie group, then $H_1(B\bar{G})$ is zero, and there is a split exact sequence

$$0 \longrightarrow H_2(B\tilde{G}) \longrightarrow H_2(BG^{\delta}) \longrightarrow H_2(BG) \longrightarrow 0 ,$$

where $H_2(BG)$ can be identified with $\pi_1 G$, since G is connected.

Next, we record the following result of Sah:

Theorem 5.5 (Theorem 5.1, [15]). Let G be any connected, simply-connected Lie group with $G_{\mathbf{C}}$ denoting its connected, simply-connected complexification. Suppose that $G_{\mathbf{C}} \neq 1$ is a simple complex group of classical type. Then the natural complexification homomorphism induces injection on Schur multipliers:

$$H_2(G) \longrightarrow H_2(G_{\mathbf{C}}) \cong K_2(\mathbf{C}) = K_2(\mathbf{C})^+ \amalg K_2(\mathbf{C})^- .$$

The image of $H_2(G)$ is $K_2(\mathbf{C})^+$, a $\mathbf{Q}$-vector space of the dimension of the continuum.

We can now prove the main result of this section:

Proof of Theorem 5.1 : By Theorem 4.1, $H_2(G_2) = H_2(SU(2))$. By 5.5, $H_2(SU(2)) = K_2(\mathbf{C})^+$, which is the positive eigen-space of $K_2(\mathbf{C})$ under complex conjugation. As shown in Sah-Wagoner[16], this is a $\mathbf{Q}$-vector space of the dimension of the continuum.

Then, since G_2 is simply connected, by 4.1 and 5.4 we see that $H_2(BG_2)$ is also a $\mathbf{Q}$-vector space.

We note that in view of Theorem 5.1, Theorem 5.5 also holds for $G = G_2$. According to Sah[15], results of Steinberg[19] show that Theorem 5.1 holds also for the split G_2 case.

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