Traces, Indices and Spectral Theory of Toeplitz Operators on Multiply Connected Domains

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Abstract of the Dissertation

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In this thesis, we study Toeplitz operators on the multiply-connected domains by employing the technique of almost commuting operator pair theory. By concretely constructing the principal current for a pair of self-adjoint Toeplitz operator and a multiplication operator, we obtain an integral representation of the trace functional for the pair. Then we represent the index formula

as the boundary current, and therefore indices as winding numbers. We are able to decompose Toeplitz operators as a direct sum of Toeplitz operators on each component of the boundary. Therefore, we can treat Toeplitz operators with unbounded real symbols and calculate the deficiency indices. Also, we calculate the von Neumann multiplicity function for the absolutely continuous part of self-adjoint Toeplitz operators and give a criterion for such an operator to have purely absolutely continuous spectrum.

To my grandparents.

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Introduction

The nature of this work is an application of the general principal current theory to the study of Toeplitz operators on multiply-connected domains. The construction of the principal current for a pair of a self-adjoint Toeplitz operator and the multiplication by a function provides an example for the explicit computation of G-currents. It turns out that the calculation of G is reduced to each connected component of the domain which is a conformal image of the unit circle. The essential part of the calculation, namely the symbol calculation, is carried out on unit circle using the invariant principle for wave operators. This idea, which is due to Professor Pincus, was used in a preliminary study [27] coauthored with him, "Self-adjoint and Symmetric Toeplitz Operators on Multiply-Connected Plane Domains". This paper will appear in the Journal of Functional Analysis.

In comparison with the results in [27], we would like to emphasize the following:

a) In [27] we studied self-adjoint Toeplitz operators T_ψ on multiply-connected domains by comparing them to the position operator on the boundary of the domain. Here, following a suggestion by Professor Pincus, we achieve a generalization when Toeplitz operators with piecewise continuous symbol replace the position operator. We show that the previously developed formalism retains its validity in this more difficult situation. The discontinuities in this case create "extra" pieces for the principle current and for the joint essential spectrum of the operator pair, the later should be compared with the results in [29].

b) We establish a symbol criterion in Chapter IV for the absolute continuity of the spectra of self-adjoint Toeplitz operators on multiply-connected domains.

c) The dificiency index result about symmetric operators in Charpter III can also be viewed as solution theory for singular integral equations on contours, see [9], [10] for detailed explainations.

In the interest of completeness, and in the hope that a detailed presentation may benefit other workers who would like to make contributions to the principal current theory, we so organize this thesis that the results in [27] are included in the text and many details which had to be omitted from [27] because of the considerations of length given. For example, we give here in detail the construction of the principal current, which is a particularization of the general construction in Carey-Pincus [28] to our situation using the invariant principle, and of which only a very abbreviated version was presented in [27].

It also should be pointed out that more important than giving an example of calculating the G-current is the study of the unitary invariants of the operator pair, or equivalently, the C*-algebra generated by the pair. It is well known that the principal function of a pair of self-adjoint operators with a rank one commutator is a complete set of unitary invariant of the pair. In general we do not expect the G-current to give all the information about the unitary invariants. But we will see that, in addition to other results, we find interesting facts that the boundary of the support of G is the joint essential spectrum of the pair which is also considered as the maximal ideal space of the Calkin algebra; that for a Fredholm element in the C*-algebra, which is represented by a non-vanishing function on the boundary, the index is minus the winding number of the function about the origin; and that the spectral multiplicity of the absolutely continuous part of the self-adjoint Toeplitz operator is given by the number of arcs, in a sense to be specified in Chapter III, in the intersection of the current with the appropriate plane. For the pairs we study in this paper, the commutator is never of rank 1. But so many properties of the G-current resemble the corresponding properties of the principal function in the rank 1 commutator case. The only underlying nature of the resemblance we have so far understood is that there is an invisible rank one perturbation problem involved, namely the commutator of the multiplication with the Cauchy projection is of rank 1 and the difference between the orthogonal projection and Cauchy projection is a trace class operator.

In his thesis, Abrahamse established a decomposition of Toeplitz operator at the Calkin algebra level, in other word, a decomposition modulo compact operators. Our study of Toeplitz operator starts from a somewhat different point of view. The decomposition theorem in Chapter III says that a Toeplitz operator on multiply-connected domain is indeed a trace class perturbation of a direct sum of Toeplitz operators on the unit circle. As a matter of fact, the perturbation being trace class is crucial for quantative analysis problems such as the spectral multiplicity. Also because of this decomposition modulo the trace class, we find that the deficiency indices of symmetric Toeplitz operator is computed in terms of the restrictions of the symbol on each individual contour.

With the intention to obtain a complete set of unitary invariant, we also study the spectrum of self-adjoint Toeplitz operator. Since the spectral multiplicity of the absolutely continuous part can be counted from the symbol, if a self-adjoint Toeplitz operator has absolutely continuous spectrum, then the symbol is a complete set of unitary invariant. In this connection, we are able to designate in Chapter IV a class of self-adjoint Toeplitz operators that do have absolutely continuous spectrum. Particularly, those operators whose symbol has harmonic conjugate and continuous derivative are in this class.

We arrange all the materials as follows. In Chapter I, we collect all necessary preliminaries. Chapter II is devoted to the study of G-currents and trace and index formulas and essential spectra. We study self-adjoint and symmetric Toeplitz operator in Chapter III. The main theorem of this chapter is the decomposition theorem. We conclude the study of Toeplitz operator by a theorem about the absolutely continuity in Chapter IV.

Chapter I. Preliminaries

I.1. Almost commuting pairs and the G-current

The determining function theory has been established for many almost commuting operator pairs $\{A,B\}$. Usually, the approach is this. If $[A,B]$ is of trace class, then we first try to establish a functional calculus at least for those functions which are smooth on the product of the spectra, so that $[F(A,B), H(A,B)]$ is a trace class operator for test functions F and H. Then the bilinear functional $(F,H) \mapsto \mathrm{tr}[F(A,B), H(A,B)]$ will be studied. If A,B are self-adjoint operator and/or unitary operator, it turns out that this functional is an integral current with the determining function being the function that the Poisson bracket $\{F,H\}$ is integrated against (cf.[7], [8], [3]). Our study of the determining current (or G-current) will essentially follow this pattern with the pair consisting of a self-adjoint operator and a normal operator. The starting point is the following new development in the general theory.

Theorem I.1.1 (Carey and Pincus [6]). If $N = x_1 + i\, x_2$ is a bounded normal operator, and Y is a bounded self-adjoint operator with $[N,Y]$ in trace class. Then trace $[F(x_1,x_2,y),$

$H(x_1, x_2, y)] = [G](dF \wedge dH)$, where the current $[G] = (\mathcal{H}^2 \lfloor \Sigma) \wedge \eta g$. Σ is an $\mathcal{H}^2$ measurable and $(\mathcal{H}^2, 2)$ rectifiable subset of R^3, η is an $\mathcal{H}^2 \lfloor \Sigma$ summable 2-vector field which is $\mathcal{H}^2$ almost everywhere simple and is such that the tangent space $\tan^{(2)}(\mathcal{H}^2 \lfloor \Sigma, x)$ is associated with $\eta(x)$. We can take $\|\eta(x)\| = 1$ a.e. with respect to $\mathcal{H}^2\ \Sigma$, two dimensional Hausdorff measure in R^3 restricted to Σ, and the principal function, g, is $\mathcal{H}^2 \lfloor \Sigma$ summable. Furthermore, this can be done so that $\Sigma \subset \sigma(N) \times (y)$.

In the proof of this theorem, Carey and Pincus provide a construction of the current G (see [6]). We adapt this construction to our present situation. Since the pairs we treat later in Chapter II have much "smoothness", the technical complexity involved in the above theorem can be avoided. Therefore, we will not bother to explain the details of this theorem. Instead, we concentrate on notations and definitions we will need later. We would like to point out that while the question of the existence of an integral current representation for the trace functional has been settled, it is still interesting to explore the methods of actually calculating the G vector field. We will demonstrate the computation of such a G for the pair $\{T_\varphi \oplus \lambda, M_f\}$ where T_φ is a

Toeplitz operator and M_f is a multiplication operator.

In this section we only introduce the aspects of the general determining function theory that are relevant to our study. The concrete calculation of G will be left for Section II.2.

All the Hilbert spaces are assumed to be complex and separable. Let H be a Hilbert space, we denote by $\mathcal{L}(H)$ and $C_1(H)$ the algebra of all bounded linear operators on H and the trace class operator ideal respectively. Let $A \in \mathcal{L}(H)$ be a self-adjoint operator and let E_λ be its spectral measure. For any $x \in H$, $\mu_x(\Delta) = (E_\Delta x, x)$ is a Borel measure on real line $\mathbb{R}$. Let $H_{ac}(A) = \{x : \mu_x$ is absolutely continuous with respect to the Lebesque measure$\}$. Then it is easy to prove that $H_{ac}(A)$ is a closed subspace of H and reduces A (see [23]). The restriction A_{ac} of A to $H_{ac}(A)$ is called the absolutely continuous part of A. The orthogonal projection from H to $H_{ac}(A)$ is denoted by $P_{ac}(A)$.

Let A, B be two self-adjoint operators on Hilbert space H. We say that the wave operators $W_\pm(A,B)$ exist if the strong limits

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itA} e^{-itB} P_{ac}(B) = W_\pm(A,B)$$

exist.

The wave operators may not always exist. But when they do exist, we know the following:

Proposition I.1.2. Suppose $W_{\pm}(A,B)$ exist, then

a) $W_{\pm}(A,B)$ are partial isometries with initial space $P_{ac}(B)H$ and final spaces $W_{\pm}(A,B)H$.

b) $W_{\pm}(A,B)H$ are invariant for A and

$$A\, W_{\pm}(A,B) = W_{\pm}(A,B)B$$

c) $W_{\pm}(A,B)H \subset P_{ac}(A)$.

Remark. Here A,B are not necessarily bounded self-adjoint operators. When one of the operators is unbounded, $W_{\pm}(A,B)$ maps the range of B into the domain of A and has the intertwining property.

Suppose $W_{\pm}(A,B)$ exist for a pair A,B. They are called complete if $W_{\pm}(A,B)H = H_{ac}(A)$. The completeness of $W_{\pm}(A,B)$ can be described by the existence of other wave operators $W_{\pm}(B,A)$.

Proposition I.1.3. Suppose $W_{\pm}(A,B)$ exist, then they are complete if and only if $W_{\pm}(B,A)$ exist.

The proofs of Proposition I.1.3 and I.1.4 are easy and can be found in [23].

Hence if the wave operators $W_{\pm}(A,B)$ exist and are complete, then the absolutely continuous parts of A and B are unitarily equivalent. For a given pair A,B, there are many criteria for the existence and completeness of wave operators, among them we only mention one that will be used in our Toeplitz operator case.

<u>Theorem I.1.4</u> (Pearson's Theorem). Let A and B be self-adjoint operators and let J be a bounded operator. Suppose that there exists $C \in C_1(H)$ so that $C = AJ - JB$ in the sence that for all x in the domain of A and y in the domain of B

$$(Cy,x) = (Jy,Ax) - (JBy,x)$$

then strong limits

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itA}\, J\, e^{-itB}\, P_{ac}(B)$$

exist.

This theorem is a generalization of a result due to Carey and Pincus (see [5]) and we also refer its proof to [23].

This theorem shows that if two self-adjoint operators defer by a trace class operator, then they have the same absolutely continuous parts.

Now we present the symbol construction which is due to Carey and Pincus. Let A be a self-adjoint operator and let T be a bounded operator. By the above theorem, if $[A,T] \in \mathcal{C}_1(H)$ then strong limits

$$S_{\pm}(A;T) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itA}\, T\, e^{-iAt}\, P_{ac}(A)$$

exist. We will call $S_+(A,T)$ and $S_-(A,T)$ the positive and negative symbols of T with respect to A respectively.

The symbols have the following properties (see [5]). If $[A,T]$ and $[A,S]$ are in $\mathcal{C}_1(H)$, then

$S_{\pm}(A;T)$ belong to the commutant of A;

$S_{\pm}(A;T^*) = [S_{\pm}(A;T)]^*$;

$S_{\pm}(A;T+S) = S_{\pm}(A;T) + S_{\pm}(A;S)$;

$S_{\pm}(A;TS) = S_{\pm}(A;T)S_{\pm}(A;S)$;

$\|S_{\pm}(A;T)\| \le \|T\|$;

$S_{\pm}(A;T) = 0$ if T is a compact operator.

Let $M(A)$ be the C*-algebra generated by all bounded operators whose commutator with A is trace class. It is easy to see that $S_{\pm}(A;\cdot)$ extended to homomorphisms from $M(A)$ to $\mathcal{L}(H)$. Indeed, we have

Proposition I.1.5 [5]. For any $T \in M(A)$, $S_{\pm}(A;T)$ exist. The kernel of homomorphism $S_{\pm}$ contains all the compact operators.

Now we introduce certain functional calculus modulo the trace class ideal. Let $M(\mathbb{R}^3)$ be the collection of finite complex valued Borel measures ω on $\mathbb{R}^3$ satisfying

$$\|\omega\| = \int_{\mathbb{R}^3} (1+|r|)(1+|s|)(1+|t|)d\omega(r,s,t) < \infty.$$

The characteristic function of such an ω is the scalar function

$$F(s,y,z) = \int_{\mathbb{R}^3} \exp[irx + isz + itz]d\omega(r,s,t).$$

Let X,Y,A be bounded self-adjoint operators on H and $[X,Y] = 0$. Then we can associate with $\{X,Y,X\}$ an element $F(X,Y,A) \in \mathcal{L}(H)$ by the iterated integral

$$F(X,Y,A) = \int_{\mathbb{R}} (\int_{\mathbb{R}} (\int_{\mathbb{R}} F(x,y,z)dE_x)dF_y)dG_z)$$

where E,F,G are the spectral resolutions of X,Y,A respectively. Note it is important that $[X,Y] = 0$ for the integral to make sense (see [7], [8]). In fact if X does not commute with Y it is not known whether the iterated integral converges or not in any sense. But when $[X,Y] = 0$, $F(X,Y,A)$ has another form:

$$F(X,Y,A) = \int_{\mathbb{R}^3} e^{irX} e^{isY} e^{itA} \, d\omega(r,s,t)$$

[7], [8]. Let $\hat{M}(\mathbb{R}^3)$ be the collection of all the characteristic functions F of measures ω in $M(\mathbb{R}^3)$ and define $\|F\| = \|\omega\|$. Suppose that $[X,A]$, $[Y,A] \in e_1(H)$.

Proposition I.1.6. $F \mapsto F(X,Y,A)$ defines a *-homomorphism from $\hat{M}(\mathbb{R}^3)$ into $\mathcal{L}(H)$ modulo $C_1(H)$ in the sense that for any $F,G \in \hat{M}(\mathbb{R}^3)$,

$$F(X,Y,A)^* - \overline{F}(X,Y,A) \in C_1(H)$$

and

$$F(X,Y,A)G(X,Y,A) - (F \cdot G)(X,Y,A) \in C_1(H).$$

Furthermore, there exists a constant $L > 0$ such that

$$\|F(X,Y,A)^* - \overline{F}(X,Y,A)\|_\tau \le L\|F\|(\|[X,A]\|_\tau + \|[Y,A]\|_\tau)$$

and

$$\|F(X,Y,A)G(X,Y,A) - (F \cdot G)(X,Y,A)\|_\tau$$

$$\le L\|F\|\|G\|(\|[X,A]\|_\tau + \|[Y,A]\|_\tau).$$

This proposition is essentially Proposition 4.2 of [5] and the proofs are similar. It follows from this proposition immediately that $[F(X,Y,A),H(X,Y,A)] \in C_1(H)$ for any

$F,H \in \hat{M}(\mathbb{R}^3)$.

The last inequality of Proposition I.1.6 describes the dependence of $\|[F(X,Y,A), G(X,Y,A)]\|_1$ on F and G when X,Y,A are fixed. Another aspect of the dependence problem is that if F and G are fixed, how the trace norm changes when X,Y and A change. This question can be discussed in more generality. First we need to generalize the functional calculus to n variables. Let $\hat{M}(\mathbb{R}^n)$ be the collection of functions

$$F(\lambda_1,\ldots,\lambda_n) = \int_{\mathbb{R}^n} \exp(i\sum_{j=1}^{n} t_j\lambda_j)d\omega(t_1,\ldots,t_n)$$

where

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} (1+|t_j|)d|\omega|(t_1,\ldots,t_n) < \infty .$$

Let $A_1,\ldots,A_n$ be n self-adjoint operator and define

$$F(A_1,\ldots,A_n) = \int_{\mathbb{R}^n} e^{it_1A_1}\cdots e^{it_nA_n}\, d\omega(t_1,\ldots,t_n) .$$

As we mentioned before, this operator can not be expressed in terms of the spectral resolutions of $A_1,\ldots,A_n$. But this definition of functional calculus still makes perfect good sense. For $F,H \in \hat{M}(\mathbb{R}^n)$, the commutator $[F(A_1,\ldots,A_n),H(A_1,\ldots,A_n)]$ has an integral representation.

In fact if H is the characteristic function of measure μ, then $[F(A_1,\dots,A_n), H(A_1,\dots,A_n)]$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [e^{it_1A_1}\dots e^{it_nA_n}e^{is_1A_1}\dots e^{is_nA_n} - e^{is_1A_1}\dots e^{is_nA_n}e^{it_1A_1}\dots e^{it_nA_n}]\,d\omega d\mu$$

$$= \sum_{j=1}^{n}\int_{\mathbb{R}^n} e^{it_1A_1}\dots[e^{it_jA_j}, e^{is_1A_1}]\dots e^{it_nA_n}e^{is_2A_2}\dots e^{is_nA_n}$$

$$d\omega(t_1,\dots,t_n)d\mu(s_1,\dots,s_n)$$

$$+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [e^{is_1A_1}e^{it_1A_1}\dots e^{it_nA_n}e^{is_2A_2}\dots e^{is_nA_n} - e^{is_1A_1}\dots e^{is_nA_n}e^{it_1A_1}\dots e^{it_nA_n}]$$

$$d\omega(t_1,\dots,t_n)d\mu(s_1,\dots,s_n)$$

$$= \dots$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{is_1A_1}\dots e^{is_{k-1}A_{k-1}}e^{it_1A_1}\dots[e^{it_jA_j}, e^{is_kA_k}]e^{it_nA_n}$$

$$e^{is_{k+1}A_{k+1}}\dots e^{is_nA_n}d\omega(t_1,\dots,t_n)d\mu(s_1,\dots,s_n)$$

Let A be a self-adjoint operator and let T be an operator, then

$$[e^{iAt},T] = (e^{iAt}Te^{-iAt}-1)e^{iAt}$$

$$= \int_0^t \frac{d}{d\tau} e^{iA\tau}Te^{-iA\tau}d\tau\, e^{iAt}$$

$$= i\int_0^t e^{iA\tau}[A,T]e^{i(t-\tau)A}d\tau.$$

If B is a self-adjoint operator, then

$$[e^{itA},e^{isB}] = i\int_0^t e^{i\tau A}[A,e^{isB}]e^{i(t-\tau)A}d\tau$$

$$= i\int_0^t e^{i\tau A}\; i\int_0^s e^{i\sigma B}[A,B]^{i(s-\sigma)B}\, d\sigma\; e^{i(t-\tau)A}d\tau$$

$$= -\int_0^t\int_0^s e^{i\tau A}\; e^{i\sigma B}[A,B]e^{i(s-\sigma)B}\; e^{i(t-\tau)A}d\sigma d\tau\ .$$

Therefore we have

$$[F(A_1,\ldots,A_n),\ H(A_1,\ldots,A_n)]$$

$$= -\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^{t_j}\int_0^{s_k} e^{is_1A_1}\ldots e^{is_{k-1}A_{k-1}}$$

$$e^{it_1A_1}\ldots e^{i\tau A_j}e^{i\sigma A_k}$$

$$[A_j,A_k]e^{i(s_k-\sigma)A_k}e^{i(t-\tau)A_j}\cdots e^{it_nA_n}e^{is_{k+1}A_{k+1}}\cdots e^{is_nA_n}$$

$$d\sigma d\tau d\omega(t_1,\ldots,t_n)d\mu(s_1,\ldots,s_n).$$

From this we can conclude that if $[A_i,A_j]$ is of trace class for each pair i,j, then so is $[F(A_1,\ldots,A_n), H(A_1,\ldots,A_n)]$. Let $\{A_1^k,\ldots,A_n^k\}_{p=1}^{\infty}$ be self-adjoint operators such that

$$\underset{p\to\infty}{\text{s-lim}}\ A_j^p = A_j, \quad j = 1,\ldots,n$$

and $[A_i^p,A_j^p] \in C_1$ for any i, j and p and

$$\lim_{p\to\infty}\|[A_i^p,A_j^p] - [A_i,A_j]\|_1 = 0$$

for each pair of i,j. It is clear that for fixed $s_1,\ldots,s_n$, $t_1,\ldots,t_n$, τ,σ,

$$e^{is_1A_1^p}\cdots e^{is_{k-1}A_{k-1}^p}e^{it_1A_1^p}\cdots e^{i\tau A_j^p}e^{i\sigma A_k^p}[A_j^p,A_k^p]e^{i(s_k-\sigma)A_k^p}$$

$$e^{i(t_j-\tau)A_j^p}\cdots e^{it_nA_n^p}$$

$$\times e^{is_{k+1}A_{k+1}^p}\cdots e^{is_nA_n^p}$$

converges to

$$e^{is_1A_1}\cdots e^{is_{k-1}A_{k-1}}e^{it_1A_1}\cdots e^{i\tau A_j}e^{i\sigma A_k}[A_j,A_k]e^{i(s_k-\sigma)A_k}$$

$$e^{i(t_j-\tau)A_j}\cdots e^{it_nA_n}e^{is_{k+1}A_{k+1}}\cdots e^{is_nA_n}$$

in trace norm. By the Lebesgue dominated convergence theorem we have

Proposition I.1.7.

$$\lim_{p\to\infty}\|[F(A_1^p,\ldots,A_n^p),\ H(A_1^p,\ldots,A_n^p)] - [F(A_1,\ldots,A_n),$$

$$H(A_1,\ldots,A_n)]\|_1 = 0 .$$

Obviously, $\mathrm{tr}[F(X,Y,A),\ H(X,Y,A)]$ is a bilinear functional defined on $\hat{M}(\mathbb{R}^3)$. We will give a representation for this functional in terms of integral and local data of operators. We start with direct integral representations of H. Suppose X and Y have spectral measures μ and ν respectively. Then the Hilbert space H has the following direct integral decompositions

$$H = \int_{\sigma(X)} \oplus\, L_\lambda d\mu(\lambda)$$

and

$$H = \int_{\sigma(Y)} \oplus\, K_\lambda d\nu(\lambda)$$

so that under the corresponding decompositions of the space, X and Y are the multiplications by the coordinates. Since Y commutes with X, we have

$$(Yf)(\lambda) = Y(\lambda)f(\lambda)$$

where $Y(\lambda) \in \mathcal{L}(L_\lambda)$ and $f \in H$. If $\alpha(\lambda)$ is in the point spectrum of $Y(\lambda)$, we denote by $F_{\lambda,\alpha(\lambda)}$ the orthogonal projection from L_λ onto the eigenspace of $Y(\lambda)$ corresponding to the eigenvalue $\alpha(\lambda)$. Because $S_\pm(X,A)$ are in the commutant of X, they have the decompositions

$$(S_\pm(X,A)f)(\lambda) = S_\pm(X,A)(\lambda)f(\lambda)$$

with $S_\pm(X,A)(\lambda) \in \mathcal{L}(L_\lambda)$. But $S_\pm(X,A)$ commute with Y too, so $S_\pm(X,A)(\lambda)$ commute with $Y(\lambda)$ for (Lebesgue) almost all λ (by the definition of $S_\pm(X,A)$). Therefore, $S_\pm(X,A)(\lambda)$ preserves the eigenspace $F_{\lambda,\alpha(\lambda)}L_\lambda$. By [5], Remark 3.1,

$$S_+(X,A)(\lambda) - S_-(X,A)(\lambda) \in e_1(L_\lambda).$$

Thus there exists a phase shift $g(\lambda,\alpha(\lambda),\cdot)$ for the perturbation problem

$$S_+(X,A)(\lambda)F_{\lambda,\alpha(\lambda)} \to S_-(X,A)(\lambda)F_{\lambda,\alpha(\lambda)}$$

for a.e. λ (see [5], [8]).

Similarly, under the decomposition

$$H = \int_{\sigma(Y)} \oplus K_\xi d\nu(\xi)$$

we have $X = \int_{\sigma(Y)} \oplus X(\xi)d\nu(\xi)$ and $S_\pm(Y,A)$

$= \int_{\sigma(Y)} \oplus S_\pm(Y,A)(\xi)d\nu(\xi)$. For each $\beta(\xi)$ in the point spectrum of $X(\xi)$, let $E_{\beta(\xi),\xi}$ be the projection from K_ξ to the corresponding eigenspace. Then there also exists a phase shift for the perturbation problem

$$S_+(Y,A)(\xi)E_{\beta(\xi),\xi} \to S_-(Y,A)(\xi)E_{\beta(\xi),\xi} .$$

Let

$$J_{13}(F,H) = \det\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial z} \end{pmatrix}(x,y,z)$$

and

$$J_{23}(F,H) = \det\begin{pmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix}(x,y,z)$$

where $F,H \in \hat{M}(\mathbb{R}^3)$. Let σ_p denote point spectrum.

<u>Theorem I.1.8</u> [8]. The trace functional has the representation

$\mathrm{tr}[F(X,Y,A),\ H(X,Y,A)]$

$$= \frac{i}{2\pi}\{\iint \sum_{\alpha(\lambda)\in\sigma_p(Y(\lambda)} J_{13}(F,H)(\lambda,\alpha(\lambda),z)g(\lambda,\alpha(\lambda),z)d\lambda dz$$

$$+ \iint \sum_{\beta(\xi)\in\sigma_p(X(\xi))} J_{23}(F,H)(\beta(\xi),\xi,z)h(\beta(\xi),\xi,z)d\xi dz\}.$$

In Chapter II we will calculate g, h, $F_{\alpha(\lambda),\lambda}$ and $E_{\beta(\xi),\xi}$ in terms of the local data of Toeplitz operators and derive a much simpler form of the representation. In fact, $X + iY$ will be the multiplication by a smooth function, $\dim L_\lambda$ and $\dim K_\xi$ will be uniformly bounded and $\alpha(\lambda)$, $\beta(\xi)$ will turn out to be the local parameters of the boundary curves.

I.2 Hardy spaces and Toeplitz operators

Let D be an open set in complex plane $\mathbb{C}$ bounded by analytic Jordan curves. In this section we summarize some basic facts about Hardy spaces and Toeplitz operators on such plane domain D. Although many results are also true for H^p spaces, we will only consider H^2. The boundary of D consists of n + 1 non-intersecting smooth curves, which are denoted as $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ where Γ_0 is the outer boundary of D. Because of the smoothness, Direchlet problem can always be solved for continuous functions on ∂D. In other words, given continuous function u on ∂D, there exists a continuous function $\hat{u}$ on $\bar{D}$ such that $\hat{u}$ restricted to D is harmonic and $\hat{u} = u$ on ∂D [9]. Pick $z_0 \in D$, then $u \mapsto \hat{u}(z_0)$ is a positive continuous functional on $C(\partial D)$. Hence there exists a measure ω_{z_0}, which is called the harmonic measure with respect to z_0, on ∂D so that

$$\hat{u}(z_0) = \int_{\partial D} u(\lambda)\, d\omega_{z_0}(\lambda)$$

ω_{z_0} and the arc length measure on ∂D are mutually boundedly equivalent [9], therefore, so are ω_{z_0} and ω_z for any $z_0, z_1 \in D$. We will henceforth fix a point $z_0 \in D$ as the base point and drop the subscript of ω.

The Hardy space $H^2(D)$ is the collection of analytic functions f on D such that $|f|^2$ has a harmonic majorant on D. The norm $\|f\|_2$ is defined to be $\inf\{u^{1/2}(z_0) : u$ is a harmonic majorant of $|f|^2\}$. There is an alternative definition for the Hardy space. Let A(D) be the algebra of analytic functions on D with continuous extension to $\bar{D}$. We restrict the functions in A(D) to the boundary ∂D and denote by $H^2(\partial D, \omega)$ the $L^2(\partial D, \omega)$ closure of these restrictions. $H^2(D)$ and $H^2(\partial D, \omega)$ can be naturally identified.

<u>Theorem I.2.1.</u> Let $f \in H^2(D)$. Then f has nontangential limiting value f^* on ∂D ω - a.e., and $f^* \in H^2(\partial D, \omega)$. $f \mapsto f^*$ is an isometric isomorphism from $H^2(D)$ to $H^2(\partial D, \omega)$. Furthermore, f is the harmonic extension of f^*. f can also be recovered from f^* by the formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f^*(\lambda)}{\lambda - z} \, d\lambda , \qquad z \in D.$$

This theorem and its proof can be found in [1]. From now on we will identify every f in $H^2(D)$ with its boundary value and use $H^2(D)$ as the only symbol for the Hardy space. Note that the correspondence

$$f \mapsto \frac{1}{2\pi i} \int_{\partial D} \frac{f(\lambda)}{\lambda - z} \, d\lambda , \qquad f \in L^2(\partial D, \omega)$$

defines the Cauchy projection which will be denoted by P.

Let g be the Green's function per D with pole z_0. Then g is a C^∞ function on ∂D and has a positive tangential derivative $\frac{\partial g}{\partial \tau}(\lambda) > 0$ on ∂D. Let h be a multi-valued harmonic conjugate of $-g$, and let ν be the derivative of $-g + ih$. ν is analytic on a neighborhood of ∂D. Let m be the arc length measure on ∂D. Then the measure $d\omega$, dm and dz have the following relation

$$d\omega(\lambda) = \frac{1}{2\pi} \frac{\partial g}{\partial \tau}(\lambda) dm(\lambda)$$

$$d\omega(\lambda) = \frac{1}{2\pi i} \nu(\lambda) d\lambda \qquad \text{(see [1])}.$$

Let $H_0^2(D)$ denote the subspace of functions f in $H^2(D)$ such that $f(z_0) = 0$. $H^2(D)$ and $\overline{H_0^2}(D)$, where bar denotes the complex conjugate, are subspaces of $L^2(\partial D, \omega)$ and it is easy to see that $H^2(D) \perp \overline{H_0^2}(D)$. Let $N = L^2(\partial D, \omega) \ominus [H^2(D) \oplus \overline{H_0^2}(D)]$.

<u>Theorem I.2.2</u> [11]. $\dim N = n$, the number of the inner-boundary components. N has a basis consisting of real valued C^∞ functions.

This theorem will be used in certain important estimates. The decomposition $L^2(\partial D, \omega) = H^2(D) \oplus N \oplus \overline{H_0^2}(D)$ will be regarded as the standard decomposition of the L^2 space.

By the Reimann mapping theorem, there is a holomorphic function π mapping the unit disk Δ onto D (0 to z_0) and Δ is the universal covering via π [1]. Let G be the group of covering transformation, the G is a subgroup of linear transformations from Δ to Δ. The set L of limiting points of G, consisting points that are accumulations of $\{\sigma_k(z)\}$ with $\{\sigma_k\} \subset G$ and $z \in D$, is a closed subset of $\partial\Delta = S^1$ and has Lebesgue measure 0. π can be analytically continuated to Δ' that contains a neighborhood of $S^1 \backslash L$. Δ is mapped by π onto the Schottky double D' of D [11].

Let H_G^2 and L_G^2, respectively, be the subspaces of H^2 and $L^2 = L^2(S^1)$ that are invariant under the action of G. Then π induces an isometry from $L^2(\partial D, \omega)$ onto L_G^2. In fact, let $f \in L^2(\partial D, \omega)$ and let u be the harmonic extension of $|f|^2$ to D, then $u \circ \pi$ is a harmonic function on Δ and obviously extends $|f \circ \pi|^2$. Hence $\|f\|_{L^2(\partial D, \omega)} = [u(z_0)]^{1/2} = [u(\pi(0))]^{1/2}$ $= \|f \circ \pi\|_{L^2(S^1)}$. Thus, $f \mapsto f \circ \pi$ is an isometry and $f \circ \pi \in L_G^2$. Let $g \in L_G^2$, then $|g|^2$ is invariant under G and so is the harmonic extension v. Define $\hat{v}(\pi(\lambda)) = v(\lambda)$ and $\hat{g}(\pi(z)) = h(z)$, where $\lambda \in \Delta$ and $z \in S^1 \backslash L$. Then $\hat{v}$ is the harmonic extension of $|\hat{g}|^2$. Therefore, $\hat{g} \in L^2(\partial D, \omega)$ and it is easy to see that

$\hat{g} \circ \pi = g$. Hence $U : f \mapsto f \circ \pi$ is an isometry from $L^2(\partial D, \omega)$ onto L^2_G. Obviously U maps $H^2(D)$ onto H^2_G.

Let $P : L^2(\partial D, \omega) \to H^2(D)$ be the orthogonal projection. For $\varphi \in L^\infty(\partial D, \omega)$, the Toeplitz operator T_φ is defined as $T_\varphi f = P(\varphi f)$ for $f \in H^2(D)$. To carry out some crucial estimates in Chapter III, we need a concrete form of P. In the case D is the unit disk, P is just the Cauchy integral. But when D is not simply connected P does not have a simple representation. Nevertheless, we will later show that the difference between P and the Cauchy projection is always trace class. But first, we will seek a representation of P in H^2_G which turns out to be very closely related to the Cauchy projection on S^1.

Corresponding to the decomposition $L^2(\partial D, \omega) = H^2(D) \oplus N \oplus \overline{H}^2_0(D)$, L^2_G has the decomposition

$$L^2_G = H^2_G \oplus N_G \oplus \overline{H}^2_{G,0}$$

Let P_G and P_1 be the orthogonal projections from L^2_G onto H^2_G and L^2 onto N_G respectively. Then for $g \in L^2_G$, $P_G g = P_G(1-P_1)g$. But $H^2_G = L^2_G = L^2_G \cap H^2$, so the Cauchy integral will serve as the projection on $H^2_G \oplus \overline{H}^2_{G,0}$. This proves

<u>Proposition I.2.3.</u> For $g \in L^2_G$,

$$(P_G g)(z) = \frac{1}{2\pi i}\int_{S'} \frac{g(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{S'} \frac{(P_1 g)(\tau)}{\tau - z}d\tau .$$

Note that P_1 has rank n.

<u>Corollary I.2.4.</u> Let $\varphi \in L^\infty(\partial D, \omega)$. Then Toeplitz operator T_φ is unitarily equivalent to $T^G_{\varphi\circ\pi}$ on H^2_G on H^2_G where

$$(T^G_{\varphi\circ\pi} f)(z) = \frac{1}{2\pi i}\int_{S'} \frac{\varphi\circ\pi(\tau) f(\tau)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{S'} \frac{(P_1 \varphi\circ\pi f)(\tau)}{\tau - z}d\tau$$

for $f \in H^2_G$. If we denote by $T_{\psi\circ\pi}$ the usual Toeplitz operator on H^2, then $T^G_{\psi\circ\pi} = (T_{\psi\circ\pi} - PP_1 M_{\psi\circ\pi})|H^2_G$.

For the purpose of decomposing Toeplitz operator on D, we introduce another decomposition of $L^2(\partial D, \omega)$.

$\partial D = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_n$, so $L^2(\partial D, \omega) = L^2(\Gamma_0, \omega_0) \oplus L^2(\Gamma_1, \omega_1)$ $\oplus \ldots \oplus L^2(P_n, \omega_n)$ where ω_i is the restriction of ω on Γ_i, $i = 0, 1, \ldots, n$. Let D_0 be the interior of Γ_0 and let D_i be the exterior of Γ_i union ∞, $i = 1, \ldots, n$. Let $A(D_i)$ be the set of holomorphic functions on D_i with continuous extension to $\overline{D_i}$. Let H_i be the closure of $A(D_i)$ in $H^2(D)$, then

$$H^2(D) = H_0 \dotplus H_1 \dotplus \ldots \dotplus H_n$$

Denote by A_i the restrictions of functions in $A(D_i)$ to P_i and let K_i be the $L^2(\Gamma_i, \omega_i)$ closure of A_i. Then

$$L^2(\partial D, \omega) = (K_0 \oplus L_0) \oplus (K_1 \oplus L_1) \oplus \dots \oplus (K_n \oplus L_n)$$

where $L_i = L^2(\Gamma_i, \omega_i) \ominus K_i$, $i = 1, \dots, n$. Each K_i can be naturally identified with H^2. In fact, let $\pi_i(\pi_0)$ be a conformal mapping from Δ onto the exterior (interior) of $\Gamma_i(\Gamma_0)$ respectively. Note π_0 preserves the positive tangent direction and π_i map reverses the positive direction on Γ_i. Thus, $\pi_0, \dots, \pi_n$ give the natural positive tangent direction on ∂D. We define X_{π_i} by $X_{\pi_i} f = f \circ \pi_i$ for $f \in L^2(\Gamma_i, \omega_i)$, $i = 0, 1, \dots, n$. By the smoothness assumption of Γ_i, $|\pi_i'|$ is bounded from above and below on S^1 and therefore, X_{π_i} is a homeomorphism. Let A be the algebra of analytic functions on Δ with continuous extension to $\overline{\Delta}$. Then clearly $X_{\pi_i} A(D_i) = A$. Hence $X_{\pi_i} K_i = H^2$. For each $f \in L^2(\Gamma_i, \omega_i)$,

$$\begin{aligned}
\|X_{\pi_i} f\|^2 &= \int_{S^1} |f(\pi_i(\lambda))|^2 dm(\lambda) \\
&= \int_{\Gamma_i} |f(\zeta)|^2 dm(\pi_i^{-1}(\zeta)) \\
&= \int_{\Gamma} |f(\zeta)|^2 |\pi_i^{-1\prime}(\zeta)| dm_i(\zeta) \\
&= \int_{\Gamma} |f(\zeta|^2 |\pi_i^{-1\prime}(\zeta)| \beta_i(\zeta) d\omega_i(\zeta)
\end{aligned}$$

where m_i is the arc length measure on Γ_i and $\beta_i d\omega_i = dm_i$.

There exist $\delta, M > 0$ such that

$$0 < \delta \leq \beta_i(\zeta)|\pi_i^{-1'}(\zeta)| \leq M < \infty .$$

Let h_i be an outer function on Δ such that

$$|h_i(\lambda)|^2 = \beta_i(\pi_i(\lambda))|\pi_i^{-1'}(\pi_i(\lambda))| .$$

Define $U_i f = h_i \chi_{\pi_i} f$ for $f \in L^2(\Gamma_i, \omega_i)$. Then it is easy to see that U_i is a unitary operator and $U_i K_i = H^2$. Thus we also have $U_i(L^2(\Gamma_i, \omega_i) \ominus K_i) = L^2 \ominus H^2$. If P_i is the orthogonal projection from $L^2(\Gamma_i, \omega_i)$ onto K_i, then $U_i P_i U_i^*$ is the orthogonal projection from L^2 onto H^2. Let $\varphi_i \in L^\infty(\Gamma_i, \omega_i)$, then $U_i P_i \varphi_i f = U_i P_i U_i^* U_i \varphi_i f$ for any $f \in K_i$. Hence, $U_i P_i M_{\varphi i} U^* = T_{\varphi \circ \pi_i}$, the usual Toeplitz operator on H^2. The above construction of U_i is due to Foias. Let $U = U_0 \oplus U_1 \oplus \ldots \oplus U_n$. Then U is a unitary operator from $L^2(\partial D, \omega)$ onto $L^2 \oplus \ldots \oplus L^2$ (n+1 copies) and

$$U[K_0 \oplus K_1 \oplus \ldots \oplus K_n] = H^2 \oplus \ldots \oplus H^2$$

On K_i, we define $T_\psi^i = P_i M_\psi$, $\psi \in L^\infty(\Gamma_i, \omega_i)$. Then if $\varphi_i \in L^\infty(\Gamma_i, \omega_i)$ for $i = 0, 1, \ldots, n$, then

$$U[T_{\varphi_0}^0 \oplus \ldots \oplus T_{\varphi_n}^n]U^* = T_{\varphi_0 \circ \pi_0} \oplus \ldots \oplus T_{\varphi_n \circ \pi_n} .$$

Let $\mathcal{J}$ be the C^* algebra generated by $\{T_\varphi : \varphi \in L^\infty(\partial D, \omega)\}$ and let $\mathcal{J}_i$ be the C^* algebra generated by $\{T^i_\varphi : \varphi_i \in L^\infty(\Gamma_i, \omega_i)\}$. Let $\mathcal{K}$ be the ideal of compact operators in $\mathcal{L}(H^2(D))$ and let $\mathcal{K}_i$ be the ideal of compact operators in $\mathcal{L}(K_i)$. Abrahamse has the following theorem which gives a decomposition of Toeplitz operator at C^* algebra level.

<u>Theorem I.2.5 [1]</u>. The correspondence $T_\varphi + \mathcal{K} \mapsto (T^0_{\varphi|\Gamma_0} + \mathcal{K}_0) \oplus (T^1_{\varphi|\Gamma_1} + \mathcal{K}_1) \oplus \ldots \oplus (T^n_{\varphi|\Gamma_n} + \mathcal{K}_n)$ is a C^*-algebra isomorphism from $\mathcal{J}/\mathcal{K}$ onto $\mathcal{J}_0/\mathcal{K}_0 \oplus \mathcal{J}_1/\mathcal{K}_1 \oplus \ldots \oplus \mathcal{J}_n/\mathcal{K}_n$.

The theory of Toeplitz operators on multiply-connected domains has been developed resembling that of Toeplitz operators on the unit disk (see [1]). However, since the topology of the domain is essentially different, new phenomena occur in the multiply-connected case. For example, on a multiply-connected domain, the spectrum of a Toeplitz operators is not necessarily connected; it is known that a self-adjoint operator can have discrete spectrun ([1]). Abrahamse's Theorem gives some explanation as to why this is so. Roughly speaking, up to a compact perturbation, a Toeplitz operator is pieced together from local data of the symbol on each of the contours.

Theorem III.3.1 shows that the interaction between the contours is actually even weaker than compact perturbation. Since the orthogonal complement of $H^2(D)$ in $L^2(\partial D, \omega)$ contains bounded real function, it is natural that self-adjoint Toeplitz operator may have point spectrum. It seems reasonable to believe that the subspace N is responsible for the existence of singular spectrum of self-adjoint Toeplitz operator. Indeed we will later see that when a real symbol is perpendicular to N, the associated Toeplitz operator does not have point spectrum.

We conclude this section by two lemmas concerning integral operators on contours. Let $k_1, k_2 \in L^\infty(\partial D \times \partial D)$ and let $f_1, \ldots, f_m, g_1, \ldots, g_m \in L^\infty(\partial D, \omega)$.

Lemma I.2.6. Integral operator defined by

$$(Kf)(z) = \int_{\partial D} k(\lambda, z) f(\lambda) d\omega(\lambda)$$

where

$$k(\lambda, z) = \int_{\partial D} k_1(\lambda, \sigma) k_2(\sigma, z) d\omega(\sigma) + \sum_{j=1}^{m} \bar{f}_j(\lambda) g_j(z)$$

is a trace class operator on $L^2(\partial D, \omega)$. Furthermore, there exists a constant $L > 0$ such that for any $\varphi, \psi \in L^2(\partial D, \omega)$, KM_φ and $M_\varphi K$ are also trace class operators and

$$\|KM_\varphi\|_1 \leq L\|\varphi\|_{L^2}, \ \|M_\varphi K\|_1 \leq L\|\varphi\|_{L^2}, \|M_\psi KM_\varphi\|_1 \leq L\|\psi\|_{L^2} \ \|\varphi\|_{L^2}.$$

Proof. Let

$$(K_i f)(z) = \int_{\partial D} k_i(\lambda,z)f(\lambda)d\omega(\lambda), \quad i = 1,2,$$

then $K = K_2K_1 + \sum_{j=1}^{m} f_j \otimes g_j$. Since $k_1, k_2 \in L^\infty(\partial D \times \partial D)$,

K_2, K_1M_φ are Hilbert-Schmidt operators for any $\varphi \in L^2(\partial D, \omega)$.

Hence KM_φ is trace class and

$$\|M_\psi KM_\varphi\|_1 \leq \|(M_\psi K_2)(K_1M_\varphi)\|_1 + \sum_{j=1}^{m} \|\overline{\varphi}f_j \otimes \overline{\psi}g_j\|$$

$$\leq \|M_\psi K_2\|_2 \ \|K_1M_\varphi\|_2 + \sum_{j=1}^{m} \|\overline{\varphi}f_j\|_{L^2} \ \|\overline{\psi}g_j\|_{L^2}$$

$$\leq (\|K_1\|_\infty \ \|K_2\|_\infty + \sum_{j=1}^{m} \|f_j\|_\infty \ \|g\|_\infty) \ \|\psi\|_{L^2} \ \|\varphi\|_{L^2}.$$

Lemma I.2.7. Let $k(\lambda,z) \in C^1(\partial D \times \partial D)$ and define

$$(Kf)(z) = \int_{\partial D} k(\lambda,z)f(\lambda)d\omega(\lambda).$$

Then for any $\varphi \in L^2(\partial D, \omega)$, KM_φ and $M_\varphi K$ are trace class operators and there exists $L > 0$ such that

$$\|KM_\varphi\|_1 \leq L\|\varphi\|_{L^2}, \ \|M_\varphi K\|_1 \leq L\|\varphi\|_{L^2}.$$

L is independent of φ

Proof. Let

$$k_j(\lambda,z) = \begin{cases} K(\lambda,z) & \text{if} \quad z \in \Gamma_j \\ 0 & \text{if} \quad z \notin \Gamma_j \end{cases}$$

$j = 0,1,\dots,n$ and define

$$(K_j f)(z) = \int_{\partial D} k_j(\lambda,z) f(\lambda) d\omega(\lambda).$$

It suffices to prove that $K_j M_\varphi$ is in trace class and $\|K_j M_\varphi\|_1 \leq L\|\varphi\|_{L^2}$. Fix $z_j \in \Gamma_j$ and let $\chi_j(\sigma,z)$ be the characteristic function of the positive arc between z_j and z. Let $\frac{\partial}{\partial\tau}$ denote the tangential derivative on Γ_j. Then

$$k_j(\lambda,z) = \int_{\Gamma_j} \frac{\partial}{\partial\tau} k_j(\lambda,\tau)\chi_j(\tau,z)d\tau + k_j(\lambda,z_j).$$

The proof follows from Lemma I.2.6 immediately.

I.3 Symmetric operator as a limit of self-adjoint operators.

It is natural to extend the study of Toeplitz operator to the case where the symbols are unbounded. But to do so, there is a question that has to be settled first. That is, what is the (or a) natural domain for such an operator and how to define it. We do not know the answer in general. But when the symbol is real and satisfies some integrability conditions, there is some special approach. In this case, we first truncate the symbol to bounded functions and then consider the sequence of resolvents of Toeplitz operator with truncated symbols. In the end we can derive a symmetric operator that behaves like a Toeplitz operator with an unbounded symbol, or to be exact, a symmetric operator that we mean by Toeplitz operator with unbounded symbol. The interesting point is that one can prescribe the dificiency indices in terms of the discontinuity of the symbol. This construction follows a pattern introduced in Pincus [19]. We sketch the scheme in this section and leave the detailed calculation to Chapter III.

Let $\{A_n\}$ be a sequence of bounded self-adjoint operators defined on a Hilbert space H. Assume that $\{A_n\}$ satisfies the following two conditions:

(i) $R(\lambda) = \lim_{n\to\infty}(A_n-\lambda)^{-1}$ exists in the weak operator topology for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$;

(ii) $-\lim_{g\to\infty, t\in\mathbb{R}} itR(it) = I$ in the weak operator topology.

Then $R(\lambda)$ is an operator valued holomorphic function on $\mathbb{C}\setminus\mathbb{R}$. Furthermore, $R(\bar{\lambda}) = R(\lambda)^*$ and $\operatorname{Im} R(\lambda) \geq 0$ if $\operatorname{Im}\lambda \gg 0$. Thus there exists a positive-contraction-valued measure F_t on $\mathbb{R}$ such that

$$R(\lambda) = \int_{\mathbb{R}} \frac{dF_t}{t-\lambda} .$$

By the Naimark dilation theorem (see Appendix I, Theorem 2, [2]) there is a Hilbert space $\hat{H} \supset H$ and an orthogonal resolution E_t of identity I such that $\hat{P}E_t|H = F_t$, where $\hat{P} : \hat{H} \to H$ is the orthogonal projection. If we require that $\vee\{E_tH : t \in \mathbb{R}\} = \hat{H}$, then $\{E_t\}$ is essentially unique. Let

$$\hat{A} = \int_{\mathbb{R}} t dE_t$$

and $\hat{R}(\lambda) = (\hat{A}-\lambda)^{-1}$, then $R(\lambda) = P\hat{R}(\lambda)|H$. Define

$N_\lambda = \hat{P}\hat{R}(\bar{\lambda})\hat{R}(\lambda)\hat{P} - R(\bar{\lambda})R(\lambda)$. For any $x, y \in H$, we have $(\hat{R}(\lambda)x, \hat{R}(\lambda)y) = (R(\lambda)x, R(\lambda)y) + (N_\lambda x, y)$. So $x \in \ker N_\lambda$ if and only if $\|R(\lambda)x\| = \|R(\lambda)x\| = \|\hat{P}\hat{R}(\lambda)x\|$. This implies that $\hat{R}(\lambda)\,|\ker N_\lambda = R(\lambda)\,|\ker N_\lambda$.

<u>Lemma I.3.1.</u>

$$\ker N_\lambda = \{x : x \in H, \lim_{n\to\infty}\|(A_n-\lambda)^{-1}x - R(\lambda)x\| = 0\}.$$

In other words, $x \in \ker N_\lambda$ if and only if $\{A_n-\lambda)^{-1}x\}$ converges to $R(\lambda)x$ in the H norm.

<u>Proof.</u> Suppose $\|(A_n-\lambda)x - R(\lambda)x\| \to 0$ as $n \to \infty$, then it is easy to see that $\lim_{n\to\infty}\|(1+(\lambda-\bar{\lambda})(A_n-\lambda)^{-1})x\| = \|(1+(\lambda-\bar{\lambda})R(\lambda))x\|$. Hence $\|P(1+(\lambda-\bar{\lambda})\hat{R}(\lambda))x\| = \|(1+(\lambda-\bar{\lambda})R(\lambda))x\| = \|x\|$ $= \|(1+(\lambda-\bar{\lambda})\hat{R}(\lambda))x\|$. From this it follows immediately that $\hat{R}(\lambda)x \in H$ and therefore $\hat{R}(\lambda)x = R(\lambda)x$, i.e. $x \in \ker N_\lambda$. Conversely, if $x \in \ker N_\lambda$, then $\|(1+(\lambda-\bar{\lambda})R(\lambda)x\|$ $= \|(1+(\lambda-\bar{\lambda})\hat{R}(\lambda))x\| = \|x\|$. Thus

$$\lim_{n\to\infty}\|(1+(\lambda-\bar{\lambda})(A_n-\lambda)^{-1})x - (1+(\lambda-\bar{\lambda})R(\lambda))x\|$$

$$2\|x\|^2 - 2\lim_{n\to\infty}\mathrm{Re}((1+(\lambda-\bar{\lambda})(A_n-\lambda)^{-1})x, (1+(\lambda-\bar{\lambda})R(\lambda))x) = 0.$$

This proves the lemma.

Now we make further assumption that

(iii) $R(\lambda)\ker N_\lambda$ is dense in H

This condition also guarantees that $\vee\{F_t H\} : t \in \mathbb{R}\} = H$, so $\ker R(\lambda) = 0$ for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Let $\mathcal{D} = R(\lambda) \ker N_\lambda$ and $V_\lambda = (1+(\lambda-\bar{\lambda})R(\lambda))|\ker N_\lambda$. Then obviously $\bar{\mathcal{D}} = H$, V_λ is an isometry and $\ker(V_\lambda - I) = \{0\}$. Define $A = (\lambda V_\lambda - \bar{\lambda})(V_\lambda - 1)^{-1}$. By the classical results of von Neumann, A is a symmetric operator.

Lemma I.3.2 [19]. $\mathcal{D}$ and A do not depend on the particular choice of λ.

Proof. Let $z \in \ker N_{\lambda'}$, then $(\hat{A}-\lambda')^{-1}z = R(\lambda')z \in H$ and $x = (\hat{A}-\lambda)(\hat{A}-\lambda')^{-1}z = [1+(\lambda'-\lambda)(\hat{A}-\lambda')^{-1}]z \in H$. Hence $(\hat{A}-\lambda)^{-1}x = (\hat{A}-\lambda')^{-1}z = P(\hat{A}-\lambda')^{-1}z = P(\hat{A}-\lambda)^{-1}x$. Thus $x \in \ker N_\lambda$ and $R(\lambda)x = R(\lambda')z$. This proves that $R(\lambda)\ker N_\lambda = R(\lambda')\ker N_{\lambda'}$. But $R(\lambda)x = (V_\lambda - 1)x/(\lambda-\bar{\lambda})$, so

$[\lambda V_\lambda - \bar{\lambda}](V_\lambda - 1)^{-1}R(\lambda)x = (\lambda V_\lambda - \bar{\lambda})x/(\lambda-\bar{\lambda}) = \lambda R(\lambda)x + x$

$= \lambda R(\lambda')z + (\hat{A}-\lambda)R(\lambda')z = \hat{A}R(\lambda')z = z + \lambda' R(\lambda')z =$

$= [\lambda' V_{\lambda'} - \bar{\lambda}'](V_{\lambda'} - 1)^{-1}R(\lambda')z$. Hence A is independent of λ.

This symmetric operator is considered as the "weak limit" of $\{A_n\}$. The deficiency spaces of A are $H \ominus \ker N_\lambda$ and

$H \ominus \ker N_{\overline{\lambda}}$. We will refer to N_λ as the deficiency operator of A.

This symmetric operator theory has been established for singular integral operators and Toeplitz operators on the unit circle (see [14], [20]). In these two cases, the space $H \ominus \ker N_\lambda$ is actually computed. For the purpose of this work, the results concerning the Toeplitz case are presented here.

Let $\varphi \in L^2(S^1)$ be a real function, define

$$\varphi_n(\tau) = \begin{cases} \varphi(\tau) & : \quad |\varphi(\tau)| \leq n \\ 0 & : \quad |\varphi(\tau)| \geq n \end{cases}$$

Operators T_{φ_n} will be used as the limiting sequence. First we have to show the weak convergence of $\{(T_{\varphi_n}-\lambda)^{-1}\}$. For $\alpha \in \Delta$, let $k_\alpha(\tau) = 1/(1-\overline{\alpha}\tau)$. Define

$$S_n(\lambda,z) = \exp\left(-\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log(\varphi_n(e^{it})-\lambda)\,dt\right)$$

Then it is easy to show that

$$((T_{\varphi_n}-\lambda)^{-1}k_\alpha, k_\beta) = \frac{S_n(\lambda,\beta)\overline{S}_n(\overline{\lambda},\alpha)}{1-\overline{\alpha}\beta}$$

It follows immediately that for $\lambda \in \mathbb{C}\backslash\mathbb{R}$,

$$\lim_{n\to\infty}((T_{\varphi_n}-\lambda)^{-1}k_\alpha,k_\beta) = \frac{S(\lambda,\beta)\overline{S}(\overline{\lambda},\alpha)}{1-\overline{\alpha}\beta}$$

where

$$S(\lambda,z) = \exp(-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log(\varphi(e^{it})-\lambda)dt).$$

Since the linear combinations of $\{k_\alpha : \alpha \in \Delta\}$ is dense in H^2, we claim that the weak limit

$$\lim_{n\to\infty}(T_{\varphi_n}-\lambda)^{-1} = R(\lambda)$$

exists for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Furthermore, computation shows that

$$R(\lambda)k_\alpha = \overline{S}(\lambda,\alpha)S(\lambda,\cdot)k_\alpha.$$

and

$$(N_\lambda k_\alpha,k_\alpha) = \frac{1}{1-|\alpha|^2}\,\frac{S(\lambda,\alpha)\overline{S}(\overline{\lambda},\alpha)-S(\overline{\lambda},\alpha)\overline{S}(\lambda,\alpha)}{\lambda-\overline{\lambda}}$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(\lambda,e^{it})}{1-\overline{\alpha}e^{it}}\right|^2 dt\,\overline{S}(\overline{\lambda},\alpha)S(\lambda,\alpha)\ \ [19].$$

Let

$$\Omega_\lambda(z) = \begin{cases} S(\lambda,z)S(\overline{\lambda},z)^{-1} & : \quad \mathrm{Im}\lambda \geq 0 \\ -S(\lambda,z)S(\overline{\lambda},z)^{-1} & : \quad \mathrm{Im}\lambda < 0 \end{cases},$$

then there exists a positive measure ν_λ such that

$$\operatorname{Im} \Omega_\lambda (z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{it}-z|^2} \, d\nu_\lambda (t)$$

The Radon-Nikodym derivative ν'_λ of η_λ with respect to the Lebesque measure is

$$\nu'_\lambda(t) = \operatorname{Im} \Omega_\lambda (e^{it}) = |S(\lambda, e^{it})|^2 .$$

Thus

$$(N_\lambda k_\alpha, k_\alpha) \overline{S}(\overline{\lambda}, \alpha)^{-1} S(\overline{\lambda}, \alpha)^{-1} = \frac{1}{2\pi} \frac{1}{\operatorname{Im} \lambda} \int \frac{d\nu_\lambda^{(s)}(t)}{|e^{it}-\alpha|^2}$$

where $\nu_\lambda^{(s)}$ is the singular part of ν_λ. Hence

$$\dim(H \ominus \ker N_\lambda) = \dim L^2(d\nu_\lambda^{(s)})$$

Now we relate $\nu_\lambda^{(s)}$ to the symbol φ as follows. Let function $\delta_\lambda(e^{it}) = \frac{1}{\pi} \arg(\varphi(e^{it}) - \overline{\lambda})$ be so determined that $0 \le \delta_\lambda \le 1$ on the unit circle. There exists a measure μ_λ such that

$$\exp\left(\int_{S^1} \delta_\lambda(\tau) \frac{d\tau}{\tau - z}\right) = 1 + \beta \int_{S^1} \frac{\tau}{\tau - z} \, d\mu_\lambda(\tau)$$

where $\beta = \exp(\int \delta_\lambda(e^{it}) dt) - 1$, (see [19], [20]). μ_λ and ν_λ have the following relation

$$\frac{1}{2} \nu_\lambda = \sin\left(\frac{1}{2} \int_0^{2\pi} \delta_\lambda(e^{it}) dt\right) \mu_\lambda \quad [19].$$

Note that $0 < \frac{1}{2} \int_0^{2\pi} \delta_\lambda(e^{it}) dt < \pi$. Hence ν_λ and μ_λ are mutually equivalent. The singular part of μ_λ is completely determined by δ_λ. For example, if φ is bounded, then δ_λ is bounded away from 0 and μ_λ does not have a singular part. In another extreme case, if $\nu_\lambda^{(s)}$ is purely atomic with $\nu_{\lambda,j}^{(s)}$ at a point $e^{it_j(\lambda)}$, $j = 1,\ldots,n(\lambda)$, then we have $S(w,e^{it})/(e^{it_j(\lambda)}-e^{it}) \in H^2$ and

$$N_\lambda = \sum_{j=1}^{n(\lambda)} \frac{1}{2\pi} \frac{\nu_{\lambda,j}^{(s)}}{|\operatorname{Im}\lambda|} \frac{S(\overline{\lambda},\cdot)}{\cdot - e^{it_j(\lambda)}} \otimes \frac{S(\overline{\lambda},\cdot)}{\cdot - e^{it_j(\lambda)}}$$

(see [19], [20]).

Chapter II. The G-Currents and the Trace and Index Formulas

II.I The difference between orthogonal and Cauchy projections

To employ the almost community pair techniques in the study of Toeplitz operator, it is necessary that the commutator of a Toeplitz operator and the multiplication by the variable be of trace class. This is clear in the case of the unit circle where a Toeplitz operator is a multiplication followed by the Cauchy integral. The key point is that on the unit circle, the Cauchy integral happens to serve as the orthogonal projection from L^2 to H^2. But on a multiply-connected domain, complications due to both the topology and the geometry of the domain merge so that the orthogonal projection can no longer be expressed as a singular integral operator. Fortunately, the difference of the orthogonal projection and the Cauchy projection is of trace class. Thus, as far as the almost commuting pair is concerned, many techniques used on the unit circle can be transplanted to the multiply-connected case. Furthermore, we have the following important theorem which is the starting point of our study.

Theorem II.1.1. Let $P, \mathcal{P} : L^2(\partial D, \omega) \to H^2(D)$ be the orthogonal and Cauchy projection respectively. Then $P - \mathcal{P}$ is a trace class operator. Furthermore, there exists a constant $L > 0$ which depends only on D such that for any $\varphi \in L^{\infty}(\partial D, \omega)$,

$$\|M_{\varphi}(P-\rho)\|_1 \leq L\|\varphi\|_{L^2}, \quad \|(P-\rho)M_{\varphi}\|_1 \leq L\|\varphi\|_{L^2}.$$

Proof. It is easy to see that P and p satisfy the following relations

$$\rho P = P, \quad P\rho = \rho, \quad \rho = P\rho, \quad \rho^* = \rho^*\rho.$$

Thus $(P-\rho)M_{\varphi} = P(\rho^*-\rho)M_{\varphi}$, $M_{\varphi}(P-\rho) = M_{\varphi}(P-\rho^*) + M_{\varphi}(\rho^*-P)$ $= [(P-\rho)M_{\overline{\varphi}}]^* + [(\rho-P)M_{\overline{\varphi}}]^*$. Hence by Lemma I.2.6, it suffices to show that $\rho - \rho^*$ is an integral operator with the kernel in $C^2(\partial D \times \partial D)$.

Let Ω be the principal value integral, i.e.

$$(\Omega f)(\tau) = \frac{1}{2\pi i} \text{ P.V. } \int_{\partial D} \frac{f(\lambda)}{\lambda-\tau}\, d\lambda$$

Ω and ρ are related by the following relations

$$\rho = \frac{1}{2} I + \Omega, \quad \rho^* = \frac{1}{2} I + \Omega \ .$$

Therefore $\rho - \rho^* = \Omega - \Omega^*$. The rest of the proof will be devoted to showing that $\Omega - \Omega^*$ has a C^{∞} kernel.

Let χ_n be the characteristic function of $\mathbb{R}\setminus[-\frac{1}{n}, \frac{1}{n}]$ and define

$$(\Omega_n u)(\tau) = \frac{1}{2\pi i} \int_{\partial D} \chi_n(|\lambda-\tau|)\, \frac{u(\lambda)}{\lambda-\tau}\, d\lambda$$

for $u \in L^2(\partial D, \omega)$. Obviously we can rewrite $\Omega_n u$ as

$$(\Omega_n u)(\tau) = \frac{1}{2\pi i} \int_{\partial D} \frac{u(\lambda)-u(\tau)}{\lambda-\tau} \chi_n(|\lambda-\tau|)d\lambda$$

$$+ u(\tau) \frac{1}{2\pi i} \int_{\partial D} \frac{\chi_n(|\lambda-\tau)}{\lambda-\tau} d\lambda \ .$$

Hence if $u \in C^1(\partial D)$, then $\{\Omega_n u\} \subset L^\infty(\partial D, \omega)$ and $\{\|\Omega_n u\|_\infty\}_{n=1}^\infty$ is bounded by the supremum norm of u and the tangential derivative u' of u. By the definition of the principal value integral, $\Omega_n u$ converges to Ωu pointwise a.e. on ∂D. Thus if $u \in C^1(\partial D)$, $\lim_{n\to\infty} \|\Omega_n u - \Omega u\|_{L^2} = 0$. Also, if $u, v \in C^1(\partial D)$, $(\Omega^* u, v) = (u, \Omega v) = \lim_{n\to\infty}(u, \Omega_n v) = \lim_{n\to\infty}(\Omega_n^* u, v)$. Hence for such a pair u, v

$$((\Omega-\Omega^*)u, v) = \lim_{n\to\infty}((\Omega_n-\Omega_n^*)u, v).$$

Let $\beta(\lambda)$ be the positive unit tangent vector to ∂D and let $S(\lambda)$ be the function defined by

$$dm(\lambda) = S(\lambda)d\omega(\lambda)$$

where $m(\lambda)$ is the arc length measure on ∂D. By Section I.2, $s \in C^\infty(\partial D)$. Thus

$$(\Omega_n u)(\tau) = \frac{1}{2\pi i} \int_{\partial D} \frac{\chi_n(|\lambda-\tau|)\beta(\tau)S(\tau)}{\lambda-\tau} u(\lambda)d\omega(\lambda)$$

and

$$(\Omega_n^* u)(\tau) = \frac{1}{2\pi i}\int_{\partial D} \frac{\chi_n(|\lambda-\tau|)\overline{\beta(\tau)}S(\tau)}{\bar{\lambda}-\bar{\tau}} u(\lambda)d\omega(\lambda).$$

Hence $\Omega_n - \Omega_n^*$ is the integral operator with kernel

$$\chi_n(|\lambda-\tau|)[\frac{\beta(\lambda)S(\lambda)}{\lambda-\tau} - \frac{\overline{\beta(\tau)}S(\tau)}{\bar{\lambda}-\bar{\tau}}]/2\pi i .$$

Let

$$k(\lambda,\tau) = \frac{1}{2\pi i}[\frac{\beta(\lambda)S(\lambda)}{\lambda-\tau} - \frac{\overline{\beta(\tau)}S(\tau)}{\bar{\lambda}-\bar{\tau}}] \quad \lambda \neq \tau.$$

Although k is not defined yet on the diagonal, if k is bounded elsewhere, then it is easy to see that

$$((\Omega-\Omega^*)u,v) = \int_{\partial D}[\int_{\partial D} k(\lambda,\tau)u(\lambda)d\omega(\lambda)]\overline{v(\tau)}d\omega(\tau) \text{ for } u,v \in C'(\partial D).$$

Since $C^1(\partial D)$ is dense in $L^2(\partial D,\omega)$, the above equality also holds for any $u,v \in L^2(\partial D,\omega)$ and $\Omega-\Omega^*$ is the integral operator with kernel k. Hence to prove the theorem we only have to show that we can assign values to k on the diagonal $\lambda = \tau$ so that it becomes a function in $C^\infty(\partial D\times\partial D)$.

For $\lambda \neq \tau$,

$$k(\lambda,\tau) = \frac{1}{2\pi i}(\beta(\lambda)\ \frac{s(\lambda)-s(\tau)}{\lambda-\tau} + s(\tau)[\frac{\beta(\lambda)}{\lambda-\tau} - \frac{\overline{\beta(\tau)}}{\bar{\lambda}-\bar{\tau}}])$$

The first term can be naturally extended to the diagonal since s is smooth.

$\partial D = \Gamma_0 \cup \ldots \cup \Gamma_n$ consists of n Jordan curves. For each Γ_j, there exist a smooth parameter

$$\alpha_j : [0, m(\Gamma_j)] \to \mathbb{C}$$

such that $\alpha_j'(t) = \beta(\lambda)$ if $\lambda = \alpha_j(t)$.

Let $\lambda, \tau \in \Gamma_j$ and $\alpha_j(t) = \lambda$, $\alpha_j(s) = \tau$. Thus $\beta(\lambda) = \alpha_j'(t)$ and $\beta(\tau) = \alpha_j'(s)$. Consider the Taylor expansion of α_j :

(1) $\alpha_j(t) = \alpha_j(s) + \alpha_j'(s)(t-s) + \frac{1}{2}\alpha_j''(s)(t-s)^2 + \varphi(t,s)(t-s)^3$

and

(2) $\alpha_j(s) = \alpha_j(t) + \alpha_j'(t)(s-t) + \frac{1}{2}\alpha_j''(t)(s-t)^2 + \psi(s,t)(s-t)^3$

where $\varphi, \psi \in C^\infty$ and $\varphi(t,s)$, $\psi(s,t) \to 0$ as $|t-s| \to 0$.

By (1),

$$\frac{\overline{\beta(\tau)}}{\overline{\lambda}-\overline{\tau}} = \frac{\overline{\alpha_j'(s)}}{\overline{\alpha_\alpha(t)}-\overline{\alpha_j(s)}}$$

$$= \frac{\overline{\alpha_j'(s)}}{\overline{\alpha_j'(s)}(t-s)[1+\frac{1}{2}(\frac{\overline{\alpha_j''(s)}}{\alpha_j'(s)})(t-s)+(t-s)^2(\frac{\overline{\varphi(t,s)}}{\alpha_j'(s)})]}$$

$$= \frac{1}{t-s}[1+\frac{1}{2}(\frac{\overline{\alpha_j''(s)}}{\alpha_j'(s)})(t-s)+(\frac{\overline{\varphi(t,s)}}{\alpha_j'(s)})(t-s)^2]^{-1}$$

Note that

$$\frac{1}{1+a+b} = 1 - a + \frac{a^2-b(1-a)}{1+a+b} .$$

Hence

$$\frac{\overline{\beta(t)}}{\bar{\lambda}-\bar{\tau}} = \frac{1}{t-s}[1 - \frac{1}{2}(\overline{\frac{\alpha_j''(s)}{\alpha_j'(s)}})(t-s)+\bar{\varphi}_1(t,s)(t-s)^2]$$

with $\varphi_1 \in C^\infty$.

Similarly, by (2) we have

$$\alpha_j(t) - \alpha_j'(s) = \alpha_j'(t)(t-s) - \frac{1}{2}\alpha_j''(t)(t-s)^2 + \psi(s,t)(t-s)^3$$

and therefore

$$\frac{\beta(\lambda)}{\lambda-\tau} = \frac{\alpha_j'(t)}{\alpha_j(t)-\alpha_j(s)}$$

$$= \frac{\alpha_j'(t)}{\alpha_j'(t)(t-s)[1-\frac{1}{2}\frac{\alpha_j''(t)}{\alpha_j'(t)}(t-s) + (\frac{\psi(s,t)}{\alpha_j'(t)})(t-s)^2]}$$

$$= \frac{1}{t-s}[1+\frac{1}{2}\frac{\alpha_j''(t)}{\alpha_j'(t)}(t-s) + \psi_1(s,t)(t-s)^2]$$

with $\psi_1 \in C^\infty$.

Hence

$$\frac{\beta(\lambda)}{\lambda-\tau} - \frac{\overline{\beta(\tau)}}{\overline{\lambda}-\overline{\tau}} = \frac{\alpha_j''(t)}{\alpha_j'(t)} + \overline{\left(\frac{\alpha_j''(s)}{\alpha_j'(s)}\right)} + [\psi_1(s,t) + \overline{\varphi}_1(t,s)](t-s).$$

Obviously, the correspondence

$$(\lambda,\tau) \mapsto \frac{\alpha_j''(\alpha_j^{-1}(\lambda))}{\alpha_j'(\alpha_j^{-1}(\lambda)} + \overline{\left(\frac{\alpha_j''(\alpha_j^{-1}(\tau))}{\alpha_j'(\alpha_j^{-1}(\tau))}\right)}$$

$$+ [\psi_1(\alpha_j^{-1}(\tau),\alpha_j^{-1}(\lambda)) + \overline{\varphi}_1(\alpha_j^{-1}(\lambda),\alpha_j^{-1}(\tau))]$$

$$(\alpha_j^{-1}(\lambda)-\alpha_j^{-1}(\tau))$$

defines a C^∞ function on $\partial D \times \partial D$. Hence $k(\lambda,\tau)$ can be extended to a C^∞ function. This completes the proof.

Corollary II.1.2. Let $\varphi \in C^2(\partial D\times\partial D)$ and $\psi \in L^\infty(\partial D,\omega)$, then $M_\psi[PM_\varphi - M_\varphi P]$, $M_\psi PM_\varphi(1-P)$ and $PM_\varphi(1-P)M_\psi$ are trace class operators and

$$\|M_\psi[PM_\varphi - M_\varphi P]\|_1 \leq L\|\psi\|_{L^2},$$

$$\|M_\psi PM_\varphi(1-P)\|_1 \leq L\|\psi\|_{L^2}$$

and

$$\|PM_\varphi(1-P)M_\psi\|_1 \leq L\|\psi\|_{L^2}$$

where L depends only on φ.

<u>Proof</u>. $M_\psi[PM_\varphi - M_\varphi P] = [M_\psi(P-\rho)]M_\varphi + M(\rho M_\varphi - M_\varphi \rho) + M_\varphi[M_\psi(\rho - P)]$. for any $f \in L^2(\partial D, \omega)$

$$(\rho M_\varphi - M_\varphi \rho)f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\varphi(\lambda)-\varphi(z)}{\lambda - z} f(\lambda)\,d\lambda$$

since $\varphi \in C^2(\partial D)$, we claim that

$$\frac{\varphi(\lambda)-\varphi(z)}{\lambda - z} \in C^2(\partial D \times \partial D).$$

Hence the first inequality follows from Lemma I.2.7 and the theorem. For the other two inequalities, we only need to note that $M_\psi PM_\varphi(1-P) = [M_\psi(PM_\varphi - M_\varphi P)(1-P)]$ and $PM_\varphi(1-P)M_\psi$ $= P[(1-P)M_\varphi - M_\varphi(1-P)]M_\psi = P[M_\varphi P - PM_\varphi]M_\psi$.

<u>Corollary II.1.3</u>. Let $\varphi \in C^2(\partial D)$ and $\psi \in L^\infty(\partial D)$ and $\psi \in L^\infty(\partial D, \omega)$, then $[T_\varphi, T_\psi]$ is a trace class operator and

$$\|[T_\varphi, T_\psi]\|_1 \leq L\|\psi\|_{L^2}$$

where L depends only on φ.

<u>Proof</u>. $[T_\varphi, T_\psi] = (PM_\varphi PM_\psi - PM_\psi PM_\varphi)|H^2(D)$

$$= (PM_\varphi(1-P)M_\psi - PM_\psi(1-P)M_\varphi P)|H^2(D).$$

Particularly, if M is the multiplication by the variable on $H^2(D)$, then $[M,T_\psi] \in C_1$ and $\|[M,T_\psi]\|_1 \leq L\|\psi\|_{L^2}$ with L depending only on the shape of the domain.

<u>Corollary II.1.4</u>. Let $\alpha, \beta \in L^\infty(\partial D, \omega)$ be such that

$$\text{distance(support } \alpha, \text{ support } \beta) > 0.$$

Then for any $\psi_1, \psi_2 \in L^\infty(\partial D, \omega)$, $[T_{\alpha\psi_1}, T_{\beta\psi_2}] \in C_1$ and

$$\|[T_{\alpha\psi_1}, T_{\beta\psi_2}]\|_1 \leq L\|\psi_1\|_{L^2} \|\psi_2\|_{L^2},$$

L depends only on α and β.

<u>Proof</u>. $[T_{\alpha\psi_1}, T_{\beta\psi_2}] = PM_{\alpha\psi_1} PM_{\beta\psi_2} P - PM_{\beta\psi_2} PM_{\alpha\psi_1}$

$$= PM_{\alpha\psi_1}(P-\mathcal{P})M_{\beta\psi_2}P - PM_{\beta\psi_2}(P-\mathcal{P})M_{\alpha\psi_1}P$$

$$+ PM_{\alpha\psi_1}\mathcal{P}M_{\beta\psi_2}P - PM_{\beta\psi_2}\mathcal{P}M_{\alpha\psi_1}P.$$

Note that $M_{\alpha\psi_1}\mathcal{P}M_{\beta\psi_2}$ is an integral operator with the kernel

$$\frac{\alpha(\tau)\psi_1(\tau)\beta(z)\psi_2(z)}{z-\tau}\, dz$$

Since distance(support α, support β) > 0, the conclusion is evident.

<u>Corollary II.1.5</u>. Let $u \in C^\infty(\partial D)$. Then $Pu \in H^2(D) \cap C^\infty(\partial D)$.

<u>Proof</u>. $Pu = (P-\rho^*)u + \rho^*u = (\rho-\rho^*)Pu + (\rho^*-\rho)u + \rho u$. By Theorem II.1.1, $(\rho-\rho^*)Pu$, $(\rho^*-\rho)u \in C^\infty(\partial D)$. ρu is also in $C^\infty(\partial D)$ since u is.

II.2 The construction of the G-current

Throughout this section, $\psi \in L^{\infty}(\partial D, \omega)$ is a real function. Let $\eta \in \mathbb{R}$ and define

$$A = T_{\psi} \oplus \eta$$

corresponding to the decomposition $L^2(\partial D, \omega) = H^2(D) \oplus [H^2(D)]^{\perp}$. Let $\varphi = x+iy \in C^2(\partial D)$ be such that

$$\omega(\{\tau : \varphi'(\tau) = 0\}) = 0$$

where $'$ indicates the tangent derivative. On $L^2(\partial D, \omega)$, define $X = Mx$ and $Y = My$. By Corollary II.1.2 and II.1.3, $[X,A]$ and $[Y,A]$ are of trace class. In the rest of this section, we will follow the notations of I.1 with the triple $\{X,Y,A\}$ being the operators specifically defined as above. Our goal is to calculate the functions $g(\lambda, \alpha(\lambda), z)$ and $h(\beta(\xi), \xi, z)$ and give the formula

$$\mathrm{tr}[F(X,Y,A), H(X,Y,A)]$$

$$= \frac{i}{2\pi}\{\iint \sum_{\alpha(\lambda)\in\sigma_p(Y(\lambda))} J_{13}(F,H)(\lambda,\alpha(\lambda),z)g(\lambda,\alpha(\lambda),z)d\lambda dz$$

$$+ \iint \sum_{\beta(\xi)\in\sigma_p(X(\xi))} J_{23}(F,H)(\beta(\xi),\xi,z)h(\beta(\xi),\xi,z)d\xi dz\}$$

a simpler form.

We begin with the decompositions

$$X = \int_{\sigma(Y)} \oplus X(\xi)d\nu(\xi) \text{ on } L^2(\partial D, \omega) = \int_{\sigma(Y)} \oplus K_\xi d\nu(\xi)$$

and $Y = \int_{\sigma(X)} \oplus Y(\lambda)d\lambda$ on $L^2(\partial D, \omega) = \int_{\sigma(X)} \oplus L_\lambda d\mu(\lambda)$.

First we need the following lemma.

<u>Lemma II.2.1</u>. There exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that

$$\omega(\{\tau : (\mathrm{Re}\ \alpha\varphi)'(\tau) = 0\}) = 0$$

and

$$\omega(\{\tau : (\mathrm{IM}\ \alpha\varphi)'(\tau) = 0\}$$

<u>Proof</u>. Let ℓ_α be the line in $\mathbb{C}$ going through 0 and α. It is easy to see that Re $\alpha\varphi'(\tau) = 0$ (Im $\alpha\varphi'(\tau) = 0$) if and only if $\tau \in (\varphi')^{-1}\ell_{i\alpha^{-1}}(\tau \in (\varphi')^{-1}\ell_{\alpha^{-1}})$. Since $\omega(\{\tau : \varphi'(\tau) = 0\}) = 0$, $\omega((\varphi')^{-1}\ell_{i\alpha^{-1}}) = \omega((\varphi')^{-1}\ell_{i\alpha^{-1}}\backslash\}\tau : \varphi'(\tau) = 0\})$. But $(\varphi'^{-1})\ell_{i\alpha^{-1}}\backslash\{\tau : \varphi'(\tau) = 0\} = (\varphi')^{-1}[\ell_{i\alpha^{-1}}\backslash\{0\}]$ and $\alpha \neq \beta$ implies $(\varphi')^{-1}[\ell_{i\alpha^{-1}}\backslash\{0\}] \cap (\varphi')^{-1}[\ell_{i\beta^{-1}}\backslash\{0\}] = \phi$. Therefore exist at most a countable number of $\{\alpha_j\} \subset S^1$ such that $\omega((\varphi')^{-1}\ell_{i\alpha}) = 0$ if $\alpha \notin \{\alpha_j\}$. Similarly, there exist $\{\beta_j\} \subset S^1$ such that $\omega((\varphi')\ell_{\beta^{-1}}) = 0$ if $\beta \notin \{\beta_j\}$. Pick $\alpha \notin S^1\backslash[\{\alpha_j\} \cup \{\beta_j\}]$, then this α will serve the purpose.

By this lemma, without loss of generality, we may assume that

$$\omega(\{\tau : x'(\tau) = 0\}) = 0$$

and

$$\omega(\{\tau : y'(\tau) = 0\}) = 0.$$

Let $S = \{\tau : \tau \in \partial D,\ x'(\tau) = 0 \text{ or } y'(\tau) = 0\}$. Then S is closed and φ restricted to $\partial D\setminus S$ is locally homeomorphic. Hence both X and Y have absolutely continuous spectrum and for almost all ξ and λ $\sigma(X(\xi))$ and $\sigma(Y(\lambda))$ are discrete. In fact we have

$$\{\lambda + i\alpha(\lambda) : \alpha(\lambda) \in \sigma_p(Y(\lambda))\} = \varphi(\partial D) \cap \{w : w \in \mathbb{C},\ \mathrm{Re}\, w = \lambda\}$$

and

$$\{\beta(\xi) + i\xi : \beta(\xi) \in \sigma_p(X(\xi))\} = \varphi(\partial D) \cap \{w : w\ \mathbb{C},\ \mathrm{Im}\, w = \xi\}$$

$\partial D\setminus S$ decomposes into disjoint union $\Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_j \cup \dots$ of arcs so that x and y restricted to each Δ_j is one to one. Let $x_j = x|\Delta_j$ and $y_j = y|\Delta_j$. Then it is easy to see that $\mathrm{tr}[F(X,Y,A),\ H(X,Y,A)] =$

$$= \sum_{j=1}^{\infty} \frac{i}{2\pi}\{\iint_{x(\Delta_j)} J_{13}(F,H)(\lambda, y_j\circ x_j^{-1}(\lambda), z)g(\lambda, y_j\circ x_j^{-1}(\lambda), z)$$

$$d\lambda dz + \iint_{y(\Delta_j)} J_{23}(F,H)(x_j\circ y_j^{-1}(\xi), \xi, z)h(x_j\circ y_j^{-1}(\xi), \xi, z)d\xi dz\}.$$

Let $\Omega_j = \varphi(\Delta_j)$, then Ω_j inherits a differential structure from Δ_j via φ. As subsets of $\mathbb{C}$, it may happen that $\Omega_i \cap \Omega_j \neq \emptyset$ for different i and j. But when differentiation is concerned, we always specify which one it means. The symbol Ω_j is understood as a set along with the differential structure. Hence Ω_j is a one dimensional C^2-manifold possibly with boundary. For each j, let $\varphi_j^{-1} : \Omega_j \to \Delta_j$ be the inverse of $\varphi : \Delta_j \to \Omega_j$. Clearly,

$$\iint_{x(\Delta_j)} J_{13}(F,H)(\lambda, y_j \circ x_j^{-1}(\lambda), x) g(\lambda, y_j \circ x_j^{-1}(\lambda), z) d\lambda dz$$

$$+ \iint_{y(\Delta_j)} J_{23}(F,H)(x_j \circ y_j^{-1}(\xi), \xi, z) h(x_j \circ y_j^{-1}(\xi), \xi, z) d\xi dz$$

$$= \iint_{\Omega_j} J_{13}(F,H)(\zeta, z) g(\xi, z) |\gamma_j(\zeta)| dm(\zeta) dz$$

$$+ \iint_{\Omega_j} J_{23}(F,H)(\zeta, z) h(\zeta, z) |\delta_j(\zeta)| dm(\zeta) dz$$

where m is the arc length measure $\gamma_j + i\delta_j$ is the positive unit tangent to Ω_j. Let $\Delta^+ = \{\tau : \frac{\partial x}{\partial \tau}(\tau) \geq 0\}$, $\Delta^- = \{\tau : \frac{\partial x}{\partial \tau}(\tau) < 0\}$ and $\Sigma^+ = \{\tau : \frac{\partial y}{\partial \tau}(\tau) \geq 0\}$, $\Sigma^- = \{\tau : \frac{\partial y}{\partial \tau}(\tau) < 0\}$.

Lemma II.2.2. On $L^2(\partial D, \omega)$ we have

$$S_+(X,A) = \eta M_{\chi_{\Delta^+}} + M_\psi M_{\chi_{\Delta^-}},$$

$$S_-(X,A) = M_\psi M_{\chi_{\Delta^+}} + \eta M_{\chi_{\Delta^-}}$$

and

$$S_+(Y,A) = \eta M_{\chi_{\Sigma^+}} + M_\psi M_{\chi_{\Sigma^-}},$$

$$S_-(Y,A) = M_\psi M_{\chi_{\Sigma^+}} + \eta M_{\chi_{\Sigma^-}}.$$

For $\lambda \in \mathfrak{x}(\Delta_j)$ and $\alpha(\lambda) = y_j \circ x_j^{-1}(\lambda)$, the perturbation problem $S_+(X,A)(\lambda)F_{\lambda, y_j \circ x_j^{-1}(\lambda)} \to S_-(X,A)(\lambda)F_{\lambda, y_j \circ x_j^{-1}(\lambda)}$ becomes

$$\eta\chi_{\Delta^+}(\lambda+iy_j\circ x_j^{-1}(\lambda)) + \psi(\varphi_j^{-1}(\lambda+iy_j\circ x_j^{-1}(\lambda))\chi_{\Delta^-}(\lambda+iy_j\circ x_j^{-1}(\lambda))$$

$$\to \psi(\varphi_j^{-1}(\lambda+iy_j\circ x_j^{-1}(\lambda))\chi_{\Delta^+}(\lambda+iy_j\circ x_j^{-1}(\lambda)) + \eta\chi_{\Delta^-}(\lambda+iy_j\circ x_j^{-1}(\lambda)).$$

In other words, $g(\zeta,\cdot)$ is the phase shift of the perturbation problem

$$\eta\chi_{\Delta^+}(\zeta) + \psi(\varphi_j^{-1})(\zeta))\chi_{\Delta^-}(\zeta) \to \psi(\varphi_j^{-1}(\zeta))\chi_{\Delta^+}(\zeta) + \eta\chi_{\Delta^-}(\zeta).$$

Similarly $h(\zeta,\cdot)$ is the phase shift of the perturbation problem

$$\eta\chi_{\Sigma^+}(\zeta) + \psi(\varphi_j^{-1}(\zeta))\chi_{\Sigma^-}(\zeta) \to \psi(\varphi_j^{-1}(\zeta))\chi_{\Sigma^+}(\zeta) + \eta\chi_{\Sigma^-}(\zeta).$$

For $a \in \mathbb{R}$, define

$$\chi_a(z) = \begin{cases} \chi_{(\eta,a)}(z) & \text{if } a \geq \eta \\ -\chi_{(a,\eta)}(z) & \text{if } a < \eta. \end{cases}$$

Then

$$g(\zeta,c) = \chi_{\Delta^+}(\zeta)\chi_{\psi(\varphi_j^{-1}(\zeta))}(z) - \chi_{\Delta^-}(\zeta)\chi_{\psi(\varphi_j^{-1}(\zeta))}(z)$$

and

$$h(\zeta,z) = \chi_{\Sigma^+}(\zeta)\chi_{\psi(\varphi_j^{-1}(\zeta))}(z) - \chi_{\Sigma^-}(\zeta)\chi_{\psi(\varphi_j^{-1}(\zeta))}(z)$$

By the definition of the differential structure on Ω_j, we have

$$g(\zeta,z)\,|\gamma_j(\zeta)| = \chi_{\psi(\varphi_j^{-1}(\zeta))}(z)\gamma_j(\zeta)$$

and

$$h(\zeta,z)\,|\delta_j(\zeta)| = \chi_{\psi(\varphi_j^{-1}(\zeta))}(z)\delta_j(\zeta)$$

Hence

$$J_{13}(F,H)(\zeta,z)g(\zeta,z)\,|\gamma_j(\zeta)| + J_{23}(F,H)(\zeta,z)h(\zeta,z)\,|\delta_j(\zeta)|$$

$$= \frac{\partial F}{\partial \tau}(\zeta,z)\frac{\partial F}{\partial z}(\zeta,z) - \frac{\partial F}{\partial z}(\zeta,z)\frac{\partial H}{\partial \tau}(\zeta,z)$$

where $\frac{\partial}{\partial\tau}$ denotes the positive tangent derivative on Ω_j.

Hence

$$\mathrm{tr}[F(X,Y,A),H(X,Y,A)]$$

$$= \sum_{j=1}^{\infty} \frac{i}{2\pi i} \iint_{\Omega_j} \left(\frac{\partial F}{\partial \tau}\frac{\partial H}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial H}{\partial \tau}\right)(\zeta,z)\ \chi_{\psi(\varphi_j^{-1}(\zeta))}(z)\,dm(\zeta)\,dz.$$

Consider Ω as the disjoint union of Ω_j's equipped with the differential structure prescribed on each Ω_j. Let φ^{-1} be the inverse of $\varphi : \partial D \to \Omega$, then $\varphi^{-1}\Omega_j = \Delta_j$. We can also introduct a measure m on Ω such that m is the arc length on each Ω_j. Thus we have the following:

Theorem II.2.3. Let $F,H \in \hat{M}(\mathbb{R}^3)$ then

$$\mathrm{tr}[F(X,Y,A),H(X,Y,A)]$$

$$= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\Omega} \langle dF \wedge dH, G\rangle(\zeta,z)\,dm(\zeta)\,dz$$

where

$$G(\zeta,z) = \chi_{\psi\circ\varphi^{-1}(\zeta)}(z)\ \frac{\partial}{\partial\tau} \wedge \frac{\partial}{\partial z}\ .$$

Corollary II.2.4. Let M be the multiplication by the complex variable on $L^2(\partial D,\omega)$. Then

$$\mathrm{tr}[F(M,T_\psi \oplus \eta),\ H(M,T_\psi \oplus \eta)]$$

$$= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\partial D} \langle dF \wedge dH,\ G\rangle(\zeta,z)\,dm(\zeta)\,dz$$

where

$$g(\zeta,z) = \chi_{\psi(\zeta)}(z)\frac{\partial}{\partial \bar{\zeta}} \wedge \frac{\partial}{\partial z}\,.$$

By the left hand side of the formula, it seems that the trace depends on φ as a function because $X = M_{\mathrm{Re}\,\varphi}$ and $Y = M_{\mathrm{Im}\,\varphi}$. But the right hand side is an area integral on a surface, to which only the image of φ rather than the function itself contributes, with one of its natural orientations. In other words, for any other function $\tilde{\varphi}$, no matter how "fast" $\tilde{\varphi}(\zeta)$ travels along as it keeps in the same track and direction as φ, replacing φ by $\tilde{\varphi}$ does not change the trace. The same is for ψ. The right hand side being purely geometric is essential for the calculation of G in the case φ has decontinuities.

Now we prove Lemma II.2.2. To calculuate these symbols we need to use the decomposition theorem of Toeplitz operator which will be proved in III.3. By this theorem, the difference

$$T_\psi \oplus \eta - [T^0_{\psi_0} \oplus \eta]\oplus\ldots\oplus[T^n_{\psi_n} \oplus \eta]$$

is a trace class operator where $T^i_{\psi_i} \oplus \eta$ corresponds to the decomposition.

$$L^2(\Gamma_i, \omega_i) = K_i \oplus L_i,$$

see I.2 for notations. Hence

$$S_\pm(X, A) = \bigoplus_{i=0}^{n} S_\pm(M_{x_i}, T^i_{\psi_i} \oplus \eta)$$

and

$$S_\pm(Y, A) = \bigoplus_{i=0}^{n} S_\pm(M_{y_i}, T^i_{\psi_i} \oplus \eta)$$

where $x_i = x|\Gamma_i$ and $y_i = y|\Gamma_i$. But

$$S_\pm(M_{x_i}, T^i_{\psi_i} \oplus \eta) = U_i^* S_\pm(M_{x_i \circ \pi_i}, T_{\psi_i \circ \pi_i} \oplus \eta) U_i$$

and $S_\pm(M_{x_i \circ \pi_i}, T_{\psi_i \circ \pi_i} \oplus \eta)$ is an operator on $L^2 = H^2 \oplus (H^2)^\perp$.

Since $\pi_0, \ldots, \pi_n$ together give the positive orientation on ∂D, the tangent derivative of χ_i at $\pi_i(\tau)$ is positive (negative) if and only if the tangent derivative of $\chi_i \circ \pi_i$ is positive (negative) at τ.

Thus we have reduced the symbol calculation to computing

$$S_\pm(M_u, T_\psi \oplus \eta)$$

on L^2 with $u \in C^2(S^1)$ and $\psi \in L^\infty(S^1)$ being a real function. First, the existence of these symbols is guaranteed by the fact that $[M_u, T_\psi \oplus \eta] \in C_1$. On S^1, define

$$\nu(\tau) = \arg \tau$$

and $-\pi \le \nu(\tau) \le \pi$. Then $u(\cdot) = u(\exp i\,\nu(\cdot)) = \tilde{u}(\nu(\cdot))$ where $\tilde{u} : \mathbb{R} \to \mathbb{R}$ is $u(i \exp(\cdot))$. By definition, the tangent derivative $\frac{\partial}{\partial \tau} u(\tau) = \frac{d}{dt}\tilde{u}(t)$, if $\tau = e^{it} \neq -1$. Hence

$$S_{\pm}(M_u, T_\psi \oplus \eta) = S_{\pm}(\tilde{u}(M_\nu), T_\psi \oplus \eta).$$

Let $w_k \in C^\infty(S^1)$ be such that $0 \le w_k \le 1$ and

$$w_k(\tau) = \begin{cases} 0 & |\tau+1| < 1/k \\ \\ 1 & |\tau+1| \ge 2/k \end{cases}$$

Then

$$S_{\pm}(\tilde{u}(M_\nu), (T_\psi \oplus \eta) M_{w_k}) = S_{\pm}(\tilde{u}(M_\nu), T_\psi \oplus \eta) S_{\pm}(\tilde{u}(M_\nu), M_{w_k})$$

$$= S_{\pm}(\tilde{u}(M_\nu), T_\psi \oplus \eta) M_{w_k} \to S_{\pm}(\tilde{u}(M_\nu), T_\psi \oplus \eta)$$

strongly as $k \to \infty$. But

$$M_\nu (T_\psi \oplus \eta) M_{w_k} - (T_\psi \oplus \eta) M_{w_k} M_\nu$$

$$= M_\nu [T_\psi \oplus \eta, M_{w_k}] + [T_\psi \oplus \eta, M_{w_k} M_\nu].$$

Since $w_k \nu \in C^\infty(S^1)$, we claim that

$$[M_\nu, (T_\psi \oplus \eta) M_{w_k}] \in C_1.$$

Let $E(\cdot)$ be the spectral resolution of M_ν and let

$$\delta^+ = \{t : t \in (-\pi,\pi), \quad \tilde{u}(t) > 0\},$$

$$\delta^- = \{t : t \in (-\pi,\pi), \quad \tilde{u}(t) < 0\}.$$

By the invariance principle (see Theorem XI.11 of [23])

$$S_\pm(\tilde{u}(M_\nu), (T_\psi \oplus \eta)M_{w_k})$$

$$= E(\delta^+)S_\pm(M_\nu, (T_\psi \oplus \eta)M_{w_k}) + E(\delta^-)S_\pm(M_\nu, (T_\psi \oplus \eta)M_{w_k})$$

$$= [E(\delta^+)S_\pm(M_\nu, (T_\psi \oplus \eta)) + E(\delta^-)S_\pm(M_\nu, (T_\psi \oplus \eta))]M_{w_k}.$$

Let

$$E^+ = \{\tau : \frac{\partial u}{\partial \tau}(\tau) > 0\}$$

$$E^- = \{\tau : \frac{\partial u}{\partial \tau}(\tau) < 0\}$$

then it is easy to see that $E(\delta^+) = M_{\chi_{E^+}}$ and $E(\delta^-) = M_{\chi_{E^-}}$.

Let V be the bileteral shif on L^2. It is easy to see that

$$S_\pm(M_\nu, T_\psi \oplus \eta) = \lim_{m\to+\infty} V^m(T_\psi \oplus \eta)V^{-m}.$$

Let $e_p(\tau) = \tau^p$ on S^1, $p = 0, \pm1, \pm2, \ldots$. Then $V^m(T_\psi \oplus \eta)V^{-m}e_p = V^m \eta\, e_{p-m} = \eta\, e_p$ if $p - m < 0$. Hence

$$S_+(M_\nu, T_\psi \oplus \eta) = \eta.$$

Suppose that $\psi = \sum_{k=-\infty}^{\infty} \psi_k e_k$. If $p - m > 0$, then

$$V^m(T_\psi \oplus \eta)V^{-m}e_p = V^m T_\psi e_{p-m} = V^m \sum_{k=m-p}^{\infty} \psi_k e_{k-m-p} = (\sum_{k=m-p}^{\infty} \psi_k e_k)e_p.$$

Taking the limit $m \to -\infty$ gives us that

$$S_-(M_\nu, T_\psi \oplus \eta) = M_\psi.$$

Hence

$$S_+(M_u, T_\psi \oplus \eta) = S_+(\tilde{u}(M_\nu), T_\psi \oplus \eta)$$

$$= \eta M_{\chi_{E^+}} + M_{\chi_{E^-}} M_\psi$$

and

$$S_-(M_u, T_\psi \oplus \eta) = S_+(\tilde{u}(M_\nu)T_\psi \oplus \eta$$

$$= M_{\chi_{E^+}} M_\psi + \eta M_{\chi_{E^-}} .$$

From these two equations Lemma II.2.2 follows immediately.

Note that the trace formulas we have derived are all for the operator on $L^2(\partial D, \omega)$. Now we consider $H^2(D)$ and Toeplitz operators. For a function

$$F(x,y,z) = \int_{\mathbb{R}^3} e^{irx} e^{isy} e^{itz} d\omega(r,s,t)$$

we have

$$F(X,Y,T_\psi \oplus \eta) = \int_{\mathbb{R}^3} e^{irX} e^{isY} [e^{itT_\psi} \oplus e^{i\eta t}] d\omega(r,s,t)$$

$$= P\int_{\mathbb{R}^3} e^{irX} e^{isY} e^{itT_\psi} d\omega(r,s,t) P + Q\int_{\mathbb{R}^3} e^{irX} e^{isY} e^{it\eta} d\omega(r,s,t) Q$$

$$+ Q\int_{\mathbb{R}^3} e^{irX} e^{isY} e^{itT_\psi} d\omega(r,s,t) P + P\int_{\mathbb{R}^3} e^{irX} e^{isY} e^{i\eta} d\omega(r,s,t) Q.$$

The last two terms are trace class operators because

$$[P, e^{irX}] = \int_0^r e^{i\sigma X}[P,X] e^{i(r-\sigma)X} d\sigma,$$

therefore $\|[P,e^{irY}]\|_1 \le |r|\,\|[P,X]\|_1$ and we have the same estimate for $\|[P,e^{isY}]\|_1$.

Furthermore, it is easy to see that

$$F(X,Y,T_\psi \oplus \eta) = \int_{\mathbb{R}^3} Pe^{irX} pe^{isY} pe^{itT_\psi} d\omega(r,s,t) P + QF(X,Y,\eta)Q$$

+ trace class operator. But

$$Pe^{irX}P - e^{irPXP} = \sum_{k=0}^{\infty} \frac{(ir)^k}{k!}[PX^kP - (PXP)^k]$$

$$= \sum_{k=0}^{\infty} \frac{(ir)^k}{k!} P \sum_{j=0}^{k-1} (PXP)^j [X,P]^{k-j-1} P \in \mathcal{C}_1 .$$

Hence if the support of $\omega(r,s,t)$ is compact, then for any operator B such that $[F(X,Y,T_\psi \oplus \eta), B] \in \mathcal{C}_1$ we have

$$\operatorname{tr}[F(X,Y,T_\psi \oplus \eta),B] = \operatorname{tr}[PF(PXP,PYP,T_\psi)P,B] + \operatorname{tr}[QF(X,Y,\eta)Q,B].$$

If H is the Fourier transform of μ with compact support, then

$$\mathrm{tr}[F(X,T_\psi \oplus \eta), H(X,Y,T_\psi \oplus \eta)]$$

$$= \mathrm{tr}[PF(PXP,PYP,T_\psi)P, PH(PXP,PYP,T_\psi)P]$$

$$+ \mathrm{tr}[QF(X,Y,\eta)Q, QH(X,Y,\eta)Q]$$

or, if we regard (x,y) as complex variable $\zeta = x + iy$

$$\mathrm{tr}[F(T_\varphi,T_\psi), H(T_\varphi,T_\psi)]$$

$$= \mathrm{tr}[F(M_\varphi,T_\psi \oplus \eta), H(M_\varphi,T_\psi \oplus \eta)]$$

$$- \mathrm{tr}[QF(M_\varphi,\eta)Q, QH(M_\varphi,\eta)Q].$$

In general, if $\omega,\mu \in M(\mathbb{R}^3)$, then we can define

$$\omega_k(\delta) = \omega(\delta \cap B_k), \mu_k(\delta) = \mu(\delta \cap B_k)$$

for any Borel set δ, where $B_k = \{(r,s,t) : \sqrt{r^2+s^2+t^2} \leq k\}$. Let F_k and H_k be the characteristic functions of ω_k and μ_k. Since $\|F_k - F\|_{\hat{M}(\mathbb{R}^3)} = \|\omega_k - \omega\|$ and $\|H_k - H\|_{M(\hat{\mathbb{R}}^3)} = \|\mu_k - \mu\|$, by Proposition I.1.6, we have that

$$\mathrm{tr}[F(T_\varphi,T_\psi), H(T_\varphi,T_\psi)] = \lim_{k\to\infty} \mathrm{tr}[F_n(T_\varphi,T_\psi), H_n(T_\varphi,T_\psi)]$$

$$= \lim_{k\to\infty}\{\mathrm{tr}[F_n(M_\varphi,T_\psi \oplus \eta), H_n(M_\varphi,T_\psi \oplus \eta)]$$

$$- \mathrm{tr}[QF_k(M_\varphi,\eta)Q, QH_k(M_\varphi,\eta)Q]\}$$

$$= \mathrm{tr}[F(M_\varphi,T_\psi \oplus \eta), H(M_\varphi,T_\psi \oplus \eta)] - \mathrm{tr}[QF(M_\varphi,\eta)Q, QH(M_\varphi,\eta)Q]$$

Following the notations of Theorem II.2.3 and using dA to denote the area measure on $\Omega \times \mathbb{R}$ and Σ the support of G, we have

Corollary II.2.5. For any $F,H \in \hat{M}(\mathbb{R}^3)$,

$$\mathrm{tr}[F(T_\varphi,T_\psi),H(T_\varphi T_\psi)]$$

$$= \frac{i}{2\pi} \int_\Sigma \langle dF \wedge dH, g\rangle (\lambda)\, dA(\lambda)$$

$$- \mathrm{tr}[QM_{F(\varphi,\eta)}, QM_{H(\varphi,\eta)}Q]$$

where
$$g(\lambda) = \begin{cases} \dfrac{\partial}{\partial \tau} \wedge \dfrac{\partial}{\partial z} & : \quad \lambda = (\zeta, z),\ z > \eta \\ -\dfrac{\partial}{\partial \tau} \wedge \dfrac{\partial}{\partial z} & : \quad \lambda = (\zeta, z), z < \eta \end{cases}$$

If $F(\cdot,\eta) = 0$ or $H(\cdot,\eta) = 0$, then

$$\mathrm{tr}[F(T_\varphi,T_\psi),H(T_\varphi,T_\psi)] = \frac{i}{2\pi} \int_\Sigma \langle dF \wedge dH, g\rangle (\lambda)\, dA(\lambda)$$

This completes our concrete calculation of the principal current.

Up to now, we have been assuming that $\varphi \in C^2(\partial D)$. But we also wish to present a trace formula like the one above for a pair Toeplitz operator $\{T_\varphi, T_\psi\}$ with φ being a function with discontinuities. Since our construction relies on the fact that $[T_\varphi, T_\psi]$ and $[T_\varphi, T_\varphi^*]$ be of trace class and the nonsmooth symbol φ can no longer provide such guarantee, we have to impose some smoothness condition on ψ and restrict the discontinuities. Let ψ be a real C^2 function. This assumption takes care of $[T_\varphi, T_\psi]$. But the problem for $[T_\varphi, T_\varphi^*]$ is much more subtle. Since φ is not smooth, $[M_\varphi, T_\psi \oplus \eta]$ is not trace class in general despite the smoothness of T_ψ. Hence rather than constructing an integral current with local data of symbols, we are forced to start with the trace formula in Corollary II.2.5 which can only be applied to operators with smooth symbols. Thus we can only expect to archive a trace formula for the nonsmooth case by limiting procedure. This is executed through delicate approximation of symbols. But when φ is altered, both the operator theory of the left hand side of the trace formula and the geometry of the right hand side change. Hence if we use a sequence $\{\varphi_n\}$ of smooth to approach φ, we have to control the trace of the commutators and the integrals so that they eventually converge. But this is not easy. Obviously, φ can not behave too badly. However, since our interest here is not the study of the pathology caused the bad behavior of φ but the exploration of new phenomena, we impose relatively strong conditions to eliminate technical complications. We assume that φ has only a finite number of jump discontinuities $\{\tau_1,\ldots,\tau_m\}$ and is a C^2 function elsewhere. Let $U_i, i = 1,\ldots,m$ be a neighborhood of τ_i such that $\overline{U}_i \cap \overline{U}_j = \phi$ if $i \neq j$. Let U_0 be an open set such that $\partial D = U_0 \cup U_1 \cup \ldots \cup U_m$ and distance $(\{\tau_0,\ldots,\tau_m\}, U_0) \geq 0$. Let $1 = \sum_{j=0}^{m} \eta_j$ be a partition of unit subordinate to the covering $\{U_0, U_1,\ldots,U_m\}$. Thus

$$[T_\varphi, T_\varphi^*] = \sum_{j=1}^{m} [T_{\varphi\eta_j}, T_{\varphi\eta_j}^*] + \sum_{\substack{i\neq j \\ i,j\neq 0}} [T_{\varphi\eta_i}, T_{\varphi\eta_j}^*]$$

$$+ [T_{\varphi\eta_0}, T_\varphi^*] + [T_\varphi, T_{\varphi\eta_0}^*].$$

Since $[T_{\varphi\eta_0}, T_\varphi^*] = PM_{\varphi\eta_0}(1-P)M_{\overline{\varphi}}P - PM_\varphi(1-P)M_{\overline{\varphi}\eta_0}P$, if $\tilde{\varphi}$ agrees with φ on the support of η_0, by Corrollary II.2

$$\|[T_{\varphi\eta_0}, T_\varphi^*] - [T_{\tilde{\varphi}\eta_0}, T_{\tilde{\varphi}}^*\|_1 \leq L\|\varphi - \tilde{\varphi}\|_{L^2}$$

where L depends only on the restriction of φ to the support of η_0. Since $\overline{U}_i \cap \overline{U}_j = \phi$ and

$$[T_{\varphi\eta_i}, T_{\varphi\eta_j}^*] - [T_{\tilde{\varphi}\eta_i}, T_{\tilde{\varphi}\eta_i}^*]$$

$$= [T_{(\varphi-\tilde{\varphi})\eta_i}, T_{\varphi\eta_j}^*] + [T_{\tilde{\varphi}\eta_i}, T_{(\varphi-\tilde{\varphi})\eta_j}^*],$$

by Corollary II.1.4, we conclude that

$$\Big\| \sum_{\substack{i\neq j \\ i,j\neq 0}} \{[T_{\varphi\eta_i}, T^*_{\varphi\eta_j}] - [T_{\tilde{\varphi}\eta_i}, T^*_{\tilde{\varphi}\eta_j}]\}\Big\|_1$$

$$\leq L(\|\tilde{\varphi}\|_{L^2} + \|\varphi\|_{L^2})\|\tilde{\varphi} - \varphi\|_{L^2}.$$

Now we make further assumption that φ' is a C^1 function on ∂D, i.e. $\lim_{\tau\to\tau_j^+} \varphi'(\tau) = \lim_{\tau\to\tau_j^-} \varphi'(\tau)$ and $\lim_{\tau\to\tau_j^+} \varphi''(\tau) = \lim_{\tau\to\tau_j^-} \varphi''(\tau)$, $j = 1,\ldots,m$, where $\tau \to \tau_j^+$ and $\tau \to \tau_j^-$ indicate the limits from different sides of τ_j. It is easy to see that there exists $\alpha_j \in S^1$ such that Re $\alpha_j\varphi\eta_j$ is continuous and therefore C^2 function on ∂D. Let $\hat{x}_j$ = Re $\alpha_j\varphi\eta_j$ and $\hat{y}_j$ = Im $\alpha_j\varphi\eta_j$. We will approximate $\hat{y}_j$ by a sequence of $C^2(\partial D)$ functions in L^2 norm. Let $\gamma_j : [-1,1] \to \partial D$ be a local parameter near τ_j and $\gamma_j(0) = \tau_j$. $\hat{y}_j\circ\gamma_j$ has the only jump at 0. Define $\tilde{y}_{j,j}$ to be a C^2 function on $[-1,1]$ such that for $|t| \geq \frac{1}{k}$, $\tilde{y}_{j,k}(t) = \hat{y}_j\circ\gamma_j(t)$ and on $[-\frac{1}{k}, \frac{1}{k}]$, $\tilde{y}_{j,k}$ joins $\hat{y}_j\circ\gamma_j(-\frac{1}{k})$ and $\hat{y}_j\circ\gamma_j(\frac{1}{k})$. We also choose $\tilde{y}_{j,k}$ such that

$$\int_{-1}^{1} |\tilde{y}'_{j,k}(t)|dt \leq M < \infty$$

for some M independent of k. We defer the proof of the existence of such $\tilde{y}_{j,k}$.

Define on ∂D

$$\hat{y}_j^k(\tau) = \begin{cases} \tilde{y}_{j,k}(t) & \text{if } \tau = \gamma_j(t) \\ \hat{y}_j(\tau) & \text{if } \tau \notin \gamma_j[-1,1] . \end{cases}$$

Obviously $\hat{y}_j^k$ is a C^2 function and $\|\hat{y}_j^k - \hat{y}_j\|_{L^2} \to 0$ ad $k \to \infty$.

Let

$$\varphi_k(\tau) = \begin{cases} \alpha_j^{-1}(\hat{x}_j + i\hat{y}_j^k)(\tau) & \tau \in \text{supp } \eta_n \\ \varphi(\tau) & \tau \in \partial D \setminus \bigcup_{j=1}^{m} \text{supp } \eta_j \end{cases}$$

then it is easy to see that $\varphi_k \in C^2$ and $\lim_{k\to\infty}\|\varphi_k - \varphi\|_{L^2} = 0$.

Moreover,

$$[T_{\varphi_k \eta_j}, T^*_{\varphi_k \eta_j}] = 2i[T_{\hat{x}_j}, T_{\hat{y}_j^k}].$$

From this we can easily conclude that there exists $L > 0$ such that

$$\|[T_{\varphi_k}, T^*_{\varphi_k}] - [T_{\varphi_{k'}}, T^*_{\varphi_{k'}}]\|_1 \leq L\|\varphi_k - \varphi_{k'}\|_{L^2}$$

for any k, k'. Since $s - \lim_{k\to\infty} T_{\varphi_k} = T_\varphi$, $[T_\varphi, T_\varphi^*] \in \mathcal{C}_1$ and

$$\lim_{k\to\infty} \| [T_{\varphi_k}, T_{\varphi_k}^*] - [T_\varphi, T_\varphi^*] \|_1 = 0.$$

Hence by Proposition I.1.7,

$$\lim_{k\to\infty} \mathrm{tr}[F(T_{\varphi_k}, T_\psi), H(T_{\varphi_k}, T_\psi)]$$

$$= \mathrm{tr}[F(T_\varphi, T_\psi), H(T_\varphi, T_\psi)].$$

Now we consider the convergence of the integrals

$$\int_{\Sigma_k} \langle dF \wedge dH, g_k \rangle (\lambda)\, dA(\lambda)$$

where Σ_k calculated in terms of $\{\varphi_k, \psi\}$. For simplicity, we assume that $\psi \geq \eta$. Then

$$\int_{\Sigma_k} \langle dF \wedge dH, g_k \rangle\, dA(\lambda) = \int_{\Sigma_k} dF \wedge dH.$$

Now let Ω be the image of φ union the line segments joining $\varphi(\tau_j^+)$ and $\varphi(\tau_j^-)$, $j = 1, \ldots, m$. Define

$$\Sigma = \begin{cases} (\varphi(\tau), t) & : \ \tau \neq \tau_j,\ \eta \leq t \leq \psi(\tau),\ j = 1, \ldots, m \\ (\zeta, t) & : \ \zeta \text{ in the line joining } \varphi(\tau_j^+) \\ & \qquad \text{and } \varphi(\tau_j^-), \\ & \qquad \eta \leq t \leq \psi(\tau_j), j = 1, \ldots, m\ . \end{cases}$$

Theorem II.2.6.

$$\operatorname{tr}[F(T_\varphi, T_\psi), H(T_\varphi, T_\psi)]$$

$$= \frac{i}{2\pi} \int_\Sigma dF \wedge dH - \operatorname{tr}[QM_{F(\varphi,\eta)}Q, QM_{H(\varphi,\eta)}Q].$$

Proof. It suffices to show that

$$\lim_{k\to\infty} \int_{\Sigma_k} dF \wedge dH = \int_\Sigma dF \wedge dH$$

and

$$\lim_{k\to\infty} \operatorname{tr}[QM_{F(\varphi_k,\eta)}Q, QM_{H(\varphi_k,\eta)}Q] = \operatorname{tr}[QM_{F(\varphi,\eta)}Q, QM_{H(\varphi,\eta)}Q]$$

By the definition of φ_k, there exists an open arc A_k^j containing τ_j such that φ and φ_k agree outside $A_k^1 \cup \ldots \cup A_k^m$ and such that $\omega(A_k^j) \to 0$ ad $k \to \infty$. Let σ_k^j be the curves consisting of $\varphi_k(A_k^j)$, $\varphi(A_k^j)$ and the line joining $\varphi(\tau_j^+)$ and $\varphi(\tau_j^-)$. If we keep the orientation on $\varphi_k(A_k^j)$ and reverse on the rest, then σ_k^j is an oriented curve and

$$\int_{\Sigma_k} dF \wedge dH - \int_\Sigma dF \wedge dH = \int_{\bigcup_{j=1}^m \Sigma_k^j} dF \wedge dH$$

where

$$\sigma_k^j \times [\eta, \min_{\tau\in A_k^j} \psi(\tau)] \subset \Sigma_k^j \subset \sigma_k^j \times [\eta, \max_{\tau\in A_k^j} \psi(\tau)]$$

By the boundedness of $\|\varphi_k'\|_\infty$ and the continuity of ψ, we claim that if H_2 is the 2 dimensional Hausdorff measure, then

$$\lim_{k\to\infty} H_2(\Sigma_k^j \setminus \sigma_k^j \times [\eta, \min_{\tau\in A_k^j} \psi(\tau)]) = 0 .$$

$$\lim_{k\to\infty}(\int_{\Sigma_k} dF \wedge dH - \int_{\Sigma} dF \wedge dH)$$

$$= \lim_{k\to\infty} \int_{\bigcup_{j=1}^{m} \sigma_k^j \times [\eta, \min_{\tau\in A_k^j} \psi(\tau)]} dF \wedge dH$$

Let V_k be the solid enclosed by $\bigcup_{j=1}^{m} \times [\eta, \min_{\tau\in A_k^j} \psi(\tau)]$ union the top and bottom, then by Stokes' theorem

$$\int_{\bigcup_{j=1}^{m} \sigma_k^j \times [\eta, \min_{\tau\in A_k^j} \psi(\tau)]} dF \wedge dH$$

$$\int_{V_k} d(dF \wedge dH) - \int_{top} dF \wedge dH - \int_{bottom} dF \wedge dH$$

A horizontal cross-section of V_k has the area no more than $\int_{\partial D} |\varphi_k - \varphi| dm$. To prove that

$$\lim_{k\to\infty} \operatorname{tr}[QM_{F(\varphi_k,\eta)}Q, QM_{H(\varphi_k,\eta)}Q]$$

$$= \operatorname{tr}[QM_{F(\varphi_k,\eta)}Q, QM_{H(\varphi_k,\eta)}Q]$$

we only need to note that like Toeplitz operators,

$[QM_{\varphi_k}Q, QM_{\bar{\varphi}}Q] \in C_1$ and

$$\lim_{k\to\infty}[QM_{\varphi_k}Q, QM_{\bar{\varphi}_k}Q] = [QM_{\varphi}Q, QM_{\bar{\varphi}}Q]$$

Then Proposition I.1.7 again enables us to conclude that the traces onverge. This proves the theorem.

Now we show how to choose those $\tilde{Y}_{j,k}$. Define

$$h(t) = \begin{cases} \exp(-\frac{1}{t^2})/[\exp(-\frac{1}{t^2}) + \exp(-\frac{1}{(t-1)^2})] & : \quad t > 0 \\ 0 & : \quad t = 0 \\ \exp(-\frac{1}{t^2})/[\exp(-\frac{1}{t^2}) + \exp(-\frac{1}{(t+1)^2})] & : \quad t < 0 \end{cases}$$

h is a C^∞ function and $h^{(p)}(0) = 0$ for any p. It is easy to see that $h(t) = 1$ when $|t| \geq 1$ and $h'(t) \geq 0$ if $t \geq 0$ and $h'(t) \leq 0$ if $t \leq 0$. Let

$$h_k(t) = h(kt)$$

and

$$\tilde{Y}_{j,k}(t) = \hat{Y}_j \circ \gamma_j(t) \cdot h_k(t).$$

Thus

$$\tilde{y}'_{j,k}(t) = (\hat{y}_j \circ \gamma_j)'(t)h_k(t) + \hat{y}_j \circ \gamma_j(t)h'_k(t)$$

and

$$\int_{-1}^{1} |\tilde{y}'_{j,k}(t)|dt \leq \int_{-1}^{1} |(\hat{y}_j \circ \gamma_j)'(t)|dt + \|\varphi\|_\infty \int_{-1}^{1} |h'_k(t)|dt.$$

But

$$\int_{-1}^{1} |h'_k(t)|dt = \int_0^1 h'_k(t)dt - \int_{-1}^{0} h'_k(t)dt$$

$$= \int_0^1 kh'(kt)dt - \int_{-1}^{0} kh'(kt)dt = 2.$$

Hence

$$\int_{-1}^{1} |y'_{j,k}(t)|dt \leq M < \infty$$

Obviously $\tilde{y}_{j,k}$ is a C^2 function and satisfies the other requirements.

Following the notations in Corollary II.2.5 and Theorem II.2.6, we will refer to the set $\Omega \times \mathbb{R}$ as the cylinder and therefore $\Sigma(=\Sigma(\varphi,\psi))$ is identified with the characteristic function that represents it on the cylinder. A point on the cylinder is said to be a discontinuity point of χ_Σ if on any neighborhood of that point χ_Σ is not exclusively (in dA measure) 0 or 1. The collection of all these points is called the discontinuity set, denoted by $\partial\Sigma = \partial\Sigma(\varphi, \psi)$, of χ_{Σ} or, equivalently, of G.

II.3 Joint essential spectra and indices.

Let $\mathfrak{D}(H) = \mathcal{L}(H)/\mathcal{K}(H)$ be the Calkin algebra on Hilbert space H. Let $C = \{A_i : i \in I\}$ be a collection of bounded operators on H and let $\hat{C} = \{\hat{A}_i : i \in I\}$ be the quitient image of C in $\mathfrak{D}(H)$. Suppose that $\hat{C}$ is a collection of commutative elements in $\mathfrak{D}(H)$ and let $\mathcal{R}(\hat{C})$ denote the C^* algebra generated by $\hat{C}$. A point $\lambda = (\lambda_i)_{i\in I} \in \mathbb{C}^I$ is said to be in the joint essential spectrum of C if $\{\hat{A}_i - \lambda_i : i \in I\}$ generates a proper ideal in $\mathcal{R}(\hat{C})$. Hence the joint essential spectrum of C and the maximal ideal space of $\mathcal{R}(\hat{C})$ are in this way naturally identified. Denote by $\mathcal{R}(C)$ the C^* algebra generated by C and $\mathcal{J}(C)$ the commutator ideal of $\mathcal{R}(C)$. If it happens that $\mathcal{J}(C) = \mathcal{K}(H)$, then $\mathcal{R}(C)/_{\mathcal{J}(C)} = \mathcal{R}(\hat{C})$ and the joint essential spectrum of C is

$$\{\lambda : \lambda \in \mathbb{C}^I,\ \hat{A}_i - \lambda_i,\ i \in I \text{ generate a proper ideal in } \mathcal{R}(C)/_{\mathcal{J}(C)}\}.$$

If we have an isomorphism

$$\alpha : \mathcal{R}(C)/_{\mathcal{J}(C)} \cong \bigoplus_{j=1}^{k} \mathcal{R}_j,$$

then the joint essential spectrum of C equals

$$\bigcup_{j=1}^{k} \{\lambda : \lambda \in \mathbb{C}^I,\ \alpha_j(\hat{A}_i) - \lambda_i,\ i \in I \text{ generate a proper ideal in } \mathcal{R}_j\}$$

where α_j is the projection of α onto $\mathcal{R}_j$. Thus the decomposition at C*-algebra level reduces the calculation of joint essential spectrum to each individual piece of maximal ideal of $\mathcal{R}_i$.

Our calculation of joint essential spectra for certain pairs of operator on $L^2(\partial D,\omega)$ will follow this strategy.

We start with the pair $\{M, T_\psi \oplus \eta\}$ where ψ is a non-constant real function.

<u>Lemma II.3.1.</u> $\mathcal{R}(M, T_\psi \oplus \eta)$ is irreducible and contains a non-zero compact operator.

<u>Proof.</u> Let H_0 be a reducing subspace for $\mathcal{R}(M, T_\psi \oplus \eta)$. Then the orthogonal projection onto H_0 must commute with M. Hence $H_0 = L^2(E,\omega)$ for some Borel set $E \subset \partial D$. But $T_{\psi-\eta} \oplus 0 = T_\psi \oplus \eta - \eta \in \mathcal{R}(M, T_\psi \oplus \eta)$ so $(T_{\psi-\eta} \oplus 0)H_0 \subset H^2(D) \cap L^2(E,\omega)$. This implies that either $\omega(\partial D \backslash E) \neq 0$, in this case $(T_{\psi-\eta} \oplus 0)H_0 = \{0\}$, or $E = \partial D$, therefore, $H_0 = L^2(\partial D,\omega)$. The former implies that $PH_0 \subset \ker T_{\psi-\eta}$. For any $X \in L^2(E,\omega)$, we have

$$X = f_x + v_x + \overline{g}_x + x(z_0)$$

where $f_x, g_x \in H_0^2(D)$ and $\nu_x \in N$, see Section I.2. $L^2(E,\omega)$ is closed under complex conjugation. Hence if $X \in L^2(E,\omega)$ then

$$f_x + x(z_0) = Px \in PH_0$$

and

$$g_x + \overline{x(z_0)} = P\overline{x} \in PH_0 \text{ .}$$

Since $\dim N = n$, if $\dim L^2(E,\omega) = \infty$, then $\dim PH_0 = \infty$. But by Section IV.1, $\dim \ker T_{\psi-\eta} \leq n$. Hence $\dim L^2(E,\omega)$ can not be infinite. This implies $\omega(E) = 0$. Therefore, $\mathcal{R}(M, T_\psi \oplus \eta)$ is irreducible.

We know that $[M, T_\psi \oplus \eta] \in C_1$. Therefore, to prove the second assertion, it suffices to show that

$$[M, T_\psi \oplus \eta] \neq 0.$$

If it were true that $[M, T_\psi \oplus \eta] = 0$, then $T_{\psi-\eta} \oplus \eta$ would belong to the commutant of M. Therefore, there would be a $\psi_1 \in L^\infty(\partial D, \omega)$ such that $T_{\psi-\eta} \oplus 0 = M_{\psi_1}$. For any $u \in \overline{H_0^2(D)}$,

$$\psi_1 u = M_{\psi_1} u = (T_{\psi-\eta} \oplus 0)u = 0.$$

Hence $\psi_1 = 0$ and $T_{\psi-\eta} = 0$. This contradicts the assumption that ψ is not a constant. Thus the proof is completed.

By a familiar lemma due to Dixmier, we conclude

Corollary II.3.2. $\mathcal{J}(M, T_\psi \oplus \eta) = \mathcal{K}(L^2(\partial D, \omega))$.

This corrollary enables us to identify the joint essential spectrum of $\{M, T_\psi \oplus \eta\}$ with the maximal ideal space of $\mathcal{R}(M, T_\psi \oplus \eta)/\mathcal{K}(L^2(\partial D, \omega))$. To explicitly calculate the latter, we need the following decomposition.

Lemma II.3.3.

$$\mathcal{R}(M, T_\psi \oplus \eta)/_{\mathcal{K}} = \bigoplus_{i=0}^{n} \mathcal{R}(M, T_\psi^i \oplus \eta)/_{\mathcal{K}_i} .$$

This is an immediate consequence of Theorem I.2.5.

Obviously,

$$\{ \bigcup_{i=0}^{n} \{\pi_i(\lambda), \mu) \ : \ (\lambda, \mu) \in \text{the joint essential spectrum of } \{M, T_{\psi \circ \pi_i} \oplus \eta\} \text{ on } H^2\}$$

is the joint essential spectrum of $\{M, T_\psi \oplus \eta\}$ on $H^2(D)$.

Now the problem is reduced to the calculation of joint essential spectrum of $\{M, T_\psi \oplus \eta\}$ on $L^2(S^1)$. This is essentially computed in Carey-Pincus [3] and [4].

On torus $T^2 = S^1 \times S^1$, let

$$\Sigma = \{(\tau, \sigma) \ : \ \tau \in S^1, \ 0 \le \arg \sigma \le \arg \frac{\psi(\tau) - \bar{\ell}}{\psi(\tau) - \ell}\}$$

where ℓ is a nonreal complex number and arg is regulated to be between 0 and 2π. According to the references mentioned above, the joint essential spectrum for the pair $\{M,(T_\psi-\ell)(T_\psi-\ell)^{-1}\oplus 1\}$ is exactly the discontinuity set $\partial\Sigma$ of χ_Σ on T^2. On the other hand we know that the joint essential spectrum for a pair $\{A,B\}$ is exactly the set of $(\tau,\sigma)\in\mathbb{C}^2$ such that there exist $\{x_k\}\subset H$ such that

$$\lim_{k\to\infty}(\|(A-\tau)x_k\|+\|(B-\sigma)x_k\|)=0,\ \|x_k\|=1,$$

if $\ker A\cap\ker B=\{0\}$ (see [15]).

$$(T_\psi-\bar{\ell})(T_\psi-\ell)^{-1}-1=(\ell-\bar{\ell})(T_\psi-\ell)^{-1} \text{ is an}$$

invertable operator on H^2. Hence the joint essential spectrum of $\{M,(T_\psi-\bar{\ell})(T_\psi-\bar{\ell})^{-1}\}$ on H^2 is

$$\partial\Sigma\backslash\{(\sigma,1)\ :\ \sigma\in S^1\}.$$

By spectral mapping, the joint essential spectrum of $\{M,T_\psi\}$ on H^2 is

$$\{(\sigma,t)\ :\ (\sigma,\frac{t-\bar{\ell}}{t-\ell}\in\partial\Sigma\}$$

and that of $\{M,T_\psi\oplus\eta\}$ on $L^2(S^1)$ is

$$\{(\sigma,t)\ :\ (\sigma,\frac{t-\bar{\ell}}{t-\ell})\in\partial\Sigma\}\cup\{(\sigma,\eta)\ :\ \sigma\in S^1\}.$$

Thus we have proved the following:

Theorem II.3.4. The joint essential spectrum of pair $\{M, T_\psi \oplus \eta\}$ on $L^2(\partial D, \omega)$ is the discontinuity set $\partial\Sigma(z, \psi)$ of $\chi_{\Sigma(z,\psi)}$ on the cylinder.

Now we consider the pair $\{M, T_\psi\}$ on $H^2(D)$. For the purpose of computing joint essential spectrum we choose $\eta < -\|\psi\|_\infty$. Thus $\mathbb{C} \times \{\eta\}$ does not intersect the joint essential spectrum of $\{M, T_\psi\}$. Since $\ker M = \{0\}$ on both $H^2(D)$ and $L^2(\partial D, \omega)$, by [15], the joint essential spectrum of $\{M, T_\psi\} \subset \partial\Sigma(z, \psi) \backslash \mathbb{C} \times \{\eta\}$. On the other hand, if $(\tau, \lambda) \in \partial\Sigma(z, \psi) \backslash \mathbb{C} \times \{\eta\}$, then there exist $\{u_k\} \subset L^2(\partial D, \omega)$ such that $\|u_k\| = 1$ and

$$\lim_{k\to\infty}[\|(M-\tau)u_k\| + (\|(T_\psi-\lambda)Pu_k\|^2 + \|(\lambda-\eta)Qu_k\|^2)^{\frac{1}{2}}] = 0$$

where $Q : L^2(\partial D, \omega) \to [H^2(D)]^\perp$ is the projection. Since $\lambda \neq \eta$, we can conclude that $\|Qu_k\| \to 0$ as $k \to \infty$. Let $v_k = Pu_k/\|Pu_k\|$, then

$$\lim_{k\to\infty}\|(T_\psi-\lambda)v_k\| = \lim_{k\to\infty}\|(T_\psi-\lambda)Pu_k\|/\|Pu_k\| = 0$$

and

$$\lim_{k\to\infty}\|(M-\tau)v_k\| = \lim_{k\to\infty}\|(M-\tau)Pu_k\|/\|Pu_k\|$$

$$\leq \lim_{k\to\infty}[(\|M-\tau)u_k\| + \|(M-\tau)Qu_k\|]/\|pu_k\| = 0.$$

Corollary II.3.5. The joint essential spectrum of $\{M, T_\psi\}$ on $H^2(D)$ is $\partial\Sigma(z,\psi)\backslash\partial D \times \{\eta\}$.

Let $\varphi \in C^2(\partial D)$. Spectral mapping theorem for C*-algebras yields the following two corollaries.

Corollary II.3.6. The joint essential spectrum of $\{M_\varphi, T_\psi \oplus \eta\}$ on $L^2(\partial D, \omega)$ is $\partial\Sigma(\varphi,\psi)$.

Corollary II.3.7. The joint essential spectrum of $\{T_\varphi, T_\psi\}$ on $H^2(D)$ is $\partial\Sigma(\varphi,\psi)\backslash \times \{\eta\}$.

For the proof of these two corollaries we only need to note that on $H^2(D)$, the difference $\varphi(M) - T_\varphi$ is a compact (indeed trace class) operator.

Denote $\quad \partial\Sigma_0(\varphi,\psi) = \partial\Sigma(\varphi,\psi)\backslash\mathbb{C} \times \{\eta\}$.

Having determined the oint essential spectrum, index is naturally the next object to be investigated. We explain the Carey-Pincus construction in [3], [6] as it applies to our situation. Let us consider a pair $\{A,B\}$. If F is a continuous function which does not vanish on the joint essential spectrum, then $F(A,B)$ is a Fredholm operator. If H is a function such that H is the inverse of F on the joint essential spectrum, then $H(A,B)$ is a pseudo inverse of $F(A,B)$.

Lemma II.3.8. If $H(A,B)F(A,B) - 1$ and $F(A,B)H(A,B) - 1$ are trace class operators, then $[F(A,B),H(A,B)] \in C_1$ and

$$\text{index } F(A,B) = \text{tr}[F(A,B),H(A,B)].$$

Proof. There exists bounded operator T such that

$$TF(A,B) = 1 - P_1$$

$$F(A,B)T = 1 - P_2$$

where P_1 and P_2 are the projections onto the kernel and co-kernel of $F(A,B)$. Therefore, P_1 and P_2 are finite rank operators. Let $S = T - H(A,B)$. Then S is compact and $SF(A,B) \in C_1$ and $F(A,B)S \in C_1$. By Theorem III.8.2 of [7], $\text{tr}[F(A,B),S] = 0$. Hence

$$\text{tr}[F(A,B),H(A,B)] = \text{tr}[F(A,B),T] = \text{tr}[P_1 - P_2]$$

$$= \text{index } F(A,B).$$

If $\{A,B\}$ is either $\{M_\varphi, T_\psi \oplus \eta\}$ on $L^2(\partial D, \omega)$ or $\{T_\varphi, T_\psi\}$ on $H^2(D)$, then we have the following:

Lemma II.3.9. For any $F,H \in \hat{M}(\mathbb{R}^3)$,

$$\mathrm{tr}[F(A,B),H(A,B)]$$

depends only on the value of F and H on the joint essential spectrum.

Proof. Let $\tilde{F}$ be another function which agrees with F on the joint essential spectrum. Then $F(A,B) - \tilde{F}(A,B)$ is compact operator and therefore so are

$$\mathrm{Re}[F(A,B) - \tilde{F}(A,B)]$$

and

$$\mathrm{Im}[F(A,B) - \tilde{F}(A,B)].$$

But by functional calculus

$$[\mathrm{Re}(F(A,B) - \tilde{F}(A,B)),H(A,B)] \in \mathcal{C}_1$$

and

$$[\mathrm{Im}(F(A,B) - \tilde{F}(A,B)),H(A,B)] \in \mathcal{C}_1$$

By Lemma 8.1 of [5], $\mathrm{tr}[(F(A,B) - \tilde{F}(A,B)),H(A,B)] = 0$. Applying the same argument to H yields the proof.

Now let F be a C^2 function defined on a neighborhood of the joint essential spectrum and let $z \in \mathbb{C}$ not in the image of F. F can be extended to a neighborhood of Σ so that z is still not in the image. Then we extend F again to a C^2 function with compact support in $\mathbb{R}^3$. Obviously the extended

F is in $\hat{M}(\mathbb{R}^3)$. Let H be a $\hat{M}(\mathbb{R}^3)$ function such that in a neighborhood of Σ, H is the inverse of F - z. On the other hand we can also extend F to $\tilde{F}$ which is zero outside a neighborhood of the joint essential spectrum. Let $\tilde{H} \in \hat{M}(\mathbb{R}^3)$ be the inverse of $\tilde{F}$ - z in a neighborhood of the joint essential spectrum. Then by Lemma II.3.8 and II.3.9,

$$\begin{aligned} \text{index}(F(A,B)-z) &= \text{tr}[F(A,B),H(A,B)] \\ &= \text{tr}[\tilde{F}(A,B),\tilde{H}(A,B)]. \end{aligned}$$

We can assume that $\eta < -\|\psi\|_\infty$ and in the case $\{A,B\}$ is $\{T_\varphi, T_\psi\}$. $\tilde{F}$ is zero on $\mathbb{C} \times \{\eta\}$. Hence

$$\text{index}(F(M_\varphi, T_\psi \oplus \eta) - z) = \frac{i}{2\pi} \int_\Sigma d\tilde{F} \wedge d\tilde{H}$$

and

$$\text{index}(F(M_\varphi, T_\psi) - z) = \frac{i}{2\pi} \int_\Sigma d\tilde{F} \wedge d\tilde{H}\,.$$

Suppose $\varphi, \psi \in C^1(\partial D)$. Then $\partial\Sigma$ is a C^1 manifold with the induced differential structure. Note that we can rewrite the (1,1) form $dF \wedge dH = d(HdF)$. By Stokes' theorem

$$\int_\Sigma dF \wedge dH = \int_\Sigma d(HdF) = \int_{\partial\Sigma} HdF. \quad \text{Hence}$$

Hence

$$\text{index}(F(M_\varphi, T_\psi \oplus \eta - z) = \frac{i}{2\pi} \int_{\partial\Sigma(\varphi,\psi)} \tilde{H}d\tilde{F}$$

$$= -\frac{1}{2\pi i}\int_{\partial\Sigma(\varphi,\psi)} \frac{dF}{F-z}$$

and

$$\text{index}(F(T_\varphi, T_\psi) - z) = \frac{i}{2\pi}\int_{\partial\Sigma_0(\varphi,\psi)} \tilde{H}d\tilde{F}$$

$$= -\frac{1}{2\pi i}\int_{\partial\Sigma_0(\varphi,\psi)} \frac{dF}{F-z}$$

The integral

$$\frac{1}{2\pi i}\int \frac{dF}{F-z}$$

gives the winding number of F about point z.

Theorem II.3.10. Let $\varphi, \psi \in C^1(\partial D)$. For $F \in \hat{M}(\mathbb{R}^3)$, $z \in \mathbb{C}$ is a Fredholm point of $F(M_\varphi, T_\psi \oplus \eta)$ or $F(T_\varphi, T_\psi)$ if and only if z is not in the image of the corresponding joint essential spectrum. If z is a Fredholm point, index $F(M_\varphi, T_\psi \oplus \eta)$ (index $F(T_\varphi, T_\psi)$) is minus the winding number of F about z. In general, if F is a continuous function on the joint essential spectrum and z is not in its image, then $F(\hat{M}_\varphi, \widehat{T_\psi \oplus \eta})$ (resp. $F(\hat{T}_\varphi, \hat{T}_\psi)$) is an invertable element in the Calkin algebra and if T is a representative of $F(\hat{M}_\varphi, \widehat{T_\psi \oplus \eta})$ (resp. $F(\hat{T}_\varphi, \hat{T}_\psi)$), then index T is minus the winding number of F about z.

The second part of the theorem follows from the first part by a simple limiting argument.

The same results can be derived for discontinuous φ. Let φ be a piecewise C^2 function as considered in Section II.2. Recall that the support set $\Sigma = \Sigma(\varphi, \psi)$ of G in this case is

$$\{\varphi(\tau), t) : \eta \le t \le \psi(\tau), \tau \in \partial D\} \cup \{(s\varphi(\tau_j^+) + (1-s)\varphi(\tau_j^-), t)$$
$$: s \in [0,1], \eta \le t \le \psi(\tau_j), j = 1, \ldots, m\}$$

(we assume that $\eta < -\|\psi\|_\infty$). It is easy to see that the joint essential spectrum of $\{T_\varphi, T_\psi\}$ is contained in

$$\partial\Sigma_0(\varphi, \psi) = \{(\varphi(\tau), \psi(\tau)) : \tau \ne \tau_j, j = 1, \ldots, m\}$$
$$\cup \{(s\varphi(\tau_j^+) + (1-s)\varphi(\tau_j^-), \psi(\tau_j)) : s \in [0,1], j=1, \ldots, m\}.$$

Use the same argument as before, we can show that if $F \in \hat{M}(\mathbb{R}^3)$ and z is not in the image of F on a neighborhood of $\partial\Sigma_0(\varphi, \psi)$, then $F(T_\varphi, T_\psi) - z$ is a Fredholm operator and

$$\text{index}(F(T_\varphi, T_\psi) - z) = -\frac{1}{2\pi i}\int_{\partial\Sigma_0(\varphi,\psi)} \frac{dF}{F-z}$$

Since the index of $F(T_\varphi, T_\psi)$ depends on the values of F on whole $\partial\Sigma_0(\varphi, \psi)$, this set must be contained in the joint essential spectrum.

<u>Theorem II.3.11</u>. For piecewise C^2 function φ and C^2 function ψ, the joint essential spectrum of $\{T_\varphi, T_\psi\}$ is $\partial\Sigma_0(\varphi, \psi)$. Let

F be a continuous function on $\partial\Sigma_0(\varphi,\psi)$, then z is a Fredholm point for $F(T_\varphi,T_\psi)$ if and only if $F - z$ does not vanish. If z is a Fredholm point, the index of $F(T_\varphi,T_\psi)$ is minus the winding number of F about z.

In the cases we considered above, $\partial\Sigma$ is a piecewise C^1 manifold, so Stokes' theorem enables us to express the index as integral on the joint essential spectrum. But Lemma II.3.9 is true regardless the smoothness of $\partial\Sigma$. In fact, if we consider the trace formula as a current for test functions, then its boundary current is representable by integral. Let $\varphi \in C^2(\partial D)$ and $\psi \in L^\infty(\partial D,\omega)$ be a real function. Then we can define the cylinder $\Omega \times \mathbb{R}$ as before. Let $C^p_{r,0}$ denote the set of r C^p forms with compact support on the cylinder. A current ℓ can be extended to a continuous functional on $C_{r,0} = C^0_{r,0}$. In this case the Riesz theorem tells us that there exists a positive measure $\|\ell\|$ such that

$$\ell(u) = \int \langle u(x), \vec{\ell}(x)\rangle d\|\ell\|(x)$$

where $\vec{\ell}$ is a r-vector field and $\|\vec{\ell}(x)\| = 1$ for almost all x.

Define

$$[G](u) = \int_\Sigma \langle u, G\rangle(\lambda)\, dA(\lambda), \quad u \in C^\infty_{2,0}$$

Let ℓ be the boundary current of $[G]$, i.e. $\ell(u) = \partial[G](u) = [G](dv)$ for $v \in C^{\infty}_{1,0}$. Suppose that the spectral multiplicity functions of $X = \text{Re } M_{\varphi}$ and $Y = \text{Im } M_{\varphi}$ are integrable.

<u>Proposition II.3.12.</u> ℓ is representable by integration. Furthermore, ℓ is supported on the joint essential spectrum.

<u>Proof.</u> Let $v = fd\tau + hdz$, then

$$\ell(v) = [G](\bar{d}v) = [G](\bar{d}f \wedge d\tau + \bar{d}h \wedge dz)$$

$$= \text{tr}[f(M_{\varphi}, T_{\psi} \oplus \eta), M_{\varphi}] + \text{tr}[h(M_{\varphi}, T_{\psi} \oplus \eta), T \oplus \eta]$$

Thus the proof is reduced to showing that

$$|\text{tr}[f(M_{\varphi}, T_{\psi} \oplus \eta), M_{\varphi}]| \text{ and } |\text{tr}[h(M, T_{\psi} \oplus \eta), T_{\psi} \oplus \eta]|$$

are domenated by the L^{∞} norm of f and h on the joint essential spectrum respectively. But

$$\text{tr}[f(M_{\varphi}, T_{\psi} \oplus \eta), X]$$

$$- \frac{i}{2\pi} \int_{\sigma(X)} \text{tr}[S_{+}(X, f(M_{\varphi}, T_{\psi} \oplus \eta))(x) - S_{-}(X, f(M_{\varphi}, T_{\psi} \oplus \eta))(x)]dx.$$

If $\tilde{f}$ agrees with f on the joint essential spectrum, then

$$S_{\pm}(X, f(M_{\varphi}, T_{\psi} \oplus \eta)) = S_{\pm}(X, \tilde{f}(M_{\varphi}, T_{\psi} \oplus \eta)).$$

Recall that $S_{\pm}(X, \cdot)$ is a C^*-algebra homomorphism. By the integrability of the spectral multiplicity function of X, we can conclude that

$$|\mathrm{tr}[f(M_\varphi, T_\psi \oplus \eta), X]| \le C \sup\{|f(\tau,t)| : (\tau,t) \in \partial\Sigma\}.$$

Applying the same argument to Y yields that

$$|\mathrm{tr}[f(M_\varphi, T_\psi \oplus \eta), M_\varphi] \le C \sup\{|f(\tau,t)| : (\tau,t) \in \partial\Sigma\}.$$

See Carey-Pincus [3] where such estimates were first introduced.

By the decomposition theorem of Toeplitz operators, we can reduce the estimate of $|\mathrm{tr}[h(M_\varphi, T_\psi \oplus \eta), T_\psi \oplus \eta]|$ to case $\partial D = S^1$. It is easy to see that

$$\begin{aligned} \mathrm{tr}[h(M_\varphi, T_\psi \oplus \eta), T_\psi \oplus \eta] &= \mathrm{tr}[h(\varphi(M), T_\psi), T_\psi] \\ &= \mathrm{tr}[h(\varphi(M), \sigma(V)), \sigma(V)] \\ &= \mathrm{tr}[h(\varphi(M), \sigma(V \oplus 1)), \sigma(V \oplus 1)] \end{aligned}$$

where $V = (T_\psi + i)(T_\psi - i)^{-1}$. The function σ is defined in the following way. Pick $e^{i\theta_0} \notin \sigma(V)$ and $e^{i\theta_0} \ne 1$. Define $\sigma(e^{it}) = i(e^{it}+1)(e^{it}-1)$ for $e^{it} \in \sigma(V)$. Since $1 \notin \sigma(V)$, we can extend σ to $S^1 \setminus$ a neighborhood of $e^{i\theta_0}$ so that σ is smooth at 1. Theorem 1.7 of [3] shows that

$$|\mathrm{tr}[h(\varphi(M), \sigma(V \oplus 1)), \sigma(V \oplus 1)]|$$

$$\le \sup\{|h(\varphi(\tau), \sigma(\xi))| : (\tau,\xi) \text{ in the joint essential spectrum of } \{M, V\}\}.$$

Simple coordinate change yields

$$|tr[h(M_\varphi, T_\psi \oplus \eta); T_\psi \oplus \eta]| \leq C \sup\{|h(\tau,t)| : (\tau,t) \in \partial\Sigma\}.$$

This prove the proposition.

By Theorem 4.2.28 of [13], [G] is also a rectifiable current. [12] asserts that the measure $\|\ell\|$ is a 1-dimensional Hausdorff measure on $\partial\Sigma$. Let H^1 be the 1-dimensional Hausdorff measure.

<u>Theorem II.3.12</u>. Let the spectral multiplicity function for X and Y be integrable. Then there is a H^1 measurable and $(H^1,1)$ rectifiable subset $\tilde{\Omega} \subset \partial\Sigma$ and an H^1 integrable vector field τ on $\tilde{\Omega}$ and a positive integer valued function $\Theta(\partial[G],x)$ such that for any $F, H \in \hat{M}(\mathbb{R}^3)$,

$$tr[F(M_\varphi, T_\psi \oplus \eta), H(M_\varphi, T_\psi \oplus \eta)]$$

$$= \int_{\tilde{\Omega}} \langle \tau(x), H(x)dF(x)\rangle \Theta(\partial[G],x)\,dH^1(x).$$

and

$$tr[F(T_\varphi T_\psi), H(T_\varphi, T_\psi)]$$

$$= \int_{\tilde{\Omega}\setminus\mathbb{C}\times\{\eta\}} \langle \tau(x), H(x)dF(x)\rangle \Theta(\partial[G],x)\,dH^1_x.$$

Particularly, if $z \notin F(\tilde{\Omega})$ $(z \notin F(\tilde{\Omega}\setminus\mathbb{C}\times\{\eta\}))$, then

$$\text{index}[F(M_\varphi, T_\psi \oplus \eta) - z] = \int_{\tilde{\Omega}} \langle \tau(x), \frac{dF(x)}{F(x)-z}\rangle \Theta(\partial[G],x)\,dH^1_x$$

(and respectively)

$$\operatorname{index}[F(T_{\varphi}, T_{\psi}) - z] \int_{\tilde{\Omega}\setminus\mathbb{C}\times\{\eta\}} \left\langle \tau(x), \frac{dF(x)}{F(x)-z} \right\rangle \Theta(\partial[G], x)\, dH^{1}_{x} .$$

The proof of this theorem is essentially that of Theorem 1.7 of [3] plus Proposition II.3.11.

Chapter III. Self-adjoint and Symmetric Toeplitz Operators

II.1. Existence of weak limits and corresponding symmetric operators

Let $\psi \in L^2(\partial D, \omega)$ be a real function and let

$$\psi_k(\tau) \begin{cases} \psi(\tau) & : \ |(\tau)| \leq k \\ \\ 0 & : \ |\psi(\tau)| > k . \end{cases}$$

In this section we derive a symmetric operator in terms of the "weak limit" of $\{T_{\psi_k}\}$ as described in Section II.3.

First we need the following lemma:

Lemma III.1.1. Let $\{A_k\}$ be a sequence of bounded self-adjoint operators on a Hilbert space H and let $\{B_k\}$ be a sequence of finite rank operators on H such that

$$R(\lambda) = \underset{k\to\infty}{\text{w-lim}}(A_k-\lambda)^{-1}$$

exists for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and

$$-i\underset{\mu\to\infty}{\text{w-lim}}\ \mu R(i\mu) = 1 \qquad (\mu \in \mathbb{R});$$

and such that rank $B_k \leq M$ and $\underset{k\to\infty}{\text{s-lim}}\ B_k = B$.

Then there exists a discrete set $S \subset \mathbb{C}\backslash\mathbb{R}$ such that

$$\sup\{|\mathrm{IM}\,\lambda| : \lambda \in S\} < \infty$$

and for each $\lambda \in (\mathbb{C}\setminus\mathbb{R})\setminus S$,

$$\mathcal{R}(\lambda) = \underset{k\to\infty}{\text{w-lim}}(A_k+B_k-\lambda)^{-1}$$

exists and

$$-i\underset{\mu\to\infty}{\text{w-lim}}\ \mu\mathcal{R}(i\mu) = 1.$$

Proof. We prove the lemma by using induction on the rank of B_k. Suppose the lemma is true for m-rank operators $\{B_k\}$. Let $\{u_k\}$, $\{v_k\} \subset H$ such that

$$\lim_{k\to\infty}\|u_k-u\| = 0, \quad \lim_{k\to\infty}\|v_k-v\| = 0.$$

Then it is easy to see that

$$(A_k+B_k+u_k\otimes v_k-\lambda)^{-1} = [1+(A_k+B_k-\lambda)^{-1}u_k\otimes v_k](A_k+B_k-\lambda)^{-1}.$$

For λ large enough, we have

$$(A_k+B_k+u_k\otimes v_k-\lambda)^{-1} = \sum_{m=0}^{\infty}(-1)^m[(A_k+B_k-\lambda)^{-1}u_k\otimes v_k]^m(A_k+B_k-\lambda)^{-1}$$

$$= (A_k+B_k-\lambda)^{-1} + \sum_{m=1}^{\infty}(-1)^m[((A_k+B_1-\lambda)v_k, u_k)]^{m-1}$$

$$\times\ (A_k+B_k-\lambda)^{-1}u_k\otimes v_k(A_k+B_k-\lambda)^{-1} = (A_k+B_k-\lambda)^{-1}$$

$$-\frac{1}{1+((A_k+B_k-\lambda)^{-1}v_k, u_k)}(A_k+B_k-\lambda)^{-1}u_k\otimes v_k$$

$$(A_k+B_k-\lambda)^{-1}.$$

Hence for $x, y \in H$,

$$((A_k+B_k+u_k \otimes v_k-\lambda)^{-1}x,y)$$

$$= ((A_k+B_k-\lambda)^{-1}x,y)$$

$$- \frac{1}{1+((A_k+B_k-\lambda)^{-1}v_k,u_k)}((A_k+B_k-\lambda)^{-1}x,u_k)((A_k+B_k-\lambda)^{-1}v_k,$$

$$y) = g_k(\lambda).$$

Since $\operatorname{s-lim}_{k\to\infty} u_k = u, \operatorname{s-lim}_{k\to\infty} v_k = v$, $(A_k+B_k-\lambda)^{-1}$ is uniformly bounded for $|\mathrm{Im}\,\lambda|$ large enough, it is easy to see that

$$\lim_{k\to\infty}((A_k+B_k+u_k\otimes v_k-\lambda)^{-1}x,y)$$

exists and is equal to

$$\lim_{k\to\infty}((A_k+B_k-\lambda)^{-1}x,y) - \frac{1}{1+f(\lambda)} \lim_{k\to\infty}(A_k+B_k-\lambda)^{-1}x,u)$$

$$\times \lim_{k\to\infty}((A_k+B_k-\lambda)^{-1}v,y) = g(\lambda)$$

where

$$f(\lambda) = \lim_{k\to\infty}((A_k+B_k-\lambda)^{-1}v,u).$$

Since the lemma is true for $\{B_k\}$, we have

$$\lim_{|\lambda|\to\infty} f(\lambda) = 0.$$

Hence $s_1 = \{\lambda \mid f(\lambda) = -1\}$ is at most a discrete subset of $\mathbb{C}\backslash\mathbb{R}$.

By the induction hypothesis $\lim_{k\to\infty}((A_k+B_k-\lambda)^{-1}x,y)$ exists and is analytic on $(\mathbb{C}\backslash\mathbb{R})\backslash s_0$ where s_0 is some discrete set.

Therefore, $\lim_{k\to\infty} g_k(\lambda)$

exists for $\lambda \in (\mathbb{C}\backslash\mathbb{R})\backslash(s_0 \cup s_1)$ and obviously the limit, the function $g(\lambda)$, is holomorphic on $(\mathbb{C}\backslash\mathbb{R})\backslash(s_0 \cup s_1)$.

Let

$$\mathcal{R}(\lambda) = w - \lim_{k\to\infty}(A_k+B_k-\lambda)^{-1},$$

and

$$\mathcal{R}_1(\lambda) = w - \lim_{k\to\infty}(A_k+B_k+u_k\otimes v_k-\lambda)^{-1}.$$

Then it is easy to see that

$$-i\mu(\mathcal{R}_1(i\mu)x,y) = -i\mu(\mathcal{R}(i\mu)x,y)$$

$$+ \frac{i\mu}{1+(\mathcal{R}(i\mu)v,u)}(\mathcal{R}(i\mu)x,u)(\mathcal{R}(i\mu)v,y)$$

$$\to (x,y) \qquad \text{as } \mu \to \infty .$$

The lemma is proved by induction

Recall that $\pi : \Delta \to D$ is the universal covering map which induces a unitary equivalence between T_φ on $H^2(D)$

and $T^G_{\varphi\circ\pi}$ on $H^2 = H^2 \cap L^2$. Thus to prove the existence of weak limit

$$w - \lim_{k\to\infty}(T_{\psi_k} - \lambda)^{-1} = R(\lambda)$$

for $\lambda \notin \mathbb{C}\backslash\mathbb{R}$, it suffices to show that the weak limit

$$w - \lim_{k\to\infty}(T^G_{\psi_k\circ\pi} - \lambda)^{-1}$$

exists. But $T^G_{\psi_k\circ\pi} = T_{\psi_k\circ\pi} - PP_1M_{\psi_k\circ\pi}$ where P and P_1 are the projection onto H^2 and N_G respectively. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of N_G consisting of bounded functions. Then $PP_1M_{\psi_k\circ\pi} = \sum_{j=1}^{n} (\psi_k\circ\pi e_j) \otimes (Pe_j)$. It is clear that

$$s - \lim_{k\to\infty} PP_1M_{\psi_k\circ\pi} = \sum_{j=1}^{n} (\psi\circ\pi e_j) \otimes (Pe_j).$$

As we mentioned in Section I.3,

$$w - \lim_{k\to\infty}(T_{\psi_k\circ\pi} - \lambda)^{-1}$$

exists for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and on H^2. Take $\{T_{\psi_k\circ\pi}\}$ to be the sequence $\{A_k\}$, then there exists a discrete set S such that

$$w - \lim_{k\to\infty}(T_{\psi_k\circ\pi} - PP_1M_{\psi_k\circ\pi} - \lambda)^{-1}$$

exists for $\lambda \in (\mathbb{C}\backslash\mathbb{R})\backslash S$. Note the above limit is on the whole H^2. Particularly,

$$w - \lim_{k\to\infty}(T^G_{\psi_k \circ \pi} - \lambda)^{-1} = R^G(\lambda)$$

exists for $\lambda \in (\mathbb{C}\backslash\mathbb{R})\backslash S$. Hence in the $H^2(D)$ space

$$R(\lambda) = w - \lim_{k\to\infty}(T_{\psi_k} - \lambda)^{-1}$$

exists for $\lambda \in (\mathbb{C}\backslash\mathbb{R})\backslash S$. Since

$$-i\ w - \lim_{\mu\to\infty} \mu R_G(\mu i) = 1$$

on H^2_G, it is also true that

$$-i\ w - \lim_{\mu\to\infty} \mu R(i\mu) = 1$$

on $H^2(D)$. But T_{ψ_k} is a self-adjoint operator, we have the estimate

$$\|(T_{\psi_k} - \lambda)^{-1}\| \le 1/|\text{Im}\ \lambda|.$$

Thus Cauchy integral formula for analytic functions enables us to conclude that the weak limit $R(\lambda)$ exists for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Therefore, the conditions (i) and (ii) of Section I.3 are satisfied. To complete the construction of the symmetric operator, we need to show that $R(\lambda)\ker N_\lambda$ is dense in $H^2(D)$.

Lemma II.1.2. Let

$$\mathcal{D}_0 = \{u : u \in H^2(D),\ \psi u \in H^2(D)\}.$$

Then $\qquad \mathcal{D}_0 \subset R(\lambda)\ker N_\lambda.$

Proof. Let $u \in \mathcal{D}_0$, then $P(\psi-\lambda)u \in H^2(D)$.

$(T_{\psi_k}-\lambda)^{-1}P(\psi-\lambda)u = u + (T_{\psi_k}-\lambda)^{-1}P(\psi-\psi_k)u.$ Since

$\psi u \in H^2(D)$, $\lim\limits_{k\to\infty}\|(\psi-\psi_k)u\| = 0$. Hence

$$\lim_{k\to\infty}\|(T_{\psi_k}-\lambda)^{-1}P(\psi-\lambda)u - u\| \le \frac{1}{|\mathrm{Im}\ \lambda|}\lim_{k\to\infty}\|(\psi-\psi_k)u\| = 0.$$

Thus $P(\psi-\lambda)u \in \ker N_\lambda$ (see Lemma I.3.1) and $R(\lambda)P(\psi-\lambda)u$

$= \lim\limits_{k\to\infty}(T_{\psi_k}-\lambda)P(\psi-\lambda)u = u$. Hence $\mathcal{D}_0 \subset R(\lambda)\ker N_\lambda$.

Thus (iii) of Section I.3 is also satisfied. We denote the symmetric operator so derived by T_ψ and its domain by $\mathcal{D}$. The next lemma tries to justify the notation T_ψ and the term "limit".

Lemma III.1.3. For any $u \in \mathcal{D}_0$,

$$T_\psi u = P\psi u.$$

Proof. Since $u = R(\lambda)P(\psi-\lambda)u$, by the definition

$$T_\psi u = (1+\lambda R(\lambda))P(\psi-\lambda)u = P\psi u$$

(see the proof of Lemma I.3.2).

Let $\alpha \in D$, then the correspondence

$$f \mapsto f(\alpha), \qquad f \in H^2(D)$$

is a bounded linear function on $H^2(D)$. Therefore, there exists a unique $k_\alpha \in H^2(D)$ such that

$$f(\alpha) = (f, k_\alpha).$$

<u>Lemma III.1.4.</u> $k_\alpha \in H^2(D) \cap C^\infty(\partial D)$.

<u>Proof.</u> For any $f \in H^2(D)$,

$$f(\alpha) = \frac{1}{2\pi i}\int_{\partial D} f(\tau)\,\frac{1}{\tau-\alpha}\,d\tau = \int_{\partial D} f(\tau)\,[\overline{\frac{\overline{h(\tau)}}{\tau-\alpha}}]\,d\omega(\tau) \quad \text{where}$$

$h \in C^\infty(\partial D)$. Hence

$$k_\alpha = P[\overline{\frac{h(\cdot)}{\cdot-\alpha}}] .$$

The lemma follows immediately from Corollary II.1.5.

The next lemma shows that how T_ψ acts on $u \in \mathcal{D}$.

<u>Lemma III.1.5.</u> For any $u \in \mathcal{D}$, the function

$$\frac{1}{2\pi i}\int_{\partial D} \frac{\psi(\tau)u(\tau)}{\tau-z}\,d\tau, \qquad z \in D$$

is in $H^2(D)$ and

$$(T_\psi u)(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\psi(\tau)u(\tau)}{\tau-z}\,d\tau + [(P-\mathcal{P})M_\psi u](z)$$

<u>Remark</u>. First, ψ, $u \in L^2(\partial D, \omega)$, therefore $\psi u \in L^1(\partial D, \omega)$ and the Cauchy integral makes sense. Second, by Theorem II.1.1 $(P-\mathcal{P})M_\psi$ is indeed a trace class operator on $H^2(D)$.

<u>Proof</u>. Let $\alpha \in D$, then

$$(T_\psi u, k_\alpha) = ((T_\psi - T_{\psi_k})u, k_\alpha) + (T_{\psi_k} u, k_\alpha)$$

$$= (u, P(\psi - \psi_k)k_\alpha) + (T_{\psi_k} u, k_\alpha).$$

Since $\lim\limits_{k\to\infty} \|P(\psi-\psi_k)k_\alpha\| = 0$,

$$(T_\psi u)(\alpha) = \lim_{k\to\infty} (T_{\psi_k} u)(\alpha).$$

But $T_{\psi_k} u = \mathcal{P}M_{\psi_k} u + (P-\mathcal{P})M_{\psi_k} u$ and

$$\lim_{k\to\infty} \|(P-\mathcal{P})M_{\psi_k} u - (P-\mathcal{P})M_\psi u\| = 0.$$

Hence

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\psi(\tau)u(\tau)}{t-\tau}\, d\tau = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{\partial D} \frac{\psi_k(\tau)u(\tau)}{\tau-\alpha}\, d\tau$$

$$= (T_\psi u)(\alpha) - [(P-\mathcal{P})M_\psi u](\alpha).$$

It is very much desired to establish a core for T_ψ. Unfortunately, this is not easy in general. What we know is that certain functions must be contained in $\mathcal{D}$ (Lemma II.1.3) and the function that $\mathcal{D}$ contains must satisfy certain condition. But when the symbol ψ is semibounded, we can actually obtain a core. Without loss of generality we assume that $\psi \geq 0$. First we prove that T_ψ is indeed in a self-adjoint operator. Note that $\psi_{k+1} \geq \psi_k$, so $T_{\psi_{k+1}} \geq T_{\psi_k}$ and

$$(T_{\psi_k}+1)^{\frac{1}{2}} [1 + (T_{\psi_k}+1)^{-\frac{1}{2}}(T_{\psi_{k+1}-\psi_k})(T_{\psi_k}+1)^{-\frac{1}{2}}](T_{\psi_k}+1)^{\frac{1}{2}} .$$

Therefore,

$$(T_{\psi_{k+1}}+1)^{-1} \leq (T_{\psi_k}+1)^{-1}.$$

Let $A_k = (T_{\psi_k}+1)^{-1}$, then for $\ell \geq k$,

$$\|(A_\ell-A_k)x\|^2 \leq \|\sqrt{A_\ell-A_k}\|^2 \ \|\sqrt{A_\ell-A_k}x\|^2$$

$$= \|\sqrt{A_\ell-A_k}\|^2((A_\ell-A_k)x,x).$$

Hence strong limit

$$\underset{k\to\infty}{\text{s-lim}}(T_{\psi_k}+1)^{-1}x = R(-1)x$$

exists for any $x \in H^2(D)$. Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ be very close to -1. Then $(T_{\psi_k}-\lambda)^{-1} = (1-(1+\lambda)(T_{\psi_k}+1)^{-1})^{-1}(T_{\psi_k}+1)^{-1}$. Thus strong limit

$$\operatorname*{s-lim}_{k\to\infty}(T_{\psi_k}-\lambda)^{-1}x = R(\lambda)x$$

exists for any $x \in H^2(D)$. This implies that $\ker N_\lambda = H^2(D)$ and therefore T_ψ is self-adjoint.

Now consider $\psi \in L^2(S^1)$ and $\psi \geq 0$. We prove that H^∞ is a core for T_ψ. Let $x \in \mathcal{D}$, then $x = R(-1)u$ for some $u \in H^2$. For any positive integer p, there exist $C_j^p \in \mathbb{C}$ and $\alpha_j^p \in \Delta$, $j = 1,\ldots,N_p$ such that $\|u-u_p\| \leq 1/p$ where

$$u_p = \sum_{j=1}^{N_p} C_j^p \, k_{\alpha_j^p} \in H^\infty.$$

Let $x_p = R(-1)u_p$, then

$$x_p = \sum_{j=1}^{N_p} C_j^p \, \overline{S}(-1,\alpha_j^p)S(-1,\cdot)k_{\alpha_j^p} \, .$$

Since $\|S(-1,\cdot)\|_\infty \leq 1$, x_p is in H^∞. Moreover,

$$\|x-x_p\| = \|R(-1)(u-u_p)\| \leq 1/p \, .$$

But $\qquad T_\psi x_p = (T_\psi+1)x_p - x_p = u_p - x_p$. Hence

$$\lim_{p\to\infty}\|T_\psi x_p - (u-x)\| = 0.$$

T_ψ is a closed operator, so $T_\psi x = u - x$. This proves that

H^∞ is a core for T_ψ.

Lemma II.1.6. The orthogonal projection

$$P^G : H^2 \to H^2_G$$

maps H^∞ into H^∞_G.

Proof. First we note that $L^2(S^1)$ has the following decomposition

$$L^2 = H^2_G \oplus N_G \oplus \overline{H^2_{G,0}} \oplus [L^2_G]^\perp.$$

Let $E : L^2 \to L^2_G$ be the condition expectation projection, then

$$EH^\infty = H^\infty_G + N_G \qquad \text{(Lemma 4 of [11])}.$$

Obviously for $u \in H^\infty$, $P^G u = PEu$. This prove the lemma.

Now consider $\psi \in L^2(\partial D, \omega)$ and $\psi \geq 0$. Under the unitary transform from $H^2(D)$ to H^2_G induced by the covering map π, T_ψ is equivalent to $T^G_{\psi\circ\pi}$ on H^2_G. Particularly for $w \in H^\infty_G$, $T^G_{\psi\circ\pi} w = P_G M_{\psi\circ\pi} w$ where $P_G : L^2_G \to H^2_G$ is the projection. It is also true that

$$T^G_{\psi\circ\pi} w = P(1-P_1)M_{\psi\circ\pi} w = PM_{\psi\circ\pi} w - PP_1 M_{\psi\circ\pi} w$$

$$= (T_{\psi\circ\pi}|H^2_G)w - PP_1 M_{\psi\circ\pi} w$$

where $P_1 : L^2_G \to N_G$ is the projection.

Let x be in the domain of $T^G_{\psi\circ\pi}$, then $x = (T^G_{\psi\circ\pi}-\lambda)^{-1}u$ for some $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $u \in H^2_G$. Note that

$$(T^G_{\psi\circ\pi}-\lambda)^{-1}u = (T_{\psi\circ\pi}-PP_1M_{\psi\circ\pi}-\lambda)^{-1}u$$

$$= (T_{\psi\circ\pi}-\lambda)^{-1}(1-PP_1M_{\psi\circ\pi}(T_{\psi\circ\pi}-\lambda)^{-1})^{-1}u.$$

Therefore, x is in the domain of $T_{\psi\circ\pi}$. Since H^∞ is a core for $T_{\psi\circ\pi}$, we can choose a sequence $\{x_p\} \subset H^\infty$ such that $\lim_{p\to\infty}\|x_p-x\| = 0$ and $\lim_{p\to\infty}\|T_{\psi\circ\pi}x_p - T_{\psi\circ\pi}x\| = 0$. Because x is an element in H^2_G, $\|P^Gx_p-x\| = \|P^G(x_p-x)\| \leq \|x_p-x\|$ and therefore,

$\lim_{p\to\infty}\|P^Gx_p-x\| = 0$. Let $v \in H^\infty_G$, then v is in the domain of $T^G_{\psi\circ\pi}$ and

$$(T^G_{\psi\circ\pi}P^Gx_p, v) = (P^Gx_p, T^G_{\psi\circ\pi}v)$$

$$= (x_p, (T_{\psi\circ\pi}-PP_1M_{\psi\circ\pi})v)$$

$$= (P^G(T_{\psi\circ\pi}-PM_{\psi\circ\pi}P_1)x_p, v).$$

This implies that $T^G_{\psi\circ\pi}P^Gx_p = P^G(T_{\psi\circ\pi}-PM_{\psi\circ\pi}P_1)x_p$. It is clear now that

$$\lim_{p\to\infty}\|T^G_{\psi\circ\pi}P^Gx_p - P^G(T_{\psi\circ\pi}x-PM_{\psi\circ\pi}P_1x)\| = 0$$

Since $P^G x_p \in H_G^\infty$, we can conclude that H_G^∞ is a core for $T^G_{\psi\circ\pi}$.

In fact we have proved:

Lemma III.1.7. Let $\psi \in L^2(\partial D, \omega)$ and $\psi \geq 0$, then T_ψ is a self-adjoint operator which has $H^\infty(D)$ as a core.

III.2. Self-adjoint extensions

For real $\psi \in L^2(\partial D, \omega)$, the symmetric operator T_ψ on $H^2(D)$ is unitary equivalent to $T^G_{\psi\circ\pi}$ on H^2_G. On the other-hand, we can define symmetric operator $T_{\psi\circ\pi}$ on H^2. It has been made clear that the domain of $T^G_{\psi\circ\pi}$ is contained in that of $T_{\psi\circ\pi}$ and that on the domain of $T^G_{\psi\circ\pi}$, $T^G_{\psi\circ\pi}$ differs from $T_{\psi\circ\pi}$ only by a finite rank operator $PP_1M_{\psi\circ\pi}$. $T_{\psi\circ\pi}$ has the Naimark dilation T on $\hat{H} \supset H^2$. Let $K = PP_1M_{\psi\circ\pi}$. Except for a discrete set $S \subset \mathbb{C}$, $(T-K-\lambda)$ is invertable for every complex number λ.

Theorem III.2.1. Let

$$\tilde{H} = V\{(T-K-\lambda_1)^{-n_1}\cdots(T-K-\lambda_k)^{-n_k}H^2_G : \lambda_j \in (\mathbb{C}\setminus\mathbb{R})\setminus S,$$

$$n_i, k \text{ are integers} \geq 0 \text{ , } 1 \leq i, j \leq k\},$$

then $\tilde{H} \supset H^2_G$, $\tilde{H}$ is invariant for $(T-K-\lambda)^{-1}$ and

$$(T-K-\lambda)^{-1} = \int_{\mathbb{R}} \frac{dE_t}{t-\lambda}$$

where $\{E_t\}$ is an orthogonal spectral resolution. Furthermore,

$$R^G(\lambda) = \underset{k\to\infty}{w\text{-}\lim}(T^G_{\psi\circ\pi_k} - \lambda)^{-1} = \tilde{P}(T-K-\lambda)^{-1}|H^2_G$$

where $\tilde{P}$ is the orthogonal projection from $\tilde{H}$ onto H_G^2. In otherwords,

$$\int_{\mathbb{R}} t dE_t = (T-K) \mid (T-K-\lambda)^{-1}\tilde{H}$$

is the Naimark extension of $T_{\psi\circ\pi}^G$.

<u>Proof</u>. Denote by T^G the Naimark extension of $T_{\psi\circ\pi}^G$ on H_1.

Let

$$H_0 = \{ \sum_{j=1}^{N} (t-K-\lambda_1^j)^{-1} \ldots (T-K-\lambda_{k_j}^j)^{-1} u_j = u_j \in H_G^2,$$

$$\lambda_i^j \neq \lambda_{i'}^{j'} \text{ if } (i,j) \neq (i',j') \text{ and } \{\lambda_i^j : i,j\} \cap \{\overline{\lambda_i^j} : i,j\} = \emptyset\}.$$

Then $\overline{H_0} = \tilde{H}$.

For any $x = \sum_{j=1}^{N} (T-K-\lambda_i^j)^{-1} \ldots (T-K-\lambda_{k_j}^j)^{-1} u_j$, by partial fraction we have

$$((T-K-\lambda)^{-1}x, x)$$

$$= \sum_{p,j=1}^{N} ((T-K-\lambda)^{-1}(T-K-\lambda_1^j)^{-1} \ldots (T-K-\lambda_{k_j}^j)^{-1} u_j,$$

$$(T-K-\lambda_1^p)^{-1} \ldots (T-K-\lambda_{k_p}^p)^{-1} u_p)$$

$$= \sum_{p,j=1}^{N} \sum_{s=0}^{k_j} \sum_{t=1}^{k_p} a_{s,t}^{p,j} ((T-K-\lambda_s^j)^{-1} u_j, (T-K-\lambda_t^p)^{-1} u_p)$$

$$= \sum_{p,j=1}^{N} \sum_{s=0}^{k_j} \sum_{t=1}^{k_p} a_{a,t}^{p,j}((T-\lambda_s^j)^{-1}(1-KB(\lambda_s^j)u_j,$$

$$(T-\lambda_t^p)^{-1}(1-KB(\lambda_t^p))u_p).$$

Since $\lambda_s^j \neq \bar{\lambda}_t^p$, and range $K \subset H^2$ (again by partial fractions) we see that the above equals

$$\sum_{p,j=1}^{N} \sum_{s=0}^{k_j} \sum_{t=1}^{k_p} a_{s,t}^{j,p}(R(\lambda_s^j)(1-KB(\lambda_s^j)u_j, R(\lambda_t^p)(1-KB(\lambda_t^p))u_p)$$

$$= \sum_{p,j=1}^{N} \sum_{s=0}^{k_j} \sum_{t=1}^{k_j} a_{s,t}^{j,p}(R^G(\lambda_s^j)u_j,\ R^G(\lambda_t^p)u_p)$$

$$= \sum_{j,p=1}^{N} \sum_{s=0}^{k_j} \sum_{t=1}^{k_p} a_{s,t}^{j,p}((T^G-\lambda_s^j)^{-1}u_j,\ (T^G-\lambda_t^p)^{-1}u_p)_{H^G}$$

$$=\ldots=((T^G-\lambda)^{-1}[\sum_{j=1}^{N}(T^G-\lambda_1^j)^{-1}\ldots(T^G-\lambda_{k_j}^j)^{-1}u_j],$$

$$[\sum_{j=1}^{N}(T^G-\lambda_1^j)\ldots(T^G-\lambda_{k_j}^j)^{-1}u_j]).$$

Hence in particular we have

$$\left\| \sum_{j=1}^{N} (T-K-\lambda_j^j)^{-1}\ldots(T-K-\lambda_{k_j}^j)^{-1}u_j \right\|$$

$$= \left\| \sum_{j=1}^{N} (T^G-\lambda_1^j)^{-1}\ldots(T^G-\lambda_{k_j}^j)^{-1}u_j \right\|.$$

So if we let

$$V \sum_{j=1}^{N} (T-K-\lambda^{j})^{-1} \ldots (T-K-\lambda^{j}_{k_j})^{-1} u_j$$

$$= \sum_{j=1}^{N} (T^G-\lambda^{j}_{1}) \ldots (T^G-\lambda^{j}_{k_j})^{-1} u_j$$

then V extends to a unitary operator from $\tilde{H}$ onto H_1 and

$$((T^G-\lambda)^{-1}Vx, Vy) = ((T-K-\lambda)^{-1}x, y).$$

Hence $(T-K-\lambda)^{-1}|H_1 = V^*(T^G-\lambda)^{-1}V$. Therefore,

$$(T-K-\lambda)^{-1}|H_1$$

is the resolvent of a self-adjoint operator. By the inversion formula, that self-adjoint operator must be

$$(T-K)\,|\,(T-K-\lambda)^{-1}H_1.$$

It is already known that

$$((T-K-\lambda)^{-1}u, v) = (R^G(\lambda), u, v)$$

for $u, v \in H^2_G$. Hence we have proved the theorem.

III.3. <u>The decomposition theorem and spectral multiplicity.</u>

Recall that $L^2(\partial D, \omega)$ has the decomposition

$$L^2(\partial D, \omega) = (K_0 \oplus L_0) \oplus (K_1 \oplus L_1) \oplus \ldots \oplus (K_n \oplus L_n)$$

and that there exist unitary operators $U_0, U_1, \ldots, U_n$ such that

$$U_i(K_i \oplus L_i) = L^2(S^1)$$

and

$$U_i K_i = H^2.$$

Let $\varphi \in L^\infty(\partial D, \omega)$ and $\varphi_j = \varphi|\Gamma_j$, $j = 0,1,\ldots,n$. Then

$$U_j T^j_{\varphi_j} U^*_j = T_{\varphi\circ\pi_j}, \quad j = 0,1,\ldots,n$$

and

$$U[T^0_{\varphi_0} \oplus \ldots \oplus T^n_{\varphi_n}]U^* = T_{\varphi\circ\pi_0} \oplus \ldots \oplus T_{\varphi\circ\pi_n},$$

see Section I.2.

Naturally, $T^j_{\varphi_j}$, $j = 0,1,\ldots,n$ are considered as Toeplitz operators on H^2 of the unit disc. A brief review of the theory of Toeplitz operators on the unit disc tells us that the development of much of the study takes the advantage that there are so many powerful analytic tools available that difficulties in operator theory can be overcomed by concrete computation. For example, for a self-adjoint Toeplitz operator, the von Neumann spectral multiplicity function can be calculated from the resolvent which, on a basis $\{k_\alpha : \alpha \in \Delta\}$, can be explicitly expressed in terms of function $S(\alpha,\cdot)$, see [25]. But unfortunately, on multiply-connected domains, the non-simply-connectedness makes it virtually impossible to carry out any direct calculation. In fact, analytic expression is not available even for a function as simple and fundamental as k_α. Conceivably, any quantitative analysis for Toeplitz operator must require well understanding about the orthogonal projection P, which, except being a trace class perturbation of ρ, still remains mysterious to us. Therefore, decomposing Toeplitz operator into direct sum of those on the unit disc seems to be the only alternative to crack the underlying nut. Essentially, Abrahamse's theorem says that $T_\varphi \oplus 0$ is a compact perturbation of $(T^0_{\varphi_0} \oplus 0)\oplus\ldots\oplus(T^n_{\varphi_n} \oplus 0)$. But for our purpose compact perturbation is not enough. What we need is the following:

<u>Theorem III.3.1.</u> For any $\varphi \in L^\infty(\partial D,\omega)$,

$$T_\varphi \oplus 0 - [(T^0_{\varphi_0} \oplus 0)\oplus\ldots\oplus(T^n_{\varphi_n} \oplus 0)] = K_\varphi$$

is a trace class operator and

$$\|K_\varphi\|_1 \leq L\|\varphi\|_{L^2(\partial D,\omega)}$$

where L is a constant which depends only on the domain D.

<u>Proof</u>. In this proof, the numbers in parentheses are that of the Lemma/Theorem we refer to.

Let χ_i be the characteristic function of Γ_i. Regarded as an operator on $L^2(L^2(\partial D,\omega)$

$$T_{\chi_i} T_{\varphi\chi_j} \oplus 0 = PM_{\chi_i} PM_{\varphi\chi_j} P$$

$$= PM_{\chi_i}(P-\wp)M_{\varphi\chi_j} P + PM_{\chi_i}\wp M_{\varphi\chi_i} P$$

$$= PM_{\chi_i} P(\wp^* - \wp)M_{\varphi\chi_i} P + P[M_{\chi_i}\wp - \wp M_{\chi_i}]M_{\chi_j\varphi} P$$

since $M_{\chi_i} M_{\varphi\chi_j} = 0$.

$M_{\chi_i}\wp - \wp M_{\chi_i}$ is a trace class operator with C^∞ kernel

$$\frac{\chi_i(z)-\chi_i(\tau)}{z-\tau}\,dz.$$

Hence $\exists\ C \geq 0$ such that

$$\|T_{\chi_i} T_{\varphi\chi_j}\|_1 \leq C\|\varphi\|_{L^2}, \quad (I.2.7,\ II.1.1).$$

Taking the adjoint, we see that

$$\|T_{\chi_j \varphi} T_{\chi_i}\|_1 \le C\|\varphi\|_{L^2}.$$

Thus

$$\begin{aligned}T_{\varphi\chi_j} &= T_{\chi_j} T_{\varphi\chi_j} + \sum_{i\neq j} T_{\chi_i} T_{\varphi\chi_j} \\ &= T_{\chi_j} T_{\varphi\chi_j} T_{\chi_j} + \sum_{i\neq j} (T_{\chi_i} T_{\varphi\chi_j} + T_{\chi_j} T_{\varphi\chi_j} T_{\chi_i}) \\ &= T_{\chi_j} T_{\varphi} T_{\chi_j} + \sum_{i\neq j} (T_{\chi_i} T_{\varphi\chi_j} + T_{\chi_j} T_{\varphi\chi_j} T_{\chi_i} - T_{\chi_j} T_{\varphi\chi_i} T_{\chi_j})\end{aligned}$$

Therefore,

$$\|T_{\varphi\chi_j} - T_{\chi_j} T_{\varphi} T_{\chi_j}\|_1 \le 3nC\|\varphi\|_{L^2}$$

and

$$\|T_{\varphi\chi_j} - T_{\chi_j} T_{\varphi\chi_j} T_{\chi_j}\| \le 2nC\|\varphi\|_{L^2}.$$

Let Y_i be the L^2 closure of A_i. We denote the projection onto K_i by P_i and the projection onto Y_i by P_{Y_i} (both considered as operators on $L^2(L^2(\partial D, \omega)$.

$$T_\varphi \oplus 0 = \sum_{j=0}^{n} T_{\chi_j} T_\varphi T_{\chi_j} \oplus 0 + \tilde{K}_\varphi$$

where $\tilde{K}_\varphi \in C_1$ and $\|\tilde{K}_\varphi\|_1 \le C_1\|\varphi\|_{L^2}$.

$$T_{\chi_j} T_\varphi T_{\chi_j} \oplus 0 = PM_{\chi_j} PM_\varphi PM_{\chi_j} P$$

for any $f \in Y_i$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in$ exterior (interior) of Γ_i if $i \neq 0 (i=0)$.

If $j \neq i$, then

$$\chi_j(z) f(z) = \frac{1}{2\pi i} \int_{\Gamma_i} \chi_j(z) \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{\chi_i(\zeta)\chi_j(z)}{\chi - z} f(\zeta) d\zeta,$$

$$PM_{\chi_j} P\, P_{y_i} M_\varphi = PM_{\chi_j} P_{Y_i} M_\varphi = PM_{\chi_j} (P_{Y_i} - \rho_i) M_\varphi + PM_{\chi_j} \rho_i M_\varphi$$

$$= PM_{\chi_j} P_{Y_i} (\rho_i^* - \rho_i) M_\varphi + PM_{\chi_j} \rho_i M_\varphi$$

where

$$(\rho_i f)(z) = \frac{1}{2\pi i} \int_{\Gamma_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and

$$P_{Y_i} = P_{Y_i} \rho^* \text{ and } P_{Y_i} \rho_i = \rho_i .$$

Since $\rho_i = \rho M_{\chi_i}$

$$\rho_i^* - \rho_i = M_{\chi_i} \rho^* - \rho M_{\chi_i} = M_{\chi_i} (\rho^* - \rho) + [M_{\chi_i}, \rho].$$

Also,

$$(M_{\chi_j} \rho_i f)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\chi_j(z)\chi_i(\tau)}{z - \tau} f(\tau) d\tau .$$

Hence for $i \neq j$,

$$\|PM_{\chi_j} P \, P_{Y_i} M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2} \quad , \quad (\text{I.2.7, II.1.1}).$$

On the other hand,

$$PM_{\chi_j} P \, P_{Y_j} M_\varphi = PM_{\chi_j} P_{Y_j} M_\varphi = P(M_{\chi_j} - 1) P_{Y_j} M_\varphi + P_{Y_j} M_\varphi$$

$$= P(M_{\chi_j} - 1)(P_{Y_j} - \rho_j) M_\varphi + P(M_{\chi_j} - 1)\rho_j M_\varphi + P_{Y_j} M_\varphi$$

$$= P(M_{\chi_j} - 1) P_{Y_j} (\rho_j^* - \rho) M_\varphi + P \sum_{k \neq j} (-M_{\chi_k} \rho_j) M + P_{Y_j} M_\varphi .$$

Thus,

$$\text{(b)} \qquad \|PM_{\chi_j} P \, P_{Y_j} M_\varphi - P_{Y_j} M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2} .$$

(a) and (b) together imply that

$$\|PM_{\chi_j} P \, P_{Y_i} M_\varphi - \delta_{ij} P_{Y_j} M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2} .$$

$$M_\varphi P_{Y_i} PM_{\chi_j} = M_\varphi (P_{Y_i} - \rho_i^*) M_{\chi_j} + M_\varphi \rho_i^* M_{\chi_j}$$

$$= M_\varphi (\rho_i - \rho_i^*) P_{Y_i} M_{\chi_j} + M_\varphi \rho_i^* M_{\chi_j} ,$$

$$M_\varphi P_{Y_j} PM_{\chi_j} = M_\varphi P_{Y_j} (M_{\chi_j} - 1) + M_\varphi P_{Y_j}$$

$$= M_\varphi (P_{Y_j} - \rho_j^*)(M_{\chi_j} - 1) + M_\varphi \rho_j^* (M_{\chi_j} - 1) + M_\varphi P_{Y_j} ,$$

so

(c) $\|M_\varphi P_{Y_j} P M_{\chi_i} - M_\varphi P_{Y_j} \delta_{ij}\|_1 \leq \text{const.}\|\varphi\|_{L^2}$, (I.2.7, II.1.1).

Let $i \neq j$. Then

$$M_\varphi P_{Y_i} P_{Y_j} = M_\varphi (P_{Y_i} - \rho_i^*) P_{Y_j} + M_\varphi \rho_i^* P_{Y_j}$$

$$= M_\varphi (\rho_i - \rho_i^*) P_{Y_i} P_{Y_j} + M_\varphi (\rho_i^* - \rho_i) P_{Y_j} + M_\varphi \rho_i P_{Y_j}$$

$$= M_\varphi (\rho_i - \rho_i^*) P_{Y_i} P_{Y_j} + M_\varphi (\rho_i^* - \rho_i) P_{Y_j} + M_\varphi \rho_i (P_{Y_j} - \rho_j^*)$$
$$+ M_\varphi \rho_i \rho_j^*$$

$$= M_\varphi (\rho_i - \rho_i^*) P_{Y_i} P_{Y_j} + M_\varphi (\rho_i^* - \rho_i) P_{Y_j} + M_\varphi \rho_i (\rho_j - \rho_j^*)$$
$$+ M_\varphi \rho_i \rho_j^*$$

But,

$\rho_i \rho_j = 0$ and $\rho_i \rho_j^* = \rho_i M_{\chi_i} \rho_j^*$, and $M_{\chi_i} \rho_j^* = (\rho_j M_{\chi_i})^* = 0$,

$$M_\varphi P_{Y_i} P_{Y_j} = M_\varphi (\rho_i - \rho_i^*) (P_{Y_i} - 1) P_{Y_j}.$$

Therefore,

$M_\varphi P_{Y_i} P_{Y_j} \in C_1$ and (by I.2.7 and II.1.1).

$$\|M_\varphi P_{Y_i} P_{Y_j}\|_1 \le \text{const.}\|\varphi\|_{L^2}.$$

Similarly

$$\|P_{Y_j} P_{Y_i} M_\varphi\|_1 \le \text{const.}\|\varphi\|_{L^2}.$$

But $\qquad M_\varphi(P - \sum_{j=0}^{n} P_{Y_j})P_{Y_i} = -\sum_{j\neq i} M_\varphi P_{Y_j} P_{Y_i}$,

so $\qquad \|M_\varphi(P - \sum_{j=0}^{n} P_{Y_j})P_{Y_i}\|_1 \le \text{const.}\|\varphi\|_{L^2}.$

Let $\tilde{Y}_0 = Y_0, \ldots, \tilde{Y}_m = Y_m \ominus [Y_j \cap (\tilde{Y}_0 + \ldots + \tilde{Y}_{m-1})]$, $\tilde{Y}_m \subset Y_m$. And $H^2, (D) = Y_0 + \ldots + Y_n = \tilde{Y}_0 + \ldots + \tilde{Y}_n$. Let $\rho^0, \ldots, \rho^n$ be the projections onto $\tilde{Y}_0, \ldots, \tilde{Y}_n$ respectively corresponding to the decomposition. Then $P_{Y_j}\rho^j = \rho^j$ and

$$\sum_{j=0}^{n} \rho^j = 1.$$

Hence

$$M_\varphi(P - \sum_{j=0} P_{Y_j}) = M_\varphi(P - \sum_{j=0}^{n} P_{Y_j}) \sum_{j=0}^{n} \rho^i$$

$$= \sum_{j=0}^{n} M_\varphi(P - \sum_{j=0}^{n} P_{Y_j})\rho^i = \sum_{j=0}^{n} M_\varphi(P - \sum_{j=0}^{n} P_{Y_j})P_{Y_i}\rho^i$$

Thus,

$$\|M_\varphi(P - \sum_{j=0}^{n} P_{Y_j})\|_1 \le \text{const.}\|\varphi\|_{L^2},$$

$$\|(P - \sum_{j=0}^{n} P_{Y_j})M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2}, \text{ and}$$

$$T_{\chi_j}T_\varphi T_{\chi_j} \oplus 0 = PM_{\chi_j}PM_\varphi PM_{\chi_j}P$$

$$= PM_{\chi_j}(P - \sum_{k=0}^{n} P_{Y_k})M_\varphi PM_{\chi_j}P + \sum_{k\neq j} PM_{\chi_j}PP_{Y_k}M_\varphi PM_{\chi_j}P$$

$$+ PM_{\chi_j}P_{Y_j}M_\varphi PM_{\chi_j}P = \ldots$$

Hence it is easy to see that

$$\|T_{\chi_j}T_\varphi T_{\chi_j} - P_{Y_j}M_\varphi P_{Y_j}\|_1 \leq \text{const.}\|\varphi\|_{L^2}.$$

$$P_iP_{Y_j}M_\varphi = P_i(P_{Y_j} - M_{\chi_j}P_{Y_j})M_\varphi + P_iM_{\chi_j}P_{Y_j}M_\varphi$$

$$= P_i(P_{Y_j} - M_{\chi_j}P_{Y_j})M_\varphi + \delta_{ij}P_{Y_j}M_\varphi$$

By (c)

$$\|P_i(P_{Y_j} - M_{\chi_j}P_{Y_j})M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2},$$

therefore, $\|P_iP_{Y_j}M_\varphi - \delta_{ij}P_{Y_j}M_\varphi\|_1 \leq \text{const.}\|\varphi\|_{L^2}$.

$$M_\varphi P_{Y_j}P_j = M_\varphi(P_{Y_j} - \rho_j^*)P_j + M_\varphi\rho_j^*P_j$$

$$= M_\varphi(\rho_j - \rho_j^*)P_{Y_j}P_j + M_\varphi(\rho_j^* - \rho_j)P_j + M_\varphi\rho_jP_j$$

$$M_\varphi\rho_jP_j = M_\varphi(\rho_j - P_j)P_j + M_\varphi P_j.$$

For any $f \in \chi_j A_j$

$$(\rho_j - P_j) f(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \sum_{k \neq j} \frac{\chi_k(z) f(\zeta)}{\zeta - z} \, d\zeta.$$

Let

$$(Kg)(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \varphi(z) \frac{\sum_{k \neq j} \chi_k(z)}{\zeta - z} g(\zeta) d\zeta.$$

Then $K \in C_1$ and $\|K\|_1 \leq \text{const.}\|\varphi\|_{L^2}$

$$M_\varphi(\rho_j - P_j) P_j = K P_j,$$

so $M_\varphi(\rho_j - P_j) P_j \in C_1$ and $\|M_\varphi(\rho_j - P_j) P_j\|_1 \leq \text{const.}\|\varphi\|_{L^2}$

$$\|M_\varphi P_{Y_j} - M_\varphi P_j\|_1 \leq \text{const.}\|\varphi\|_{L^2}.$$

Hence

$$\|M_\varphi P_i - M_\varphi P_{Y_i}\| \leq \text{const.}\|\varphi\|_{L^2}.$$

Now we can conclude that

$$T_\varphi \oplus 0 - \sum_{j=0}^{n} P_j M_\varphi P_j \in C_1$$

and

$$\|T_\varphi \oplus 0 - \sum_{j=0}^{n} P_j M_\varphi P_j\|_1 \leq \text{const.}\|\varphi\|$$

$P_j M_\varphi P_j = T^j_{\varphi_j}$. Thus we have proved the theorem.

The first application of this theorem is the calculation of the spectral multiplicity for self-adjoint Toeplitz operators. Let $\psi \in L^{\infty}(\partial D, \omega)$ be a real function. The existence of wave operators asserts that the absolutely continuous part of $T_{\psi} \oplus 0$ is unitarily equivalent to the absolutely continuous part of $(T^{0}_{\psi_0} \oplus 0) \oplus (T^{1}_{\psi_1} \oplus 0) \oplus \ldots \oplus (T^{n}_{\psi_n} \oplus 0)$. The restrictions of $T_{\psi} \oplus 0$ to $[H^2(D)]^{\perp}$ and $(T^{0}_{\psi_0} \oplus 0) \oplus \ldots \oplus (T^{n}_{\psi_n} \oplus 0)$ to $L_0 \oplus \ldots \oplus L_n$ have singular spectra, therefore the absolutely continuous parts of T_{ψ} and $T^{0}_{\psi_0} \oplus \ldots \oplus T^{n}_{\psi_n}$ are unitarily equivalent. On the unit disc, $T_{\psi \circ \pi_i}$ is of purely absolutely continuous spectrum if and only if $\psi \circ \pi_i$ is not a constant or, equivalently, ψ is not a constant on Γ_i.

<u>Theorem III.3.2</u>. Let $\psi \in L^{\infty}(\partial D, \omega)$ be a real function and let $\{\Gamma_{i_1}, \ldots, \Gamma_{i_m}\}$ be the contours such that on each Γ_{i_k}, $k = 1, \ldots, m$, ψ is not a constant. Then the absolutely continuous part of T_{ψ} is unitarily equivalent to

$$T_{\psi \circ \pi_1} \oplus \ldots \oplus T_{\psi \circ \pi_{i_m}}$$

on the orthogonal sum of m copies of H^2.

The spectral multiplicity of bounded self-adjoint Toeplitz operators on the unit disc was first found by R.S. Ismagilov, then by M. Rosenblum (see [24]). These results are actually special cases of the multiplicity theory of singular integral operators found previously by Pincus as a part of the principal function theory of pairs of self-adjoint or unitary operators with a rank one commutator. We refer the reader to [24] in this connection. Let φ be a real function on S^1. Then the actual counting of the spectral multiplicity of T_φ is the following. For any $\xi \in \mathbb{R}$, let $m(\xi)$ be the spectral multiplicity of T_φ at ξ. Then $m(\xi)$ is equal to k if $\{\tau : \tau \in S^1, \varphi(\tau) < \xi\}$ is (upto a set of Lebesgue measure 0) k but not k - 1 arcs; 0 if this set is empty or all of S^1; and ∞ if neither of the previous is the case (see [24]). Using the mappings $\pi_0, \pi_1, \ldots, \pi_n$, we can give a description for the spectral multiplicity of self-adjoint Toeplitz operators on D. By a proper arc of ∂D we mean a nonempty connected open subset of ∂D which is not any full Γ_i, $i = 0, 1, \ldots, n$. Let E be a subset of ∂D, we define $n(E)$ to be k if E consists of, upto a subset of Lebesgue measure 0, k but not k - 1 proper arcs and some full Γ_i's; 0 if E is either empty of the union of some full Γ_i's; and ∞ if none of these two is the case.

Corollary III.3.3. Let $m(\xi)$ be the spectral multiplicity function for the absolutely continuous part of T_ψ. Then

$$m(\xi) = n(\{\tau : \tau \in \partial D, \psi(\tau) < \xi\}).$$

Another description of $m(\xi)$ can be given in terms of the G current. Let $\eta < -\|\psi\|_\infty$ and let Σ be the support set of the G current for the pair $\{M, T_\psi \oplus \eta\}$. let $P_z = \{(x,y,z) : (x,y) \in \mathbb{R}^2\}$. Then $m(\xi) = n(P_\xi \cap \Sigma)$. Note that this property of the G current resembles that of the principal current for a pair of self-adjoint or unitary operators with rank one self-commutator even though in this case the rank of the commutator is not one.

The following observation is a simple soncequence of known facts, but since it can not be found in the literature, we would like to present here. Let $\varphi \in L^2(S^1)$ be a real function and let T_φ be the symmetric operator. Let $\hat{T}_\varphi$ be the Naimark dilation of T_φ to $\hat{H}$. Since

$$((T-\lambda)^{-1}k_\alpha, k_\beta) = (R(\lambda)k_\alpha, k_\beta) = \lim_{k\to\infty}((T_{\varphi_k}-\lambda)^{-1}k_\alpha, k_\beta)$$

for any $\alpha, \beta \in \Delta$ and $v\{(\hat{T}_{\varphi}-\lambda)^{-1}x : \lambda \in H^2, \lambda \in \mathbb{C}\setminus\mathbb{R}\} = \hat{H}$, from [23] one can easily conclude that the spectrum of $\hat{T}_{\varphi}$ is purely absolutely continuous and the spectral multiplicity for T at ξ is

$$n(\{\tau : \tau \in S^1, \varphi(\tau) < \xi\}).$$

III.4. The dificiency indices of symmetric Toeplitz operators

In this section we give the dificiency indices of a symmetric Toeplitz operator on D in terms of the local data of the symbol on each individual contour. First we prove the following elementary lemmas.

Lemma III.4.1. Let $\{A_m\}$ be a sequence of compact operators and $\{B_m\}$ a sequence of bounded operator on Hilbert space H such that

$$\lim_{m\to\infty}\|A_m - A\| = 0$$

and

$$\text{w-}\lim_{m\to\infty} B_m = B$$

for some $A,B \in \mathcal{L}(H)$. Then

$$\text{s-}\lim_{m\to\infty} A_m B_m$$

exists and is equal to AB.

Proof. By uniform boundedness principle, there is an up-bound M for $\{\|A_m\|\}$ and $\{\|B_m\|\}$. Let $x \in H$ and $\|x\| = 1$, then

$$\|ABx - A_m B_m x\| \le \|A(B-B_m)x\| + \|A-A_m\|\ \|B_m x\|.$$

Given $\varepsilon > 0$, let A_ε be a finite rank operator such that

$$\|A_\varepsilon - A\| < {}^{\varepsilon}/_{3}(2M+1).$$

Suppose

$$A_\varepsilon = \sum_{j=1}^{N} x_j \otimes y_j,$$

then

$$A_\varepsilon (B-B_m)x = \sum_{j=1}^{N} ((B-B_m)x, x_j)y_j.$$

Since $\|A - A_m\| \to 0$ and $\underset{m\to\infty}{\text{w-lim}}\, B_m = B$, there exists $L \geqslant 0$ such that for any $m \geq L$, $\|A - A\| < {}^{\varepsilon}/_{3}(M+1)$ and

$$|((B-B_m)x, x_j)| < {}^{\varepsilon}/_{3}N(\|y_j\| + 1).$$

Therefore for $m \geq L$,

$$\|ABx - A_m B_m x\| \leq \|(A-A_\varepsilon)(B-B_m)x\| + \|A_\varepsilon(B-B_m)x\|$$

$$+ \|A - A_m\| \; \|B_m x\| < \varepsilon.$$

This proves the strong convergence of $\{A_m B_m\}$.

<u>Lemma III.4.2</u>. Let $\{A_m\}$, $\{B_m\}$ and $\{C_m\}$ be bounded operators on H such that $A_m C_m = B_m$,

$$\underset{m\to\infty}{\text{w-lim}}\, A_m = A, \quad \underset{m\to\infty}{\text{w-lim}}\, B_m = B$$

and

$$\underset{m\to\infty}{\text{s-lim}}\, C_m = C$$

Then for any $x \in H$, $\lim_{m\to\infty}\|(A_m C-AC)x\| = 0$ if and only if $\lim_{m\to\infty}\|(B_m-B)x\| = 0$.

Proof. First it is easy to see that $AC = B$.

Obviously, there exists $M > 0$ such that $\|A_m\| \leq M$ for all m. Let $x \in H$, then

$$(AC-A_m C)x = A_m(C_m-C)x + (AC-A_m C_m)x = A_m(C_m-C)x + (B-B_m)x.$$

Since $\|A_m(C_m-C)x\| \leq M\|(C_m-C)x\|$ and $\operatorname{s-lim}_{m\to\infty} C_m = C$, the conclusion becomes obvious.

Now let $\psi \in L^2(\partial D,\omega)$ be a real function and let T_ψ be the symmetric Toeplitz operator defined in Section III.1. Let $\hat{T}_\psi$ be the Naimark dilation of T_ψ to the space $\hat{H}$. Recall that for non-real λ,

$$N_\lambda = \hat{P}(\hat{T}_\psi-\bar{\lambda})^{-1}(\hat{T}_\psi-\lambda)^{-1}\hat{P} - \hat{P}(\hat{T}_\psi-\bar{\lambda})^{-1}\hat{P}(\hat{T}_\psi-\lambda)^{-1}\hat{P}.$$

The dificiency spaces of T_ψ are

$$H^2(D) \ominus \ker N_\lambda \text{ and } H^2(D) \ominus \ker N_{\bar{\lambda}}\ .$$

Suppose $\mathcal{D}$ is the domain of T_ψ and define $T_1 = T_\psi \oplus 0$ on $\mathcal{D} + [H^2(D)]^\perp$. Obviously, T_1 is a symmetric operator with Naimark dilation $\hat{T}_1 = \hat{T}_\psi \oplus 0$ on $\hat{H} \oplus [H^2(D)]^\perp = \hat{H}_1$. Let $\hat{P}_1$

be the orthogonal projection from $\hat{H}_1$ onto $L^2(\partial D, \omega)$ and let

$$N_{1,\lambda} = \hat{P}_1(\hat{T}_1-\lambda)^{-1}(\hat{T}_1-\lambda)^{-1}\hat{P}_1 - \hat{P}_1(\hat{T}_1-\bar{\lambda})^{-1}\hat{P}_1(\hat{T}_1-\lambda)^{-1}\hat{P}_1.$$

Since $(\hat{T}_1-\lambda)^{-1} = (\hat{T}_\psi-\lambda)^{-1} \oplus (-{}^1/_\lambda)$ and $\hat{P}_1(\hat{T}_1-\lambda)^{-1}\hat{P}_1$ $= R(\lambda) \oplus (-{}^1/_\lambda)$,

$$L^2(\partial D, \omega) \ominus \ker N_{1,\lambda} = H^2(D) \ominus \ker N_\lambda$$

for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Hence T_1 and T_ψ have the same dificiency spaces.

On the otherhand, on each k_i, we can define symmetric operator $T^i_{\psi^i} = U_i^* T_{\psi\circ\pi_i} U_i^*$ where $\psi^i = \psi|\Gamma_i$. Let $\hat{T}^i$ be the Naimark dilation of $T^i_{\psi^i}$ on $\hat{K}_i$ and let $\hat{P}^i : \hat{K}_i \to K_i$ be the projection. Then

$$N^i_\lambda = \hat{P}^i(\hat{T}^i-\bar{\lambda})^{-1}(\hat{T}^i-\lambda)\hat{P}^i - \hat{P}^i(\hat{T}^i-\bar{\lambda})^{-1}\hat{P}^i(\hat{T}^i-\lambda)\hat{P}^i$$

is the dificiency operator and

$$K_i \ominus \ker N^i_\lambda, \quad K_i \ominus \ker N^i_{\bar{\lambda}}$$

are the dificiency spaces for $T^i_{\psi^i}$. But

$$U_i^*(T_{\psi_k\circ\pi_i}-\lambda)^{-1}U_i = (T^i_{\psi^i_k}-\lambda)^{-1}$$

and $U_iK_i = H^2$, so $\{(T_{\psi_k\circ\pi_i}-\lambda)^{-1}x\}$ converges in the H^2 norm if and only if $\{(T^{i}_{\psi^{i}_{k}}-\lambda)^{-1}U_i^*x\}$ converges in the K_i norm. Therefore,

$$U_i[K_i \ominus \ker N^i_\lambda] \text{ and } U_i[K_i \ominus \ker N^i_{\bar{\lambda}}]$$

are the dificiency spaces of $T_{\psi\circ\pi_i}$. Obviously, the dificiency spaces of $\bigoplus_{i=0}^{n} T^{i}_{\psi^{i}}$ are

$$\bigoplus_{i=0}^{n}[K_i \ominus \ker N^i_\lambda] \text{ and } \bigoplus_{i=0}^{n}[K_i \ominus \ker N^i_{\bar{\lambda}}].$$

Let $T_2 = \bigoplus_{i=0}^{n}(T^{i}_{\psi^{i}} \oplus 0)$ corresponding to the decomposition $L^2(\partial D,\omega) = (K_0 \oplus L_0)\oplus\ldots\oplus(K_n \oplus L_n)$. Then T_2 is a symmetric operator and $\bigoplus_{i=0}^{n}(\hat{T}^{i} \oplus 0)$ defined on $(\hat{K}_0 \oplus L_0)\oplus\ldots\oplus(\hat{K}_n \oplus L_n)$ is the Naimark dilation. T_2 and $\bigoplus_{i=0}^{n} T^{i}_{\psi^{i}}$ have the same dificiency spaces. Let $N_{2,\lambda}$ be the dificiency operator for T_2. Then by the definition of T_2,

$$\ker N_{2,\lambda} = \{x : \{(\bigoplus_{i=0}^{n}[T^{i}_{\psi^{i}} \oplus 0]-\lambda)^{-1}x\} \text{ converges in the } L^2 \text{ norm}\}.$$

But $\bigoplus_{i=0}^{n}[T^{i}_{\psi^{i}_{k}} \oplus 0] = T_{\psi_k} \oplus 0 - K_{\psi_k}$, therefore,

$$\ker N_{2,\lambda} = \{x : \{(T_{\psi_k} \oplus 0 - K_{\psi_k}-\lambda)^{-1}x\} \text{ converges in the } L^2 \text{ norm}\}.$$

Obviously, $\varphi \mapsto K_\varphi$ is a linear map from $L^\infty(\partial D, \omega)$ to $C_1(L^2(\partial D, \omega))$. Hence there exists $K_\psi \in C_1$ such that

$$\lim_{k\to\infty} \|K_{\psi_k} - K_\psi\|_1 = 0 .$$

Since

$$(T_{\psi_k} \oplus 0 - K_{\psi_k} - \lambda)^{-1}(1 - K_{\psi_k}[(T_{\psi_k} - \lambda)^{-1} \oplus (\tfrac{-1}{\lambda})])$$

$$= (T_{\psi_k} \oplus 0 - \lambda)^{-1} = (T_{\psi_k} - \lambda)^{-1} \oplus (\tfrac{-1}{\lambda})$$

and, by Lemma III.4.1,

$$\text{s-}\lim_{k\to\infty} K_{\psi_k}[(T_{\psi_k} - \lambda)^{-1} \oplus (\tfrac{-1}{\lambda})] = K_\psi[R(\lambda) \oplus (\tfrac{-1}{\lambda})],$$

Lemma III.4.2 enables us to conclude that the sequence $\{(T_{\psi_k} \oplus 0 - K_{\psi_k} - \lambda)^{-1}(1 - K_\psi[R(\lambda) \oplus (-\tfrac{1}{\lambda})])x\}$ converges if and only if $\lim_{k\to\infty}\|[(T_{\psi_k} - \lambda)^{-1} \oplus (-\tfrac{1}{\lambda})]x - [R(\lambda) \oplus (-\tfrac{1}{\lambda})]x\| = 0$.

Let $W_\lambda = 1 - K_\psi[R(\lambda) \oplus - \tfrac{1}{\lambda})]$.

<u>Lemma III.4.3</u>. For any $\lambda \in \mathbb{C}\setminus\mathbb{R}$

$$W_\lambda \ker N_{1,\lambda} = \ker N_{2,\lambda}.$$

If, in particular, we choose $\lambda \in \mathbb{C}$ such that $|\mathrm{Im}\,\lambda| > 2(1+L\|\psi\|_{L^2})$, where L is the constant introduced in the decomposition theorem, then W_λ is invertible. Hence we have proved

<u>Theorem III.4.4</u>. The symmetric operators T_ψ on $H^2(D)$ and $T_{\psi\circ\pi_0} \oplus T_{\psi\circ\pi_1} \oplus\ldots\oplus T_{\psi\circ\pi_n}$ on $H^2 \oplus H^2 \oplus\ldots\oplus H^2$ have the same dificiency indices.

Furthermore, since for nonreal λ

$$(\hat{T}_1-K_\psi-\lambda)^{-1} = (\hat{T}_1-\lambda)^{-1}(1-K_\psi(\hat{T}_1-\lambda)^{-1})^{-1},$$

it is obvious that

$$\hat{P}_1(\hat{T}_1-K_\psi-\lambda)^{-1}|L^2(\partial D,\omega) = R_1(\lambda)(1-K_\psi R_1(\lambda))^{-1}$$

where

$$R_1(\lambda) = \hat{P}_1(\hat{T}_1-\lambda)^{-1}|L^2(\partial D,\omega) = R(\lambda) \oplus (-\tfrac{1}{\lambda}).$$

On the other hand,

$$(\bigoplus_{i=0}^{n}[T^i_{\psi^i}\oplus O]-\lambda)^{-1} = (T_{\psi_k}\oplus O-K_{\psi_k}-\lambda)^{-1}$$

$$= (T_{\psi_k}\oplus O-\lambda)^{-1}(1-K_{\psi_k}(T_{\psi_k}\oplus O-\lambda)^{-1})^{-1}$$

and therefore,

$$\operatorname*{w-lim}_{k\to\infty}(\bigoplus_{i=0}^{n}[T^i_{\psi^i}\oplus O]-\lambda)^{-1} = R_1(\lambda)(1-K_\psi R_1(\lambda))^{-1} = R_2(\lambda).$$

This implies that $\hat{T}_\psi \oplus O - K_\psi = \hat{T}_1 - K_\psi$ is the Naimark dilation of $T_2 = \bigoplus_{i=0}^{n}[T^i_{\psi^i}\oplus O]$. Since $R_1(\lambda) = R_2(\lambda)W_\lambda$ and $W_\lambda \ker N_{1,\lambda} = \ker N_{2,\lambda}$, T_1 and T_2 have the same domain $\tilde{\mathfrak{D}}$.

For any $y \in \tilde{\mathfrak{D}}$, there exists $x \in \ker N_{1,\lambda}$ such that $R_1(\lambda)x = y = R_2(\lambda)W_\lambda x$. By the definition

$$T_1 y = \lambda y + x \text{ and } T_2 y = \lambda y + W_\lambda x.$$

Hence

$$T_1 y - T_2 y = x - W_\lambda x = K_\psi R_1(\lambda)x = K_\psi y.$$

Theorem III.4.5. The symmetric operator $T_\psi \oplus 0$ and $\bigoplus_{i=0}^{n} [T^i_{\psi^i} \oplus 0]$ have the same domain $\tilde{\mathfrak{D}}$ and on this domain

$$T_\psi \oplus 0 - \bigoplus_{i=0}^{n} [T^i_{\psi^i} \oplus 0] = K_\psi \in C_1.$$

Furthermore, the difference of the Naimark dilations of these two symmetric operators is also K_ψ and in particular, they defer only on $L^2(\partial D, \omega)$.

The following corollary is obvious.

Corollary III.4.6. Let $m(\xi)$ be the spectral multiplicity function for the absolutely continuous part of $\hat{T}_\psi$, the Naimark dilation of symmetric Toeplitz operator T_ψ, then

$$m(\xi) = n(\{\tau : \tau \in \partial D, \ \psi(\tau) < \xi\})$$

Particularly, if T_ψ itself is self-adjoint, then the above number is the spectral multiplicity of the absolutely continuous part of T_ψ at ξ.

Chapter IV. The Spectra of Self-adjoint Toeplitz Operators

IV.1. The singular spectra

It is well known that on the unit disc, every self-adjoint Toeplitz operator has purely absolutely continuous spectrum if the symbol is not a constant. But when the domain D is not simply-connected, singular spectra for self-adjoint Toeplitz operators do occur. For example, if $\psi = c_j$ on Γ_j, $j = 0,1,\ldots,n$, then we know that $T_\psi \oplus 0$ on $L^2(\partial D,\omega)$ is a compact perturbation of $(c_0 \oplus 0)\oplus\ldots\oplus(c_n \oplus 0)$ corresponding to the decomposition $L^2(\partial D,\omega) = (K_0 \oplus L_0)\oplus\ldots\oplus(K_n \oplus L_n)$, the later having the spectrum consisting at most $n + 1$ points. Hence $\sigma(T_\psi)$ must be discrete. This example was observed by Abrahamse, from the view point that the essential spectrum of T_ψ consists at most n points, see [1]. Another example of singular spectrum is the following. The space N contains real bounded functions. If we choose real $\psi \in N$, since $\psi \perp H^2(D)$, 0 is an eigenvalue of T_ψ. But note that so far we do not have an example of continuous singular spectrum. In fact it is doubtful that such spectrum really exists. The only thing we know about the continuous singular spectrum is the following:

Theorem IV.1.1. No self-adjoint Toeplitz operators have purely continuous singular spectrum.

Proof. If the spectrum of self-adjoint Toeplitz operator T_ψ is purely singular, then it does not contain absolutely continuous part and therefore, by Theorem III.3.2, ψ must be constant on each of the contours. Hence the spectrum of T_ψ is discrete, not continuous.

Theorem IV.1.2. Let $\psi \in L^\infty(\partial D, \omega)$ be real nonconstant function. Then

$$\dim \ker T_\psi \leq n.$$

Proof. T_ψ is unitarily equivalent to

$$T^G_{\psi\circ\pi} = (T_{\psi\circ\pi} - PP_1 M_{\psi\circ\pi})|H^2_G$$

where $T_{\psi\circ\pi}$ is the usual Toeplitz operator on H^2 and

$$\operatorname{rank} PP_1 M_{\psi\circ\pi} = n.$$

Simple linear algebra shows that if it were true that

$$\dim \ker T^G_{\psi\circ\pi} > n,$$

then $\ker T_{\psi\circ\pi} \neq \{0\}$, which is impossible.

Corollary I.V.1.3. If nonconstant $\psi \in L^\infty(\partial D, \omega)$ is real, then the multiplicity of each $\lambda \in \sigma_p(T_\psi)$ is not more than n, the number of holes in D.

Using Theorem III.2.1, we can similarly reach the conclusion that if T is the Naimark dilation of a symmetric Toeplitz operator, then the multiplicity for each eigenvalue of T is not more than n.

We can show that some self-adjoint Toeplitz operators do not have point spectrum. Recall that for an $x \in L^2(\partial D, \omega)$, the harmonic conjugate is a function $y \in L^2(\partial D, \omega)$ the harmonic extension of $x + iy$ is in $H^2(D)$. If D is not simple-connected, not all $x \in L^2(\partial D, \omega)$ have harmonic conjugate. In fact x has a harmonic conjugate if and only if $x \perp N$, in other words, $x \in H^2(D) \oplus \overline{H_0^2(D)}$.

<u>Theorem IV.1.4.</u> If non-constant $x \in L^\infty(\partial D, \omega)$ has a harmonic conjugate, then $\sigma_p(T_x) = \phi$.

<u>Proof.</u> It suffices to show that $\ker T_x = \{0\}$. Let $y \in L^2(\partial D, \omega)$ by the harmonic conjugate of x, so $\varphi = x + iy \in H^2(D)$. Let $u \in \ker T_x$, then there exist $v \in N$ and $w \in H_0^2(D)$ such that

$$xu = v + \overline{w} .$$

Thus

$$yu = i\varphi u + ixu = -\varphi u + iv + i\overline{w}$$

and

$$y\overline{u} = i\overline{\varphi u} - i\overline{v} - iw.$$

Hence

$$|yu|^2 = yu\cdot\overline{yu} = [-i\varphi u + iv + i\overline{w}][i\overline{\varphi u} - i\overline{v} - iw]$$

$$= |\varphi u|^2 + |v|^2 + |w|^2 - [\varphi u\overline{v} + \overline{\varphi u}v] + [\overline{wv} + wv]$$

$$- [\varphi uw + \overline{\varphi uw}]$$

Since $|\varphi u|^2 = |\varphi|^2 \; |u|^2 = (x^2+y^2)|u|^2$, we have

$$x^2|u|^2 + |v|^2 + |w|^2 = [\varphi u\overline{v} + \overline{\varphi u}v] - [\overline{wv} + wv] + [\varphi uw + \overline{\varphi uw}].$$

It is clear that in the above equality, the left hand side and the first and second terms of the right hand side are in $L^1(\partial D, \omega)$, therefore, so is $\varphi uw + \overline{\varphi vw}$. Because $w \in H^2(D)$, $\varphi(z_0)u(z_0)w(z_0) = 0$ where z_0 is the base point for ω. Hence

$$\int_{\partial D} [\varphi uw(\tau) + \overline{\varphi uw}(\tau)]d\omega(\tau) = 0.$$

Furthermore, since $\varphi u \in H^1(D)$, $w \in H_0^2(D)$ and $v \in N$, it is easy to see that

$$\int_{\partial D} \{[\varphi u\overline{v} + \overline{\varphi}\overline{u}v] - [\overline{w}\overline{v} + wv]\}d\omega = 0.$$

Therefore,

$$u = v = w = 0.$$

IV.2. Self-adjoint Toeplitz operators with purely absolutely continuous spectrum

In Section III.4, we characterized the spectral multiplicity for the absolutely continuous part of self-adjoint Toeplitz operators in terms of the symbol. Hence for those operators whose spectrum is absolutely continuous we obtained a complete set of unitary invariants from the symbol. In some sense, this is a kind of diagonalization for these operators. So it is natural to investigate which self-adjoint Toeplitz operators have purely absolutely continuous spectrum. On the unit disc, it is well known that all of them do. But on multiply-connected domain D, again due to the lack of analytical tools, we so far can only designate a class of self-adjoint Toeplitz operators which have purely absolutely continuous spectrum. Let A,B be bounded self-adjoint operators on Hilbert space H. Let $H_a(A)$ and $H_a(B)$ respectively be the absolutely continuous space of A and B. Suppose that $AB - BA = iC$ where $C \geq 0$. Let L be the smallest invariant subspace for both A and B that contains the range of C. The following theorem is due to Putnam.

Theorem IV.2.1 [21], [22].

$$L \subset H_a(A) \cap H_a(B).$$

Using this theorem, we can prove the following:

Theorem IV.2.2. Let $x \in [H^2(D) \oplus \overline{H_0^2(D)}] \cap L^\infty(\partial D, \omega)$ be a real function. If the harmonic conjugate y of x is semi-bounded, the spectrum of T_x is purely absolutely continuous.

Proof. Since y is semi-bounded, without loss of generality, we can assume $y \geq 0$. Let $\varphi = x + iy$, then $\varphi \in H^2(D)$. By Section III, we know that the symmetric operator T_y is actually a self-adjoint operator and $T_y u = Pyu$ for any $u \in H^\infty(D)$. Hence for $u \in H^\infty(D)$, we have

$$-2i[(T_x u, t_y u) - (T_y u, T_x u)] = -2i[(Pxu, Pyu) - (Pyu, Pxu)]$$

$$= [(P(x+iy)u, P(x+iy)u) - (P(x-iy)u, P(x-iy)u)]$$

$$= \|\varphi u\|^2 - \|P\bar{\varphi}u\|^2 = \|\bar{\varphi}u\|^2 - \|P\bar{\varphi}u\|^2 \geq 0 .$$

Since $H^\infty(D)$ is a core for T_y, it is true for any u in the domain $\mathcal{D} = \mathcal{D}_{T_y}$ of T_y

$$-i[(T_x u, T_y u) - (T_y u, T_x u)] \geq 0.$$

It is also true that

$$-i[(T_x u, (T_y+1)u) - ((T_y+1)u, T_x u)] \geq 0.$$

Hence for any $v \in H^2(D)$,

$$-i[(T_x(T_y+1)^{-1}v,v) - ((T_y+1)^{-1}T_x v,v)] \geq 0.$$

Let $C = i[(T_y+1)^{-1}, T_x]$, then $C \geq 0$. Thus by Putnam's theorem $H_a(T_x)$ contains the least invariant subspace for both T_x and $(T_y+1)^{-1}$ that contains the range of C. Denote this subspace by K. We now prove that $K = H^2(D)$. Let $L = H^2(D) \ominus K$, then L is also invariant for T_x and $(T_y+1)^{-1}$ and $L \subset \ker C$. Let $u \in L$, then $(T_y+1)^{-1}u = v \in \mathcal{D}_{T_y}$. By the definition of T_y

$$T_y v = T_y(T_y+1)^{-1}u = u - (T_y+1)^{-1}u = u - v .$$

Hence

$$\begin{aligned}(T_x v, T_y v) &- (T_y v, T_x v) \\ &= (T_x v, u-v) - (u-v, T_x v) \\ &= (T_x v, u) - (u, T_x v) = (T_x(T_y+1)^{-1}u, u) \\ &- (u, T_x(T_y+1)^{-1}u) = i(Cu,u) = 0.\end{aligned}$$

From this, it easily follows that

$$\|(T_x+iT_y)v\| - \|(T_x-iT_y)v\|^2 = 0.$$

Since $H^\infty(D)$ is a core for T_y, there exists a sequence

$$\{v_m\} \subset H^\infty(D)$$

such that $\lim_{m\to\infty}\|v_m-v\| = 0$ and $\lim_{m\to\infty}\|T_y v_m - T_y v\| = 0$. But $T_y v_m = Pyv_m$ since $v_m \in H^\infty(D)$.

Hence

$$\lim_{m\to\infty}[\|P(x+iy)v_m\|^2 - \|P(x-iy)v_m\|^2] = \|(T_x+iT_y)v\|^2 - \|(T_x-iT_y)v\|^2 = 0 .$$

On the other hand

$$\|(1-P)(x-iy)v_m\|^2 = \|(x-iy)v_m\|^2 - \|P(x-iy)v_m\|^2$$

$$= \|(x+iy)v_m\|^2 - \|P(x-iy)v_m\|^2 = \|P(x+iy)v_m\|^2 - \|P(x-iy)v_m\|^2 \to 0 \quad \text{as } m \to \infty.$$

Therefore,

$$2xv = \lim_{m\to\infty} 2xv_m = \lim_{m\to\infty}[(x+iy)v_m + (x-iy)v_m]$$

$$= \lim_{m\to\infty}\{P[(x+iy)v_m + (x-iy)v_m] + (1-P)(x-iy)v_m\}.$$

Since $\|(1-P)(x-iy)v_m\| \to 0$, we have $xv \in H^2(D)$ and therefore, $xv = T_x v \in L$.

So for $u \in L$

$$x(T_y+1)^{-1}u = T_x(T_y+1)^{-1}u \in L.$$

But $T_x(T_y+1)^{-1}u = (T_y+1)^{-1}T_x u$ for $u \in L$, so

$$(T_y+1)^{-1}T_x u = T_x(T_y+1)^{-1}u = x(T_y+1)^{-1}u.$$

Therefore, $x(T_y+1)^{-1}u \in \mathcal{D}$ and

$$(T_y+1)x(T_y+1)^{-1}u = T_x u.$$

For any n, we also have

$$T_x^n u = (T_y+1)x^n(T_y+1)^{-1}u$$

and therefore,

$$x^n(T_y+1)^{-1}u = (T_y+1)^{-1}T_x^n u.$$

So

$$(*) \qquad p(x)(T_y+1)^{-1}u = (T_y+1)^{-1}p(T_x)u \in L$$

for any polynomial p. There are constants a and b so that

$-\|x\|_\infty < a < b < \|x\|_\infty$, and such that the sets

$x^{-1}[-\|x\|_\infty, a)$, $x^{-1}(b, \|x\|_\infty]$ have positive measure.

Let f be a continuous function which is equal to one in $[-\|x\|_\infty, a]$, which decreases to zero smoothly in $[a,b]$, and which is zero on $[b, \|x\|_\infty]$.

Let $\{p_n\}$ be a sequence of polynomials that converges to f uniformly on $[-\|x\|_\infty, \|x\|_\infty]$. It is a consequence of (*) above that $f(x(\tau))[(T_y+1)^{-1}u](\tau) = [(T_y+1)^{-1}f(T_x)u](\tau)$, where $(T_y+1)^{-1}f(T_x)u \in L \subset H^2(\Gamma)$. On $x^{-1}(b, \|x\|_\infty$, the left-hand side is 0, so $(T_y+1)^{-1}f(T_x)u = 0$. But $f(x(\tau))$ is 1 on $x^{-1}[-\|x\|_\infty, a]$; hence we have

$$[(T_y+1)^{-1}u](\tau) = [(T_y+1)^{-1}f(T_x)u](\tau) = 0$$

for $\tau \in x^{-1}[-\|x\|_\infty, a]$.

Hence

$$(T_y+1)^{-1}u = 0.$$

It follows immediately that $L = \{0\}$. This completes the proof.

Having a semi-bounded harmonic conjugate is certainly not easy to be checked. But if x itself has certain smoothness, then its harmonic conjugate is semi-bounded and is automatically guaranteed. Recall that the covering map $\pi : \Delta \to D$ carries $S^1\backslash L(G)$ onto ∂D, where $L(G)$ is the limit set of the automorphic group G. The inverse image of each Γ_i, $i = 0,1,\ldots,n$, is the union of infinitely many connected components (if $n \geq 1$) or one component (if $n = 1$) of $S^1\backslash L(G)$. Also, under π, each component of $S^1\backslash L(G)$ is mapped wrapping some Γ_i infinitely many times. Hence if x is a C^1 function on ∂D, then there are open arcs $C_0, C_1 \ldots C_n$ on S^1 such that $x \circ \pi$ C^1 on a neighborhood of each C_i. If x is real, then the harmonic conjugate $\hat{y}$ of $x \circ \pi$ is continuous on each $\overline{C_i}$ (see page 79 of [16]). If x has a harmonic conjugate y on D, then it is clear that $y \circ \pi - \hat{y}$ is a constant. Therefore, y is semi-bounded.

<u>Corollary IV.2.3</u>. Let $x \in C^1(\partial D)$ be a real function. If x has a harmonic conjugate, then the spectrum of T_x is purely absolutely continuous.

References

1. B. Abrahamse, Toeplitz operators on multiply-connected domains, Amer. J. Math., 96 (1974), 261-297.

2. M.S. Brodskii, Triangular and Jordan Representation of Linear Operators, English translation, Amer. Math. Soc., 1971.

3. R.W. Carey and J.D. Pincus, Almost commuting pairs of unitary operators and flat currents, Integral Eq. and Operator Theory, 4 (1981), 45-122.

4. R.W. Carey and J.D. Pincus, An invariant for certain operator algebras, Proc. Nat. Acad. Sci. U.S.A., 71 (1974), 1952-1956.

5. R.W. Carey and J.D. Pincus, Commutators, symbols, and determining functions, J. Functional Analysis, 19 (1975), 50-80.

6. R.W. Carey and J.D. Pincus, Commuting algebras of operators and holomorphic one chains, to appear.

7. R.W. Carey and J.D. Pincus, Mosaics, principal functions and mean motion in von Neumann algebras, Acta. Math., 138 (1977), 153-218.

8. R.W. Carey and J.D. Pincus, Almost Commuting Algebras, Lecture Notes in Mathematics 575, Springer-Verlag, 1977, 19-43.

9. J.B. Conway, Subnormal Operators, Pitman Advanced Publishing Program, London, 1980.

10. J. Dixmier, C*-Algebras, North Holland, New York, 1977.

11. C. Earle and A. Marden, On Poincaré series with applications to H^p spaces on bordered Riemann surfaces, Illinois J. Math., 13 (1969), 202-219.

12. H. Federel and W.H. Fleming, Normal and integral currents, Ann. of Math., 72 (1960), 458-520.

13. H. Federel, Geometric Measure Theory, Springer-Verlag, New York, 1969.

14. F. Forelli, Bounded holomorphic functions and projections, Illinois J. Math., 10 (1966), 367-380.

15. L.A. Coburn and M. Schechter, Joint spectra and interpolation of operators, J. Functional Analysis, 2(1968), 226-237.

16. K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

17. M.A. Naimark, Linear Differential Operators, Pari II: Linear Differential Operators in Hilbert Spaces, English translation, Frederic Ungar Publishing Co., New York, 1968.

18. J.D. Pincus, On the spectral theory of singular integral operators, Trans. Amer. Math. Soc., 113 (1964), 101-128.

19. J.D. Pincus, Symmetric singular integral operators with arbitrary dificiency, Advances in Math. Suppl. series 3, (1978), 241-281.

20. J.D. Pincus, Symmetric Wiener Hopf and Toeplitz Operators, Integral Eq. and Operator Theory, 4 (1981), 123-133.

21. C.R. Putnam, Commutators and absolutely continuous operators, Trans. Amer. Math. Soc., 87 (1958), 513-525.

22. C.R. Putnam, Commutation Properties of Hilbert Space Operators and Related Topics, Springer-Verlag, New York, 1967.

23. M. Reed and B. Simon, Methods of Modern Mathematical Physics III: Scattering Theory, Academic Press, New York, 1979.

24. M. Rosenblum, The absolute continuity of Toeplitz matrices, Pacific J. Math., 10 (1960), 987-996.

25. M. Rosenblum, A concrete spectral theory for self-adjoint Toeplitz operators, Amer. J. Math., 87 (1965), 709-718.

26. M.G. Krein, Perturbation determinant formula for the traces of unitary and self-adjoint operators, Dokl. Akad. Nauk SSSR, 144 (1962), 267-271.

27. J.D. Pincus and J. Xia: Symmetric and self-adjoint Toeplitz operators on multiply-connected plane domains, to appear in the Journal of Functional Analysis.

28. R.W. Carey and J.D. Pincus: Operator theory and boundaries of complex curves, to appear in the Proc. National Academy of Sciences, U.S.A.

29. I.Ts. Gokhberg and N. Ya. Krupnik: On the algebra generated by Toeplitz matrices, English translation from Funksional'nyi Analiz i Ego Prilozheniya, Vol.3,No.2,pp.46-56,April-June,1969