Curvature Inequality and Certain Toeplitz-like Operators

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Abstract of the Dissertation

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For $T \in B_n(\Omega)$, there is an associated Hermitian holomorphic vector bundle E_T over Ω of rank n. In the case $n = 1$, the curvature $K_T(w)$ of the associated Hermitian holomorphic vector bundle E_T is the unitary invariant of T. It is known that $U^* \in B_1(D)$, where U^* is the adjoint of unilateral shift and D is the open unit disc, and the closure of D is a spectral set for U^*. In $B_1(D)$, we have a curvature inequality namely if $T \in B_1(D)$ and closure of D is a spectral set for T, then $K_T(w) \leq K_{U^*}(w)$, $w \in D$.

In this paper, we obtain a generalization of the curvature inequality for $B_1(\Omega)$, where Ω is a simply-connected domain with Jordan curve boundary. We consider a Riemann map $\varphi : D \to \Omega$, which exists since Ω is simply-connected, and show that $T^*_{\tilde{\varphi}} \in B_1(\Omega^*)$, where $\tilde{\varphi} \in H^\infty(T)$ and is the w^*-limit of $\varphi_r \in H^\infty(T)$ $(0 < r < 1)$. We also show that closure of Ω^* is a spectral set for $T^*_{\tilde{\varphi}}$ and if $T \in B_1(\Omega^*)$ such that closure of Ω^* is a spectral set for T, then $K_T(w) \leq K_{T^*_{\tilde{\varphi}}}(w)$, $w \in \Omega^*$.

Next, we consider co-subnormal operators T quasi-similar to U^* and show that $T \in B_1(D)$. We also show that if M is an invariant subspace of T^* then the compression $P_M T|M \in B_1(D)$ and is a quasi-affine transform of T. In the case M is finite co-dimensional we show that $P_M T|M$ is similar to T.

Finally, for any bounded operator X on $H^2(T)$, we introduce a bounded operator T^X on $H^2(T) \oplus H^2(T)$. We show that $T^X \in B_2(D)$. We consider the special cases by taking X as the adjoint of analytic Toeplitz operator T_φ and denote T^X by T^φ. In this paper, we have determined the necessary and sufficient conditions for T^φ and T^ψ to be unitarily equivalent, similar or quasi-similar.

Dedicated to the memory of my late grandfather Narayana Panda.

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CHAPTER I

Introduction

In this dissertation we study a class of operators that possess an open set of eigenvalues. This class of operators were first introduced by Cowen and Douglas [2,3]. Later, the same authors have presented a detail study of these operators in [4] and [5]. They have shown that these operators associate with themselves certain geometric objects, namely Hermitian holomorphic vector bundles over some open connected subset of the complex plane. And as expected, many of the properties of these operators are shown to be translated into the geometric properties of the associated Hermitian holomorphic vector bundles. For example, unitarily equivalent operators give rise to equivalent Hermitian holomorphic vector bundles. Using this result, the above authors have further shown that unitary equivalence of these operators reduces to unitary equivalence of certain finite dimensional local operators associated with the operator. These local operators are nilpotent operators and there is existing literature on the unitary equivalence of finite dimensional nilpotent operators [12]. When the associated Hermitian holomorphic vector bundle is one dimensional, it is not too difficult to determine the unitary invariants of the operator, for one dimensional Hermitian holomorphic vector bundles are characterized by their curvature [15]. Also, given a suitable nonzero holomorphic cross-section, it is fairly stright forward to determine the curvature in the one dimensional case.

This paper consists of three chapters, the first two of which deal mainly with the case $n = 1$ and in the last chapter we deal with the case $n = 2$. In the first chapter we consider a generalization of the curvature inequality and show that many of the properties of operators in $B_1(\Omega)$ for a reasonable simply connected Ω, may well be translated into the case $\Omega = D$, the open unit disc. The most important of the operators in $B_1(D)$ is U^*, the adjoint of the unilateral shift on the Hilbert space of square summable sequences. One of the striking properties of U^*, as a member of $B_1(D)$, is that it leads the class of operators in $B_1(D)$, which have the closed unit disc as a spectral set. More specifically, the curvature of its associated bundle E_{U^*} dominates the curvature of E_T, for any $T \in B_1(D)$ which have the closed unit disc as a spectral set. And this is what we mean by the curvature inequality.

In Chapter I, we consider a simply connected open set Ω with Jordan curve boundary [14] and show that the adjoint of the Toeplitz operator $T_{\tilde{\varphi}}$ induced by the Riemann map $\varphi : D \to \Omega$ is in $B_1(\Omega^*)$, where $\Omega^* = \{\overline{w} : w \in \Omega\}$ and that the closure of Ω^* is a spectral set for $T^*_{\tilde{\varphi}}$. Further we show that if T is any other operator in $B_1(\Omega^*)$ such that the closure of Ω^* is a spectral set for T, then the curvature of E_T is dominated by the curvature of $E_{T^*_{\tilde{\varphi}}}$ at every point in Ω^*. We also show that if Ω_1 and Ω_2 are two simply connected open sets with Jordan curve boundaries and $T \in B_1(\Omega_1)$ and closure of Ω_1 is a spectral set for T and if we consider the Riemann map $\varphi : \Omega_1 \to \Omega_2$ then $\varphi(T)$ is defined and $\varphi(T) \in B_1(\Omega_2)$, furthermore the closure of Ω_2 is a spectral set for $\varphi(T)$. Then we deduce a relation between the curvatures of E_T and $E_{\varphi(T)}$.

In the second chapter, we show that co-subnormal operators quasi-similar [11] to U^* also belong in $B_1(D)$. This follows from a more general result. We do not claim any originality on this result for it follows from the works of Clary [1] and more recently of Raphael [13]. Later in the chapter, we consider the invariant subspaces of the adjoint of co-subnormal operators quasi-similar to U^*. We show that if T is co-subnormal and quasi-similar to U^* and M is any invariant subspace of T^*, then the compression $P_M T|M$ is a quasi-affine transform of T.

Finally, in the last chapter we introduce certain class of operators T^X for any bounded operator X. We show that $T^X \in B_2(D)$. In our study, we consider only special cases taking X as the adjoint of analytic Toeplitz operator. When X is the adjoint of analytic Toeplitz operator T_φ, we denote T^X by T^φ. We have been able to determine the necessary and sufficient conditions for two such operators T^φ and T^ψ, $\varphi, \psi \in H^\infty(D)$, to be unitary equivalent, similar or quasi-similar. At this moment, we haven't been able to determine necessary and sufficient conditions for the equivalence of general operators T^X and T^Y, where X and Y are arbitrary bounded operators. However, we have been able to show that if T^X is unitarily equivalent to T^φ, where X is an arbitrary bounded operator and $\varphi \in H^\infty(D)$, then X is indeed the adjoint of an analytic Toeplitz operator. Again, we don't know if the same thing is true if unitary equivalence is replaced by similarity or quasi-similarity.

Before we go on to the next chapter, we give a few relevant definitions and some known results with little or no proof.

Throughout the paper, H will denote a complex separable Hilbert space and L(H) will denote the space of bounded operators on H. As is usual, we set $\ker T = \{x \in H : Tx = 0\}$ and $\text{Rng}\,T = \{x \in H : x = Ty$ for some $y \in H\}$ for any $T \in L(H)$.

Definition 1.1. Let Ω be a connected open subset of $\mathbb{C}$ and n be a positive integer. Then we define $B_n(\Omega)$ as the class of operators satisfying the following properties:

(i) $\Omega \subset \sigma(T) = \{w \in \mathbb{C} : (T-w)$ not invertible$\}$;

(ii) $\text{Rng}\,(T-w) = H$ for $w \in \Omega$;

(iii) $\bigvee_{w\in\Omega} \ker(T-w) = H$; and

(iv) $\dim \ker(T-w) = n$ for $w \in \Omega$.

Note that the first two properties imply that $(T-w)$ is semi-Fredholm on a connected set and hence the Fredholm index is constant all throughout the connected set. We choose this constant to be a finite number n and consequently property (iv) follows.

Definition 1.2. The map $w \to \ker(T-w)$ defines a rank n Hermitian holomorphic vector bundle E_T over Ω. Given a manifold M with a complex structure, we define a rank n holomorphic vector bundle as a complex manifold E together with a holomorphic map $\pi: E \to M$ such that each fibre $E_\lambda = \pi^{-1}(\lambda)$, $\lambda \in M$, is isomorphic to $\mathbb{C}^n$ and for each point $\lambda_0 \in M$ there exists a neighborhood U of λ_0 in M with n holomorphic function $\gamma_1(\lambda), \ldots, \gamma_n(\lambda)$ defined on U such that their values at any point λ in U form a basis of the fibre E_λ. These n functions are called a frame of E over U. If U can be choosen as all of M, the holomorphic vector bundle is said to be trivial. It is known that any holomorphic vector bundle over Ω is trivial. So, it is the Hermitian structure in E_T, that we will be interested in. A holomorphic vector bundle E is said to have a Hermitian structure if each fibre E_λ is an inner-product space. A bundle map between two bundles E_1 and E_2 over M is a holomorphic map $\phi : E_1 \to E_2$, which is linear transformation between each corresponding fibres $\pi_1^{-1}(\lambda)$ and $\pi_2^{-1}(\lambda)$, $\lambda \in M$. If in addition, this linear transformation is isometric, the Hermitian holomorphic vector bundles E_1 and E_2 are said to be equivalent.

Definition 1.3. For $T \in B_n(\Omega)$, we define the local operators

N_w, $w \in \Omega$ as the restriction of $(T-w)$ on $\ker(T-w)^{n+1}$. Clearly, the local operators are nilpotent operators of order $(n+1)$.

Theorem 1.4. Let $T \in B_1(\Omega)$, then the unitary invariant for the local operator N_w is the trace of $N_w^* N_w$.

Proof. See Cowen and Douglas [4].

Theorem 1.5. Let $T \in B_1(\Omega)$ and $\gamma(w)$ be a nonzero holomorphic cross-section for E_T, then the curvature $K_T(w)$ of E_T is given by

$$K_T(w) = -\frac{\partial^2}{\partial\bar{w}\partial w} \operatorname{Ln}\|\gamma(w)\|^2 = {}^{-1}/\operatorname{trace}(N_w^* N_w)$$

Proof. See Cowen and Douglas [4].

Definition 1.6. Let $T \in B_1(\Omega)$ and $\gamma(w)$ be a non-zero holomorphic cross-section for E_T. Then there is a natural representation Γ of H as a space of holomorphic functions on $\Omega^* = \{\bar{w} : w \in \Omega\}$ defined by $(\Gamma x)w = \langle x, \gamma(\bar{w})\rangle$ for $x \in H$, $w \in \Omega^*$. And according to this representation, T serves as the adjoint of the multiplication by w. If we denote $K(\gamma, w) = \langle r(\bar{w}), \gamma(\bar{\lambda})\rangle$, then K is a reproducing kernel for this space of holomorphic functions. However, there is no canonical representation of H as a space of holomorphic function on Ω^*, since there is no canonical holomorphic cross-section of E_T. But in the case of U^*, the Szegö kernel $K(\lambda,w) = (1-\lambda\overline{w})^{-1}$ corresponds to the canonical cross-section $\gamma(w) = (1,w,w^2,\ldots)$ and in the case of B_z^*, the adjoint of Bergman operator, the Bergman kernel $K(\lambda,\overline{w}) = (1-\lambda\overline{w})^{-2}$ corresponds to a canonical cross-section. Using these cross-section, we can calculate that

$$K_{U^*}(w) = -(1-|w|^2)^{-2} \text{ and}$$

$$K_{B_z^*}(w) = -2(1-|w|^2)^{-2}.$$

Thus, U^* and B_z^* are not unitarily equivalent.

Definition 1.7. Let $T \in B_1(\Omega)$ and X be a bounded operator that commutes with T. So, if $w \in \Omega$ we have $X(T-w) = (T-w)X$ and hence $X \ker(T-w) \subset \ker(T-w)$. Thus, $X\gamma(w) = \varphi(w)\gamma(w)$, for some holomorphic function φ on Ω, where $\gamma(w)$ is a holomorphic cross-section for E_T. Now since

$$|\varphi(w)|\,\|\gamma(w)\| = \|\varphi(w)\gamma(w)\| = \|X\gamma(w)\| \leq \|X\|\,\|\gamma(w)\|$$

We see that $\varphi \in H^\infty(\Omega)$, the space of bounded holomorphic functions on Ω.

If we denote the collection of bounded operators that commute with T as $(T)'$, then we have a contractive map

$$\Gamma_T : (T)' \to H^\infty(\Omega) \qquad \text{defined by}$$

$$\Gamma_T(X) = \varphi$$

Using property (iii) of $T \in B_1(\Omega)$ one can easily show that Γ_T is one-to-one. But Γ_T is not always onto. However if Ω is finitely connected and closure of Ω is a spectral set for T, then Γ_T is onto. Instead of taking Ω finite connected, we may assume that for any $\varphi \in H^\infty(\Omega)$, there exists a sequence of rational functions r_n with poles off the closure of Ω such that $\|r_n\|_\infty \leq \|\varphi\|_\infty$ and $r_n \to \varphi$ pointwise. We call such connected set Ω as reasonable.

<u>Theorem 1.8</u>. Let $T \in B_1(\Omega)$ and Ω be reasonable and closure of Ω be a spectral set for T, then Γ_T is an isometric isomorphism onto $H^\infty(\Omega)$.

<u>Proof</u>. See Cowen and Douglas [4].

Finally, we state and provide a proof of the curvature inequality.

<u>Theorem 1.9</u> (Curvature Inequality). Let $T \in B_1(D)$ and $\bar{D}$ be a spectral set for T, then

$$K_T(w) \leq K_{U^*}(w) \qquad w \in D.$$

<u>Proof</u>. Since $\bar{D}$ is a spectral set for T, it is a contraction and so $\|T\| \leq 1$.

Let $w \in D$. Now consider the local operator N_w. With appropriate basis the matrix of N_w can be shown to be $\begin{pmatrix} 0 & h(w) \\ 0 & 0 \end{pmatrix}$, where $h^2(w) = -{}^1/K_T(w)$, [4]. Hence the matrix of T restricted to $\ker(T-w)^2$ with respect to the above basis is $\begin{pmatrix} w & h(w) \\ 0 & w \end{pmatrix}$. Since T is a contraction, we have

$$\left\|\begin{pmatrix} w & h(w) \\ 0 & w \end{pmatrix}\right\| \leq 1 \Rightarrow \left\|\begin{pmatrix} w & h(w) \\ 0 & w \end{pmatrix}\right\|^2 \leq 1$$

$$\Rightarrow \left\|\begin{pmatrix} \bar{w} & 0 \\ h(w) & \bar{w} \end{pmatrix}\begin{pmatrix} w & h(w) \\ 0 & w \end{pmatrix}\right\| \leq 1$$

$$\Rightarrow \left\|\begin{pmatrix} |w|^2 & \bar{w}h(w) \\ wh(w) & |w| + h^2(w) \end{pmatrix}\right\| \leq 1.$$

Since, $\begin{pmatrix} |w|^2 & \bar{w}h(w) \\ wh(w) & |w|^2 + h^2(w) \end{pmatrix}$ is hermitian, its norm is given by the maximum of its eigenvalues.

The eigenvalues, in this case, can be calculated to be

$$\frac{(2|w|^2 + h^2(w)) \pm \sqrt{4|w|^2h^2(w) + h^4(w)}}{2}$$

hence,

$$\frac{2|w|^2 + h^2(w) + \sqrt{4|w|^2 h^2(w) + h^4(w)}}{2} \leq 1 .$$

This can be simplified as

$$h^2(w) \leq (1-|w|^2)^2$$

$$\text{or} \quad K_T(w) \leq K_{U^*}(w) .$$

And this proves the theorem. ∎

CHAPTER II

Toeplitz Operator and Curvature Inequality

Let Ω be a simply-connected bounded open subset of $\mathbb{C}$ with Jordan curve boundary. And let $\varphi : D \to \Omega$ be a Riemann map. Then consider the function $\varphi_r : T \to \mathbb{C}$ defined by $\varphi_r(e^{i\theta}) = \varphi(re^{i\theta})$ for $0 < r < 1$. It is well known that $\varphi_r \in H^\infty(T)$, $\|\varphi_r\|_\infty \leq \|\varphi\|_\infty$ and φ_r converges to $\tilde{\varphi} \in H^\infty(T)$ in the w*-topology, where the Poisson integral of $\tilde{\varphi}$ is φ. Also, φ_r converges to $\tilde{\varphi}$ pointwise almost everywhere on T with respect to the Lebesgue measure on T. For details on this topic see Hoffman [9].

In this chapter we will show that $T^*_{\tilde{\varphi}} \in B_1(\Omega^*)$, where $\Omega^* = \{\overline{w} : w \in \Omega\}$ and that closure of Ω^* is a spectral set for T^*_{φ}. Then we obtain a generalization of the Curvature Inequality Theorem. More specifically, we will show that if $T \in B_1(\Omega^*)$ and closure of Ω^* is a spectral set for T, then $K_T(w) \leq K_{T^*_{\tilde{\varphi}}}(w)$, $w \in \Omega^*$. Then we show that $T^*_{\tilde{\varphi}}$ is same as $\Phi(U^*)$, where $\Phi : D \to \Omega^*$ is the Riemann map given by $\Phi(z) = \overline{\varphi(\overline{z})}$. Then we go on to show that if Ω_1 and Ω_2 are simply-connected bounded open subsets of $\mathbb{C}$ with Jordan curve boundaries and $\varphi : \Omega_1 \to \Omega_2$ is a Riemann map and if $T \in B_1(\Omega_1)$ such that closure of Ω_1 is a spectral set for T, then $\varphi(T)$ is defined and closure of Ω_2 is a spectral set for $\varphi(T)$. Furthermore, $\varphi(T) \in B_1(\Omega_2)$. Then we deduce a relation between the curvatures K_T and $K_{\varphi(T)}$.

First we begin with a simple lemma.

Lemma 2.1. Let Ω and φ be as described above. And let $w_0 \in \Omega$, then there exist $\varepsilon > 0$ and $0 < \delta < 1$ such that $|\varphi(z) - w_0| \geq \varepsilon$ whenever $1 - \delta < |z| < 1$.

Proof. Let $\psi : \Omega \to D$ be the inverse map of φ. And let $\psi(w_0) = z_0$. Then consider the function $\Psi(w) = \psi(w) - z_0$. Then clearly $\Psi(w_0) = 0$ and thus by the continuity of Ψ we have a, b > 0 such that $|\Psi(w)| < b$ whenever $|w-w_0| < a$. In other words, $|w-w_0| \geq a$ whenever $|\Psi(w)| \geq b$ or $|\psi(w) - z_0| \geq b$. Writing $\varphi(z) = w$ we have $|\varphi(z) - w_0| \geq a$ whenever $|z - z_0| \geq b$. Let $E = \{z : |z - z_0| < b\}$ and set $\delta_1 = \sup_{z \in E}|z|$. Let $\varepsilon = a$ and $\delta = 1 - \delta_1$. Then $0 < \delta < 1$ and $1 - \delta < |z| < 1$ implies $z \notin E$ or, $|z - z_0| \geq b$ which in turn implies that $|\varphi(z) - w_0| \geq \varepsilon$. And this completes the proof of the lemma. ∎

Next, we state a theorem and refer to the book of Douglas [7] for its proof.

<u>Theorem 2.2.</u> Let $\tilde{\varphi} \in H^{\infty}(T)$, then $T_{\tilde{\varphi}}$ is a Fredholm operator if and only if there exist $\varepsilon > 0$ and $0 < \delta < 1$ such that $1 - \delta < |z| < 1$ implies $|\varphi(z)| \geq \varepsilon$ where φ $(= \hat{\tilde{\varphi}})$ is the Poisson integral of $\tilde{\varphi}$. Moreover, the Fredholm index of $T_{\tilde{\varphi}}$ equals the negative of the winding number of the curve $\varphi(re^{i\theta})$ around the origin for $1 - \delta < r < 1$.

<u>Lemma 2.3.</u> Let Ω and φ be as described above and $w_0 \in \Omega$. Then $(T_{\tilde{\varphi}} - w_0)$ is Fredholm and $j(T_{\tilde{\varphi}} - w_0) = -1$, where j denotes the Fredholm index.

<u>Proof.</u> First we note that $(\tilde{\varphi} - w_0) \in H^{\infty}(T)$ and the Poisson integral of $(\tilde{\varphi} - w_0)$ is $\varphi - w_0$. Since $(\varphi - w_0)$ is conformal, its winding number around the origin is 1. Now from Lemma 2.1 and Theorem 2.2 we obtain that $(T_{\tilde{\varphi}} - w_0)$ is Fredholm and $j(T_{\tilde{\varphi}} - w_0) = -1$. ∎

<u>Lemma 2.4.</u> Let $z_0 \in D$ and $f \neq 0 \in H^2(T)$ such that $U^* f = z_0 f$. Then $T^*_{\tilde{\varphi}} f = \overline{\varphi(\bar{z}_0)}\, f$.

<u>Proof.</u> Consider the holomorphic cross-section $\gamma(z) = (1, z, z^2, \ldots)$ of E_{U^*}, then $f = c\,\gamma(z_0)$, for some constant c. Let $g \in H^2(T)$, then clearly $\langle g, \gamma(\bar{z}_0) \rangle = \hat{g}(z_0)$ where $\hat{g}$ denotes the harmonic extension of g to the open unit disc. So, we have

$$\left\langle g, T^*_{\tilde{\varphi}}\gamma(\bar{z}_0)\right\rangle = \left\langle T_{\tilde{\varphi}}g, \gamma(\bar{z}_0)\right\rangle = \left\langle \tilde{\varphi}g, \gamma(\bar{z}_0)\right\rangle$$

$$= \hat{\tilde{\varphi}g}(z_0) = \varphi(z_0)\hat{g}(z_0) = \varphi(z_0)\left\langle g, \gamma(\bar{z}_0)\right\rangle$$

$$= \left\langle g, \overline{\varphi(z_0)}\gamma(\bar{z}_0)\right\rangle$$

and hence

$$T^*_{\tilde{\varphi}}\gamma(\bar{z}_0) = \overline{\varphi(z_0)}\gamma(\bar{z}_0)$$

or
$$T^*_{\tilde{\varphi}}\gamma(z_0) = \overline{\varphi(\bar{z}_0)}\gamma(z_0)$$

or
$$T^*_{\tilde{\varphi}}f = \overline{\varphi(\bar{z}_0)}f$$

<u>Theorem 2.5.</u> $T^*_{\tilde{\varphi}} \in B_1(\Omega^*)$

<u>Proof.</u> Let $\bar{w} \in \Omega^*$. Then by Lemma 2.3, $(T_{\tilde{\varphi}}-w)$ is Fredholm and $j(T_{\tilde{\varphi}}-w) = -1$. From Lemma 2.4, we see that $\ker(T_{\tilde{\varphi}}-w)^*$ is nonempty, so by F and M Riesz Theorem $\ker(T_{\tilde{\varphi}}-w) = \{0\}$. Hence

$$\dim \ker(T^*_{\tilde{\varphi}}-\bar{w}) = 1 \qquad - (2.5.1)$$

and $(T^*_{\tilde{\varphi}}-\bar{w})$ has dense range. But since $(T_{\tilde{\varphi}}-w)$ is Fredholm, it follows that

$$\text{Rng}\,(T^*_{\tilde{\varphi}}-\bar{w}) = H^2(T) \qquad - (2.5.2)$$

From Lemma 2.4 we have

$$\ker(U^*-z) \subset \ker(T^*_{\varphi}-\overline{\varphi(\bar{z})}), \; z \in D.$$

But both the spaces have dimension equal to one. Thus we have

$$\ker(U^*-z) = \ker(T^*_{\tilde{\varphi}}-\overline{\varphi(\overline{z})}).$$

Hence,

$$\vee_{\overline{w}\in\Omega} \ker(T^*_{\tilde{\varphi}}-\overline{w}) = \vee_{z\in D} \ker(U^*-z) = H^2(T)$$

or, $\vee_{\overline{w}\in\Omega^*} \ker(T^*_{\tilde{\varphi}}-\overline{w}) = H^2(T) \quad - \quad (2.5.3)$

And this completes the proof. ∎

<u>Theorem 2.6</u>. Closure of Ω^* is a spectral set for $T^*_{\tilde{\varphi}}$.

<u>Proof</u>. Let $R : \Omega^* \to \mathbb{C}$ be a rational function with poles off the closure of Ω^*.

Consider the function

$$S : \Omega \to \mathbb{C} \text{ defined by}$$
$$S(w) = \overline{R(\overline{w})}.$$

Clearly S is a rational function with poles off the closure of Ω. Let

$$S(w) = \frac{a_K(w-w_1)(w-w_2)\ldots(w-w_K)}{a_\ell(w-w_1')(w-w_2')\ldots(w-w_\ell')}$$

where $w_i' \notin \operatorname{clos}(\Omega)$ $1 \leq i \leq \ell$. By Wintner Theorem, we have

$$\sigma(T_{\tilde{\varphi}}) = \operatorname{clos}(\hat{\tilde{\varphi}}(D)) = \operatorname{clos}(\varphi(D)) = \operatorname{clos}(\Omega)$$

hence $(T_{\tilde{\varphi}}-w_i')$ is invertible for $1 \leq i \leq \ell$ and $S(T_{\tilde{\varphi}})$ is defined.

Next, we note that $S(T_{\tilde{\varphi}}) = T_{\widetilde{S\circ\varphi}}$, where the Poisson integral of $\widetilde{S\circ\varphi}$ is $S\circ\varphi$. Thus, we have

$$\|S(T_{\tilde{\varphi}})\| = \|T_{\widetilde{S\circ\varphi}}\| \leq \|\widetilde{S\circ\varphi}\|_\infty \leq \|S\circ\varphi\|_\infty$$

$$\leq \|S\|_\infty = \|R\|_\infty .$$

Since,

$$R(T^*_{\tilde{\varphi}}) = S(T_{\tilde{\varphi}})^*$$

we have

$$\|R(T^*_{\tilde{\varphi}})\| \leq \|R\|_\infty .$$

Hence closure of Ω^* is a spectral set for $T^*_{\tilde{\varphi}}$. ∎

<u>Theorem 2.7.</u>

$$K_{T^*_{\tilde{\varphi}}}(w) = |\underline{\Psi}'(w)|^2 K_{U^*}(\underline{\psi}(w)), \quad w \in \Omega^*$$

where $\underline{\Psi}(w) = \overline{\varphi^{-1}(\overline{w})}$

<u>Proof.</u> Let $\gamma(z)$ be a nonzero holomorphic cross-section for U^*. In Theorem 2.5 we have shown that

$$\ker(U^*-z) = \ker(T^*_{\tilde{\varphi}}-\overline{\varphi(\overline{z})}), \quad z \in D.$$

So, $\tilde{\gamma}(w) = \gamma(\psi(w))$ is a nonzero holomorphic cross-section for $T^*_{\tilde{\varphi}}$.

Now by Theorem 1.5, we have

$$K_{T^*_{\tilde{\varphi}}}(w) = -\frac{\partial^2}{\partial\bar{w}\partial w}\operatorname{Ln}\|\gamma(\bar{\Psi}(w))\|^2$$

$$= -\frac{\partial}{\partial\bar{w}}\frac{\langle\bar{\Psi}'(w)\gamma'(\bar{\Psi}(w)),\gamma(\bar{\Psi}(w))\rangle}{\langle\gamma(\bar{\Psi}(w)),\gamma(\bar{\Psi}(w))\rangle}$$

(By chain rule)

$$= -\frac{\|\gamma(\bar{\Psi}(w))\|^2\|\bar{\Psi}'(w)\gamma'(\bar{\Psi}(w))\|^2-|\langle\bar{\Psi}'(w)\gamma'(\bar{\Psi}(w)),\gamma(\bar{\Psi}(w))\rangle|^2}{\|\gamma(\bar{\Psi}(w))\|^4}$$

$$= |\bar{\Psi}'(w)|^2\frac{|\langle\gamma'(\bar{\Psi}(w),\gamma(\bar{\Psi}(w))\rangle|^2-\|\gamma(\bar{\Psi}(w))\|^2\|\gamma'(\bar{\Psi}(w))\|^2}{\|\gamma(\bar{\Psi}(w))\|^4}$$

$$= |\bar{\Psi}'(w)|^2 K_{U^*}(\bar{\Psi}(w))$$ ∎

<u>Theorem 2.8.</u> Let $T \in B_1(\Omega^*)$ and closure of Ω^* is a spectral set for T, then

$$K_T(w) \leq K_{T^*_{\tilde{\varphi}}}(w), \quad w \in \Omega^*.$$

<u>Proof.</u> Consider the function

$\bar{\Psi} : \Omega^* \to D$ defined by

$\bar{\Psi}(w) = \overline{\varphi^{-1}(\bar{w})}$, where

$\varphi : D \to \Omega$ is the Riemann map.

Then $\overline{\Psi} \in H^\infty(\Omega^*)$. Now since Ω^* is simply-connected and is a spectral set for T, the map $\Gamma_T : (T)' \to H^\infty(\Omega^*)$ is an isometric isomorphism by Theorem 1.8. Let

$$\overline{\Psi}(T) = \Gamma_T^{-1}(\overline{\Psi}), \text{ then}$$

$$\|\overline{\Psi}(T)\| = \|\overline{\Psi}\|_\infty \leq 1$$

Next we claim that

$$\overline{\Psi}(T) \in B_1(D).$$

Let $z_0 \in D$ and $\overline{\Psi}(w_0) = z_0$. Then $\overline{\Psi}(w) - z_0$ is a holomorphic function and has a zero at w_0. Thus $\overline{\Psi}(w) - z_0 = (w-w_0)P(w)$, where $P(w) \in H^\infty(\Omega^*)$. But since Ω^* has Jordan curve boundary, the Riemann map $\overline{\Psi} : \Omega^* \to D$ extends homeomorphically to a map from the closure of Ω^* onto the closure of D, [14]. Hence $\overline{\Psi}(w) - z_0$ is not zero at any point inside closure of Ω^* except at w_0. Thus $P(w)$ is invertible in $H^\infty(\Omega^*)$. Again by Theorem 1.8, $P(T)$ is defined and invertible. And we have

$$\overline{\Psi}(T) - z_0 = (T-w_0)P(T), \; P(T) \text{ invertible.}$$

Thus,

$$\ker(\overline{\Psi}(T)-z_0) = \ker(T-w_0) \qquad - \quad (2.8.1)$$

and $\quad Rng\,(\Psi(T)-z_0) = H \quad - \quad (2.8.2).$

And from (2.8.1) we have

$$\bigvee_{z\in D} \ker(\Psi(T)-z) = \bigvee_{w\in\Omega^*} (T-w) = H \quad - \quad (2.8.3).$$

So, $\quad \Psi(T) \in B_1(D).$

As $\Psi(T)$ is a contraction, closure of D is a spectral set for $\Psi(T)$. Hence by Theorem 1.9, we have

$$K_{\Psi(T)}(z) \leq K_{U^*}(z), \quad z \in D \quad - \quad (2.8.4).$$

Next let $\gamma(z)$ be a nonzero holomorphic cross-section for $\psi(T)$, then from (2.8.1), $\tilde{\gamma}(w) = \gamma(\Psi(w))$ is a nonzero holomorphic cross-section for T. Hence by Theorem 1.5, we have

$$K_T(w) = -\frac{\partial^2}{\partial\bar{w}\partial w} \operatorname{Ln}\|\gamma(\Psi(w))\|^2$$

$$= |\Psi'(w)|^2 K_{\Psi(T)}(\Psi(w))$$

(By chain rule)

$$\leq |\Psi'(w)|^2 K_{U^*}(\Psi(w)). \quad \text{(From (2.8.4))}$$

$$= K_{T^*_{\tilde{\varphi}}}(w) \quad \text{(By Theorem 2.7.)}$$

Thus,

$$K_T(w) \leq K_{T^*_{\tilde{\varphi}}}(w), \quad w \in \Omega^*. \qquad \blacksquare$$

Theorem 2.9. $\underline{\Psi}(T^*_{\tilde{\varphi}}) = U^*$ and $\Phi(U^*) = T^*_{\tilde{\varphi}}$, where

$\Phi(z) = \overline{\varphi(\overline{z})}$ and $\underline{\Psi}(w) = \overline{\varphi^{-1}(\overline{w})}$.

Proof. Ω^* is simply-connected and closure of Ω^* is a spectral set for $T^*_{\tilde{\varphi}}$. Hence by Theorem 1.8,

$\Gamma_{T^*_{\tilde{\varphi}}} : (T^*_{\tilde{\varphi}})' \to H^\infty(\Omega^*)$ is onto. So $\underline{\Psi}(T^*_{\tilde{\varphi}}) = \Gamma^{-1}_{T^*_{\tilde{\varphi}}}(\psi)$ is defined.

And for similar reason $\Phi(U^*)$ is defined.

Let $\gamma(z)$ be a nonzero holomorphic cross-section for U^*. Then by Theorem 2.5, $\tilde{\gamma}(w) = \gamma(\underline{\Psi}(w))$ is a nonzero holomorphic cross-section for $T^*_{\tilde{\varphi}}$. Hence

$$\begin{aligned} \underline{\Psi}(T^*_{\tilde{\varphi}})\gamma(z) &= \underline{\Psi}(T^*_{\tilde{\varphi}})\gamma(\underline{\Psi}(w)) \\ &= \underline{\Psi}(T^*_{\tilde{\varphi}})\tilde{\gamma}(w) \\ &= \underline{\Psi}(w)\tilde{\gamma}(w) \\ &= \underline{\Psi}(w)\gamma(\underline{\Psi}(w)) \\ &= z\gamma(z) = U^*\gamma(z). \end{aligned}$$

Thus, $\underline{\Psi}(T^*_{\tilde{\varphi}}) = U^*$.

Similarly,

$$T^*_{\tilde{\varphi}}\gamma(z) = T^*_{\tilde{\varphi}}\gamma(\psi(w)) = T^*_{\tilde{\varphi}}\tilde{\gamma}(w) = w\,\tilde{\gamma}(w)$$

$$= \Phi(z)\Upsilon(z) = \Phi(U^*)\Upsilon(z).$$

Thus,

$$T^*_{\tilde{\varphi}} = \Phi(U^*)$$

In the above, we have assumed that $\Psi(w) = z$ and and $w = \Phi(z)$.

∎

Theorem 2.10. Let Ω_1 and Ω_2 be two simply-connected bounded open subsets of the complex plane $\mathbb{C}$, with Jordan curve boundaries. And let $\varphi : \Omega_1 \to \Omega_2$ be a Riemann map. Then if $T \in B_1(\Omega_1)$ and closure of Ω_1 is a spectral set for T, then $\varphi(T)$ is defined and $\varphi(T) \in B_1(\Omega_2)$. Furthermore, closure of Ω_2 is a spectral set for $\varphi(T)$ and the following curvature relation holds.

$$K_{\varphi(T)}(w) = |\Psi'(w)|^2 K_T(\Psi(w)), \quad w \in \Omega_2$$

$$\text{where } \psi(w) = \varphi^{-1}(w).$$

Proof. Since Ω_2 is bounded, we have $\varphi \in H^\infty(\Omega_1)$. Again, since Ω_1 is simply-connected and closure of Ω_1 is a spectral set for T, $\Gamma_T : (T)' \to H^\infty(\Omega_1)$ is onto. Thus $\varphi(T) = \Gamma_T^{-1}(\varphi)$ is defined.

Let $w_0 \in \Omega_2$ and $\varphi(z_0) = w_0$, $z_0 \in \Omega_1$. Then $\varphi(z) - w_0 = (z-z_0)P(z)$, where $P \in H^\infty(\Omega_1)$. Since Ω_1 and Ω_2 have

Jordan boundaries, φ can be extended homeomorphically from closure of Ω_1 onto the closure of Ω_2. Thus $\varphi - w_0$ is not zero on closure of Ω_1 except at z_0, and consequently P is not zero on closure of Ω_1. Hence P is invertible in $H^\infty(\Omega_1)$. So P(T) is defined and invertible and we have

$$\varphi(T) - w_0 = (T-z_0)P(T),\ P(T) \text{ invertible.}$$

Hence,

$$\ker(\varphi(T)-w_0) \quad = \quad \ker(T-z_0) \qquad - \quad (2.10.1).$$

and $\quad Rng\ (\varphi(T)-w_0) = H \quad - \quad (2.10.2)$

From (2.10.1) we have

$$\vee_{w\in\Omega_2} \ker(\varphi(T)-w) = \vee_{z\in\Omega_1} \ker(T-z) = H \quad - \quad (2.10.3).$$

Thus,

$$\varphi(T) \in B_1(\Omega_2).$$

To prove that closure of Ω_2 is a spectral set for $\varphi(T)$, let $R : \Omega_2 \to \mathbb{C}$ be a rational function with poles off the closure of Ω_2. Then $(R\circ\varphi) \in H^\infty(\Omega_1)$ and $R\circ\varphi(T)$ is defined. And we have

$$\|R\circ\varphi(T)\| = \|R\circ\varphi\|_\infty \le \|R\|_\infty$$

or, $\quad \|R(\varphi(T))\| \le \|R\|_\infty.$

Finally, let $\gamma(z)$ be a nonzero holomorphic cross-section for T, then from (2.10.1) $\tilde{\gamma}(w) = \gamma(\varphi^{-1}(w)) = \gamma(\Psi(w))$ is a nonzero holomorphic cross-section for $\varphi(T)$. Hence by Theorem 1.5, we have

$$K_{\varphi(T)}(w) = -\frac{\partial^2}{\partial\bar{w}\partial w} \operatorname{Ln}\|\tilde{\gamma}(w)\|^2$$

$$= -\frac{\partial^2}{\partial\bar{w}\partial w} \operatorname{Ln}\|\gamma(\Psi(w))\|^2$$

$$= |\Psi'(w)|^2 K_T(\Psi(w)). \qquad \text{(By chain rule)}$$

This completes the proof. ∎

Note that in the above theorem, we could have taken Ω_1 and Ω_2 finitely-connected instead of simply-connected as long as they are conformally equivalent via the Riemann map $\varphi : \Omega_1 \to \Omega_2$.

Lastly, after we had obtained the results on curvature inequality, Misra [10] has worked on it independently and has provided a different proof of Theorem 2.8.

CHAPTER III

Co-subnormal Operators Quasi-similar to U^* and Their Invariant Subspaces

Let $T \in B_n(\Omega)$ and S be similar to T. Then it is easy to show that $S \in B_n(\Omega)$. But it is not known if the same thing can be said when the similarity is replaced by quasi-similarity [11]. The problem lies in whether the approximate point spectrum [8] of S^* is same as that of T^*. Recently Raphael [13] has shown that quasi-similar cyclic subnormal operators have the same approximate point spectrum. Using this result we show that if T^* is cyclic subnormal and $T \in B_n(\Omega)$ then co-subnormal operators quasi-similar to T belong to $B_n(\Omega)$. And as a corollary to this we deduce that co-subnormal operators quasi-similar to U^* belong to $B_1(D)$.

Later in the chapter we consider an invariant subspace problem. In [6], Douglas has shown that if $T \in B_n(\Omega)$ and M is a finite co-dimensional invariant subspace of T^*, then compression of T to M, $P_M T|M$, where P_M is the projection operator onto M, belongs to $B_n(\Omega)$. The finite co-dimensionality of M plays a key role in the technique of his proof. For M of arbitrary dimension, this is still a conjecture. In this chapter, we consider co-subnormal operators T quasi-similar to U^* and consider invariant subspace M of T^* of arbitrary dimension, and show that $P_M T|M \in B_1(D)$. It is known that in the case of U^*, $P_M U^*|M$ is unitarily equivalent to U^* for any invariant subspace M of U. Though, however, the same is not true for co-subnormal operators quasi-similar to U^*, we have been able to show that $P_M T|M$ is similar to T in the finite co-dimensional case and quasi-affine transform of T for M of arbitrary dimension.

<u>Theorem 3.1</u>. Let T^* be cyclic subnormal and $T \in B_n(\Omega)$ and S quasi-similar to T, then $S \in B_n(\Omega)$.

<u>Proof</u>. Let A and B be quasi-invertible operators (one-to-one and dense range) such that

$$AT = SA \quad \text{and} \quad TB = BS.$$

So, if $w \in \Omega$, then we have

$$A(T-w) = (S-w)A \quad \text{and} \quad (T-w)B = B(S-w).$$

Clearly, $A\ker(T-w) \subset \ker(S-w)$ as

$$x \in \ker(T-w)$$

$$\Rightarrow \quad (T-w)x = 0$$

$$\Rightarrow \quad A(T-w)x = 0$$

$$\Rightarrow \quad (S-w)Ax = 0$$

$$\Rightarrow \quad Ax \in \ker(T-w)$$

But, A is one-to-one, so we have

$$\dim \ker(T-w) \le \dim \ker(S-w)$$

similarly, we have

$$\dim \ker(S-w) \le \dim \ker(T-w).$$

Thus,

$$\dim \ker(S-w) = \dim \ker(T-w) = n, \quad w \in \Omega \quad - \quad (3.1.1)$$

and in the same way as above we can also show

$$\dim \ker(S-w)^* = \dim \ker(T-w)^* = 0$$

and this shows that $(S-w)$ has dense range.

Due to Raphael [13], we have the approximate point spectrum of S^* is same as that of T^*.

Hence if $w_0 \in \Omega$, then $(S-w_0)^*$ is bounded from below as $(T-w_0)^*$ is bounded from below. Thus, $(S-w_0)^*$ has closed range which in turn implies that $(S-w_0)$ has closed range. This together with the fact that $(S-w_0)$ has dense range implies that $(S-w_0)$ is onto. Thus we have

$$\text{Rng}\,(S-w) = H, \quad w \in \Omega \quad - \quad (3.1.2)$$

To complete the proof, we observe that

$$A \ker(T-w) = \ker(S-w).$$

Thus,

$$AH = A \vee_{w\in\Omega} \ker(T-w) \subset \vee_{w\in\Omega} A \ker(T-w)$$

$$= \vee_{w\in\Omega} \ker(S-w).$$

But AH is dense in H, hence

$$\vee_{w\in\Omega} \ker(S-w) = H \quad - \quad (3.1.3)$$

and we conclude that $S \in B_n(\Omega)$. ∎

Corollary 3.2. Co-subnormal operators quasi-similar to U^* belong to $B_1(D)$.

Let T be co-subnormal and quasi-similar to U^* and M be an invariant subspace of T^*. According to the classification due to Clary [1], $M^\perp$, the orthogonal complement of M, takes the form as $\ker \varphi(T)$, where φ is an inner-function in $H^\infty(D)$. Next, we use this fact to show that $P_M T|M \in B_1(D)$.

Theorem 3.3. Let T be co-subnormal quasi-similar to U^* and M be a nontrivial invariant subspace of T^*, then $P_M T|M \in B_1(D)$.

Proof. Let

$$M^\perp = \ker \varphi(T),\ \varphi \in H^\infty(D) \text{ is inner.}$$

Define

$$A : M \to H \text{ by}$$

$$AX = \varphi(T)x$$

then A is clearly one-to-one and has dense range. To show that A is one-to-one, let $x \in M$ and $Ax = 0$. Then

$$\varphi(T)x = 0 \Rightarrow x \in \ker \varphi(T) \Rightarrow x \in M^{\perp}$$

and $x \in M \cap M^{\perp} \Rightarrow x = 0$.

To show that A has dense range, let $\gamma(w)$ be a nonzero holomorphic cross-section for E_T. Suppose $x \perp \mathrm{Rng}\, \varphi(T)$. Then

$$\langle \varphi(T)\gamma(w), x\rangle = 0 \qquad \forall\, w \in D$$

$$\Rightarrow \langle \varphi(w)\gamma(w), x\rangle = 0 \qquad \forall\, w \in D$$

$$\Rightarrow \langle \gamma(w), x\rangle = 0 \qquad \forall\, w \in D, \text{ as } \varphi \not\equiv 0$$

$$\Rightarrow x = 0, \text{ since } \bigvee_{w \in D} \gamma(w) = H.$$

Thus $\varphi(T)$ has dense range and consequently A has dense range.

Next, we observe that

$$A\, P_M T|M = TA\ .$$

Let $w_0 \in D$ and $x \in \ker(P_M T|M - w_0)$, then

$$(P_M T|M - w_0)x = 0$$

$$\Rightarrow A(P_M T|M - w_0)x = 0$$

$$\Rightarrow (T - w_0)Ax = 0 \Rightarrow Ax \in \ker(T - w_0).$$

Thus $A \ker(P_M T|M-w_0) \subset \ker(T-w_0)$. As A is one-to-one, we have

$$\dim \ker(P_M T|M-w_0) \leq \dim \ker(T-w_0) = 1.$$

Next, we will show that $\ker(P_M T|M-w_0)$ is nonempty from which it will follow that $\dim \ker(P_M T|M-w_0) = 1$.

Let $\gamma^{(P)}(w_0)$ denote Pth derivative of $\gamma(w)$ at $w = w_0$.

We claim that $\gamma^{(P)}(w_0)$ cannot belong to $M^{\perp}$ for all nonnegative integers P.

To show this, we observe that, in a small neighborhood W of w_0 in D, $\gamma(w)$ can be expanded as a Taylor series as follows:

$$\gamma(w) = \gamma(w_0) + (w-w_0)\frac{\gamma^{(1)}(w_0)}{1!} + (w-w_0)^2\frac{\gamma^{(2)}(w_0)}{2!} + \cdots$$

If $\gamma^{(P)}(w_0) \in M^{\perp}$ for all nonnegative integers then $\gamma(w) \in M^{\perp}$ in a small neighborhood of w_0. And it follows that $M^{\perp} = H$ contradicting the fact that M is nontrivial.

Thus, there exists a smallest nonnegative integer P_{w_0} such that $\gamma^{(P_{w_0})}(w_0) \notin M^{\perp}$.

Denote $$P_M \gamma^{(P_{w_0})}(w_0) = \gamma_M^{(P_{w_0})}(w_0)$$

Clearly, $\gamma_M^{(P_{w_0})}(w_0) \neq 0$ and

$$(P_M T|M-w_0)\gamma_M^{(P_{w_0})}(w_0) = 0$$

for $(T-w_0)\gamma^{(P_{w_0})}(w_0) = P_{w_0}\gamma^{(P_{w_0}-1)}(w_0)$, for reference see Douglas [3]. Hence,

$$(T-w_0)\gamma_M^{(P_{w_0})}(w_0) = -(T-w_0)\gamma_{M^\perp}^{(P_{w_0})}(w_0) + P_{w_0}\gamma^{(P_{w_0}-1)}(w_0)$$

since $M^\perp$ is invariant under T and P_{w_0} is the smallest non-negative integer such that $\gamma^{P_{w_0}}(w_0) \notin M^\perp$, we have $\gamma^{(P_{w_0}-1)}(w_0) \in M^\perp$. Also, $(T-w_0)\gamma_{M^\perp}^{(P_{w_0})}(w_0) \in M^\perp$. Hence

$$(T-w_0)\gamma_M^{(P_{w_0})}(w_0) \in M^\perp$$

and hence $\gamma_M^{(P_{w_0})}(w_0) \in \ker\,(P_M T|_M - w_0)$. So, we have proved that

$$\dim \ker(P_M T|M-w_0) = 1 \quad - \quad (3.3.1)$$

w_0 is arbitrary point in D, so we have

$$\dim \ker(P_M T|M-w) = 1, \quad w \in D \quad - \quad (3.3.1)$$

For some $w \in D$, $\gamma_M(w)$ may be zero, but $\gamma_M(w) \in \ker(P_M T|M-w)$.

So, to show that $\ker(P_M T|M-w)$ span M, let $x \in M$ and $x \perp \gamma_M(w)$, $w \in D$. Then $x \perp \gamma(w)$, $w \in D$ and consequently $x = 0$ since $\gamma(w)$, $w \in D$ span H. Thus

$$\bigvee_{w \in D} \ker(P_M T|M-w) = M \quad - \quad (3.3.2).$$

To prove that $(P_M T|M-w)$ is onto, let $x \in M$ and let $x = (T-w)y$ for some $y \in H$. We can choose y as $(T-w)$ is onto. Let $y = y_M + y_{M^\perp}$. Then

$$(T-w)y_M + (T-w)y_{M^\perp} = x$$

or, $P_M(T-w)y_M = x$, since $(T-w)y_{M^\perp} \in M^\perp$. Thus

$$(P_M T|M-w)y_M = x$$

and $(P_M T|M-w)$ is onto for $w \in D$. This completes the proof of the theorem. ∎

Note that in the proof of the above theorem, we first showed that $P_M T|M$ is a quasi-affine transform of T, from which the rest of the proof followed. As a matter of fact, whenever $P_M T|M$ is a quasi-affine transform of $T \in B_1(\Omega)$, we have $P_M T|M \in B_1(\Omega)$, and in this case $M^\perp = \ker \varphi(T)$, for some $\varphi \in H^{\infty}(\Omega)$.

Theorem 3.4. Let $T \in B_1(\Omega)$ and M be an invariant subspace of T^*. Then $P_M T|M$ is a quasi-affine transform of T if and only if $M^{\perp} = \ker \varphi(T)$ for some $\varphi \in H^{\infty}(\Omega)$. And, in this case $P_M T|M \in B_1(\Omega)$.

Proof. We need only show that if $P_M T|M$ is a quasi-affine transform of T then $M^{\perp} = \ker \varphi(T)$ for some $\varphi \in H^{\infty}(\Omega)$. The rest of the result is contained in the proof of Theorem 3.3.

Let

$$A : M \to H \text{ be quasi-invertible}$$

such that

$$A\, P_M T|M = TA\ .$$

Define

$$\tilde{A} : H \to H \text{ by}$$

$$\begin{aligned} \tilde{A}x &= Ax \qquad x \in M \\ &= 0 \qquad x \in M^{\perp} \end{aligned}$$

then $\tilde{A}$ is bounded and commutes with T.

Let $\tilde{A} = \varphi(T)$ for some $\varphi \in H^{\infty}(\Omega)$, then clearly $M^{\perp} = \ker \varphi(T)$.

∎

Theorem 3.5. Let $T \in B_n(\Omega)$ such that the point spectrum

$\sigma_p(T)$ of T exactly equals to Ω and M be a finite co-dimensional invariant subspace of T^*, then $P_M T|M$ is similar to T and consequently $P_M T|M \in B_n(D)$.

Proof. We first observe that the orthogonal complement $M^\perp$ is an invariant subspace of T.

Now, consider the restriction of T on $M^\perp$,

$$T : M^\perp \to M^\perp.$$

Let $P(X) = (X-\lambda_1)^{n_1}(X-\lambda_2)^{n_2}\ldots(X-\lambda_K)^{n_K}$ be the characteristic polynomial of $T|M^\perp$. Clearly, $n_1 + n_2+\ldots+n_K = \dim M^\perp$ and $\lambda_i \in \Omega$ $(1 \leq i \leq K)$.

Consider the operator $P(T)$ on H. We claim that $\ker P(T) = M^\perp$.

First, we note that for $1 \leq i \leq K$, $(T-\lambda_i)$ is a Fredholm operator and the Fredholm index $j(T-\lambda_i) = +1$. Hence

$$j(T-\lambda_i)^{n_i} = n_i \quad \text{and consequently}$$

$$j(P(t) = n_1 + n_2 + \ldots + n_K = \dim M^\perp$$

since $\ker(T-\lambda_i)^* = \{0\}$ for $1 \leq i \leq K$, we have $\ker P(T)^* = \{0\}$. So,

$$\dim \ker(P(T)) = \dim M^\perp$$

but $M^\perp \subset \ker P(T)$ as $P(X)$ is the characteristic polynomial of $T|M^{\perp}$. Hence

$$\ker P(T) = M^{\perp}.$$

Consider the operator

$$A : M \to H \text{ defined by}$$

$$Ax = P(T)x.$$

Clearly, A is a bounded operator, we claim that A is invertible and

$$A\, P_M T|M = TA.$$

To prove that A is invertible let $x \in M$ and $Ax = 0$. Then

$$P(T)x = 0 \Rightarrow x \in \ker P(T) = M^{\perp}$$

$$\Rightarrow x \in M \cap M^{\perp} \Rightarrow x = 0.$$

So, A is one-to-one.

Next, since each $(T-\lambda_i)$ is onto, $(T-\lambda_i)^{n_i}$ is onto and consequently, $P(T)$ is onto. Thus for $x \in H$ we can get $y \in H$ such that $P(T)y = x$. Let $y = y_M + y_{M^{\perp}}$. Then

$$P(T)y_M + P(T)y_{M^{\perp}} = x .$$

But $P(T)y_{M^{\perp}} = 0$ as $\ker P(T) = M^{\perp}$. So

$$P(T)y_M = x$$

$$\text{or, } Ay_M = x \quad \text{and } A \text{ is onto.}$$

And by open map theorem A is invertible. To prove that

$$A\, P_M T|M = TA$$

let $x \in M$, then

$$\begin{aligned} TAx = TP(T)x = P(T)Tx &= P(T)[(Tx)_M + (Tx)_{M^\perp}] \\ &= P(T)(Tx)_M = A(Tx)_M \\ &= A\, P_M T|M\, x\,. \end{aligned}$$

Thus, T and $P_M T|M$ are similar. After similarity, it will be routine to check $P_M T|M \in B_n(\Omega)$. ∎

Corollary 3.6. Let T be co-subnormal and quasi-similar to U^* and M be a finite co-dimensional invariant subspace of T^*, then $P_M T|M$ is similar to T and consequently $P_M T|M \in B_1(D)$.

Before closing this chapter, we remark that in most of the invariant subspace results we have indirectly assumed that $M^\perp = \ker \varphi(T)$ for some $\varphi \in H^\infty(\Omega)$. We don't know any condition on $T \in B_1(\Omega)$ which will assure us that any invariant subspace of T will be the kernel of some operator commuting with T.

Though we have not been able to prove it, we believe that most of the results may be obtained assuming that closure of Ω is a spectral set for T. The technique certainly will be different since it is unlikely that invariant subspaces will have the above form.

CHAPTER IV

Certain Toeplitz-like Operators in $B_2(D)$

Let X be a bounded operator on $H^2(T)$. Then consider the operator

$$T^X = \begin{pmatrix} U^* & 0 \\ X & U^* \end{pmatrix} \quad \text{On } H^2(T) \oplus H^2(T)$$

It is not very difficult to show that $T^X \in B_2(D)$. But it is rather difficult to determine the necessary and sufficient conditions for two such operators T^X and T^Y to be unitarily equivalent, similar or quasi-similar. In this chapter, we consider only a particular case, namely we take X as the adjoint of analytic Toeplitz operator. Since the adjoint of analytic Toeplitz operator takes the form $\varphi(U^*)$ for some $\varphi \in H^\infty(D)$, we denote T^X for $X = \varphi(U^*)$ as T^φ.

If we consider the two dimensional nilpotent complex matrices

$$T^a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \; a \in \mathbb{C},$$

then it can be easily seen that two such matrices T^a, T^b ($a,b \in \mathbb{C}$) are unitarily equivalent if and only if

$|a| = |b|$ and similar if and only if both a and b are simultaneously zero or nonzero. In finite dimensional case quasi-similarity is equivalent to similarity. As we show later in this chapter, it turns out that two operators T^{φ}, T^{ψ} ($\varphi, \psi \in H^{\infty}(D)$) are unitarily equivalent if and only if $|\varphi(w)| = |\psi(w)|$, $w \in D$ if and only if $\varphi = C\psi$, where C is an unimodular constant. Further we find that T^{φ}, T^{ψ} are similar if and only if there exist constant K_1 and K_2 such that $K_1 \leq |\varphi(w)|/|\psi(s)| \leq K_2$, $w \in D$. However, quasi-similarity doesn't turn out to be equivalent to similarity. We find that T^{φ} and T^{ψ} are quasi-similar if and only if there exist outer functions f and g in $H^{\infty}(D)$ such that $f(w)\varphi(w) = g(w)\psi(w)$, $w \in D$.

At this point we do not know the necessary and sufficient conditions for two general operators T^X, T^Y to be unitarily equivalent, similar or quasi-similar. We do, however, show that if T^X is unitarily equivalent to T^{φ}, then X is the adjoint of an analytic Toeplitz operator and $X = C\varphi(U^*)$ where C is a unimodular constant. Again, we do not know if the same thing can be said if unitary equivalence is replaced by similarity or quasi-similarity equivalence.

Before considering the particular case, we first present the proof that $T^X \in B_2(D)$ for any bounded operator X on $H^2(T)$.

Theorem 4.1. $T^X \in B_2(D)$.

Proof. Let $w \in D$ and suppose $T^X(f,g) = w(f,g)$ for $(f,g) \in H^2(T) \oplus H^2(T)$. Then we have

$$\begin{pmatrix} U^* & 0 \\ X & U^* \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} U^*f \\ Xf + U^*g \end{pmatrix} = \begin{pmatrix} wf \\ wg \end{pmatrix}$$

and equating the terms, we get

$$U^*f = wf \qquad - \quad (4.1.1)$$

and

$$Xf + U^*g = wg \qquad - \quad (4.1.2).$$

To determine f and g, we note that $U^* \in B_1(D)$. Hence (U^*-w) is a Fredholm operator for $w \in D$ and there exists an operator S_w such that

$$(U^*-w)S_w = I \qquad - \quad (4.1.3)$$

and

$$S_w(U^*-w) = I + P_{\ker(U^*-w)} \qquad - \quad (4.1.4).$$

Let $\gamma(w)$ be a nonzero holomorphic cross-section for E_{U^*}. Then from equation (4.1.1) we get

$$f = a\,\gamma(w) \text{ for some constant } a \in \mathbb{C} \qquad - \quad (4.1.5)$$

and from equation (4.1.2) we get

$$(U^*-w)g = -Xf = -a X \gamma(w)$$

Now, applying S_w on the left, we get

$$S_w(U^*-w)g = -a S_w X \gamma(w)$$

or, $[I + P_{\ker(U^*-w)}]g = -a S_w X \gamma(w)$ from equation (4.1.4)

or, $g + b\,\gamma(w) = -a S_w X \gamma(w)$, where b is some constant in $\mathbb{C}$.

So, $g = -a S_w X \gamma(w) - b\,\gamma(w)$ — (4.1.6).

From equations (4.1.5) and (4.1.6) we get

$$(f,g) = a(\gamma(w), -S_w X \gamma(w)) - b(0, \gamma(w))$$

where a and b are some arbitrary constant. Thus $\ker(T^*-w)$ is two dimensional and spanned by $(\gamma(w), -S_w X \gamma(w))$ and $(0, \gamma(w))$ and we have

$$\dim \ker(T^X-w) = 2, \quad w \in D \quad - \quad (4.1.7).$$

To prove that (T^X-w) is onto, let $(f,g) \in H^2(T) \oplus H^2(T)$. Now, since (U^*-w) is onto, there exists $f_1 \in H^2(T)$ such that $(U^*-w)f_1 = f_2$ and there exists $f_2 \in H^2(T)$ such that $(U^*-w)f_2 = -Xf_1$. It is easy to show that

$$(T^X-w)(f_1, g_1) = (f,g)$$

Thus,

$$(T^X-w) \text{ is onto for } w \in D \quad - \quad (4.1.8)$$

Finally, to show that $\ker(T^X - w)$, $w \in D$ span $H^2(T) \oplus H^2(T)$. Let $(f,g) \in H^2(T) \oplus H^2(T)$ and $(f,g) \perp \ker(T^X - w)$ $\forall\, w \in D$. Since $\ker(T^X - w)$ is spanned by $(\gamma(w), -S_w X\gamma(w))$ and $(0, \gamma(w)$ we have

$$(f,g) \perp (\gamma(w), -S_w X\gamma(w) \text{ and } (0,\gamma(w)) \ \forall\, w \in D$$

$(f,g) \perp (0,\gamma(w))$ $\forall\, w \in D$ implies $\langle g, \gamma(w)\rangle = 0$ $\forall\, w \in D$ and since $\gamma(w)$, $w \in D$ span $H^2(T)$ we have $g = 0$. So, $(f,g) \perp (\gamma(w), -S_w X\gamma(w))$, $w \in D \Rightarrow \langle f, \gamma(w)\rangle = 0$ $\forall\, w \in D$ and for the same reason as above, $f = 0$. Thus, $(f,g) = 0$ and we conclude that

$$\vee_{w \in D} \ker(T^X - w) = H^2(T) \oplus H^2(T) \quad - \quad (4.1.9)$$

and consequently $T^X \in B_2(D)$. ∎

Next, we state and prove a very useful lemma, which has been a key for the proof of subsequent results.

<u>Lemma 4.2.</u> Let B be a bounded operator such that the commutator $(U^*B - BU^*)$ commutes with U^*. Then B commutes with U^*.

<u>Proof.</u> Since $(U^*B - BU^*)$ commutes with U^*, we can identify it with $\varphi(U^*)$ for some $\varphi \in H^\infty(D)$ through the isometric isomorphism

$$\Gamma_{U^*} : (U^*)' \to H^\infty(D).$$

So, we have

$$U^*B - BU^* = \varphi(U^*), \quad \varphi \in H^\infty(D).$$

If we define

$$\psi : D \to \mathbb{C} \text{ by}$$

$$\psi(z) = \overline{\varphi(\overline{z})}$$

then ψ is holomorphic and bounded on D. It can be easily shown that $\varphi(U^*) = T^*_{\tilde{\psi}}$. Applying problem 184 of Halmos [8], we obtain that $T^*_{\tilde{\psi}}$ is quasi-nilpotent and thus $T_{\tilde{\psi}}$ is also quasi-nilpotent. But there is no quasi-nilpotent Toeplitz operator other than zero by Corollary 2 of Problem 196 in Halmos [8]. Thus we have $U^*B - BU^* = 0$ and B commutes with U^*.

∎

Next, one by one, we state and prove the results on the necessary and sufficient conditions of unitary equivalence, similarity and quasi-similarity of T^φ, T^ψ, $\varphi, \psi \in H^\infty(D)$.

Theorem 4.3. T^{φ} and T^{ψ}, $\varphi, \psi \in H^{\infty}(D)$ are unitarily equivalent if and only if $|\varphi(w)| = |\psi(w)|$ $\forall$ $w \in D$ if and only if $\varphi = C\psi$, where C is a unimodular constant.

Proof. It is easy to show that the condition $|\varphi(w)| = |\psi(w)|$ $\forall$ $w \in D$ is equivalent to saying $\varphi = C\psi$ for some unimodular constant C.

Suppose, $\varphi = e^{i\Theta}\psi$ for some real number Θ. Then we have

$$\varphi(U^*) = e^{i\Theta}\psi(U^*).$$

Now, consider the matrix W given by

$$W = \begin{pmatrix} e^{-i\frac{\Theta}{2}} I & 0 \\ 0 & e^{i\frac{\Theta}{2}} I \end{pmatrix} \text{ on } H^2(T) \oplus H^2(T).$$

Clearly, W is a unitary operator and $WT^{\psi} = T^{\varphi}W$ as can be seen by

$$\begin{pmatrix} e^{-i\frac{\Theta}{2}} I & 0 \\ 0 & e^{i\frac{\Theta}{2}} I \end{pmatrix} \begin{pmatrix} U^* & 0 \\ \psi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ \varphi(U^*) & U^* \end{pmatrix} \begin{pmatrix} e^{-i\frac{\Theta}{2}} I & 0 \\ 0 & e^{i\frac{\Theta}{2}} I \end{pmatrix}$$

Thus T^{φ} and T^{ψ} are unitarily equivalent.

Conversely, suppose T^{φ} and T^{ψ} are unitarily equivalent and let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{be a unitary operator}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad - (4.3.1)$$

simplifying (4.3.1) we get

$$\begin{pmatrix} AU^* + B\varphi(U^*) & BU^* \\ CU^* + D\varphi(U^*) & DU^* \end{pmatrix} = \begin{pmatrix} U^*A & U^*B \\ \psi(U^*)A+U^*C & \psi(U^*)B + U^*D \end{pmatrix}$$

Now equating the coefficients we get

$$AU^* + B\varphi(U^*) = U^*A \quad - (4.3.2)$$

$$BU^* = U^*B \quad - (4.3.3)$$

$$CU^* + D\varphi(U^*) = \psi(U^*)A + U^*C \quad - (4.3.4)$$

and

$$DU^* = \psi(U^*)B + U^*D \quad - (4.3.5)$$

From (4.3.2), (4.3.3) and (4.3.4), we see that (AU^*-U^*A) and (DU^*-U^*D) commute with U^* and so by Lemma 4.2, we obtain that A and D commute with U^* and $B = O$. Now, from (4.3.4) we obtain that (CU^*-U^*C) commutes with U^* and

$$D\varphi(U^*) = \psi(U^*)A \quad - (4.3.6).$$

Next, since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{is a unitary operator,}$$

we can write (4.3.1) as

$$\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix}$$

As before, we conclude that A^* and D^* commute with U^* and $C^* = O$.

Since A, A^*, D and D^* commute with U^* and the only operators that commute with both U and U^* are scalar operators, we have both A and D are scalar operators. Further, since

$$\begin{pmatrix} A & O \\ O & D \end{pmatrix} \quad \text{is unitary,}$$

we conclude that A and D are unimodular scalar operators. And from (4.3.6) we obtain

$$\varphi(U^*) = A\bar{D}\ \psi(U^*)$$

and hence

$$\varphi(w) = A\bar{D}\ \psi(w), \qquad w \in D$$

and this completes the proof. ∎

<u>Theorem 4.4</u>. T^{φ} and T^{ψ} are similar if and only if there

exist constants K_1 and K_2 such that $K_1 \le |\varphi(w)|/|\psi(w)| \le K_2$ $\forall\, w \in D$.

Proof. Suppose that there exist constants K_1 and K_2 such that the above inequality holds. Define

$$f(w) = \varphi(w)/\psi(w) \quad \text{on } D.$$

Then, clearly $f \in H^\infty(D)$ and invertible in $H^\infty(D)$. So we have

$$\varphi(w) = f(w)\psi(w) \quad \text{and consequently}$$

$$\varphi(U^*) = f(U^*)\psi(U^*),\ f(U^*) \text{ invertible.}$$

Thus,

$$\begin{pmatrix} I & O \\ O & f(U^*) \end{pmatrix} \text{ is an invertible operator}$$

and

$$\begin{pmatrix} I & O \\ O & f(U^*) \end{pmatrix}\begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix}\begin{pmatrix} I & O \\ O & f(U^*) \end{pmatrix}$$

and hence T^φ and T^ψ are similar.

Conversely, suppose T^φ and T^ψ are similar and let

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \text{ be invertible with}$$

inverse $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ and

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

From this, as in Theorem 4.3, we obtain that $B_1 = O$ and A_1, C_1 and D_1 all commute with U^*. Also,

$$\varphi(U^*)A_2 = D_2\psi(U^*) \quad - \quad (4.4.1)$$

Similarly, considering

$$\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix}$$

we have $B_2 = O$ and A_2, C_2 and D all commute with U^*.

Next, since

$$\begin{pmatrix} A_1 & O \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & O \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix} = \begin{pmatrix} A_2 & O \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} A_1 & O \\ C_1 & D_1 \end{pmatrix}$$

we have

$$A_1A_2 = I = A_2A_1$$

and

$$D_1D_2 = I = D_2D_1 \ .$$

Thus, A_2 and D_2 are invertible and from equation (4.4.1) we have

$$\varphi(U^*) = A_1D_2\psi(U^*).$$

Note that A_1D_2 commutes with U^*. We denote A_1D_2 by $f(U^*)$ for some invertible $f \in H^\infty(D)$. So, we have

$$\varphi(U^*) = f(U^*)\psi(U^*)$$

and hence $$\varphi(w) = f(w)\psi(w).$$

Next, since f is invertible there exist constants K_1 and K_2 such that

$$K_1 \leq |f(w)| \leq K_2, \quad w \in D.$$

or $$K_1 \leq |\varphi(w)|/|\psi(w)| \leq K_2 \quad \forall\, w \in D. \qquad \blacksquare$$

<u>Theorem 4.5.</u> T^φ and T^ψ are quasi-similar if and only if there exist outer functions f and g in $H^\infty(D)$ such that $f(w)\varphi(w) = g(w)\psi(w)\ \forall\, w \in D$.

<u>Proof.</u> Suppose there exist outer functions f and g in $H^\infty(D)$ such that

$$f(w)\varphi(w) = g(w)\psi(w) \quad \forall\, w \in D.$$

Then we have

$$f(U^*)\varphi(U^*) = g(U^*)\psi(U^*).$$

We note that $f(U^*)$ and $g(U^*)$ are the adjoint of the multiplication operators M_f and M_g. And since f and g are outer functions, the multiplication operators M_f and M_g have dense range. Thus $f(U^*)$ and $g(U^*)$ are one-to-one. Further, $f(U^*)$ and $g(U^*)$ have dense range since if x is any vector perpendicular to range of $f(U^*)$, then

$$x \perp f(U^*)\gamma(w) = f(w)\gamma(w) \ \forall\, w$$

where $\gamma(w)$ is a nonzero holomorphic cross-section for U^*. And hence $x = 0$.

So, both $f(U^*)$ and $g(U^*)$ are one-to-one and have dense range. In other words, they are quasi-invertible.

Now, consider the operators

$$\begin{pmatrix} f(U^*) & 0 \\ 0 & g(U^*) \end{pmatrix} \text{ and } \begin{pmatrix} g(U^*) & 0 \\ 0 & f(U^*) \end{pmatrix}.$$

Clearly, they are quasi-invertible and

$$\begin{pmatrix} f(U^*) & 0 \\ 0 & g(U^*) \end{pmatrix}\begin{pmatrix} U^* & 0 \\ \psi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ \varphi(U^*) & U^* \end{pmatrix}\begin{pmatrix} f(U^*) & 0 \\ 0 & g(U^*) \end{pmatrix}$$

and

$$\begin{pmatrix} g(U^*) & 0 \\ 0 & f(U^*) \end{pmatrix}\begin{pmatrix} U^* & 0 \\ \varphi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ \psi(U^*) & U^* \end{pmatrix}\begin{pmatrix} g(U^*) & 0 \\ 0 & f(U^*) \end{pmatrix}.$$

Hence, T^{φ} and T^{ψ} are quasi-similar.

Conversely, suppose T^{φ} and T^{ψ} are quasi-similar and let

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \text{ and } \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \text{ be quasi-invertible}$$

operators such that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad - (4.5.1)$$

and

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} U^* & O \\ \psi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \quad - (4.5.2)$$

From (4.5.1) and (4.5.2), as in Theorem 4.3, we obtain that $B_1 = O = B_2$ and A_1, C_1, D_1 and A_2, C_2, D_2 all commute with U^* and also

$$D_1\varphi(U^*) = \psi(U^*)A_1 \quad - (4.5.3)$$

and

$$D_2\psi(U^*) = \varphi(U^*)A_2 \quad - (4.5.4).$$

From (4.5.3) and (4.5.4) we obtain

$$D_1D_2\varphi(U^*) = A_1A_2\varphi(U^*)$$

or

$$D_1D_2 = A_1A_2 \quad - (4.5.3).$$

Next, since

$$\begin{pmatrix} A_i & O \\ C_i & D_i \end{pmatrix} \text{ (i=1,2) are one-to-one,}$$

we have

$$\begin{pmatrix} A_i & 0 \\ C_i & D_i \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ D_i x \end{pmatrix} \neq 0 \quad \text{if} \quad X = 0.$$

Thus D_i $(i=1,2)$ are one-to-one. We claim that A_i $(i=1,2)$ are also one-to-one. For example, if $A_1 x = 0$ then

$$\begin{aligned} A_2 A_1 x = 0 \;&\Rightarrow\; D_1 D_2 x = 0 \text{ , from (4.5.3)} \\ &\Rightarrow\; D_2 x = 0 \text{ , since } D_1 \text{ is one-to-one} \\ &\Rightarrow\; x = 0 \text{ , since } D_2 \text{ is one-to-one.} \end{aligned}$$

Similarly, A_2 is also one-to-one.

If we denote A_i, D_i as $A_i(U^*)$ and $D_i(U^*)$, then clearly, they are one-to-one and have dense range. Consequently, the analytic functions A_i and D_i are outer in $H^\infty(D)$.

From (4.5.3) we have

$$D_1(w)\varphi(w) = \psi(w)A_2(w).$$

And this completes the proof. ∎

Finally, as we had noted, we shall provide the proof that if T^X is unitarily equivalent to T^φ, then X is the adjoint of an analytic Toeplitz operator and $X = C\varphi(U^*)$ for some unimodular constant C. We suspect that if T^X

is similar or quasi-similar to T^{φ} then it may be possible to show that X is the adjoint of an analytic Toeplitz operator.

Theorem 4.6. Let T^X be unitary equivalent to T^{φ}, then X is the adjoint of an analytic Toeplitz operator and $X = C\varphi(U^*)$, where C in some unimodular constant.

Proof. Let

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ be a unitary

operator such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ X & U^* \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then we have

$$AU^* + B\varphi(U^*) = U^*A \qquad - \quad (4.6.1)$$

$$BU^* = U^*B \qquad - \quad (4.6.2)$$

$$CU^* + D\varphi(U^*) = XA + U^*C \qquad - \quad (4.6.3)$$

and

$$DU^* = XB + U^*D \qquad - \quad (4.6.4).$$

Using Lemma 4.2 and using the above equations we obtain that

$B = O$ and A, C, D commute with U^*.

Similarly, considering

$$\begin{pmatrix} A^* & C^* \\ O & D^* \end{pmatrix}\begin{pmatrix} U^* & O \\ X & U^* \end{pmatrix} = \begin{pmatrix} U^* & O \\ \varphi(U^*) & U^* \end{pmatrix}\begin{pmatrix} A^* & C^* \\ O & D^* \end{pmatrix}$$

we have

$$A^*U^* + C^*X = U^*A^* \quad - \quad (4.6.5)$$

$$C^*U^* = U^*C^* \quad - \quad (4.6.6)$$

$$D^*X = \varphi(U^*)A^* \quad - \quad (4.6.7)$$

and $D^*U^* = \varphi(U^*)C^* + U^*D$ — (4.6.8).

From (4.6.6) and (4.6.8) we see that

$(D^*U^*-U^*D^*)$ commutes with U^*, so by Lemma 4.2,

$$D^*U^* - U^*D^* = \varphi(U^*)C^* = O$$

if $\underset{-}{c}^* \in H^\infty(D)$ correspond to $C^* \in (U^*)'$, we

$$\varphi(w)c^*(w) = 0.$$

Now, since $\varphi \not\equiv 0$, we have $c^* \equiv 0$. Thus $C^* = 0$ and from (4.6.5), we see that A^* commutes with U^* and we have

A, A^*, D, D^* all commute with U^*.

Since the only operators that commute with U and U^* both are the scalar multiples of identity, we conclude that A and D are scalar multiples of identity.

Next, from the fact that

$$\begin{pmatrix} A & O \\ O & D \end{pmatrix} \text{ is unitary, we have}$$

A and D unimodular constant multiples of identity.

Next from (4.6.3) we have

$$D\ \varphi(U^*) = XA$$

or,
$$X = \bar{A}D\ \varphi(U^*).$$

This completes the proof. ∎

Bibliography

1. W.S. Clary, Quasi-similarity and subnormal operators, Ph.D. Dissertation, University of Michigan, 1973.

2. M.J. Cowen and R.G. Douglas, Complex geometry and operator theory, Bull. Amer. Math. Soc., 83 (1977), 131-133.

3. M.J. Cowen and R.G. Douglas, Operator theory and complex geometry, Proc. Sympos. Pure Math., Vol. 30, Part II, Amer. Math. Soc., Providence, R.I. (1977), 229-236.

4. M.J. Cowen and R.G. Douglas, Complex geometry and operator theory, Acta. Math., Vol. 141 (1978), 187-262.

5. M.J. Cowen and R.G. Douglas, On operators possessing an open set of eigenvalues, Proc. Fejer-Riesz Conference, Budapest (1980).

6. M.J. Cowen and R.G. Douglas, On moduli for invariant subspace, Operator Theory Advances and Application, Vol. 6, 65-73.

7. R.G. Douglas, Banach algebra techniques in operator theory, Academic Press, New York (1972).

8. P.R. Halmos, Hilbert space problem book, D. Van Nostrand, Princeton, New Jersey (1967).

9. K. Hoffman, Banach spaces of analytic functions, Prentice Hall, Englewood Cliffs, New Jersey (1962).

10. G. Misra, Curvature inequalities and Extremal Properties of Bundle shifts, Ph.D. Dissertation, SUNY at Stony Brook, 1982.

11. B. Sz.-Nagy and C. Foias, Harmonic analysis of operators on Hilbert space, North Holland, Amsterdam (1970).

12. C. Pearcy, A complete set of unitary invariants for operators generating W*-algebras of type I, Pacific J. Math., 12 (1962), 1405-1416.

13. M. Raphael, Quasi-similarity and essential spectra for subnormal operators, Indiana Univ. Math. J., Vol. 31 (1982), 243-246.

14. W. Rudin, Real and Complex analysis, McGraw-Hill, New York (1974).

15. R.O. Wells, Differential analysis on complex manifolds, Springer Verlag, New York (1980).