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BI-QUOTIENTS OF COMPACT LIE GROUPS AND THEIR CURVATURE

A Dissertation presented

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Julio Vidaurrazaga

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Julio Vidaurrazaga

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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After a brief survey of properties of homomorphism of Lie groups, bi-quotients are defined. Some necessary and/or sufficient conditions for existence or a biquotient manifold are given. An outline of classification for the most important cases is given.

Finally, a formula for the sectional curvature of a bi-quotient is given and the properties are exploited to get conditions for an existence of strictly positive sectional curvature.

To my mother, my grandmother, the memory of my father and grandparents (Vičnaja Pamjat), to Ynigo Lopez de Loyola and especially to all-holy, spotless and glorious Lady, Theotokos and ever-virgin Mary.

Čestnjejšuju Herubim i slavnejsuju bez sravnjenija Serafim, bez istljenija Boga Slova roždšuju suščuju Bogorodicu tja veličajem.

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Estella Shivers typed the finished version and succeeded to make my shapeless notes a finished product.

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Pauca, sed a pleno ve ientia pectore veri.

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0. INTRODUCTION

The complete simply-connected Riemannian manifolds of negative sectional curvature are well-known. They are diffeomorphic to the euclidean spaces of the same dimension.

On the other hand, no general result is known for manifolds of strictly positive sectional curvature. Such manifolds are necessarily compact if $K > \varepsilon > 0$ (Myers Theorem). For a long time, the only known examples have been the symmetric spaces of rank 1, i.e., spheres and projective spaces.

Berger [B] succeeded in classifying all the homogeneous simply-connected manifolds of strictly positive sectional curvature. Besides the well-known examples of symmetric spaces of rank 1, with their normal metric, he found some isolated examples of homogeneous manifolds diffeomorphic but not isometric to symmetric spaces of rank 1. They correspond to exceptional Lie group homorphisms $\varphi : H \to G$, if the manifold M can be written as $M = G/"H"$. (The quotation marks point to the fact that there is a particular embedding of H into G. The precise notation should be $G/\varphi(H)$.)

Besides those cases, Berger found two new examples: $SU(5)/Sp(2)\times T$ with the usual notation for classical Lie groups, and a seven-dimensional example $Sp(2)/"SU(2)"$, for a particular homomorphism of SU(2) into Sp(2). This manifold has the rational homology of the 7-sphere but a different homology for characteristics 2 and 5. The interest in this example is related to the fact that it shows that the topology of the quotient manifold depends not only on the pair $(\mathfrak{g}, \mathfrak{f})$ of Lie algebras but on the Lie algebra homomorphism $\varphi : \mathfrak{f} \to \mathfrak{g}$ as well.

The results of Berger assume that there is a bi-invariant metric on G. Up to a constant on every simple factor of G, such a metric will be the metric given by the Killing form.

It is natural, in this context, to look for other isometric actions of a Lie group on a compact group.

A problem will arise, however, because the action to be defined is not necessarily free. The simplest example would be the following: Let $\varphi : H \to G$ be a Lie group homomorphism, and define an action of H on G by $g \mapsto \varphi(h)g\varphi(h^{-1})$. Clearly, any central element of G is a fixed point and the isotopy subgroups of fixed points of the action are different, if g is a regular or a singular element (in the sense used in the theory of compact Lie groups, as it can be found in [Helgason]).

Given a compact Lie group, the existence of a bi-invariant metric is classical and correspond to an Ad(G)-invariant inner-product of $\mathfrak{g}$, the Lie algebra. If $\langle , \rangle$ is the inner-product, the group of isometries is, therefore, $O(\langle , \rangle, \mathfrak{g})$, the orthogonal group of the form $\langle , \rangle$.

In the homogeneous case, we take a Lie group H and a homomorphism $\varphi : H \to G$. We may define an isometric action, as we already said, by $g \mapsto \varphi(h)g$ (or, equivalently, $g \to g\varphi(h^{-1})$, $h \in H$. It is clear that this action is fixed point free, and we get the usual theory of homogeneous spaces.

Let us notice that, usually, the theory is presented by taking a manifold M, where G acts transitively, and one proves that $M = {}^{G}/_{H}$, H the isotropy group of a point. In this approach, the quotient manifold is the basic object. However, if we want to classify the possible quotients, we are forced to look at G as the basic object for the construction. (There is some lack of precision in the literature about this simple fact.)

Though recently, Wallach replaced the hypothesis of a bi-invariant metric on G (equivalently Ad(G)-invariant on the Lie algebra by the requirement that the metric is) left invariant on G, and Ad(H)-invariant in the Lie algebra. As in the case of the two new examples of Berger, his examples are isolated.

In the case of the symmetric spaces of rank 1, we have an example for each dimension of certain type. The two examples of Berger and the new examples of Wallach are particular cases, corresponding to highly specific pairs of Lie groups.

L. Berard-Bergery [BB] completed the work of Wallach by showing that the classification of homogeneous manifolds of positive sectional curvature was complete. Here there are no new examples, besides those of Berger and Wallach.

Let us return to the original question. It has been known for many years that a compact Lie group admits a metric of non-negative sectional curvature. More precisely, the curvature of a Lie group with respect to a bi-invariant metric is non-negative. However, this curvature has many zero sections. In particular, the curvature will be zero at the origin for pairs of vectors tangent to the same maximal torus. For higher ranks of the Lie groups more and more pairs of vector will give zero sections.

In fact, Wallach [WA1] has also proved that the only simply-connected Lie group that accepts a left invariant metric of strictly positive sectional curvature is SU(2) (which is diffeomorphic to the sphere S^3).

It is natural, therefore, to ask for new ways to define a manifold by taking a Lie group as the starting point. The reason is O'Neill's well-known formula for Riemannian submersions which implies that, for a Riemannian submersion, the projection does not decrease (horizontal) sectional curvature. We look back, therefore, at the original situation. If G is a Lie group with a bi-invariant metric and M a manifold such that $\pi : G \to M$ is a Riemannian submersion, we have $K_p \geq 0$ for $p = \pi(g)$, $g \in G$. The basic fact that was used in the construction of homogeneous manifolds of positive curvature is that the Lie group H acts by isometries on G via the homomorphism $\varphi : H \to G$. The action is defined, of course, as $g \mapsto \varphi(h)g$ where $h \in H$, $g \in G$. (It would be completely equivalent, in case of bi-invariant metrics, to take a right action with the obvious modifications. For a left invariant metric, it would be necessary to modify the conditions of invariance. The appropriate condition is the metric being the left invariant by G and $Ad(\varphi(H))$-invariant, via the homomorphism φ.)

As we observed above, in the usual definition of homogeneous spaces, we do not pay attention to the homomorphism $\varphi : H \to G$ and we consider H as a subgroup of G. The reason for this apparent inconsistency is that, in most of the interesting cases, the homomorphism φ is unique. We shall return to this point of the classification theory of Lie groups. A much more general situation whether it is possible to define a Riemannian submersion with total space G is to use a representation of a Lie group H into the isometry group of G with respect to a given metric. Let $G \to M$ be such a Riemannian submersion. The curvature of M at $p = \pi(g)$ will be greater or equal to some curvature of G at $g \in G$. This fact is the basic idea in the study of homogeneous spaces as sources of examples of positively curved spaces.

Our main purpose here is to consider, among the actions of a Lie group on other Lie groups, those that can be defined by $\rho(h) : g \mapsto \varphi(h)g\psi(h^{-1})$, where $\varphi : H \to G$ and $\varphi : H \to G$ are

Lie groups homomorphisms and G,H compact.

There is a slightly more general situation that can be described as follows: Let H_1, H_2, G be compact Lie groups, and $\varphi : H_1 \to G$ be Lie group $\psi : H_2 \to G$ homomorphisms.

We may define an action of $H_1 \times H_2$ on G by $g \mapsto \varphi(h_1)g\psi(h_2^{-1})$, where $h_1 \in H_2$.

If $H_1 = H_2$, this product action admits a restriction to the diagonal, i.e. H considered as the diagonal of $H \times H$, and we are in the situation described first.

Any classification effort will have two parts: To get conditions such that the action is free, that will imply that the quotient of G by the action is a manifold.

After finding some necessary or sufficient conditions for a free action, we shall look at the problem of determining the sectional curvature of the quotient, considered as a Riemannian submersion. We shall do it here by putting on G a bi-invariant metric.

Let us notice that the two parts are essentially independent. The conditions that we shall obtain to decide if the action is free or not, do not depend on the particular metric. Of course, this will be different when we study the curvature. However, at least bi-invariant metrics are unique up to a factor on a simple Lie group. Therefore, in most cases results will not depend on choice of a particular bi-invariant metric.

Let us notice that there is an <u>essential</u> difference between homomorphism of semi-simple Lie group and other Lie groups, even in the compact case. The simplest situation will show the difference.

A homomorphism φ between abelian groups, $\varphi : S^1 \to S^1$ is represented by $S^1 = \{u \in \mathbb{C},\ u = \exp 2\pi i t,\ t \in \mathbb{R}\}$, $\exp 2\pi i t \mapsto \exp\ 2\pi i\, nt$, or $u \mapsto u^n, n \in \mathbb{Z}$. Those homomorphisms are essentially different, the degree of the map φ being enough to classify them.

On the other hand, for the 3-dimensional simple Lie group SU(2), a homomorphism φ : SU(2) $\to$ SU(2), is unique up to an inner-automorphism. This comes from the necessary commutation relation on the Lie algebra.

Our main interest being the geometry of a bi-quotient and the analogy between our situation and that of the classical homogeneous spaces, we have restricted our attention mainly to the semi-simple case. The non-semi-simple case will almost always produce metrics with not strictly positive curvature in the bi-quotient. On the other hand, most of the interesting geometrical examples of homogeneous spaces depend on the existence of a fibration $H \to G \xrightarrow{\pi} M$, where G is semi-simple (even simple, as for symmetric spaces of rank 1 and Stiefel manifolds).

Our work has the parts after a brief review of the basic properties of Lie groups, we look at the important properties of homomorphism from a simple Lie group to a simple Lie group.

A basic tool will be the index of a homomorphism, as introduced by Dynkin [DY1]. The index can be described approximately as the ratio between the metric induced by the embedding $\varphi : H \to G$ and the metric induced by the Cartan-Killing form. In fact, our main goal is to find simple conditions on the Lie algebra to analyze the problem.

Next, we look at the necessary or sufficient conditions for a free action. Up to a discrete subgroup, all the conditions can be expressed in terms of the Lie algebras and their root systems.

Our main result for the diagonal action is that if $\varphi, \psi : H \to G$, H,G compact simple, and index $\varphi \neq$ index ψ, the action will be free. This criterion is simple enough and in practice will often suffice to prove the action is free.

We have not succeeded to prove the converse. In fact, there may be a counterexample. However, we do have two partial converses ($\varphi_*, \psi_* : \underline{H} \to \underline{G}$).

If $\varphi_*, \psi_* : \underline{H} \to \underline{G}$ are Lie algebra homomorphisms, both of 1, or if index φ_* = index ψ_* and $\varphi_*(\underline{H})$ and $\psi_*(\underline{H})$ are what Dynkin [DY1] calls regular subalgebras of $\underline{G}$, then there exist fixed points.

(A regular subalgebra can be defined by the condition that $\varphi_* : \underline{H} \to \underline{G}$ transforms the root system of $\underline{H}$ into a subset of the root system of $\underline{G}$.)

The existence of what Dynkin calls singular subalgebras does not allow us to produce a completely general converse to our proposition about the index. As it stands, however, we can cover most cases of geometrical interest.

The proposition "index φ_* = index ψ = 1 implies the action has fixed points" is in fact, extremely useful. It applies to the product action as well as to the diagonal action, and it covers many cases of subgroups that give manifold of geometrical interest.

We repeat that the study we make of fixed points does not depend on any particular metric on the Lie group. The Cartan-Killing form does induce a bi-invariant metric or a semi-simple Lie group and it is helpful to prove some results. However, the Cartan-Killing form is defined purely in terms of the Lie algebra of the Lie group. It may be considered, up to a positive factor, as being the natural metric for a semi-simple Lie group.

In the third and last part we study geometrical properties of the bi-quotients. Our main concern is to decide whether or not it is possible to obtain <u>strictly</u> positive curvatures.

The example of Gromoll and Meyer [GM] of an exotic sphere of non-negative curvature and the fact that O'Neill's formula for a Riemannian submersion guarantees non-negative curvature, at least makes this a natural question.

After deriving a formula for the sectional curvature in terms of the bi-invariant metric and the different invariant of the Lie algebra, we will discuss an important condition that says positive curvature implies that the <u>difference</u> of the ranks of both Lie groups is at most 1.

This proposition radically reduces the possible bi-quotients. If rank G > 2, we prove, by applying our previous results to this case, that regular subalgebras will not give any bi-invariant, because the action must have fixed points.

The general path to follow will be:

> First the analysis is done for regular subalgebras of a Lie algebra. The fact that the roots of the subalgebra are the roots of the algebra makes the proof extremely simple.

In fact, the generalization to a regular non-semi-simple subalgebra will be more or less immediate.

In the case of a non-regular subalgebra, the general classification theory and the work of Dynkins [DY1,2] will have to be used. Nevertheless, the result is analogous. Given the structure of the possible subalgebras of rank n or n - 1 (where n is the rank of $\mathfrak{g}$), the different actions always have a fixed point that can be considered the identity of G (as usual, a more careful description of fixed points would be possible in terms of singular and regular elements of G).

Let us remark again that our analysis of fixed points does not depend on the metric of G. We conjecture that the condition rank H ≥ rank G - 1 is necessary for any other reasonable (not necessarily bi-invariant) metric on G, as has been proved in the homogeneous case by Wallach. Then, our results on fixed points would complete the classification of bi-quotients.

It remains to consider the case rank G = 2. This case is essentially different, (c.f., the example [GM]).

The essential difference is that, if rank H = 1, there are no restrictions in the different commutation relations of the Lie algebra. However, the case rank G = 2 can be settled easily by a case by case examination. It has been done by Gromoll and Meyer to $B_2 \simeq C_2$, it is simple for A_2 (where, anyway, all roots have the same length) and it will be done, again, by a case by case examination to G_2, the exceptional Lie group of rank 2.

The non-semi-simple case, recently [E] has produced new examples, using the pair $(SU(3), S^1 \times S^1)$. His examples are a generalization of the Wallach examples. However, the abelian character of S^1 implies that this example should remain isolated. As we said before, our conjecture in the semi-simple case is, there will be no new examples, also for not bi-variant metrics. The reason being that we prove that the pairs of Wallach do give fixed points for the double action.

Some questions remain open. We would like to have a better understanding of the index of a homomorphism. It has been a very useful invariant here, and its topological interest is obvious.

Also, there may be some other invariants that may be of interest in the topology of homogeneous spaces and bi-quotients. (The topology of homogeneous spaces is poorly understood if the difference of the ranks is big.)

Finally, it is difficult to see how it would be possible, starting with a compact Lie group, to define other actions of a continuous group that may give rise to manifolds of positive curvature. The basic problem is the character of the O'Neill's formula. Even though curvature is non-decreasing, under the projections usually it does not increase for many pairs of vectors. The few examples of manifolds of non-negative curvature, besides Lie groups and homogeneous spaces, makes any progress in that direction difficult, but also very important.

At the end of our work we examine the examples of Wallach from our viewpoint. Given the particular groups and subgroups used in his work, with the exception of $(SU(3), S^1 x S^1)$ (that has been studied in [E]), we will see that no general section seems possible.

We may ask about other possible actions of a Lie group on a Lie group. The essential point is to find a representation of the acting Lie group in the (connected component of the) group of isometries of G with respect to some left-invariant metrics. We have mostly cover the bi-invariant case. The problem of dealing with a "φ,ψ-invariant metric", i.e. an innerproduct such that $L_{\varphi(h)} \circ R_{\psi(h^{-1})}$ is an isometry, would be a possible generalization. However, our results about fixed points of the double action remain the same, and there is a serious restriction in the number of possibilities.

Other metrics, in general, do not have non-negative curvature. However, we think that our results will, at least, help to clarify the real difficulties. A compact Lie group, even if it carries a metric of non-negative sectional curvature, always has a very big flat submanifold, the maximal torus (whose dimension increases with the dimension of G).

On the other hand, a semi-simple Lie group is a very rigid object. In fact, the theory of semi-simple Lie groups offers one of the few instances in mathematics where a complete classification is possible. The results of Dynkin [DY1,2] also show that homomorphisms between two semi-simple Lie groups can be completely described, up to innerautomorphism. These results make the beauty of the theory. They also confronts to the difficulty of producing essentially new examples of positively curved spaces via Lie group theory.

PRELIMINARIES

(This chapter has the character of a survey of basic facts about Lie groups and Lie algebras that will be used in the next chapter. We refer to the Bibliography for the necessary proofs.)

(See [SA], [HE], [VA], [LO], [T1], [DY1,2])

Section P.1. Let G be a compact semi-simple Lie group

Associated with $\underline{G}$ there exists $\underline{G}^{\mathbb{C}} = \underline{G} \otimes \mathbb{C}$, the tensor product over the reals. $\underline{G}^{\mathbb{C}}$ is called the complexification of $\underline{G}$. The theory of Lie algebras is usually developed in terms of complex Lie algebras. Our interest has been the real compact Lie groups, so we shall describe some relations between both objects. Also, we denote $T \subset G$ a maximal torus of G (sometimes T_G, if there are several Lie groups) and $\underline{T}_G$ its (abelian) Lie algebra, the Cartan subalgebra. (We may also denote this algebra by $\underline{C}_G$.)

On $\underline{G}$ we have the Cartan-Killing form K.

(P.1.1) $K(X,Y) := \mathrm{trace}(ad_x \circ ad_y)$.

The Cartan-Killing form is negative definite and ad-invariant (i.e. $ad_X K(Y,Z) = K(ad_X Y,Z) + K(Y,ad_X Z)$). We define an invariant positive definite product by

$$\langle X,Y\rangle = -K(X,Y) \quad \text{for } X,Y \in \underline{G}.$$

We remember also that a complex semi-simple Lie algebra $\underline{G}$ has the canonical decomposition

(P.1.2) $\underline{G} = T \oplus \bigoplus_{\alpha} G^{\alpha}$ where $\alpha \in R$, the roots,

$\underline{T}$ is the Cartan subalgebra, and the spaces $\underline{G}^{\alpha}$ are of (complex) dimension 1.

We have also, (for the sake of completeness)

(P.1.3) $[\underline{G}^{\alpha},\underline{G}^{\beta}] = \underline{G}^{\alpha+\beta}$

if $\alpha + \beta$ is a root = 0 otherwise.

This decomposition and the properties of the root systems implies that, in general, any theorem of semi-simple Lie groups may be stated in terms of roots. We shall use freely this fact in the next chapters.

Given a complex semi-simple Lie algebra, there canonically is associated a real compact Lie algebra, i.e., a real Lie algebra whose Cartan-Killing form is negative definite.

Also, given a semi-simple complex Lie group G, there is a maximal compact Lie group K, unique up to inner-automorphism, such that

(P.1.4) $\underline{K} \otimes \mathbb{C} = \underline{G}$, i.e., the complexification of $\underline{K}$ is the Lie algebra of G ("Weyl's unitary trick").

Section P.2. Real Lie algebras and Roots

The real Lie algebra $\underline{K}$ has the decomposition

(P.2.1) $\underline{K} = T_K \oplus \bigoplus_{\alpha \in R^+} K^\alpha$

where R^+ denotes the positive roots.

$$T_K \otimes \mathbb{C} = \underline{T}_G$$

$$[\underline{K}^\alpha, \underline{K}^\beta] \subset \underline{K}^{\alpha+\beta} \oplus \underline{K}^{\alpha-\beta}$$

$\underline{K}^\alpha$ is of real dimension 2. $\alpha \neq \pm \beta$. Also $[K^\alpha, K^\alpha] = \mathbb{R} H_\alpha$ where H_α is defined by

(P.2.2) $H \in \underline{T}_K \quad \alpha(H) = -K(H_\alpha, H) = \langle H_\alpha, H \rangle$

H_α is called the inverse root of α. We have (see [VA]) that $\underline{K}$ generates $\underline{T}$ over $\mathbb{C}$ and $T_{K^-} = i \, \Sigma \, \mathbb{R} H_\alpha$; i.e., iH_α span the Lie algebra of the maximal torus of $K \subset G$).

If G is simple, it is well known that, up to normalization by a positive factor, the H_α have length 1 or 2 (with the exception of G_2, (length 1 and 3).

We have also other set of vectors

(P.2.3) $\overline{H}_\alpha = \dfrac{2}{\langle \alpha, \alpha \rangle} H_\alpha$, the basic translations.

(P.2.4) In fact ([O], [HE]), there are three lattices

(1) Λ_0, the lattice generated by the $\overline{H}_\alpha$

(2) $\Lambda_1 = \bigcap_{\alpha \in R} T_\alpha$

$T_\alpha := \{x \in \underline{T}_K | \alpha(x) \in \mathbb{Z}\}$

Λ_1 is called the <u>central</u> <u>lattice</u>

(3) $\Lambda(K) = \exp^{-1}(e)$ where e is the identity of K.

We have ([LO], [HE]) that

(P.2.5) $\Lambda_0 \subset \Lambda(K) \subset \Lambda_1$.

If K is simple-connected then

(P.2.6) $\Lambda_0 = \Lambda(K)$.

(Our main interest being the simple-connected case, we shall assume at least implicitely, this hypothesis very often. In particular, there is in this case a cannonical isomorphism between the maximal torus and $\underline{T}_K / \Lambda(K)$.)

In general we have the isomorphisms:

(P.2.7) $\pi_1(K) \simeq \Lambda(K) / \Lambda_0$

and, where Z(K) is the center of K,

(P.2.8) $Z(K) \simeq \Lambda_1 / \Lambda_0$

We shall exploit these properties. In fact, we must reduce our conditions to conditions that can be expressed only in terms of the H_α. (Or, equivalently, the $\bar{H}_\alpha$ on α, the roots).

For a complex semi-simple Lie group G with Lie algebra $\underline{G}$ it always is possible to find a basis (Weyl's basis) H_α, X_α with the properties

(P.2.9) (1) $X_\alpha \in \underline{G}^\alpha$

(2) $[X_\alpha, X_{-\alpha}] = H_\alpha$

(3) $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$

$N_{\alpha,\alpha} \in \mathbb{Z}$

if $\alpha + \beta$ a root.

Also, for the compact form $\underline{K}$ of G we have

(1) $Y_\alpha = X_\alpha - X_{-\alpha}$

(2) $Z_\alpha = i(X_\alpha - X_{-\alpha})$

It is known (see [HE]) that

(P.2.10) (1) $K(X_\alpha, X_{-\alpha}) = 1$

(2) $K(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = 2$

(3) $K(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) = -2$

(4) $K(i(X_\alpha + X_{i\alpha}), X_\alpha - X_{-\alpha}) = 0$

(5) $K(iH_\alpha, iH_\alpha) = -\alpha(H_\alpha) < 0$.

We remember also that

$$\overline{H}_\alpha = \frac{2}{\langle\alpha,\alpha\rangle} H_\alpha \text{ is proportional}$$

to H_α by a real factor.

Even if we shall not have occasion to use explicitely all these facts, they will be assumed. They will, in effect, allow us to use the results of classification theory as in [DY1,DY2] in the compact case.

Note. There is, in the literature, some inconsistency with the factor $2\pi i$. This factor comes, of course, from the necessary changes to pass from the real compact Lie algebra to the complex one, and in ceversa.

Also, the factor 2π is related to the fact that we want the innerproduct given by the Cartan-Killing form to agree with the canonical bi-invariant metric on S^3 (with constant curvature +1).

(Let us remember

(P.2.11) $S^3 \simeq SU(2) \simeq Sp(1) \simeq Spin(3)$)

Section P.3.

We shall use freely the convention of classification theory of Lie groups in particular the notations

$$A_n, B_n, C_n, D_n$$

for the classification Lie algebras and

$$G_2, F_4, E_7, E_8$$

for the five exceptional ones.

(Even if it may look inconsistent we may speak of G_2, say, as a real Lie algebra or a compact Lie group. This "Abus de langage" is common in the literature.)

CHAPTER I

Section 1.1.

Given $\varphi : H \to G$, a Lie group homomorphism, it is always possible to define a new one by taking an inner-automorphism of G, ig : $x \mapsto gxg^{-1}$, and composing it with φ:

$$\tilde{\varphi} = i_g \circ \varphi$$

is clearly a new homomorphism. Any classification of the homomorphism between $\neq 0$ Lie groups will be done, therefore, up to equivalence by an inner-automorphism.

Analogously, for the Lie algebra H and G, we have that $Ad(g) \circ \varphi_*$ is Lie algebra homomorphism.

We may also modify φ by composing at the source with an inner-automorphism of H. Thus will be a particular case of the first (conjugation by $\varphi(h)$ in G) and shows again that any classification should be understood up to inner-automorphism.

Given a system of simple roots, π, we may introduce an order in the real vector space of linear combinations of the roots. We have a linear function φ^* from the span of the roots of G to the span H of the roots of H.

We say that an ordering of the span of roots of H is consistent with the ordering respect to φ^*.

If the relation $\varphi^*(x) < \varphi^*(y)$ implies the relation

$x < y$, $x,y \in \mathrm{Span}(\pi)$ is G. We have the following proposition.

Proposition 1.1 [DY1]. For every homomorphism $\varphi : H \to G$ there exist an equivalent homomorphism that respects the order defined by π.

Proof. (See [DY], page 122).

Section 1.2. Index of a Homorphism [DY1]

Let $\underline{H}$ and $\underline{G}$ be simple Lie algebras. The Cartan-Killing form defines on $\underline{H}$ (respectively $\underline{G}$) an inner-product $\langle , \rangle$ that is invariant in the sense (1) $\langle [X,Y,Z \rangle + \langle X,[Y,Z] \rangle = 0$. If $\underline{H}$, $\underline{G}$ are simple then this innerproduct is unique up to a factor.

It is clear enough, in the semi-simple case, to classify the subalgebras of the simple Lie algebras. We also may consider the subgroups of the simple Lie groups. Both problems are intimately related and it is helpful sometimes to interpret Lie algebra results in Lie group terms and reciprocally.

The compact non-semi-simple case requires slightly different methods and will be studied only occasionally.

The problem of finding the semi-simple subalgebras of a semi-simple Lie algebra was solved by Dynkin (DY1] for complex Lie algebras. With minor modifications we may use his results and methods. This section has, therefore, the character of a survey of results. We refer to the important papers of Dynkin [DY1,2] for complete development of the theory. We shall not hesitate to use his results and tables when suitable.

Let H,G be two compact Lie groups and $\varphi : H \to G$ be a group homomorphism.

It is clear that φ induces a Lie algebra homomorphism

$$\varphi_* : \underline{H} \to \underline{G} .$$

(Occasionally, if there is no risk of confusion we shall use the same letter for both homomorphisms.)

The problem of finding and classifying the subgroups (resp. subalgebras) of a compact Lie group (resp. semi-simple compact Lie algebra) is equivalent to the problem of finding such homomorphisms.

Given two Lie algebras $\underline{H}$, $\underline{G}$ and simply-connected Lie groups H,G with Lie algebras $\underline{H}$, $\underline{G}$ respectively, $(\underline{H},\underline{G})$ any homomorphism $\varphi : \underline{H} \to \underline{G}$ defines a homomorphism "φ" form H to G, $\varphi : H \to G$.

The proposition is not, in general, true if H and G are not simply-connected. For instance, there is not a homomorphism between SO(n) and Spin(n) even though the Lie algebras are isomorphic.

Let us notice that Dynkin's definition makes sense for a real compact Lie algebra, our main interest.

Let us consider $\varphi : \underline{H} \to \underline{G}$, a Lie algebra homomorphism. Let $\langle , \rangle_H$, respectively, $\langle , \rangle_G$ be invariant innerproducts on $\underline{H}$ (respectively $\underline{G}$).

We may define a "new" innerproduct $\langle , \rangle$ on $\underline{H}$ on $x,y \in \underline{H}$ $\langle\langle x,y\rangle\rangle = \langle\varphi(x),\varphi(y)\rangle_G$ i.e., the "pull back" of $\langle , \rangle_G$ to $\underline{H}$ by φ.

The innerproduct on $\underline{H}$ being unique up to a factor, there exist a positive number j_φ such that

$$\langle\langle x,y\rangle\rangle = j_\varphi \langle x,y\rangle_{\underline{H}}.$$

Definition 1.2.1. j_φ is the index of the homomorphism φ.

Example 2.1. Let $\underline{G} = sp(2)$

$\underline{H} = sp(1)$

realized as 2x2 quaterniomic antihermitian matrices and purely quaternionic matrices respectively.

An innerproduct on sp(2) is given by Re tr AB, $A,B \in Sp(2)$ and an innerproduct on sp(1) is given by Re $q_1 q_2$, q_1, q_2 pure imaginary quaternions. We have the two representations

$$\varphi : q \mapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \in sp(2)$$

$$\psi : g \mapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \in sp(1)$$

It is easy to see that φ is of index 1 and ψ of index 2.

Let us notice that $\varphi(sp(1))$ and $\psi(sp(1))$ are isomorphic, as Lie algebras, but not conjugated by an inner automorphism of sp(2).

The index of a homomorphism (equivalently, a subalgebra) will be an invariant of fundamental importance in the study of subalgebras, subgroups and bi-quotients.

It is clear that, given two compact simple Lie groups, analogous definitions make sense with the bi-invariant metric.

Let us remember too that the Cartan-Killing form a canonical invariant associated with the semi-simple Lie group (Lie algebra), defines an invariant innerproduct.

Also, the existence of such bi-linear invariant has strong implications about the topology of the (semi-simple compact) Lie group G. In particular the existence of such a product implies that the group has a finite fundamental group and that $H_2(G,\mathbb{R}) = 0$.

Moreover, in a semi-simple compact Lie group there exist an invariant 3-form ω, the Cartan form defined by [M1]

$$\omega(X,Y,Z) = \langle X[Y,Z]\rangle .$$

The Cartan form will allow us to give a different interpretation of the index of a homomorphism.

Let us remember the following facts from topology [HE] for a compact semi-simple Lie group:

$\pi_1(G)$ is finite

$\pi_2(G) = 0$

$H_2(G,\mathbb{Z}) = 0.$

If G simple $\pi_3(G) \simeq \mathbb{Z}$

$$(1) \quad \begin{cases} H_3(G,\mathbb{Z}) \simeq \mathbb{Z} \\ \\ H^3(G,\mathbb{Z}) \simeq \mathbb{Z} . \end{cases}$$

Proposition 2.1. Let $\varphi : H \to G$, H,G simple, φ Lie group homomorphism.

Let ω and ω' be generators of $H^3(G,\mathbb{Z})$ and $H^3(H,\mathbb{Z})$, respectively. Then $\varphi^*\omega = \pm j_\varphi \omega'$. In particular j_φ is an integer.

Proof. Let $\tilde{\omega},\tilde{\omega}'$, the 3-forms defined by

$\tilde{\omega}(X,Y,Z) = \langle X,[Y,Z]\rangle \quad X,Y,Z \in \underline{G} \quad \langle , \rangle$ an invariant innerproduct in $\underline{G}$

$\tilde{\omega}'(X_1,X_2,X_3) = \langle\langle X_1,[Y_1,Z_1]\rangle\rangle \quad X_1,Y_1,Z_1 \in \underline{H} \langle\langle , \rangle\rangle$ an invariant innerproduct in $\underline{H}$.

It is classical that $\tilde{\omega}(\tilde{\omega}')$ defines a cohomology class in $H^3(G,\mathbb{R})$, $(H^3(H,\mathbb{R}))$ [de Rham cohomology].

Now, $\quad H^3(G,\mathbb{Z}) \otimes \mathbb{R} = \mathbb{R}$

$\quad H^3(H,\mathbb{Z}) \otimes \mathbb{R} = \mathbb{R} \quad$ from (1).

We have the pull-back of $\tilde{\omega}$ defined by

$$\varphi^*\tilde{\omega}(X_1,Y_1,Z_1) = \langle \varphi X_1, [\varphi Y_1, \varphi Z_1]\rangle$$

$$= \langle \varphi X_1, \varphi[Y_1, Z_1]\rangle$$

$$= j_\varphi \langle\langle X_1, [Y_1, Z_1]\rangle\rangle$$

$$= j_\varphi \tilde{\omega}' \qquad (2).$$

We have $\varphi^*\omega = A\omega'$ and $\varphi^* : H^3(G,\mathbb{Z}) \to H^3(H,\mathbb{Z})$ is a homomorphism (multiplication by $A \in \mathbb{Z}$). $H^3(G,\mathbb{Z}) \simeq H^3(H,\mathbb{Z}) \simeq \mathbb{Z}$ is free. A homomorphism of free modules is determined by its tensor product with $\mathbb{R}$ (over $\mathbb{Z}$)

(2) implies $j_\varphi = \pm A$

QED

Note. G,H being semi-simple

$$[\underline{G},\underline{G}] = \underline{G}$$

$$[\underline{H},\underline{H}] = \underline{H} .$$

This implies that the definition of j_φ by $\varphi^*\tilde{\omega} = j_\varphi\tilde{\omega}'$ is completely equivalent to the defintion given above.

Remarks. Let us notice that, from our proposition, it is clear that j_φ is an integer. This fact has been proved by [DY1] using more complicated techniques of Lie algebra representations.

It would be interesting to see if it is possible to find a simpler algebraic proof of the fact that the index is an integer. The cohomological nature of our proof suggests, that a simpler proof in terms of Lie algebra cohomology may be possible.

Section 1.2.3. Regular Subalgebras and Homomorphisms

Let $\underline{G}$ be a (complex) semi-simple Lie algebra.

Definition 1.3.1. $\underline{\tilde{G}} \subset \underline{G}$ is called regular if there exist a basis consisting of elements of some Cartan subalgebra $\underline{C} \subset \underline{G}$ and root vectors of the algebra G respectively to C ([DY1] page 142).

For every Cartan subalgebra $\underline{C} \subset \underline{G}$ there exist a canonical decomposition of $\underline{G}$.

$$\underline{G} = \underline{C} \oplus \bigoplus_{R} \underline{G}^{\alpha} \qquad \alpha \in R, \text{ R the roots.}$$

Also, there is a canonical decomposition of $\tilde{G} = \underline{\tilde{C}} \oplus \bigoplus_{R'} \underline{G}^{\alpha}$, $\alpha \in R'$ $\underline{\tilde{G}}$ is regular if $\underline{\tilde{C}} \subset \underline{C}$ and $R' \subset R$. (This definition is due to Dynkin [DY1].) Equivalently taking $\varphi : \underline{\tilde{G}} \to \underline{G}$, $\underline{G}, \underline{\tilde{G}}$ semi-simple Lie algebras, we have $\varphi^{*}(R) \subset R'$ and

$$\varphi(\underline{\tilde{C}}) \subset \underline{C}$$

$$\varphi(\tilde{H}_{\alpha}) = H_{\alpha} \qquad \alpha \in \tilde{R}$$

$$\alpha \in R.$$

Regular subalgebras are, of course, easier to describe and study.

In the case of a real compact Lie algebra, the definition should be modified as follows: [BB]

let $\quad \underline{G} = \underline{C} \oplus \bigoplus_{R^{+}} \underline{G}^{\alpha}$.

R^+ a positive system of roots $\underline{G}^\alpha$ a subspace of real dimension two

$$\underline{\tilde{G}} = \underline{C} \oplus \bigoplus_{\tilde{R}^+} \underline{\tilde{G}}^\alpha .$$

Definition 1.2.2. A subalgebra $\underline{\tilde{G}}$ is regular if $\tilde{R} \subset R^+$.

Among the regular subalgebras, the 3-dimensional Lie algebras

$$\underline{G}(\alpha) = \mathbb{C}H_\alpha \oplus \underline{G}^\alpha \oplus \underline{G}^{-\alpha}$$

generated by

$$\overline{H}_\alpha, \; X_\alpha, \; X_{-\alpha}$$

are of fundamental importance. (We have, of course, also the 3-dimensional compact Lie algebra generated by $i\overline{H}_\alpha$, $i(X_\alpha + X_{-\alpha})$, $i(X_\alpha - X_{-\alpha})$ of the real compact Lie algebra whose complexified is $\underline{G}$.)

In general, from the decomposition

$$\underline{G} = \underline{C} \oplus \bigoplus_{R} \underline{G}^\alpha$$

$$\underline{H} = \underline{\tilde{C}} \oplus \bigoplus_{R} \underline{H}^\alpha$$

we have that $\varphi_*(\underline{H}(\alpha))$ is a 3-dimensional Lie algebra of $\underline{G}$ and, is such generated by

$$F_\alpha = \varphi(\overline{H}_\alpha)$$

$$Y_\alpha = \varphi(X_\alpha)$$

$$Y_{-\alpha} = \varphi(X_{-\alpha})$$

CHAPTER II

Section 2.1. Introduction

Let us remember the definition of a homogeneous space [HE].

Let M be a manifold. Suppose that a Lie group G acts transitively on M. Then M is diffeomorphic to the quotient G/H, where H is the isotopy group of a point $p \in G$. M is called a homogeneous space.

The usal definition does not put emphasis on the fact that H, as an abstract Lie group, can be often realized as <u>different</u> subgroups of G. Two examples will show the importance of the observation.

Example 2.1. Lets take $G = U(n+1)$

$H = U(n)$

we may define $\varphi : H \to G$ by (representing G by (n+1) complex matrices H by (n) complex matrices)

$A \in U(n)$

$$\varphi(A) = \left[\begin{array}{ccc|c} & A & & 0 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

i.e., the matrix with 1 in the (n+1), (n+1) entry.

The quotient space $G/_{\varphi(H)}$ is, as it is well known, diffeomorphic to S^{2n+1}.

Let us, now, take $A \in U(n)$ and ψ a homomorphism defined by

$$A \stackrel{\psi}{\longmapsto} \begin{bmatrix} A & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \; 0 & \bar{\alpha} \end{bmatrix} \quad \text{where } \bar{\alpha} = \overline{\text{Det}[A]}$$

A simple exercise shows that $G/_{\psi(H)}$ is diffeomorphic to $CP^n \times S^1$ clearly non-diffeomorphic to S^{2n+1}. In this case G is compact but non-semi-simple.

<u>Example 2.2</u> [BE]. The example of Berger of a homogeneous space quotient of Sp(2) by a particular embedding of Sp(1) shows that even in the semi-simple case, different homomorphisms may give different quotient manifolds (the example of Berger is a <u>real</u> cohomology sphere that is not an integral cohomology sphere. There is 2- and 5-torsion).

Section 2.2. Bi-quotients

In all our considerations the homomorphisms

$$\varphi : H \to G \quad \text{and}$$

$$\psi : H \to G$$

will play an important role.

In our context, a homogeneous space is defined as the quotient of a (in general compact) Lie group by the action of a (compact) Lie group H, via a homomorphism φ. The action will be defined by

$$H \times G \xrightarrow{m} G$$

$$(h,g) \overset{m}{\longmapsto} \varphi(h)g \in G \; .$$

This definition suggests the following definition.

Definition 2.2.1. Let G, H_1, H_2 (compact) Lie groups

$$\varphi : H_1 \to G$$

$$\psi : H_2 \to G \qquad \text{Lie group homomorphism.}$$

We define a double action

$$\rho : (H_1 \times H_2) \times G \to G$$

by

$$(h_1,h_2,y) \mapsto \varphi(h_1)g\psi(h_2^{-1})$$

for $h_1 \in H_1$, $h_2 \in H_2$ and $g \in G$.

The quotient space, if the action is free (that is, without fixed points) will be called a bi-quotient.

A simple example of bi-quotient appears in [GM].

Example 2.1. Let us take $G = Sp(2)$

$$H_1 = Sp(1)$$

$$H_2 = Sp(1).$$

We define $\varphi : H_1 \to G$ by q, a quaternion of module 1,

$$\varphi(q) = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

and $\psi : H_2 \to G$ by

$$\psi(q) = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} .$$

The bi-quotient of $Sp(2)$ by the action

$$(q_1 \times q_2, Q) \mapsto \begin{pmatrix} q_1 & 0 \\ 0 & q_1 \end{pmatrix} Q \begin{pmatrix} \bar{q}_2 & 0 \\ 0 & 1 \end{pmatrix}$$

is a manifold diffeomorphic to S^4. A similar example with $H_1 \neq H_2$ would be

Example 2.2.

$$H_1 = Sp(n)$$

$$H_2 = Sp(1)$$

$$G = Sp(n+1)$$

$$\varphi : A \mapsto \left[\begin{array}{ccc|c} & & & 0 \\ & A & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\psi : q \mapsto \begin{bmatrix} q & 0 & & 0 \\ 0 & q & & 0 \\ & & q & \\ & & & 0 \\ 0 & & 0 & q \end{bmatrix}$$

The double action is free and the bi-quotient is a manifold diffeomorphic to $\mathbb{H}P(n)$, the quaterniomic projective space.

Our main interest will be, however, if $H_1 = H_2$. In this case we can restrict the action to the diagonal $\Delta(H) \subset H \times H$, a Lie group isomorphic to H. In this case

<u>Definition 2.2.2</u>. Let $\varphi : H \to G$ and $\psi : H \to G$ be Lie group homomorphisms.

An action $\rho : H \times G \to G$ is defined by $g \mapsto \varphi(h)g\psi(h^{-1})$ where $h \in H$ and $g \in G$.

If the action is free, the bi-quotient will be a manifold. In the example of [GM] they obtained a manifold Σ, homomorphic but not diffeomorphic to S^7, i.e. one of Milnor's exotic spheres. Our first purpose is to obtain conditions to decide if the action is free. In general, we should study the case where H and G are compact and semi-simple. (We may have occasional allusions to the <u>compact</u> non-semi-simple case.

Anyway, the compact case depends on the study of the semi-simple case from the known fact that any compact Lie group is locally isomorphic to the product of a semi-simple compact group and a toral group.)

We should try to reduce our problem to some number of necessary (and eventually sufficient) Lie algebra conditions; we should use the result sketched in the previous sections, and, more specially, the important results of Dynkin [DY1].

<u>Proposition 2.2.1</u>. Let $\varphi, \psi : H \to G$ be homomorphisms. The topology of the bi-quotient depends only on the conjugacy class of φ and ψ.

<u>Proof</u>. Let us take
$$\tilde{\varphi} = Ad(g_1)\,\varphi$$
$$\tilde{\psi} = Ad(g_2)\,\psi\ .$$

There are loops $C_1, C_2 : [0,1] \to G$ such that
$$C_1(0) = e \qquad C_2(0) = e$$
$$C_1(1) = g_1 \qquad C_2(1) = g_2\ .$$

Therefore $\tilde{\varphi}$ is homotopically equivalent to φ, $\tilde{\psi}$ to ψ. By a standard result of fibration the proposition follows.

Let T_H (resp. T_G) be a fixed maximal torus of H (resp. G). The proposition allows us to consider, in general,

$$\varphi : H \to G \qquad \varphi(T_H) \subset T_G$$

$$\psi : H \to G \qquad \psi(T_H) \subset T_G$$

Observation 1. In the case of the production of $H_1 \times H_2$ on G, we may still take T_{H_1}, T_{H_2}, T_G as before with

$$\varphi(T_{H_1}) \subset T_G$$

$$\psi(T_{H_2}) \subset T_G \ .$$

Observation 2. All maximal tori of H being conjugated by inner-automorphisms, we can take an equivalent representation

$$\tilde{\varphi} = \varphi \circ Ad(h) : H \to G$$

such that

$$\tilde{\varphi}(T_H) \subset T_G$$

Proposition 2.2.2. Let $\varphi, \psi : H \to G$ and let $h \in H$, $g \in G$ be such that g is fixed by h, i.e., $\varphi(h)g\psi(h^{-1}) = g$. Then there exist equivalent homorphisms $\tilde{\varphi}, \tilde{\psi}$ such that

$$\tilde{\varphi}(h)e\overline{\psi}(h^{-1}) = e \ ; \text{ i.e.,}$$

the neutral element is a fixed point.

Proof. Let us write $\quad g = g_1 g_2$

$$g^{-1} = g_2^{-1} g_1^{-1}$$

we have

$$\varphi(h)g_1 g_2 \psi(h^{-1}) = g_1 g_2$$

$$\Leftrightarrow \quad (g_1^{-1}\varphi(h)g_1)e(g_2\psi(h^{-1})g_2^{-1}) = e$$

i.e., we have to take

$$\tilde{\varphi} = Ad(g_1^{-1})\circ\varphi$$

$$\tilde{\psi} = Ad(g_2^{-1})\circ\psi$$

QED

Observation. We have proved that we may change both φ, ψ. Clearly, it would be equally good to modify φ or ψ by composition with an inner-automorphism.

Corollary 2.2.3. Under the same hypothesis, if e is a fixed point so is every element of the center of G.

Observation. It is clear that, if $\varphi(h) = \psi(h)$ for some $h \in H$ then h fixes e, and so will any power of h. This observation has a more useful form in terms of Lie algebras.

Proposition 2.2.3. Let $\varphi, \psi : H \to G$ and let $\varphi_*, \psi_* : H \to G$ be the correspondong Lie algebra homomorphisms.

If there is an $X \in \underline{H}$ such that $\psi_*(X) = \varphi_*(X)$, then the identity is fixed by every element of the toral group $\{\overline{\exp tX}\}$ $t \in \mathbb{R}$.

Proof. Let us take $h_t = \exp(tX)$. $\varphi_*(X) = \varphi_*(X)$ implies, for all $t \in \mathbb{R}$ $t\varphi_*(X) = \varphi_*(tX) = \psi_*(tX) = t\psi_*(X)$.

By a basic result of Lie groups theory

$$\varphi(\exp tY) = \exp t\varphi_*(Y)$$

$$\psi(\exp tY) = \exp t\psi_*(Y)$$

therefore, $\varphi(h_t) = \psi(h_t)$ fixes the identity. The rest follows by continuity.

Proposition 2.2.4. Let $\varphi, \psi : H \to G$. If $h \in H$ fixes g, then the cyclic groups generated by $\varphi(h)$ and $\psi(h)$ are conjugated by g.

Proof.
$$\varphi(h)g\psi(h^{-1}) = g \quad \Leftrightarrow$$

$$g^{-1}\varphi(h)g = \psi(h)$$

$$g^{-1}\varphi(h^n)g = \psi(h^n) \quad \forall\, n \in \mathbb{Z} \quad (A)$$

Observation. If the group generated by $\varphi(h)$ is continuous, the relation (A) reads $g^{-1}T_{\varphi(h)}g = T_{\psi(h)}$ where T_g the torus generated (topologically) by g.

By differentiation, we obtain $\varphi(X) = Ad(g)\psi_*$ for an X such that $\exp X = h$. In the semi-simple case it is possible to say more. We have, in fact, that C_H, the Cartan subalgebra of $\underline{H}$ is generated by the $i\overline{H}_\alpha$.

$$\underline{T_{\varphi(h)}} \subset \varphi_*(C_H)$$

we have that

$$\underline{T_{g^{-1}\varphi(h)g}} \subset g^{-1}\varphi_*(C_H)g$$

If T_φ contains some defining vector f corresponding to a 3-dimensional Lie algebra, $\varphi_*(f)$ contains the other two vectors as well.

By the same reasoning applied to $Ad(g^{-1})\circ\psi$, we obtain that there is a 3-dimensional Lie algebra fixing the point g.

Section 2.3.

We shall try to obtain conditions for the existence of fixed points.

Our first proposition shows the importance of the Index as an invariant of a representation.

Proposition 2.3.1. Let H,G be simple Lie groups and $\varphi,\psi : H \to G$ such that $j_\varphi \neq j_\psi$. Then the diagonal action is fixed point free, up to a discrete subgroup of H.

Proof. Let us assume $\varphi(h)g\psi(h^{-1}) = g$. If h generates a toral group the relation implies as we know that for some vector $X \in \underline{H}$

$$\varphi_*(X) = Ad(g)\psi_*(X).$$

By the canonical bi-variance of the innerproduct on $\underline{H},\underline{G}$ we have $\langle\varphi_*(X),\varphi_*(X)\rangle = \langle\psi_*(X),\psi_*(X)\rangle$ in contradiction with the hypothesis: $j_\varphi\langle X,X\rangle = \langle\varphi_*(X)\varphi_*(X)\rangle = \langle\psi_*(X),\psi_*(X)\rangle = j_\psi\langle X,X\rangle$.

Corollary 2.3.2. Let $\underline{H}$ be semi-simple

i.e., $\underline{H} = \underline{H}_1 \oplus H_2 \oplus \ldots \oplus H_n$

and let $\varphi : H \to G$

and $\psi : H \to G$.

Let us call φ_i (resp. ψ_i) the restriction of φ_* to $\underline{H}_i$. If $j_{\varphi i} \neq j_{\psi j}$ $\forall i,j$ then the action is fixed point free.

Proof. The proof is completely analogous to the previous proposition up to the fact that we may permute the $\underline{H}_i$ by a representation of the Weyl group.

The generalization of the proposition to a semi-simple Lie group depends on a careful definition of the various index.

Remark. As in the study of homogeneous spaces we shall assume in the future that $\varphi_*(\underline{H}) \subset \underline{G}$ and $\psi_*(\underline{H}) \subset \underline{G}$ do not contain any non-trivial ideal in common with $\underline{G}$. (For the analogous condition in homogeneous, see [B] and [BB].)

Observation. Let $G = G_1 \times G_2$ and H, be as usual, but with G_1, G_2 simple and let

$$\varphi = (\varphi_1, 1)$$

and

$$\psi = (1, \psi_2).$$

We may define a double action on G by

$$\rho(h) : (g_1, g_2) \mapsto (\varphi_1(h)g_1, g_2\psi_2(h^{-1})).$$

The action is free and we can define a bi-quotient. However, the bi-quotient is an ordinary homogeneous space. In fact, if M is the quotient manifold, we define an action of G on M by

$$\text{Orbit}(g_1, g_2) * (\tilde{g}_1, \tilde{g}_2) = \text{Orbit}(g_1\tilde{g}_1, \tilde{g}_2 g_2) \; \tilde{g}_1, \tilde{g}_2 \in G$$

The isotropy group of a point is $\varphi_1(H) \times \psi_2(H)$.

<u>Observation</u>. By the equivalence up to inner-automorphism of φ (resp. ψ) we may always suppose that

$$\left.\begin{array}{l}\varphi(T_H) \subset T_G \\ \psi(T_H) \subset T_G\end{array}\right. \qquad (A)$$

where T_G, T_H are maximal tori of G and H, respectively.

Let us notice that we still have a choice; we may modify φ and ψ by an inner-automorphism that fixes either T_H or T_G, i.e., a representative for the Weyl group of H or G. We shall use this fact several times. Usually, we will assume (A) without explicitly saying that the condition holds.

The results to follow in this section shall depend on the result in [DY1, DY2].

<u>Proposition 2.3.3</u>. Let $\varphi, \psi : H \to G$, where H,G are simple, be such that $\varphi_*(\underline{H})$ and $\psi_*(\underline{H})$ are regular subalgebras of the same index. Then there exists fixed points.

<u>Proof</u>. Let $\varphi_*(\underline{H}) = \varphi_*(\underline{C}) \oplus \bigoplus_{R^+} \varphi_*(\underline{G}^\alpha)$ and

$$\psi_*(\underline{H}) = \psi_*(\underline{C}) \oplus \bigoplus_{R^+} \psi_*(\underline{G}^\alpha)$$

(with the notations of the previous section)

then
$$\varphi_*(iH_\beta) = iH_\alpha$$
$$\psi_*(iH_\beta) = iH_\alpha$$

β root on $\underline{H}$ and α roots of $\underline{G}$ contained in R^+ by the definition of H_α and the definition of regularity.

Therefore, (the Weyl group acts transitively on roots of some length) we can replace ψ_* by an equivalent representation with $\psi_*(iH_\beta) = iH_\alpha$. We see that there are elements $\exp(tiH_\alpha) \in H$ that fix the identity element.

Corollary 2.3.4. With the appropriate condition in the index of

$$\begin{matrix} \varphi_i : H_i \to \underline{G} \\ \psi_i : H_i \to \underline{G} \end{matrix} \quad \text{the proposition}$$

is still valid if $\underline{H}, \underline{G}$ are semi-simple.

Proposition 2.3.5. Let $\varphi, \psi : H \to G$ where G, H are simple Lie groups. If $j_\varphi = j_\psi = 1$, then there are fixed points.

Proof. By [DY1] up to equivalence by inner-automorphism we may suppose

$$\varphi_*(H_\alpha) = \psi_*(H_\alpha) = H_\beta$$

where α is the highest root of $\underline{H}$ and β is the longest root of $\underline{G}$.

Therefore, we shall have fixed points by our usual criterion: $\psi_*(H_\alpha) = \psi_*(H_\alpha)$.

Observation. The proof will be equally valid for the double action of $H_1 \times H_2$ as it is for the diagonal action.

Proof. We apply exactly the same proof of the previous proposition about bi-quotients, remembering that the Weyl group acts transitively on roots of the same length.

Observation. If G is semi-simple and compact and H compact, the proposition will still be true by the remarks of [DY1] and [B,B]. $[\underline{H},\underline{H}]$, the semi-simple part of $\underline{H}$ is a regular subalgebra if rank $\underline{H}$ = rank $\underline{G}$.

Proposition 2.3.6 [DY1]. If rank H = rank G, H is regular.

Proof. See [DY1].

Proposition 2.3.7. If H_1, H_2, G semi-simple Lie groups and rank H_1 = rank H_2 = rank G, then there always exists fixed points for the quotient by

$$\varphi : H_1 \to G$$

$$\psi : H_2 \to G.$$

Proof. Let us apply the previous propositions and the proposition about regular subalgebras. Up to a representative of the Weyl group we must have $\psi_*(H_\alpha) = \varphi_*(H_\alpha)$ for some α.

<u>Proposition 2.3.8.</u> If $\psi, \varphi : H \to G$, rank H = rank G, and rank $G > 2$, then there are fixed points for the diagonal action.

<u>Proof.</u> Again we use the fact that $\underline{H} \subset \underline{G}$ is a regular subalgebra. The rest is as in Proposition 2.3.8.

Section 2.4. Symmetric spaces and bi-quotients

If we look at the list of compact symmetric spaces, as in [HE], we observe that all pairs of Lie algebras ($\underline{G},\underline{H}$) with a specific homomorphism $\varphi : H \to \underline{G},\underline{H}$ contains an ideal of index 1. Therefore, if we take

$$\varphi : H^1 \to G$$

$$\psi : H^2 \to G$$

such that $G/\varphi(H_1)$ and $G/\varphi(H_2)$ are symmetric spaces, the double action <u>will</u> have fixed points and the index of $\varphi_i : \underline{H}_i \to \underline{G}$ is 1 ($\oplus\underline{H}_i^j = \underline{H}^j, \underline{H}_i^j$ simple $j = 1,2$) [DY1]. (References: [HE], [LO], and [WO].)

The observation can be explained by the fact that $\underline{H} \subset \underline{G}$ is the set of fixed points of an automorphism σ such that $\sigma^2 = I$. σ, therefore, does not change the length of the roots and the index must be one.

Symmetric spaces being the most interesting examples of homogeneous spaces, this fact radically reduces the possibilities of bi-quotients that are related to well-known symmetric spaces.

Section 2.5. The case rank $\underline{H}$ = rank $\underline{G}$ - 1

(For the necessary facts about Lie algebras, see [DY1,2], [BB], and [WO].)

[A] The case $\varphi_*(\underline{H})$ and $\psi_*(\underline{H})$, rank $\underline{G} > 2$ are regular subalgebras.

If rank $\underline{H} > 1$, $\varphi_*(\underline{H})$ and $\psi_*(\underline{H})$ both contains a root of the same length (let us remember that up to normalization the roots are of length 1 or 2).

The Weyl group acts transitively on roots of same lengths. Therefore, we should have, for equivalent Lie algebra homomorphism $\tilde{\varphi}_*$, $\tilde{\psi}_*$, that for some $H_\alpha \in \underline{H}$, $\tilde{\varphi}_*(H_\alpha) = \tilde{\psi}_*(H_\alpha)$, implying the existence of fixed points.

[B] The case rank $\underline{G} = 2$, rank H.

If rank $\underline{H} = 1$, $\underline{H} \simeq Sp(1) \simeq A_1$ simple.

(1) The case G = Sp(2) has been studied by [GM].

The other possible cases are:

(2) $$G = Spin(4) = SU(2) \times SU(2) = Sp(1) \times Sp(1).$$

The only possibility is

$$\varphi : q \to (q,1)$$

$$\psi : q \to (1,q)$$

and we get a particular realization of the sphere $S^3 \simeq Sp(1) = SU(2)$.

(3) $G = Spin(5) \simeq Sp(2)$ and we are in the case (1). (Classical isomorphism between C_2 and B_2.)

The case G_2 deserves special attention.

$\underline{G_2}$ contains <u>two</u> non-conjugated 3-dimensional Lie algebras. In fact, let $\{\alpha, \beta\}$ be a simple system of roots with $\|\alpha\| = 1$, $\|\beta\| = 3$. $\overline{H}_\alpha$ and $\overline{H}_\beta$ may be considered as defining vectors of a 3-dimensional Lie algebra that will be distinguished by the index. The index of the algebra defined by H_α is 1, the index of the algebra defined by H_β is 3. Therefore, the diagonal action is without fixed points.

[C] The case $\varphi_*(\underline{H})$, $\psi_*(\underline{H})$ regular $\underline{G}$, $\underline{H}$ semi-simple rank $\underline{H}$ = rank $\underline{G}$ - 1, rank $\underline{H} > 1$.

In this case, we may apply the theorem of Dynkin [DY1, Theorem 5.5, page 148].

The algebras $\varphi_*(\underline{H})$, $\psi_*(\underline{H})$ being regular, up to conjugation by a representative of the Weyl group, again we have

$$\varphi_*(H_\alpha) = \psi_*(H_\alpha), \ \alpha \text{ a root},$$

and we have fixed points.

<u>Theorem 2.5.1</u>. If G, H are semi-simple with rank G = rank H + 1 and $\varphi_*(\underline{H})$, $\psi_*(\underline{H})$ are regular subalgebras with rank $\underline{H} > 1$, then the diagonal action <u>always</u> has fixed points.

Observation. In [B] we discussed the case rank $\underline{H} = 1$. The theorem does not apply to this case that we settled by a case examination.

Section 2.6. $\varphi_*(\underline{H})$ non-regular, semi-simple, rank $\underline{H}$ = rank $\underline{G}$ - 1

In this case, the solution depends strongly on classification arguments. Again we refer mainly to the articles of Dynkin [DY1,2].

Definition 2.6.1 [Dynkin DY1,2]. A subalgebra that is not regular is called an S-subalgebra.

We remember the following lemma.

Lemma 2.6.1 [BB]. If $\underline{H}$ is a S-subalgebra of $\underline{G}$, then $\underline{H}^{\mathbb{C}}$ is a S-subalgebra of $\underline{G}^{\mathbb{C}}$.

Proof [BB]. Let $\underline{K}$ be a subalgebra of $\underline{G}^{\mathbb{C}}$, K regular and $\underline{H}^{\mathbb{C}} \subset \underline{K}$. If $K^{\mathbb{C}}$ is semi-simple, then by Theorem 7.7 of [DY1] there exists a $\underline{L}$ semi-simple complex in $\underline{G}^{\mathbb{C}}$, $\underline{L} \supset \underline{H}^{\mathbb{C}}$.

If rank($\underline{L}$) = rank($\underline{H}^{\mathbb{C}}$), $\underline{H}^{\mathbb{C}}$ would be regular in L, and therefore regular in $\underline{G}$; then rank($\underline{L}$) = rank($\underline{G}^{\mathbb{C}}$).

H being a compact subalgebra, there exist compact forms $\underline{U}$ of $\underline{L}$ and $\underline{B}$ of $\underline{G}^{\mathbb{C}}$ such that $\underline{H} \subset \underline{U} \subset \underline{B}$. We know that $\underline{B}$ is conjugated of $\underline{G}$ in $\underline{G}^{\mathbb{C}}$. By that conjugation we have that $\underline{H} \subset \underline{U} \subset \underline{B}$ becomes $\underline{H}' \subset \underline{U}' \subset \underline{G}$.

$\underline{H}$, $\underline{H}'$ are subalgebras of $\underline{G}$ conjugated in $\underline{G}^{\mathbb{C}}$. They are, therefore, conjugated in $\underline{G}$ and we obtain $\underline{H} \subset \underline{U}'' \subset \underline{G}$.

Finally, rank(U'') = rank($\underline{U}$) = rank($\underline{B}$) = rank($\underline{G}$) and $\underline{U}'' \neq \underline{G}$ becomes $U \neq B$. $\underline{U}''$ would be a regular subalgebra of $\underline{G}$.

The lemma allows us to use the results of [DY1,2] for S-subalgebras ([DY1,2] has classified them in the complex case).

Proposition 2.6.2. The possible S-subalgebras $\underline{H}$ of $\underline{G}$, $\underline{G}$ semi-simple, rank $\underline{H}$ = rank $\underline{G}$ - 1 are given by the following table.

(1)	$\underline{G} = D_n$	$\underline{H} = B_p \oplus B_{n-p-1}$ $n \geq 3,\ 0 \leq p \leq [\frac{n-1}{2}]$
(2)	$\underline{G} = C_3$	$\underline{H} = A_1 \oplus A_1$
(3)	$\underline{G} = C_4$	$\underline{H} = A_1 \oplus A_1 \oplus A_1$
(4)	$\underline{G} = G_2$	$\underline{H} = A_1$
(5)	$\underline{G} = F_4$	$\underline{H} = A_1 \oplus G_2$
(6)	$\underline{G} = A_2$	$\underline{H} = A_1$
(7)	$\underline{G} = A_1 \oplus A_1$	$\underline{H} = A_1$ (diagonal inclusion)
(8)	$\underline{G} = C_2$	$\underline{H} = A_1$
(9)	$\underline{G} = B_3$	$\underline{H} = G_2$
(10)	$\underline{G} = D_4$	$\underline{H} = \overline{B}_3$
(11)	$\underline{G} = D_4$	$\underline{H} = \overline{B_1 \oplus B_2}$

Observations.

(1) The classes (10) and (11) correspond to (1) for $n = 4$, $p = 0$ and $p = 1$, but by an exterior automorphism of D_4 (triality automorphism).

(2) The inclusions are different from the usual (regular) inclusions.

The proof of the proposition depends on the lemma and of the Theorem 15.1 ([DY1], page 235), Table 3 of [DY1], page 233, and if $\underline{G}$ is a classical algebra we refer to [DY2] Theorems 1.1, 1.2, 1.3, 1.4 (pages 252-253). For $\underline{G}^{\mathbb{C}}$ a classical algebra and $\underline{H}^{\mathbb{C}}$ simple, Theorem 1.5, page 253 of [DY2]. Finally, the table 1, page 364 of [DY2] gives (G_2, A_1) and $(D_4, \overline{B_1 \oplus B_2})$.

The conclusion is:

Theorem 2.6.3. If $\varphi_*(\underline{H})$ and $\psi_*(\underline{H})$ are S-subalgebras, there are no bi-quotients by the diagonal action (i.e., the diagonal action is not fixed point free).

Proof. The table of possibilities gives only $(D_4, B_3, \overline{B}_3)$ and $(D_4, B_1 \oplus B_2, \overline{B_1 \oplus B_2})$; i.e., the case (1) for $n = 4$, $p = 0$ and (10) and case (1) for $n = 0$, $p = 1$. The exterior automorphism is an isometry and has fixed points. For some elements on $\underline{H}$ both homomorphisms agree and we have fixed points.

Theorem 2.6.4. If $\varphi_*(\underline{H})$ is a regular subalgebra and $\psi_*(\underline{H})$ is a S-subalgebra, the possibilities for free actions are:

(1) $(A_1 \oplus A_1, \overline{A}_1$ (diagonal inclusion)

$A_1)$

(2) $(G_2, A_1, \varphi_*(A_1))$, the example of [G-M], φ_* of index 2

(3) $(G_2, A_1, \varphi_*(A_1))$, φ_* of index 4, [DY1]

(4) $(G_2, A_1, \varphi_*(A_1))$, φ_* of index 28 [DY]

(5) $(C_3, A_1^8 \oplus A_1^3, A_1^1 \oplus A_1^1)$

(6) $(C_4, A_1^4 \oplus A_1^4 \oplus A_1^4, A_1^1 \oplus A_1^1 \oplus A^1)$.

(In (5) and (6), the exponent tells the index of the subalgebra.)

Proof. By Proposition 2.6.2, (1), (10), (11) correspond to the case where both subalgebras contain a factor of index 1. (The embedding are obtained via an exterior automorphism which leaves the innerproduct invariant.)

The cases of rank G = 2 will be analysed at the end of this paper. In (5) and (9) of 2.6.2 there is not a regular subalgebra to define a bi-quotient. In (9) and (10) we may observe that the index of the singular subalgebras are different of 1, therefore our criteria for free action holds. ((8) and (8) correspond to very particular homomorphic in the symplectic group.)

CHAPTER III

Section 3.1. The curvature of bi-quotients

Motivation. In the article of [GM], D. Gromoll and W. Mayer gave an example of an exotic sphere of non-negative curvature, however this exotic sphere has sections of zero curvature at some points. It is natural to try to generalize this construction to other bi-quotients of Lie groups and this is the purpose of this chapter.

Let G be a Compact Lie Group. It is classical that G admits a bi-invariant metric that is given, if G is simple, up to a multiple, by the Cartan-Killing form. In general, if $\underline{G} = \underset{i}{\oplus} \underline{G}_i \oplus \mathbb{R}^n$ an ad-invariant innerproduct in $\underline{G}$ is given by $\langle x,y\rangle = \Sigma\, k_i(x_i,y_i) + (x_0,y_0)$ where $x_i,y_i \in G_i$, $x_0,y_0 \in \mathbb{R}^n$, and k_i a positive multiple of the Cartan-Killing form. If G is semi-simple, it is known that the sectional curvature K is nonnegative.

O'Neill's formula for a Riemannian submersion implies that the curvature of the bi-quotient space, with the induced metric, is also nonnegative. It is natural to ask if it is possible to use this construction to obtain manifolds of strictly positive curvature that may give other examples. The basic reference in this section is the article of [GM].

Our results are, essentially, a generalization for (semi-simple) compact Lie groups of their results.

As before, we shall make the hypothesis

$$\varphi_*(\underline{H}) \text{ and } \underline{G}$$

$$\psi_*(\underline{H}) \text{ and } \underline{G} \qquad \text{do not}$$

contain a common ideal. This hypothesis, we remark, corresponds in the homogeneous case to assuming that the action of G on $^{G}/_{\varphi(H)}$ is effective.

The hypothesis that $(\underline{G}, \psi_*(\underline{H}))$ do not contain a common ideal $\underline{G}$ compact Lie algebra, is always assumed by [B], [BB] or [WA1].

<u>Notation</u>. Let M_1, M_2 be Riemannian manifolds with $\dim M_1 \geq \dim M_2$ and let π be a submersion $\pi : M_1 \to M_2$; i.e., π surjective and of maximal rank.

The tangent space $T_q(M_1)$ at q splits into an orthogonal sum

$$V_q \oplus H_q \qquad \text{where } V_q \text{ is the tangent}$$

space to the fiber $\pi^{-1}(\pi(q))$ and H_q the orthogonal complement to V_q. V and H are the vertical and horizontal distribution of the Riemannian submersion.

For a vector field Z on M_1 let us denote by Z^V its vertical component. Any vector field X on M_2 has a unique horizontal lift $\tilde{X}$ on $\tilde{H}$, i.e. $\tilde{X}^V = 0$ and $\pi_* \tilde{X} = X \circ \pi$ (see [GM] or [CE] for the details).

The sectional curvatures $\overline{K}$ of M_1 and K of M_2 are related by O'Neill's formula ([ON] or [CE]).

If X,Y are orthogonal vector fields on an open subset of M_2, then

$$K(X,Y)\circ\pi = \widetilde{K}(\widetilde{X},\widetilde{Y}) + \frac{3}{4}\|[X,Y]^v\|.$$

Following [GM], we consider a Lie group G of Lie algebra $\underline{G}$.

We denote by L and R the left and right translations and by $Ad = R_*^{-1}\circ L_*$ the adjoint representation. For $X \in \underline{G}$ we consider the right and left invariant vector fields L_*X and R_*Y defined by

$$(L_*X)_g = L_g{*}X$$
$$(R_*X)_g = R_g{*}X$$

and we remember the lemma proved in [GM], (page 402).

<u>Lemma 3.1.1</u>. If $\langle,\rangle$ is a bi-invariant metric on G and $X,Y \in \underline{G}$, we define

$$f : G \to \mathbb{R} \text{ by } f(g) = \langle X, Ad(g)Y\rangle$$

then $(L_g{*}Z)f = \langle X, Ad(g)[Z,Y]\rangle$. With this fact present we may generalize the formula of [GM] (page 404).

Let H,G be (compact) Lie groups. [It would be enough that G has a bi-invariant metric] and $\underline{H},\underline{G}$ their Lie algebras.

We consider φ,ψ : H → Lie group homomorphism and $\varphi_*,\psi_* : \underline{H} \to \underline{G}$ the corresponding Lie algebra homomorphisms.

We have then the possibility of defining the bi-quotient by the diagonal action.

(With obvious modifications we may derive an entirely analogous formula for the double product action.)

We have the fibration $H \to G \to M(1)$, M being the bi-quotient, and we take G with a bi-invariant metric $\langle , \rangle$. With the help of the metric we may give the fibration (1) the structure of a Riemannian Fibration. Then we have

Theorem 3.1.2. Let $A \in \underline{H}$, $g \in G$.

(1) The tangent space to the fibre $\pi^{-1}(\pi(g)$ of $\pi : G \to G$ is spanned by

$R_{g*}\varphi_*(A) - L_{g*}\psi_*(A) \qquad A \in \underline{H}$

(2) Let $u,v \in \underline{G}$ be orthonormal vector such that

$\tilde{u} = L_{g*}u \in H_g$

$\tilde{v} = L_{g*}v \in H_g$ then

$$K(\pi_*\tilde{u},\pi_*\tilde{v}) = \frac{1}{4}\|[u,v]\|^2 + \frac{3}{4} \operatorname*{Max}_{\substack{A\neq 0\\ A\in \underline{H}}} \frac{\langle Ad_g^{-1}(\varphi_*(A))+\psi_*(A),[u,v]\rangle}{\|Ad(g^{-1})\varphi_*(A)-\psi_*(A))\|^2}$$

In particular, $K(\pi_*\tilde{u},\pi_*\tilde{v}) = 0$ if and only if $[u,v] = 0$.

Proof.

(1) Let us consider the curve

$$\Phi_A(t) = (\exp t\varphi_*(A))g(\exp -t\psi_*(A))$$

then $\Phi'_A(0) = R_{g*}\varphi_*(A) - L_{g*}\psi_*(A)$

by Leibniz rule.

$\Phi'_A(0)$ is clearly a vertical vector tangent to the orbit of g. Also, $\{\exp tA(A \in \underline{H})\} = H$.

(2) Let $A \in \underline{H}$. We define a 1-form on G by

$$\omega_A(x)_g = \langle R_{g*}\varphi_*(A) - L_{g*}\psi_*(A), X_g\rangle$$

then $H_g = \{w \mid \omega_A(w) = 0 \quad \forall A \in \underline{H}\}$.

We know that, on G with the bi-invariant metric $\tilde{K}(\tilde{X},\tilde{Y})_g = \overline{K}(u,v) = \frac{1}{4}\|[u,v]\|^2$, the second term in the O'Neill formula is $\frac{3}{4}\|[X,Y]^v\|^2$. By the same method as that of [GM], we shall compute this term. We have, by the properties of Linear functionals

$$\|w^H\|^2 = \underset{A\in\underline{H}}{\text{Max}} \frac{\omega_A(w)^2}{\|R_{g*}\varphi_*(A) - L_{g*}\psi_*(A)\|^2}$$

$$= \underset{A\in\underline{H}}{\text{Max}} \frac{\omega_A(w)^2}{\|Ad_g^{-1}\ \varphi_*(A) - \psi_*(A)\|^2}$$

and therefore $\|[\tilde{X},\tilde{Y}]_g^v\|^2 = \underset{A\in\underline{H}}{\text{Max}} \dfrac{\omega_A([\tilde{X},\tilde{Y}]_g)^2}{\|Ad(g^{-1})\varphi_*(A) - \psi_*(A)\|^2}$.

As in [GM], we have by the properties of the exterior derivative

$$2d\omega_A(\tilde{X},\tilde{Y}) = \tilde{X}\omega_A(\tilde{Y}) - \tilde{Y}\omega_A(\tilde{X}) - \omega_A([\tilde{X},\tilde{Y}])$$

$$= -\omega_A([\tilde{X},\tilde{Y}])$$

since $\tilde{X},\tilde{Y}$ are horizontal, and

$$d\omega_A(\tilde{X},\tilde{Y})_g = d\omega_A(\tilde{u},\tilde{v}).$$

On the other hand, we have

$$2d\omega_A(\tilde{u},\tilde{v}) = 2d\omega_A(L_{g*}u, L_{g*}v)$$

$$= (L_{g*}u)\omega_A(L_*v) - (L_{Q*}\omega_A)(L_*u)$$

$$- \omega_A(L_g[u,v]).$$

We get from the definition of ω_A and the relation $(L_{g*}u)\langle a, Ad_g b\rangle = \langle a, Ad_g[u,b]\rangle$ already mentioned:

$$(L_{g*}u)\omega_A(L_*v) = (L_{g*}u)\langle \varphi_*(A), Ad_g v\rangle$$

$$= \langle \varphi_*(A),\ Ad_g[u,v]\rangle$$

and $(L_{g*}v)\omega_A(L_*u) = \langle \varphi_*(A), Ad_g[u,v]\rangle$

Finally:

$$\omega_A(L_{g*}[u,v]) = \langle \varphi_*(A) - Adg\,\psi_*(A),\ Adg[u,v]\rangle,$$

using the fact that the innerproduct is adjoint-invariant.

Combining the different terms we get

$$\omega_A([\tilde{X},\tilde{Y}])_g = -\langle Adg^{-1}\varphi_*(A) + \psi_*(A), [u,v]\rangle$$

and we get the announced formula.

GED

Observations

(1) We should get a completely equivalent formula with $Ad_g \psi_*$ in the corresponding place and the obvious changes.

(2) It is clear that $K(u,v) = 0$ with $u = \pi \tilde{u}$, $v = \pi_* \tilde{v}$ if and only if $\tilde{u}, \tilde{v}$ are horizontal at the point g and $[\tilde{u}, \tilde{v}] = 0$.

Our problem will be, in favorable cases, to compute the curvature to verify if it is zero for some sections. Essentially, the method is an adaption of the method of [B] to our purposes. Let us notice that, in general, the curvature of a bi-quotient is more complicated that the curvature of a homogeneous space.

Commentary. If we observe the formula for the curvature, we notice the denominator $\|Ad_g^{-1}\varphi_*(A) - \psi_*(A)\|^2$.

This expression will not be defined if there is $A \in \underline{H}$ such that $\|Ad_g \varphi_*(A) - \psi_*(A)\| = 0$; i.e., $Ad_g \varphi_*(A) = \psi_*(A)$. But this is precisely the Lie algebra condition for the existence of fixed points.

Section 3.2.

We shall see in this section that the necessary conditions for the existence of positive curvature are incompatible, in the semi-simple case, with the free action of the group H via the two different homomorphism, with the exception of some isolated cases in low dimensions.

Our basic result will be analogous of one of the proposition of [B].

Proposition 3.2.1. Let $\varphi, \psi : H \to G$ be as usual and M the bi-quotient. If rank H < rank G-1 then $K_M = 0$ for some pair of tangent vectors at some point $\pi(g) \in M$.

Proof. By our original analysis (see Chapter II) we may take φ, ψ such that

$$\varphi(T_H) \subset T_G$$

$$\psi(T_H) \subset T_G$$

for a fixed maximal torus of G, T_G, and a fixed maximal torus of H, T_H.

(Let us remember that, if it were not the case, we may replace φ and/or ψ by equivalent homomorphism up to an inner-automorphism and the equivalent homomorphism will satisfy the condition.)

The proof now, will be very similar to Berger's result [R]. We need the following lemma.

Lemma 3.2.2. Let $\underline{G}$ be a compact Lie algebra with ad-invariant metric $\underline{H} \subset \underline{G}$ subalgebra, C_G, C_H the Cartan subalgebras of $\underline{G}$ and $\underline{H}$ respectively, such that $C_H \subset C_G$. Take $\underline{P}$ an orthogonal complement of C_H in $\underline{H}$; i.e., $\underline{H} = C_H \oplus \underline{P}$. Then $\underline{P}$ is orthogonal to C_H on $\underline{G}$ [the essential point will be the ad-invariance of the metric].

Proof. Let us take $X \in C_{\underline{G}}$ and X orthogonal to $C_{\underline{H}}$. We have $(C_H \subset C_G)$, $[X, C_H] = 0$.

Choose (as in [B]) $\underline{M}$ an orthogonal complement of $\underline{H}$, respectively to the metric; i.e., $\underline{G} = \underline{H} \oplus \underline{M}$ and decompose orthogonally

$$X = X_H \oplus X_M \qquad X_H \in \underline{H},\ X_M \in \underline{M}\ .$$

We have for $h \in \underline{H}$, $h' \in \underline{H}$, $m \in \underline{M}$

(1) $[h, h'] \in \underline{H}$ (Lie algebra condition)

$\langle [h,m], h' \rangle = \langle m, [h,h'] \rangle = 0$

(ad-invariance of the metric)

$\Rightarrow [h,m] \in \underline{M}$

now $[X, C_H] = 0 \Rightarrow [X_M, C_H] = 0$

$$[X_H, C_H] = 0$$

because $\qquad [X_M, C_H] \subset \underline{M}$

$$[X_H, C_H] \subset \underline{H}$$

Therefore $X_M \in C_H$ (C_H is a maximal abelian subalgebra of $\underline{H}$).

Since X is orthogonal to $C_{\underline{H}}$ we have

$$X_H = 0$$

$$X = X_M \in M$$

$$\Rightarrow X_M \text{ orthogonal } \underline{H} .$$

With the help of the Lemma 3.2.2 we may prove the proposition. Let us take C_H, C_G Cartan subalgebras corresponding to T_H, T_G and φ, ψ such that

$$\varphi_*(C_H) \subset C_G$$

$$\psi_*(C_H) \subset C_G$$

$\underline{H}$ being reductive, $\varphi_*(C_H)$ is a Cartan subalgebra of $\varphi_*(\underline{H})$ for any homomorphism $\varphi_* : \underline{H} \to \underline{G}$ so if $\underline{H} = C_H \oplus \underline{P}$, $x \in \underline{P} \Rightarrow \varphi_*(X) \in \varphi_*(\underline{P})$ (invariance of the metric). Let us take $y \in C_G$, y orthogonal to $(\varphi_* - \psi_*)(C_G)$. By the previous observation and the lemma, y is orthogonal to $\varphi_*(P)$ and $\psi_*(P)$.

Now, if $t \in C_H$, $h \in \underline{P}$ we have

$$\langle \varphi_*(h) - \psi_*(h), \varphi_*(t) - \psi_*(t) \rangle$$

$$= -\langle \varphi_*(h), \psi_*(t) \rangle - \langle \psi_*(h) \varphi_*(t) \rangle = 0$$

because $\varphi_*(t)$ and $\psi_*(t) \in C_G$ and $\varphi_*(t)$, $\psi_*(t)$ are orthogonal to $\varphi_*(\underline{P})$, $\psi_*(\underline{P})$.

Therefore, $y \in \underline{G}$ and y orthogonal to $(\varphi_* - \psi_*)(\underline{H})$. But $(\varphi_* - \psi_*)(\underline{H})$ is the vertical subspace of the submersion at the identity, i.e., $y \in H_e$.

Since the orthogonal elements to $(\varphi_* - \psi_*)(C_H)$ in C_G are an abelian subalgebra of $\underline{G}$, $K > 0$ implies there are no linearly independent vectors t_1, t_2 in H_e such that $[t_1, t_2] \neq 0$ and so the dimension of such subalgebra should be 0 or 1.

GED

Observations.

(1) With trivial modifications the proposition would be due also for the double product action.

(2) If we replace φ or ψ by homomorphism $\tilde{\varphi}$ or $\tilde{\psi}$ such that, say $\tilde{\varphi}(T_H) \subset T_G$, the conclusion $K = 0$ at the projection of the identity $\pi(e)$ will still follow.

(3) It is clear how to prove the proposition if $\varphi(C_H) \subset C_G$ but $\psi(C_H) \not\subset C_G$, from the fact that all maximal tori are conjugate.

In general, we should take specific homomorphism, in the equivalent class in order to make possible a conclusion without excess of computation. In most of the cases, we shall assume that φ_*, ψ_* are such that they preserve the ordering induced by a system of simple roots.

Section 3. Classification of bi-quotients of positive curvature

Proposition 3.3.1. Let $\varphi, \psi : H \to G$ where rank H = rank G and G, H are semi-simple.

Our construction does not give any bi-quotient with $K_M > 0$.

Proof. From our results of Chapter 2, a subalgebra of rank equal to rank $\underline{G}$ is always regular and, therefore, there are fixed points.

Proposition 3.3.2. Let $\varphi, \psi : H \to G$, where H, G are semi-simple, be such that $\varphi_*(\underline{H})$, $\psi_*(\underline{H})$ are regular subalgebras of $\underline{G}$.

If rank $G > 2$, there are no bi-quotients of positive curvature, with the possible exception of $(C_3, A_1 \oplus A_1)$ and $(C_4, A_1 \oplus A_1)$.

Proof. We apply our results for the case rank $\underline{H}$ = rank $\underline{G} - 1$ rank $\underline{G} > 2$ as in Chapter 2 and we get, again, that the condition $K_M > 0$ is incompatible with a free action.

The case $n = 2$ requires a particular analysis:

(1) In fact, there is the example of [GM]. This manifold, an exotic sphere Σ^7 has $K > 0$ at the projection of the identity. However, there is a lower dimensional subset with 0 sections.

(2) The case $G = Spin(4) = S^3 \times S^3 = Sp(1) \times Sp(1)$ does not give any new result. The diagonal inclusion of S^3 and a homomorphism of the kind $q \mapsto (q,1)$ (a) gives an ordinary sphere, (b) in fact $\varphi_* : A \to (A,0)$ is such that $\varphi_*(A_1) \cap (A_1 \oplus A_1)$ does have a common ideal.

(3) The case $Sp(1) \times Sp(1)$

with $\varphi : Sp(1) \to Sp(1) \times Sp(1)$
$\psi : Sp(1) \to Sp(1) \times Sp(1)$

$\varphi : q \mapsto (q,1)$
$\psi : q \mapsto (1,q)$ has the same conclusion.

(4) Spin(5) is, as it is well-known, isomorphic to Sp(2) [HE] and we have a different presentation of [G-M], example Σ.

Note [E] (preprint) has succeeded in generalizing the construction of [Wallach] to a double action of $S^1 \times S^1$ on SU(3), and he got positive curvature with a $S^1 \times S^1$ invariant metric (no bi-invariance), i.e., the Lie algebra pair $(A_2, \mathbb{R}^2)$.

The case A_2

A_2 does not fall in our previous considerations because rank $A_2 = 2$.

A semi-simple subalgebra is necessarily isomorphic to A_1. There are two such subalgebras. The first (up to conjugation) regular subalgebra described by SU(1) → SU(3)

$$\begin{bmatrix} i\alpha & \beta+i\gamma \\ \beta-i\gamma & -i\alpha \end{bmatrix} \to \begin{bmatrix} -i\alpha & \beta+i\gamma & 0 \\ \beta-i\gamma & -i\alpha & \\ 0 & 0 & 0 \end{bmatrix}$$

(Subalgebra of index 1)

The second subalgebra can be described by the representation of A_1 of degree 3 given by (up to conjugation)

$$\to \begin{bmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2i & 0 \\ -i & 0 & 2i \\ -0 & -i & 0 \end{bmatrix}$$

Both representations being of different index (1 and 4 respectively) we do not get fixed points at the Lie algebra level.

A discrete set of fixed points can be discarded in the following way. Up to conjugation such points are necessary in the maximal torus. Our representations restricted to it are given by

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \rightarrow \begin{bmatrix} z & 0 & 0 \\ 0 & \bar{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \rightarrow \begin{bmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{z}^2 \end{bmatrix}$$

(Up to conjugations)

We would have a fixed point if

$$\begin{bmatrix} z & 0 & 0 \\ 0 & \bar{z} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} z^2 & 0 & 0 \\ 0 & \bar{z}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $z = 1$.

The element $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is obviously central.

Also, $$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \varphi(-I_2)$$

$\neq \psi(-I_2) = I_3$.

The manifold has the same homotopy sequence that the sphere S^5 and its curvature at the origin is positive.

The Case $B_2 = C_2$

The case $B_2 = C_2$ has been studied by [G-M]. We remember that, for the Lie algebra A_1 they get an exotic sphere of positive curvature at the projection of the identity with sections of zero curvature at some points.

Their main case is described by

$$q \in Sp(1) \qquad q \in \mathbb{H} \qquad q\bar{q} = 1$$

$$\varphi : q \mapsto \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \in Sp(2)$$

$$\psi : q \mapsto \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \in Sp(2).$$

There is other non-equivalent representations that can be better described in terms of the complex realization of $Sp(2) \subset SU(4)$.

The representation is the third complex representation of $Sp(1) = SU(2)$ and it has been used by Berger [B].

In terms of the Lie algebra the representation is given by

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \mapsto \begin{bmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 3i & 0 & 0 \\ i & 0 & 4i & 0 \\ 0 & i & 0 & 3i \\ 0 & 0 & i & 0 \end{bmatrix}$$

Let us call it θ

Index $\theta = 7$

Index $\varphi = 1$

Index $\psi = 2$

so well for the pairs (θ,φ) as (ψ,θ) the index does not detect fixed points. We shall use the same method with A_2.

If there are fixed points, we may, up to conjugation find them in a maximal torus. We restrict to a maximal torus. Thereby the representation φ is given

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} z & 0 & 0 & 0 \\ 0 & \bar{z} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ψ by

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} z & 0 & 0 & 0 \\ 0 & \bar{z} & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & \bar{z} \end{bmatrix}$$

and θ by

$$\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} z^3 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & \bar{z} & 0 \\ 0 & 0 & 0 & \bar{z}^3 \end{bmatrix}.$$

For the pair φ, θ we should have (up to conjugation)

$z^3 = 1$ i.e. $z = 1$, $w = \frac{1}{2}(1+\sqrt{3}i)$

$(z = z)$

$\bar{w} = w^2 = \frac{1}{2}(1-\sqrt{3}i)$

The element $\begin{bmatrix} w & 0 \\ 0 & \bar{w} \end{bmatrix}$ has the same image in $Sp(2)$ by φ or ψ.

The subset fixed by $\begin{bmatrix} w & 0 \\ 0 & \overline{w} \end{bmatrix}$ may be described in a precise way.

Similarly, for ψ, θ we have to find z such that

$$z^3 = z \qquad \text{i.e. } z^2 = 1 \qquad z = \pm 1.$$

The element $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has as image either by ψ or θ the central element $-I_4$. We can pass to the quotient $Sp(2)/_{\mathbb{Z}_2} = SO(5)$ and we get an action of SU(2) on SU(5).

The quotient manifold by this action is an ordinary sphere. (Oral communication of Professor W. Meyer.)

The Case G_2

Let us notice in the first place that (G_2, A_1) is not one of the pairs of Berger [B]. Neither is $(G_2, A_1 \oplus A_1)$.

G_2 contains four 3-dimensional Lie algebras:

1) A^1 generated by the shortest root
2) A^3 generated by the longest root
3) A^4 the diagonal subalgebra of index 4
   $A^1 \oplus A^3 \simeq A_1 \oplus A_1$ and
4) A^{28}, a singular subalgebra

(see [D1] page 172, 176 and Chapter 9).

(The exponent correspond to the index of the subalgebral)

The fact that $(G_2, A_1 \oplus A_1)$ is not one of the pairs of Berger, eliminates immediately the possibilities

$$\varphi(A_1) = A^1 \qquad \psi(A) = A^3$$
$$\varphi(A_1) = A^1 \qquad \psi(A_1) = A^4$$
$$\varphi(A_1) = A^3 \qquad \psi(A_1) = A^4 .$$

In all cases the horizontal subspace contains the ortogonal complement to $A^1 \oplus A^3$ and [B] such complement contains a couple of commuting vectors.

For the singular subalgebra A^{28}, we use the description of [BB] (page 65) and [DY1] (page 163-173). The Cartan subalgebra is generated by a vector proportional to $H_{3\alpha_1+\alpha_2}$ with the usual description of the roots of G_2 (see [H]).

The decomposition of G_2 can be written down explicitely. There are 6 positive roots with the usual relations, i.e. $G_2 = C \oplus \sum_{i=1}^{6} G(\alpha_i)$ where α_1 and α_2 are the simple roots.

We can always assume

$A^1 \subset C \oplus G(\alpha_1)$ (equivalently any root of length 1)

$A^3 \subset C \oplus G(\alpha_2)$ (equivalently any root of length 3).

Also A^{28} has a basis H, X^+, Y^- with $X^+ = X_1 + X_2$

$$X^- = X_1' + X_2'$$

$(X_1, X') \in \alpha_1 (X_2, X_2') \in \alpha_2$ (with the notation of [B]) ([DY1], page 163). In the same way of [B] or [BB] we find a pair of vectors X,Y that

$$X \perp A^1 \quad \text{and} \quad X \perp A^{28}$$

$$Y \perp A^1 \quad \text{and} \quad X \perp A^{28}$$

and $[X,Y] = 0$. We have, therefore, zero curvature at the projection of the identity for these vectors.

The case of the subalgebra $A^4 \subset A^1 \oplus A^3$ is completely analogous and the conclusion is similar. (We have, however, take $A^4 \subset G(\alpha_1 1 \oplus G(3\alpha_1+2\alpha_2)$ and a slightly different description of A^{28}.)

This analysis concludes the study of diagonal actions of A_1 if the rank of the Lie group is 2.

FINIS

P.S. We expect to cover, in the near future the cases $(C_3, A_1 \oplus A_1, A_1 \oplus A_1)$ and $(C_4, A_1 \oplus A_1 \oplus A_1, A_1^4 \oplus A_1^4 \oplus A_1^4)$.

Appendix

Wallach found three new pairs that give homogeneous spaces of strictly positive curvature. They are:

$(SU(3), T^2)$

$(Sp(3), SU(2) \times SU(2) \times SU(2))$,

and $(F_4, \text{Spin } 8)$.

For all of them, rank G = rank H. The first example does not fit in our considerations, because T^2 is not semi-simple. It has been studied by [E] (preprint).

The last two do fit into our general considerations. We have proved that if rank H = rank G, then there are always fixed points (for the diagonal and the product action and for any pair of semi-simple Lie groups). Therefore, it is not possible to define a bi-quotient for those pairs, independently of the metric that we put on G.

Behrard-Berger [BB] showed that there are no new odd dimensional examples, with G's that arise from a left invariant metric. Our analysis of fixed points has used his pairs and we conclude that, besides the case rank G = 2, there are always fixed points.

The fact that O'Neill's formula is simply curvature non-decreasing and the existence of many pairs of vectors on G such that [X,Y] = 0 makes it unlikely that more general metrics will provice new examples of positive curvature.

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