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Characteristic Classes for Modules over Cyclic Groups

by

Iris Cox Hayslip

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

Characteristic Classes for Modules over Cyclic Groups

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In this paper we study the Charlap-Vasquez characteristic classes $v^i(M) \in H^2(\Phi; H^{i-1}(M, H_1(M,k)))$ for the case where M is an n-dimensional vector space over the field $\mathbb{F}_p$, p prime, and $\Phi \subseteq GL(n,p)$. We employ a special resolution of $\mathbb{Z}$ for M, particular choices of subgroups Φ, and the chain homotopy techniques of Charlap and Vasquez to obtain results on the classes $v^i(M) \in H^2(\Phi; H^{i-1}(M, H_1(M,k)))$ for $(n,p) = (2,3)$ and $(3,2)$. These together with the Naturality Theorem of Charlap and Vasquez provide some results on the universal classes $\bar{v}^i(M) \in H^2(GL(n,p); H^{i-1}(M, H_1(M,k)))$.

In memory of Innes Warner Cox

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INTRODUCTION

Suppose Φ is a group, M a module over $\mathbb{Z}[\Phi]$ the integral group ring of Φ and let D_* be any resolution of $\mathbb{Z}$ for M. One cannot in general define an action of Φ on the modules D_i in a natural way, but the notion of a Φ-system approximates an action of Φ on D_i. One part of a Φ-system consists for each i and each $\sigma \in \Phi$ of a $\mathbb{Z}$-homomorphism $A_i(\sigma) : D_i \to D_i$ which is a chain map satisfying an appropriate semi-linearity condition. Now $A_i(\sigma) \circ A_i(\tau) \neq A_i(\sigma\tau)$ in general, but they are chain homotopic via $U_i(\sigma,\tau) : D_i \to D_{i+1}$ and these maps U_i are a measure of the obstruction to the existence of an action of Φ on D_i. For each $\sigma,\tau \in \Phi$, $U_i(\sigma,\tau) \in \mathrm{Hom}(D_i, D_{i+1})$ defines an element in $\mathrm{Hom}(H_i(M,k), H_{i+1}(M,k))$ which is $H^i(M; H_{i+1}(M,k))$ if the coefficient group k is suitably chosen. This defines a 2-cochain w^{i+1} for Φ with coefficients in $H^i(M; H_{i+1}(M,k))$ which is a cocycle, if the resolution D_* is suitably chosen (i.e. small), and the cohomology class corresponding to w^{i+1} is

$$v^{i+1}(M) \in H^2(\Phi; H^i(M, H_{i+1}(M,k)))$$

and called the characteristic class of M. The characteristic classes $v^i(M)$ depend only on Φ, M and the action of Φ on M.

Now suppose we have the extension $0 \to M \to \pi \to \Phi \to 1$ where π is the semidirect product of Φ and M. In [2]

Charlap and Vasquez showed that the map $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ in the Hochschild-Serre spectral sequence for this extension can be obtained by forming the cup product with the class $v^r(M)$. When M is free Abelian and finitely generated so we can consider $\Phi \subseteq GL(n,\mathbb{Z})$, Charlap and Vasquez [3] have computed the classes $v^r(M)$ and proved various theorems about characteristic classes in this case including:

(*) Theorem: If M is $\mathbb{Z}$-free, then $v^2(M) = 0$ iff $v^n(M) = 0$ for all n.

The problem considered in this paper is that of determining the classes $v^i(M)$ where M is a vector space of dimension n over the finite field $\mathbb{Z}_p$, p prime, and where $\Phi \subseteq GL(n,p)$. Some results on this problem were obtained by Sah in [8] and [9] by methods other than the use of the chain homotopies mentioned above; in particular $v^2(M)$ has another group theoretical interpretation which yields results including the fact that $v^2(M) \neq 0$ for $\Phi = GL(n,2)$ where $n > 3$. However, no such interpretation is known for $v^i(M)$ with $i > 2$.

In this paper we develop some computational techniques for $v^i(M)$ when $\Phi \subseteq GL(n,p)$, for small n and p, using a special choice of resolution D_* and direct construction of Φ-systems in low dimensions. These techniques are carried out to compute:

i) $v^2(M)$ and $v^3(M)$ where $\Phi \subseteq GL(2,3)$ and

ii) $v^2(M)$ and $v^3(M)$ where $\Phi \subseteq GL(3,2)$.

We hope the procedures used in i) could be extended (with use of a computer) to find some $v^i(M)$ for $i > 2$, $\Phi \subseteq GL(2,p)$ and $p > 3$. Moreover, it may be possible to extend the procedures used in ii) to find $v^i(M)$ for $\Phi \subseteq GL(n,2)$ with $n > 3$, $i > 2$. The main result obtained from ii) is the fact that no theorem analogous to (*) exists in the case where M is finite.

The calculations rely on particular choices of subgroups $\Phi \subseteq GL(n,p)$. Results on the classes $\bar{v}^i(M) \in H^2(GL(n,p); H^i(M; H_{i+1}(M,k)))$ for the whole group $GL(n,p)$ are obtained by using the naturality theorem in [3], which says that if $v^i(M) \in H^2(\Phi; H^{i-1}(M, H_i(M,k)))$, $\Phi \subset GL(n,p)$, $v^i(M) \neq 0$ then $\bar{v}^i(M)$ is also non-zero. Moreover, if Φ is chosen to be a p-Sylow subgroup of $GL(n,p)$ then $v^i(M) = 0$ will imply the universal class $\bar{v}^i(M) = 0$.

Preliminaries and the Free Abelian Case

Let Φ be a group, and M a module over $\mathbb{Z}[\Phi]$, the integral group ring of Φ; i.e. there is a homomorphism $\phi : \Phi \to \mathrm{Aut}(M)$. M is called a Φ-module and the action of Φ on M is written $\sigma \circ m$ where $\sigma \cdot m = \phi(\sigma)(m)$. Since M is itself a group, it also can act on another group to form an M-module. Let $\overline{D} = (D_*, d_*)$ be a projective resolution of the trivial M-module $\mathbb{Z}$, i.e. an exact sequence of projective M-modules

$$\overline{D} : \cdots \to D_n \xrightarrow{d_n} D_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d} D_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Definition 1: (Φ-system). A Φ-system for $\overline{D}$ consists of two sequences of functions:

$$A_n : \Phi \to \mathrm{Hom}_{\mathbb{Z}}(D_n, D_n) \text{ and}$$

$$U_n : \Phi \times \Phi \to \mathrm{Hom}_{\mathbb{Z}}(D_n, D_{n+1})$$

with the following properties:

i) $\varepsilon A_0(\sigma) = \varepsilon$

ii) $d_n A_n(\sigma) = A_{n-1}(\sigma) d_n$, $n \geq 1$

iii) $A_n(\sigma)$ is σ-linear, i.e.

$$A_n(\sigma)(r \cdot x) = \sigma(r) \circ A_n(\sigma)(x) \text{ for } x \in D_n,\ r \in \mathbb{Z}[M]$$

i') $d_1 U_0(\sigma, \tau) = A_0(\sigma\tau) - A_0(\sigma) \circ A_0(\tau)$

ii') $d_{n+1} U_n(\sigma, \tau) + U_{n-1}(\sigma, \tau) d_n = A_n(\sigma\tau) - A_n(\sigma) \circ A_n(\tau)$, $n \geq 1$.

iii') $U_n(\sigma, \tau)$ is $\sigma\tau$-linear, i.e.

$$U_n(\sigma, \tau)(rx) = \sigma\tau(r) \cdot U_n(\sigma, \tau)(x) \text{ for } x \in D_n,\ r \in \mathbb{Z}[M].$$

If Γ is both an M-module and a Φ-module such that $\sigma(m\cdot\gamma) = \sigma(m)\cdot\sigma(\gamma)$ for all $\sigma\in\Phi$, $m\in M$, $\gamma\in\Gamma$, then A_* can be extended from D_* to $\mathrm{Hom}_{\mathbb{Z}[M]}(D_*,\Gamma)$ by

$$(A_n(\sigma)\cdot f)(x_n) = \sigma(f(A_n(\sigma^{-1})x_n)).$$

This map induces an action of Φ on $H^n(M,\Gamma)$ which is independent of the Φ-system and the choice of resolution $\overline{D}$. Analogously, there is an action of Φ on $H_n(M,\Gamma)$ induced from

$$\sigma(x_n\otimes\gamma) = [A_n(\sigma)(x_n)] \otimes \gamma \text{ for } x_n\in D_n,\ \gamma\in\Gamma.$$

This action of Φ on $x\in H_n(M,\Gamma)$ will be denoted by $\sigma_*(x)$.

Now let k be a principal ideal domain on which M acts trivially. The universal coefficient theorem yields

$$H^n(M,A) \cong \mathrm{Hom}_k(H_n(M,k),A)$$

where A is a k-module on which M acts trivially. The group $H_n(M,k)$ can be considered as a trivial k-module and trivial M-module so is a possible choice for coefficient group A.

Let $f^n\in \mathrm{Hom}_{\mathbb{Z}[M]}(D_n,H_n(M,k))$ be a cocycle representing the cohomology class in $H^n(M,H_n(M,k))$ corresponding to the identity map in $\mathrm{Hom}_k(H_n(M,k), H_n(M,k))$. Then $A_n(\sigma)\cdot f^n$ represents the same cohomology class so for each $\sigma\in\Phi$ there exists $F_\sigma^{n-1}\in \mathrm{Hom}_{\mathbb{Z}[M]}(D_{n-1},H_n(M,k))$ such that $A_n(\sigma)f^n - f^n = F_\sigma^{n-1}d_n$. Define $u^n(\sigma,\tau)\in\mathrm{Hom}_{\mathbb{Z}[M]}(D_{n-1},H_n(M,k))$ by

Definition 2:

$$u^n(\sigma,\tau) = A_{n-1}(\sigma)\cdot F_\tau^{n-1} - F_{\sigma\tau}^{n-1} + F_\sigma^{n-1} + (\sigma\tau)_*[f^n U_{n-1}(\tau^{-1},\sigma^{-1})].$$

Theorem I: $u^n(\sigma,\tau)$ is a cocycle representing an element $w^n(\sigma,\tau)\epsilon H^{n-1}(M,H_n(M,k))$. And $w^n : \Phi \times \Phi \to H^{n-1}(M,H_n(M,k))$ is a 2-cocycle for Φ and represents an element $v^n \epsilon H^2(\Phi;H^{n-1}(M,H_n(M,k)))$. Furthermore, the cohomology class of w^n depends only on the action of Φ on M. (It is independent of choices of f^n, F_σ^{n-1}, the Φ-system and resolution $\overline{D}$.)

Definition 3 (Characteristic Class): The cohomology class of $w^n \epsilon H^2(\Phi;H^{n-1}(M,H_n(M,k)))$ is called the n^{th} characteristic class of M and is denoted $v^n(M)$ or v^n.

Theorem II (Naturality): Let $h : \Phi' \to \Phi$ be a group homomorphism. Then Φ' acts on M via h and the action of Φ, i.e. $\sigma'(m) = (h(\sigma'))m$. Denote by M' the Φ'-module M under this action. Then h induces a homomorphism

$$h^* : H^2(\Phi,H^{n-1}(M,H_n(M,k)) \to H^2(\Phi',H^{n-1}(M',H_n(M',k)))$$

and $h^*(v^n(M)) = v^n(M')$.

The proofs of theorems I and II can be found in [3].

Suppose M is taken to be free Abelian of rank n and Φ a subgroup of $GL(n,\mathbb{Z})$. Then Theorem II implies that a nonzero characteristic class $v^i \epsilon H^2(\Phi;H^{i-1}(M,H_i(M,k)))$ will lift to a non-zero class $\overline{v}^{i} \epsilon H^{2}(GL(n,\mathbb{Z});\ H^{i-1}(M,H_{i}(M,k)))$. The classes $\overline{v}^{i} \epsilon H^{2}(GL(n,\mathbb{Z});\ H^{i-1}(M,H_{i}(M,k)))$ will be called universal. Also, M may be taken to be a module of rank n over a finite field, $\mathbb{F}_{p}$, of prime characteristic and Φ a subgroup of $GL(n,p)$.

The principal importance of the characteristic classes arises from the interpretation of the second differential in the Hochschild-Serre spectral sequence for a split group extension. Let G = the semidirect product of M and Φ. Given the extension

$$0 \to M \to G \to \Phi \to 1$$

there is a spectral sequence (Hochschild-Serre) $E_{r}^{p,q}(B) \Rightarrow H^{n}(G;B)$ where $E_{2}^{p,q}(B) = H^{p}(\Phi;H^{q}(M,B))$ for a suitable coefficient module B. In [2] and [3] Charlap and Vasquez showed that the second differential $d_{2}^{p,q} : E_{2}^{p,q}(B) \to E_{2}^{p+2,q-1}(B)$ in this spectral sequence may be interpreted as a cup product with the characteristic class v^{n}.

Another interpretation of the characteristic classes is as follows. Let M a Φ-module and $\overline{D} : \cdots D_{n} \xrightarrow{d_n} D_{n-1} \to \cdots \to D_{0} \to \mathbb{Z} \to 0$ any resolution of $\mathbb{Z}$ for M. Then Φ doesn't act on the component modules D_{i} in any natural way. But a Φ-system for $\overline{D}$ is an approximation to an action of Φ on D_{i}, i.e. if there are no $\mathbb{Z}$-homomorphisms $A_{i}(\sigma) : D_{i} \to D_{i}$ which satisfy $A_{i}(\sigma) \circ A_{i}(\tau) = A_{i}(\sigma\tau)$

then the homotopy part of the Φ-system $U_i(\sigma,\tau)$ is non-zero and measures the obstruction to an action of Φ on D_i.

Also in [3], Charlap and Vasquez computed $v^1(M)$ for M a $\mathbb{Z}$-free module and as a result proved the following.

<u>Theorem III</u>: If M is $\mathbb{Z}$-free, $2v^1(M) = 0$.

<u>Theorem IV</u>: If M is $\mathbb{Z}$-free, then $v^2(M) = 0$ iff $v^i(M) = 0$ for all i.

(An alternate proof of Theorem III which doesn't use Φ-systems or specific calculation of $v^1(M)$ appears in [8].)

Finite Case

We consider in the following, some cases where M is a vector space of dimension n over a field with p elements, p a prime, and Φ is a subgroup of GL(n,p). We wish to investigate the characteristic classes $v^i(M)$ and to determine if a theorem analogous to Theorem IV exists in this case.

No formula for computing v^n in this case is known, however, some of the groups $H^2(GL(n,p);\ H^{n-1}(M,H_n(M,k)))$ which contain the universal classes have been computed in [8] by other methods, (see Appendix). We compute v^2 and v^3 in some examples for finite M and particular subgroups of GL(2,3) and GL(3,2). The procedure constructing the Φ-system in these cases in theory works for GL(n,p) with arbitrary p and n if Φ is chosen to be particularly simple but the technique is very unwieldy. The question arises: which subgroups of GL(n,p) may be chosen to insure that a non-zero universal class $\overline{v}^i \epsilon H^2(GL(n,p);\ H^{i-1}(M;H_i(M,k)))$ will restrict to the class $v^i \epsilon H^2(\Phi;H^{i-1}(M;H_i(M,k)))$? Let Φ^* = the subgroup of GL(n,p) consisting of upper triangular matrices with 1's on the diagonal. Then Φ^* is a p-Sylow subgroup of GL(n,p) and a non-zero universal class must be detected on this subgroup. However, construction of the Φ-system even for $\Phi = \Phi^*$, $n \geq 2$ is too cumbersome. For $n = 2$, $\Phi^* \cong \mathbb{Z}_p$ and the same procedure as in Example 1 below should work. In Example 2, the case for GL(3,2), $\Phi^* \cong D_8$ (dihedral group) and the construction of a Φ^*-system is too difficult. Therefore, we choose a subgroup Φ = the group of all matrices in GL(3,2) generated by

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so $\Phi \cong \mathbb{Z}_2$. It turns out that this choice of Φ is sufficient to detect the universal class $\bar{v}^3$, i.e. show $\bar{v}^3 \neq 0$.

The calculation of v^n for the free Abelian case relied upon the choice of a particularly simple resolution for M which facilitated the construction of the chain map A_n and homotopy U_n in the Φ-system for arbitrarily large n. We are unable to do this in the finite case. Perhaps there is a more efficient resolution than the one employed in these examples.

Example 1: (GL(2,3))

Let $M = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and let Φ be the subgroup of GL(2,3) consisting of upper triangular matrices with 1's on the diagonal,

$$\Phi = \{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \}: \quad * \in \mathbb{Z}_3$$

so $\Phi \cong \mathbb{Z}_3$. Define action of Φ on M in the natural way but write it multiplicatively:

$$\sigma(s^i,t^j) = (s^{i+j},t^j)$$

$$\sigma^2(s^i,t^j) = (s^{i+2j},t^j)$$

for σ the generator of Φ, $(s^i,t^j)\varepsilon M$, s and t generators of $\mathbb{Z}_3$.

Now let $\overline{X} = (X_*,\partial_*)$ be the usual free resolution of $\mathbb{Z}$ for $\mathbb{Z}_3$, i.e.

$$\overline{X} : \cdots X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$X_i \cong \mathbb{Z}[\mathbb{Z}_3]$$

for all i.

$$\partial_i = \begin{cases} \text{multiplication by } 1 - t \text{ when } i \text{ is odd} \\ \text{multiplication by } 1 + t + t^2 \text{ when } i \text{ is even.} \end{cases}$$

Now construct a resolution $(D_*,d_*) = \overline{D}$ for M by the tensor product of resolutions $\overline{D} = \overline{X} \otimes \overline{X}$, specifically

1) $$D_n = \bigoplus_{i+j=n} X_i \otimes X_j$$

$$d_n = \oplus(\partial_i \otimes id_j) \oplus (-1)^i(id_i \otimes \partial_j)$$

where id_j denotes the identity map on X_j. The augmentation $\overline{\varepsilon} : D_0 \to \mathbb{Z}$ is obtained from the composition

$$X_0 \otimes_{\mathbb{Z}} X_0 \xrightarrow{\epsilon\otimes\epsilon} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

where $f(n\otimes m) = nm$ and ϵ is the usual augmentation in $\overline{X}$ namely if $X_0 = \sum_{\sigma\epsilon\mathbb{Z}_3} n_\sigma \sigma$ then $\epsilon(X_0) = \sum_{\sigma\epsilon\mathbb{Z}_3} n_\sigma$.

Each D_n is an M-module via the action $(m,n)(a\otimes b) = (ma) \otimes (nb)$ for $(m,n)\epsilon M$; $a,b\epsilon\mathbb{Z}[\mathbb{Z}_3]$. We fix notation for elements in the components D_i of $\overline{D}$ as follows. Since $X_i = \mathbb{Z}[\mathbb{Z}_3]$ for all i and $D_i = \bigoplus_{p+q=i} X_p \otimes X_q$, denote by s_p the generator of $\mathbb{Z}_3$ inside X_p and by t_q the generator of $\mathbb{Z}_3$ inside X_q. So $S_p \otimes t_q$ denotes a generating element of the summand of D_i corresponding to $X_p \otimes X_q$.

<u>Lemma 1</u>: The following $\mathbb{Z}$-homomorphisms $A_i : D_i \to D_i$ satisfy (i), (ii) and (iii) of Definition 1:

$$A_0(1) = A_1(1) = A_2(1) = \text{identity map}$$

$$A_0(\sigma)(s_0^i\otimes t_0^j) = s_0^{i+j}\otimes t_0^j$$

$$A_0(\sigma^2)(s_0^i\otimes t_0^j) = s_0^{i+2j}\otimes t_0^j$$

$$A_1(\sigma)(s_1^i\otimes t_0^j) = s_1^{i+j}\otimes t_0^j$$

$$A_1(\sigma^2)(s_1^i\otimes t_0^j) = s_1^{i+2j}\otimes t_0^j$$

$$A_1(\sigma)(s_0^i\otimes t_1^j) = s_0^{i+j}\otimes t_1^j + s_1^{i+j}\otimes t_0^{j+1}$$

$$A_1(\sigma^2)(s_0^i\otimes t_1^j) = s_0^{i+2j}\otimes t_1^j + s_1^{i+2j}\otimes t_0^{j+1} + s_1^{i+2j+1}\otimes t_0^{j+1}$$

$$A_2(\sigma)(s_1^i\otimes t_1^j) = s_1^{i+j}\otimes t_1^j$$

$$A_2(\sigma)(s_2^i \otimes t_0^j) = s_2^{i+j} \otimes t_0^j$$

$$A_2(\sigma^2)(s_1^i \otimes t_2^j) = s_1^{i+2j} \otimes t_1^j$$

$$A_2(\sigma^2)(s_2^i \otimes t_0^j) = s_2^{i+2j} \otimes t_0^j$$

$$A_2(\sigma)(s_0^i \otimes t_2^j) = s_0^{i+j} \otimes t_2^j - s_1^{i+j} \otimes t_1^{j+1} - s_1^{i+j} \otimes t_1^{j+2} - s_1^{i+j+1} \otimes t_1^{j+2} + s_2^{i+j} \otimes t_0^j$$

$$A_2(\sigma^2)(s_0^i \otimes r_2^j) = s_0^{i+2j} \otimes t_2^j + s_1^{i+2j+1} \otimes t_1^{j+2} + s_1^{i+2j+2} \otimes t_1^{j+1} + s_1^{i+2j+2} \otimes t_1^{j+2} + s_2^{i+2j} \otimes t_0^{j+1} + s_2^{i+2j} \otimes t_0^{j+2}$$

$$A_3(\sigma)(s_3^i \otimes t_0^j) = s_3^{i+j} \otimes t_0^j$$

$$A_3(\sigma^2)(s_3^i \otimes t_0^j) = s_3^{i+2j} \otimes t_0^j$$

$$A_3(\sigma)(s_2^i \otimes t_1^j) = s_2^{i+j} \otimes t_1^j + s_3^{i+j} \otimes t_0^{j+1}$$

$$A_3(\sigma^2)(s_2^i \otimes t_1^j) = s_2^{i+2j} \otimes t_1^j + s_3^{i+2j} \otimes t_0^{j+1} + s_3^{i+2j+1} \otimes t_0^{j+1}$$

$$A_3(\sigma)(s_1^i \otimes t_2^j) = s_1^{i+j} \otimes t_2^j + s_3^{i+j} \otimes t_0^j$$

$$A_3(\sigma^2)(s_1^i \otimes t_2^j) = s_1^{i+2j} \otimes t_2^j + s_3^{i+2j} \otimes t_0^{j+1} + s_3^{i+2j} \otimes t_2^{j+2}$$

$$A_3(\sigma)(s_0^i \otimes t_3^j) = s_0^{i+j} t_3^j + s_1^{i+j} \otimes t_2^{j+1} + s_2^{i+j} \otimes t_1^j + s_3^{i+j} \otimes t_0^{j+1}$$

$$A_3(\sigma^2)(s_0^i \otimes t_3^j) = s_0^{i+2j} \otimes t_3^j + s_1^{i+2j} \otimes t_2^{j+1} + s_1^{i+2j+1} \otimes t_2^{j+1} + s_2^{i+2j+1} \otimes t_1^{j+1} + s_2^{i+2j+1} \otimes t_1^{j+2} + s_3^{i+2j} \otimes t_0^{j+1} + s_3^{i+2j} \otimes t_0^{j+2} + s_3^{i+2j+1} \otimes t_0^{j+2} + s_3^{i+2j+1} \otimes t_0^j .$$

<u>Proof:</u> Define $A_0(\sigma) : D_0 \to D_0$ by $A_0(\sigma)(s_0^i \otimes t_0^j) = s_0^{i+j} \otimes t_0^j$. This clearly satisfies (i) in Definition 1. The other A_i are defined by starting in dimension 0 and solving equations

(ii) for A_1 subject to condition (iii). Then define A_i by solving (ii) using A_{i-1} subject to (iii). For example we compute $A_3(\sigma)(s_1^i \otimes t_2^j)$. A_3 must satisfy (ii)

$$
\begin{aligned}
d_3A_3(s_1^i\otimes t_2^j) &= A_2d_3(s_1^i\otimes t_2^j) \\
&= A_2(\sigma)[s_0^i\otimes t_2^j - s_0^{i+1}\otimes t_2^j - s_1^i\otimes t_1^j - s_1^i\otimes t_1^{j+1} - s_1^i\otimes t_1^{j+2}] \\
&= s_0^{i+j}\otimes t_2^j - s_1^{i+j}\otimes t_1^{j+1} - s_1^{i+j}\otimes t_1^{j+2} - s_1^{i+j+1}\otimes t_1^{j+2} \\
&+ s_2^{i+j}\otimes t_0^j - s_0^{i+1+j}\otimes t_2^j + s_1^{i+j+1}\otimes t_1^{j+1} \\
&+ s_1^{i+j+1}\otimes t_1^{j+2} + s_1^{i+j+2}\otimes t_1^{j+2} - s_2^{i+j+1}\otimes t_0^j \\
&- s_1^{i+j}\otimes t_1^j - s_1^{i+j+1}\otimes t_1^{j+1} - s_1^{i+j+2}\otimes t_1^{j+2} \\
&= s_0^{i+j}\otimes t_2^j - s_0^{i+j+1}\otimes t_2^j - s_1^{i+j}\otimes t_1^j - s_1^{i+j}\otimes t_1^{j+1} \\
&- s_1^{i+j}\otimes t_1^{j+2} + s_2^{i+j}\otimes t_0^j - s_2^{i+j+1}\otimes t_0^j \\
&= d_3(s_1^{i+j}\otimes t_2^j + s_3^{i+j}\otimes t_0^j).
\end{aligned}
$$

The following is a typical check for σ-linearity. Let $r = (s^k, t^\ell)\epsilon M$, then $\sigma(r) = (s^{k+\ell}, t^\ell)$, and let $d = s_1^i\otimes t_2^j \epsilon D_3$. Then

$$
\begin{aligned}
A_3(\sigma)(rd) &= A_3(\sigma)(s_1^{i+k}\otimes t_2^{j+\ell}) \\
&= s_1^{i+j+k+\ell}\otimes t_2^{j+\ell} + s_3^{i+j+k+\ell}\otimes t_0^{j+\ell} \\
&= (s^{k+\ell}, t^\ell)(s_1^{i+j}\otimes t_2^j + s_3^{i+j}\otimes t_0^j) \\
&= \sigma(r)\cdot A_3(\sigma)(d).
\end{aligned}
$$

Other σ-linearity checks are done in a similar fashion. ∎

It is possible that computation of A_i could be done by computer. The difficult case in all examples is for $A_i(\tau)(s_0 \otimes t_1^m)$.

Given A_i as above, $A_i(\sigma\tau) \neq A_i(\sigma) \circ A_i(\tau)$ in general and the next step is to find a chain homotopy U.

<u>Lemma 2</u>: The following maps $U_i(\sigma,\tau) : D_i \to D_{i+1}$ satisfy (i'), (ii') and (iii') in Definition 1:

$$U_0(\sigma,\tau) = 0 \text{ for all } \sigma,\tau\epsilon\Phi$$

$$U_1(1,\tau) = U_1(\tau,1) = 0 \text{ for all } \tau\epsilon\Phi$$

$$U_1(\sigma,\tau)(s_1^i \otimes t_0^j) = 0 \qquad \forall\ \sigma,\tau\epsilon\Phi$$

$$U_1(\sigma,\sigma)(s_0^i \otimes t_1^j) = 0$$

$$U_1(\sigma,\sigma^2)(s_0^i \otimes t_1^j) = -s_2^i \otimes t_0^{j+1}$$

$$U_1(\sigma^2,\sigma)(s_0^i \otimes t_1^j) = -s_2^i \otimes t_0^{j+1}$$

$$U_1(\sigma^2,\sigma^2)(s_0^i \otimes t_1^j) = -s_2^{i+j} \otimes t_0^{j+1}$$

$$U_2(\sigma,\tau)(s_2^i \otimes t_0^j) = 0 \qquad \forall\ \sigma,\tau\epsilon\Phi$$

$$U_2(1,\tau)(x) = U_2(\tau,1)(x) = 0 \qquad \forall\ \tau\epsilon\Phi,\ x\epsilon D_2$$

$$U_2(\sigma,\sigma)(s_1^i \otimes t_1^j) = 0$$

$$U_2(\sigma,\sigma^2)(s_1^i \otimes t_1^j) = s_3^i \otimes t_0^{j+1}$$

$$U_2(\sigma^2,\sigma)(s_1^i \otimes t_1^j) = s_3^i \otimes t_0^{j+1}$$

$$U_2(\sigma^2,\sigma^2)(s_1^i \otimes t_1^j) = s_3^{i+j} \otimes t_0^{j+1}$$

$$U_2(\sigma,\sigma)(s_0^i \otimes t_2^j) = s_2^{i+2j} \otimes t_1^{j+1} + 2(s_2^{i+2j} \otimes t_1^{j+2})$$

Now choose $k = \mathbb{Z}_3$ a trivial Φ-module. Consider $H_n(M,k)$ as a trivial M-module and trivial k-module. We need $f^n \epsilon \mathrm{Hom}_{\mathbb{Z}[M]}(D_n, H_n(M,k))$ corresponding to the identity map in $\mathrm{Hom}_K(H_n(M,k), H_n(M,k))$. Then, by the Künneth theorem $H_n(M,k)$ is isomorphic to a direct sum of copies of k, one copy for each pair (p,q) such that p+q = n, i.e.

$$H_n(M,k) = \bigoplus_{p+q=n} \langle g_{pq} \rangle$$

where $\langle g_{pq} \rangle = k = \mathbb{Z}_3$ and g_{pq} is a generator. Now define:

$$f^n(s_i^{\ell} \otimes t_j^m) = g_{ij} \qquad \text{for } s_i^{\ell} \otimes t_j^m \epsilon D_n.$$

Note that $f^n(s_i^{\ell} \otimes t_j^m) = f^n(1_i \otimes 1_j)$ since $H_n(M,k)$ is a trivial M-module and f^n is an M-homomorphism $[s_i^{\ell} \otimes t_j^m = (s^{\ell}, t^m) \cdot (1_i \otimes 1_j)$ where $(s^{\ell}, t^m) \epsilon M$ and $\cdot$ denotes the M-action on $D_*]$.

The action of Φ on $H_n(M,k)$ is given by the following:

$$\begin{bmatrix} \sigma_*(g_{01}) \\ \sigma_*(g_{10}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_{01} \\ g_{10} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_*^2(g_{01}) \\ \sigma_*^2(g_{10}) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_{01} \\ g_{10} \end{bmatrix}$$

$= f^3(s_2^i \otimes t_1^j).]$

Consequently $F_\tau^n = 0$ for $n \le 2$, $\tau \epsilon \Phi$, so for this example $u^n(\sigma,\tau) = (\sigma\tau)_*[f^n U_{n-1}(\tau^{-1},\sigma^{-1})]$.

<u>Lemma 3</u>: For the resolution (1); $\sigma,\tau\epsilon\Phi$, $u^n(\sigma,\tau)\epsilon$ $\mathrm{Hom}_{\mathbb{Z}[M]}(D_n;H_{n+1}(M,k))$ for $n = 2,3$ are given by the following:

$u^2(\rho,\tau)(s_1^i \otimes t_0^j) = 0$ for all $\rho,\tau\epsilon\Phi$

$u^2(\tau,1) = u^2(1,\sigma) = 0$ for all $\tau\epsilon\Phi$

$u^2(\sigma,\sigma)(s_0^i \otimes t_1^j) = -g_{20}$

$u^2(\sigma,\sigma^2)(s_0^i \otimes t_1^j) = -g_{20}$

$u^2(\sigma^2,\sigma)(s_0^i \otimes t_1^j) = -g_{20}$

$u^2(\sigma^2,\sigma^2)(s_0^i \otimes t_1^j) = 0$

$u^3(\rho,\tau)(s_2^i \otimes t_0^j) = 0$ for all $\rho,\tau\epsilon\Phi$

$u^3(\tau,1) = u^3(1,\tau) = 0$ for all $\tau\epsilon\Phi$

$u^3(\sigma,\sigma)(s_1^i \otimes t_1^j) = g_{30}$

$u^3(\sigma,\sigma^2)(s_1^i \otimes t_1^j) = g_{30}$

$u^3(\sigma^2,\sigma)(s_1^i \otimes t_1^j) = g_{30}$

$u^3(\sigma^2,\sigma^2)(s_1^i \otimes t_1^j) = 0$

$u^3(\tau,\rho)(s_0^i \otimes t_2^j) = 0$ for all $\tau,\rho\epsilon\Phi$.

<u>Proof</u>: The following is a typical calculation:

$$u^2(\sigma^2,\sigma)(s_0^i\otimes t_1^j) = 1_* f^2(U_1(\sigma^{-1},(\sigma^2)^{-1})(s_0^i\otimes t_1^j))$$
$$= 1_* f^2(U_1(\sigma^2,\sigma)(s_0^i\otimes t_1^j)$$
$$= 1_* f^2[-s_2^i\otimes t_0^{j+1}]$$
$$= 1_*(-g_{20}) = -g_{20}.$$

Other calculations are done similarly. ∎

Proposition 1: For $\Phi \subset GL(2,3)$, $M = \mathbb{Z}_3\oplus\mathbb{Z}_3$, $v^2(M)$ and $v^3(M)$ are both zero.

Proof: We have $u^2(\sigma,\tau)\in \mathrm{Hom}_M(D_1,H_2(M,k))$ representing the class $w^2(\sigma,\tau)\in H^1(M;H_2(M,k))$. And $w^2 : \Phi \times \Phi \to H^1(M;H_2(M,k))$ is a Φ-cocycle which represents $v^2\in H^2(\Phi;H^1(M,H_2(M,k)))$. To determine whether $v^2 = 0$, we need to know if w^2 is a coboundary, i.e. if there is a map $\psi : \Phi \to H^1(M,H_2(M,k))$ such that

3) $u^2(\sigma,\tau) = \sigma\psi(\tau) - \psi(\sigma\tau) + \psi(\sigma)$

where $\sigma\cdot\psi(\tau)$ denotes the action of Φ on $H^1(M;H_2(M,k))$. For any $\sigma\in\Phi$, $\psi(\sigma)\in H^1(M;H_2(M,k))$. Let ψ_σ represent this class; $\psi_\sigma\in \mathrm{Hom}_{\mathbb{Z}[M]}(D_1,H_2(M,k))$. Now, if $u^2(\sigma,\tau)$ represents a coboundary then

4) $u^2(\sigma,\tau)(x_1) = (\sigma\psi_\tau-\psi_{\sigma\tau}+\psi_\sigma)(x_1)$ for any $x_1\in D_1$.

$H_2(M,k) = \langle g_{02}\rangle + \langle g_{20}\rangle + \langle g_{11}\rangle$ (where $\langle g_{ij}\rangle \cong k = \mathbb{Z}_3$) so for any $\sigma\in\Phi$, there exist n_σ^i, $m_\sigma^i\in\mathbb{Z}_3$ such that

3) $\psi_\sigma(1_0 \otimes 1_1) = n_\sigma^1 g_{02} + n_\sigma^2 g_{20} + n_\sigma^3 g_{11}$

$$\psi_\sigma(1_1 \otimes 1_0) = m_\sigma^1 g_{02} + m_\sigma^2 g_{20} + m_\sigma^3 g_{11}$$

and since $H^2(M,k)$ is a trivial M-module, we have

$$\psi_\sigma(s_0^i \otimes t_1^j) = \psi_\sigma(1_0 \otimes 1_1)$$

$$\psi_\sigma(s_1^i \otimes t_0^j) = \psi_\sigma(1_1 \otimes 1_0) \qquad \text{for } \sigma \epsilon \Phi.$$

Now $v^2 = 0$ because ψ can be defined by setting (in equations (5)):

$$m_1^i = n_1^i = 0 \qquad \text{for } i = 0,1,2$$

$$m_\sigma^1 = 1, \; n_\sigma^1 = 1$$

$$m^1_{\sigma^2} = 2, \; n^1_{\sigma^2} = 1$$

$$m_\sigma^2 = 0, \; n_\sigma^2 = 0$$

$$m^2_{\sigma^2} = 1, \; n^2_{\sigma^2} = 1 .$$

We have $u^3(\sigma,\tau) \epsilon \mathrm{Hom}_{\mathbb{Z}[M]}(D_2, H_3(M,k))$ which represents $w^3(\sigma,\tau) \epsilon H^2(M, H_3(M,k))$. The cocycle $w^3 : \Phi \times \Phi \to H^2(M,H_3(M,k))$ represents the characteristic class $v^3 \epsilon H^2(\Phi; H^2(M,H_3(M,k)))$. The class $v^3 = 0$ if there exists a map $\psi : \Phi \to H^2(M;H_3(M,k))$ whose boundary is v^3, i.e. $w^3(\sigma,\tau) = \sigma\psi(\tau) - \psi(\sigma\tau) + \psi(\sigma)$. Denote by $\psi_\sigma : D_2 \to H_3(M,k)$ a representative cocycle of $\psi(\sigma)$.

Now $H_3(M,k) = \langle g_{03}\rangle + \langle g_{30}\rangle + \langle g_{12}\rangle + \langle g_{21}\rangle$ where $\langle g_{ij}\rangle \cong \mathbb{Z}_3$. We will define ψ_τ on generators of D_2 by determining elements of $\mathbb{Z}_3$; ℓ^i_τ, m^i_τ, n^i_τ for any $\tau\epsilon\Phi$, $i = 1,2,3,4$ such that

$$\psi_\tau(1_0\otimes 1_2) = \ell^1_\tau g_{03} + \ell^2_\tau g_{30} + \ell^3_\tau g_{12} + \ell^4_\tau g_{21}$$

$$\psi_\tau(1_1\otimes 1_1) = m^1_\tau g_{03} + m^2_\tau g_{30} + m^3_\tau g_{12} + m^4_\tau g_{21}$$

$$\psi_\tau(1_2\otimes 1_0) = n^1_\tau g_{03} + n^2_\tau g_{30} + n^3_\tau g_{12} + n^4_\tau g_{21}$$

and satisfying

6) $$u^3(\tau,\rho)(x_2) = (\tau\psi_\rho - \psi_{\tau\rho} + \psi_\tau)(x_2)$$

for any $\tau,\rho\epsilon\Phi$, $x_2\epsilon D_2$. In this case the map ψ can be defined by taking:

7) $n^i_\tau = \ell^i_\tau = 0$ for all $\tau\epsilon\Phi$ and for all i

$m^1_\sigma = 1 \quad m^1_{\sigma^2} = 2$

$m^2_{\sigma^2} = 0 \quad m^2_{\sigma^2} = 2$

$m^3_\sigma = 1 \quad m^3_{\sigma^2} = 0$

$m^4_\sigma = 1 \quad m^4_{\sigma^2} = 0$ where σ generates Φ

$m^i_1 = 0$ for all i.

The n^i, m^i, ℓ^i are determined by solving a system of equations determined by 6). For instance, if we write out 6) for $x_2 = (1_1 \otimes 1_1)$ and $\rho = \tau = \sigma$, the generator of Φ we obtain:

8) $u^3(\sigma,\sigma)(1_1 \otimes 1_1) = (\sigma\psi_\sigma - \psi_{\sigma^2} + \psi_\sigma)(1_1 \otimes 1_1)$.

The left hand side is equal to g_{30} by (2) and the right hand side can be expanded to obtain:

$$g_{30} = (\sigma\psi_\sigma - \psi_{\sigma^2} + \psi_\sigma)(1_1 \otimes 1_1)$$

$$= \sigma\psi_\sigma(1_1 \otimes 1_1) - \psi_{\sigma^2}(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)$$

$$g_{30} = \sigma_*(\psi_\sigma(A_2(\sigma^2)(1_1 \otimes 1_1)) - \psi_{\sigma^2}(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)$$

$$g_{30} = \sigma_*(\psi_\sigma(1_1 \otimes 1_1)) - \psi_{\sigma^2}(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)$$

$$g_{30} = \sigma_*(m_\sigma^1 g_{03} + m_\sigma^2 g_{30} + m_\sigma^3 g_{12} + m_\sigma^4 g_{21})$$

$$- (m^1_{\sigma^2} g_{03} + m^2_{\sigma^2} g_{30} + m^3_{\sigma^2} g_{11} + m^4_{\sigma^2} g_{21})$$

$$+ (m_\sigma^1 g_{03} + m_\sigma^2 g_{30} + m_\sigma^3 g_{12} + m_\sigma^4 g_{21})$$

$$g_{30} = m_\sigma^1(g_{03} + g_{12} + g_{21} + g_{30}) + m_\sigma^2 g_{30} + m_\sigma^3(g_{12} + g_{30})$$

$$+ m_\sigma^4(g_{21} + g_{30}) - (m^1_{\sigma^2} g_{03} + m^2_{\sigma^2} g_{30} + m^3_{\sigma^2} g_{11} + m^4_{\sigma^2} g_{21})$$

$$+ m_\sigma^1 g_{03} + m_\sigma^2 g_{30} + m_\sigma^3 g_{11} + m_\sigma^4 g_{21} .$$

Equating coefficients of g_{ij} on both sides yields:

$$m_\sigma^1 - m^1_{\sigma^2} + m_\sigma^1 = 0$$

$$m_\sigma^1 + 2m_\sigma^2 + m_\sigma^3 + m_\sigma^4 - m^2_{\sigma^2} = 1$$

$$m_\sigma^1 + 2m_\sigma^3 - m_{\sigma^2}^3 = 0$$

$$m_\sigma^1 + 2m_\sigma^4 - m_{\sigma^2}^4 = 0 .$$

Writing out 6) for all $\tau, \rho \in \Phi$ and other $x_2 \in D_2$ yields a system of equations in ℓ_τ^1, m_τ^1, n_τ^1, for which 7) is a solution. ∎

In this case v^2 is known to be zero since the group $H^2(GL(2,3);H^1(M,H_2(M,k))$ containing the universal class $\bar{v}^2$ is zero, (see Appendix). We have shown that $v^3 = 0$ for Φ the p-Sylow subgroup above. Since $\bar{v}^3$ must be detected on a Sylow subgroup $v^3 = 0$ implies $\bar{v}^3 = 0$. In other words no non-zero element of $H^2(SL(2,3);H^2(M;H_3(M,k)))$ arises from a characteristic class.

The procedure of Example I could be used to explicitly compute v^2, v^3 for $p > 3$, $\Phi \subseteq GL(2,p)$ but calculations would be more difficult. It is known, (see Appendix) that $\bar{v}^2(M) \neq 0$ for $p \geq 5$.

<u>Example 2</u>: (GL(3,2))

Let $M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ and let Φ be the subgroup of GL(3,2) generated by

$$\sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

Then $\sigma^2 = 1$ so $\Phi \cong \mathbb{Z}_2$. Define action of Φ on M by

$$\sigma(r^{\ell}, s^n, t^m) = (r^{\ell+m}, s^{n+m}, t^m) \quad r,s,t \text{ generators of } \mathbb{Z}_2.$$

Now suppose $\overline{X} : \cdots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \cdots \to X_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is the usual resolution of $\mathbb{Z}$ for $\mathbb{Z}_2$, i.e. $X_i = \mathbb{Z}[\mathbb{Z}_2]$ with ∂_i = multiplication by $(1+t)$ for i even and ∂_i = multiplication by $(1-t)$ for i odd. Now let $\overline{D}$ be defined as the three-fold tensor product of the resolution $\overline{X}$:

9) $\overline{D} = \overline{X} \otimes \overline{X} \otimes \overline{X}$

$$D_n = \bigoplus_{p+q_1+q_2=n} X_p \otimes X_{q_1} \otimes X_{q_2}$$

$$d_n = \oplus \partial_p \otimes id_q \oplus (-1)^p id_p \otimes [\partial_{q_1} \otimes id_{q_2} + (-1)^{q_1} id_{q_1} \otimes \partial_{q_2})$$

where $q = q_1 + q_2$ and id_j denotes the identity map on X_j.

We can now construct a Φ-system for this resolution. Here $r_p^i \otimes s_{q_1}^j \otimes t_{q_2}^k$ will denote an element of D_i, $i = p+q_1+q_2$, corresponding to the summand $X_p \otimes X_{q_1} \otimes X_{q_2}$ and extended to be M-linear.

Lemma 2.1: The following $\mathbb{Z}$-homomorphisms $A_n(\tau) : D_n \to D_n$ satisfy the conditions of Definition 1.

$A_i(1)$ = identify map for all i

$$A_0(\sigma)(r_0^i \otimes s_0^j \otimes t_0^k) = r_0^{i+k} \otimes s_0^{j+k} \otimes t_0^k$$

$$A_1(\sigma)(r_1^i \otimes s_0^j \otimes t_0^k) = r_1^{i+k} \otimes s_0^{j+k} \otimes t_0^k$$

$$A_1(\sigma)(r_0^i \otimes s_1^j \otimes t_0^k) = r_0^{i+k} \otimes s_1^{j+k} \otimes t_0^k$$

$$A_1(\sigma)(r_0^i \otimes s_0^j \otimes t_1^k) = r_0^{i+k} \otimes s_0^{j+k} \otimes t_1^k + r_1^{i+k} \otimes s_0^{j+k+1} \otimes t_0^{k+1} + r_0^{i+k} \otimes s_1^{j+k} \otimes t_0^{k+1}$$

$$A_2(\sigma)(r_2^i \otimes s_0^j \otimes t_0^k) + r_2^{i+k} \otimes s_0^{j+k} \otimes t_0^k$$

$$A_2(\sigma)(r_0^i \otimes s_2^j \otimes t_0^k) = r_0^{i+k} \otimes s_2^{j+k} \otimes t_0^k$$

$$A_2(\sigma)(r_1^i \otimes s_1^j \otimes t_0^k) = r_1^{i+k} \otimes s_1^{j+k} \otimes t_0^k$$

$$A_2(\sigma)(r_0^i \otimes s_1^j \otimes t_1^k) = r_0^{i+k} \otimes s_1^{j+k} \otimes t_1^k + r_1^{i+k+1} \otimes s_1^{j+k+1} \otimes t_0^{k+1} + r_2^{i+k+1} \otimes s_0^{i+k+1} \otimes t_0^{k+1} - r_2^{i+k} \otimes s_0^{j+k} \otimes t_0^{k+1}$$

$$A_2(\sigma)(r_1^i \otimes s_0^j \otimes t_1^k) = r_1^{i+k} \otimes s_0^{j+k} \otimes t_1^k + r_1^{i+k} \otimes s_1^{j+k} \otimes t_0^{k+1}$$

$$A_2(\sigma)(r_0^i \otimes s_0^j \otimes t_2^k) = r_0^{i+k} \otimes s_0^{j+k} \otimes t_2^k + r_0^{i+k} \otimes s_1^{j+k+1} \otimes t_1^{k+1} + r_1^{i+k+1} \otimes s_0^{j+k+1} \otimes t_1^{k+1} + r_1^{i+k+1} \otimes s_1^{j+k+1} \otimes t_0^k + r_2^{i+k+1} \otimes s_0^{j+k+1} \otimes t_0^{k+1} + r_0^{i+k} \otimes s_2^{j+k} \otimes t_0^{k+1}$$

$$A_3(\sigma)(r_3^i \otimes s_0^j \otimes t_0^k) = r_3^{i+k} \otimes s_0^{j+k} \otimes t_0^k$$

$$A_3(\sigma)(r_0^i \otimes s_3^j \otimes t_0^k) = r_0^{i+k} \otimes s_3^{j+k} \otimes t_0^k$$

$$A_3(\sigma)(r_2^i \otimes s_1^j \otimes t_0^k) = r_2^{i+k} \otimes s_1^{j+k} \otimes t_0^k$$

$$A_3(\sigma)(r_1^i \otimes s_2^j \otimes t_0^k) = r_1^{i+k} \otimes s_2^{j+k} \otimes t_0^k$$

$$A_3(\sigma)(r_2^i \otimes s_0^j \otimes t_1^k) = r_2^{i+k} \otimes s_0^{j+k} \otimes t_1^k + r_3^{i+k} \otimes s_0^{j+k} \otimes t_0^{k+1} + r_2^{i+k+1} \otimes s_1^{j+k} \otimes t_0^{k+1}$$

$$\begin{aligned}
A_3(\sigma)(r_1^i\otimes s_1^j\otimes t_1^k) = {} & r_1^{i+k}\otimes s_1^{j+k}\otimes t_1^k + r_2^{i+k}\otimes s_1^{j+k+1}\otimes t_0^{k+1} \\
& - r_2^{i+k+1}\otimes s_1^{j+k+1}\otimes t_0^{k+1} + r_3^{i+k+1}\otimes s_0^{j+k+1}\otimes t_0^{k+1} \\
& - r_3^{i+k}\otimes s_0^{j+k+1}\otimes t_0^{k+1}
\end{aligned}$$

$$\begin{aligned}
A_3(\sigma)(r_0^i\otimes r_2^j\otimes t_1^k) = {} & r_0^{i+k}\otimes s_2^{j+k}\otimes t_1^k + r_0^{i+k}\otimes s_3^{j+k}\otimes t_0^{k+1} \\
& + r_1^{i+k}\otimes s_2^{j+k+1}\otimes t_0^{k+1} + r_2^{i+k}\otimes s_1^{j+k}\otimes t_0^{k+1} \\
& + r_2^{i+k+1}\otimes s_1^{j+k+1}\otimes t_0^{k+1} + r_3^{i+k+1}\otimes s_0^{j+k+1}\otimes t_0^{k+1} \\
& + r_3^{i+k+1}\otimes s_0^{j+k}\otimes t_0^{k+1}
\end{aligned}$$

$$\begin{aligned}
A_3(\sigma)(r_1^i\otimes s_0^j\otimes t_2^k) = {} & r_1^{i+k}\otimes s_0^{j+k}\otimes t_2^k - r_1^{i+k}\otimes s_1^{j+k}\otimes t_1^{k+1} \\
& + r_1^{i+k}\otimes s_2^{j+k}\otimes t_0^k - r_0^{i+k+1}\otimes s_2^{j+k+1}\otimes t_1^{k+1} \\
& + r_0^{i+k}\otimes s_2^{j+k}\otimes t_1^{k+1} + r_0^{i+k+1}\otimes s_3^{j+k+1}\otimes t_0^{k+1} \\
& + r_0^{i+k+1}\otimes s_3^{j+k}\otimes t_0^k + r_3^{i+k+1}\otimes s_0^{j+k+1}\otimes t_0^{k+1}
\end{aligned}$$

$$\begin{aligned}
A_3(\sigma)(r_0^i\otimes s_1^j\otimes t_2^k) = {} & r_0^{i+k}\otimes s_1^{j+k}\otimes t_2^k + r_1^{i+k}\otimes s_1^{j+k+1}\otimes t_1^{k+1} \\
& + r_2^{i+k}\otimes s_0^{j+k+1}\otimes t_1^{k+1} - r_2^{i+k}\otimes s_0^{j+k}\otimes t_1^{k+1} \\
& - r_2^{i+k}\otimes s_1^{j+k+1}\otimes t_0^{k+1} + r_2^{i+k}\otimes s_1^{j+k+1}\otimes t_0^k \\
& - r_2^{i+k}\otimes s_1^{j+k}\otimes t_0^k + r_0^{i+k}\otimes s_3^{j+k}\otimes t_0^{k+1} \\
& + r_3^{i+k}\otimes s_0^{j+k}\otimes t_0^{k+1} + r_3^{i+k+1}\otimes s_0^{j+k+1}\otimes t_0^k
\end{aligned}$$

$$\begin{aligned}
A_3(\sigma)(r_0^i\otimes s_0^j\otimes t_3^k) = {} & r_0^{i+k}\otimes s_0^{j+k}\otimes t_3^k + r_0^{i+k}\otimes s_1^{j+k}\otimes t_2^{i+1} \\
& + r_1^{i+k}\otimes s_0^{j+k+1}\otimes t_2^{k+1} + r_0^{i+k}\otimes s_2^{j+k}\otimes t_1^{k+1} \\
& + r_1^{i+k+1}\otimes s_1^{j+k+1}\otimes t_1^{k+1} + r_2^{i+k}\otimes s_0^{j+k+1}\otimes t_1^{k+1} \\
& + r_1^{i+k}\otimes s_2^{j+k}\otimes t_0^k + r_2^{i+k}\otimes s_1^{j+k+1}\otimes t_0^k \\
& + r_0^{i+k+1}\otimes s_3^{j+k}\otimes t_0^k + r_3^{i+k+1}\otimes s_0^{j+k+1}\otimes t_0^{k+1}
\end{aligned}$$

With the above definitions, $A_n(1) - A_n(\sigma)\circ A_n(\sigma) \neq 0$ in general and a homotopy $U_n(\rho,\tau) : D_n \to D_{n+1}$ may be defined as follows:

<u>Lemma 2.2</u>: The following maps U_i satisfy the conditions of Definition 1.

$$U_i(\sigma,1) = U_i(1,\sigma) = 0 \qquad \text{for all } i$$

$$U_0(\sigma,\sigma) = 0$$

$$U_1(\sigma,\sigma)(r_1^i \otimes s_1^j \otimes t_0^k) = 0$$

$$U_1(\sigma,\sigma)(r_0^i \otimes s_1^j \otimes t_0^k) = 0$$

$$U_1(\sigma,\sigma)(r_0^i \otimes s_0^j \otimes t_1^k) = -r_1^i \otimes s_1^j \otimes t_0^{k+1} - r_0^{i+k} \otimes s_2^j \otimes t_0^{k+1} - r_2^i \otimes s_0^j \otimes t_0^{k+1}$$

$$U_2(\sigma,\sigma)(r_0^i \otimes s_1^j \otimes t_1^k) = -r_2^i \otimes s_1^{j+1} \otimes t_0^{k+1} + r_0^{i+1} \otimes s_3^j \otimes t_0^{k+1}$$

$$U_2(\sigma,\sigma)(r_2^i \otimes s_0^j \otimes t_0^k) = 0$$

$$U_2(\sigma,\sigma)(r_0^i \otimes s_2^j \otimes t_0^k) = 0$$

$$U_2(\sigma,\sigma)(r_1^i \otimes s_1^j \otimes t_0^k) = 0$$

$$U_2(\sigma,\sigma)(r_1^i \otimes s_0^j \otimes t_1^k) = r_1^{i+1} \otimes s_2^j \otimes t_0^{k+1} + r_3^i \otimes s_0^j \otimes t_0^{k+1}$$

$$U_2(\sigma,\sigma)(r_0^i \otimes s_0^j \otimes t_2^k) = -r_2^i \otimes s_1^{j+1} \otimes t_0^k - r_1^i \otimes s_2^j \otimes t_0^{k+1} - r_2^i \otimes s_1^{j+1} \otimes t_0^{k+1} + r_1^{i+1} \otimes s_1^{j+1} \otimes t_1^{k+1}$$

(continued)

$$- r_2^i \otimes s_0^j \otimes t_1^{k+1} - r_0^{i+1} \otimes s_2^{j+1} \otimes t_1^{k+1}$$

$$+ r_0^{i+1} \otimes s_3^j \otimes t_0^k + r_3^{i+1} \otimes s_0^j \otimes t_0^k$$

$$+ r_3^i \otimes s_0^{j+1} \otimes t_0^{k+1}$$

The proofs of Lemmas 2.1 and 2.2 are omitted since the computations involved are analogous to those appearing in the proofs of Lemmas 1.1 and 1.2.

We now let the coefficient group $k = \mathbb{Z}_2$. Define $f^n(r_a \otimes s_b \otimes t_c) = g_{abc}$ where $r_a \otimes s_b \otimes t_c \in X_a \otimes X_b \otimes X_c$ and where g_{abc} is the corresponding generator of $H_n(M;k) = \bigoplus_{a+b+c=n} \langle g_{abc} \rangle$ and each $\langle g_{abc} \rangle = \mathbb{Z}_2$, i.e. $H_2(M,k)$ is isomorphic to six copies of $\mathbb{Z}_2$ and written:

$$H_2(M,k) = \langle g_{002} \rangle + \langle g_{020} \rangle + \langle g_{200} \rangle$$

$$+ \langle g_{011} \rangle + \langle g_{101} \rangle + \langle g_{110} \rangle.$$

Now by the same argument as that in Example 1 we may take $F^n = 0$ and $u^n(\sigma,\tau) \in \mathrm{Hom}_{\mathbb{Z}[M]}(D_{n-1}, H_n(M,k))$ are defined by the following lemma. The proof is omitted since the procedure imitates that used in proof of Lemma 1.3.

<u>Lemma 2.3</u>: The cocycles $u^n(\sigma,\tau) \in \mathrm{Hom}_{\mathbb{Z}[M]}(D_{n-1}, H_n(M,k))$ for $n = 1,2,3$ can be defined as follows:

$$u^n(\sigma,\tau) = (\sigma\tau)_*[f^n U_{n-1}(\tau^{-1},\sigma^{-1})]$$

$$u^n(\sigma,1) = u^n(1,\sigma) = 0 \qquad \forall n$$

$$u^1(\sigma,\sigma) = 0$$

$$u^2(\sigma,\sigma)(r_1^i \otimes s_0^j \otimes t_0^k) = 0$$

$$u^2(\sigma,\sigma)(r_0^i \otimes s_1^j \otimes t_0^k) = 0$$

$$u^2(\sigma,\sigma)(r_0^i \otimes s_0^j \otimes t_1^k) = -g_{110} - g_{020} - g_{200}$$

$$u^3(\sigma,\sigma)(r_0^i \otimes s_1^j \otimes t_1^k) = -g_{210} + g_{030}$$

$$u^3(\sigma,\sigma)(r_1^i \otimes s_0^j \otimes t_1^k) = g_{120} + g_{300}$$

$$u^3(\sigma,\sigma)(r_0^i \otimes s_0^j \otimes t_2^k) = -g_{210} - g_{120} - g_{210} + g_{111} - g_{201} - g_{201} - g_{021} + g_{030} + g_{300}$$

$$u^3(\sigma,\sigma)(r_2^i \otimes s_0^j \otimes t_0^k) = 0$$

$$u^3(\sigma,\sigma)(r_0^i \otimes s_2^j \otimes t_0^k) = 0$$

$$u^3(\sigma,\sigma)(r_1^i \otimes s_1^j \otimes t_0^k) = 0$$

<u>Proposition 2</u>: For $\Phi \subset GL(3,2)$ generated by $\sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and $M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$, $v^2(M) = 0$.

<u>Proof</u>: We have $u^2(\rho,\tau) \in \mathrm{Hom}_{\mathbb{Z}[M]}(D_1, H_2(M,k))$ defining $w^2(\rho,\tau) \in H^1(M;H_2(M,k))$ and $w^2 : \Phi \times \Phi \to H^1(M;H_2(M,k))$

is a cocycle which represents $v^2 \epsilon H^2(\Phi; H^1(M;H_2(M,k)))$. The class v^2 is a coboundary if there exists a map $\psi : \Phi \to H^1(M;H_2(M,k))$ such that

$$v^2(\rho,\tau) = \rho\psi(\tau) - \psi(\rho\tau) + \psi(\rho).$$

Denote by ψ_ρ an element of $\mathrm{Hom}_{\mathbb{Z}[M]}(D_1,H_2(M,k))$ which represents $\psi(\rho)\epsilon H^1(M;H_2(M,k))$. If $w^2(\sigma,\tau)$ represents a coboundary then

11) $u^2(\rho,\tau)(x_1) = (\rho\psi_\tau - \psi_{\rho\tau} + \psi_\rho)(x_1)$ for all $x_1 \epsilon D_1$ and all $\rho,\tau\epsilon\Phi$.

To determine if such a ψ exists we follow an analogous procedure as in Example 1. We determine how ψ_τ must be defined on $1_p \otimes 1_{q_1} \otimes 1_{q_2} \epsilon D_1$ and use the fact that $H_2(M,k)$ is a trivial M-module so

$$\psi_\tau(1_p \otimes 1_{q_1} \otimes 1_{q_2}) = \psi_\tau(r_p^i \otimes s_{q_1}^j \otimes t_{q_2}^k).$$

If such a ψ exists there are elements ℓ^i, m^i, n^i in $\mathbb{Z}_2$ such that:

$$\psi_\tau(1_0 \otimes 1_0 \otimes 1_1) = \ell_\tau^1 g_{002} + \ell_\tau^2 g_{020} + \ell_\tau^3 g_{200} + \ell_\tau^4 g_{011} + \ell_\tau^5 g_{101} + \ell_\tau^6 g_{110}$$

$$\psi_\tau(1_0 \otimes 1_1 \otimes 1_0) = m_\tau^1 g_{002} + m_\tau^2 g_{020} + m_\tau^3 g_{200} + m_\tau^4 g_{011} + m_\tau^5 g_{101} + m_\tau^6 g_{110}$$

$$\psi_\tau(1_1 \otimes 1_0 \otimes 1_0) = n_\tau^1 g_{002} + n_\tau^2 g_{020} + n_\tau^3 g_{200}$$
$$+ n_\tau^4 g_{011} + n_\tau^5 g_{101} + n_\tau^6 g_{110}.$$

Writing out equation 11) for various $x_1 \epsilon D_1$ and $\rho, \tau \epsilon \Phi$ and substituting u^2 values from 10) yields a set of equations in ℓ_τ^i, m_τ^i, n_τ^i. The action of Φ on $H_2(M,k)$ is used in computing $\rho\psi_\tau(x_1)$ on the right hand side of 11):

$$\rho\psi_\tau(x_1) = \rho_*(\psi_\tau(A_1(\rho^{-1})(x_1))).$$

The action of Φ on $H_2(M,k)$ is given by:

$$1_* = \text{id}$$

$$\begin{bmatrix} \sigma_*(g_{002}) \\ \sigma(g_{020}) \\ \sigma_*(g_{200}) \\ \sigma_*(g_{011}) \\ \sigma_*(g_{101}) \\ \sigma_*(g_{110}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{002} \\ g_{020} \\ g_{200} \\ g_{011} \\ g_{101} \\ g_{110} \end{bmatrix}$$

The system of equations in ℓ_τ^i, m_τ^i, n_τ^i has a solution in this case:

$$m_1^i = n_1^i = \ell_1^i = 0 \qquad \text{for all } i$$

$m_\sigma^i = 0 \qquad i \neq 2,3$

$m_\sigma^2 = 1$

$n_\sigma^i = 0 \qquad i \neq 5$

$n_\sigma^5 = 1$

$\ell_\sigma^i = 0 \qquad$ for all i .

Therefore $v^2 = 0$ since this explicitly exhibits an M-homomorphism ψ whose coboundary is v^2. ∎

We now compute $v^3 \varepsilon H^2(\Phi; H^2(M; H_3(M,k)))$. Again v^3 is represented by $w^3 : \Phi \times \Phi \to H^2(M; H_3(M,k))$ where $w^3(\sigma,\tau) \varepsilon H^2(M; H_3(M,k))$ is represented by $u^3(\sigma,\tau) \varepsilon \mathrm{Hom}_{\mathbb{Z}[M]}(D_2, H_3(M,k))$ and $u^3(\sigma,\tau)$ are tabulated in 10). We must determine whether $v^3 = 0$. We follow a procedure similar to the previous cases but write out a few more dtails.

$H_3(M,k)$ is a direct sum of ten copies of $\mathbb{Z}_2$ written with generators:

$$\begin{aligned} H_3(M,k) = \langle g_{003}\rangle + \langle g_{030}\rangle + \langle g_{300}\rangle \\ + \langle g_{012}\rangle + \langle g_{102}\rangle + \langle g_{201}\rangle \\ + \langle g_{111}\rangle + \langle g_{021}\rangle + \langle g_{120}\rangle \\ + \langle g_{210}\rangle . \end{aligned}$$

The action of Φ on $H_3(M,k)$ is given by:

$$\sigma_*(g_{003}) = g_{003} + g_{030} + g_{300} + g_{012} + g_{102} + g_{201} + g_{111} + g_{021} + g_{120} + g_{210}$$

$$\sigma_*(g_{030}) = g_{030}$$

$$\sigma_*(g_{300}) = g_{300}$$

$$\sigma_*(g_{012}) = g_{012} + g_{111} - g_{210} + g_{030}$$

$$\sigma_*(g_{102}) = g_{102} - g_{111} + g_{120} + g_{300}$$

$$\sigma_*(g_{201}) = g_{201} + g_{210} + g_{300}$$

$$\sigma_*(g_{111}) = g_{111}$$

$$\sigma_*(g_{021}) = g_{021} + g_{030} + g_{120}$$

$$\sigma_*(g_{120}) = g_{120}$$

$$\sigma_*(g_{210}) = g_{210}\,.$$

We recall how this action was determined: an element of $H_3(M,k)$ is represented by an element $x \otimes \gamma \in D_3 \otimes_{\mathbb{Z}[M]} k$ and $\sigma \in \Phi$ acts on $x \otimes \gamma$ diagonally,

$$\sigma_*(x \otimes \gamma) = (A_3(\sigma)(x)) \otimes \gamma$$

Now $v^3 = 0$ if there exists a homomorphism $\psi : \Phi \to H^2(M; H_3(M,k))$ such that

12) $u^3(\rho,\tau)(x_2) = [\rho\psi(\tau) - \psi(\rho\tau) + \psi(\rho)](x_2)$

$$\forall\ x_2 \in D_2;\ \forall\ \rho,\tau\in\Phi.$$

Represent $\psi(\tau)\in H^2(M;H_3(M,k))$ by $\psi_\tau : D_2 \to H_3(M,k)$ so ψ_τ is an M-homomorphism, $H_3(M,k)$ is a trivial M-module. Here again $\rho\psi_\tau$ is computed by

$$\rho\psi_\tau(x) = \rho_*(\psi_\tau(A_2(\rho^{-1})(x))).$$

Again for all $\tau\in\Phi$, $\psi_\tau(x)\in H_2(M,k)$ so there exist elements a_τ^i, b_τ^i, c_τ^i, d_τ^i, e_τ^i, f_τ^i in $\mathbb{Z}_2$ such that:

$$\psi_\tau(1_0\otimes 1_0\otimes 1_2) = a_\tau^1(g_{003}) + a_\tau^2(g_{030}) + a_\tau^3(g_{300}) + a_\tau^4(g_{012}) + a_\tau^5(g_{102}) + a_\tau^6(g_{201}) + a_\tau^7(g_{111}) + a_\tau^8(g_{021}) + a_\tau^9(g_{120}) + a_\tau^{10}(g_{210})$$

$$\psi_\tau(1_0\otimes 1_2\otimes 1_0) = b_\tau^1(g_{003}) + b_\tau^2(g_{030}) + b_\tau^3(g_{300}) + b_\tau^4(g_{012}) + b_\tau^5(g_{102}) + b_\tau^6(g_{201}) + b_\tau^7(g_{111}) + b_\tau^8(g_{021}) + b_\tau^9(g_{120}) + b_\tau^{10}(g_{210})$$

$$\psi_\tau(1_2\otimes 1_0\otimes 1_0) = c_\tau^1 g_{003} + c_\tau^2 g_{030} + c_\tau^3 g_{300} + c_\tau^4 g_{012} + c_\tau^5 g_{102} + c_\tau^6 g_{201} + c_\tau^7 g_{111} + c_\tau^8 g_{021} + c_\tau^9 g_{120} + c_\tau^{10} g_{210}$$

$$\psi_\tau(1_0\otimes 1_1\otimes 1_1) = d_\tau^1 g_{003} + d_\tau^2 g_{030} + d_\tau^3 g_{300} + d_\tau^4 g_{012}$$
$$+ d_\tau^5 g_{102} + d_\tau^6 g_{201} + d_\tau^7 g_{111} + d_\tau^8 g_{021}$$
$$+ d_\tau^9 g_{120} + d_\tau^{10} g_{210}$$

$$\psi_\tau(1_1\otimes 1_0\otimes 1_1) = e_\tau^1 g_{003} + e_\tau^2 g_{030} + e_\tau^3 g_{300} + e_\tau^4 g_{012}$$
$$+ e_\tau^5 g_{102} + e_\tau^6 g_{201} + e_\tau^7 g_{111} + e_\tau^8 g_{021}$$
$$+ e_\tau^9 g_{120} + e_\tau^{10} g_{210}$$

$$\psi_\tau(1_1\otimes 1_1\otimes 1_0) = f_\tau^1 g_{003} + f_\tau^2 g_{030} + f_\tau^3 g_{300}$$
$$+ f_\tau^4 g_{012} + f_\tau^5 g_{102} + f_\tau^6 g_{201} + f_\tau^7 g_{111}$$
$$+ f_\tau^8 g_{021} + f_\tau^9 g_{120} + f_\tau^{10} g_{210}.$$

Again $\psi_\tau(r_p^i\otimes s_{q_1}^j\otimes t_{q_2}^k) = \psi_\tau(1_p\otimes 1_{q_1}\otimes 1_{q_2})$ since M acts trivially on $H_3(M,k)$.

We write 12) using generators:

13) $u^3(\rho,\tau)(x_2) = (\rho\psi_\tau - \psi_{\rho\tau} + \psi_\rho)(x_2)$ for $x_2 \epsilon D_2$.

Proposition 3: For Φ the given subset of GL(3,2) and using resolution 9) we obtain a non-zero characteristic class $v^3(M)\epsilon H^2(\Phi;H^2(M;H_3(M,k)))$.

Proof: We write out 13) for various $\rho,\tau\epsilon\Phi$ and $x_2\epsilon D_2$ (note that $\psi_1 = 0$ since ψ is a homomorphism) to get a system of equations in a_τ^i, b_τ^i, c_τ^i, d_τ^i, e_τ^i, f_τ^i, $\tau\epsilon\Phi$. Since in this example $\Phi \cong \mathbb{Z}_2$ generated by σ, we drop the subscripts for simplicity, i.e. $d^1 = d^1_\sigma$. The values for $u^3(\rho,\tau)$ are substituted into the left hand side of 13) using 10). Recall that

$$\rho\psi_\tau(x) = \rho_*(\psi_\tau(A_n(\rho^{-1})(x)))$$

where σ_* denotes the action of Φ on $H_*(M,k)$.

14) $$0 = u^3(\sigma,\sigma)(1_0 \otimes 1_2 \otimes 1_0) = (\sigma\psi_\sigma - \psi_1 + \psi_\sigma)(1_0 \otimes 1_2 \otimes 1_0)$$

$$= \sigma_*(\psi_\sigma(A_2(\sigma^{-1})(1_0 \otimes 1_2 \otimes 1_0))) + \psi_\sigma(1_0 \otimes 1_2 \otimes 1_0)$$

$$= b^1(g_{003}+g_{030}+g_{300}+g_{012}+g_{102}+g_{201}+g_{111}+g_{021}+g_{120}+g_{210})$$

$$+ b^2(g_{030}) + b^3(g_{300})$$

$$+ b^4(g_{012}+g_{111}-g_{210}+g_{030})$$

$$+ b^5(g_{102}-g_{111}+g_{120}+g_{300})$$

$$+ b^6(g_{201}+g_{210}+g_{300}) + b^7(g_{111})$$

$$+ b^8(g_{021}+g_{030}+g_{120}) + b^9(g_{120})$$

$$+ b^{10}(g_{210}) + b^1(g_{003}) + b^2(g_{030})$$

$$+ b^3(g_{300}) + b^4(g_{012}) + b^5(g_{102}) + b^6(g_{201})$$

$$+ b^7(g_{111}) + b^8(g_{021}) + b^9(g_{120}) + b^{10}(g_{210}).$$

Equating coefficients of g_{102} on left and right hand sides of 14) yields:

$$0 = 2b^5 + b^1$$

or $b^1 = 0$ since all calculations are mod 2. Equating coefficients of g_{300} in 14) yields:

$$2b^3 + b^1 + b^5 + b^6 = 0$$

which implies

15) $$b^5 + b^6 = 0$$

Writing out equation 13) for $u^3(\sigma,\sigma)(1_2 \otimes 1_0 \otimes 1_0)$ results in the same equation as 14) except "b" is replaced by "f". And writing out equation 13) for $u^3(\sigma,\sigma)(1_1 \otimes 1_0 \otimes 1_0)$ yields the same equation as 14) except "b" is replaced by "c". This is true since A_2 is defined on all of $1_0 \otimes 1_2 \otimes 1_0$, $1_2 \otimes 1_0 \otimes 1_0$, and $1_1 \otimes 1_1 \otimes 1_0$ in the same way and $u^3 = 0$ in each case. Therefore, equating coefficients for g_{102} (and g_{300}) as above yields:

16) $f^1 = 0$

$f^5 + f^6 = 0$

17) and $c^1 = 0$

$c^5 + c^6 = 0.$

Now

$$
\begin{aligned}
(18) \quad - g_{210} + g_{030} &= u^3(\sigma,\sigma)(1_0 \otimes 1_1 \otimes 1_1) = (\sigma\psi_\sigma - \psi_1 + \psi_\sigma(1_0 \otimes 1_1 \otimes 1_1) \\
&= \sigma_*\psi_\sigma(A_2(\sigma)(1_0 \otimes 1_1 \otimes 1_1)) + \psi_\sigma(1_0 \otimes 1_1 \otimes 1_1) \\
&= \sigma_*\psi_\sigma(\ 1_0 \otimes 1_1 \otimes 1_1 \ + \ r_1 \otimes s_1 \otimes t_0 \\
&\qquad + \ r_2 \otimes s_0 \otimes t_0 - 1_2 \otimes 1_0 \otimes t_0) + \psi_\sigma(1_0 \otimes 1_1 \otimes 1_1) \\
&= \sigma_*(\psi_\sigma(1_0 \otimes 1_1 \otimes 1_1 \ + \ r_1 \otimes s_1 \otimes t_0)) \\
&\qquad + \psi_\sigma(1_0 \otimes 1_1 \otimes 1_1) \\
&= \sigma_*(d^1 g_{003} + d^2 g_{030} + d^3 g_{300} + d^4 g_{012} + d^5 g_{102} \\
&\qquad + d^6 g_{201} + d^7 g_{111} + d^8 g_{021} + d^9 g_{120} \\
&\qquad + d^{10} g_{210} + f^1 g_{003} + f^2 g_{030} + f^3 g_{300} \\
&\qquad + f^4 g_{012} + f^5 g_{102} + f^6 g_{201} + f^7 g_{111} \\
&\qquad + f^8 g_{021} + f^9 g_{120} + f^{10} g_{210}) + \psi_\sigma(1_0 \otimes 1_1 \otimes 1_1) \\
&= (2d^1 + f^1) g_{003} \\
&\qquad + (2d^2 + f^2 + d^1 + f^1 + d^4 + f^4 + d^8 + f^8) g_{030} \\
&\qquad + (2d^3 + f^3 + d^1 + f^1 + d^5 + f^5 + d^6 + f^6) g_{300} \\
&\qquad + (2d^4 + f^4 + d^1 + f^1) g_{012} \\
&\qquad + (2d^5 + f^5 + d^1 + f^1) g_{102}
\end{aligned}
$$

$$+ (2d^6+f^6+d^1+f^1)g_{201}$$

$$+ (2d^7+f^7+d^1+f^1+d^4+f^4-d^5-f^5)g_{111}$$

$$+ (2d^8+f^8+d^1+f^1)g_{021}$$

$$+ (2d^9+f^9+f^1+d^1+d^5+f^5+d^8+f^8)g_{120}$$

$$+ (2d^{10}+f^{10}+d^1+f^1-d^4-f^4+d^6+f^6)g_{210}\ .$$

Equating coefficients of g_{300} yields:

19) $2d^3 + f^3 + d^1 + f^1 + d^5 + f^5 + d^6 + f^6 = 0.$

Writing out 13) for $u^3(\sigma,\sigma)(1_1 \otimes 1_0 \otimes 1_1)$ yields the same right hand side as 18) but replacing "d" with "e" and the left hand side will equal $g_{120} + g_{300}$. So equating coefficients of g_{300} in this equation yields:

20) $2e^3 + f^3 + e^1 + f^1 + e^5 + f^5 + e^6 + f^6 = 1.$

Writing out 13) for $u^3(\sigma,\sigma)(1_0 \otimes 1_0 \otimes 1_2)$ and equating coefficients of g_{102} and g_{201} yield:

$$b^5 + c^5 + d^5 + e^5 + f^5 + a^1 + b^1 + c^1 + d^1 + e^1 + f^1 = 0$$

and $b^6 + c^6 + d^6 + e^6 + f^6 + a^1 + b^1 + c^1 + d^1 + e^1 + f^1 = 0$

which implies (adding)

$$b^5 + b^6 + c^5 + c^6 + d^5 + d^6 + e^5 + e^6 + f^5 + f^6 = 0$$

which with 15), 16) and 17) above yields

21) $d^5 + d^6 + e^5 + e^6 = 0.$

Adding equations 19) and 20) yields

$$d^1 + e^1 + d^5 + e^5 + d^6 + e^6 = 1$$

and this with 21) implies

22) $d^1 + e^1 = 1.$

Now equating coefficients of g_{012} in 18) yields

23) $f^4 + d^1 = 0$

and equating coefficients of g_{012} in 13) for $u^3(\sigma,\sigma)(1_1 \otimes 1_0 \otimes 1_1)$ yields

24) $f^4 + e^1 = 0.$

Adding 23) and 24) yields $d^1 + e^1 = 0$ which contradicts 22).

We conclude that $v^3 \neq 0$ since it is impossible to find a^i, b^i, c^i, d^i, e^i, f^i which could define a map ψ such that

$$u^3(\rho,\tau)(x_2) = [\rho\psi(\tau) - \psi(\rho\tau) + \psi(\rho)](x_2)$$

for all $x_2 \epsilon D_2$; and for all $\rho,\tau \epsilon \Phi$. ∎

Proposition 3 implies that the universal class $\overline{v}^3(M) \epsilon H^2(GL(3,2); H^2(M; H_3(M,k)))$ is also non-zero by the naturality theorem (Theorem II). But since Φ in this example was not chosen a Sylow subgroup, Proposition 2 does not imply that the universal class $v^2 \in H^2(GL(3,2);H^1(M,H_2(M,k)))$ is zero. However, Proposition 2 and 3 prove that no analog to Theorem IV exists for finite M. Actually, we have shown that no analog to Theorem IV exists where M is a sum of copies of $\mathbb{Z}_2$. It is still possible that such a theorem holds for M a sum of copies of $\mathbb{Z}_p$ with $p > 2$ since $p = 2$ is often an anomalous case. However, only the case of $p = 3$ would matter since $v^2(M)$ is known to be non-zero when M is a sum of copies of $\mathbb{Z}_p$, $p > 5$ (see Appendix).

The same procedure as that in Example 2 above might be useful for computing $v^2(M)$ and $v^3(M)$ where $M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ since one could choose $\Phi \subseteq GL(4,2)$ to be the subgroup generated by $\sigma = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

so $\sigma^2 = id$ and Φ would again be isomorphic to $\mathbb{Z}_2$ making construction of a Φ-system manageable. It is not clear whether this choice would detect non-zero universal classes. The class $v^2(M)$ is not zero for GL(4,2) (see Appendix) but $v^3(M)$ is not known.

The choice of resolution in calculating $v^i(M)$ in these cases was made such that the F^i_σ in Definition 2 are zero.

(The resolution chosen for calculating $v^i(M)$ for free Abelian M also had this property see [3].) The definition of the cocycle $u^n(\sigma,\tau)$ which leads to v^n can be thought of as splitting into two parts: one containing the F_σ^i's and the other containing U_i's. If the resolution chosen is "small" the F_σ^i are zero and only the U_i's contribute to the characteristic class. If the resolution chosen for M is "large" (i.e. the standard resolution) then the U_i are zero (Φ acts on D_i) and only the F_σ^i contribute to the characteristic classes. Computation of $v^i(M)$ is easiest when the resolution is chosen in order to force $F_\sigma^i = 0$.

Appendix

The following is a compilation of some other known results concerning characteristic classes in the finite case, almost all of which are due to Sah [8, 9]. The methods, some of which are outlined below, do not entail the use of the chain homotopy arguments. Described are two approaches which can be used for gaining information regarding the characteristic classes $v^1(M)$ (especially v^2) where $\Phi = GL(n,p)$, and M is n-dimensional vector space over $\mathbb{F}_p$. First, the groups $H^2(\Phi; H^{n-1}(M, H_n(M,k)))$ which contain the $v^n(M)$ can sometimes be computed directly or at least vanishing or nonvanishing results obtained. Secondly, the class $v^2(M)$ has an alternate interpretation in terms of group extensions and automorphisms. These extensions can be studied and the $v^2(M)$ can be realized as obstructions to the lifting of group actions. The following contains outlines of some of these methods and lists consequential results.

Some computations of the groups $H^2(\Phi; H^{n-1}(M, H_n(M,k)))$ rely on the description of $H^*(M,k)$ as a tensor product of polynomial rings on M^* and exterior algebras on M^* (recall M is a vector space of dimension n over the finite field $\mathbb{F}_p$ and $k = \mathbb{F}_p$). M^* denotes the vector space dual of M, Hom(M,k). The formula for p > 2 is given by:

i) $H^i(M,k) \cong \bigoplus_{0 \le i \le j/2} [\wedge^{j-2i}(M_1^*) \otimes S^i(M_2^*)]$

where $M_1 \cong M_2 \cong M$ but M_i lives only in degree i. Here $\wedge^i$ denotes the i^{th} exterior power and S^i denotes homogeneous polynomials of degree i in n-variables.

<u>Example</u>: $H^0(M,k) \cong \wedge^0(M^*)$

$$H^1(M,k) \cong \wedge^1 \otimes S^0 = \wedge^1$$

$$H^2(M,k) \cong \wedge^2 \otimes S^0 + \wedge^0 \otimes S^1 = \wedge^2 + M^*$$

$$H^3(M,k) \cong \wedge^3 \otimes S^0 + \wedge^1 \otimes S^1$$

$$= \wedge^3(M^* \otimes M^*)$$

using $\wedge^0 = S^0 = \mathbb{F}_p$, $\wedge^1 = S^1 = M^*$. All exterior powers and symmetric algebras above are over the dual space M^*, i.e. $\wedge^i$ means $\wedge^i(M^*)$. However, the homology groups $H_*(M)$ can be written as vector space duals of $H^*(M)$ so there are formulas dual to those above.

Now $v^m(M) \in H^2(G,H^{m-1}(M,H_m(M,k)))$ and we rewrite the group before using the above splitting. The coefficients k may be suppressed since k is $\mathbb{Z}_p$,

$$H^2(G,H^{m-1}(M,H_m(M,k)))$$

ii) $\cong H^2(G,\mathrm{Hom}(H_{m-1}(M),H_m(M)))$

iii) $\cong H^2(G,\mathrm{Hom}(H^m(M),H^{m-1}(M)))$

using only Universal Coefficient Theorem or the fact that

H^* is dual of H_*.

The next step is to break up the coefficients in ii) or iii) using i). For instance if H^m and H^{m-1} are rewritten in iii) in terms of symmetric and exterior algebras then one can use a weight (eigenvalue) argument to determine possible homomorphisms in $\mathrm{Hom}(H^m(M), H^{m-1}(M))$. Sometimes such a procedure will show that no non-trivial homomorphisms can exist, forcing the group to be zero.

The center of $\Phi = GL(n,p)$ is of order prime to p. Since Φ acts trivially on its cohomology and $\sigma\epsilon\Phi$ operates on cochains by

$$(\sigma f)(c) = \sigma f(\sigma^{-1}(c)) \quad \text{with } \sigma^{-1}(c) = \sigma^{-1}c\sigma,$$

the coefficient module for Φ can be replaced by its fixed points under the action of the center. In other words, after splitting the coefficients $H^{n-1}(M, H_n(M))$ up according to i) some parts of the resulting sums may be ignored since they can contribute nothing to the cohomology $H^2(\Phi)$.

<u>Example</u>: Suppose $p > 2$.

$$\bar{v}^2 \epsilon H^2(GL(n,\mathbb{F}_p); H^1(M, H_2(M)))$$

$$= H^2(\Phi; \mathrm{Hom}(H_1(M), H_2(M)))$$

$$= H^2(\Phi; \mathrm{Hom}(M, M \oplus \Lambda^2(M)))$$

iv) $$= H^2(\Phi; \mathrm{Hom}(M, M))$$

since the center removes the part of coefficients corresponding to $Hom(M,\Lambda^2(M))$.

Now when $p = 3$ and $n = 2$, $Hom(M,M)$ is the direct sum of $\mathbb{F}_3$ and a free module for a 3-Sylow subgroup of $GL(2,3)$, so $H^2(GL(2,3),\mathbb{F}_3) = 0$ (with trivial action), so the group containing $\overline{v}^2$ for $\Phi = GL(2,3)$ is zero. This argument (due to Sah) serves as an example of how such calculations can yield vanishing results for characteristic classes. Equation iv) is valid for any prime $p \geq 3$, and the group was computed for $n = 2$ [Prop. 4.5, Sah [9]]:

$$H^2(GL(2,p),Hom(M,M)) \cong \begin{cases} \mathbb{F}_p & p > 3 \\ 0 & p = 2,3 \end{cases} .$$

In fact, it was shown in [8, 9] that $v^2 \neq 0$ for $GL(2,p)$ with $p \geq 5$.

The above procedures don't work for $p = 2$. The problem occurs since

$$0 \to \Lambda^2(M) \to H_2(M,k) \to M \to 0$$

is exact but does not split (it splits for $p \geq 3$). Therefore, the splitting of the coefficients as above doesn't hold. In this case the class $v^2 \in H^2(GL(n,2);H^1(M,H_2(M)))$ and there is a mapping

$$H^2(GL(n,p),H^1(M,H_2(M))) \to H^2(GL(n,2),H^1(M,M))$$

and the image under this map is the characteristic class associated to

$$0 \to M \to (\mathbb{Z}/4\mathbb{Z})^n \to M \to 0.$$

This was shown [9] to be non-zero for $n \geq 4$. Therefore, $v^2 \neq 0$ for $GL(n,2)$, $n \geq 4$. When $n = 2$ the coefficient modules are free for a 2-Sylow subgroup. Namely, $H^i(M) \cong S^i(M^*) =$ homogeneous polynomials of degree i in two variables. The monomials $x_1^a x_2^b$, $a+b = i$ form a basis, $\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ is a 2-Sylow subgroup. Therefore, $H^i(M)$ is free when i is odd and $H^{m-1}(M, H_m(M)))$ is always free. Therefore, $\bar{v}^m = 0$ for all m when $\Phi = GL(2,2)$.

In the correspondence of v^2 to d_2 in the Hochschild-Serre spectral sequence for the split extension

v) $$0 \to M \to \pi \to \Phi \to 1$$

$v^2 = d_2^{0,2}(f^2)$ where $f^2 \in H^0(GL(n,p), H^2(M, H_2(M,k)))$; and $[f^2] \in H^2(M, H_2(M,k))$ corresponds to identity in $Hom(H_2(M), H_2(M))$.

Let L = any π module which is M trivial (i.e. $H_2(M,k)$ will work). Define $E = \{(\ell, m) | \ell \in L, m \in M\}$ and define for each 2-cocycle f a group structure E(f) on E:

$$E(f) : (\ell_1, x)(\ell_2, y) = (\ell_1 + \ell_2 + f(x,y), xy).$$

This leads to an exact sequence

$$0 \to L \to E(f) \to M \to 1$$

which is a central extension of L by M. Each $\sigma\epsilon\Phi = GL(n,p)$ defines a bijective map $E \to E$ by $\sigma(\ell,x) = (\sigma(\ell),\sigma(x))$, which induces a new group structure on E:

$$\sigma : E(f) \to E(\sigma f).$$

For a 1-cochain g (in $H^1(M,L)$), define $g : E \to E$ by $g(\ell,x) = (\ell+g(x),x)$. Now suppose $[f]\epsilon H^0(\Phi;H^2(M,L))$. For each $\sigma\epsilon\Phi$, there exists a 1-cochain $g\epsilon H^1(M,L)$ such that $\sigma f-f = \partial_L g_\sigma$, or equivalently $g_\sigma : E(\sigma f) \to E(f)$ is a group isomorphism which is the identity on each of M and L. The action of Φ on the 1-cochain g is $\sigma g = \sigma o g o \sigma^{-1}$ where o denotes composition of maps in E. Now $g_\sigma o \sigma(g_\tau) o g_{\sigma\tau}^{-1} = \sigma(g_\tau) - g_{\sigma\tau}+g_\sigma$ so we have an automorphism E(f)which is the identity map on both L and M. Let A = the group of all automorphisms of E(f) which induce identity on L and M. Let B = the group of automorphisms of E(f) which map L onto L. Then $g_\sigma o\sigma\epsilon B$ so there exists an action:

$$h : \Phi \to A/B.$$

We can identify B with $H^1(M,L)$ and in the Hochschild-Serre spectral sequence for (v), $d_2^{0,2}[f]$ is the cohomology class in $H^2(\Phi;H^1(M,L))$ determined by $\partial_L g$. The map h defines a group extension:

vi) $$0 \to H^1(M,L) \to X \to \Phi \to 1$$

corresponding to the class $d_2^{0,2}[f]$. There is a homomorphism $\eta : X \to B$ and an exact sequence

$$1 \to E(f) \to E(f) \times X \to X \to 1.$$

Now $d_2^{0,2}[f]$ is 0 if vi) splits, and this is equivalent to finding a split

$$1 \to E(f) \to Ef \times \Phi \to \Phi \to 1,$$

considering Φ as a subgroup of X. In such a case the action h can be lifted to an action $\bar{h} : \Phi \to B$. Now, in the above procedure take $L = H_2(M,k)$, $k = \mathbb{F}_p$. Then $[f] = [f^2]$ corresponds to identity in $Hom(H_2(M),H_2(M))$. This yields a way of determining $v^2(M)$ by checking to see if the Φ-action $h : \Phi \to A/B$ lifts to $\bar{h} : \Phi \to B$.

For $p > 2$ combine the above sequence

$$0 \to H_2(M,k) \to E(f) \to M \to 1$$

with the splitting $H_2(M,k) \cong M \oplus \wedge^2(M)$ to get two exact sequences

vii) $\quad 0 \to M \to E(f_1) \to M \to 0$

viii) $\quad 0 \to \wedge^2(M) \to E(f_2) \to M \to 0.$

The part of v^2 corresponding to viii) is killed by the center so the determination of v^2 reduces to checking liftings of Φ-action on both ends of vii). Actually in this case $E(f_1) = (\mathbb{Z}/p^2\mathbb{Z})^n$, and in [8] Sah shows that $\Phi = GL(n,p)$ action can't be lifted for $p > 3$ on vii) in this case. Therefore, $v_2^2 \neq 0$ for $p > 3$.

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