Curvature Inequalities and Extremal Properties of Bundle Shifts

A Dissertation presented

by

Gadadhar Misra

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

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Gadadhar Misra

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Irwin Kra — Committee Chairman

Ronald G. Douglas — Thesis Advisor

Daryl Geller

Vaclav Dolezal, Applied Mathematics (outside member)

The dissertation is accepted by the Graduate School

Dean of the Graduate School

Abstract of the Dissertation

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To some Hilbert space operators T possessing an open set of eigenvalues Ω, it is possible to associate a hermitian holomorphic vector bundle E_T. The curvature of E_T then provides a unitary invariant for the operator T. Our problem is to find reasonable estimates for the curvature $\mathcal{K}_T(\omega)$.

We first obtain necessary and sufficient conditions for a certain 2×2 matrix to admit $\mathrm{Cl}\,\Omega$ as a spectral set. This characterization enables us to estimate the curvature $\mathcal{K}_T(\omega)$ in terms of the Szegö kernel for the region Ω or

more precisely $\mathcal{K}_T(\omega) \leq -\hat{K}_\Omega(\omega,\bar{\omega})^2$. As an application, we produce examples to show that $\|T\| \leq 1$ and $\|T^{-1}\| \leq {}^1/r$ are not sufficient for T to admit the annulus $\{z|r \leq |z| \leq 1\}$ as a spectral set.

Next we prove that the curvature inequality is sharp. If Ω is simply connected, it is possible to compute the curvature of the operator $M_z^* : H^2(\bar{\Omega}) \to H^2(\bar{\Omega})$, which happens to be equal to $-\hat{K}_\Omega(\omega,\bar{\omega})^2$. When Ω is not simply connected we can find no single operator with this extremal property. Never the less given a fixed ζ in Ω, we find an operator T depending on ζ so that $\mathcal{K}_T(\zeta) = -\hat{K}_\Omega(\zeta,\bar{\zeta})^2$. We apply this result to show that the two notions of spectral set and complete spectral set are actually the same for certain 2 x 2 matrices. Or, equivalently if $\mathrm{Cl}\Omega$ is a spectral set for such a matrix, then it possesses a normal $R(\mathrm{Cl}\Omega)$-dilation.

To my grandparents, Apsara and Biswanath.

Table of Contents

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Lastly I wish to thank my parents who bravely endured my rather wild childhood.

is a nonzero nilpotent operator of order two. Relative to an appropriate basis we have the matrix representation

$\begin{pmatrix} \omega & h_T(\omega) \\ 0 & \omega \end{pmatrix}$. Now $\mathcal{K}_T(\omega)$ bears a simple relationship to $h_T(\omega)$, namely $\mathcal{K}_T(\omega) = -h_T(\omega)^{-2}$.

a. Extremal operators in a simply connected region.

Given an operator T in $B_1(\Omega)$, how are we going to estimate the curvature $\mathcal{K}_T(\omega)$? Assuming $\mathrm{Cl}\Omega$ is a spectral set for T seems to provide some estimate on $\mathcal{K}_T(\omega)$. But then one has to ask how good are these estimates? We will show that they are indeed best possible. First, we state an elementary

Proposition 0.1. Suppose T is in $\mathcal{L}(\mathcal{H})$ and $\mathrm{Cl}\Omega$ is a spectral set for T. If $\mathfrak{M} \subset \mathcal{H}$ is a rationally invariant subspace of T and $S = T|_{\mathfrak{M}}$ then $\mathrm{Cl}\Omega$ is also a spectral set for S.

Let $\tilde{B}_1(\Omega)$ consist of the operators T in $B_1(\Omega)$ that admit $\mathrm{Cl}\Omega$ as a spectral set. Since $\ker(T-\omega)^2$ is a rationally invariant subspace of T, it follows that $\mathrm{Cl}\Omega$ is a spectral set for the operator $N_\omega = T|_{\ker(T-\omega)^2}$. Once we know that $\mathrm{Cl}\Omega$ is a spectral set for N_ω then the curvature inequality follows. In particular, if Ω happens to be the unit disk $\mathbb{D}$ and $T \in \tilde{B}_1(\mathbb{D})$ then we obtain $\mathcal{K}_T(\omega) \le -(1-|\omega|^2)^{-2}$.

Since we intend to prove this statement in a somewhat more general setting later (Theorem 1.1), we do not produce a proof now. However, we observe that the backward shift operator U_+^* which is in $\tilde{B}_1(\mathbb{D})$ has curvature precisely equal to $-(1-|\omega|^2)^{-2}$. This is merely a computation and resists generalization to more complicated domains.

Now, let Ω be any simply connected region and T be in $\tilde{B}_1(\Omega)$. As before we obtain $\mathcal{K}_T(\omega) \le -\hat{K}_\Omega(\omega,\bar{\omega})^2$, where $\hat{K}_\Omega(\omega,\bar{\omega})$ is the Szegö kernel for the region Ω. At this point it's natural to ask if $\mathrm{Cl}\,\Omega$ has to be a spectral set for T, whenever T satisfies the curvature inequality. We are unable to resolve this question.

To show that the estimate $\mathcal{K}_T(\omega) \le -\hat{K}_\Omega(\omega,\bar{\omega})^2$ is best possible, we must look for an operator T in $\tilde{B}_1(\Omega)$ satisfying $\mathcal{K}_T(\omega) = -\hat{K}_\Omega(\omega,\bar{\omega})^2$. To carry out the search, let us see if we can compute the curvature of an operator $T \in B_1(\Omega)$ in some reasonable manner. For any T in $B_1(\Omega)$, if γ is a nonzero holomorphic cross-section of E_T, then corresponding to γ there is a natural representation Γ of the Hilbert space $\mathcal{H}$ as a space of holomorphic functions on $\bar{\Omega} = \{\omega | \bar{\omega} \in \Omega\}$ defined by $(\Gamma x)(\omega) = \langle x, \gamma(\bar{\omega})\rangle$ for $x \in \mathcal{H}$. Moreover, since

$$(\Gamma T^* x)(\omega) = \langle x, T\gamma(\bar{\omega})\rangle = \langle x, \bar{\omega}\gamma(\bar{\omega})\rangle = \omega(\Gamma x)(\omega) \text{ for } \omega \in \bar{\Omega},$$

it follows that T is the adjoint of multiplication on $\Gamma(\mathcal{H})$.

If we set $K(\lambda,\overline{\omega}) = \langle \gamma(\overline{\omega}), \gamma(\overline{\lambda}) \rangle$, then K is the reproducing kernel for $\Gamma(\mathcal{H})$. We can express $\mathcal{K}_T(\omega)$ in terms of K by means of the formula

$$\mathcal{K}_T(\omega) = \frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log K(\omega,\overline{\omega})^{-1}.$$

The map $\omega \xrightarrow{\gamma} K(\overline{z},\omega)$ is a holomorphic section of the bundle E_T and $\mathcal{K}_T(\omega)d\omega\wedge d\overline{\omega}$ is the curvature defined with respect to the metric $\langle K,K \rangle_\omega = K(\overline{\omega},\omega)$. Since all holomorphic bundles over an open set of $\mathbb{C}$ are trivial, our problem lies in choosing an appropriate metric on the trivial bundle over Ω so that the curvature with respect to this metric equals $-\hat{K}^2_\Omega(\omega,\overline{\omega})$. How do we choose this metric? The following lemma is very suggestive.

Let $d^2s = h^2 d\omega d\overline{\omega}$ be a metric on Ω. The Gaussian curvature of the region Ω with respect to the metric d^2s is then given by the formula

$$C(h^2) = -\Delta(\log h)/h^2.$$

If Ω is simply connected and $\hat{K}_\Omega(\omega,\overline{\omega})$ is the Szegö kernel for the region Ω, then $d^2s = \hat{K}_\Omega(\omega,\overline{\omega})^2 d\omega d\overline{\omega}$ is the Poincaré metric for Ω (Ahlfors [4]).

<u>Lemma</u>. $$\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \hat{K}_\Omega(\omega,\overline{\omega})^{-1} = -\hat{K}_\Omega(\omega,\overline{\omega})^2.$$

Proof. $\dfrac{\partial^2}{\partial\omega\partial\bar{\omega}} \log \hat{K}_\Omega(\omega,\bar{\omega})^{-1}$

$$= -{}^1/_4\ \Delta \log \hat{K}_\Omega(\omega,\bar{\omega})$$

$$= {}^1/_4\ C(\hat{K}_\Omega^2)\hat{K}_\Omega^2(\omega,\bar{\omega}).$$

Since $\hat{K}_\Omega(\omega,\bar{\omega})^2 d\omega d\bar{\omega}$ is the Poincaré metric for Ω, we know that $C(\hat{K}_\Omega^2) = -4$, therefore $\dfrac{\partial^2}{\partial\omega\partial\bar{\omega}} \log \hat{K}_\Omega(\omega,\bar{\omega})^{-1} = -\hat{K}_\Omega(\omega,\bar{\omega})^2$.

The reproducing kernel for the usual Hardy space $H^2(\bar{\Omega})$ is the Szegö kernel function. Now, we can consider the adjoint of multiplication M_z^* on $H^2(\bar{\Omega})$ as a candidate for the extremal operator. One must first show that M_z^* is in $\tilde{B}_1(\Omega)$. Once this is done, it follows that

$$\kappa_{M_z^*}(\omega) = \frac{\partial^2}{\partial\omega\partial\bar{\omega}} \log \hat{K}_{\bar{\Omega}}(\omega,\bar{\omega})^{-1} = -\hat{K}_{\bar{\Omega}}(\bar{\omega},\omega)^2,\ \omega \in \Omega.$$

To complete the proof that M_z^* is the extremal operator, we have to verify the relation $\hat{K}_{\bar{\Omega}}(\bar{\omega},\omega) = \hat{K}_\Omega(\omega,\bar{\omega})$. We will prove all this (Proposition 2.1 and 2.2) later for an arbitrary region Ω. Some time back, D. Purohit used the Riemann map π of an arbitrary simply connected region $\bar{\Omega}$ to show that the extremal problem for such a region can be reduced to that of the disk. He found the operator $M_\pi^* : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ to be extremal in the class $\tilde{B}_1(\Omega)$. It is easy to verify that the operator M_π^* is unitarily equivalent to the operator $M_z^* : H^2(\bar{\Omega}) \to H^2(\bar{\Omega})$.

b. Extremal operator for a region that is not simply connected.

As one might suspect, when the region Ω is not simply connected, the situation is rather complicated. Fortunately the proof of the inequality $\mathcal{K}_T(w) \le -\hat{K}_\Omega(w,\overline{w})^2$ does not depend on the connectivity of the region Ω. One of the difficulties is that the Szegö kernel for a non-simply connected region Ω does not yield the Poincaré metric for it and our previous techniques fail. Perhaps, even more surprising is the inequality $\frac{\partial^2}{\partial w \partial \overline{w}} \log \hat{K}_\Omega(w,\overline{w})^{-1} \lneq -\hat{K}_\Omega(w,\overline{w})^2$ which essentially says that $\mathcal{K}_{M_z^*}(w) \lneq -\hat{K}_\Omega(w,\overline{w})^2$. So the operator $M_z^* : H^2(\overline{\Omega}) \to H^2(\overline{\Omega})$ definitely fails to be extremal in $\tilde{B}_1(\Omega)$. At present we do not know if there is any single operator in $\tilde{B}_1(\Omega)$ that would be extremal! What is the next best thing we can do? Let us ask if the inequality is sharp pointwise, that is, given a point ζ in Ω does there exist an operator T in $\tilde{B}_1(\Omega)$ so that at least

$$\mathcal{K}_T(\zeta) = -\hat{K}_\Omega(\zeta,\overline{\zeta})^2.$$

For each given point ζ, we are able to produce an operator in $\tilde{B}_1(\Omega)$ that satisfies the curvature equality at least at the point ζ. First, we define certain Hilbert space $H^2(\partial\overline{\Omega},m)$ of analytic functions on $\overline{\Omega}$ determined by a positive measure m on the boundary $\partial\overline{\Omega}$ depending on the given point ζ. As before, we consider the operator $M_z^* : H^2(\partial\overline{\Omega},m) \to H^2(\partial\overline{\Omega},m)$ as a candidate for the extremal operator. Once we show that M_z^* is in $\tilde{B}_1(\Omega)$, we would like to establish the equality, $\mathcal{K}_{M_z^*}(\omega) = -\hat{K}_\Omega(\omega,\bar{\omega})^2$. This involves some work, which we will take up later. These operators have been studied in a more general setting (Abrahamse-Douglas [2]), where they are known as bundle shifts.

c. An application to generalized dilations.

At present one may say that the theory of operators related to the unit disk $\mathbb{D}$ is well understood. One early result that can be quoted in this context is due to Von-Neumann and states that (Abrahamse-Douglas [3], Douglas [11]) any operator T in $\mathcal{L}(\mathcal{H})$ is a contraction if and only if $\mathrm{Cl}\,\mathbb{D}$ is a spectral set for it, which means $\|p(T)\| \le \|p\|_{\mathbb{D}}$ for all polynomials p. Shortly afterwards Sz-Nagy proved that if the operator T is a contraction then there is a strong unitary dilation U on a superspace $\mathcal{K}$ containing $\mathcal{H}$. That is to say, for every polynomial p the operator p(T) is the compression of the operator p(U) to $\mathcal{H}$ namely, $p(T) = P_{\mathcal{H}}p(U)|_{\mathcal{H}}$. Now we obtain Von Neumann's inequality as a Corollary, since

$$\begin{aligned}\|p(T)\| \le \|p(U)\| &= \sup\{|p(z)| : z \in \sigma(U)\} \\ &\le \sup\{|p(z)| : z \in \mathbb{D}\} = \|p\|_{\mathbb{D}}.\end{aligned}$$

Thus the following three statements are equivalent for an operator T in $\mathcal{L}(\mathcal{H})$.

(i) $C\mathbb{D}$ is a spectral set for T.

(ii) T is a contraction.

(iii) T has a strong unitary dilation.

In an attempt to extend the equivalence of (i) and (iii) to more general sets X, one is led to consider normal R(X)-dilation N on the superspace $\mathcal{K} \supset \mathcal{H}$ of an operator T on $\mathcal{H}$, where we assume the operator N is normal with its spectrum contained in the boundary ∂X and $\varphi(T) = P_{\mathcal{H}}\varphi(N)|_{\mathcal{H}}$ for every φ in Rat(X). It is immediate that if T has a normal R(X)-dilation then X is a spectral set for T. The nontrivial converse when X is simply connected was established by Berger, Foias and Lebow (Douglas [11]). When the region X is not simply connected, it is not known if the existence of normal R(X)-dilation for T is guaranteed whenever X is a spectral set for T.

We can reformulate many of these results into statements about operator valued representation of Rat(X). Such representations need not be continuous. The following example is due to Abrahamse [1].

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $X = \{z \mid |z| \le 1\}$ and $f_n(z) = z^n$.

Then $\|f_n\| = 1$ and $\|f_n(T)\| = \left\| \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\| \ge n.$

We notice that as soon as we assume X is a spectral set for T, not only $f \to f(T)$ is continuous but indeed it is contractive. However, we do not know if this hypothesis is enough to guarantee a dilation for the operator T. So, what is necessary is an appropriate strengthening of the notion of the spectral set. One such concept is due to Arveson [5], which he calls complete spectral set. He is then able to show that the region X is a complete spectral set for the operator T if and only if it has a normal R(X)-dilation. It remains to see when these two notions are the same. We show that a region X is a spectral set for certain 2 x 2 matrices if and only if it is a complete spectral set for these matrices. One half of this assertion is trivial, to prove the other half we use the existence of extremal operators in $\widetilde{B}_1(\Omega)$.

Some of the results (Theorem 2.3 and Corollary 2.1) we have discussed here were reported without proof by Bruce Abrahamse in a private communication (February, 1979) to Ronald Douglas.

1. CURVATURE INEQUALITIES.

We begin with a well known definition of spectral set which we reformulate in various ways suitable for our purpose. We suppose Ω is an open bounded set in $\mathbb{C}$. Let $\mathrm{Hol}(\Omega,\mathbb{D})$ and $\mathrm{Rat}(\mathrm{Cl}\Omega)$ be the holomorphic functions mapping the region Ω into the disk $\mathbb{D}$ and the rational functions on $\mathrm{Cl}\Omega$ without poles respectively.

Definition 1.1. The set $\mathrm{Cl}\Omega$ is a spectral set for T in $\mathfrak{L}(\mathfrak{H})$ if $\sigma(T) \subset \mathrm{Cl}\Omega$ and $\|f(T)\| \le \|f\|_\Omega$ for all f in $\mathrm{Rat}(\mathrm{Cl}\Omega)$.

Let us call an Ω reasonable if for f in $H^\infty(\Omega)$ there is a sequence of rational functions γ_n with poles outside the set $\mathrm{Cl}\Omega$ so that $\|\gamma_n\|_\Omega \le \|f\|_\Omega$ and $\gamma_n(\omega) \to f(\omega)$ for each ω in Ω. If Ω is a finitely connected Jordan region, that is, the boundary of Ω consists of simple analytic curves, then Ω is reasonable (Gamelin [12]). Since we can define $f(T)$ whenever f is in $H^\infty(\Omega)$ and $\sigma(T) \subset \Omega$, the following remark is self-evident.

Remark 1. Suppose $\sigma(T) \subset \Omega$ and Ω is a Jordan region. Then $\mathrm{Cl}\Omega$ is a spectral set for an operator T if and only if $\|f(T)\| \le 1$, for all f in $\mathrm{Hol}(\Omega,\mathbb{D})$.

Remark 2. Suppose Ω is again a Jordan region and $\sigma(T) \subset \Omega$. Then $\|f(T)\| \le 1$ for all f in $\mathrm{Hol}(\Omega,\mathbb{D})$ is equivalent to saying $\|f(T)\| \le 1$, only for those f in $\mathrm{Hol}(\Omega,\mathbb{D})$ that vanish at any one given point ζ in Ω. For our convenience we let $\mathrm{Hol}(\Omega,\zeta,\mathbb{D})$ denote the set $\{f \in \mathrm{Hol}(\Omega,\mathbb{D}) : f(\zeta) = 0\}$.

One half of the assertion is trivial, to prove the other half let us take an f in $\mathrm{Hol}(\Omega,\mathbb{D})$ that does not vanish at ζ. Suppose $f(\zeta) = \alpha$ then $|\alpha| < 1$ by the maximum modulus principle, so that $\varphi_\alpha(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}$ is a conformal map of the unit disk $\mathbb{D}$. The function $g = \varphi_\alpha \circ f$ lies in $\mathrm{Hol}(\Omega,\mathbb{D})$ and it vanishes at ζ. It follows that $\|g(T)\| \leq 1$ and Von-Neumann's Theorem now implies that the unit disk $\mathbb{D}$ is a spectral set for $g(T)$. By the same token, the unit disk is a spectral set for $\varphi_\alpha^{-1}(g(T))$. Equivalently $\|f(T)\| = \|\varphi_\alpha^{-1}(g(T))\| \leq 1$.

Now, we are ready to prove our main

<u>Theorem 1.1</u>. Let Ω be a Jordan region in $\mathbb{C}$. For each fixed ω in Ω, if $h(\omega)$ is a positive number depending on ω then $\mathrm{Cl}\Omega$ is a spectral set for the matrix $\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix}$ if and only if

$$h(\omega) \leq [\sup\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega,\omega,\mathbb{D})\}]^{-1}.$$

<u>Proof</u>. For any polynomial p one can easily verify that

$$p\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} p(\omega) & p'(\omega)h(\omega) \\ 0 & p(\omega) \end{pmatrix}.$$

If p and q are polynomials such that $q \neq 0$ on $\mathrm{Cl}\Omega$, then for the rational function $f = p/q$, the usual functional calculus yields

$$f\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix} = (p\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix})(q\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix})^{-1},$$

which in turn leads to

$$f\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} f(\omega) & f'(\omega)h(\omega) \\ 0 & f(\omega) \end{pmatrix}.$$

Since the spectrum of the matrix under consideration consists of the single eigenvalue ω, which is contained in Ω, it follows that the earlier remarks 1 and 2 apply to it. We then find, the set $\mathrm{Cl}\Omega$ is going to be a spectral set for $\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix}$ if and only if

$$\left\| f\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix} \right\| \le 1 \text{ for all } f \text{ in } \mathrm{Hol}(\Omega,\omega,\mathbb{D}).$$

or equivalently we must have

$$\left\| \begin{pmatrix} 0 & f'(\omega)h(\omega) \\ 0 & 0 \end{pmatrix} \right\| \le 1 \text{ for all } f \text{ in } \mathrm{Hol}(\Omega,\omega,\mathbb{D}),$$

which is the same thing as saying

$$|f'(\omega)|h(\omega) \le 1 \text{ for all } f \text{ in } \mathrm{Hol}(\Omega,\omega,\mathbb{D}),$$

or $\quad h(\omega) \le [\sup\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega,\omega,\mathbb{D})\}]^{-1}$.

Corollary 1.1. For any T in $B_1(\Omega)$ if $\mathrm{Cl}\Omega$ is a spectral set then

$$\varkappa_T(\omega) \leq -[\sup\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega,\omega,\mathbb{D})\}]^2.$$

Proof. Since $\mathrm{Ker}(T-\omega)^2$ is a rationally invariant subspace of T therefore, $N_\omega = T|_{\mathrm{Ker}(T-\omega)^2}$ admits $\mathrm{Cl}\Omega$ as a spectral set, which was pointed out in Proposition 0.1. It can be shown that N_ω has the matrix representation $\begin{pmatrix} \omega & h_T(\omega) \\ 0 & \omega \end{pmatrix}$ with respect to an appropriate basis. The Theorem applies to this matrix and consequently we obtain the desired inequality.

$$(-\varkappa_T(\omega))^{-1/2} = h_T(\omega) \leq [\sup\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega,\omega,\mathbb{D})\}]^{-1}.$$

We wish to reformulate this inequality in terms of the Szegö kernel for the domain Ω. Let us proceed with the relevant definitions and theorems, most of which appear in Bergman [6].

Let m be a positive measure defined on the boundary $\partial\Omega$, that is mutually absolutely continuous with respect to the arc length. Let $L^2(\partial\Omega,m)$ be the space consisting of complex functions on $\partial\Omega$ that are square integrable with respect to the measure m, where the inner product is the usual one. We also introduce the Hardy class $H^2(\Omega)$ consisting of analytic functions f on the region Ω so that $|f|^2$ admits a harmonic majorant. The properties of $H^2(\Omega)$ are well known (Rudin [13]), in particular each f in $H^2(\Omega)$ possess a well behaved non-tangential boundary value in $L^2(\partial\Omega,|dz|)$. It is natural to define the class $H^2(\partial\Omega,m)$ to be the one consisting of functions f^* in $L^2(\partial\Omega,m)$ that are the boundary value of some f in $H^2(\Omega)$. The class $H^2(\partial\Omega,m)$ is a closed subspace of a separable Hilbert space, namely $L^2(\partial\Omega,m)$. Therefore, $H^2(\partial\Omega,m)$ itself must also be a separable Hilbert space. Now, the existence of the reproducing kernel for a Hilbert space is equivalent to boundedness of the point evaluation functional $\phi_\omega : f \to f(\omega)$ for each ω in Ω. For the Hilbert space $H^2(\partial\Omega,|dz|)$ the boundedness of the functional ϕ_ω follows from the Cauchy integral formula. Consequently $H^2(\partial\Omega,|dz|)$ possesses a well defined reproducing kernel say, $\hat{K}_\Omega(z,\bar{\zeta})$. This is the classical Szegö kernel function, it has the representation

$$\hat{K}_\Omega(z,\bar{\zeta}) = \Sigma\, e_n(z)\overline{e_n(\zeta)},$$

where $\{e_n\}$ is an orthonormal basis for $H^2(\partial\Omega,|dz|)$.

On our domain Ω there exists another kernel, $\hat{L}(z,\zeta)$ determined by the following properties:

(i) $\hat{L}(z,\zeta)$ is regular in $Cl\Omega$ with the exception of a simple pole with the residue at $z = \zeta$.

(ii) For ω in the boundary $\partial\Omega$, $\hat{K}_\Omega(z,\bar{\zeta})$ and $\hat{L}(z,\zeta)$ are connected by the relation

$$\hat{K}_\Omega(z,\bar{\zeta})\,|dz| = {}^1/_i\,\hat{L}(z,\zeta)\,dz \ldots\ldots\ldots\ldots (*).$$

We record here one more relationship which will be useful to us later

$$1/_i \hat{L}(z,\zeta)\hat{K}_\Omega(z,\overline{\zeta})dz > 0 \quad \text{on } \partial\Omega \ldots\ldots\ldots\ldots (**)$$

If $F_\zeta(z) = F(z,\zeta)$ denotes the function

$$F(z,\zeta) = \hat{K}_\Omega(z,\overline{\zeta})/\hat{L}(z,\zeta),$$

then F_ζ maps the region Ω onto the n times covered unit disk $\mathbb{D}$. We now consider the problem of finding among all functions f in $\mathrm{Hol}(\Omega,\omega,\mathbb{D})$ the one that maximizes $|f'(\omega)|$.

<u>Schwarz Lemma</u> (for multiplying connected domains): If the function f is in $\mathrm{Hol}(\Omega,\omega,\mathbb{D})$, then

$$|f'(\omega)| \leq F'_\omega(\omega) = \hat{K}_\Omega(\omega,\overline{\omega}),$$

where equality holds if and only if $f(z) = e^{i\theta}F_\omega(z)$.

Whenever Ω_1 is a subset of Ω, it follows that $\mathrm{Hol}(\Omega,\omega,\mathbb{D})$ is a subset of $\mathrm{Hol}(\Omega_1,\omega,\mathbb{D})$ and we obtain the monotonicity of kernel functions, that is,

$$\hat{K}_\Omega(\omega,\overline{\omega}) = \sup_{f\in \mathrm{Hol}(\Omega,\omega,\mathbb{D})} |f'(\omega)| \leq \sup_{f\in \mathrm{Hol}(\Omega_1,\omega,\mathbb{D})} |f'(\omega)| = \hat{K}_{\Omega_1}(\omega,\overline{\omega}).$$

Since both $\hat{K}_\Omega(\omega,\overline{\omega})$ and $\hat{K}_{\Omega_1}(\omega,\overline{\omega})$ are real analytic functions, they can not be equal on any open set. Therefore, we can find a point ω in any open subset Ω_0 of Ω_1 so that $\hat{K}_\Omega(\omega,\overline{\omega}) \lneq \hat{K}_{\Omega_1}(\omega,\overline{\omega})$.

We now restate Theorem 1.1 and its Corollary using the Szegö kernel.

<u>Theorem 1.1'</u>. For ω in Ω and $F_\omega(z) = \hat{K}_\Omega(z,\bar{\omega})/\hat{L}(z,\bar{\omega})$, the set $\mathrm{Cl}\Omega$ is a spectral set for $\begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix}$ if and only if

$$\left\| F_\omega \begin{pmatrix} \omega & h(\omega) \\ 0 & \omega \end{pmatrix} \right\| \le 1 \text{ or equivalently } h(\omega) \le \hat{K}_\Omega(\omega,\bar{\omega})^{-1}.$$

<u>Corollary 1.1'</u>. For any T in $B_1(\Omega)$ if $\mathrm{Cl}\Omega$ is a spectral set, then

$$\kappa_T(\omega) \le -(\hat{K}_\Omega(\omega,\bar{\omega}))^2 ..$$

Our next task is to find if the inequality $\kappa_T(\omega) \le -(\hat{K}_\Omega(\omega,\bar{\omega}))^2$ is sufficient for T to admit $\mathrm{Cl}\Omega$ as a spectral set. Once the question is asked one begins to wonder if the hypothesis of being a spectral set was really necessary in obtaining the curvature inequality. The following proposition shows that we can get away with much less. Since the function $F_\zeta(z) = \hat{K}_\Omega(z,\bar{\zeta})/\hat{L}(z,\zeta)$ extends to a holomorphic function on a region containing $\mathrm{Cl}\Omega$, it follows that we can define $F_\zeta(T)$ by the Riesz calculus, whenever $\sigma(T) \subset \mathrm{Cl}\Omega$. In particular, for T in $B_1(\Omega)$ and $\sigma(T) \subset \mathrm{Cl}\Omega$ we have

<u>Proposition 1.1(1)</u>. If $\|F_\zeta(T)\| \le 1$ for some fixed ζ in Ω then $\kappa_T(\omega) \le -\hat{K}_\Omega(\omega,\bar{\omega})^2$ for all ω in Ω.

(2). If $\|F_\omega(T)\| \le k$ for all ω in Ω then

$\kappa_T(\omega) \leq -1/{K^2 \hat{K}_\Omega(\omega,\bar{\omega})^2}$ for all ω in Ω.

<u>Proof.</u> Note that $F_\omega(T)|_{\mathrm{Ker}(T-\omega)^2} = F_\omega(T|_{\mathrm{Ker}(T-\omega)^2})$. Therefore, if $\|F_\omega(T)\| \leq k$ for each ω, we would obtain

$$\left\|\begin{pmatrix} 0 & F'_\omega(\omega)h_T(\omega) \\ 0 & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} F_\omega(\omega) & F'_\omega(\omega)h_T(\omega) \\ 0 & F_\omega(\omega) \end{pmatrix}\right\| = \|F_\omega(T|_{\mathrm{Ker}(T-\omega)^2})\|$$

$$= \|F_\omega(T)|_{\mathrm{Ker}(T-\omega)^2}\| \leq \|F_\omega(T)\| \leq k.$$

Equivalently $h_T(\omega) \leq k/|F'_\omega(\omega)| = k\,\hat{K}_\Omega(\omega,\bar{\omega})^{-1}$ for all ω in Ω.

Next, suppose we only know that $\|F_\zeta(T)\| \leq 1$ for some fixed ζ in Ω. If F_ω is any other Riemann map of Ω then there is a Möbius transformation φ of the disk $\mathbb{D}$, so that $\varphi F_\omega = F_\zeta$. By our hypothesis $\|\varphi(F_\omega(T))\| = \|F_\zeta(T)\| \leq 1$. We can apply Von-Neumann's Theorem to conclude that $\|F_\omega(T)\| = \|\varphi^{-1}(\varphi F_\omega(T))\| \leq 1$, for all ω in Ω. Thus we are able to apply the techniques of previous paragraph with $k = 1$.

Let us apply the results we have obtained so far, to show that $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1/r$ are not sufficient for T to admit $A = \{z \mid r \leq |z| \leq 1\}$ as a spectral set. We begin with a lemma, that can be found in Williams [18].

Lemma 1.1. Consider the two dimensional shift A_2 whose matrix relative to an orthonormal basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\|\alpha + \beta A_2\| = {}^1/2\ \{|\beta| + \sqrt{4|\alpha|^2 + |\beta|^2}\}.$$

Proof. If $\varphi_\alpha(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}$ for $|\alpha| < 1$, then $|\varphi_\alpha(z)| < 1$ for $|z| < 1$ and $|\varphi_\alpha(z)| = 1$ for $|z| = 1$. Hence by the usual functional calculus $\|\varphi_\alpha(T)\| < 1$ and $\|\varphi_\alpha(T)\| = 1$ if $\|T\| = 1$.

Now we have $\|\varphi_\alpha(A_2)\| = 1$ for all $|\alpha| < 1$. Since

$$\varphi_\alpha(A_2) = (A_2-\alpha)(1-\bar{\alpha}A_2)^{-1} = (A_2-\alpha)(1+\bar{\alpha}A_2) = -\alpha + (1-|\alpha|^2)A_2$$

which yields

$$\|A_2 - \alpha(1-|\alpha|^2)\| = (1-|\alpha|^2)^{-1}$$

we put $\lambda = \alpha(1-|\alpha|^2)^{-1}$ and compute

$$(1-|\alpha|^2)^{-1} = {}^1/2\ \{1 + \sqrt{4|\lambda|^2 + 1}\}$$

to arrive at

$$\|A_2 - \lambda\| = {}^1/2\ \{1 + \sqrt{4|\lambda|^2 + 1}\}.$$

The proof is now complete if $\beta \neq 0$. For $\beta = 0$, the result is obvious.

We remark that, the norm can be computed directly or by computing the eigenvalues of $(\alpha+\beta A_2)(\alpha+\beta A_2)^*$. Now to produce the desired examples, let us consider

$$T = \begin{pmatrix} \omega & 1 - |\omega|^2 \\ 0 & \omega \end{pmatrix},$$ where $\omega \in A_0 = \{z | \sqrt{r} \le 1\}$. We can apply the lemma to verify that

$\|T\| = 1$, whereas $T^{-1} = \begin{pmatrix} 1 & \frac{1}{\omega^2}(1-|\omega|^2) \\ 0 & \frac{1}{\omega} \end{pmatrix}$ and $\|T^{-1}\| \le {}^1/r$

is equivalent to $^1/2\ \{\frac{1}{|\omega|^2}(1-|\omega|^2) + \sqrt{\frac{4}{|\omega|^2} + \frac{1}{|\omega|^4}(1-|\omega|^2)^2}\}$

$$\le {}^1/r.$$

$$\Leftrightarrow 4|\omega|^2 + (1-|\omega)^2)^2 \le (2|\omega|^2/r - (1-|\omega|^2))^2.$$

$$\Leftrightarrow 4r^2|\omega|^2 + r^2(1-|\omega|^2)^2 \le 4|\omega|^2 - 4|\omega|^2 r(1-|\omega|^2) + r^2(1-|\omega|^2)^2$$

$$\Leftrightarrow |\omega|^2 - r(1-|\omega|^2) - r^2 \ge 0$$

$$\Leftrightarrow (|\omega|^2 - r) + (r|\omega|^2 - r^2) \ge 0.$$

The last statement is correct since $|\omega|^2 \ge r$.

As we have pointed out, there is an ω in every open subset of A_0, satisfying $1 - |\omega|^2 = \hat{K}_{\mathbb{D}}(\omega,\overline{\omega})^{-1} \gneq \hat{K}_A(\omega,\overline{\omega})^{-1}$. For all such ω in A_0, we can apply Theorem 1.1 to show that A is not a spectral set for T.

In particular, let us fix $T = \begin{pmatrix} \omega & 1 - r \\ 0 & \omega \end{pmatrix}$, $|\omega|^2 = r$.

Then one may also apply the following theorem due to Williams [18].

<u>Theorem.</u> If an operator T on a finite dimensional vector space is completely non-normal and $\|T\| = 1$, then the unit disk is a minimal spectral set for T.

Since $\|T\| = 1$ and T is completely non-normal it follows that $\mathbb{D}$ is a minimal spectral set for T. But $\|T^{-1}\| = {}^1/r$ and the annulus A can not be a spectral set for T, the disc $\mathbb{D}$ being minimal.

2. EXTREMAL OPERATORS.

Some extremal problems arising in classical Hilbert space theory will be useful to us, which we will discuss first (Bergman [6]). Let Ω be a Jordan region and m be a positive measure on the boundary $\partial\Omega$, that is mutually boundedly absolutely continuous with respect to the arc length measure. As before we define the Hardy class $H^2(\partial\Omega,m)$ with respect to the measure m. For any point ω in Ω we let $\mathfrak{M}_0 = \{f \in H^2(\partial\Omega,m) : f(\omega) = 1\}$ and $\mathfrak{M}_1 = \{f \in H^2(\partial\Omega,m) : f(\omega) = 0$ and $f'(\omega) = 1\}$. We now consider two minimum problems.

0. To find the minimum $\|f\|^2$ over the subspace $\mathfrak{M}_0$.

1. To find the minimum $\|f\|^2$ over the subspace $\mathfrak{M}_1$.

Even though the existence and uniqueness of the solution to both of these problems are well known, we quote here a lemma from Suita [16], which immediately establishes the extremality of certain functions.

Lemma 2.1. The function F in $H^2(\partial\Omega,m)$ is a solution to

(i) Problem 0. if and only if F is orthogonal to $\{f \in H^2(\partial\Omega,m) : f(\omega) = 0\}$

(ii) Problem 1. if and only if F is orthogonal to $\{f \in H^2(\partial\Omega,m) : f(\omega) = f'(\omega) = 0\}$.

<u>Proof</u>. Since the proofs of (i) and (ii) are similar, we only prove part (ii) here. Assume F is the extremal function and let F_1 be any other function in $H^2(\partial\Omega,m)$ satisfying $F_1(\omega)$ and $F_1'(\omega) = 1$. We set $g = F - F_1$ to obtain

$$\|F+\varepsilon g\|^2 = \|F\|^2 + \text{Re}\{\overline{\varepsilon}(F,G)\} + |\varepsilon|^2\|g\|^2 \geq \|F\|^2.$$

Since ε is arbitrary we have $\langle F,g\rangle = 0$. Conversely we find from $\langle F_1,g\rangle = 0$ that,

$$\|F_1\|^2 = \|F\|^2 + \|F_1 - F\|^2$$

which implies the extremality of the function F.

Since we have chosen the measure m to be mutually absolutely continuous with respect to arc length measure on the boundary $\partial\Omega$ it follows that the point evaluation functional on $H^2(\partial\Omega,m)$ is continuous for each point ω in the region Ω. We are now assured of the existence of a reproducing kernel function $K_m(\cdot,\omega)$ for the Hilbert space $H^2(\partial\Omega,m)$. We now define two functions f_0 and f_1, which turn out to be the extremal function we are looking for.

Suppose K_m^{jk} are the various partial derivations $\frac{\partial^{j+k}}{\partial z^j \partial\overline{z}^k} K_m(z,\overline{z})|_{z=\omega}$. Set $f_0(z) = K_m(z,\overline{\omega})$ and

$$f_1(z) = \begin{vmatrix} K_m(z,\overline{\omega}) & \frac{\partial}{\partial\overline{\zeta}} K_m(z,\overline{\zeta})|_{\zeta=\omega} \\ K_m^{00} & K_m^{10} \end{vmatrix} \Big/ \begin{vmatrix} K_m^{00} & K_m^{01} \\ K_m^{10} & K_m^{11} \end{vmatrix}.$$

Clearly if g is any function in $H^2(\partial\Omega,m)$ satisfying $g(\omega) = 0$ then $(g,f_0) = 0$ and similarly if h is any function in $H^2(\partial\Omega,m)$ satisfying $h(\omega) = h'(\omega) = 0$ then $\langle h,f_1\rangle = 0$. In view of Lemma 2.1, we see that f_0 and f_1 are the extremal functions for the Problems 0. and 1. respectively. If we denote the solution to Problems 0. and 1. by $\lambda_0(\omega)$ and $\lambda_1(\omega)$, then one can compute without much difficulty, the value

$$\lambda_0(\omega) = \|f_0\|^2 = 1/K_m(\omega,\overline{\omega}) \text{ and}$$

$$\lambda_1(\omega) = \|f_1\|^2 = K_m^{00}/(K_m^{00}K_m^{11}-|K_m^{01}|^2) = K_m^{00}\left[\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \log K_m(\zeta,\zeta)^{-1}\Big|_{\zeta=\omega}\right].$$

Let $M_z^* : H^2(\partial\Omega,m) \to H^2(\partial\Omega,m)$ be the adjoint of the multiplication operator. Assuming it is in the class $B_1(\overline{\Omega})$, we obtain the curvature inequality. For each point ω in Ω we have,

$$\mathcal{K}_{M_z^*}(\omega) = -\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \log K_m(\zeta,\overline{\zeta})\Big|_{\zeta=\omega} \le -\hat{K}_{\overline{\Omega}}(\omega,\omega)^2 = \hat{K}_{\Omega}(\omega,\overline{\omega})^2$$

We now wish to prove this inequality in a completely different manner using ideas that can be found in Suita [16] and Burbea [7].

Theorem 2.1. For all points ω in Ω we have, $\mathcal{K}_{M_z^*}(\omega) \le -\hat{K}_{\Omega}^2(\omega,\overline{\omega})^2$.

Proof. Let F be the mapping function, that is, $F(z,\omega) = \hat{K}_{\Omega}(z,\overline{\omega})/\hat{L}(z,\omega)$. Now, define a function φ so that

$$\varphi(z) = F(z,\omega)K_m(z,\overline{\omega})/\hat{K}_\Omega K_m,$$

where $\hat{K}_\Omega$ and K_m denote the numbers $\hat{K}_\Omega(\omega,\overline{\omega})$ and $K_m(\omega,\overline{\omega})$ respectively. Since F has a zero at $z = \omega$, therefore $\varphi(\omega) = 0$. Also

$$\varphi'(\omega) = F'(\omega,\overline{\omega})K_m(\omega,\overline{\omega})/\hat{K}_\Omega K_m + F(\omega,\overline{\omega})K_m'(\omega,\overline{\omega})/\hat{K}_\Omega K_m.$$

Since $F'(\omega,\overline{\omega}) = \hat{K}_\Omega(\omega,\overline{\omega})$, the first term reduces to 1 whereas $F(\omega,\overline{\omega}) = 0$ and the second term vanishes, so $\varphi'(\omega) = 1$. Thus the function φ lies in the subspace $\mathfrak{m}_1$. But $\|f_1\|^2$ being a solution to, $\min\{\|f\|^2 | f \in \mathfrak{m}_1\}$ forces

$$\|\varphi\|^2 \geq \|f_1\|^2 = [K_m^{00} \frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} \log K_m(\zeta,\overline{\zeta})]^{-1}.$$

On the other hand,

$$\|\varphi\|^2 = 1/_{\hat{K}_\Omega^2 K_m^2} \int_{\partial\Omega} |F(z,\overline{\omega})|^2 |K_m(z,\overline{\omega})|^2 dm = [\hat{K}_\Omega(\omega,\overline{\omega})K_m(\omega,\overline{\omega})]^{-1},$$

since $|F(z,\overline{\omega})| = 1$ on the boundary $\partial\Omega$ and $\|K_m(z,\overline{\omega})\|^2 = K_m(\omega,\overline{\omega})$. Putting everything together we obtain

$$\|\varphi\|^2 = [\hat{K}_\Omega(\omega,\overline{\omega})^2 K_m(\omega,\overline{\omega})]^{-1} \geq -[K_m(\omega,\overline{\omega})\varkappa_{M_z^*}(\omega)]^{-1}.$$

It is interesting that the inequality can be shown to be strict if the region Ω is not simply connected and m is the usual arc length measure on the boundary $\partial\Omega$ (Burbea [7] and Suita [16]). We now proceed to find out when equality can occur. For each fixed point ω in the region Ω, let m be the measure $|\hat{K}_\Omega(z,\omega)|^2|dz|$. For this particular choice of the measure m we have

Theorem 2.2. $\varkappa_{M_z^*}(\omega) = -\hat{K}_\Omega(\omega,\bar{\omega})^2.$

Proof. As before, let φ be the function defined by

$$\varphi(z) = F(z,\bar{\omega})K_m(z,\bar{\omega})/\hat{K}_\Omega K_m.$$

Again we find

$$\|\varphi\|^2 = [\hat{K}_\Omega(\omega,\bar{\omega})^2 K_m(\omega,\bar{\omega})]^{-1} \geq -[K_m(\omega,\bar{\omega})\varkappa_{M_z^*}(\omega)]^{-1}$$

where we will have equality, if φ can be shown to be the extremal function. Now Lemma 2.1 plays an important role and reduces our problem to merely showing that φ is orthogonal to the subspace $\{f \in H^2(\partial\Omega,m) : f'(\omega) = f(\omega) = 0\}$. So for any function f in this subspace let us compute,

$$\langle f,\varphi\rangle = \frac{1}{(2\pi\hat{K}_\Omega K_m)} \int_{\partial\Omega} f(z)F(z,\omega)K_m(z,\bar{\omega}|\hat{K}_\Omega(z,\bar{\omega})|^2|dz|$$

$$= \frac{1}{(2\pi\hat{K}_\Omega K_m)} \int_{\partial\Omega} f(z)\,\frac{1}{\overline{F(z,\omega)}}\,\overline{K_m(z,\bar{\omega})}\hat{K}_\Omega(z,\bar{\omega})\,\frac{\hat{L}(z,\bar{\omega})}{i}|dz|$$

$$= \frac{1}{(2\pi i\hat{K}_\Omega K_m)} \int_{\partial\Omega} f(z)\,\frac{\hat{L}(z,\bar{\omega})}{\hat{K}_\Omega(z,\bar{\omega})}\,\overline{K_m(z,\bar{\omega})}\hat{K}_\Omega(z,\bar{\omega})\hat{L}(z,\omega)|dz|$$

$$= \frac{1}{(2\pi i\hat{K}_\Omega K_m)} \int_{\partial\Omega} f(z)\hat{L}(z,\omega)^2\,\overline{K_m(z,\bar{\omega})}|dz|.$$

But for every function f in $H^2(\partial\Omega,m)$, we also have

$$\langle f,\frac{1}{\hat{K}_\Omega(\omega,\bar{\omega})}\rangle = \frac{1}{(2\pi\hat{K}_\Omega(\omega,\bar{\omega}))} \int_{\partial\Omega} f(z)|\hat{K}_\Omega(z,\bar{\omega})|^2|dz|$$

$$= \frac{1}{(2\pi i\hat{K}_\Omega(\omega,\bar{\omega})} \int_{\partial\Omega} f(z)\hat{K}_\Omega(z,\bar{\omega})\hat{L}(z,\omega)dz.$$

Since the functions f and $\hat{K}_\Omega(\ ,\bar{\omega})$ are both holomorphic in the region Ω and the function $\hat{L}(\ ,\omega)$ is meromorphic there except for a simple pole at $z = \omega$ with residue 1, it follows that

$$\langle f, 1/\hat{K}_\Omega(\omega,\overline{\omega})\rangle = f(\omega).$$

Uniqueness of the kernel function now implies that $K_m(z,\overline{\omega}) = 1/\hat{K}_\Omega(\omega,\overline{\omega})$ for all z in Ω. Continuing, we obtain

$$\langle f,\varphi\rangle = 1/(2\pi i\hat{K}_\Omega K_m) \int_{\partial\Omega} f(z)\hat{L}(z,\omega)^2 \overline{K_m(z,\overline{\omega})}dz$$

$$= 1/(2\pi i\hat{K}_\Omega^2 K_m) \int_{\partial\Omega} f(z)\hat{L}(z,\omega)^2 dz.$$

Now, $\hat{L}(z,\omega)^2$ is again meromorphic on the region Ω, but this time it has a double pole at the point $z = \omega$. At this point the function f has a zero of order two at the least; therefore, the product $f(z)\hat{L}(z,\omega)^2$ remains holomorphic on all of the region Ω. It follows that $\langle\varphi,f\rangle = 0$.

This theorem suggests that the operator M_z^* is a natural candidate for our extremal operator. In fact, we have the following

<u>Theorem 2.3.</u> For each fixed ω in the region Ω, if m be the measure $|\hat{K}_\Omega(z,\overline{\omega})|^2|dz|$, then the operator M_z^* on the Hilbert space $H^2(\partial\Omega,m)$ satisfies

$$\kappa_{M_z^*}(\omega) = -\hat{K}_{\overline{\Omega}}(\omega,\overline{\omega})^2.$$

We reiterate, it only remains for us to show that the operator M_z^* is in the class $B_1(\overline{\Omega})$ and $\hat{K}_\Omega(\omega,\overline{\omega})$ equals $\hat{K}_{\overline{\Omega}}(\overline{\omega},\omega)$. Let us recall that every operator T in $B_1(\Omega)$ is unitarily equivalent to the adjoint of the multiplication operator on a certain Hilbert space of analytic functions. With a little effort we can prove a converse.

Proposition 2.1. Let $\mathcal{H}$ be a Hilbert space of analytic functions on the region Ω, equipped with a reproducing kernel function $K : \Omega \times \overline{\Omega} \to \mathbb{C}$. Suppose the operator M_z maps $\mathcal{H}$ into itself. Then it is bounded and each $\overline{\omega}$ in $\overline{\Omega}$ is an eigenvalue for the operator M_z^*. If, in addition, each $\overline{\omega} \in \overline{\Omega}$ is a simple eigenvalue, then the operator M_z^* lies in $B_1(\overline{\Omega})$.

Proof. Suppose that $\langle f_n, g_n\rangle$ is in the graph of the operator M_z and suppose that $\langle f_n, g_n\rangle \to \langle f, g\rangle$. Since convergence in the Hilbert space $\mathcal{H}$ implies pointwise convergence, therefore, $f_n(z) \to f(z)$ and $g_n(z) \to g(z)$ for all $z \in \Omega$. Now, $g_n = M_z f_n = zf_n$ and $zf_n(z) \to zf(z)$ for all points $z \in \Omega$, it follows that $g = zf = M_z f$. An application of the closed graph theorem shows that the operator M_z is bounded. For then, we have

$$\begin{aligned}\langle (M_z^* - \overline{\omega})K(\ ,\overline{\omega}), f\rangle &= \langle M_z^* K(\ ,\overline{\omega}), f\rangle - \langle(\overline{\omega}K(\ ,\omega), f\rangle \\ &= \langle K(\ ,\overline{\omega}), M_z f\rangle - \overline{\omega}\langle K(\ ,\overline{\omega}), f\rangle \\ &= \overline{\omega f(\omega)} - \overline{\omega f(\omega)} = 0 \text{ for all } \omega \text{ in } \Omega.\end{aligned}$$

Thus each $\overline{\omega}$ in the region $\overline{\Omega}$ is an eigenvalue for the operator M_z^*. Now, $\operatorname{ran}(M_z-\omega) = \{f \in \mathcal{H} \mid f(\omega) = 0\}$ and since point evaluations are continuous, it follows that $\operatorname{ran}(M_z-\omega)$ is closed. Also since the operator $(M_z-\omega)$ is one-to-one, it follows that the operator $(M_z^*-\overline{\omega})$ is surjective. Since the function $K(\ ,\overline{\omega})$ is the eigenvector corresponding to the eigenvalue $\overline{\omega} \in \overline{\Omega}$, we have $\underset{\overline{\omega}\in\overline{\Omega}}{\underline{\operatorname{Span}}} \operatorname{Ker}(M_z^*-\overline{\omega}) = \underset{\overline{\omega}\in\overline{\Omega}}{\underline{\operatorname{Span}}} K(\ ,\overline{\omega}) = \mathcal{H}$.

It is now clear that if we assume $\overline{\omega}$ to be a simple eigenvalue for the operator M_z^*, then it will lie in the class $B_1(\overline{\Omega})$.

<u>Corollary 2.1</u>. The operator M_z^* on the Hilbert space $H^2(\partial\Omega,m)$ is in the class $B_1(\overline{\Omega})$.

<u>Proof</u>. We need only show that each $\overline{\omega}$ in $\overline{\Omega}$ is a simple eigenvalue. Since each function f in $H^2(\partial\Omega,m)$ can be written as $f(z) = f(\omega) + (z-\omega)\dfrac{f(z) - f(\omega)}{z - \omega}$, where $\dfrac{f(z) - f(\omega)}{z - \omega} \in H^2(\partial\Omega,m)$, it follows that $\operatorname{ran}(M_z-\omega)$ has codimension 1.

<u>Proposition 2.2</u>. For all $\overline{\omega}$ in Ω the function $\hat{K}_\Omega(\overline{\omega},\omega)$ equals $\hat{K}_{\overline{\Omega}}(\omega,\overline{\omega})$.

<u>Proof</u>. Let F be the function that maps Ω onto the n-times covered disk. The corresponding function F^* mapping the region $\overline{\Omega}$ is given by the formula $F^*(z,\omega) = \overline{F(\overline{z},\overline{\omega})}$, where z and ω are now in $\overline{\Omega}$. Since the function $\hat{K}_\Omega(\overline{\omega},\omega)$ can be defined

as $\frac{d}{dz} F(\overline{z},\omega)\Big|_{z=\omega}$, it follows that $\hat{K}_{\overline{\Omega}}(\omega,\overline{\omega}) = \frac{d}{dz} F^*(z,\omega)\Big|_{z=\omega} =$

$\frac{d}{dz} \overline{F(\overline{z},\overline{\omega})}\Big|_{z=\omega} = \hat{K}_{\Omega}(\overline{\omega},\omega)$.

Now that we have solved the extremal problem, let us apply it to obtain some results on complete spectral sets. Before defining this we offer a generalization of Theorem 1.1 that we proved earlier.

Let $\alpha(r,s) = (\gamma^{-1}-1)^{-1/2}|r-s|^{-1}$ where

$$\gamma = \max\{|\varphi(r)| : \varphi \in \mathrm{Hol}(\Omega,s,\mathbb{D})\}.$$

<u>Theorem 2.4</u>. $\mathrm{Cl}\Omega$ is a spectral set for the matrix $\begin{bmatrix} r & t \\ 0 & s\end{bmatrix}$ if and only if $|t| \le \alpha^{-1}(r,s)$.

<u>Proof</u>. Let $p(x) = a_0 + a_1(x)+\ldots+a_nx_n$, then

$$p\begin{pmatrix} r & t \\ 0 & s\end{pmatrix} = a_0\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + a_1\begin{pmatrix} r & t \\ 0 & s\end{pmatrix} + a_2\begin{pmatrix} r^2 & (r+s)t \\ 0 & s^2\end{pmatrix}+\ldots+$$

$$a_n\begin{pmatrix} r^n & (r^{n-1}+r^{n-2}s+\ldots+s^{n-1})t \\ 0 & s^n\end{pmatrix}.$$

$$= \begin{pmatrix} p(r) & a_1+a_2(r+s)+\ldots+a_n(r^{n-1}+r^{n-2}s+\ldots+s^{n-1})t \\ 0 & p(s)\end{pmatrix}$$

$$= \begin{pmatrix} p(r) & (a_1 \frac{r-s}{r-s} + a_2 \frac{r^2-s^2}{r-s} + \dots + a_n \frac{r^n-s^n}{r-s})t \\ 0 & p(s) \end{pmatrix}$$

$$= \begin{pmatrix} p(r) & \frac{t}{r-s}(p(r)-p(s)) \\ 0 & p(s) \end{pmatrix}$$

Now, let $\varphi = {}^{p}/_{q}$ and $q \neq 0$ in Ω. Then via the usual functional calculus we obtain

$$\varphi\begin{pmatrix} r & t \\ 0 & s \end{pmatrix} = \left(p\begin{pmatrix} r & t \\ 0 & s \end{pmatrix}\right)\left(q\begin{pmatrix} r & t \\ 0 & s \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} p(r) & \frac{t}{r-s}(p(r)-p(s)) \\ 0 & p(s) \end{pmatrix} \begin{pmatrix} q^{-1}(r) & -\frac{t}{q(r)q(s)(r-s)}(q(r)-q(s)) \\ 0 & q^{-1}(s) \end{pmatrix}$$

$$= \begin{pmatrix} p(r)/q(r) & \frac{t}{r-s}\,\frac{p(r)q(s)-q(r)p(s)}{q(r)q(s)} \\ 0 & p(s)/q(s) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi(r) & \frac{t}{r-s}(\varphi(r)-\varphi(s)) \\ 0 & \varphi(s) \end{pmatrix} .$$

Now, $\mathrm{Cl}\Omega$ will be a spectral set for $\begin{bmatrix} r & t \\ 0 & s \end{bmatrix}$ if and only if

$$\left\| \varphi\begin{pmatrix} r & t \\ 0 & s \end{pmatrix} \right\| \leq 1 \qquad \text{for all } \varphi \text{ in } \mathrm{Hol}(\Omega, s, \mathbb{D})$$

which in turn is equivalent to

$$\left\| \begin{pmatrix} \varphi(r) & \frac{t}{r-s}\varphi(r) \\ 0 & 0 \end{pmatrix} \right\| \le 1 \text{ for all } \varphi \text{ in } \mathrm{Hol}(\Omega, s, \mathbb{D}).$$

That is, $|t|^2 \le |r-s|^2\left(\frac{1}{|\varphi(r)|^2} - 1\right)$ for all φ in $\mathrm{Hol}(\Omega,s,\mathbb{D})$.

We say that the region $\mathrm{Cl}\Omega$ is a complete spectral set for the operator T if the map $\sigma \otimes I_n : \mathrm{Rat}(\mathrm{Cl}\Omega) \otimes M_n \to \mathcal{L}(\mathcal{H}) \otimes M_n$ is contractive for each n. Here σ is the map $\varphi \to \varphi(T)$ and M_n is the C^* algebra of $n \times n$ complex matrices. Now, we are ready to prove the following

Theorem 2.5. If $\mathrm{Cl}\Omega$ is a complete spectral set for the matrix $\begin{bmatrix} r & t \\ 0 & s \end{bmatrix}$, then $\mathrm{Cl}\Omega$ is also a complete spectral set for $\begin{bmatrix} r & \omega \\ 0 & s \end{bmatrix}$ wherever $|\omega| \le |t|$.

Proof. First, for $|\omega| \le |t|$ and any three $k \times k$ matrices A, B and C, we show

$$\left\| \begin{pmatrix} A & \omega B \\ 0 & C \end{pmatrix} \right\| \le \left\| \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix} \right\| .$$

If $\begin{bmatrix} X \\ Y \end{bmatrix}$ in $\mathbb{C}^{2k}$ is a unit vector, then

$$\left\| \begin{pmatrix} A & \omega B \\ 0 & C \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right\|^2 = \|AX\|^2 + \|CY\|^2 + |\omega|\|BY\|^2 + \mathrm{Re}\, \omega \langle AX, BY \rangle .$$

Let $\omega = |\omega|\overline{\alpha}$, $t = |t|\overline{\beta}$ and $\langle AX, \alpha BY\rangle = |\langle AX, \alpha BY\rangle|\overline{\delta}$, then

$$\|AX\|^2 + \|CY\|^2 + |\omega|\|BY\|^2 + 2\,\mathrm{Re}\,\omega\langle AX, BY\rangle$$

$$= \|AX\|^2 + \|CY\|^2 + |\omega|\|BY\|^2 + 2|\omega|\,\mathrm{Re}\langle AX, \alpha BY\rangle$$

$$\leq \|AX\|^2 + \|CY\|^2 + |\omega|\|BY\|^2 + 2|\omega|\,|\langle AX, \alpha BY\rangle|$$

$$= \|AX\|^2 + \|CY\|^2 + |\omega|\|BY\|^2 + 2|\omega|\langle AX, {}^{\alpha}/_{\delta}\, BY\rangle$$

$$\leq \|AX\|^2 + \|C({}^{\alpha}/_{\beta\delta}\, Y)\|^2 + |t|\|B({}^{\alpha}/_{\beta\delta}\, Y)\|^2 + 2|t|\overline{\beta}\langle AX, {}^{\alpha}/_{\delta\beta}\, BY\rangle$$

$$= \left\| \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix} \begin{pmatrix} X \\ ({}^{\alpha}/_{\delta\beta})Y \end{pmatrix} \right\|^2 .$$

For each unit vector $[{X \atop Y}]$ in $\mathbb{C}^{2k}$ we have produced another unit vector $\begin{bmatrix} X \\ ({}^{\alpha}/_{\delta\beta})\,Y \end{bmatrix}$ satisfying

$$\left\| \begin{pmatrix} A & \omega B \\ 0 & C \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix} \begin{pmatrix} X \\ ({}^{\alpha}/_{\beta\delta})Y \end{pmatrix} \right\| .$$

Now we take the supremum on both sides over all the unit vectors to obtain the desired inequality.

Now, to prove the statement about complete spectral sets we need only show that $\|(\varphi_{ij}\begin{pmatrix} r & \omega \\ 0 & s \end{pmatrix})\| \leq \|(\varphi_{ij}\begin{pmatrix} r & t \\ 0 & s \end{pmatrix})\|$, for all $(\varphi_{ij}) \in \mathrm{Rat}(\mathrm{Cl}\,\Omega) \otimes M_n$.

Let us set $A = (\varphi_{ij}(r))_{k\times k}$, $B = (\varphi_{ij}(r) - \varphi_{ij}(s))_{k\times k}$

and $C = (\varphi_{ij}(s))_{k\times k}$. As shown in the proof of Theorem 2.4,

$$\varphi_{ij}\begin{pmatrix} r & \omega \\ 0 & s \end{pmatrix} = \begin{pmatrix} \varphi_{ij}(r) & \frac{\omega}{r-s}(\varphi_{ij}(r) - \varphi_{ij}(s)) \\ 0 & \varphi_{ij}(s) \end{pmatrix},$$ so that

applying some elementary row operators to $\left(\varphi_{ij}\begin{pmatrix} r & \omega \\ 0 & s \end{pmatrix}\right)_{2k\times 2k}$

we obtain

$$\left(\varphi_{ij}\begin{pmatrix} r & \omega \\ 0 & s \end{pmatrix}\right) \sim \begin{pmatrix} (\varphi_{ij}(r))_{k\times k} & {}^{\omega}/_{(r-s)}(\varphi_{ij}(r) - \varphi_{ij}(s))_{k\times k} \\ 0 & (\varphi_{ij}(s))_{k\times k} \end{pmatrix}$$

$$= \begin{bmatrix} A & {}^{\omega}/_{(r-s)}\, B \\ 0 & C \end{bmatrix}.$$

Similarly, $\left(\varphi_{ij}\begin{pmatrix} r & t \\ 0 & s \end{pmatrix}\right) \sim \begin{pmatrix} A & {}^{t}/_{(r-s)}\, B \\ 0 & C \end{pmatrix}$.

Since ${}^{|\omega|}/_{|r-s|} \le {}^{|t|}/_{|r-s|}$, it follows that $\left\| \begin{pmatrix} A & {}^{\omega}/_{(r-s)}\, B \\ 0 & C \end{pmatrix} \right\|$ $\le \left\| \begin{pmatrix} A & {}^{t}/_{(r-s)}\, B \\ 0 & C \end{pmatrix} \right\|$ and the proof is complete.

We wish to determine whether the region $\mathrm{Cl}\Omega$ is a complete spectral set for $\begin{bmatrix} r & t \\ 0 & s \end{bmatrix}$ whenever $\mathrm{Cl}\Omega$ is a spectral set for it. The two theorems we have proved so far reduce the problem to finding whether the region $\mathrm{Cl}\Omega$ is a complete spectral set for the matrix $\begin{bmatrix} r & \alpha(r,s) \\ 0 & s \end{bmatrix}$. We are able to answer this question only if $r = s$ and $\alpha(r,r) = \hat{K}_\Omega(\omega,\overline{\omega})^{-1}$.

<u>Corollary 2.1.</u> If the region $\mathrm{Cl}\Omega$ is a spectral set for $\begin{bmatrix} r & t \\ 0 & r \end{bmatrix}$ then it is also a complete spectral set.

<u>Proof.</u> We need only prove that the region $\mathrm{Cl}\Omega$ is a complete spectral set for $\begin{pmatrix} r & \hat{K}_\Omega(r,\overline{r})^{-1} \\ 0 & r \end{pmatrix}$. Recall that, for any point r in Ω there is a subnormal operator T in $\tilde{B}_1(\Omega)$ with curvature equal to $-\hat{K}_\Omega(r,\overline{r})^2$. The normal extension N of the operator T provides a dilation for the matrix $\begin{pmatrix} r & \hat{K}_\Omega(r,\overline{r})^{-1} \\ 0 & r \end{pmatrix}$, Arveson's result [5] now implies that $\mathrm{Cl}\Omega$ is a complete spectral set for it.

3. THE ANNULUS.

For many simple domains, computations involving the curvature are quite complicated. One can easily verify that U_+^* is an extremal operator in $\widetilde{B}_1(\mathbb{D})$ and $\varkappa_{U_+^*}(\omega) = -(1-|\omega|^2)^{-2}$.

To illustrate the difficulties involved, let us outline an alternative procedure. According to Theorem 2.3, the operator $M_z^* : H^2(\mathbb{D},m) \to H^2(\mathbb{D},m)$, where $dm = |\hat{K}_{\mathbb{D}}(s,\overline{\zeta})|^2|dz|$ is extremal in $\widetilde{B}_1(\mathbb{D})$. Since,

$\int \hat{K}_{\mathbb{D}}(z,\overline{\zeta})^{-1} z^n \overline{\hat{K}_{\mathbb{D}}(z,\overline{\zeta})^{-1}\, z^m} |\hat{K}_{\mathbb{D}}(z,\overline{\zeta})|^2 |dz| = \int z^n \overline{z}^m |dz| = 0$, for $n \neq m$ and similarly $\int |\hat{K}(z,\overline{\zeta})|^{-2} |z^n|^2 |\hat{K}(z,\overline{\zeta})|^2 |dz| = \int |z^n|^2 |dz| = 1$, it follows that $\{\hat{K}_{\mathbb{D}}(z,\overline{\zeta})^{-1} z^n\}$ form a complete orthonormal set in $H^2(\mathbb{D},m)$. Now, the kernel function $K_m(z,\overline{\zeta})$ for the space $H^2(\mathbb{D},m)$ can be expressed as

$$\begin{aligned} K_m(z,\overline{\omega}) &= \Sigma\, \hat{K}_{\mathbb{D}}(z,\overline{\zeta})^{-1} z^n\, \overline{\hat{K}_{\mathbb{D}}(\omega,\overline{\zeta})}^{-1} \overline{\omega}^n \\ &= \hat{K}_{\mathbb{D}}(z,\overline{\zeta})^{-1}\, \overline{\hat{K}_{\mathbb{D}}(\omega,\overline{\zeta})}^{-1}\, \hat{K}_{\mathbb{D}}(z,\overline{\omega}). \end{aligned}$$

We are ready to compute the curvature of the operator M_z^* using the formula

$$\varkappa_{M_z^*}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log K_m(\omega,\overline{\omega})$$

or,
$$\varkappa_{M_z^*}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log|\hat{K}_{\mathbb{D}}(\omega,\overline{\zeta})|^2\, \hat{K}_{\mathbb{D}}(\omega,\overline{\omega}).$$

Since $\hat{K}_{\mathbb{D}}(\ ,\bar{\zeta})$ is a non-vanishing holomorphic function in $\mathbb{D}$, it follows that $\log|\hat{K}_{\mathbb{D}}(\omega,\bar{\zeta})|^2$ is harmonic. Consequently,

$$\mathcal{K}_{M_z^*}(\omega) = \frac{\partial^2}{\partial\omega\partial\bar{\omega}} \log \hat{K}_{\mathbb{D}}(\omega,\bar{\omega})^{-1} = -(1-|\omega|^2)^{-2}.$$

As one might expect the curvature of M_z^* does not depend on the point ζ. Indeed for any pair of points ζ and ζ' in $\mathbb{D}$, the curvature of the corresponding operators are equal in $\mathbb{D}$. Therefore, the Cowen-Douglas Theorem implies that they must be unitarily equivalent. In particular, if $\zeta = 0$, then $H^2(\mathbb{D},m)$ is the usual Hardy space and M_z^* is the backward shift operator U_+^*. For each ζ in $\mathbb{D}$ we now have

$$\mathcal{K}_{M_z^*}(\omega) = \mathcal{K}_{U_+^*}(\omega) = -(1-|\omega|^2)^{-2}.$$

The situation for any arbitrary simply connected domain is more or less the same. The trick is to show that $\hat{K}_\Omega(z,\bar{\zeta})^{-1}\ \overline{\hat{K}_\Omega(\omega,\bar{\zeta})^{-1}}\ \hat{K}_\Omega(z,\omega)$ is the kernel function for $H^2(\Omega,m)$ without having to find a complete orthonormal set in $H^2(\Omega,m)$. In spite of considerable effort it has not been possible to carry out similar computations for an annulus. Fortunately we can compute the curvature for a family of operators $\{Z_\alpha\}$ related to an annulus. It is known that the operators Z_α^* are all similar to one another (Sarason [14]). For any pair α and β we wish to prove $\mathcal{K}_{Z_\alpha^*}(\omega)/\mathcal{K}_{Z_\beta^*}(\omega) \to 1$ as ω approaches the boundary. Once proved, it would provide evidence in favor of a conjecture announced in Cowen and Douglas [9]. The operators Z_α were first defined in Sarason [14], we reproduce some of his work here.

One can choose a covering space of the annulus $A = \{z \mid r_0 < |z| < 1\}$ to be a subsurface of the logarithmic surface where the logarithm surface consists of all pairs of real numbers (r,t) such that $0 < r < \infty$ and $-\infty < t < \infty$ with local conformal coordinate given by $(r,t) \to r\,e^{it}$. Let $\hat{A}$ be the subsurface defined by $r_0 < r < 1$, φ be the map of $\hat{A}$ onto A defined by $\varphi(r,t) = r\,e^{it}$. The pair $(\hat{A},\varphi)$ constitutes a covering space for A. Any meromorphic function F on $\hat{A}$ will be called modulus automorphic if it has the same absolute value at all points of $\hat{A}$ lying above the same point of A. For such function F the minimum modulus principle implies that $F(r,t+2\pi)/F(r,t) = \lambda$ is a constant of unit modulus, called the multiplier of F. The unique real number α in the interval $[0,1)$ satisfying $\lambda = e^{2\pi i\alpha}$ is called the index of F.

For $0 \le \alpha < 1$ and $1 \le p < \infty$ we define $H^p_\alpha(A)$ to be the collection of all holomorphic modulus automorphic functions F of index α satisfying

$$\sup_{r_0<r<1} \int_0^{2\pi} |F(r,t)|^p dt < \infty .$$

The collection $H^p_\alpha(A)$ is a complex vector space. It can be shown that the elements of $H^p_\alpha(A)$ possess nontangential limits at almost every boundary point of $\hat{A}$. Hence with each F in $H^p_\alpha(A)$ we can associate a boundary function x defined almost everywhere on the boundary of A by

$$x(e^{it}) = F(1,t) \text{ and } x(r_0e^{it}) = F(r_0,e^{it}) \text{ for } 0 \leq t < 2\pi.$$

It follows from Fatou's lemma that $x \in L^p(\partial A)$. We denote by $H^p_\alpha(\partial A)$ the linear manifold in $L^p(\partial A)$ consisting of boundary functions of elements in $H^p_\alpha(A)$. The class $H^p_0(A)$ consists of precisely all of functions on $\hat{A}$ that can be obtained by lifting functions of class $H^p(A)$. Consequently $H^p_0(A)$ coincides with $H^p(\partial A)$, and so is a subspace of $L^p(\partial A)$. Let us define the modulus automorphic function E_α on $\hat{A}$ by setting $E_\alpha(r,t) = r^\alpha e^{i\alpha t}$. The index of E_α is α. For each α we let ω_α be the boundary function of E_α that is, $\omega_\alpha(e^{it}) = e^{i\alpha t}$ and $\omega_\alpha(r_0e^{it}) = r_0^\alpha e^{i\alpha t}$ for $\leq t < 2\pi$. Then $H^p_\alpha(A)$ consists precisely of all products $E_\alpha F$ with F in $H^p_0(A)$ and thus $H^p_\alpha(\partial A)$ consists of all products $\omega_\alpha x$ with x in $H^p_0(\partial A)$. Since the transformation $x \rightarrow \omega_\alpha x$ of $L^p(\partial A)$ onto itself is a bounded linear transformation, we conclude that $H^p_\alpha(\partial A)$, the image of $H^p_0(\partial A)$ under this transformation is a subspace of $L^p(\partial A)$ and hence a Banach space. Similarly, $H^2_\alpha(A)$ is a Hilbert space under the inner-product

$$\langle F,G\rangle = {}^1/_{2\pi} \int_0^{2\pi} F(1,t)\,\overline{G(1,t)}dt + {}^1/_{2\pi} \int_0^{2\pi} F(r_0,t)\,\overline{G(r_0,t)}dt\,.$$

The functions $E_{\alpha+n}$, $n = 0, \pm 1, \pm 2, \ldots$, form an orthogonal set, which is complete in $H^2_\alpha(A)$. For $0 \leq \alpha < 1$, define the operator Z_α on $H^2_\alpha(A)$ by $(Z_\alpha F)(r,t) = re^{it}F(r,t)$.

Any holomorphic modulus automorphic function F of index α has a "Laurent expansion"

$$F(r,t) = \sum_{-\infty}^{\infty} c_n E_{\alpha+n}(r,t),$$

which converges at each point of $\hat{A}$. For $r_0 < r < 1$

$$1/2\pi \int_0^{2\pi} |F(r,t)|^2 dt = \sum_{-\infty}^{\infty} |c_n|^2 r^{2(\alpha+n)}.$$

Therefore, F belongs to $H^2_\alpha(A)$ if and only if the sums

$$\sum_{-\infty}^{\infty} |c_n|^2 \text{ and } \sum_{-\infty}^{\infty} |c_n|^2 r_0^{2(\alpha+n)}$$

are finite. We can now think of $H^2_\alpha(A)$ as a weighted sequence space (Shields [15]) with weights $\beta(n) = (1+r_0^{2(\alpha+n)})^{1/2}$, $n = 0, \pm 1, \pm 2, \ldots$. The multiplication operator Z_α on $H^2_\alpha(A)$ is therefore unitarily equivalent to a weighted shift operator on $\ell^2(\mathbb{Z})$, with weights $\alpha_n = \{(1+r_0^{2(\alpha+n+1)})^{1/2}/(1+r_0^{2(\alpha+n)})^{1/2}\}$ $n = 0, \pm 1, \pm 2, \ldots$. Suppose ω is any point in the annulus A and $(a_n)_{n=-\infty}^{\infty}$ in $\ell^2(\mathbb{Z})$ is an eigenvector of Z^*_α corresponding to the point ω. We can then show that $\bar{\alpha}_n a_{n+1} = \omega a_n$, which leads to

$$a_n = \omega^n/\alpha_0 \cdots \alpha_{n-1} \cdot a_0$$

$$n \geq 1 \quad .$$

$$a_{-n} = \alpha_{-1} \cdots \alpha_{-n}/\omega^n \cdot a_0 \quad .$$

Therefore each ω in A is a simple eigenvalue for the operator Z_α^* with eigenvector $\{(1+r_0^{2(\alpha+n)})^{-1/2}\omega^n\}_{n=-\infty}^{\infty}$, if we choose $a_0 = (1+r_0^{2\alpha})^{-1/2}$. Define the lower bound of an operator T (Shields [15])

$$m(T) = \inf\{\|Af\| : \|f\| = 1\}.$$

If the operator T is injective and is represented as M_z on $L^2(\beta)$ then

$$m(T^n) = \inf_k \frac{\beta(n+k)}{\beta(k)} \qquad n = 1,2,\ldots$$

In particular, $m(Z_\alpha) = \inf_k (1+r_0^{2(\alpha+k+1)})^{1/2}/(1+r_0^{2(\alpha+k)})^{1/2} = 1.$ Since, $\|(Z_\alpha-\omega)f\| \geq \|Z_\alpha f\| - \|\omega f\| \geq (m(Z_\alpha)-|\omega|)\|f\| = (1-|\omega|)\|f\| \gneq 0$ for each ω in the annulus A, it follows that $(Z_\alpha-\omega)$ is bounded below. Consequently the operator $(Z_\alpha-\omega)$ has closed range which in turn implies that the operator $(Z_\alpha^*-\overline{\omega})$ is surjective. Lastly if $(a_n)_{n=-\infty}^{\infty}$ be any sequence in $\ell^2(\mathbb{Z})$ that is orthogonal to the space spanned by the eigenvectors of the operator Z_α^* then for each ω in the annulus A, we must have

$$\langle((1+r_0^{2(\alpha+n)})^{-1/2}\omega^n),(a_n)\rangle = 0$$

or,
$$\sum_{n=-\infty}^{\infty} (1+r_0^{2(\alpha+n)})^{-1/2}\,\overline{a}_n\omega^n = 0.$$

It now follows that $a_n = 0$ for all n and therefore the eigenvectors of the operator Z_α^* span $H_\alpha^2(A)$. The proof that

Z_α^* lies in $B_1(A)$ is now complete. The map

$\omega \xrightarrow{\gamma} ((1+r_0^{2(\alpha+n)})^{-1/2}\omega^n)_{n=-\infty}^{\infty}$, defines a nonzero holomorphic

section of the bundle $E_{Z_\alpha^*}$. We compute the curvature of $E_{Z_\alpha^*}$

using the induced metric $\omega \to \|\gamma(\omega)\|^2$ as follows.

$$\mathcal{K}_{Z_\alpha^*}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log\|\gamma(\omega)\|^2$$

$$= -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log(\sum_{-\infty}^{\infty} (1+r_0^{2(\alpha+n)})^{-1}(\omega\overline{\omega})^n).$$

$$= -(2|\omega|^2)^{-1} \frac{(\sum_{-\infty}^{\infty} n^2(1+r_0^{2(\alpha+n)})^{-1}|\omega|^{2n})(\sum_{-\infty}^{\infty}(1+r_0^{2(\alpha+n)})^{-1}|\omega|^{2n}) - (\sum_{-\infty}^{\infty} n(1+r_0^{2(\alpha+n)})^{-1}|\omega|^{2n})^2}{(\sum_{-\infty}^{\infty}(1+r_0^{2(\alpha+n)})^{-1}|\omega|^{2n})^2}.$$

Finally we wish to verify that $\mathcal{K}_{Z_\alpha^*}(\omega)/\mathcal{K}_{Z_\beta^*}(\omega) \to 1$ as $|\omega| \to 1$ or r_0. To simplify notation, let us write x for $|\omega|^2$ and x_0 for r_0^2.

$$\sum_0^\infty \frac{x^n}{1+x_0^{\alpha+n}} - \sum_0^\infty x^n = -\sum_0^\infty \frac{x_0^\alpha}{1+x_0^{\alpha+n}}(x_0x)^n$$

$$\sum_{-\infty}^{-1} \frac{x^n}{1+x_0^{\alpha+n}} - \sum_1^\infty ({}^{x_0}/_x)^n = \sum_1^\infty \frac{1-x_0^n-x_0^\alpha}{x_0^n+x_0^\alpha}({}^{x_0}/_x)^n .$$

Adding together we obtain

$$\sum_{-\infty}^{\infty} \frac{x^n}{1+x_0^{\alpha+n}} = \frac{x(1-x_0)}{(x-x_0)(1-x)} + \sum_{1}^{\infty} \frac{1-x_0^{\alpha}-x_0^{n}}{x_0^{n}+x_0^{\alpha}} (x_0/x)^n - \sum_{0}^{\infty} \frac{x_0^{\alpha}}{1+x_0^{\alpha+n}} (x_0 x)^n .$$

Let us simplify notation further by setting

$$f(x) = \frac{x(1-x_0)}{(x-x_0)(1-x)} \quad \text{and}$$

$$B_1(x,\alpha) = \sum_{1}^{\infty} \frac{1-x_0^{\alpha}-x_0^{n}}{x_0^{n}+x_0^{\alpha}} (x_0/x)^n - \sum_{0}^{\infty} \frac{x_0^{\alpha}}{1+x_0^{\alpha+n}} (x_0 x)^n$$

so that we can write

$$\sum_{-\infty}^{\infty} \frac{x^n}{1+x_0^{\alpha+n}} = f(x) + B_1(x,\alpha).$$

The crucial point is that the function f does not depend on α and approaches ∞ as $x \to 1$ or x_0, whereas the series $B_1(x,\alpha)$ depends on α but remains bounded as $x \to 1$ or x_0. Similarly one can show

$$\sum_{-\infty}^{\infty} n \frac{x^n}{1+x_0^{\alpha+n}} = g(x) + B_2(x,\alpha)$$

$$\sum_{-\infty}^{\infty} n^2 \frac{x^n}{1+x_0^{\alpha+n}} = h(x) + B_3(x,\alpha)$$

It follows that

$$\mathcal{K}_{Z_\alpha^*}(\omega)/\mathcal{K}_{Z_\beta^*}(\omega) = \left\{\frac{[\{h(x)+B_3(x,\alpha)\}\{f(x)+B_1(x,\alpha)\}]-[g(x)+B_2(x,\ )]^2}{[\{h(x)+B_3(x,\beta)\}\{f(x)+B_1(x,\alpha)\}]-[g(x)+B_2(x,\beta)]^2}\right\} \times \left\{\frac{[f(x)+B_1(x,\beta)]^2}{[f(x)+B_1(x,\alpha)]^2}\right\}$$

Now, $$\frac{[f(x)+B_1(x,\beta)]^2}{[f(x)+B_1(x,\alpha)]^2} = \frac{[1+B_1(x,\beta)/f(x)]^2}{[1+B_1(x,\alpha)/f(x)]^2} \to 1$$

as $x \to$ or x_0.

It is not difficult to verify that the other factor also approaches 1 as $x \to 1$ or x_0. One way to make sure everything works is to divide both numerator and denominator by the factor $h(x)f(x)$ and then observe that $\lim g(x)^2/h(x)f(x)$ exists and is finite as $x \to 1$ or x_0.

<u>The Cowen-Douglas Conjecture</u>: If T and $\tilde{T}$ are operators in $B_1(\Omega)$ each having $\mathrm{Cl}\Omega$ as a k-spectral set, then T and $\tilde{T}$ are similar if and only if

$$\lim_{|\omega|\to\partial\Omega} \mathcal{K}_T(\omega)/\mathcal{K}_{\tilde{T}}(\omega) = 1.$$

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