997-64
(3)

Normal Two Dimensional Triple Point Singularities

A Dissertation presented

by

Chunghyuk Kang

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

May 1982

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

Chunghyuk Kang

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Bernard Maskit — Committee Chairman

Henry B. Laufer — Thesis Advisor

Walter Parry

Yung Ming Chen, Applied Mathematics (Outside Member)

The dissertation is accepted by the Graduate School.

Dean of the Graduate School

Abstract of the Dissertation

Normal Two Dimensional Triple Point Singularities

by

Chunghyuk Kang

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1982

Let (V',P') be a triple point singularity of a purely two-dimensional analytic space V'. Then there exists (V,P) with V a hypersurface, P a triple point singularity of V and such that V and V' have isomorphic normalizations and resolutions. Suppose that V has a projection to $\mathbb{C}^2$ with a suitably simple branch locus Q. Let $\tilde{\Gamma}$ be the topological type of the minimal resolution of P. Let Γ be the topological type of the resolution of P using Q. Then $\tilde{\Gamma}$ is determined from Q via an explicit computation. An algorithm is given for finding the equisingular type of the plane curve singularity Q in terms of Γ.

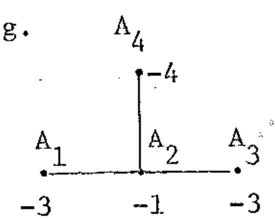
TABLE OF CONTENTS

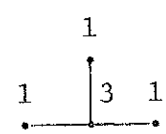
LIST OF SYMBOLS

V	=	two dimensional analytic space
${}_V\mathcal{O}$	=	the sheaf of germs of holomorphic functions on V.
${}_V\mathcal{O}_P$	=	the stalk of the sheaf $\mathcal{O}$ over P.
Z	=	fundamental cycle
m	=	maximal ideal of $\mathcal{O}_P$.
supp D	=	support of the divisor of D
Let D	=	$\Sigma d_i A_i$ be a cycle, an integral combination of the A_i
$\mathcal{O}(-D)$	=	the sheaf of germs of holomorphic functions on N whose divisors are at least D where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on N.

Convention of weighted dual graphs: Vertices without specifying genera are of genus zero. We write the multiplicity d_i of A_i in a cycle $D = \Sigma d_i A_i$ by placing that integer in the corresponding position of the vertex.

e.g.



D = 

$= A_1 + 3A_2 + A_3 + A_4$

Let $D = \Sigma d_i A_i$ be a positive cycle. Let $B \subseteq$ supp D. Then $D|_B = \Sigma e_i A_i$ is a cycle where $e_i = d_i$ if $A_i \subset B$ and $e_i = 0$ if $A_i \not\subset B$.

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Professor Henry B. Laufer. He has always been extremely kind and patient with me in our many conversations concerning my work. He pointed me in the right direction many times.

I would also like to thank my wife, Youngsang, for her constant patience under difficult circumstances and for her helping me to complete my work.

INTRODUCTION

The classification of normal two dimensional singularities can be studied by the resolution of singularities. The resolution problem has been studied by Zariski [Z2], Hirzebruch [Hz], Hironaka [Hr], Brieskorn [B1] and Abhyankar [Ab]. In resolving a two dimensional singularity P, one replaces P by a compact analytic space A. Because P is a normal two dimensional singularity, A is one-dimensional. Let $A = \cup A_i$ be the decomposition of A into irreducible components. Thus each A_i is a (possibly singular) Riemann surface. It is easy to reduce all considerations to the case where the A_i are nonsingular, intersect transversely, and no three meet at a point. There is a purely topological but very important criterion due to Grauert [Gr] and Mumford [M] which says that A comes from a resolution if and only if the intersection matrix $(A_i \cdot A_j)$ is negative definite.

The classification problem of isolated singularities of complex surfaces have been studied from various standpoints. Taut singularities in the sense of Tyurina [T2] have been studied by Grauert [Gr], Brieskorn [B2], Laufer [L3] and Wagreich [W2]. The analytic structures of the taut singularities are, by definition, determined by the topological information of their weighted dual graphs. The topological classification of normal two dimensional singularities has been studied by Mumford [M], Wagreich [W1, W2] and Brieskorn [B2].

Let P be a singularity of a normal two dimensional analytic space V. In 1964, M. Artin introduced a definition for P to be rational. Rational singularities have also been studied by, for instance, DuVal [D], Tyurina [T1], Lipman [Li] and Laufer [L2]. In 1970, Wagreich introduced a definition for P to be weakly elliptic. Weakly elliptic singularities have been studied by Wagreich [W1], Laufer [L4], Karras [K1, K2] and Saito [Sa].

One of the important questions in normal two dimensional singularities is "the classification of all weighted dual graphs for hypersurface singularities." Double points are hypersurface singularities. It is known by [Ki] that double points determine plane curve singularities analytically. Wagreich [W1] proved that for double points $Z \cdot Z \geqq -2$. Then Laufer [L5] showed how to translate the topological classification of double points into the topological classification of the corresponding plane curves. It is a natural question to ask for a theory about normal triple point singularities. Let (V',P') be a singularity of any purely two dimensional analytic space V'. Then there exists a hypersurface singularity (V,P) such that V and V' have isomorphic normalizations [G & R, Chapter 3]. Note that the multiplicity of V at P and the multiplicity of V' at P' can be chosen to be the same.

Now let P be a triple point of V. Then by the Weierstrass Preparation Theorem, V may be locally defined by

$$V = \{(x,y,z) : f = z^3 + az^2 + bz + c = 0\} \text{ and } P = (0,0,0);$$

$a = a(x,y)$, $b = b(x,y)$ and $c = c(x,y)$ are holomorphic near $Q = (0,0)$ and f is of total order 3 at P. Also with a nonsingular change of local coordinates, we may eliminate the coefficient in z^2 of f. Therefore hereafter we assume that V is locally defined by

$$V = \{(x,y,z) : f = z^3 + 3pz + 2q = 0\} \quad \text{and} \quad P = (0,0,0)$$

where $p = p(x,y)$ and $q = q(x,y)$ are holomorphic near $Q = (0,0)$ and f is of total order 3 at P. Let $\rho : V \to \mathbb{C}^2$ be given by $\rho(x,y,z) = (x,y)$ and let $B = \{(x,y) : -108(p^3 + q^2) = 0\}$. Note that $-108(p^3 + q^2)$ is the z-discriminant of f for V. Then B is the branch locus of ρ. Q is a plane curve singularity. Plane curve singularities have been studied extensively. Their equisingular or topological classification is well understood [Z2]. We shall use the topology of the resolution of Q by quadratic transformations to describe the equisingular type of Q. For a normal two-dimensional singularity P of V, there is the minimal resolution $\tilde{r} : \tilde{N} \to V$. Let $r : N \to V$ be a resolution induced by the projection using the resolution to Q (See [L1, chapter 2] or it will be done later in this thesis). Let $\tilde{\Gamma}$ and Γ denote the topological type of the enbeddings of $\tilde{r}^{-1}(P)$ and $r^{-1}(P)$ in $\tilde{N}$ and N respectively. In this paper we shall relate $\tilde{\Gamma}$ and Γ to the equisingular type of Q where P is a triple point and Q is the associated plane curve singularity. In Section 2 we give numerical criteria for which components of the discriminant locus are part of the branch locus. We also determine the order of the branching. Γ determines $\tilde{\Gamma}$ but not conversely. The examples of Proposition 3.6 show that a given $\tilde{\Gamma}$ may have more than one Γ coming from normal triple points.

In Section 4 we impose condition (4.1). This seems to be a reasonable condition. Suppose that the above V satisfies (4.1). Let $\tilde{r} : \tilde{N} \to V$ be the minimal good resolution. Then N is obtained from $\tilde{N}$ by at most 5-time quadratic transformations at each s_j, $1 \leq j \leq \ell$ in $A = \tilde{r}^{-1}(P)$ (Theorem 4.7). Also given V with (4.1), there is an algorithm to determine the equisingular type of the plane curve singularity from Γ, the topological type of the embedding of $A = r^{-1}(P)$ in N (Theorem 4.8).

§1 Preliminaries

Gunning and Rossi [G&R] provides a good general reference. Let V be a complex analytic subvariety of a domain in $\mathbb{C}^m$ given by $V = \{z = (z_1, z_2, \ldots, z_m) : f_i(z) = 0,\ i = 1,2,\ldots,r\}$. We assume that V is reduced, i.e., that $\{f_i(z) : i = 1,2,\ldots,r\}$ generate the ideal id(V) at each point in V.

<u>Definition 1.1</u> A point $P \in V$ is a regular or nonsingular point of V if the jacobian $\left(\frac{\partial f_i}{\partial z_j}\right)(P)$, $1 \leq j \leq m$, $i \in I$ where I is a subset of $\{1,2,\ldots,r\}$ and $\{f_i\}$, $i \in I$ is a minimal set of defining equations for V at P has maximal rank. If P is not a regular point of V, then P is called a singular point of V. Note that [G&R, Proposition 9, p. 159] the set of singular points of V is a nowhere dense subvariety. If P is a regular point of rank k, then k is called the dimension of V at P.

<u>Definition 1.2</u> Let $V = \cup V_i$, $1 \leq i \leq k$, be the decomposition of V into irreducible components. By [G&R] note that for each i the set of regular points of V_i is connected and the dimension of V_i at any regular point P of V_i is constant. The number is denoted by dim V_i. Then dim V is defined by max dim V_i, $1 \leq i \leq k$. We shall say that V is pure dimensional if all components of V are of the same dimension. A singular point of V is a two dimensional singularity of V if V is purely two dimensional.

<u>Definition 1.3</u> A quadratic transformation at a point Q in a two dimensional manifold M consists of a new manifold M' and a map $\pi : M' \to M$ such that π is biholomorphic on $\pi^{-1}(M - Q)$ and π is given near $\pi^{-1}(Q)$ as follows. Let (x,y) be a coordinate system for a polydisc neighborhood $\Delta(0;r) = \Delta$ of Q, with $Q = (0,0)$. $\Delta' = \pi^{-1}(\Delta)$ has two coordinate patches $U_1 = (u,v)$ and $U_2 = (u',v')$ with $u' = \frac{1}{u}$ and $v' = uv$. $U_1 \cap U_2 = \{u \neq 0\}$. $\pi(u,v) = (uv,v)$ and $\pi(u',v') = (v',u'v')$. Thus $\Delta = \{(x,y) : |x| < r_1,\ |y| < r_2\}$, $U_1 = \{(u,v) : |uv| < r_1,\ |v| < r_2\}$ and $U_2 = \{(u',v') : |v'| < r_1,$ $|u'v'| < r_2\}$. A quadratic transformation as defined above is often called a monoidal transformation, a σ-process or a blowing-up at Q. Quadratic transformations are canonical. Namely, let $\phi : M \to L$ be a biholomorphic map between the two-dimensional manifolds M and L and let $\pi' : L' \to L$ be a quadratic transformation at $\phi(Q)$. Then there is a unique induced biholomorphic map $\phi' : M' \to L'$ such that $\phi \circ \pi = \pi' \circ \phi'$. Let $B \subset M$ be any analytic subvariety of M. Then we define the proper transform W of B to be the closure in M' of the inverse image of B away from Q, i.e., $W = \overline{\pi^{-1}(B - Q)}$.

<u>Definition 1.4</u> A germ h of a function defined on the regular points of V near P is said to be weakly holomorphic at P if h is holomorphic on the regular points near P and locally bounded near P. Let $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ be respectively the sheaf of germs of weakly holomorphic functions and the sheaf of germs of holomorphic functions on V. There is a natural inclusion $\mathcal{O} \subset \widetilde{\mathcal{O}}$. V is normal at P if $\mathcal{O}_p \subset \widetilde{\mathcal{O}}_p$ is an isomorphism. V is normal if $\mathcal{O} \simeq \widetilde{\mathcal{O}}$, i.e., if V is normal at each of its points.

Definition 1.5 If V is an analytic space, then a normalization (Y,π) of V is a normal analytic space Y and a holomorphic map $\pi : Y \to V$ such that

(i) $\pi : Y \to V$ is proper and has finite fibres.

(ii) If S is the singular set of V and $A = \pi^{-1}(S)$, then $Y - A$ is dense in Y and $\pi \mid Y - A$ is biholomorphic.

If $P \in V$ and (Y,π) is a normalization of V, then the number of points in $\pi^{-1}(P)$ equals the number of irreducible components of V near P [L1, p. 37]. A normalization of any two dimensional analytic space V always exists and is unique [L1, p. 38].

Definition 1.6 If V is an analytic space, then a resolution of the singularities of V consists of a manifold M and a proper holomorphic map $\pi : M \to V$ such that π is biholomorphic on the inverse image of R, the regular points of V, and such that $\pi^{-1}(R)$ is dense in M.

Definition 1.7 A nowhere discrete compact analytic subset A of an analytic space G is called exceptional (in G) if there exists an analytic space Y and a proper holomorphic map $\Phi : G \to Y$ such that $\Phi(A)$ is discrete, $\Phi : G - A \to Y - \Phi(A)$ is biholomorphic and such that for any open set $U \subset Y$, with $V = \Phi^{-1}(U)$, $\Phi^* : \Gamma(U,\mathcal{O}) \to \Gamma(V,\mathcal{O})$ is an isomorphism. If A is exceptional in G, then we shall sometimes say that A can be "blown down" or Φ blows down A.

Definition 1.8 A resolution $\pi : M \to V$ of the singularities of V is a minimal resolution if for any other resolution $\pi' : M' \to V$ there is a unique holomorphic map $\rho : M' \to M$ such that $\pi' = \pi \circ \rho$.

Assume that V is a normal two dimensional analytic space with P its only singularity. Then there is a unique minimal (good) resolution $\pi : M \to V$ among all resolutions satisfying conditions (i), (ii) and (iii) below. Let $\pi^{-1}(P) = A = \cup A_i$ be the decomposition of $\pi^{-1}(P)$ into irreducible components.

(i) Each A_i is nonsingular

(ii) A_i and A_j, $i \neq j$, intersect transversely whenever they intersect

(iii) No three distinct A_i meet [L1, p. 91].

<u>Definition 1.9</u> A branched analytic covering is a triple (V,π,U) such that

(i) V is a complex analytic variety

(ii) U is a domain in $\mathbb{C}^n$

(iii) π is a proper holomorphic mapping of V onto U and has discrete fibres

(iv) there exists a complex analytic subvariety $D \subset U$ and an integer λ such that π is a λ-sheeted covering map from $V - \pi^{-1}(D)$ onto $U - D$

(v) $V - \pi^{-1}(D)$ is dense in V.

The subvariety $D \subset U$ will be called the branch locus of the branched analytic covering. If the above integer $\lambda = 3$, then $\pi : V \to U$ is called a three-fold branched covering.

Definition 1.10 Let $f(z)$ be holomorphic in a domain $U \subset \mathbb{C}^m$ with $w \in U$. Let $f(z) = \sum_{n=k}^{\infty} f_n(z)$ near w where f_n is the homogeneous polynomial of degree n. If $f_k(z)$ is the homogeneous polynomial of lowest degree in this expansion which does not vanish identically, then $f(z)$ is said to have total order k at the point w; if $f(z) \equiv 0$, then the function is said to be of total order ∞.

Definition 1.11 Suppose that V is a complex subvariety in $\mathbb{C}^r$ and V is pure dimensional near $P \in V$. Let m be its maximal ideal at P. Then it is well-known [S] that $h(n) = \dim_{\mathbb{C}} m^n/m^{n+1}$ is a polynomial for sufficiently large n. Suppose $h(n) = a_0 + a_1 n + \ldots + a_d n^d$ for sufficiently large n where $a_d \neq 0$. Then the a_i are rational and $d = \dim V - 1$. The polynomial h is called the Hilbert polynomial of V. Recall that the multiplicity of V is at P, by definition, is $d! a_d$. The multiplicity is a positive integer. Let V be an analytic subvariety of a polydisc in $\mathbb{C}^3$ given by $\{(x,y,z) : f(x,y,z) = 0\}$ with $P = (0,0,0) \in V$. Let m be its maximal ideal at P. Then $h(n)$ is a polynomial of degree 1. If $f(x,y,z)$ has a total order k, then the multiplicity of V at P is $a_1 = k$. Therefore if V has a multiplicity 3 at P, then by the Weierstrass Preparation Theorem we may assume that V is locally defined by $\{(x,y,z) : f = z^3 + az^2 + bz + c = 0\}$ and $P = (0,0,0)$; $a = a(x,y)$, $b = b(x,y)$ and $c = c(x,y)$ are holomorphic near $Q = (0,0)$ and f is of total order 3 at P. Also with a nonsingular change of local coordinates, we may eliminate the coefficient in z^2 of f (by replacing z by $z - a/3$). Thus hereafter V may be locally defined by $V = \{(x,y,z) : f = z^3 + pz + q = 0\}$ and $P = (0,0,0)$ where $p = p(x,y)$ and $q = q(x,y)$ are holomorphic near $Q = (0,0)$ and f is of total order 3 at P. If $P \in V$ is singular then we call P a two dimensional triple point singularity or a triple point.

<u>Lemma 1.12</u> Let $V = \{(x,y,z) \mid z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ be an analytic subvariety of a polydisc in $\mathbb{C}^3$ with $P = (0,0,0) \in V$ and P a singular point. Let $\rho : V \to \mathbb{C}^2$ be given by $\rho(x,y,z) = (x,y)$. Let $\pi : M' \to \mathbb{C}^2$ be the quadratic transformation of $\mathbb{C}^2$ at $Q = (0,0)$. Let (u_1,v_1) and (u_2,v_2) be coordinate patches for M' with $\pi(u_1,v_1) = (x,y) = (u_1v_1,v_1)$ and $\pi(u_2,v_2) = (x,y) = (v_2,u_2v_2)$. Then π may be extended to $\pi' : M' \times \mathbb{C} \to \mathbb{C}^3$ with $\pi' : (u_1,v_1,z) \to (u_1v_1,v_1,z)$ and $\pi' : (u_2,v_2,z) \to (v_2,u_2v_2,z)$. Let $V' = (\pi')^{-1}(V)$. Let $\omega : N' \to V'$ be the normalization of V'. There is a map ρ_1 such that the following diagram is commutative:

$$
\begin{array}{ccc}
N' & & \\
\omega\downarrow & & \\
V' & \xrightarrow{\pi'} & V \\
\rho_1\downarrow & & \rho\downarrow \\
M' & \xrightarrow{\pi} & \mathbb{C}^2
\end{array}
\qquad (1.1)
$$

ρ_1 and $\rho' = \rho_1 \circ \omega$ locally represent V' and N' respectively as three-fold branched covering spaces of M'.

Proof Let λ and μ be the total order of the zero of $p(x,y)$ and $q(x,y)$ at $(0,0)$ respectively. Then the local defining equations for V' are the following:

$$(1.2) \qquad \begin{aligned} & z^3 + 3p(u_1v_1,v_1)z + 2q(u_1v_1,v_1) \\ & = z^3 + 3v_1^{\lambda}p_1(u_1,v_1)z + 2v_1^{\mu}q_1(u_1,v_1) \end{aligned}$$

$$(1.3) \qquad \begin{aligned} & z^3 + 3p(v_2,u_2v_2)z + 2q(v_2,u_2v_2) \\ & = z^3 + 3v_2^{\lambda}p_2(u_2,v_2)z + 2v_2^{\mu}q_2(u_2,v_2) \end{aligned}$$

where p_1, q_1 are holomorphic near $(u_1,v_1) = (0,0)$ and p_2, q_2 are holomorphic near $(u_2,v_2) = (0,0)$. Then ρ_1 is given by $\rho_1(u_1,v_1,z) = (u_1,v_1)$ and $\rho_1(u_2,v_2,z) = (u_2,v_2)$. So the diagram (1.1) is commutative. Also, ρ_1 and ρ' are three-fold branched covering maps by Definition 1.5.

§2 Resolution of Triple Point Singularities

Recall [B&C, p. 180] that the discriminant of $z^3 + 3p(x,y)z + 2q(x,y) = 0$ with respect to z is $D \equiv -108[p^3(x,y) + q^2(x,y)]$ in $\mathbb{C}^2$. Recall that V of Lemma 1.12 may have a nonisolated singular point $P = (0,0,0)$. Let $D \not\equiv 0$, otherwise a resolution of V would be trivial. Let $B = \operatorname{loc} D$. B contains all points above which ρ may fail to be a covering map, where $\rho : V \to \mathbb{C}^2$ is given by $\rho(x,y,z) = (x,y)$. B is the branch locus if ρ is thought of as a branched covering map. B is a plane curve. Let $Q = (0,0)$ be the singularity of B. Let us look at a resolution process in terms of iterations of Lemma 1.12. Without loss of generality, we may work near just $(u,v) = (u_1,v_1) = (0,0)$ in M' of (1.1). Let B_1 be the branch locus for ρ_1 and B', the branch locus for ρ'. Then clearly $B' \subset B_1$. Also if B' is singular at some point, then so is B_1 at that point. The branch locus B_1 for (1.2) is $\{v^{3\lambda}p_1^3(u,v) + v^{2\mu}q_1^2(u,v) \equiv v^m b_1(u,v) = 0\}$ where m is the total order of the zero of $v^{3\lambda}p_1^3(u,v) + v^{2\mu}q_1^2(u,v)$ over $v = 0$. Then we get the following.

Lemma 2.1 If N' has a singular point of $(\rho')^{-1}(0,0)$, then (0,0) is a singular point of $\{b_1(u,v) = 0\}$ or $\{v = 0\}$ is part of the branch locus of ρ' and $b_1(0,0) = 0$.

Proof Since N' is a normalization of V', it is enough to show that if Q is a regular point of B', then N' is regular above Q. For proof see [G, Theorem, p. 48].

Now suppose that $(u,v) = (0,0)$ is a singular point of B'. Let $M^{(2)}$ be the blow-up of M' at $(0,0)$ and apply Lemma 1.12. Also using (1.1) from our blow-up of $\mathbb{C}^2$ we get (2.1).

$$
\text{(2.1)} \qquad
\begin{array}{ccccc}
N^{(2)} & & N' & & \\
\omega_2 \downarrow & & \omega \downarrow & & \\
V^{(2)} & \xrightarrow{\pi^{(2)}} & V' & \xrightarrow{\pi'} & V \\
\rho_2 \downarrow & & \rho_1 \downarrow & & \rho \downarrow \\
M^{(2)} & \xrightarrow{\pi_2} & M' & \xrightarrow{\pi} & \mathbb{C}^2
\end{array}
$$

We may iterate the process of going from (1.1) to (2.1) for so long as the branch locus $B^{(n)}$ of $\rho^n = \rho_n \circ \omega_n : N^{(n)} \to M^{(n)}$ has singular points. To see that a nonsingular $B^{(n)}$ exists for some n, we may proceed as follows. After m iterations, let $\tau_m = \pi \circ \pi_2 \circ \ldots \circ \pi_{m-1} \circ \pi_m : M^{(m)} \to \mathbb{C}^2$. Let $W^{(m)}$ be the proper transform of $B = \{p^3(x,y) + q^2(x,y) = 0\}$ under τ_m. Let $E^{(m)} = \tau_m^{-1}(0,0)$. Then $E^{(m)}$ is, by definition, an exceptional set of the first kind. Let $E^{(m)} = \cup E_i^{(m)}$, $1 \leq i \leq m$, be the decomposition of $E^{(m)}$ into irreducible components. Let $(B_m) \equiv (-108(p^3 + q^2) \circ \tau_m) = W^{(m)} + \Sigma e_i E_i^{(m)}$, $1 \leq i \leq m$, be the divisor of $(p^3 + q^2) \circ \tau_m$. Then first to find the branch locus $B^{(m)}$ for $\rho^{(m)}$, we need the following propositions.

<u>Lemma 2.2</u> Let $W = \{(x,y,z) : f(x,y,z) = z^3 + 3p(x)z + 2q(x) = 0\}$ be an analytic subvariety of a polydisc in $\mathbb{C}^3$ with $P = (0,0,0) \in W$ and P a singular point where p(x) and q(x) are holomorphic in x.

Note that W is independent of y. Let $\rho : W \to \mathbb{C}^2$ be given by $\rho(x,y,z) = (x,y)$. Let D be the discriminant of $f(x,y,z) = 0$ with respect to z. Then $D = \prod_{i<j} (t_i - t_j)^2 = -108[p^3(x) + q^2(x)]$ where the $t_i = t_i(x)$, $i = 1,2,3$, are solutions of $f(x,y,z) = 0$ for z. Assume $D \not\equiv 0$. Let $\omega : \tilde{W} \to W$ be the normalization of W. Let $\rho' = \rho \circ \omega$. Let $\mathbb{C}\langle x\rangle[z]$ be the polynomial ring in z with coefficients in $\mathbb{C}\langle x\rangle$ where $\mathbb{C}\langle x\rangle$ is the ring of power series expansion of x. Factoring $f(x,y,z)$ in $\mathbb{C}\langle x\rangle[z]$, we have the following cases: Observe that irreducibility in $\mathbb{C}\langle x\rangle[z]$ is the same as irreducibility in $\mathbb{C}\langle x,z\rangle$ (See [G&R, Lemma 5, p. 71]) which is equivalent to irreducibility in $\mathbb{C}\langle x,y,z\rangle$ by connectivity of regular set.

(i) If $f(x,y,z)$ has three distinct linear factors in $\mathbb{C}\langle x\rangle[z]$, then $\rho' = \rho \circ \omega$ is 3 - 1 over the locus $x = 0$.

(ii) If $f(x,y,z)$ has one linear factor and one irreducible quadratic factor in $\mathbb{C}\langle x\rangle[z]$, then ρ' is 2 - 1 over the locus $x = 0$.

(iii) If $f(x,y,z)$ is irreducible in $\mathbb{C}\langle x\rangle[z]$, then ρ' is 1 - 1 over the locus $x = 0$.

Moreover, in case (i) D has a zero of even order in x and in case (ii) D has a zero of odd order in x over the locus $x = 0$.

<u>Proof</u> By Definition 1.5, it remains to show that in case (i) D has a zero of even order in x and in case (ii) D has a zero of odd order in x over the locus $x = 0$. In case (i) note that D is a square of a holomorphic function in x and so the proof is obvious.

In case (ii) we may assume that $f(x,y,z) = z^3 + 3p(x)z + 2q(x) = (z-r_1)(z^2+r_1z+r_2)$ where $r_1 = r_1(x)$, $r_2 = r_2(x)$ are in $\mathbb{C}\langle x\rangle$ and $z^2 + r_1z + r_2$ is irreducible in $\mathbb{C}\langle x\rangle[z]$. Note that $3p(x) = -r_1^2 + r_2$ and $2q(x) = -r_1r_2$. Then $D = -108[p^3(x) + q^2(x)] = (2r_1^2 + r_2)^2(r_1^2 - 4r_2)$. But $r_1^2 - 4r_2$ must have a zero of odd order in x over the locus $x = 0$, otherwise $z^2 + r_1z + r_2$ is not irreducible over $\mathbb{C}\langle x\rangle$. Thus D has a zero of odd order in x over the locus $x = 0$.

Proposition 2.3 Assume that the hypotheses in Lemma 1.12 are satisfied. Let λ and μ be the total order of the zero of $p(x,y)$ and $q(x,y)$ at $(0,0)$ respectively. Without loss of generality we assume that $V' = \{(u,v,z) \mid f_1 = f_1(u,v,z) = z^3 + 3v^\lambda p_1(u,v)z + 2v^\mu q_1(u,v) = 0\}$. Then the z-discriminant of $f_1 = -108[v^{3\lambda}p_1^3(u,v) + v^{2\mu}q_1^2(u,v)] = -108\, v^m b_1(u,v)$ where $b_1(u,v)$ is holomorphic near $(0,0)$ and $v \nmid b_1(u,v)$. Assume this discriminant is not identically zero. Then $b_1(u,0) \not\equiv 0$. Since $b_1(u,0)$ is a polynomial in u, there may exist $u_1,\ldots,u_k$ in $\mathbb{C}$ such that $b_1(u_i,0) = 0$. Let $B^* = \{v = 0\} - \{(u_i,0) \mid b_1(u_i,0) = 0\}$. There are three cases:

(i) $m \geq 3\lambda = 2\mu$

(a) If m is even, then ρ' is three to one over the locus B^*.

(b) If m is odd, then ρ' is two to one over the locus B^*.

(ii) Let $m = 3\lambda < 2\mu$

(a) If $m = 3\lambda$ is even, then ρ' is three to one over the locus B^*.

(b) If $m = 3\lambda$ is odd, then ρ' is two to one over the locus B^*.

(iii) Let $m = 2\mu < 3\lambda$

(a) If $m = 2\mu \equiv 0 \pmod 3$, then ρ' is three to one over the locus B^*.

(b) If $m = 2\mu \not\equiv 0 \pmod 3$, then ρ' is one to one over the locus $v = 0$.

Proof Note that V' is equisingular along the locus $v = 0$ except possibly for those $(u_i,0)$ where $b_1(u_i,0) = 0$, $i = 1,2,\ldots k$ [Z2, Theorem 7, p. 529]. Let u_0 be fixed such that $b_1(u_0,0) \neq 0$. Then it is enough to prove the above cases over $(u_0,0)$ locally. We write $(0,0)$ for such $(u_0,0)$. Also we need the following theorem.

Theorem 2.4 (Hensel's Lemma)

Let $h = z_n^s + a_1 z_n^{s-1} + \ldots + a_n$ where the coefficients $a_i = a_i(z_1,\ldots,z_{n-1})$ are holomorphic near $(0,\ldots,0)$, $i = 1,\ldots,n$. Let h have the decomposition $h(0,\ldots,0,z_n) = \prod_{\alpha=1}^{\ell} (z_n - c_\alpha)^{s_\alpha}$ into linear factors (with the c_α distinct and $s_1 + s_2 + \ldots + s_\ell = s$). Then there are uniquely determined polynomials $h_1,\ldots,h_\ell \in {}_{n-1}\mathcal{O}[z_n]$ with $\deg(h_\alpha) = s_\alpha$ and $h_\alpha(0,\ldots,0,z_n) = (z_n - c_\alpha)^{s_\alpha}$ for $\alpha = 1,\ldots,\ell$ such that $h = h_1 \cdots h_\ell$ where ${}_{n-1}\mathcal{O}[z_n]$ is a polynomial ring in z_n with coefficients holomorphic near $(z_1,\ldots,z_{n-1}) = (0,\ldots,0)$.

Proof of Theorem. See [G&F, p. 82].

(i) Let $m > 3\lambda = 2\mu$

To apply Hensel's lemma, substitute $z \cdot v^{\lambda/2}$ for z in the equation $f_1(0,v,z)$. Then we get

$$L' = \{g_1(0,v,z) = z^3 + 3p_1(0,v)z + 2q_1(0,v) = 0\}$$

Note that $g_1(0,v,z)$ is reducible if and only if $f_1(0,v,z)$ is reducible in $\mathbb{C}<v>[z]$, a polynomial ring in z with coefficients holomorphic near $v = 0$. If $m = 3\lambda = 2\mu$, then observe that the z-discriminant of $g_1(0,v,z)$ is $-108[p_1^3(0,v) + q_1^2(0,v)] = -108b_1(0,v)$. So if $b_1(0,0) \neq 0$, then clearly $g_1(0,0,z)$ has three distinct roots and by Hensel's lemma $g_1(0,v,z)$ has three distinct linear factors in $\mathbb{C}<v>[z]$. So does $f_1(0,v,z)$. Now if $m > 3\lambda = 2\mu$, then the z-discriminant of $g_1(0,v,z)$ is $-108[p_1^3(0,v) + q_1^2(0,v)] = -108v^{m-3\lambda}b_1(0,v)$. We shall prove later that if $b_1(0,0) \neq 0$, then $p_1(0,0) \neq 0$ and $q_1(0,0) \neq 0$. Consider the equation $g_1(0,0,z) = z^3 + 3p_1(0,0)z + 2q_1(0,0)$. Note that the z-discriminant of $g_1(0,0,z)$ is zero. Since $p_1(0,0) \neq 0$ and $q_1(0,0) \neq 0$, $g_1(0,0,z)$ has one root of multiplicity 1 and the other root of multiplicity 2. Then by Hensel's lemma $g_1(0,v,z)$ is reducible in $\mathbb{C}<v>[z]$. If m is even, then $m - 3\lambda$ is even and by Lemma 2.2 $g_1(o,v,z)$ has three distinct linear factors in $\mathbb{C}<v>[z]$. So does $f_1(0,v,z)$. If m = odd, then $m - 3\lambda$ is odd and by Lemma 2.2 $g_1(0,v,z)$ has one linear factor and one irreducible quadratic factor in $\mathbb{C}<v>[z]$. So does $f_1(0,v,z)$. Now we are going to prove that if $m > 3\lambda = 2\mu$ and $b_1(0,0) \neq 0$ then $p_1(0,0) \neq 0$ and $q_1(0,0) \neq 0$. Since $-108(p_1^3(u,v) + q_1^2(u,v)) = -108v^{m-3\lambda}b_1(u,v)$, $p_1^3(u,0) + q_1^2(u,0)$

is identically zero. So if $p_1(0,0) = 0$, then $q_1(0,0) = 0$ and conversely. Therefore it is enough to show that if $p_1(0,0) = 0$ and $q_1(0,0) = 0$ then $b_1(0,0) = 0$. Note that $p_1(u,v)$ and $q_1(u,v)$ may be written as $p_1(u,v) = F + v^s \cdot G$ and $q_1(u,v) = H + v^t \cdot J$ respectively, where $F = F(u)$, $H = H(u)$ are polynomials in u, $G = G(u,v)$, $J = J(u,v)$ are holomorphic near (0,0) and $v \nmid G$ and $v \nmid J$. Also observe that $F = p_1(u,0) \not\equiv 0$ and $H = q_1(u,0) \not\equiv 0$ otherwise it would contradict to the fact that $v \nmid p_1(u,v)$ and $v \nmid q_1(u,v)$. But $u \mid F$ and $u \mid H$ because $p_1(0,0) = q_1(0,0) = 0$. Now writing $p_1^3(u,v)$ and $q_1^2(u,v)$ in increasing order of degree of v, we have $p_1^3(u,v) = F^3 + 3F^2Gv^s + 3FG^2v^{2s} + G^3v^{3s}$ and $q_1^2(u,v) = H^2 + 2HJv^t + J^2v^{2t}$. Since $p_1^3(u,0) + q_1^2(u,0) \equiv 0$, then $F^3 + H^2 \equiv 0$. Therefore if $s \neq t$ then the first term of $p_1^3(u,v) + q_1^2(u,v)$ has a factor u and the remaining terms of $p_1^3(u,v) + q_1^2(u,v)$ has a factor either v or u because $u \mid F$ and $u \mid H$. Thus $b_1(0,0) = 0$. If $s = t$ and the first term of $p_1^3(u,v) + q_1^2(u,v)$ is not identically zero, then it is trivial. So if $s = t$ and $(3F^2G + 2HJ)v^s \equiv 0$, then it suffices to show that $3FG^2 + J^2 \not\equiv 0$ but that $3FG^2 + J^2$ vanishes at (0,0). But we know that $F^3 + H^2 \equiv 0$ and $3F^2G + 2HJ \equiv 0$. Thus $3F^2G + 2HJ \equiv 0$ implies $9F^4G^2 \equiv 4H^2J^2 \equiv -4F^3J^2$. Hence we get $9FG^2 = -4J^2$. Therefore $3FG^2 + J^2 \equiv 3FG^2 - 9/4 \cdot FG^2 \equiv 3/4 \cdot FG^2 \not\equiv 0$ because $G \not\equiv 0$. Since $u \mid F$ it is trivial.

(ii) Let $m = 3\lambda < 2\mu$

Let $L' = \{(0,v,z) : g_1(0,v,z) = z^3 + 3v^\lambda p_1(u,v)z = 0\}$. Note that the z-discriminants of $f_1(0,v,z)$ and $g_1(0,v,z)$ are $-108v^{3\lambda}b_1(0,v) = -108[v^{3\lambda}p_1^3(0,v) + v^{2\mu}q_1^2(0,v)]$ and $-108v^{3\lambda}p_1^3(0,v)$ respectively and that $b_1(0,0) = p_1^3(0,0)$. So by [Z2, Theorem 7, p. 529] $\{f_1(0,v,z) = 0\}$ and L' are equisingular. So it remains to consider the equation $g_1(0,v,z)$. Since $b_1(0,0) \neq 0$, then $p_1(0,0) \neq 0$. Therefore if $\lambda \equiv 0 \pmod 2$ then $g_1(0,v,z)$ has three distinct linear factors in $¢\langle v\rangle[z]$ and so does $f_1(0,v,z)$. Also if $\lambda \not\equiv 0 \pmod 2$, then $g_1(0,v,z)$ has one linear factor and one irreducible quadratic factor in $¢\langle v\rangle[z]$. So does $f_1(0,v,z)$.

(iii) Let $m = 2\mu < 3\lambda$

Let $L' = \{(0,v,z) \mid g_1(0,v,z) = z^3 + 2v^\mu q_1(0,v) = 0\}$. Note that the z-discriminant of $f_1(0,v,z)$ and $g_1(0,v,z)$ are $-108[v^{3\lambda}p_1^3(o,v) + v^{2\mu}q_1^2(0,v)]$ and $-108v^{2\mu}q_1^2(0,v)$ respectively and that $b_1(0,0) = q_1^2(0,0)$. Then by [Z2, Theorem 7, p. 529] $\{f_1(0,v,z) = 0\}$ and L' are equisingular. So it is enough to consider the equation $g_1(0,v,z)$. Since $b_1(0,0) \neq 0$, then $q_1(0,0) \neq 0$. Therefore if $\mu \equiv 0 \pmod 3$ then $g_1(0,v,z)$ has three distinct linear factors in $¢\langle v\rangle[z]$. So does $f_1(0,v,z)$. If $\mu \not\equiv 0 \pmod 3$ then $g_1(0,v,z)$ is irreducible in $¢\langle v\rangle[z]$ by Lemma 2.2. So is $f_1(0,v,z)$.

Corollary 2.5 Under the hypotheses of Proposition 2.3, in the following cases $\{v = 0\}$ is part of the branch locus of ρ' as three-fold covering map.

(i) If $m > 3\lambda = 2\mu$ and m is odd, then ρ' is $2-1$ over the locus $v = 0$ except possibly for $(u_i,0)$ where $b_1(u_i,0) = 0$, $i = 1,2,\ldots,k$.

(ii) If $m = 3\lambda < 2\mu$ and m is odd, then ρ' is $2-1$ over the locus $v = 0$ except possibly for $(u_i,0)$ where $b_1(u_i,0) = 0$, $i = 1,2,\ldots,k$.

(iii) If $m = 2\mu < 3\lambda$ and $m \not\equiv 0 \pmod 3$, then ρ' is $1-1$ over the locus $v = 0$.

Therefore, by Corollary 2.5, $B^{(m)}$ consists of some irreducible components of $W^{(m)}$ and $E^{(m)} = \cup E_i^{(m)}$ satisfying one of the conditions in that Corollary. So to get a nonsingular $B^{(n)}$, we may first perform quadratic transformations on $\mathbb{C}^2$ until the plane curve singularity, that is, the singularity of $\{(x,y) \mid p^3(x,y) + q^2(x,y) = 0\}$ is just resolved. Then the irreducible components of the branch locus of $B^{(m)}$ are submanifolds. $E^{(m)}$ has normal crossings. Thus after additional quadratic transformations, any component of $W^{(m)}$ which meets an $E_i^{(m)}$ will meet that $E_i^{(m)}$ with normal crossings and moreover, no three distinct components of $W^{(m)}$ and $E_i^{(m)}$, $1 \leq i \leq m$ meet. But some components of the branch locus $B^{(m)}$ may happen to intersect with normal crossings. Before resolving this situation, let us recall that $\tau_m = \pi \circ \pi_2 \circ \ldots \circ \pi_{m-1} \circ \pi_m$ and $(B_m) = ((p^3 + q^2) \circ \tau_m) = W^{(m)} + \Sigma e_i E_i^{(m)}$, $1 \leq i \leq m$, is the divisor of $(p^3 + q^2) \circ \tau_m$. Under this mapping τ_m, then $(p^3)|F_i = (p^3 \circ \tau_m)|F_i = 3\lambda_i F_i$ and $(q^2)|F_i = (q^2 \circ \tau_m)|F_i = 2\mu_i F_i$, where F_i is an irreducible component of $W^{(m)}$ and $E^{(m)}$. Then observe that we have the only three cases below:

(i) $e_i \geq 3\lambda_i = 2\mu_i$

(ii) $e_i = 3\lambda_i < 2\mu_i$

(iii) $e_i = 2\mu_i < 3\lambda_i$

If $p(x,y) \equiv 0$ in V, then note that we have only one case (iii) since $p(x,y)$ is thought as of total order ∞ near Q.

Lemma 2.6 Let F_i and F_j be irreducible components of $B^{(m)}$ with normal crossings and $F_i \cap F_j \neq \phi$. Then after at most three time quadratic transformations at $F_i \cap F_j$, F_i and F_j can be chosen with $F_i \cap F_j = \phi$ satisfying the same conditions among (i), (ii) and (iii) in Corollary 2.5.

Proof After restricting to $F_i \cup F_j$ we may write

$$(B_m) = e_i F_i + e_j F_j$$

$$(p^3) = 3\lambda_i F_i + 3\lambda_j F_j \quad \text{and}$$

$$(q^2) = 2\mu_i F_i + 2\mu_j F_j$$

Let π_{m+1} be the quadratic transformation of $M^{(m)}$ at $F_i \cap F_j$. Let $\tau_{m+1} = \pi_{m+1} \circ \tau_m$. Let F_k be the new exceptional curve and let

$$(B_{m+1}) = ((p^3 + q^2) \circ \tau_{m+1}) = e_k F_k$$

$$(p^3) = ((p^3) \circ \tau_{m+1}) = 3\lambda_k F_k$$

$$(q^2) = ((q^2) \circ \tau_{m+1}) = 2\mu_k F_k,$$

after restricting to F_k. Observe that $e_k = e_i + e_j$ but that $3\lambda_k \geq 3\lambda_i + 3\lambda_j$ and $2\mu_k \geq 2\mu_i + 2\mu_j$ because there might be additional components of (p^3) and (q^2) at $F_i \cap F_j$ other than F_i and F_j. Now consider the following three cases:

(a) Assume that F_i satisfies the condition (i) in Corollary 2.5. Then $e_i > 3\lambda_i = 2\mu_i$ and e_i is odd.

(a1) If F_j also satisfies the same condition, then

$$e_j > 3\lambda_j = 2\mu_j \text{ and } e_j \text{ is odd.}$$

Then we claim that (p^3) and (q^2) have no additional components at $F_i \cap F_j$, which will be proved later. If so, $e_k = e_i + e_j > 3\lambda_k = 3\lambda_k + 3\lambda_j = 2\mu_i + 2\mu_j = 2\mu_k$. Since e_k is even and $e_k > 3\lambda_k = 2\mu_k$, then by Proposition 2.3, F_k is not part of the branch locus $B^{(m+1)}$ and F_k separates these two components. To prove our claim, suppose that there are additional components of (p^3) or (q^2) at $F_i \cap F_j$. Let the local defining equation of $V^{(m)}$, say f, near $F_i \cap F_j$ be $\{f = z^3 + 3v^{\lambda_i}u^{\lambda_j}pz + 2v^{\mu_i}u^{\mu_j}q = 0\}$, where $p = p(u,v)$ and $q = q(u,v)$ is holomorphic near $(0,0)$, $v \nmid p$, $v \nmid q$, $u \nmid p$ and $u \nmid q$ and $F_i = \{v = 0\}$ and $F_j = \{u = 0\}$. $F_i \cap F_j = (0,0)$.

If $p(0,0) = 0 \neq q(0,0)$, then $e_k = e_i + e_j > 2\mu_k = 2\mu_i + \mu_j$ but $3\lambda_k > 3\lambda_i + 3\lambda_j = 2\mu_i + 2\mu_j = 2\mu_k$. So it would be a contradiction otherwise this implies $e_k = \mathrm{Min}(3\lambda_k, 2\mu_k) = 2\mu_k$. If $p(0,0) \neq 0 =$

$q(0,0)$ then similarly we get a contradiction. Now let $p(0,0) = q(0,0) = 0$. Note that the z-discriminant of $f = -108v^{3\lambda_i}u^{3\lambda_j}[p^3+q^2] = -108v^{e_i}u^{e_j}b$, where $b = b(u,v)$ is holomorphic near $(0,0)$, $v \nmid b$, $u \nmid b$. Also observe that $b(0,0) \neq 0$ by assumption. Since $e_i > 3\lambda_i$ and $e_j > 3\lambda_j$, $v^m u^k b = p^3 + q^2$ for some $m > 0$, $k > 0$. As in the proof of (i) of Proposition 2.3, we write

$$p = F + v^s G$$

$$q = H + v^t J$$

where $F = F(u)$, $H = H(u)$ are holomorphic near $u = 0$, $G = G(u,v)$, $J = J(u,v)$ are holomorphic near $(0,0)$, $v \nmid G$, $v \nmid J$ and s and t are integers. Note that $F \not\equiv 0$ and $H \not\equiv 0$ otherwise it would contradicts to $v \nmid p$ and $v \nmid q$. Now writing p^3 and q^2 in increasing order of degree of v, we have

$$p^3 = F^3 + 3F^2Gv^s + 3FG^2v^{2s} + G^3v^{3s}$$

$$q^2 = H^2 + 2HJv^t + J^2v^{2t}.$$

Since $p^3(u,0) + q^2(u,0) \equiv 0$ then $F^3 + H^2 \equiv 0$. But $u \mid F$ and $u \mid H$ because $p(0,0) = q(0,0) = 0$. If $G \equiv 0$ then $p^3(0,v) + q^2(0,v) \equiv 0$ implies $J \equiv 0$ because $F^3 + H^2 \equiv 0$. Thus we get $G \not\equiv 0$ and $J \not\equiv 0$. Then similarly as in the proof of Proposition 2.3, we get $b(0,0) = 0$. Thus we get a contradiction.

(a2) If F_j satisfies (ii) in Corollary 2.5, then

$$e_j = 3\lambda_j < 2\mu_j \text{ and } e_j \text{ is odd.}$$

Let the local defining equation f for $V^{(m)}$ be defined as in case (a1). Now if $p(0,0) \neq 0$, then $3\lambda_k = 3\lambda_i + 3\lambda_j < 2\mu_i + 2\mu_j \leq 2\mu_k$ but also $3\lambda_k = 3\lambda_i + 3\lambda_j < e_i + e_j = e_k$, which is a contradiction. If $p(0,0) = 0$, then similarly as in case (a1) we have $v^m b(u,v) = [p^3 + u^{2\mu_j - 3\lambda_j} q^2]$ for $m > 0$. Then similarly as in the proof of Proposition 2.3, we get $b(0,0) = 0$, which is impossible.

(a3) If F_j satisfies (iii) in Corollary 2.5, then

$$e_j = 2\mu_j < 3\lambda_j \text{ and } e_j = 2\mu_j \not\equiv 0 \pmod 3.$$

Then similarly as in case (a2), we get a contradiction.

(b) Assume that F_i satisfies the condition (ii) in Corollary 2.5. Then $e_i = 3\lambda_i < 2\mu_i$ and e_i is odd.

(b1) If F_j satisfies (ii) in Corollary 2.5, then

$$e_j = 3\lambda_j < 2\mu_j \text{ and } e_j \text{ is odd.}$$

Then $e_k = e_i + e_j = 3\lambda_i + 3\lambda_j = 3\lambda_k < 2\mu_i + 2\mu_j \leq 2\mu_k$ and e_k is even. Thus, by Proposition 2.3, F_k is not part of the branch locus $B^{(m+1)}$ and separates F_i and F_j.

(b2) If F_j satisfies (iii) in Corollary 2.5, then

$$e_j = 2\mu_j < 3\lambda_j,\ e_j = 2\mu_j \not\equiv 0 \pmod 3 \text{ and}$$

$$e_k = e_i + e_j = 3\lambda_i + 2\mu_j.$$

But $e_k < 3\lambda_i + 3\lambda_j \leq 3\lambda_k$ and $e_k < 2\mu_i + 2\mu_j \leq 2\mu_k$.

It is absurd, because $(B^{m+1}) < \min((p^3),(q^2))$ over F_k.

(c) Assume that F_i and F_j satisfy the same condition (iii) in Corollary 2.5. Then

$$e_i = 2\mu_i < 3\lambda_i \text{ with } e_i \not\equiv 0 \pmod 3 \text{ and}$$
$$e_j = 2\mu_j < 3\lambda_j \text{ with } e_j \not\equiv 0 \pmod 3.$$

So $e_k = e_i + e_j = 2\mu_i + 2\mu_j = 2\mu_k < 3\lambda_i + 3\lambda_j \leq 3\lambda_k$.

If $e_k \equiv 0 \pmod 3$, then by Proposition 2.3, F_k is not part of the branch locus $B^{(m+1)}$ and separates these two components. If $e_k \not\equiv 0 \pmod 3$, then by Corollary 2.5, F_k is still part of the branch locus $B^{(m+1)}$. Then by the same argument as above, after one additional blow-up at $F_i \cap F_k$ and $F_j \cap F_k$, respectively, these three components F_i, F_j and F_k will be separated by the new two exceptional curves which are not part of the branch locus $B^{(m+3)}$.

Therefore, after performing quadratic transformations n times, by the previous discussion, we may assume the following:

(2.2)
$$
\begin{array}{cccccccccc}
 & N^{(n)} & & & & N^{(2)} & & N' & & \\
 & \downarrow \omega_n & & & & \downarrow \omega_2 & & \downarrow \omega & & \\
\rho^{(n)} & V^{(n)} & \xrightarrow{\pi^{(n)}} & \cdots & \longrightarrow & V^{(2)} & \xrightarrow{\pi^{(2)}} & V' & \xrightarrow{\pi'} & V \\
 & \downarrow \rho_n & & & & \downarrow \rho_2 & & \downarrow \rho_1 & & \downarrow \rho \\
 & M^{(n)} & \xrightarrow{\pi_n} & \cdots & \longrightarrow & M^{(2)} & \xrightarrow{\pi_2} & M' & \xrightarrow{\pi} & \mathbb{C}^2
\end{array}
$$

(i) We may iterate the process of going from (1.1) to (2.1) until the branch locus $B^{(n)}$ of $\rho^{(n)} = \rho_n \circ \omega_n$ is nonsingular.

(ii) Let $\tau = \tau_n = \pi \circ \pi_2 \circ \ldots \circ \pi_{n-1} \circ \pi_n : M = M^{(n)} \to \mathbb{C}^2$ and

and let $(B_n) = W^{(n)} + \Sigma e_i E_i^{(n)}$, $1 \leq i \leq n$, be the divisor of $(p^3+q^2) \circ \tau$. Then $B^{(n)}$ consists of those $W^{(n)}$ and those $E_i^{(n)}$ satisfying the conditions in Corollary 2.5. Any two distinct components which are part of the branch locus $B^{(n)}$ do not intersect. Any two distinct components of $W^{(n)}$ and $\cup E_i^{(n)}$, $1 \leq i \leq n$, meet with normal crossings and moreover, no three distinct component of them will intersect.

Observe that as long as $B^{(n)}$ satisfies the above conditions (i) and (ii) we can stop this process.

Given V of Lemma 1.12, we assume that V may have a non-isolated singular point P. Now for such a resolution $r : N = N^{(n)} \to V$ where $r = \pi' \circ \pi^{(2)} \circ \ldots \circ \pi^{(n)} \circ \omega_n$, there is associated the topological type of the embedding of $A = r^{-1}(P)$ in N. We shall use Γ to denote this topological type. Let $A = \cup A_j$, $1 \leq j \leq m$, be the decomposition of A into irreducible components. Γ gives the geometric genus g_j of each A_j, the $A_j \cdot A_j$ or self-intersection numbers in N and how each A_j intersects A_k, $j \neq k$. Note that each A_j is nonsingular by construction. When V of Lemma 1.12 may have a nonisolated singular point P and $r : N \to V$ is such a resolution, henceforth we use the terminology "a resolution by (2.2)".

Now we are going to describe Γ from the topological type of Q, the corresponding plane curve singularity, that is, the singularity of the branch locus B of ρ where $\rho : V \to \mathbb{C}^2$ is given by $\rho(x,y,z) = (x,y)$. Let $Z_E = \Sigma z_i E_i$, $1 \leq i \leq n$, be the fundamental cycle where $E_i = E_i^{(n)}$. Then Z_E is also the pull-back under $\tau = \pi \circ \pi_2 \circ \ldots \circ \pi_n : M \to \mathbb{C}^2$

of the maximal ideal (x,y). Let $X = \Sigma m_j A_j$, $1 \leq j \leq m$, be the divisor of the pull-back under $r : N \to V$ of the ideal (x,y) on V (induced by the projection $\rho : V \to \mathbb{C}^2$). For describing the A_j on N in terms of the E_i on M, there are three cases, (I), (II) and (III) below. $\rho = \rho^{(n)} : N \to M$ is the three-fold covering map. Let us recall that $(B) = (B_n) = W^{(n)} + \Sigma e_i E_i$, $1 \leq i \leq n$ where $E_i = E_i^{(n)}$ and $(B)\big|_{F_i} = e_i F_i$, $(p^3)\big|_{F_i} = 3\lambda_i F_i$ and $(q^2)\big|_{F_i} = 2\mu_i F_i$ where F_i is any irreducible component of $\cup E_i^{(n)}$ and $W^{(n)}$. For brevity we write $o(B) = e_i$, $o(p^3) = 3\lambda_i$ and $o(q^2) = 2\mu_i$ along F_i. We may assume that $E_i = \{v = 0\}$, $\rho_n^{-1}(E_i) = \{f_i = z^3 + 3v^{\lambda_i} p_i(u,v)z + 2v^{\mu_i} q_i(u,v) = 0\}$ and the z-discriminant of $f_i = -108(v^{3\lambda_i} p_i^3 + v^{2\mu_i} q_i^2) = -108 v^{e_i} b_i(u,v)$ where $v \nmid p_i$, $v \nmid q_i$ and $v \nmid b_i$.

(I) Assume that $E_i = \{v = 0\}$ satisfies the codition (iii) in Corollary 2.5.

Let the local defining equation of $V^{(n)}$ over E_i be $\{f_i = z^3 + 3v^{\lambda_i} p_i(u,v)z + 2v^{\mu_i} q_i(u,v) = 0\}$. Then $o(B) = e_i = 2\mu_i < 3\lambda_i$ and $2\mu_i \not\equiv 0 \pmod 3$ along E_i. So $C_i = \rho^{-1}(E_i)$ is irreducible, nonsingular and of genus 0. $C_i = A_j$ for some j. To compute $A_j \cdot A_j$ in terms of $E_i \cdot E_i$ let F be a tubular neighborhood of E_i. Let $\rho^{-1}(F) = G$. By [M, pp. 6-13] the fundamental group $\pi_1(F - E_i)$ is isomorphic to a cyclic group of order $-E_i \cdot E_i$. Since ρ is a three-fold covering above $F - E_i$, $\pi_1(G - A_j)$ is a cyclic group whose order is equal to the index 3 of $\pi_1(F - E_i)$. Also [M, pp. 6-13] $\pi_1(G - A_j)$ is cyclic of order $A_j \cdot A_j$. Thus we get $E_i \cdot E_i = 3A_j \cdot A_j$.

$3z_i = m_j$ because if A_j is locally defined by $\{t = 0\}$ then $v = t^3$.

(II) Assume that $E_i = \{v = 0\}$ satisfies the condition (i) or (ii) in Corollary 2.5.

Let the local defining equation of $V^{(n)}$ over E_i be

$$\{f_i = z^3 + 3v^{\lambda_i}p_i(u,v)z + 2v^{\mu_i}q_i(u,v) = 0\}.$$

Then (i) $o(B) = e_i = \text{odd} > 3\lambda_i = 2\mu_i$ or

(ii) $o(B) = e_i = \text{odd} = 3\lambda_i < 2\mu_i$.

Since N is nonsingular above E_i and ρ is two to one over E_i by construction, $\rho^{-1}(E_i)$ must have two disjoint irreducible components, that is, two disjoint spheres using the fact that spheres are simply connected. Take a tubular neighborhood F of E_i. By [M] $\pi_1(F - E_i)$ is a cyclic group of order equal to $-E_i \cdot E_i$. Let $G = \rho^{-1}(F)$. Then $\rho|_{G - C_i} : G - C_i \to F - E_i$ is a three fold covering map. Note that G is disconnected because the set of regular points of $V^{(n)}$ over any neighborhood of E_i is not connected. Let $G = G_j \cup G_k$ where $G_j \supset A_j$, $G_k \supset A_k$, $\rho|_{G_j} : G_j \to F$ is one to one and $\rho|_{G_k} : G_k \to F$ is a two-fold connected branched cover with the branch locus E_i where $\rho^{-1}(E_i) = C_i = A_j \cup A_k$. Then by [M] $2A_k \cdot A_k = E_i \cdot E_i$, and $A_j \cdot A_j = E_i \cdot E_i$. $m_j = z_i$ and $m_k = 2z_i$ as we see in the proof of case (I).

(III) Let $E_i = \{v = 0\}$ be not part of the branch locus of ρ. Since by Proposition 2.3, ρ is $3-1$ over the E_i except possibly for those $(u,v) = (u_j,0)$ with $b_i(u_j,0) = 0$, $j = 1,2,\ldots,k$, $C_i = \rho^{-1}(E_i)$ need not be connected. Note that $\rho^{-1}(E_i)$ is connected if and only if

$\rho^{-1}(E_i - \{(u_1,0),\ldots,(u_k,0)\})$ is connected. We shall prove later that $C_i \cdot C_i = 3E_i \cdot E_i$. Then we have the following three cases:

(i) If $\rho^{-1}(E_i)$ consists of globally three topological components, then let $C_i = A_j \cup A_k \cup A_\ell$. Then similarly as in (I) $m_j = m_k = m_\ell = z_i$. $A_j \cdot A_j = A_k \cdot A_k = A_\ell \cdot A_\ell = E_i \cdot E_i$. Each of C_i is nonsingular of genus 0.

(ii) If $\rho^{-1}(E_i)$ consists of globally two topological components, then let $C_i = A_j \cup A_k$ where ρ is one to one near a neighborhood of A_j and ρ is a two-fold branched cover near a neighborhood of A_k. Then $A_j \cdot A_j = E_i \cdot E_i$. Since $C_i \cdot C_i = 3E_i \cdot E_i$, $A_k \cdot A_k = 2E_i \cdot E_i$. Also $m_j = m_k = z_i$ which can be proved similarly as we see in case (I). A_j is nonsingular of genus 0. Let ℓ be the number of irreducible components of the branch locus of ρ which intersect E_i. Then as in the Riemann-Hurwitz formula, A_k is of genus $(\ell-2)/2$, for $2A - 2B + 2C - \ell = 2 - 2g$ and $A - B + C - 2 = 0$ where A is the number of 0-cell, B is the number of 1-cell, C is the number of 2-cell in a triangulation of E_i and g is the geometric genus of A_k.

(iii) If $\rho^{-1}(E_i)$ is irreducible, then $C_i = A_j$ for some j. Similarly $m_j = z_i$, and $A_j \cdot A_j = 3E_i \cdot E_i$. Let k be the number of irreducible components of the branch locus over which ρ are one to one and which intersect E_i. Let ℓ be the number of irreducible components of the branch locus over which ρ are two to one and which intersect E_i. Then by the Riemann-Hurwitz formula A_j is of genus $(2k + \ell - 4)/2$.

Now to prove $C_i \cdot C_i = 3E_i \cdot E_i$, we may proceed as follows. Recall a resolution by (2.2). Let h be any generic function in the maximal ideal (x,y). Let (h) be the divisor of the pull-back of h under $\tau: M \to \mathbb{C}^2$ in (2.2). Let $(h) = Z_E + W_h$. Recall that $Z_E = \Sigma z_i E_i$ is the fundamental cycle. $Z_E \cdot Z_E = -1$ [Ar, Corollary, p. 135]. In fact by induction, $E_i \cdot Z_E = 0$ except for i = 1 where E_1 is the curve appearing at the initial quadratic transformation at Q. So $z_1 = 1$ and $E_1 \cdot Z_E = -1$. Recall the map $\rho = \rho^{(n)}: N \to M$. Let $(\rho^* h) = X + W_{\rho^* h}$ be the divisor of the pull-back of (h) under ρ. Let us recall that $X = \Sigma m_j A_j$. Then either $X \cdot C_i = 0$ if $W_h \cap E_i = \phi$ or $X \cdot C_i = -3$ if $W_h \cap E_i \neq \phi$, because $C_i \cdot (X + W_{\rho^* h}) = 0$ and ρ is a three-fold branched cover over E_i. If $W_h \cap E_i \neq \phi$, then $E_i = E_1$. Thus we proved $3Z_E \cdot C_i = X \cdot C_i$. If there is part of the branch locus E_k of ρ which intersects E_i let $\rho^{-1}(E_k) = C_k$. Then in case (I) $C_k = A_s$ for some s and note that X has coefficient $3z_k$ on A_s. Thus $3z_k A_s \cdot C_i = 3z_k$. In Case (II) $C_k = A_s \cup A_t = A_s + A_t$ as divisor and X may be assumed to have coefficient z_k on A_s and $2z_k$ on A_t. Thus $(z_k A_s + 2z_k A_t) \cdot C_i = 3z_k$. If there is not part of the branch locus, E_k which intersects E_i then $C_i \cdot C_k = 3$ where $\rho^{-1}(E_k) = C_k$. Therefore $X \cdot C_i = z_i C_i \cdot C_i + \Sigma 3z_k$ where the sum Σ is taken over the set $\{E_i \cap E_k \neq \phi\}$. Since $3Z_E \cdot E_i = X \cdot C_i$ and $Z_E \cdot E_i = z_i E_i \cdot E_i + \Sigma z_k$, $C_i \cdot C_i = 3E_i \cdot E_i$. In fact we also proved that $X \cdot X = -3$.

Let us discuss the case (III) in terms of Proposition 2.3. Then we have the subcases (a), (b), (c) and (d) below;

(a) $o(B) = 3\lambda_i < 2\mu_i$ and $3\lambda_i \equiv 0 \pmod 2$

(b) $o(B) = e_i > 3\lambda_i = 2\mu_i$ and $e_i \equiv 0 \pmod 2$

(c) $o(B) = 2\mu_i < 3\lambda_i$ and $2\mu_i \equiv 0 \pmod 3$

(d) $o(B) = 3\lambda_i = 2\mu_i$

(a) Let $o(B) = o(p^3) = 3\lambda_i < 2\mu_i = o(q^2)$ and $3\lambda_i \equiv 0 \pmod 2$ along E_i.

(a1) If there is no part of the branch locus of ρ which intersects E_i, then $C_i = \rho^{-1}(E_i)$ has globally three components in N. Thus the case (a1) belongs to the case (i) of (III).

(a2) If there is part of the branch locus, E_k which intersects E_i, then we claim that

(1) $o(B) = o(p^3) = 3\lambda_k < 2\mu_k = o(q^2)$, $3\lambda_k \not\equiv 0 \pmod 2$ along E_k

(2) $\rho^{-1}(E_i)$ has globally two irreducible components.

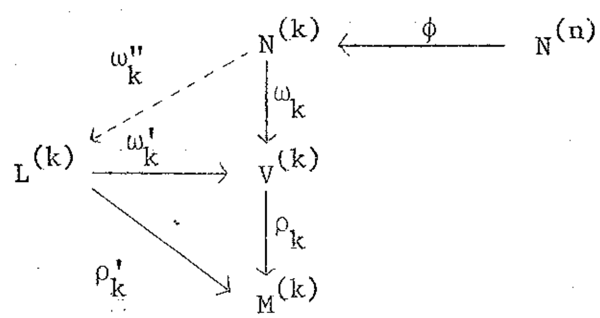
To show (1) is trivial as we see from the proof of Lemma 2.6. To show (2), there is a partial order between those E_i induced by the order of their appearance in a resolution by (2.2). So after blowing down those E_j for which E_j appears later than E_i then we get a local defining equation for $V^{(k)}$, some k, say

$$V^{(k)} = \{z^3 + 3v^\lambda p_k(u,v)z + 2v^\mu q_k(u,v) = 0\}$$

where $E_i = \{v = 0\}$, $\lambda = \lambda_i$ and $\mu = \mu_i$. Note that connectedness of $\rho^{-1}(E_i)$ is just dependent on the regular set of $(\rho^{(k)})^{-1}(E_i)$

in $N^{(k)}$ by (2.2), that is, independent of local coordinates for $V^{(k)}$ over E_i. Since $\lambda \equiv 0 \pmod 2$, let $\lambda = 2\alpha$. Replacing z by $z \cdot v^{\alpha}$ in the local defining equation for $V^{(k)}$ we get $L^{(k)} = \{z^3 + 3p_k(u,v)z + 2v^{\mu-3\alpha}q_k(u,v) = 0\}$. Then recalling (2.2), consider the following diagram (2.3).

(2.3)



Clearly $L^{(k)}$ and $V^{(k)}$ have the same normalization $N^{(k)}$ since the fact that ρ_k and ρ'_k are proper implies that ω'_k is proper, finite and biholomorphic over $V^{(k)} - E_i$. So there exists ω''_k such that $(N^{(k)}, \omega''_k)$ is the normalization of $L^{(k)}$. Since $N^{(k)}$ is the normalization of $V^{(k)}$ and $N^{(n)}$ is a resolution of $V^{(k)}$, there exists $\phi : N^{(n)} \to N^{(k)}$ such that ϕ is holomorphic and unique. But $(\omega'_k)^{-1}(E_i) = \{z^3 + 3p_k(u,0)z = 0\}$ in $L^{(k)}$ is reducible and contains $\{z = 0\}$ in $L^{(k)}$. Since $\{z = 0\}$ is independent of local coordinates needed for blowing up process to get a resolution by (2.2), $(\omega'_k)^{-1}(E_i)$ should have globally at least two components and so $\rho^{-1}(E_i)$ has globally at least two components. But $E_i \cap E_k \neq \phi$ implies that $\rho^{-1}(E_i)$ has at most two components. Thus the case (a2) belongs to (ii) of (III).

(b) Let $o(B) = e_i = \text{even} > 3\lambda_i = 2\mu_i$ along E_i.

(b1) If there is no part of the branch locus of ρ which intersects E_i, then $C_i = \rho^{-1}(E_i)$ has globally three components in N. Thus the case (b1) belongs to (i) of (III).

(b2) If there is part of the branch locus, E_k which intersects E_i, then we claim that

(1) $o(B) = \text{odd} > o(p^3) = o(q^2)$ along E_k

(2) $\rho^{-1}(E_i)$ has globally two irreducible components.

To show (1), it is trivial as we see from the proof of Lemma 2.6. To show (2), we follow the same technique as in the case (a2) and use the same diagram (2.3). Then in this case $L^{(k)} = \{g_k = z^3 + 3p_k(u,u)z + 2q_k(u,v) = 0\}$. Note that g_k has the z-discriminant $-108(p_k^3 + q_k^2)$, $v \nmid p_k$ $v \nmid q_k$ but $v | (p_k^3 + q_k^2)$ by assumption. So $p_k^3(u,0) + q_k^2(u,0) \equiv 0$. Let us look at $(\omega_k')^{-1}(E_i) = \{g_k(u,0,z) = z^3 + 3p_k(u,0)z + 2q_k(u,0) = 0\}$ by (2.3). Since the z-discriminant of $g_k(u,0,z)$ is identically zero, $(\omega_k')^{-1}(E_i)$ is reducible. Thus $\rho^{-1}(E_i)$ has globally two irreducible components because ρ is two to one over E_k. Thus the case (b2) belongs to (ii) of (III).

(c) Let $o(B) = o(q^2) = 2\mu_i < 3\lambda_i = o(p^3)$ and $2\mu_i \equiv 0 \pmod 3$ along with E_i.

(c1) If there is no part of the branch locus of ρ which intersects E_i, then $C_i = \rho^{-1}(E_i)$ has globally three components in N. Thus the case (c1) belongs to (i) of (III).

(c2) If there is part of the branch locus, E_k which intersects E_i, then we claim that

(1) $o(B) = o(q^2) = 2\mu_k < 3\lambda_k = o(p^3)$, $2\mu_k \not\equiv 0 \pmod 3$ along E_k

(2) $\rho^{-1}(E_i)$ is irreducible.

To show (1) is trivial as we see from the proof of Lemma 2.6 and (2) follows immediately from (1) by Corollary 2.5. This case belongs to (iii) of (III).

(d) Let $o(B) = 3\lambda_i = 2\mu_i$ along E_i.

(d1) If there is no part of the branch locus of ρ which intersects E_i, then $C_i = \rho^{-1}(E_i)$ has globally three components in N. Thus the case (d1) belongs to (i) of (III).

(d2) If there is part of the branch locus of ρ which intersects E_i, then there are two possibilities:

(1) There is at least one irreducible component of the branch locus of ρ over which ρ is one to one.

(2) There is no such a component.

Consider the case (1). It is obvious that $C_i = \rho^{-1}(E_i)$ is irreducible. So the case (1) belongs to (iii) of (III). Now consider the case (2). In this case, there might be globally one component or two components of $\rho^{-1}(E_i)$. To discuss it more, we follow the same technique as in (a2) and use the same diagram (2.3). Then in this case $L^{(k)} = \{g_k = z^3 + 3p_k(u,v)z + 2q_k(u,v) = 0\}$. Similarly as in the case (a2), $L^{(k)}$ and $V^{(k)}$ have the same normalization $N^{(k)}$. Note that $\rho^{-1}(E_i)$ is connected if and only if the regular set of $(\omega_k')^{-1}(E_i)$ is connected by (2.3). $(\omega_k')^{-1}(E_i) = \{z^3 + 3p_k(u,0)z + 2q_k(u,0) = 0\}$. Thus connectedness of $\rho^{-1}(E_i)$ is just dependent on the global irreducibility of the plane curve $(\omega_k')^{-1}(E_i)$. Thus the case (2) of (d2) belongs to either (ii) or (iii) of (III). For the case (2) of (d2), examples of Proposition 3.4 will be seen to satisfy that $\rho^{-1}(E_i)$ is connected, and on the other hand if P is a triple point of V but V is reducible near P then observe that $\rho^{-1}(E_i)$ is not connected.

Now, let us recall that $C_1 = \rho^{-1}(E_1)$ where E_1 is the curve appearing at the initial quadratic transformation at Q. For $A_j \not\subset C_1$, $A_j \cdot X = 0$. In case (I) with $A_1 = C_1$, $m_1 = 3$. In case (II) with $C_1 = A_1 \cup A_2$ we may say that $m_1 = 1$ and $m_2 = 2$ where $A_1 \cdot A_1 = E_1 \cdot E_1$ and $2A_2 \cdot A_2 = E_1 \cdot E_1$. In case (III) for each $A_j \subset C_1$, $m_j = 1$.

There is a partial ordering < between the A_j induced by the order of their appearance in a resolution by (2.2). For each A_j in N, there is a least m, call it m(j), for which A_j is the proper transform of a curve in $N^{(m)}$.

<u>Definition 2.7</u> $A_j < A_k$ if $m(j) < m(k)$ for all possible resolution processes by (2.2).

<u>Definition 2.8</u> A_k follows A_j if $A_j < A_k$ and there does not exist A_i with $A_j < A_i < A_k$.

§3. From Resolution to Triple Point

<u>Proposition 3.1</u> Let $\omega : \tilde{V} \to V$ be a normalization of V, a two-dimensional analytic space. V may have a nonisolated irreducible singular point P and $\omega^{-1}(P)$ is the only singular point of $\tilde{V}$. Let $r : N \to V$ be a resolution of V. Let $A = r^{-1}(P)$ be the exceptional set. A is connected. Let $A = \cup A_i$, $1 \leq i \leq n$, be its decomposition into irreducible components. Let m be the maximal ideal of P. Suppose that $r^*(m)$ is locally principal, i.e. that the sheaf $r^*(m) = r^*(m)\mathcal{O}$ on N is locally free of rank 1. Let $X = \Sigma m_i A_i$, $1 \leq i \leq n$, be the divisor of $r^*(m)$. Let N' be obtained from V by blowing up V at P and then normalizing. Then N' may also be obtained by blowing down those A_i in N such that $A_i \cdot X = 0$.

<u>Proof</u> The same proof as that of Proposition 5.1 of Laufer's paper [L5, p. 322].

Recall $\omega_1 : N' \to V'$, $\rho_1 : V' \to M'$ and $\rho' = \rho_1 \circ \omega_1 : N' \to M'$ defined as in Lemma 1.12. Assume that P' is a singular point of V' and the corresponding Q' (induced by ρ_1) is a singular point of the branch locus of ρ'. But N' may be regular above Q or may have a double or triple point above Q. Note that P' is still a triple point of V'. Let $r' : N^* \to V'$ be a resolution by (2.2) near P' (may be induced by $r : N \to V$). If X^* is the divisor of $(r')^*(m)$ where m is its maximal ideal at P', then we want $-X^* \cdot X^* = 3$ which can be proved in (*) later. If V' is irreducible near P' then supp X^* is connected.

Anyhow by Proposition 3.1 and (2.2) we may study a resolution by (2.2) by studying $r^*(m)$ on any resolution $r : N \to V$ near P for which $r^*(m)$ is principal where P is a triple point of V and m is its maximal ideal. Recall the result of Wagreich [W1, p. 426], if V is normal at P and X is the divisor of $r^*(m)$ with $r^*(m)$ principal, then $-X \cdot X$ is the multiplicity of P. Therefore we need to extend this result to the following form (*).

(*): Let V be a purely two-dimensional analytic space near $P = (0,0,0)$ in $\mathbb{C}^3$. V may not be normal at P. Let m be its maximal ideal and let $r : N \to V$ be a resolution near P. If $r^*(m)$ is principal and X is the divisor of $r^*(m)$, we claim that $-X \cdot X$ is the multiplicity of P.

Proof of (*) By Definition 1.11, the multiplicity of V near P is defined to be the degree of the cover of a generic projection λ on a domain in $\mathbb{C}^2$. Let f and g be generic elements of m. Consider $(f,g) : V \to \mathbb{C}^2$. Let $(f) = X + W_f$ and $(g) = X + W_g$ with $W_f \cap W_g = \phi$ where (f) and (g) are the divisors under the pull-back r^*. To find the number of points in $\lambda^{-1}(a,b)$ with (a,b) small and generic we may choose a new local coordinate (f',g') by linear change of coordinates for which $(f'(z),g'(z)) = (0,\epsilon)$ instead of $(f(z),g(z)) = (a,b)$. Note that f' and g' are still generic elements of m. Then we may write $(f') = X + W_{f'}$ and $(g') = X + W_{g'}$ with $W_{f'} \cap W_{g'} = \phi$ under the pull-back r^*. Note that the number of components of $W_{f'}$ meeting A_i is $-X \cdot A_i$. But $g' \circ r$ has a zero of order x_i on A_i. Thus

$g' \circ r = \epsilon$ appears x_i times on each component of W_f meeting A_i. Therefore the number of points in $\lambda^{-1}(0,\epsilon)$ is $\Sigma(-X \cdot A_i)x_i = -\Sigma X \cdot x_i A_i$ $-X \cdot X$.

Now Z be the fundamental cycle for an arbitrary resolution of P. $r^*(m) \subset \mathcal{O}(-Z)$. Also [W1, Theorem 2.7, p. 426], the multiplicity of P is at least $-Z \cdot Z$. Since for P a triple point $-X \cdot X = 3 \geq -Z \cdot Z$, $Z \cdot Z$ may be -1, -2, or -3. Consider a resolution such that $r^*(m)$ is principal. $X = Z + D$ with $D \geq 0$. $X \cdot X = Z \cdot Z + 2Z \cdot D + D \cdot D$. $D \cdot D \leq 0$, since the intersection matrix for the A_j is negative definite. $D \cdot D = 0$ if and only if $D = 0$. $Z \cdot D \leq 0$ by the definition of Z.

<u>Lemma 3.2</u> If P is a triple point and $Z \cdot Z = -3$, then on any resolution, $r^*(m)$ is principal and equal to $\mathcal{O}(-Z)$.

<u>Proof</u> If $r^*(m)$ is not principal, let $Y = \Sigma m_i A_i$, where m_i is the order to which functions $g \circ r$, $g \in m$, generically vanish on A_i. Let $r_1 : N_1 \to V$ be a resolution on which $r_1^*(m)$ is principal. Let $r_1 = r \circ \pi$ Then on N_1, letting π^* denote the pull-back, $X > \pi^* Y \geq \pi^* Z = Z_1$. Since $X \cdot X = -3 < \pi^* Y \cdot \pi^* Y \leq Z_1 \cdot Z_1 = -3$, it is a contradiction. So $r^*(m)$ is principal and $X = Z$.

Thus to describe $r^*(m)$ for triple points, there only remain the cases $Z \cdot Z = -2$ and $Z \cdot Z = -1$. If $r^*(m)$ is principal, then $X > Z$. Recall that $X = \Sigma m_j A_j$, $1 \leq j \leq n$, is the divisor of $r^*(m)$ and that $A = \cup A_j$, $1 \leq j \leq n$.

Definition 3.3 A cycle D on A is a integral combination of the A_i, i.e., $D = \Sigma d_i A_i$, $1 \leq i \leq n$, with d_i an integer. In the following cycle will always mean a cycle on A. There is a natural partial ordering, denoted by $<$, between cycles defined by comparing the coefficients. We shall only be considering cycles $D \geq 0$. Let $\text{supp}\, D = \cup A_i$, $d_i > 0$, denote the support of D. Let L and K be two cycles on A where $L = \Sigma \ell_i A_i$ and $K = \Sigma k_i A_i$. The cycle Min(L,K) is defined by $\Sigma \min(\ell_i, k_i) A_i$ where $\min(\ell_i, k_i)$ is the minimum of ℓ_i and k_i. For any integer λ, λD is defined by $\Sigma \lambda d_i A_i$. $D|_{\text{supp}\, E}$ means the cycle restricted to the supp E where E is a cycle on A.

Proposition 3.4 Let $V = \{(x,y,z) : z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ be an analytic space with $P = (0,0,0) \in V$ and P an irreducible singular point. Let $\omega : \tilde{V} \to V$ be the normalization of V and $\omega^{-1}(P)$, the only singular point of $\tilde{V}$. Let $r : N \to V$ be a resolution by (2.2). Let $A = r^{-1}(P)$ be the exceptional set. A is connected. Let $A = \cup A_i$, $1 \leq i \leq n$, be its decomposition into irreducible components. Let $X = \Sigma m_j A_j$, $1 \leq j \leq m$, be the divisor of the pull-back under $r : N \to V$ of the ideal (x,y) on V. Then $X \cdot X = -3$. Let Z be the fundamental cycle on $\cup A_i$, $1 \leq i \leq n$. Assume that $Z \cdot Z = -2$ or -1. Let us discuss the structure Z and X in terms of the A_i. There are two cases, (I) and (II) below.

(I) If $Z \cdot Z = -2$, then we have the subcases (A), (B) and (C).

(A) There exists A_1 such that $Z \cdot A_1 = -2$ and A_1 has coefficient 1 in Z. Let $X = Z + D$. Then D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1$. $D \cdot D = -1$. Let A_2 be such that $D \cdot A_2 = -1$. Then A_2 has coefficient 1 in D. So A_1 and A_2 have coefficients 1 and 2 in X, respectively. $X \cdot A_1 = X \cdot A_2 = -1$.

(B) There is $A_1 \neq A_2$ such that $Z \cdot A_1 = Z \cdot A_2 = -1$ and A_1 and A_2 have coefficients 1 in Z. Then $X = Z + D$ where D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1,2$. $D \cdot D = -1$. Let A_3 be such that $D \cdot A_3 = -1$. Then $X \cdot A_3 = -1$. So A_3 has coefficient either 2 or 3 in X.

(B1) If A_3 has coefficient 3 in X, then $X \cdot A_1 = X \cdot A_2 = 0$. A_1, A_2 and A_3 have coefficients 1, 1 and 3 in X, respectively.

(B2) If A_3 has coefficient 2, then either ($X \cdot A_1 = -1$ and $X \cdot A_2 = 0$) or ($X \cdot A_1 = 0$ and $X \cdot A_2 = -1$). A_1, A_2 and A_3 have coefficients 1, 1 and 2 in X, respectively.

(B1) Consider the case that A_3 has coefficient 3 in X. Then A_1 and A_2 have coefficient 1 in $Z - 2D$. $A_3 \not\subset \mathrm{supp}(Z - 2D)$. $(Z - 2D) \cdot A_1 = (Z - 2D) \cdot A_2 = -3$ and $(Z - 2D)^2 = -6$. $\mathrm{Supp}(Z - 2D)$ is not connected. In fact, $\mathrm{supp}(Z - 2D)$ is a union of two disjoint connected components of $\cup A_i$, $i \neq 3$ where one component C_1 contains A_1 and the other component C_2 contains A_2. Also $\mathrm{supp}\, D$ is a unique component of $\cup A_i$, $i \neq 1,2$ which intersects both A_1 and A_2. Let Z_1 and Z_2 be the fundamental cycles on C_1 and C_2, respectively. Then $Z_1^2 = Z_1 \cdot A_1 = -3$ and $Z_2^2 = Z_2 \cdot A_2 = -3$.

(B2) Consider the case that A_3 has coefficient 2 in X. Then we may assume without loss of generality that $X \cdot A_1 = -1$ and

$X \cdot A_2 = 0$. Then $\operatorname{supp} D$ is a connected component of $\cup A_i$, $i \neq 2$ which does not contain A_1. Also there exists a unique component C of $\cup A_i$, $i \neq 1,3$ which intersects A_1 and A_3 both. Then $A_2 \subset C$. Let Z_* be the fundamental cycle on C. Then $Z_*^2 = -3$ and A_2 has coefficient 1 in Z_*. Either $Z_* \cdot A_2 = -2$, $Z_* \cdot A_t = -1$, where A_2 and A_t have coefficients 1 in Z_* or $Z_* \cdot A_2 = -3$. If $Z_* \cdot A_2 = -2$ and $Z_* \cdot A_t = -1$, then let D_* be the fundamental cycle of the connected component of $\cup A_i$, $i \neq 1,2$ which contains A_t (and intersects both A_1 and A_2). Then $D_* \cdot D_* \neq -1$.

(C) There exists A_1 such that $Z \cdot A_1 = -1$ and A_1 has coefficient 2 in Z. Then $X = Z + D$ where D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1$. $D \cdot D = -1$. Let A_2 be such that $D \cdot A_2 = -1$. Then $X \cdot A_2 = -1$ with A_2 coefficient 3 in X. A_1 has coefficient 2 in $Z - 2D$ and A_2 is not contained in $\operatorname{supp}(Z-2D)$. $(Z-2D) \cdot A_1 = -3$ and $(Z-2D)^2 = -6$. So $\operatorname{supp}(Z-2D)$ is the connected component of $\cup A_i$, $i \neq 2$ which contains A_1. Let Z_* be the fundamental cycle on $\operatorname{supp}(Z-2D)$. Then $Z_*^2 = -2$ or -3 and also $Z_* \cdot A_1 = 0$ or -1. Let X_* be the cycle on $\operatorname{supp}(Z-2D)$ such that $X_* \cdot A_i \leq 0$ for all $A_i \subseteq \operatorname{supp}(Z-2D)$ and $X_* \cdot X_* = -3$. Then $X_* \cdot A_1 = 0$. So we have the following possibilities (C1), (C2), (C3), (C4) and (C5).

(C1) Let $A_s \neq A_1$ be such that $X_* \cdot A_s = -3$. Then $X_* \cdot A_2 = 1$. A_s has coefficient 1 in X and then $A_s \cap \operatorname{supp} D = \phi$. Since $Z - 2D > X_*$, let $Z - 2D = X_* + F$. Then $F^2 = F \cdot A_1 = -3$.

$F \cdot A_2 = 1$ and $F \cdot A_s = 3$. So F is the fundamental cycle on the connected component of $\cup A_i$, $i \neq 2, s$, which contains A_1. Consider the cycle $X_* - F$. $(X_* - F) \cdot A_s = -6$. A_s has coefficient 1 in $X_* - F$ and $(X_* - F)^2 = -6$. Thus $\mathrm{supp}(X_* - F)$ is the connected component of $\cup A_i$, $i \neq 1$ which contains A_s and $X_* - F$ is the fundamental cycle on its support.

(C2) Let $A_s \neq A_t$ be such that $X_* \cdot A_s = -2$ and $X_* \cdot A_t = -1$. Then $X_* \cdot A_2 = 1$. A_s and A_t have coefficients 1 in X. So A_s and A_t are not contained in $\mathrm{supp}\, D$. Since $Z - 2D > X_*$, let $Z - 2D = X_* + F$. Then $F^2 = F \cdot A_1 = -3$, $F \cdot A_2 = 1$, $F \cdot A_s = 2$ and $F \cdot A_t = 1$. So F is the fundamental cycle on the connected component of $\cup A_i$, $i \neq 2, s, t$ which contains A_1. Let us consider the cycle $X_* - F$. Then $(X_* - F) \cdot A_s = -4$, $(X_* - F) \cdot A_t = -2$ and $(X_* - F)^2 = -6$ since A_s and A_t have coefficient 1 in $X_* - F$. So there is no fundamental cycle with its self-intersection number -1 on any component of $\mathrm{supp}(X_* - F)$.

(C3) Let A_{s1}, A_{s2} and A_{s3} be distinct with $X_* \cdot A_{s1} = X_* \cdot A_{s2} = X_* \cdot A_{s3} = -1$. So A_{s1}, A_{s2} and A_{s3} have coefficient 1 in X and are not contained in $\mathrm{supp}\, D$. Since $Z - 2D > X_*$, let $Z - 2D = X_* + F$. Then $F^2 = F \cdot A_1 = -3$, $F \cdot A_2 = 1$ and $F \cdot A_{s1} = F \cdot A_{s2} = F \cdot A_{s3} = 1$. So $\mathrm{supp}\, F$ is the connected component of $\cup A_i$, $i \neq 2, s1, s2, s3$ which contains A_1. Consider the cycle $X_* - F$. Then $(X_* - F) \cdot A_{s1} = (X_* - F) \cdot A_{s2} = (X_* - F) \cdot A_{s3} = -2$, $(X_* - F)^2 = -6$ and A_{s1}, A_{s2} and A_{s3} have coefficient 1 in $X_* - F$. So there is no fundamental cycle with its self-intersection number -1 on any component of $\mathrm{supp}(X_* - F)$.

(C4) Let A_s be such that $X_*\cdot A_s = -1$ with coefficient 3 in X_*. Then $X_*\cdot A_2 = 1$. So A_s has coefficient 3 in X and then A_s is not contained in supp D. Since A_s has coefficient 3 in $Z-2D$, $Z-2D > X_*$. Let $Z-2D = X_*+F$. Then $F^2 = F\cdot A_1 = -3$, $F\cdot A_2 = 1$ and $F\cdot A_s = 1$. So supp F is the connected component of $\cup A_i$, $i \neq 2,s$ which contains A_1. Consider the cycle X_*-F. Then $(X_*-F)\cdot A_s = -2$, $(X_*-F)\cdot A_1 = 3$ and A_s has coefficient 3 in X. A_1 is not contained in $\mathrm{supp}(X_*-F)$. $(X_*-F)^2 = -6$. So $\mathrm{supp}(X_*-F)$ is the connected component of $\cup A_i$, $i \neq 1$ which contains A_s. Note that there is no fundamental cycle L with $L^2 = -1$ on $\mathrm{supp}(X_*-F)$.

(C5) Let $A_s \neq A_t$ be such that $X_*\cdot A_s = X_*\cdot A_t = -1$ where A_s and A_t have coefficients 2 and 1 in X respectively. Then $X_*\cdot A_2 = 1$. So A_s and A_t have coefficients 2 and 1 in X respectively. A_s and A_t are not contained in supp D. Since A_s and A_t have coefficients 2 and 1 in $Z-2D$, $Z-2D > X_*$. Let $Z-2D = X_*+F$. Then $F^2 = F\cdot A_1 = -3$, $F\cdot A_2 = F\cdot A_s = F\cdot A_t = 1$. So supp F is the connected component of $\cup A_i$, $i \neq 2,s,t$ which contains A_1. Consider the cycle X_*-F. Then $(X_*-F)\cdot A_s = (X_*-F)\cdot A_t = -2$ and $(X_*-F)^2 = -6$ since A_s and A_t have coefficient 2 and 1 in X_*-F respectively. Moreover X_* is the fundamental cycle on $\mathrm{supp}(Z-2D)$ and there is no fundamental cycle L with $L\cdot L = -1$ on any connected component of $\mathrm{supp}(X_*-F)$.

(II) If $Z \cdot Z = -1$, then let A_1 be such that $A_1 \cdot Z = -1$. Then $X = Z + D$ where D is a cycle on a connected component of $\cup A_i$, $i \neq 1$. $D \cdot D = -2$. Then there exists A_2 such that $D \cdot A_2 = -1$ and that A_2 has coefficient 2 in D. $X \cdot A_2 = -1$ and A_2 has coefficient 3 in X. $A_1 \cap A_2 = \phi$. Let $G = \min(Z|_{\mathrm{supp}\, D}, D)$. Then $G \cdot G = -1$ and so G is the fundamental cycle on supp D. Let Z_* be the fundamental cycle on the connected component of $\cup A_i$, $i \neq 2$ which contains A_1. Then $Z_* \cdot Z_* = -2$, $Z_* \cdot A_1 = -1$ and A_1 has coefficient 1 in Z_*. Let A_p be such that $G \cdot A_p = -1$ and then A_p has coefficient 1 in G. $Z_* \cdot A_p = -1$ and A_1 and A_p have coefficient 1 in Z_*. Let X_* be the cycle on $\mathrm{supp}\, Z_*$ such that $X_* \cdot A_i \leq 0$ for all $A_i \subset \mathrm{supp}\, Z_*$ and $X_* \cdot X_* = -3$. Then we have the following two possibilities:

(A) Let A_t be such that $X_* \cdot A_t = -1$ and that A_t has coefficient 3 in X_*. $\mathrm{Supp}(D - G)$ is a connected component of $\cup A_i$, $i \neq p$ which does not contain A_1 and $D - G$ is the fundamental cycle on its support with $(D - G)^2 = (D - G) \cdot A_2 = -1$. Also $X_* - Z_*$ is the fundamental cycle with $(X_* - Z_*)^2 = (X_* - Z_*) \cdot A_t = -1$ on the connected component of $\cup A_i$, $i \neq 1, p$ which intersects both A_1 and A_p. A_t follows A_2, A_1 follows A_t and A_p follows A_t.

(B) Let X_* be such that $X_* \cdot A_t = X_* \cdot A_1 = -1$ and that A_1 and A_t have coefficients 1 and 2 in X, respectively. Let $X_* = Z_* + G_*$. Then $\mathrm{supp}\, G_*$ is a connected component of $\cup A_i$, $i \neq p$ which does not contain A_1 and A_2 both. $G_*^2 = G_* \cdot A_t = -1$. Also

$\text{supp}(D - G)$ is a connected component of $\cup A_i$, $i \neq p$ which does not contain A_1 and A_t. $(D - G)^2 = (D - G)\cdot A_2 = -1$. A_t and A_1 follow A_p at the same time and A_p follows A_t and A_1.

Also, we can prove the existence of such cases by providing examples.

Proof of Proposition 3.4

(I) Let $Z\cdot Z = -2$.

(A) Let A_1 be such that $Z\cdot A_1 = -2$ with coefficient 1 in Z. By the definition of Z, $X = Z + D$ with $D > 0$. Since $-3 = X\cdot X = Z\cdot Z + 2Z\cdot D + D\cdot D$ and $Z\cdot Z = -2$, $Z\cdot D \leq 0$ and $D\cdot D < 0$ imply that $Z\cdot D = 0$ and $D\cdot D = -1$. So $A_1 \not\subset \text{supp}\, D$ since $Z\cdot D = 0$. For $j \neq 1$, $Z\cdot A_j = 0$ implies $D\cdot A_j = (X - Z)\cdot A_j \leq 0$. So $D\cdot A_j \leq 0$ for $A_j \subset \text{supp}\, D$. Since $D\cdot D = -1$, let A_2 be such that $D\cdot A_2 = -1$. $X\cdot A_2 = (Z + D)\cdot A_2 = -1$. Since $X\cdot X < X\cdot 2A_2 = -2$, A_2 has coefficient either 3 or 2 in X. But we claim that A_2 must have coefficient 2 in X. If A_2 had coefficient 3 in X, then $X\cdot X = X\cdot 3A_2 = -3$ and so $X\cdot A_i = 0$ for $i \neq 2$. Therefore $0 = X\cdot A_1 = (Z + D)\cdot A_1 = -2 + D\cdot A_1$ would imply $D\cdot A_1 = 2$. Since $\text{supp}\, D$ is a component of $\cup A_i$, $i \neq 1$, let $G = \min(Z|_{\text{supp}\, D}, 2D)$. Note that A_2 has coefficient 2 in Z and 1 in D. Then $G \leq 2D$ and $G\cdot A_2 \leq 2D\cdot A_2 = -2$. Since $G\cdot G \leq G\cdot 2A_2 \leq -4$ and $4D\cdot D = -4$, $G\cdot G = -4$ and thus $G = 2D$. Thus $Z|_{\text{supp}\, D} \geq 2D$. Consider $Z - 2D$. Then $\text{supp}(Z - 2D)$ is the connected component of $\cup A_i$, $i \neq 2$ containing A_1, because $(Z - 2D)^2 = -6$, $(Z - 2D)\cdot A_1 = -6$,

$(Z-2D)\cdot A_i = 0$, $i \neq 1,2$, $(Z-2D)\cdot A_2 = 2$ and $A_2 \not\subset \mathrm{supp}(Z-2D)$. Since A_1 has coefficient 1 in $Z-2D$, $Z-2D$ would be the fundamental cycle on that component and thus it leads to a contradiction because $(Z-2D)^2$ must be ≥ -3 by Proposition 3.1 and by (*) below Proposition 3.1. Let A_2 have coefficient 2 in X. Then there is A_k with $k \neq 2$ such that $X\cdot A_k = -1$ and so A_k has coefficient 1 in X. If $Z\cdot A_k = 0$, then $X = Z + D$ would imply $D\cdot A_k = -1$ for $k \neq 2$. So $Z\cdot A_k < 0$. Therefore $A_k = A_1$. Since $X\cdot A_1 = -1$ and $Z\cdot A_1 = -2$, $D\cdot A_1 = 1$. In fact, D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1$.

(B) Let $A_1 \neq A_2$ be such that $Z\cdot A_1 = Z\cdot A_2 = -1$. Then A_1 and A_2 have coefficient 1 in Z. By the definition of Z, $X = Z + D$ with $D > 0$. Then by the same argument as in the case (A), D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1,2$ with $D\cdot D = -1$. Let A_3 be such that $D\cdot A_3 = -1$. Then $X\cdot A_3 = -1$. Since $X\cdot X < X\cdot 2A_3 = -2$, A_3 has coefficient either 3 or 2 in X.

(B1) If A_3 has coefficient 3 in X, then $X\cdot X = X\cdot 3A_3 = -3$ and so $X\cdot A_1 = X\cdot A_2 = 0$ implies $D\cdot A_1 = D\cdot A_2 = 1$. Let $G = \min(Z|_{\mathrm{supp}\,D}, 2D)$. Then $G \leq 2D$. Since A_3 has coefficient 2 in $Z|_{\mathrm{supp}\,D}$ and also in 2D, $G\cdot A_3 \leq 2D\cdot A_3 = -2$. $G\cdot G \leq G\cdot 2A_3 \leq -4$ and $G\cdot G \geq 4D\cdot D = -4$ imply $G = 2D$. Thus $Z|_{\mathrm{supp}\,D} \geq 2D$. Consider the cycle $Z-2D$. Note that A_1 and A_2 have coefficient 1 in $Z-2D$.
$(Z-2D)\cdot A_1 = (Z-2D)\cdot A_2 = -3$, $A_3 \not\subset \mathrm{supp}(Z-2D)$ and $(Z-2D)\cdot A_3 = 2$. So $\mathrm{supp}(Z-2D)$ cannot be connected. If not,

$Z-2D$ would be the fundamental cycle on a connected component of $\cup A_i$, $i \neq 3$ with $(Z-2D)^2 = -6$ and it is impossible by Proposition 3.1 and (*). Since the intersection matrix for the A_i is negative definite, $\mathrm{supp}(Z-2D)$ is a union of two disjoint connected components of $\cup A_i$, $i \neq 3$ where one component C_1 contains A_1 and the other component C_2 contains A_2. Let Z_1 and Z_2 be the fundamental cycles on C_1 and C_2 respectively. Then clearly $Z_1^2 = Z_1 \cdot A_1 = -3$ and $Z_2^2 = Z_2 \cdot A_2 = -3$. Moreover, $\mathrm{supp}\, D$ is the unique component of $\cup A_i$, $i \neq 1,2$ which intersects both A_1 and A_2, otherwise $\mathrm{supp}(Z-2D)$ would be connected.

(B2) If A_3 has coefficient 2 in X, then $X \cdot A_j = -1$ for some $j \neq 3$ because $X \cdot X < X \cdot 2A_3 = -2$. If $X \cdot A_j < 0$ for some $j \neq 1,2,3$ then $X = Z + D$ would imply that $D \cdot A_j < 0$ for some $j \neq 1,2,3$. It is impossible. So either $(X \cdot A_1 = -1$ and $X \cdot A_2 = 0)$ or $(X \cdot A_1 = 0$ and $X \cdot A_2 = -1)$. Thus without loss of generality we may assume that $X \cdot A_1 = -1$ and $X \cdot A_2 = 0$. Note that A_1 and A_2 have coefficient 1 in X and A_1 and A_2 are not contained in $\mathrm{supp}\, D$. $X = Z + D$ implies $D \cdot A_1 = 0$ and $D \cdot A_2 = 1$. $\mathrm{supp}\, D \cap A_1 = \phi$ but $\mathrm{supp}\, D \cap A_2 \neq \phi$. Therefore $\mathrm{supp}\, D$ is a connected component of $\cup A_i$, $i \neq 2$ which does not meet A_1 and intersects A_2. So there is a unique component C of $\cup A_i$, $i \neq 1,3$ which intersects both A_1 and A_3 because $D \cdot A_1 = 0$. $A_2 \subset C$. Let Z_* be the fundamental cycle on C. Since $Z_* < Z$ and A_2 has coefficient 1 in Z, A_2 has coefficient 1 in Z_* and $Z \cdot A_2 = -1$ implies $Z_* \cdot A_2 < 0$. So we have the following possibilities:

(i) $Z_*\cdot Z_* = -1$, (ii) $Z_*\cdot Z_* = -2$ and (iii) $Z_*\cdot Z_* = -3$.

But we claim that $Z_*\cdot Z_* = -3$. Note that $Z_*\cdot A_1 > 0$ and $Z_*\cdot A_3 > 0$.

(i) If $Z_*\cdot Z_* = -1$, then $Z_*\cdot A_2 = -1$.

$0 = X\cdot Z_* = Z_*\cdot x = Z_*\cdot(A_1 + A_2 + 2A_3) = Z_*\cdot A_1 - 1 + 2Z_*A_3 > 0$.

It is a contradiction.

(ii) If $Z_*\cdot Z_* = -2$, then $Z_*\cdot A_2 = -1$ or -2.

If $Z_*\cdot A_2 = -2$, then $0 = X\cdot Z_* = Z_*\cdot X = Z_*\cdot(A_1 + A_2 + 2A_3) = Z_*\cdot A_1 - 2 + 2Z_*\cdot A_3 > 0$. It is a contradiction. If $Z_*\cdot A_2 = -1$, then $Z_*\cdot A_s = -1$ for some $s \neq 2$. Consider $X - Z_*$. Then $A_2 \not\subset \mathrm{supp}(X - Z_*)$ and $X\cdot A_2 = 0$ implies $(X - Z_*)\cdot A_2 = 1$. Now we claim that $A_s \subset \mathrm{supp}(X - Z_*)$. Let $X = \Sigma m_i A_i$, $1 \leq i \leq n$. Note that $(X - Z_*)^2 = X\cdot X - 2Z_*\cdot X + Z_*\cdot Z_* = -5$, $(X - Z_*)\cdot A_1 = -1 - Z_*\cdot A_1$ and $(X - Z_*)\cdot A_3 = -1 - Z_*\cdot A_3$. So $(X - Z_*)^2 = (X - Z_*)\cdot[A_1 + 2A_3 + (m_s - 1)\cdot A_s] = -1 - Z_*\cdot A_1 - 2 - 2Z_*\cdot A_3 + (m_s - 1) = -5$. Since $Z_*\cdot A_1 > 0$ and $Z_*\cdot A_3 > 0$, $m_s - 1 > 0$. Now A_1 and A_3 belong to the different components of $\cup A_i$, $i \neq 2$. Therefore $(X - Z_*)\cdot A_2$ would be greater than 1 because A_1 and A_3 have coefficients 1 and 2 in $X - Z_*$, respectively and $A_s \subset \mathrm{supp}(X - Z_*)$. Thus we would get $1 = (X - Z_*)\cdot A_2 > 1$. It is absurd.

(iii) If $Z_*\cdot Z_* = -3$, then $Z_*\cdot A_2 = -1, -2$, or -3. Then consider the following subcases:

(iiia) $Z_*\cdot A_2 = -1$, $Z_*\cdot A_t = -2$ and A_2 and A_t have coefficient both 1 in Z_*.

(iiib) $Z_*\cdot A_2 = Z_*\cdot A_t = -1$ and A_2 and A_t have coefficients 1 and 2 in Z_* respectively.

(iiic) $Z_*\cdot A_2 = Z_*\cdot A_{21} = Z_*\cdot A_{22} = -1$ and A_2, A_{21} and A_{22} have coefficient 1 in Z_*.

(iiid) $Z_*\cdot A_2 = -2$, $Z_*\cdot A_t = -1$ and A_2 and A_t have coefficient 1 in Z_*.

(iiie) $Z_*\cdot A_2 = -3$ and A_2 has coefficient 1 in Z_*.

(iiia) Since A_2 has coefficient 1 in X, A_t also has coefficient 1 in X from case (III) of section 2. $0 = X\cdot Z_* = Z_*\cdot X$ $= Z_*\cdot(A_2 + A_t + A_1 + 2A_3) = -1 - 2 + Z_*\cdot A_1 + 2Z_*\cdot A_3$ implies $Z_*\cdot A_1 = Z_*\cdot A_3 = 1$. Note that A_2 and A_t are not in supp D. Consider $X - Z_*$. Then A_1 and A_3 have coefficients 1 and 2 in $X - Z_*$ respectively, $A_t \not\subset \text{supp}(X - Z_*)$ and $A_2 \not\subset \text{supp}(X - Z_*)$. Observe that $(X - Z_*)^2 = -6$, $(X - Z_*)\cdot A_1 = -1 - Z_*\cdot A_1 = -2$, $(X - Z_*)\cdot A_t = 2$, $(X - Z_*)\cdot A_3 = -1 - Z_*\cdot A_3 = -2$ and $(X - Z_*)\cdot A_2 = 1$. Also $X - Z_* = (Z + D) - Z_* = Z - Z_* + D > D$. Since A_3 has coefficient 2 in $X - Z_*$, $[(X - Z_*)|_{\text{supp } D}]^2 = -4$. Thus $X - Z_* = 2D$ because $(X - Z_*)\cdot A_3 = -2$. But $(X - Z_*)\cdot A_2 = 1$ and $2D\cdot A_2 = 2$. It is absurd.

(iiib) Let $Z_*\cdot A_2 = Z_*\cdot A_t = -1$ and A_2 and A_t have coefficients 1 and 2 in Z_* respectively. Since A_2 has coefficient 1, A_t has coefficient 2 in X, too, from case (II) of section 2. $0 = X\cdot Z_* = Z_*\cdot X = Z_*\cdot(A_1 + 2A_3 + A_2 + 2A_t) = Z_*\cdot A_1 + 2Z_*\cdot A_3 - 3.$

Thus $Z_* \cdot A_1 = Z_* \cdot A_3 = 1$. Also $A_2 \not\subset \text{supp } D$ and $A_t \not\subset \text{supp } D$ since $A_2 \not\subset \text{supp}(X - Z_*)$, $A_t \not\subset \text{supp}(X - Z_*)$ and $Z \geq Z_*$. $(X - Z_*)^2 = -6$, $(X - Z_*) \cdot A_1 = -1 - Z_* \cdot A_1 = -2$, $(X - Z_*) \cdot A_3 = -1 - Z_* \cdot A_3 = -2$ and $(X - Z_*) \cdot A_2 = (X - Z_*) \cdot A_t = 1$. Also $X - Z_* = (Z + D) - Z_* = Z - Z_* + D > D$. So $\text{supp}(X - Z_*)$ is a sunion of two disjoint components, since supp D is a connected component of $\cup A_i$, $i \neq 2$ which does not contain A_1 and contains A_3. Since $(X - Z_*) \cdot A_3 = -2$ and A_3 has coefficient 2 in $X - Z_*$, $(X - Z_*)\big|_{\text{supp } D} = 2D$. But $(X - Z_*) \cdot A_2 = 1$ and $2D \cdot A_2 = 2$. It is impossible.

(iiic) Let $Z_* \cdot A_2 = Z_* \cdot A_{21} = Z_* \cdot A_{22} = -1$ where A_2, A_{21} and A_{22} have coefficient 1 in Z_*. Then A_2, A_{21} and A_{22} have coefficient 1 in X by case (III) of section 2. $(X - Z_*)^2 = -6$. $0 = X \cdot Z_* = Z_* \cdot X = Z_* \cdot (A_1 + 2A_3 + A_2 + A_{21} + A_{22}) = Z_* \cdot A_1 + 2Z_* \cdot A_3 - 3$. Thus $Z_* \cdot A_1 = Z_* \cdot A_3 = 1$. Consider $X - Z_*$. A_2, A_{21} and A_{22} are not in $\text{supp}(X - Z_*)$. $(X - Z_*) \cdot A_1 = (X - Z_*) \cdot A_3 = -2$ and $(X - Z_*) \cdot A_2 = (X - Z_*) \cdot A_{21} = (X - Z_*) \cdot A_{22} = 1$. Consider $(X - Z_*)\big|_{\text{supp } D}$. Since A_3 has coefficient 2 in $X - Z_*$ and $(X - Z_*) \cdot A_3 = -2$, $X - Z_* = 2D$ on supp D. But $(X - Z_*) \cdot A_2 = D \cdot A_2 = 1$. It is contradiction.

(iiid) Let $Z_* \cdot A_2 = -2$ and $Z_* \cdot A_t = -1$. Then A_2 and A_t have coefficient 1 in Z_*. So A_2 and A_t have the same coefficient 1 in X by case (III) of section 2. $0 = X \cdot Z_* = Z_* \cdot X = Z_* \cdot (A_2 + A_t + A_1 + 2A_3) = -2 - 1 + Z_* \cdot A_1 + 2Z_* \cdot A_3$. $Z_* \cdot A_1 = Z_* \cdot A_3 = 1$. $(X - Z_*)^2 = -6$. $(X - Z_*) \cdot A_1 = (X - Z_*) \cdot A_3 = -2$, $(X - Z_*) \cdot A_2 = 2$ and $(X - Z_*) \cdot A_t = 1$.

Since A_3 has coefficient 2 in $X-Z_*$ and $(X-Z_*)\cdot A_3 = -2$, $X-Z_* = 2D$ on supp D. Let D_* be the fundamental cycle of the connected component of $\cup A_i$, $i \neq 1,2$ which contains A_t. Note that $A_t \not\subset$ supp D since A_t has coefficient 1 in X. Let $Z_* = D_* + K$. We claim that $D_* \cdot D_* \neq -1$. Assume the contrary. Since A_2 and A_t have coefficient 1 in Z_*, A_t has coefficient 1 in D_* and $A_t \not\subset$ supp K. If $D_* \cdot A_t = 0$, then $K \cdot A_t = -1$ and it is absurd. Thus $D_* \cdot A_t = -1$ and $K \cdot A_t = 0$. Note that A_2 has coefficient 1 in K since $A_2 \subset$ supp Z_* and $A_2 \not\subset$ supp D_*. By case (II) and (III) of section 2, $D_* \cdot A_2 > 0$ and $D_* \cdot A_1 > 0$. So $Z_* \cdot A_1 = 1$ implies $D_* \cdot A_1 = 1$ and $K \cdot A_1 = 0$. Also $K \cdot A_2 = (Z_* - D_*) \cdot A_2 = -2 - D_* \cdot A_2 < 0$. Moreover, $K \cdot A_j \leq 0$ for all $j \neq 3$ since $K \cdot A_3 = 1$. So $A_1 \subset$ supp K. That is impossible.

(iiie) Let $Z_* \cdot A_2 = -3$. Then A_2 has coefficient 1 both in Z_* and in X by case (III) of section 2. $0 = X \cdot Z_* = Z_* \cdot X = Z_* \cdot (A_1 + A_2 + 2A_3) = Z_* \cdot A_1 - 3 + 2Z_* \cdot A_3$. Thus $Z_* \cdot A_1 = Z_* \cdot A_3 = 1$. Consider $X - Z_*$. $(X - Z_*)^2 = -6$. $(X - Z_*) \cdot A_1 = (X - Z_*) \cdot A_3 = -2$. $(X - Z_*) \cdot A_2 = 3$. Since A_3 has coefficient 2 in $X - Z_*$, $X - Z_* = 2D$ on supp D.

(C) Let A_1 be such that $Z \cdot A_1 = -1$ with coefficient 2 in Z. By the definition of Z, $X = Z + D$ which $D > 0$. Since $-3 = X \cdot X = Z \cdot Z + 2Z \cdot D + D \cdot D$ and $Z \cdot Z = -2$, $Z \cdot D \leq 0$ and $D \cdot D < 0$ imply that $Z \cdot D = 0$ and $D \cdot D = -1$. So $A_1 \not\subset$ supp D since $Z \cdot D = 0$. For $j \neq 1$

$D \cdot A_j = (X - Z) \cdot A_j \leq 0$. Thus $D \cdot A_j \leq 0$ for $A_j \subset \operatorname{supp} D$. Since $D \cdot D = -1$, let A_2 be such that $D \cdot A_2 = -1$. $X \cdot A_2 = (Z + D) \cdot A_2 = -1$. Since $X \cdot X < X \cdot 2A_2 = -2$, $X \cdot A_j = -1$ for some j. If $j \neq 1,2$, then $X \cdot A_j = -1$ would imply either $Z \cdot A_j < 0$ or $D \cdot A_j < 0$. It is a contradiction. If $j = 1$, then $X \cdot X \leq X \cdot (2A_1 + 2A_2) = -4$ and it is impossible. So $A_j = A_2$ and thus A_2 has coefficient 3 in X. Also $D \cdot A_1 = (X - Z) \cdot A_1 = 1$ because $X \cdot A_1 = 0$. Also D is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1$. Let $G = \operatorname{Min}(Z|_{\operatorname{supp} D}, 2D)$. Since A_2 has coefficient 2 in Z and 1 in D, $G = 2D$. Consider $Z - 2D$. Then $(Z - 2D)^2 = -6$, $(Z - 2D) \cdot A_1 = -3$ and $(Z - 2D) \cdot A_i = 0$ for $i \neq 1,2$. Note that $A_2 \not\subset \operatorname{supp}(Z - 2D)$ and $(Z - 2D) \cdot A_2 = 2$. Thus $\operatorname{supp}(Z - 2D)$ is a connected component of $\cup A_i$, $i \neq 2$ because A_1 has coefficient 2 in $\operatorname{supp}(Z - 2D)$. Let Z_* be the fundamental cycle on $\operatorname{supp}(Z - 2D)$. If A_1 has coefficient 2 in Z_*, then $Z_* \cdot A_1 \leq (Z - 2D) \cdot A_1 = -3$ and so $Z_* \cdot Z_* \leq Z_* \cdot 2A_1 \leq -6$. It is a contradiction. So A_1 has coefficient 1 in Z_*. Let $K = \operatorname{Min}(2Z_*, Z-2D)$. Since A_1 has coefficient 2 both in $2Z_*$ and in $Z - 2D$, $K \cdot A_1 \leq (Z - 2D) \cdot A_1 = -3$ and $K \cdot K \leq K \cdot 2A_1 = -6$. Since $K \leq Z - 2D$, $K = Z - 2D$. Therefore we get $Z_* < Z - 2D < 2Z_*$ and $2Z_* \cdot A_1 \geq (Z - 2D) \cdot A_1 = -3$. So $Z_* \cdot A_1 = 0$ or -1. Since $Z_*^2 > (Z - 2D)^2 = -6 > 4Z_*^2$, $Z_*^2 = -2$ or -3. Let X_* be the cycle on $\operatorname{supp}(Z - 2D)$ such that $X_* \cdot A_1 \leq 0$ for all $A_i \subset \operatorname{supp}(Z - 2D)$ and that $X_*^2 = -3$. If A_1 has coefficient not equal to 1 in X_*, then let $K = \operatorname{Min}(X_*, Z-2D)$. Then $K \cdot A_1 \leq (Z - 2D) \cdot A_1 = -3$ and $K \cdot K \leq K \cdot 2A_1 = -6$. But $K \cdot K \leq X_* \cdot X_* = -3$

and it is impossible. So A_1 has coefficient 1 in X_*. Now repeating the same argument for X_* as just above for Z_*, we get $X_* \cdot A_1 = 0$ or -1. We claim that $X_* \cdot A_1 = 0$. If $X_* \cdot A_1 = -1$ and X_* is the fundamental cycle, then $Z - 2D > X_*$. Let $Z - 2D = X_* + E$. Then we would have the following possibilities:

$$(Z - 2D)^2 = X_*^2 + 2X_* \cdot E + E^2$$

$$-6 = -3 + 0 - 3$$

$$-6 = -3 - 2 - 1$$

Since $(Z - 2D) \cdot A_1 = -3$ and $X_* \cdot A_1 = -1$, $E \cdot A_1 = -2$ and A_1 has coefficient 1 in E. Since $X_* \cdot E \leq X_* \cdot A_1 = -1$, $X_* \cdot E$ would be -1. Then $X_* \cdot A_j = 0$ for $A_j \subset \operatorname{supp} E$, $j \neq 1$. So $E \cdot A_j = (Z - 2D - X_*) \cdot A_j = -X_* \cdot A_j = 0$ for $A_j \subset \operatorname{supp} E$, $j \neq 1$. Therefore it would be $E \cdot E = E \cdot A_1 = -2$. Thus we get a contradiction. If $X_* \cdot A_1 = -1$ and X_* is not the fundamental cycle on $\operatorname{supp}(Z - 2D)$ then note that $Z_* \cdot Z_* = -2$ and by the results of case (I) of this Proposition we got so far, there exists $A_s \neq A_1$ such that $X_* \cdot A_s = -1$ and that A_s has coefficient 2 in X_*. Let $\rho^{-1}(E_2) = C_2 = A_1 \cup A_s$. Note that ρ is $2-1$ over E_2 by case (II) of section 2. Since A_1 has coefficient 2 in X, A_s would have coefficient 4 in X by case (II) of section 2. Recall that $Z_E = \Sigma z_i E_i$ is the fundamental cycle where $(B) = (-108(p^3+q^2) \circ \tau_n) = W^{(n)} + \Sigma e_i E_i$, $1 \leq i \leq n$, be the divisor of $B \circ \tau_n$. Let $\rho^{-1}(E_2) = A_1 \cup A_s$. If E_1 is the curve appearing at the initial quadratic transformation at Q, then E_2 follows E_1 by a partial ordering induced by the order of appearance of the E_i in a resolution by (2.2). So E_2 has coefficient 1 in Z_E. Therefore by case (II) of section 2, A_1 and A_s have coefficients 1 and 2 in X respectively. Thus we get a contradiction. Hence $X_* \cdot A_1 = 0$. Now consider the following subcases (C1), (C2), (C3), (C4) and (C5).

(C1) Assume that $X_* \cdot A_s = -3$ for $s \neq 1$. Since X_* is the fundamental cycle on supp$(Z - 2D)$, $Z - 2D > X_*$. Let $\rho^{-1}(E_2) = A_s$. Since E_2 has coefficient 1 in Z_E, A_s has coefficient 1 in X by case (III) of section 2. $0 = X \cdot X_* = X_* \cdot X = X_* \cdot (A_s + 3A_2) = -3 + 3X_* \cdot A_2$. Thus $X_* \cdot A_2 = 1$. Since A_s has coefficient 1 in X and $X = Z + D$, A_s has coefficient 1 in Z and $A_s \not\subset \text{supp}\, D$. Let $Z - 2D = X_* + F$. Since A_s has coefficient 1 both in $Z - 2D$ and in X_*, $A_s \not\subset \text{supp}\, F$ and so $X_* \cdot F = 0$. $(Z - 2D)^2 = X_*^2 + 2X_* \cdot F + F^2$ and $(Z - 2D) \cdot A_1 = -3$ imply $F^2 = F \cdot A_1 = -3$ and A_1 has coefficient 1 in F because A_1 has coefficient 2 in $Z - 2D$ and 1 in X_*. $F \cdot A_s = 3$ and $F \cdot A_2 = 1$ since $(Z - 2D) \cdot A_2 = 1$ and $X_* \cdot A_2 = 1$. So F is the fundamental cycle with $F^2 = -3$ on a connected component of $\cup A_i$, $i \neq 2, s$. Also $X_* > F$. Consider $X_* - F$. Then A_s has coefficient 1 and $A_1 \not\subset \text{supp}(X_* - F)$. $(X_* - F)^2 = -6$, $(X_* - F) \cdot A_s = -6$ and $(X_* - F) \cdot A_1 = 3$. Since $A_1 \not\subset \text{supp}(X_* - F)$, $X_* - F$ is the fundamental cycle with $(X_* - F)^2 = -6$ on a component of $\cup A_i$, $i \neq 1$.

(C2) Assume that $X_* \cdot A_s = -2$ and $X_* \cdot A_t = -1$. Then A_s and A_t have coefficient both 1 in X_*. Let $\rho^{-1}(E_2) = A_s \cup A_t$. Since E_2 has coefficient 1 in Z_E, A_s and A_t have coefficient both 1 in

X by case (III) of section 2. $0 = X \cdot X_* = X_* \cdot X = X_* \cdot (A_s + A_t + 3A_2)$ $= -2 - 1 - 3X_* \cdot A_2$. Thus $X_* \cdot A_2 = 1$. Since A_s and A_t have coefficient 1 in X and $X = Z + D$, $A_s \not\subset \operatorname{supp} D$ and $A_t \not\subset \operatorname{supp} D$. Since X_* is fundamental on $\operatorname{supp}(Z - 2D)$, let $Z - 2D = X_* + F$. Note that $A_s \not\subset \operatorname{supp} F$ and $A_t \not\subset \operatorname{supp} F$. So $X_* \cdot F = 0$. $-6 = (Z - 2D)^2 = X_*^2 + 2X_* \cdot F + F^2 = -3 + F^2$ implies $F^2 = -3$. $F \cdot A_1 = -3$ and A_1 has coefficient 1 in F since A_1 has coefficient 2 in $Z - 2D$ and 1 in X_*. $F \cdot A_s = 2$, $F \cdot A_t = 1$ and $F \cdot A_2 = 1$ because $(Z - 2D) = X_* + F$, $(Z - 2D) \cdot A_2 = 2$ and $X_* \cdot A_2 = 1$. So F is the fundamental cycle with $F^2 = -3$ on the connected component of $\cup A_i$, $i \neq 2,s,t$ which contains A_1. Since F is fundamental, $X_* > F$. Consider $X_* - F$. Then $(X_* - F) \cdot A_s = -4$, $(X_* - F) \cdot A_t = -2$ and $(X_* - F)^2 = -6$. Since A_1 has coefficient 1 in both X_* and F, $A_1 \not\subset \operatorname{supp}(X_* - F)$ and $(X_* - F) \cdot A_1 = 3$. Since A_s and A_t have coefficient both 1 in $X_* - F$, there is no fundamental cycle with its self-intersection number -1 on any component of $\operatorname{supp}(X_* - F)$.

(C3) Assume that $X_* \cdot A_{s1} = X_* \cdot A_{s2} = X_* \cdot A_{s3} = -1$ where A_{s1}, A_{s2} and A_{s3} are distinct. Let $\rho^{-1}(E_2) = A_{s1} \cup A_{s2} \cup A_{s3}$. Since E_2 has coefficient 1 in Z_E, A_{s1}, A_{s2} and A_{s3} have coefficient 1 in X. $0 = X \cdot X_* = X_* \cdot X = X_* \cdot (A_{s1} + A_{s2} + A_{s3} + 3A_2) = -3 + 3X_* \cdot A_2$. Thus $X_* \cdot A_2 = 1$. Note that A_{s1}, A_{s2} and A_{s3} are not contained in $\operatorname{supp} D$, for $X = Z + D$. Since X_* is fundamental on $\operatorname{supp}(Z - 2D)$, $Z - 2D > X_*$. Let $Z - 2D = X_* + F$. Observe that A_{s1}, A_{s2} and A_{s3} are not in $\operatorname{supp} F$. So $X_* \cdot F = 0$. $-6 = (Z - 2D)^2 =$

$X_*^2 + 2X_* \cdot F + F^2 = -3 + F^2$. $F^2 = F \cdot A_1 = -3$. A_1 has coefficient 1 in F since A_1 has coefficient 2 in $Z - 2D$ and 1 in X_*. $F \cdot A_{s1} = F \cdot A_{s2} = F \cdot A_{s3} = 1$. $F \cdot A_2 = 1$ because $(Z - 2D) \cdot A_2 = 1$ and $X_* \cdot A_2 = 1$. Therefore F is the fundamental cycle with $F^2 = -3$ on the connected component of $\cup A_i$, $i \neq 2, s1, s2, s3$ which contains A_1. Since $X_* > F$, consider the cycle $X_* - F$. Then A_{s1}, A_{s2} and A_{s3} have coefficient 1 in $X_* - F$ and $A_1 \not\subset \mathrm{supp}(X_* - F)$. $(X_* - F) \cdot A_{s1} = (X_* - F) \cdot A_{s2} = (X_* - F) \cdot A_{s3} = -2$ and $(X_* - F) \cdot A_1 = 3$. $(X_* - F)^2 = -6$. So there is no fundamental cycle with its self-intersection number -1 on any component of $\mathrm{supp}(X_* - F)$.

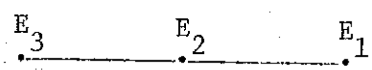
(C4) Let A_s be such that $X_* \cdot A_s = -1$ with coefficient 3 in X_*. Let $\rho^{-1}(E_2) = A_s$. Since E_2 has coefficient 1 in Z_E, A_s has coefficient 3 in X by case (I) of section 2. $0 = X \cdot X_* = X_* \cdot X = X_* \cdot (3A_s + 3A_2) = -3 + 3X_* \cdot A_2$. $X_* \cdot A_2 = 1$. Since A_s has coefficient 3 in X, $X = Z + D$ and $A_s \subset \mathrm{supp}(Z - 2D)$, A_s has coefficient 3 in Z and $A_s \not\subset \mathrm{supp}\, D$. Thus A_s has coefficient 3 in $Z - 2D$. So $Z - 2D > X_*$. Let $Z - 2D = X_* + F$. Note that $A_s \not\subset \mathrm{supp}\, F$. $X_* \cdot F = 0$. $-6 = (Z - 2D)^2 = X_*^2 + 2X_* \cdot F + F^2 = -3 + F^2$. $F^2 = F \cdot A_1 = -3$. A_1 has coefficient 1 in F. $F \cdot A_s = 1$. $F \cdot A_2 = 1$ because $(Z - 2D) \cdot A_2 = 2$ and $X_* \cdot A_2 = 1$. Therefore F is the fundamental cycle with $F^2 = -3$ on the connected component of $\cup A_i$, $i \neq 2, s$ which contains A_1. Since $X_* > F$, consider the cycle $X_* - F$. Then A_s has coefficient 3 in $X_* - F$, $A_1 \not\subset \mathrm{supp}(X_* - F)$, $(X_* - F) \cdot A_s = -2$ and $(X_* - F) \cdot A_1 = 3$. Since

$(X_* - F)^2 = -6$, $\mathrm{supp}(X_* - F)$ is a connected component of $\cup A_i$, $i \neq 1$. We claim that there is no fundamental cycle with its self-intersection number -1 on $\mathrm{supp}(X_* - F)$. If there would be the fundamental cycle L with $L \cdot L = -1$ on $\mathrm{supp}(X_* - F)$, then let $X_* - F = L + M$. Then we would have the following table:

$$\begin{aligned}(X_* - F)^2 &= L^2 + 2L\cdot M + M^2\\ -6 &= -1 + \ \ 0 \ + 5\\ -6 &= -1 - \ \ 2 \ - 3\\ -6 &= -1 - \ \ 4 \ - 1\end{aligned}$$

If $L \cdot A_s = -1$, then A_s has coefficient 1 in L and 2 in M. Note that $M \cdot A_s = -1$. But $L \cdot M = L \cdot 2A_s = -2$ and $M^2 = M \cdot 2A_s = -2$. It is impossible. If $L \cdot A_t = -1$ for $A_t \neq A_s$, then $M \cdot A_s = -2$. Moreover, A_s would have coefficient either 1 or 2 in M. If A_s would have coefficient 1 in M, then $M \cdot M \geq M \cdot A_s = -2$ because $M \cdot A_i \geq 0$ for all $i \neq s$. According to the above table, $M^2 = -1$ and $L \cdot M = -2$. Thus A_t has coefficient 2 in M. But $M^2 = M \cdot (A_s + 2A_t) = -2 + 2 = 0$. It is absurd. If A_s has coefficient 2 in M, $M \cdot M \geq M \cdot 2A_s = -4$. $M^2 = -3$ or -1 according to the above table. If $M^2 = -1$, then $L \cdot M = -2$ and so A_t has coefficient 2 in M. But $M^2 = M \cdot (2A_s + 2A_t) = -4 + 2 = -2$. It is absurd. If $M^2 = -3$, then $L\ M = -1$ and A_t has coefficient 1 in M. Now we claim that $M > L$. Note that A_s has coefficient 2 in M and 1 in L. Let $K = \mathrm{Min}(L,M)$. Since A_t has coefficient 1 in both L and M, $K \cdot A_t \leq L \cdot A_t = -1$. Also $K \cdot A_s \leq L \cdot A_s \leq 0$. Since $M \cdot A_j \leq 0$ for $A_j \subset \mathrm{supp}\, M$, $j \neq t$, $K \cdot A_i \leq 0$ for $A_i \subset \mathrm{supp}\, L$.

So $K = L$. Consider the cycle $M - L$. Then A_s has coefficient 1 in $M - L$, $(M-L)\cdot A_s = -2$, $A_t \not\subset \operatorname{supp}(M-L)$ and $(M-L)\cdot A_t = 1$. $(M-L)\cdot A_1 = 1$ since $(X_* - F)\cdot A_1 = 3$, $X_* - F = M + L$ and $M > L$. So $M - L$ is the fundamental cycle with $(M-L)^2 = -2$ on the connected component of $\cup A_i$, $i \neq 1, t$ which contains A_s. So $M - L < L$. Next consider $2L - M$. Then A_t has coefficient 1, $A_s \not\subset \operatorname{supp}(2L - M)$, $(2L-M)\cdot A_t = -3$ and $(2L-M)\cdot A_s = 2$. $(2L-M)\cdot A_1 = 0$ and so $\operatorname{supp}(2L-M) \cap A_1 = \phi$. Note that $(2L-M)^2 = (2L-M)\cdot A_t = -3$. Therefore $2L-M$ is the fundamental cycle on a connected component of $\cup A_i$, $i \neq s$. Observe that $\operatorname{supp}(2L-M) \cap \operatorname{supp} \bar{D} = \phi$ since $\operatorname{supp}(2L-M) \cap A_1 = \phi$.
Let $\rho^{-1}(E_1) = A_2$, $\rho^{-1}(E_2) = A_s$ and $\rho^{-1}(E_3) = A_t$. Note that $(2L-M)^2 = (2L-M)\cdot A_t = -3$. Since A_s follows A_2, A_t follows A_s and E is an exceptional set of the first kind, after a finite suitable number of collapse [L1, Corollary 5.8, p. 86], then we may assume the following diagram:



where $E_3\cdot E_3 = -1$ and $E_2\cdot E_2 = E_1\cdot E_1 = -2$. Since Z_E is independent of such collapse, E_3 has coefficient 1 in Z_E. So A_t has coefficient 1 in X by case (III) of section 2. But

$0 = X\cdot(2L-M) = (2L-M)\cdot X = (2L-M)\cdot(A_t + 3A_s) = -3+6 = 3.$

Thus we get a contradiction.

(C5) Let $A_s \neq A_t$ be such that $X_*\cdot A_s = X_*\cdot A_t = -1$ where A_s and A_t have coefficients 2 and 1 in X_* respectively. Let

$\rho^{-1}(E_2) = A_s \cup A_t$. Since E_2 has coefficient 1 in Z_E. A_s has coefficient 2 in X and A_t, 1 in X. $0 = X \cdot X_* = X_* \cdot X =$ $X_* \cdot (2A_s + A_t + 3A_2) = -2 - 1 + 3X_* \cdot A_2$. Thus $X_* \cdot A_2 = 1$. Since A_s, $A_t \subset \mathrm{supp}(Z - 2D)$, A_s and A_t have coefficients 2 and 1 in Z, $A_s \not\subset \mathrm{supp}\, D$ and $A_t \not\subset \mathrm{supp}\, D$. So A_s and A_t have coefficient 2 and 1 in $Z - 2D$ respectively. $Z - 2D > X_*$. Let $Z - 2D = X_* + F$. Note that $A_s \not\subset \mathrm{supp}\, F$ and $A_t \not\subset \mathrm{supp}\, F$. So $X_* \cdot F = 0$. $-6 = (Z - 2D)^2 = X_*^2 + 2X_* \cdot F + F^2 = -3 + F^2$ implies $F^2 = -3$. $F \cdot A_1 = -3$. A_1 has coefficient 1 in F. $F \cdot A_s = F \cdot A_t = 1$. $F \cdot A_2 = 1$ because $(Z - 2D) \cdot A_2 = 2$ and $X_* \cdot A_2 = 1$. So F is the fundamental cycle with $F^2 = -3$ on a connected component of $\cup A_i$, $i \neq 2, s, t$. $X_* > F$. Consider $X_* - F$. Then A_s and A_t have coefficients 2 and 1 in $X_* - F$, $A_1 \not\subset \mathrm{supp}(X_* - F)$, $(X_* - F) \cdot A_s =$ $(X_* - F) \cdot A_t = -2$, $(X_* - F) \cdot A_1 = 3$ and $(X_* - F)^2 = -6$. First we claim that X_* is the fundamental cycle on $\mathrm{supp}(Z - 2D)$. Assume the contrary. Let us recall that Z_* is the fundamental cycle on $\mathrm{supp}\, X_* = \mathrm{supp}(Z - 2D)$. Then $Z_*^2 = -2$ and $Z_* \cdot A_1 = 0$ or -1. If $Z_* \cdot A_1 = 0$, then we would have the following three possibilities:

(i) $Z_* \cdot A_p = Z_* \cdot A_q = -1$ for $A_p \neq A_q$, $A_p \neq A_1$ and $A_q \neq A_1$

(ii) $Z_* \cdot A_p = -2$

(iii) $Z_* \cdot A_p = -1$ where A_p has coefficient 2 in Z_*.

Recall that $2Z_* > Z - 2D$. Let $2Z_* = Z - 2D + E$. Then we would have the following table:

$$4Z_*^2 = (Z - 2D)^2 + 2(Z - 2D) \cdot E + E^2$$

$$-8 = -6 + 0 - 2.$$

Case (i): Since $Z_* \cdot A_p = Z_* \cdot A_q = -1$, then $E \cdot A_p = E \cdot A_q = -2$. Since A_1 has coefficient 2 both in $2Z_*$ and $Z - 2D$, $A_1 \not\subset \operatorname{supp} E$. So $E^2 \leq E \cdot A_p + E \cdot A_q = -4$. Contradiction.

Case (ii): Since $Z_* \cdot A_p = -2$, then $E \cdot A_p = -4$. Since $A_1 \not\subset \operatorname{supp} E$, $E^2 \leq E \cdot A_p = -4$. Contradiction.

Case (iii): Since $Z_* \cdot A_p = -1$ and A_p has coefficient 2 in Z_*, by the arguments in the beginning of case (C), A_s should have coefficient 3 in X_*. Contradiction to the assumption of X_*.

Now, if $Z_* \cdot A_1 = -1$, then $Z_* \cdot A_t = -1$ and A_s has coefficient 1 in Z_* because $Z_* \cdot Z_* = -2$, A_1 has coefficient 1 in Z_*, $X_* \cdot A_t = X_* \cdot A_s = -1$ and A_s and A_t have coefficients 2 and 1 in X_* respectively and so by case (B2) of (I) of this proposition. Let $2Z_* = Z - 2D + E$ as before. Then in this case, $E \cdot E = E \cdot A_t = -2$ and $E \cdot A_1 = 1$ since $Z_* \cdot A_1 = -1$ and $(Z - 2D) \cdot A_1 = -3$. Note that $A_1 \not\subset \operatorname{supp} E$. If we put $X_* = Z_* + G_*$, then $-3 = X_*^2 = Z_*^2 + 2Z_* \cdot G_* + G_*^2$, $Z_* \cdot G_* \leq 0$ and $G_*^2 < 0$ imply $G_*^2 = -1$ and $G_* \cdot A_s = -1$. Since $X_* \cdot A_1 = 0$ and $Z_* \cdot A_1 = -1$, $G_* \cdot A_1 = 1$. Note that $A_1 \not\subset \operatorname{supp} G_*$ and A_s has coefficient 1 in G_*. Therefore E is the fundamental cycle of the component of $\cup A_i$, $i \neq 1$ which contains A_t and G_* is the fundamental cycle of the component of $\cup A_i$, $i \neq 1$ which contains A_s. So by negative definiteness of the intersection matrix for the A_i, $\operatorname{supp}(X_* - F) = \operatorname{supp} E \cup \operatorname{supp} G_*$. Let $Z' = E + G_* + F + D$. Note that $A_1 \subset \operatorname{supp} F$. Since $\operatorname{supp} F \cap \operatorname{supp} E \neq \phi$, $\operatorname{supp} F \cap \operatorname{supp} G_* \neq \phi$ and $\operatorname{supp} F \cap \operatorname{supp} D \neq \phi$, $\operatorname{supp} Z' = \operatorname{supp} Z$. Since A_1, A_s, A_t

and A_2 have coefficient 1 in Z',

$$Z'\cdot A_1 = (E + G_* + F + D)\cdot A_1 = 1 + 1 - 3 + 1 = 0$$
$$Z'\cdot A_2 = (E + G_* + F + D)\cdot A_2 = 1 - 1 = 0$$
$$Z'\cdot A_s = (E + G_* + F + D)\cdot A_s = -1 + 1 = 0 \text{ and}$$
$$Z'\cdot A_t = (E + G_* + F + D)\cdot A_t = -2 + 1 = -1.$$

Thus $Z'\cdot A_j \leq 0$ for all j and $Z'^2 = -1 > Z^2 = -2$. Since Z is fundamental, it is impossible. Next, we claim that there is no fundamental cycle with its self-intersection number -1 on any component of $\mathrm{supp}(X_* - F)$. Recall that A_s and A_t have coefficients 2 and 1 in $X_* - F$, $A_1 \not\subset \mathrm{supp}(X_* - F)$, $(X_* - F)\cdot A_s = (X_* - F)\cdot A_t = -2$, $(X_* - F)\cdot A_1 = 3$ and $(X_* - F)^2 = -6$. If $\mathrm{supp}(X_* - F)$ is not connected, then by negative definiteness of the intersection matrix for the A_i, $\mathrm{supp}(X_* - F)$ is two disjoint union of connected components of $\cup A_i$, $i \neq 1$. One component C_1 contains A_s and the other component C_2 contains A_t. But the self-intersection number of the fundamental cycle on C_2 is -2 since A_t has coefficient 1 in $X_* - F$. Therefore it suffices to show that there is no fundamental cycle L with $L\cdot L = -1$ on C_1. If it exists, let $(X_* - F)|_{C_1} = L + M$. Then we would have the following table:

$$-4 = [(X_* - F)|_{C_1}]^2 = L^2 + 2L\cdot M + M^2$$
$$-4 \qquad = -1 + 0 - 3$$
$$-4 \qquad = -1 - 2 - 1$$

If $L\cdot A_p = -1$ for $A_p \neq A_s$, then $L\cdot A_s = 0$ and $M\cdot A_s = -2$. Since A_s has coefficient 2 in $X_* - F$, A_s has coefficient 1 in both L and M. Since $M \cdot A_s = -2$ and $M \cdot A_p = 1$, $M^2 \geq M \cdot A_s = -2$. $M^2 = -1$ and A_p has coefficient 1 in M by the above table. So supp L = supp M. $L \cdot A_1 = M \cdot A_1 = 1$ since $(X_* - F) \cdot A_1 = 3$ and $\mathrm{supp}(X_* - F) = C_1 \cup C_2$. Let $K = \mathrm{Min}(L,M)$. Since A_p has coefficient 1 in both L and M and A_s, 1 in both L and M, $K \cdot A_p \leq L \cdot A_p = -1$ and $K \cdot A_s \leq M \cdot A_s = -2$. So $K^2 \leq K \cdot A_p + K \cdot A_s = -3$. This would be a contradiction. If $L \cdot A_s = -1$, then $M \cdot A_s = -1$. Since A_s has coefficient 1 in both L and M, L = M. Let $Z' = L + (X_* - F)\big|_{C_2} + F + D$. Then A_1, A_s, A_t and A_2 have coefficient 1 in Z', respectively. $Z' \cdot A_1 = 0$, $Z' \cdot A_s = 0$, $Z' \cdot A_t = -1$ and $Z' \cdot A_2 = 0$. Since supp Z' = supp Z and $Z' \cdot Z' = -1 > Z \cdot Z = -2$, it is a contradiction. If $\mathrm{supp}(X_* - F)$ is connected and there exists such L, then let $X_* - F = L + M$. Note that A_t has coefficient 1 in $X_* - F$ and $A_t \not\subset \mathrm{supp}\, M$. If $L \cdot A_t = 0$, then $M \cdot A_t = -2$. It is absurd. If $L \cdot A_t = -1$, then $M \cdot A_t = -1$. Still it is impossible.

(II) Let $Z \cdot Z = -1$. Then there exists A_1 such that $Z \cdot A_1 = -1$. By the definition of Z, $X = Z + D$ with $D > 0$. Since $-3 = X \cdot X = Z \cdot Z + 2Z \cdot D + D \cdot D$, $Z \cdot Z = -1$, $Z \cdot D \leq 0$ and $D \cdot D < 0$, $Z \cdot D = 0$ and $D \cdot D = -2$. So $A_1 \not\subset \mathrm{supp}\, D$ since $Z \cdot D = 0$. For $j \neq 1$, $D \cdot A_j = (X - Z) \cdot A_j \leq 0$. Thus $D \cdot A_j \leq 0$ for $A_j \subset \mathrm{supp}\, D$. Since $D \cdot D = -2$, we have the following possibilities:

(a) There exists A_2 such that $D \cdot A_2 = -2$.

(b) There exist $A_2 \neq A_3$ such that $D \cdot A_2 = D \cdot A_3 = -1$.

(c) There exists A_2 such that $D \cdot A_2 = -1$ and A_2 has coefficient 2 in D.

If (a) is true, then $X \cdot A_2 = (Z+D) \cdot A_2 = -2$. Since $X \cdot X \leq X \cdot 2A_2 = -4$, this is a contradiction. If (b) is true, then $X \cdot A_2 = (Z+D) \cdot A_2 = -1$ and $X \cdot A_3 = (Z+D) \cdot A_3 = -1$. But $X \cdot X \leq X \cdot (2A_2 + 2A_3) = -4$. It is impossible. Now consider the case (c). $X \cdot A_2 = (Z+D) \cdot A_2 = -1$. Since A_2 has coefficient 2 in D, $X \geq 3A_2$. So $X \cdot X \leq X \cdot 3A_2 = -3$. A_2 has coefficient 3 in X and $X \cdot A_j = 0$ for $j \neq 2$. $D \cdot A_1 = 1$. Let us consider the cycle D in more detail. Supp D is a connected component of $\cup A_i$, $i \neq 1$. Let $Z = A_1 + \Sigma F_i$ where the supp F_i are connected components of $\cup A_i$, $i \neq 1$. Then supp F_i = supp D for some i. $Z|_{\text{suppD}} \cdot D = -1$ because $0 = Z \cdot D = (A_1 + \Sigma F_i) \cdot D = A_1 \cdot D + Z|_{\text{suppD}} \cdot D = 1 + Z|_{\text{suppD}} \cdot D$. Now we claim that $A_1 \cap A_2 = \phi$. If $A_1 \cap A_2 \neq \phi$, then $(Z|_{\text{suppD}} + A_1) \cdot A_2 = 0$ would imply $Z|_{\text{suppD}} \cdot A_2 = -A_1 \cdot A_2 = -1$ since $D \cdot A_1 = 1$. But $Z|_{\text{suppD}} \cdot D \leq Z|_{\text{suppD}} \cdot 2A_2 = -2A_1 \cdot A_2 = -2$. This leads to a contradiction for $Z|_{\text{suppD}} \cdot D = -1$. Let $G = \text{Min}(Z|_{\text{suppD}}, D)$. Since A_2 has coefficient 1 in Z and 2 in D, $G < D$. So $G \cdot G = -1$. G is the fundamental cycle on supp D. To prove $2Z > D$, let $K = \text{Min}(2Z|_{\text{suppD}}, D)$. Since A_2 has coefficient 2 in both 2Z and D, $K \cdot A_2 \leq D \cdot A_2 = -1$ $K \cdot K \leq K \cdot 2A_2 = -2$. $K = D$ because $K \leq D$. Consider $2Z - D$. $(2Z - D) \cdot A_1 = -3$, $(2Z - D) \cdot A_2 = 1$, A_1 has coefficient 2 in $2Z - D$ and $A_2 \not\subset \text{supp}(2Z - D)$. Since $(2Z - D)^2 = -6$, $\text{supp}(2Z - D)$ is the connected component of $\cup A_i$, $i \neq 2$ which contains A_1. Let Z_* be the fundamental cycle on $\text{supp}(2Z - D)$. $Z_* < 2Z - D$. Then A_1 has coefficient 1 in Z_* otherwise the fact that A_1 has coefficient 2 in both Z_* and $2Z - D$ imply $Z_* \geq 2Z - D$. Since $G \cdot G = -1$, let A_p be such that $G \cdot A_p = -1$. We claim that $Z_* \cdot A_p = -1$. Let $D = G + F$. $-2 = D^2 = G^2 + 2G \cdot F + F^2$, $G \cdot F \leq 0$ and $F^2 < 0$ imply $F^2 = -1$. If $G \cdot A_2 = -1$, then $D = G + F$ would imply $F \cdot A_2 = 0$ and also $F \cdot A_i = (D - G) \cdot A_i = 0$ for all $A_i \subset \operatorname{supp} D$. This is a contradiction. So $G \cdot A_p = -1$ for $A_p \neq A_2$ and $G \cdot A_2 = 0$. $F \cdot A_2 = -1$. Since $G \cdot F = 0$, $A_p \not\subset \operatorname{supp} F$ and $F \cdot A_p = (D - G) \cdot A_p = 1$. Since $D \cdot A_1 = G \cdot A_1 = 1$, $A_1 \cap \operatorname{supp} F = \phi$. Therefore F is the fundamental cycle with $F^2 = F \cdot A_2 = -1$ on a connected component of $\cup A_i$, $i \neq p$. Let $Z = \Sigma z_i A_i$. $0 = Z \cdot G = G \cdot (A_1 + z_p A_p) = 1 - z_p$. So $z_p = 1$. Let $Z' = Z - F = Z - (D - G) = Z + G - D$. Then A_1 and A_p have coefficient 1 in Z' and $A_2 \not\subset \operatorname{supp} Z'$. $Z' \cdot A_1 = (Z + G - D) \cdot A_1 = -1$. $Z' \cdot A_p = (Z + G - D) \cdot A_p = -1$ and $Z' \cdot A_2 = (Z + G - D) \cdot A_2 = 1$. Note that $\operatorname{supp} Z' = \operatorname{supp}(2Z - D)$. $Z' \cdot Z' = (Z + G - D)^2 = -2$. So Z' is the fundamental cycle on $\operatorname{supp}(2Z - D)$. So $Z_* = Z'$ and $Z_* \cdot A_p = -1$. Let X_* be the cycle on $\operatorname{supp} Z_*$ such that $X_* \cdot X_* = -3$ and $X_* \cdot A_i \leq 0$ for all $A_i \subset \operatorname{supp} Z_*$. Since $Z_* \cdot A_1 = Z_* \cdot A_p = -1$ with $A_1 \neq A_p$ and $Z_*^2 = -2$, by case (B) of (I) of this proposition, we have the following three cases:

(A) $X_* \cdot A_t = -1$ where A_t has coefficient 3 in X_*.

(B) $X_* \cdot A_1 = -1$ and $X_* \cdot A_t = -1$ with $A_1 \neq A_t$ where A_1 and A_t have coefficients 1 and 2 in X_* respectively.

(C) $X_* \cdot A_p = -1$ and $X_* \cdot A_t = -1$ with $A_p \neq A_t$, $A_p \neq A_1$ and $A_t \neq A_1$ where A_p and A_t have coefficients 1 and 2 in X_* respectively.

(A) Consider the case that $A_t \cdot X_* = -1$ and that A_t has coefficient 3 in X_*. It is trivial by case (B1) of (I).

(B) Let $A_1 \neq A_t$ be such that $X_* \cdot A_1 = X_* \cdot A_t = -1$, A_1 and A_p have coefficient 1 in X_* and A_t has coefficient 2 in X_*. Then $X_* \cdot A_p = 0$. Since A_1 has coefficient 1 in X, A_t has coefficient 2 in X. $0 = X \cdot X_* = X_* \cdot X = X_* \cdot (A_1 + 2A_t + 3A_2) = -3 + 3X_* \cdot A_2$. $X_* \cdot A_2 = 1$. Let $X_* = Z_* + G_*$. Since A_1, A_p and A_t have coefficients 1, 1 and 2 in X_* respectively and A_1, A_p and A_t have the same coefficient 1 in Z_*, $G_* \cdot A_t = -1$, $G_* \cdot A_p = 1$, $G_* \cdot A_1 = 0$. $A_p \not\subset \text{supp } G_*$ and $A_1 \cap \text{supp } G_* = \phi$. $G_* \cdot A_2 = 0$ and $A_2 \cap \text{supp } G_* = \phi$ because $(2Z - D) \cdot A_2 = X_* \cdot A_2 = Z_* \cdot A_2 = 1$. Since $G_*^2 = -1$ and $Z_* \cdot G_* = 0$, G_* is the fundamental cycle on the connected component of $\cup A_i$, $i \neq p$ which does not intersect A_1 and A_2 both and contains A_t. Now we claim that $A_t \subset \text{supp } D$. Note that $2Z_* \geq 2Z - D$ because A_1 has coefficient 2 in both $2Z - D$ and $2Z_*$, $(2Z - D)^2 = -6$ and $(2Z_*)^2 = -8$. If $A_t \not\subset \text{supp } D$, then $X = Z + D$ implies that A_t has coefficient 2 in Z since A_t has coefficient 2 in X. So A_t has coefficient 4 in $2Z - D$. Since $2Z_* \geq 2Z - D$, A_t has coefficient ≥ 4 in $2Z_*$. Thus A_t has coefficient ≥ 2 in Z_*. In fact A_t has coefficient 1 in Z_*. Therefore $A_t \subset \text{supp } D$. Let $2Z_* = 2Z - D + E$. $-8 = 4Z_*^2 = (2Z - D)^2 + 2(2Z - D) \cdot E + E^2$ and $(2Z - D)^2 = -6$ imply that $(2Z - D) \cdot E = 0$ and $E^2 = -2$ by $(2Z - D) \cdot E \leq 0$ and negative definiteness of the intersection matrix for the A_j. Since A_1, A_p and A_t have the same coefficient 2 in $2Z_*$ and have coefficients 2, 1 and 1 in $2Z - D$ respectively, A_p and A_t have the same coefficient 1 in E, $A_1 \not\subset \text{supp } E$, $E \cdot A_p = -2$, $E \cdot A_1 = E \cdot A_2 = 1$. So E is the fundamental cycle on the connected component of $\cup A_i$, $i \neq 1,2$ which contains A_p and A_t. Therefore $E > G_*$. $(E - G_*) \cdot A_1 = (E - G_*) \cdot A_2 = 1$. $(E - G_*) \cdot A_p = -3$. $A_t \not\subset \text{supp}(E - G_*)$ and $(E - G_*) \cdot A_t = 1$ because A_t has coefficient 1 in both E and G_*. Since $(E - G_*)^2 = E^2 - 2E \cdot G_* + G_*^2 = -3$ and $(E - G_*) \cdot A_p = -3$, $E - G_*$ is the fundamental cycle on a connected component of $\cup A_i$, $i \neq 1,2$ and t. Observe that A_t and A_1 follow A_p at the same time and A_p follows A_t and A_1 at the same time. It is clear that $D - G$ is the fundamental cycle on a connected component of $\cup A_i$, $i \neq p$ which does not contain A_1 and A_t with $(D - G)^2 = (D - G) \cdot A_2 = -1$.

(C) Let $A_p \neq A_t$ be such that $X_* \cdot A_p = X_* \cdot A_t = -1$ and A_p and A_t have coefficients 1 and 2 in X_* respectively. Note that $A_t \neq A_1$ and $X_* \cdot A_1 = 0$. Recall that A_p has coefficient 1 in both Z and D. So A_p has coefficient 2 in X. Let $\rho^{-1}(E_1) = A_2$. Let $\rho^{-1}(E_2) = A_p \cup A_t$. Then by case (II) of section 2, A_t has coefficient 4 in X. Note that E_2 has coefficient 1 in Z_E because E_2 follows E_1. Again by case (II) of section 2, A_p and A_t have coefficients 1 and 2, respectively. It is impossible.

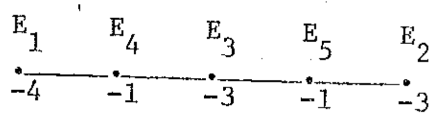
Examples of Proposition 3.4

(I) $Z \cdot Z = -2$

(A) $Z \cdot Z = Z \cdot A_1 = -2$

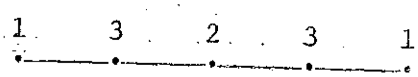
Let $V = \{z^3 + 3x^5 z + 2y^{10} = 0\}$

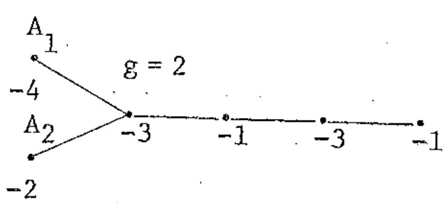
$B = \{-108(x^{15} + y^{20}) = 0\}$

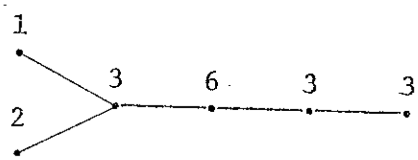
E:


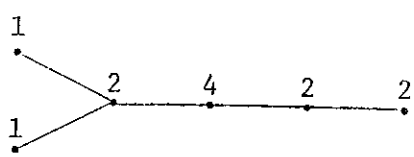
$(B) = 15E_1 + 20E_2 + 40E_3 + 60E_4 + 60E_5 + W^{(5)}$

where $W^{(5)}$ meets E_4 in five points

Z_E =


A:


X =


Z =


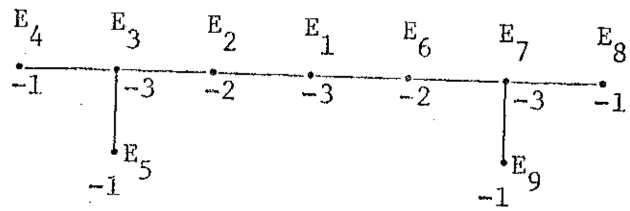
(B) $Z \cdot Z = -2$, $Z \cdot A_1 = Z \cdot A_2 = -1$

(B1) $X \cdot A_3 = -1$ and A_3 has coefficient 3 in X

Let $V = \{z^3 + 2(x^2 + y^6)(x^6 + y^2) = 0\}$

$B = \{-108(x^2 + y^6)^2 \cdot (x^6 + y^2)^2 = 0\}$

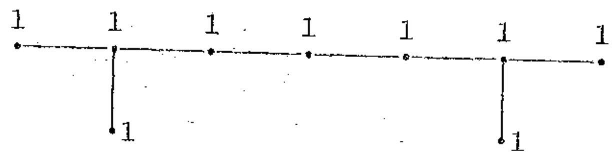
E:



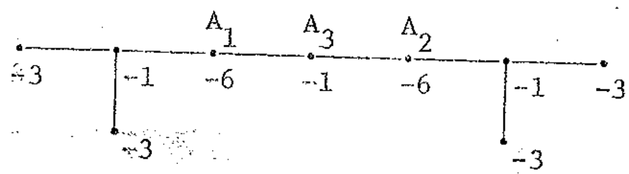
$(B) = 8E_1 + 12E_2 + 16E_3 + 18E_4 + 18E_5 + 12E_6 + 16E_7 + 18E_8 + 18E_9 + W^{(9)}$

where $W^{(9)}$ meets E_4, E_5, E_8 and E_9 in one point with multiplicity 2, respective.y.

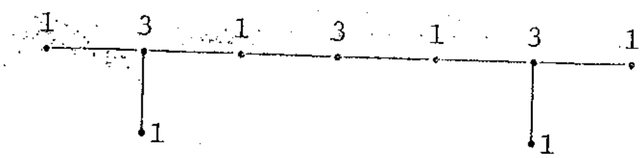
$Z_E =$



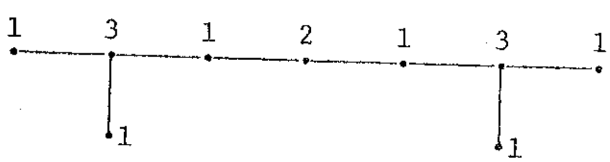
A:



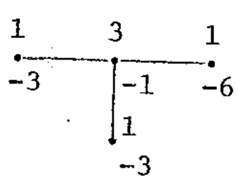
$X =$



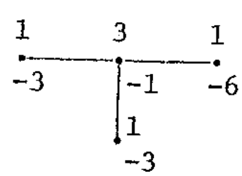
$Z =$



$Z_1 =$



$Z_2 =$



$Z_1 \cdot Z_1 = Z_1 \cdot A_1 = -3$

$Z_2 \cdot Z_2 = Z_2 \cdot A_2 = -3$

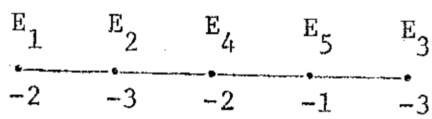
(B2) $X \cdot A_1 = X \cdot A_3 = -1$ and A_1 and A_3 have coefficients 1 and 2 in X respectively.

(iiid) $Z_* \cdot A_t = -1$, $Z_* \cdot A_2 = -2$ and $Z_* \cdot Z_* = -3$.

Let $V = \{z^3 + 2x^3 z + 2y^{12} = 0\}$

$B = \{-108(x^9 + y^{24}) = 0\}$

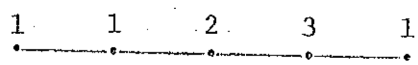
E:



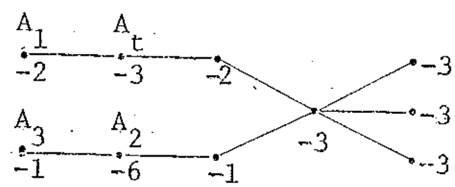
$(B) = 9E_1 + 18E_2 + 24E_3 + 45E_4 + 72E_5 + W^{(5)}$

where $W^{(5)}$ meets E_5 in three points.

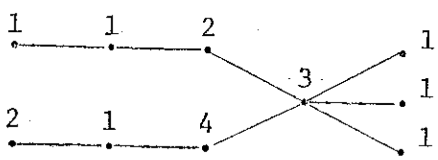
Z_E =



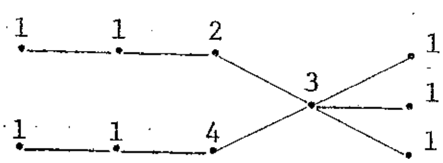
A:



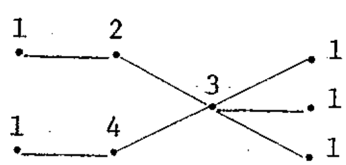
X =



Z =



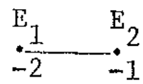
Z_* =



(iiie) $Z_* \cdot Z_* = Z_* \cdot A_2 = -3$

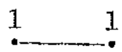
Let $V = \{z^3 + 3x^3 z + 2y^9 = 0\}$

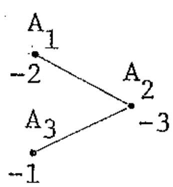
$B = \{-108(x^9 + y^{18}) = 0\}$

E: 

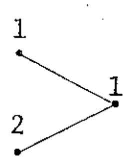
$(B) = 9E_1 + 18E_2 + W^{(2)}$

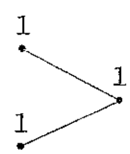
where $W^{(2)}$ meets E_2 in nine points.

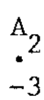
$Z_E =$ 

A: 

where A_2 is of genus 3.

$X =$ 

$Z =$ 

$Z_* =$ 

where A_2 is of genus 3.

(C) $Z \cdot Z = -2$, $Z \cdot A_1 = -1$ and A_1 has coefficient 2 in Z

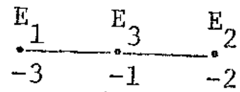
$X \cdot A_2 = -1$ and A_2 has coefficient 3 in Z.

(C1) $X_* \cdot X_* = X_* \cdot A_s = -3$

Let $V = \{x^3 + 2(x^6 + y^3)(x^6 + y^4) = 0\}$

$B = \{-108(x^6 + y^3)^2 \cdot (x^6 + y^4)^2 = 0\}$

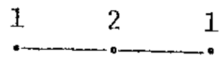
E:



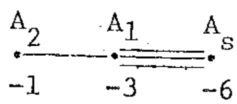
$(B) = 14E_1 + 24E_2 + 42E_3 + W^{(3)}$

where $W^{(3)}$ meets E_2 in three points with each point multiplicity 2 and E_3 in two points with each point multiplicity 2.

Z_E =

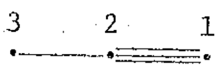


A:

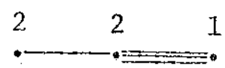


where A_1 and A_s are of genus 1.

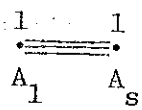
X =



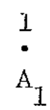
Z =



X_* =



F =

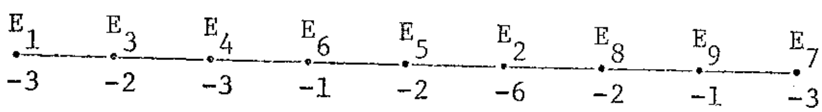


(C2) $X_* \cdot X_* = -3$, $X_* \cdot A_s = -2$ and $X_* \cdot A_t = -1$.

Let $V = \{z^3 + 3x^4yz + 2(y^5 + x^{10}y^2 + x^{11}) = 0\}$

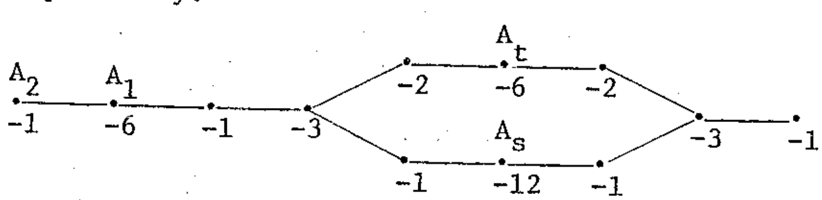
$B = \{-108[x^{12}y^3 + (y^5 + x^{10}y^2 + x^{11})^2] = 0\}$

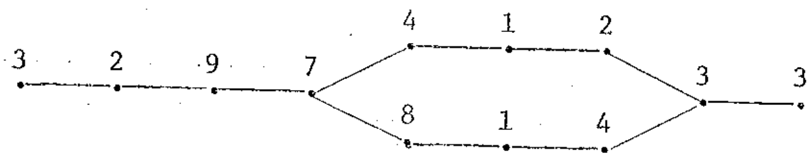
E:

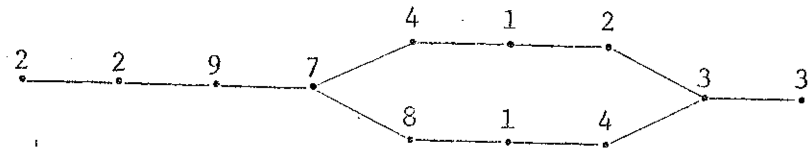


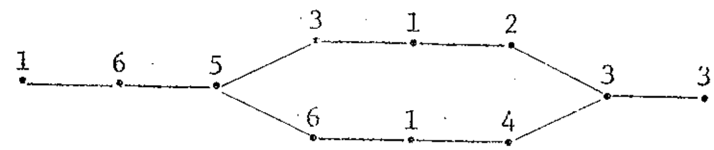
$(B) = 10E_1 + 18E_2 + 30E_3 + 50E_4 + 69E_5 + 120E_6 + 20E_7 + 39E_8 + 60E_9 + W^{(9)}$ where $W^{(9)}$ meets E_6 and E_9 in one point respectively.

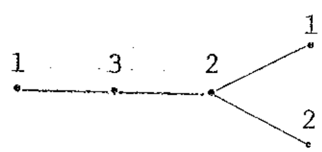
A:



X =


Z =


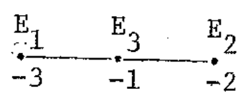
X_* =


F =


(C3) $X_* \cdot A_{s1} = X_* \cdot A_{s2} = X_* \cdot A_{s3} = -1$

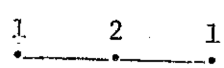
Let $V = \{z^3 + 3y^4 z + 2x^4 = 0\}$

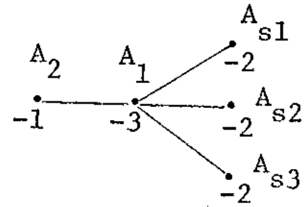
$B = \{-108(y^{12} + x^8) = 0\}$

E:


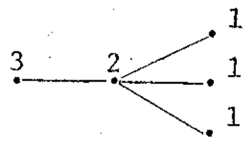
$(B) = 8E_1 + 12E_2 + 24E_3 + W^{(3)}$

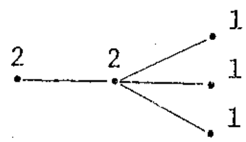
where $W^{(3)}$ meets E_3 in four points.

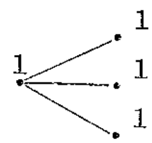
Z_E =


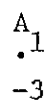
A:


where A_1 is of genus 1.

X =


Z =


X_* =


F =


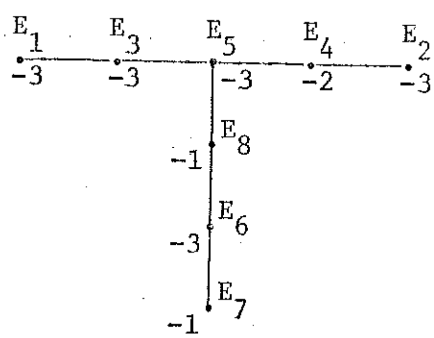
where A_1 is of genus 1.

(C4) $X_* \cdot A_s = -1$ and A_s has coefficient 3 in X_*.

(i) Let X_* be not a fundamental cycle on supp(Z - 2D).

Let $V = \{z^3 + 2(x^8 + y^5) = 0\}$

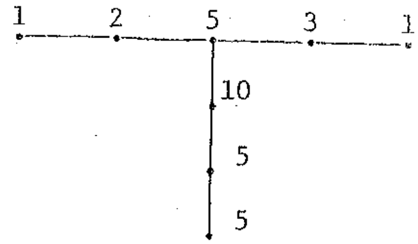
$B = \{-108(x^8 + y^5)^2 = 0\}$

E =


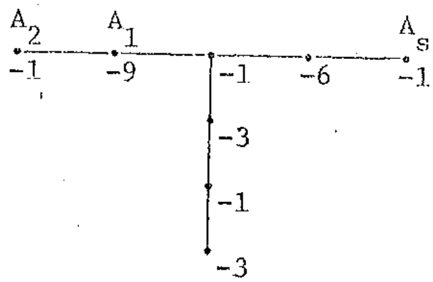
$(B) = 10E_1 + 16E_2 + 30E_3 + 48E_4 + 80E_5 + 82E_6 + 84E_7 + 162E_8 + W^{(8)}$

where $W^{(8)}$ meets E_7 in one point with multiplicity 2.

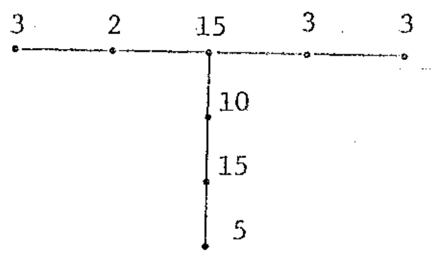
Z_E =


1
2
5
3
1
10
5
5

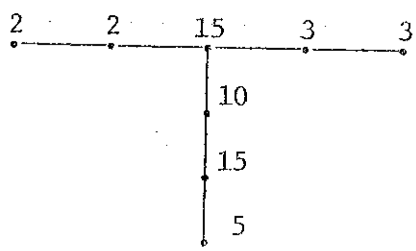
A:


A2
A1
As
-1
-9
-1
-6
-1
-3
-1
-3

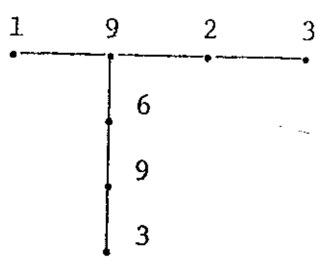
X =


3
2
15
3
3
10
15
5

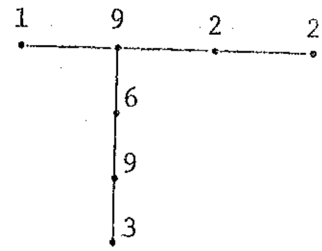
Z =


2
2
15
3
3
10
15
5

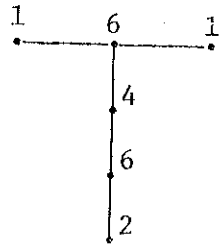
X_* =


1
9
2
3
6
9
3

$Z_* =$



$F =$

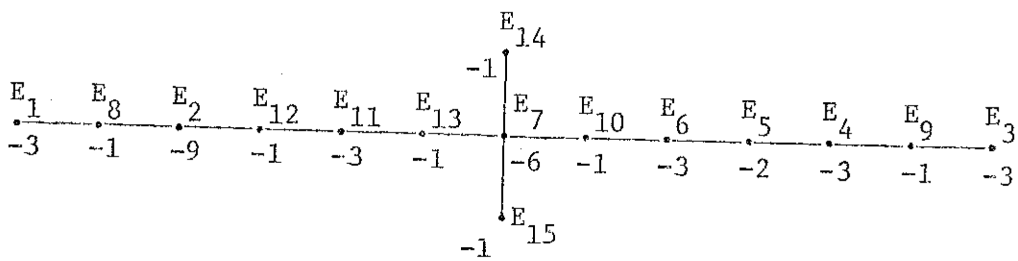


(ii) Let X_* be the fundamental cycle on $\mathrm{supp}(Z - 2D)$.

Let $V = \{z^3 + 2(x^{10} + y^{22}) = 0\}$

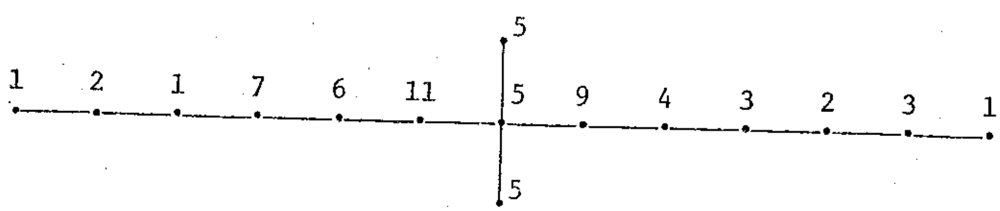
$B = \{-108(x^{10} + y^{22})^2 = 0\}$

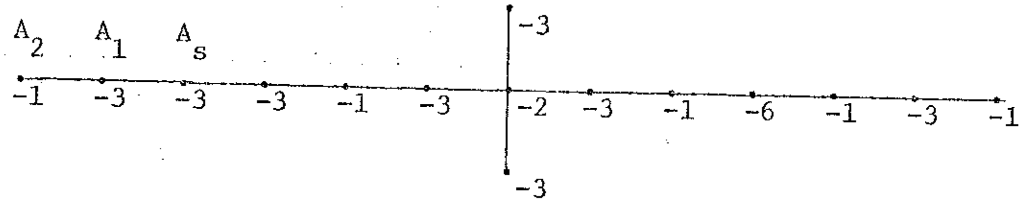
E:

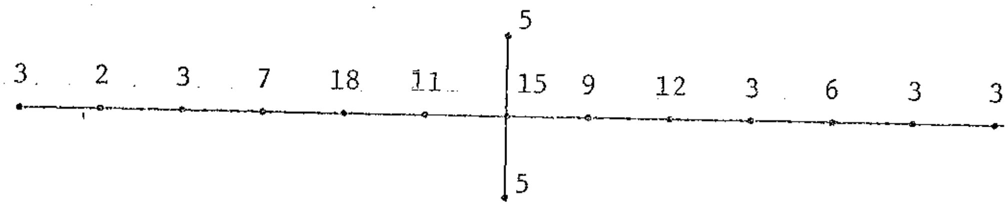


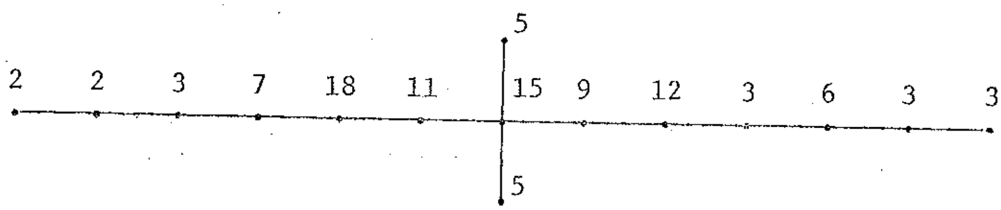
$(B) = 20E_1 + 40E_2 + 44E_3 + 88E_4 + 132E_5 + 176E_6 + 220E_7 + 60E_8$
$+ 132E_9 + 396E_{10} + 260E_{11} + 300E_{12} + 480E_{13} + 222E_{14}$
$+ 222E_{15} + W^{(15)}$ where $W^{(15)}$ meets E_{14} and E_{15} in one point with each point multiplicity 2, respectively.

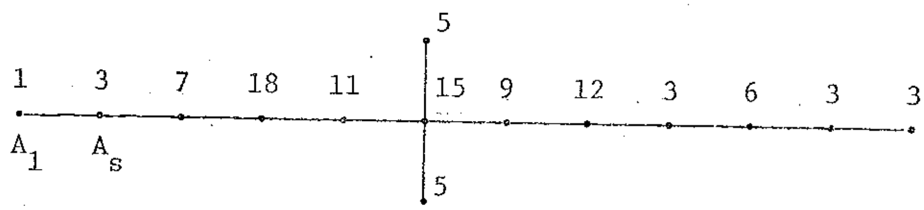
$Z_E =$

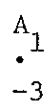


A:


X =


Z =


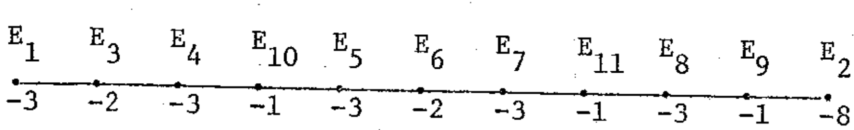
X_* =


F =


(C5) $X_* \cdot A_s = X_* \cdot A_t = -1$ and A_s and A_t have coefficients 2 and 1 in X_* respectively. Recall that X_* is the fundamental cycle on supp(Z - 2D).

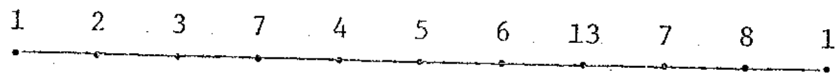
Let $V = \{z^3 + 3y^5z + 2x^4 = 0\}$

$B = \{-108(y^{15} + x^8) = 0\}$

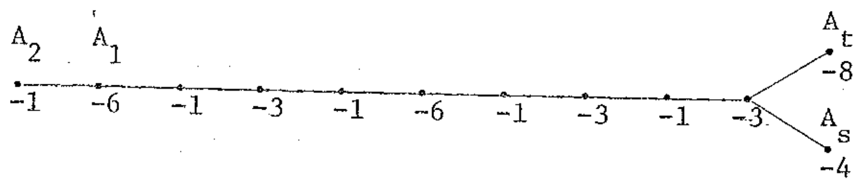
E:


$$(B) = 8E_1 + 15E_2 + 24E_3 + 40E_4 + 56E_5 + 72E_6 + 88E_7 + 104E_8 + 120E_9 + 96E_{10} + 192E_{11} + W^{(11)}$$ where $W^{(11)}$ meets E_9 in one point.

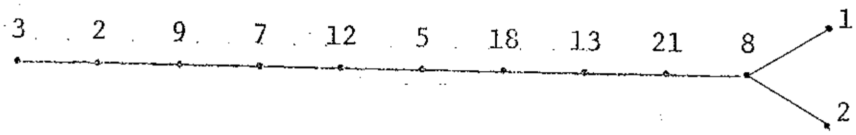
$Z_E =$



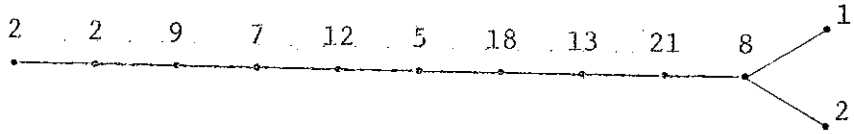
A:



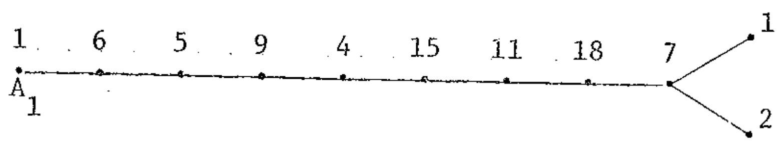
X =



Z =



$X_* =$



F =

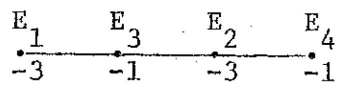
(II) $Z \cdot Z = Z \cdot A_1 = -1$.

$X \cdot A_2 = -1$ and A_2 has coefficient 3 in X.

(A) $X_* \cdot A_t = -1$ and A_t has coefficient 3 in X_*.

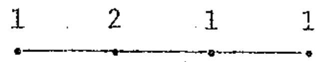
Let $V = \{z^3 + 2(x^7 + y^{21}) = 0\}$

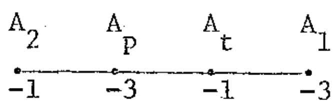
$B = \{-108(x^7 + y^{21})^2 = 0\}$

E: 

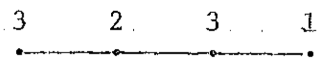
$(B) = 14E_1 + 28E_2 + 42E_3 + 42E_4 + W^{(4)}$

where $W^{(4)}$ meets E_4 in seven points with each point multiplicity 2.

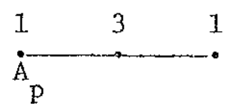
Z_E = 

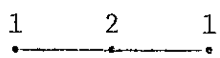
A: 

where A_1 is of genus 6.

X = 

Z = 

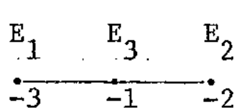
X_* = 

Z_* = 

(B) $X_* \cdot A_1 = X_* \cdot A_t = -1$ and A_1 and A_t have coefficients 1 and 2 in X_* respectively.

Let $V = \{z^3 + 3y^5z + 2x^5 = 0\}$

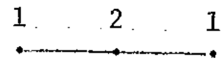
$B = \{-108(y^{15} + x^{10}) = 0\}$

E: 

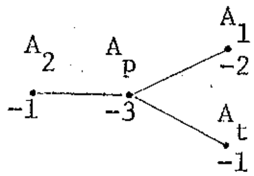
$(B) = 10E_1 + 15E_2 + 30E_3 + W^{(3)}$

where $W^{(3)}$ meets E_3 in five points.

Z_E:

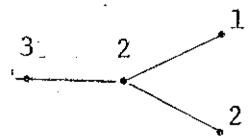


A:

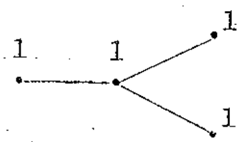


where A_p is of genus 2.

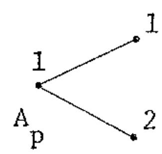
X =



Z =

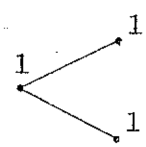


X_* =



where A_p is of genus 2.

Z_* =



<u>Corollary 3.5</u> Suppose that the hypotheses of Proposition 3.4 are satisfied. Consider the following subcases as we discussed in Proposition 3.4. We will follow notations and numberings of Proposition 3.4.

(I) Let $Z \cdot Z = -2$

(A) A_1 and A_2 appear first in a resolution process by (2.2).

(B)

(B1) A_3 appears first in that resolution. A_1 follows A_3 and A_2 follows A_3.

(B2) Either (i) A_1 and A_3 first appear in that resolution and A_2 follows A_1 and A_3 at the same time

or (ii) A_1 and A_3 first appear in that resolution and A_2 and A_t follow A_3 and A_1 at the same time.

(C) A_2 appears first in that resolution.

(C1) A_s follows A_2 and A_1 follows A_s.

(C2) A_s and A_t follow A_2 at the same time and A_1 follows A_s and A_t at the same time.

(C3) A_{s1}, A_{s2} and A_{s3} follow A_2 at the same time and A_1 follows A_{s1}, A_{s2} and A_{s3} at the same time.

(C4) A_s follows A_2 and A_1 follows A_s.

(C5) A_s and A_t follows A_2 at the same time and A_1 follows A_s and A_t at the same time.

(II) Let $Z \cdot Z = -1$. A_2 appear first in that resolution.

(A) A_t follows A_2 and then A_1 follows A_t and A_p follows A_t.

(B) A_1 and A_t follow A_2 at the same time and A_p follows A_1 and A_t at the same time.

Now consider the minimal resolution $r : N \to V$. Thus to describe $r^*(m)$ for irreducible triple points, let us consider the case that $r^*(m)$ is not principal. If $r^*(m)$ is not principal, let $Y = \Sigma m_i A_i$, where m_i is the order to which functions $g \circ r$, $g \in m$, generically vanish on A_i. Let $r_1 : N_1 \to V$ be a resolution on which $r_1^*(m)$ is principal. Let $r_1 = r \circ \pi$. Then on N_1, letting π^* denote the pull-back,

$$X > \pi^* Y \geq \pi^* Z = Z_1$$

where X is the divisor of the pull-back $r_1^*(m)$ of the maximal ideal at $P \in V$. Since $Z \cdot Z = -2$ or -1, consider the following cases:

(I) If $Z_1 \cdot Z_1 = -2$, then $\pi^* Y = Z_1$. Then Y is the fundamental cycle Z on N. If $A_i \cdot Z = 0$ on N, then as in the proof of [L5, Proposition 5.1, p. 323] a generic function $g \circ r$, $g \in m$, generates $r^*(m)$ in a neighborhood of A_i. So $r^*(m)$ is locally principal near A_i for $A_i \cdot Z = 0$. Since $Z \cdot Z = -2$, consider the subcases below:

(i) There exists a A_1 with $z_1 = 1$ such that $A_1 \cdot Z = -2$ where $Z = \Sigma z_i A_i$. On A_1, $\mathcal{O}(-Z)/\mathcal{O}(-Z - A_1)$ is the sheaf of germs of a section of a line bundle L_1 of chern class $-A_1 \cdot Z = 2$. Let $g \in m$ be generic, so that $g \circ r$ vanishes to order $m_i = z_i$ on each of the A_i. Then as a section of L_1, $g \circ r$ has either one double zero at some point $s \in A_1$ or two distinct simple zeros at s_1 and s_2 in A_1. First of all, observe the following facts: Suppose that there exists a resolution $r_1 : N_1 \to V$ on which $r_1^*(m)$ is principal. Let X be its divisor. Then by (*) below Proposition 3.1, $X \cdot X = -3$. So we may assume that $r_1 : N_1 \to V$ is a resolution by (2.2), without loss of generality. Then by the explicit computation of such a resolution in section 2 and by case (A) of (I) in Proposition 3.4, we see that any connected component of $\cup A_i$, $i \neq 1$ of $\text{supp}\, X$ which does not contain A_2 is a exceptional set of the first kind, and so can be collapsed down, become to the empty graph and moreover, does not produce any singularity of A_1 for the minimal resolution except possibly a singularity of A_1 resulted from blowing down the component containing A_2, because we assumed that $r^*(m)$ is not principal. So we get for the minimal resolution that $\text{supp}\, Z = A_1$, $A_1 \cdot A_1 = -2$ and A_1 is a rational curve. By following the notations of case (A) of (I) in Proposition 3.4, observe that $X = Z + D$, $D \cdot A_1 = 1$, $A_2 \subset \text{supp}\, D$, $\text{supp}\, D$ is the connected component with $D \cdot D = -1$, $X \cdot A_1 = X \cdot A_2 = -1$ and A_1 and A_2 have coefficients 1 and 2 in X, respectively. Therefore for the minimal resolution we claim that $\text{supp}\, Z = A_1$ is nonsingular. If not, that singularity could be resulted from blowing down the connected component of $\cup A_i$, $i \neq 1$ which does contain A_2 but we would get $D \cdot A_1 \geq 2$. Thus we proved that $\text{supp}\, Z = A_1$ is nonsingular and $A_1 \cdot A_1 = -2$. So we might have two subcases below.

(a) Assume that $g \circ r$ have one double zero at $s \in A_1$, as a section of L_1. Since $r^*(m)$ is assumed not to be principal, all such $g \circ r$ have double zeros at the same point s. Now s is a regular point. Let $\pi' : N' \to N$ be the blow-up of N at s. Let $A_0 = (\pi')^{-1}(s)$ and let $r' = r \circ \pi'$. Then still all such $g \circ r'$,

$g \in m$ have common simple zeros at $A_1 \cap A_0$. So $(r')^*(m)$ is not principal. But observe that $A_0 = A_2$, following the notation of case (A) of (I) in Proposition 3.4 and so $(r')^*(m)$ would be principal. Thus we get a contradiction.

(b) Assume that $g \circ r$ has two simple zeros at s_1 and s_2 in A_1 respectively with $s_1 \neq s_2$. If all such $g \circ r$ have two simple zeros at the same points s_1 and s_2 then $X \cdot X$ would be ≤ -4. It is a contradiction. Since $r^*(m)$ is assumed not to be principal we may assume that all such $g \circ r$ have one and only one common simple zeros at the same point s_1. Moreover, since $Z \cdot Z = -2$, s_1 is a regular point of A. Let $\pi : N_1 \to N$ be the blow-up of N at s_1. Let $A_0 = \pi^{-1}(s)$ and let $r_1 = r \circ \pi$.
Then $r_1^*(m) \subset \mathcal{O}(-\pi^* Z - A_0)$ and $(\pi^* Z + A_0) \cdot (\pi^* Z + A_0) = -3$.
Thus $r_1^*(m) = \mathcal{O}(-\pi^* Z - A_0)$ and $r_1^*(m)$ is principal. Moreover, in this case we will prove later in Theorem 4.7 that V has a normalization isomorphic to the variety $\{z^3 + 3xz + 2y^2 = 0\}$. That has a rational double point at (0,0,0).

(ii) There exist $A_1 \neq A_2$ such that $A_1 \cdot Z = A_2 \cdot Z = -1$. On A_1, $\mathcal{O}(-Z)/\mathcal{O}(-Z - A_1)$ is the sheaf of germs of a section of a line bundle L_1 of chern class $-A_1 \cdot Z = 1$. On A_2, $\mathcal{O}(-Z)/\mathcal{O}(-Z - A_2)$ is the sheaf of germs of a section of a line bundle L_2 of chern class $-A_2 \cdot Z = 1$. Let $g \in m$ be generic, so that $g \circ r$ vanishes to order $m_i = z_i$ on each of the A_i. Then, as a section of L_1, $g \circ r$ has a simple zero at some point $s_1 \in A_1$, and also, as a section of L_2, $g \circ r$ has a simple zero at some point $s_2 \in A_2$.

But either $A_1 \cap A_2 = \phi$ or $A_1 \cap A_2 \neq \phi$

(a) Assume that $A_1 \cap A_2 = \phi$. Since $r^*(m)$ is assumed not to be principal, we claim that either all such $g \circ r$ have simple zeros at the same point $s_1 \in A_1$ or all such $g \circ r$ have simple zeros at the same point $s_2 \in A_2$, not both. If all such $g \circ r$ have simple zeros at the same point $s_1 \in A_1$ and also have simple zeros at the same point $s_2 \in A_2$, then $X \cdot X$ would be ≤ -4. It would be a contradiction.

(a1) Let us assume that all such $g \circ r$ have simple zeros at the same point $s_1 \in A_1$. Since $Z \cdot Z = -2$, s_1 is a regular point. Let $\pi : N_1 \to N$ be the blow-up of N at s_1. Let $A_0 = \pi^{-1}(s_1)$ and let $r_1 = r \circ \pi$. Then $r_1^*(m) \subset \mathcal{O}(-\pi^* Z - A_0)$ and $(\pi^* Z + A_0) \cdot (\pi^* Z + A_0) = -3$. Thus $r_1^*(m) = \mathcal{O}(-\pi^* Z - A_0)$ and $r_1^*(m)$ is principal.

(a2) Let us assume that all such $g \circ r$ have simple zeros at the same point $s_2 \in A_2$. Since $Z \cdot Z = -2$, s_2 is a regular point. Let $\pi : N_1 \to N$ be the blow-up of N at s_2. Let $A_0 = \pi^{-1}(s_2)$ and let $r_1 = r \circ \pi$. Then $r_1^*(m) \subset \mathcal{O}(-\pi^* Z - A_0)$ and $(\pi^* Z + A_0) \cdot (\pi^* Z + A_0) = -3$. Thus $r_1^*(m) = \mathcal{O}(-\pi^* Z - A_0)$ and $r_1^*(m)$ is principal.

(b) Assume that $A_1 \cap A_2 \neq \phi$. Since $r^*(m)$ is assumed not to be principal, we claim that either (b1) all such $g \circ r$ have simple zeros at the same point $s_1 \in A_1 - A_2$ or (b2) all such $g \circ r$ have simple zeros at the same point $s_2 \in A_2 - A_1$ or (b3) all such $g \circ r$ have simple zeros at the same point $s = A_1 \cap A_2$. In the first two cases (b1) and (b2), we have the same result as in (a). So we may assume that all such $g \circ r$ have simple zeros at the same point $s = A_1 \cap A_2$. Since $Z \cdot Z = -2$, s is a regular point of A_1 and A_2 because of case (B1) of (I) of Proposition 3.4. Let $\pi : N_1 \to N$ be the blow-up at s. Let $A_0 = \pi^{-1}(s)$ and let $r_1 = r \circ \pi$. Then $r_1^*(m) \subset \mathcal{O}(-\pi^* Z - A_0)$ and $(\pi^* Z + A_0) \cdot (\pi^* Z + A_0) = -3$. Thus $r_1^*(m) = \mathcal{O}(-\pi^* Z - A_0)$ and $r_1^*(m)$ is principal.

(iii) There is A_1 with $z_1 = 2$ such that $A_1 \cdot Z = -1$ where $Z = \Sigma z_i A_i$. On A_1, $\mathcal{O}(-Z)/\mathcal{O}(-Z - A_1)$ is the sheaf of germs of a section of a line bundle L_1 of chern class $-A_1 \cdot Z = 1$. Let $g \in m$ be generic, so that $g \circ r$ vanishes to order $m_i = z_i$ on each of the A_i. Then as a section of L_1, $g \circ r$ has a simple zero at some point $s \in A_1$. Since $r^*(m)$ is assumed not to be principal, all such $g \circ r$ have simple zeros at the same point s. Moreover, since $Z \cdot Z = -2$, s is a regular point of A. Let $\pi : N_1 \to N$ be the blow-up of N at s. Let $A_0 = \pi^{-1}(s)$ and let $r_1 = r \circ \pi$. Then $r_1^*(m) \subset \mathcal{O}(-\pi^* Z - A_0)$ and $(\pi^* Z + A_0) \cdot (\pi^* Z + A_0) = -3$. Thus $r_1^*(m) = \mathcal{O}(-\pi^* Z - A_0)$ and $r_1^*(m)$ is principal.

(II) If $Z_1 \cdot Z_1 = -1$, then $-3 = X \cdot X \leq \pi^* Y \cdot \pi^* Y \leq Z_1 \cdot Z_1 = -1$. So $\pi^* Y \cdot \pi^* Y = -2$ or -1. Since $Z \cdot Z = -1$, then let A_1 be such that $A_1 \cdot Z = -1$.

(i) Assume that $\pi^* Y \cdot \pi^* Y = -2$. Since $Y > Z$, let $Y = Z + G$. Then $-2 = Y \cdot Y = Z^2 + 2Z \cdot G + G^2$, $G \cdot Z \leq 0$ and $G^2 \leq 0$ imply $G^2 = -1$ and $G \cdot Z = 0$. Since $G \cdot Z = 0$, $A_1 \not\subset \operatorname{supp} G$ and $G \cdot A_1 = 1$. For

$A_i \subset \operatorname{supp} G$, $G \cdot A_i = (Y - Z) \cdot A_i \leq 0$. Since $G^2 = -1$, let A_p be such that $G \cdot A_p = -1$. Then $Y \cdot A_p = -1$ and $Y \cdot Y \leq Y \cdot 2A_p = -2$ imply that A_p has coefficient 2 in Y. So Y replaces Z with the previous arguments as in case (iii) of (I).

(ii) Assume that $\pi^* Y \cdot \pi^* Y = -1$. Then $\pi^* Y = Z_1$ and also Y is the fundamental cycle Z on N. Since $A_i \cdot Z = 0$ for $i \neq 1$, then as in the proof of [L5, Proposition 5.1, p. 323], a generic function $g \circ r$, $g \in m$, generates $r^*(m)$ in a neighborhood of A_i. So $r^*(m)$ is locally principal near A_i for $A_i \cdot Z = 0$. On A_1, $\mathcal{O}(-Z)/\mathcal{O}(-Z - A_1)$ is the sheaf of germs of a section of a line bundle L_1 of chern class $-A_1 \cdot Z = 1$. Let $g \in m$ be generic, so that $g \circ r$ vanishes to order $m_i = z_i$ on each of the A_i where $Z = \Sigma z_i A_i$. Then as a section of L_1, $g \circ r$ has a simple zero at some point $s_1 \in A_1$. Since $r^*(m)$ is assumed not to be principal, all such $g \circ r$ have simple zeros at the same point $s_1 \in A_1$. Since $Y \cdot Y = Z \cdot Z = -1$, s_1 is a regular point of A. Let $\pi' : N' \to N$ be the blow-up at s_1. Let $A_0 = (\pi')^{-1}(s_1)$ and let $r' = r \circ \pi'$. Then $(r')^*(m) \subset \mathcal{O}(-\pi'^* Z - A_0)$ and $(\pi'^* Z + A_0) \cdot (\pi'^* Z + A_0) = -2$. Let $Z' = \pi'^* Z + A_0$. Then $Z' \cdot A_j = 0$ for $A_j \neq A_0$. $Z' \cdot A_0 = -1$ and A_0 has coefficient 2 in Z'. Therefore, as in the proof of the previous case (i), let $\pi'' : N'' \to N'$ be the blow-up at $s_2 \in A_0$. Let $A_0' = (\pi'')^{-1}(s_2)$ and let $r'' = r' \circ \pi''$. Then $(r'')^*(m) = \mathcal{O}(-\pi''^* Z' - A_0')$ and $(r'')^*(m)$ is principal.

If we summarize the previous results, we have the following:

<u>Proposition 3.6</u> Let $r : N \to V$ be the minimal resolution of a two-dimensional irreducible triple point P. Let m be the maximal ideal at P. Let Z be the fundamental cycle on N and let us assume that $Z \cdot Z = -1$ or -2. If $r^*(m)$ is principal, then the divisor X of $r^*(m)$ satisfies $X > Z$. If $r^*(m)$ is not principal, consider the following cases:

(I) Suppose that $Z \cdot Z = -2$ on N. Then there exist the following three subcases.

(i) There exists A_1 with $z_1 = 1$ such that $A_1 \cdot Z = -2$ where $Z = \Sigma z_i A_i$. Then $\mathcal{O}(-Z)/r^*(m)$ is the structure sheaf for an embedded point $s \in A_1$. s is a regular point. Blowing up N at s makes $r_1^*(m)$ principal where $\pi : N_1 \to N$ is the blow-up of N at s and $r_1 = r \circ \pi$. Moreover, V has a normalization isomorphic to the variety $\{z^3 + 3xz + 2y^2 = 0\}$, which will be proved in Theorem 4.7.

(ii) There exist $A_1 \neq A_2$ such that $Z \cdot A_1 = Z \cdot A_2 = -1$. Then $\mathcal{O}(-Z)/r^*(m)$ is the structure sheaf for an embedded point s where $s = A_1 \cap A_2$, $s \in A_1 - A_2$ or $s \in A_2 - A_1$. s is a regular point of $A_1 \cup A_2$. Blowing up N at s makes $r_1^*(m)$ principal where $\pi : N_1 \to N$ is the blow-up of N at s and $r_1 = r \circ \pi$.

(iii) There exists A_1 with $z_1 = 2$ such that $A_1 \cdot Z = -1$ where $Z = \Sigma z_i A_i$. Then $\mathcal{O}(-Z)/r^*(m)$ is the structure sheaf for an embedded point $s \in A_1$. s is a regular point. Blowing up N at s makes $r_1^*(m)$ principal where $\pi : N_1 \to N$ is the blow-up of N at s and $r_1 = r \circ \pi$.

(II) Suppose that $Z \cdot Z = -1$ on N. Let $Y = \Sigma m_i A_i$, where m_i is the order to which functions g o r, $g \in m$, generically vanish on A_i. Then there exist the following two subcases.

(i) Let $Y \cdot Y = -2$. Then there is A_p such that $Y \cdot A_p = -1$, $m_p = 2$. $\mathcal{O}(-Y)/r^*(m)$ is the structure sheaf for an embedded point $s \in A_p$. s is a regular point of A. Blowing up N at s makes $r_1^*(m)$ principal where $\pi : N_1 \to N$ is the blow-up of N at s and $r_1 = r \circ \pi$.

(ii) Let $Y \cdot Y = -1$, i.e., $Y = Z$. Then there is A_1 such that $Z \cdot A_1 = -1$. $\mathcal{O}(-Z)/r^*(m)$ is the structure sheaf for an embedded point $s \in A_1$. s is a regular point. Let $\pi : N_1 \to N$ be the blow-up of N at s. Let $r_1 = r \circ \pi$. Let $A_0 = \pi^{-1}(s)$ and $Z_1 = \pi^* Z + A_0$. Then $r_1^*(m) \subset \mathcal{O}(-Z_1)$ and $Z_1 \cdot Z_1 = -2$. Again $\mathcal{O}(-Z_1)/r_1^*(m)$ is the structure sheaf for an embedded point $t \in A_0$. t is a regular point of supp Z_1. Blowing up N_1 at t makes $r_2^*(m)$ principal where $\pi_1 : N_2 \to N_1$ is the blow-up of N_1 at t and $r_2 = r_1 \circ \pi_1 = r \circ \pi \circ \pi_1$.

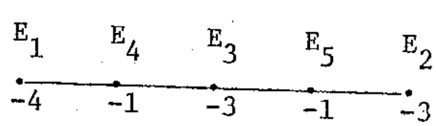
<u>Examples of Proposition 3.6</u>

(I) $Z \cdot Z = -2$

Let $Z \cdot A_1 = -2$

Let $V = \{z^3 + 3x^3 z + 2y^2 x^3 = 0\}$.

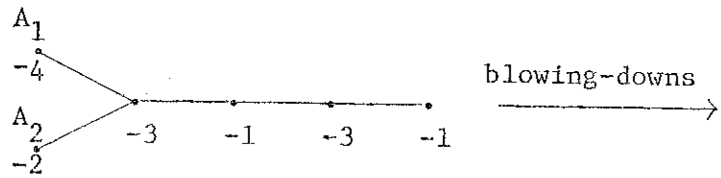
$B = \{-108x^6(x^3 + y^4) = 0\}$.

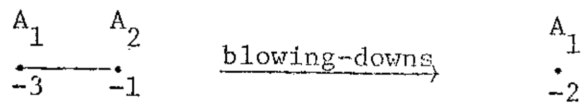
E: 

$(B) = 9E_1 + 16E_2 + 26E_3 + 36E_4 + 42E_5 + W^{(5)}$

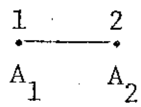
where $W^{(5)}$ meets E_4 in one point and meets E_2 in one point with multiplicity 6.

Its resolution is

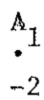




X =



Z =

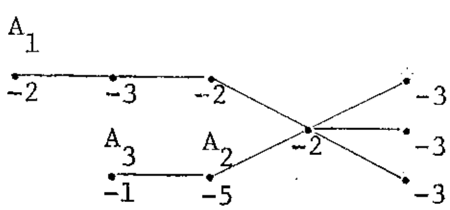


Note that V is not normal at (0,0,0) and the normalization of V is given by $V_1 = \{z^3 + 3xz + 2y^2 = 0\}$ which has a rational double point singularity at (0,0,0).

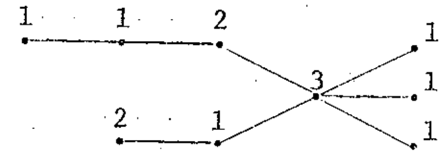
(ii) Let $Z \cdot A_1 = Z \cdot A_2 = -1$

(a) Let $V = \{z^3 + 3x^3 z + 2y^{12} = 0\}$ (an example of Proposition 3.4)

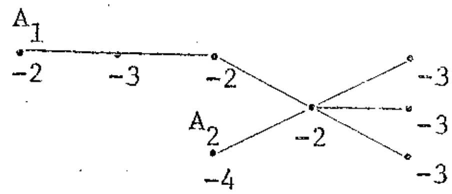
$B = \{-108(x^9 + y^{24}) = 0\}$



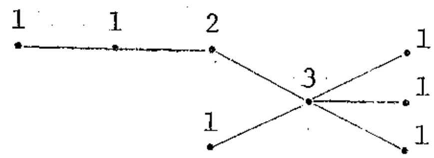
X =



blowing down →



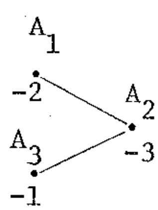
Z =



Note that $s \in A_2 - \cup A_i$, $i \neq 2$ and $A_1 \cap A_2 = \phi$

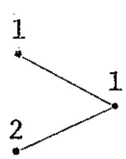
(b) Let $V_1 = \{z^3 + 3x^3z + 2y^9 = 0\}$ (an example of Proposition 3.4)

$B = \{-108(x^9 + y^{18}) = 0\}$

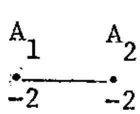


where A_2 is of genus 3.

X =

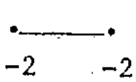


blowing down →



where A_2 is of genus 3.

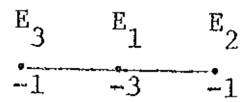
Z =



Note that $s \in A_2 - A_1$ and $A_1 \cap A_2 \neq \phi$

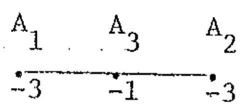
Let $V_2 = \{z^3 + 2y(x^7 + y^{14}) = 0\}$

$B = \{-108y^2(x^7 + y^{14})^2 = 0\}$

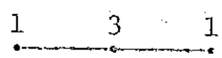
E: 

$(B) = 16E_1 + 30E_2 + 18E_3 + W^{(3)}$

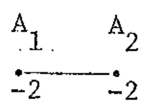
where $W^{(3)}$ meets E_2 in seven points with each point multiplicity 2 and meets E_3 in one point with multiplicity 2.



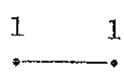
where A_2 is of genus 6.

X = 

blowing down →



where A_2 is of genus 6.

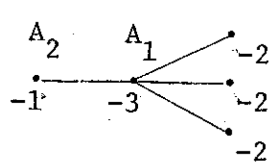
Z = 

Note that $s = A_1 \cap A_2$ and $A_1 \cap A_2 \neq \phi$.

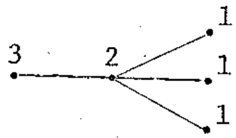
(iii) Let $Z \cdot A_1 = -1$ where A_1 has coefficient 2 in Z.

Let $V = \{z^3 + 3y^4z + 2x^4 = 0\}$ (an example of Proposition 3.4)

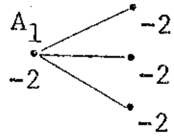
$B = \{-108(y^{12} + x^8) = 0\}$



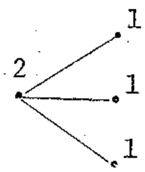
where A_1 is of genus 1.

X = 

blowing-down →



where A_1 is of genus 1.

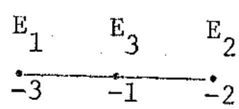
Z = 

(II) $Z \cdot Z = Z \cdot A_1 = -1$

(i) Let $Y \cdot Y = -2$ and $Y \cdot A_p = -1$ where A_p has coefficient 2 in A_p.

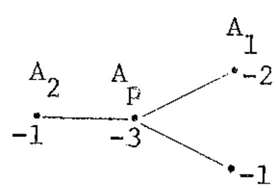
Let $V = \{z^3 + 3y^7z + 2x^7 = 0\}$

$B = \{-108(y^{21} + x^{14}) = 0\}$

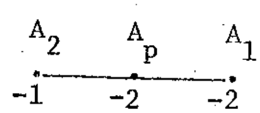
E: 

$(B) = 14E_1 + 21E_2 + 42E_3 + W^{(3)}$

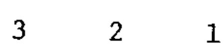
where $W^{(3)}$ meets E_3 in seven points.

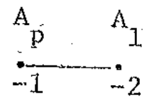
Its resolution is 

where A_p is of genus 3.

blowing-down → 

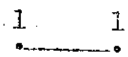
where A_p is of genus 3.

X = 

blowing-down $\rightarrow$ 

where A_p is of genus 3.

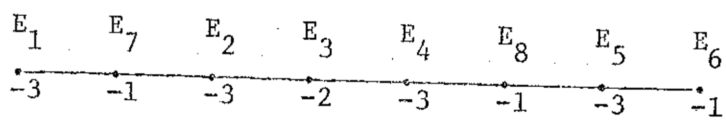
Y = 

Z = 

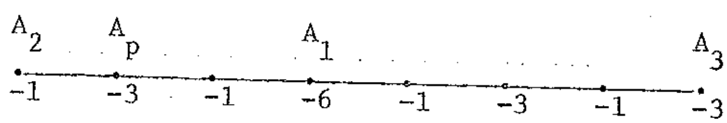
(ii) Let Y = Z

Let $V = \{z^3 + 2(x^4 + y^{24}) = 0\}$

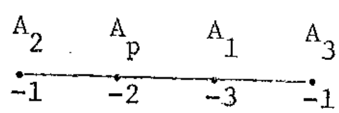
$B = \{-108(x^4 + y^{24})^2 = 0\}$

E: 

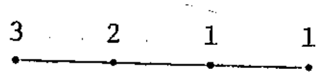
$(B) = 8E_1 + 16E_2 + 24E_3 + 32E_4 + 40E_5 + 48E_6 + 24E_7 + 72E_8 + W^{(8)}$
where $W^{(8)}$ meets E_6 in four points with each point multiplicity 2.
Its resolution is:

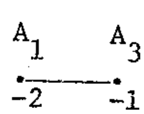


where A_3 is of genus 3.

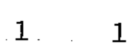
blowing-downs $\rightarrow$ 

where A_3 is of genus 3.

X = 

blowing-downs $\rightarrow$ 

where A_3 is of genus 3.

Y = Z = 

Note that the above examples (i) and (ii) of (II) have the same topological minimal resolutions but they have different equisingular types, i.e., the multiplicity μ are different [M & O]. Example (i) has $\mu = 114$ and example (ii) has $\mu = 138$.

§4 From Resolution to Normal Triple Point with the Condition (4.1)

Given V of Lemma 1.12, let $V = \{(x,y,z) : f = z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ be a normal two-dimensional analytic space with $P = (0,0,0)$ its only singularity. Let us recall that $\tau : M \to \mathbb{C}^2$ and $(B) = W^{(n)} + \Sigma e_i E_i$, $1 \leq i \leq n$, is a divisor of $-108(p^3 + q^2)$ in (2.2). Note that $-108(p^3 + q^2)$ is the z-discriminant of the above f. Throughout the rest of this paper, we can now consider our main concern. But in order to avoid complicated and difficult situations, we are going to impose the condition (4.1) which is described below:

(4.1): If $p(x,y) \not\equiv 0$ in V, then we assume that $-108(p^3 + q^2)$ is a product of distinct prime factors up to a unit near $Q = (0,0)$ and $o(B) = \text{Min}[o(p^3), o(q^2)]$ along E_i, $1 \leq i \leq n$.

If $p(x,y) \equiv 0$, then $q(x,y)$ in V must be square free near Q. Observe that all examples given for Propositions 3.4 and 3.6 satisfy (4.1). Moreover, by Lemma 2.6 and (2.2) we can stop a resolution process by (2.2) if the branch locus satisfies (i) and (ii) in (2.2). Note that (4.1) is preserved under additional blow-ups provided we do not blow up again at a point where an irreducible curve of the proper transform of B and an exceptional curve intersect transversely. Here is an example which fails to satisfy (4.1). Let $V = \{z^3 - 3x^4z + 2(x^6 + x^9 + y^9) = 0\}$. Note that V is normal at P because $-108(p^3 + q^2) = -108(x^9 + y^9)\cdot(2x^6 + x^9 + y^9)$ is a product of distinct prime factors near (0,0). But observe that $o(B) = 15 > o(p^3) = o(q^2) = 12$ along E_1 where E_1 is the curve appearing at the initial blow-up at (0,0). Now given $\tilde{r} : \tilde{N} \to V$, the minimal good resolution of a normal two-dimensional triple point singularity P, when can we get a resolution by (2.2)? Let $r : N \to V$ be a resolution by (2.2). Let $\tilde{\Gamma}$ and Γ denote topological embeddings of $\tilde{r}^{-1}(P)$ and $r^{-1}(P)$ respectively. The examples of (II) of Proposition 3.6 show that topologically different P can yield the same $\tilde{\Gamma}$ for the minimal resolution. If Γ is found, then what can be said about the topological type of the singularity Q of the plane curve which is the discriminant locus determined by P independently of choice of coordinates? Then we need the following propositions.

Lemma 4.1 Let $V = \{(x,y,z) \mid z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ be a two-dimensional analytic space with $P = (0,0,0)$ its only singularity. Suppose that we have completed t of the n-steps needed for a resolution process by (2.2). Let $V^{(t)}$ be locally defined by $\{f_t = z^3 + 3v^{\lambda}p_t(u,v)z + 2v^{\mu}q_t(u,v) = 0\}$ where $v \nmid p_t$ and $v \nmid q_t$. Let $E_t = \{v = 0\}$. Note that the z-discriminant of $f_t =$ $-108(v^{3\lambda}p_t^3 + v^{2\mu}q_t^2) = -108v^e b_t(u,v)$ where $v \nmid b_t(u,v)$ and $b_t(u,v)$ is holomorphic in u and v. Recall that $o(p^3) = 3\lambda$, $o(q^2) = 2\mu$ and $o(B) = e$ along E_t. If f_t is reducible in $\mathbb{C}\langle u,v\rangle[z]$, the polynomial ring in z with coefficients holomorphic in $\mathbb{C}\langle u,v\rangle$, then f_t can be written $f_t = (z - r_1)(z^2 + r_1 z + r_2)$ where r_1 and r_2 are holomorphic near $(u,v) = (0,0)$. Let $o(r) = k$ be an integer such that $v^k \mid r$ and $v^{k+1} \nmid r$ where r is holomorphic near $(u,v) = (0,0)$. Note that $3p = 3v^{\lambda}p_1 = -r_1^2 + r^2$, $2q = 2v^{\mu}q_1 = -r_1 r_2$ and the z-discriminant of f_t is $-108[v^{3\lambda}p_t^3 + v^{2\mu}q_t^2] = (2r_1^2 + r_2)^2(r_1^2 - 4r_2)$. Then compare $o(2r_1^2 + r_2)$ and $o(r_1^2 - 4r_2)$ in terms of $o(r_1^2)$ and $o(r_2)$.

(1) If $o(r_1^2) < o(r_2)$ along E_t, then $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2)$ and $o(p^3) < o(q^2)$ with $o(p^3) \equiv 0 \pmod 2$.

(2) If $o(r_1^2) > o(r_2)$ along E_t, then $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2)$ and $o(p^3) < o(q^2)$.

(3) If $o(r_1^2) = o(r_2)$ along E_t, then $o(p^3) \geq o(q^2)$ with $o(q^2) \equiv 0 \pmod 3$.

(i) If $o(p^3) > o(q^2)$, then $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2)$.

(ii) If $o(B) = o(p^3) = o(q^2)$, then $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2)$.

(iii) If $o(B) = \text{odd} > o(p^3) = o(q^2)$, then $o(r_1^2 - 4r_2) > 2 \cdot o(r_1) = o(r_2) = o(2r_1^2 + r_2)$.

(iv) If $o(B) = \text{even} > o(p^3) = o(q^2)$, then either $o(r_1^2 - 4r_2) > 2 \cdot o(r_1) = o(r_2) = o(2r_1^2 + r_2)$ or $o(2r_1^2 + r_2) > 2 \cdot o(r_1) = o(r_2) = o(r_1^2 - 4r_2)$, not both.

Proof

(1) Since $o(r_1^2) < o(r_2)$, trivially $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2) = o(-r_1^2 + r_2) = o(p) = 2 \cdot o(r_1)$. But $o(q) = o(r_1) + o(r_2) > 3 \cdot o(r_1)$. Thus $o(q^2) > 6 \cdot o(r_1) = o(p^3)$.

(2) It is trivial that $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2) = o(-r_1^2 + r_2) = o(p) = o(r_2)$. But $o(q) = o(r_1) + o(r_2) > \frac{3}{2} \cdot o(r_2)$. Thus $o(q^2) > 3 \cdot o(r_2) = o(p^3)$.

(3) Since $o(p) = o(-r_1^2 + r_2)$ and $o(r_1^2) = o(r_2)$, then $o(p) \geq 2 \cdot o(r_1) = o(r_2)$. So $o(p^3) \geq 6 \cdot o(r_1) = 3 \cdot o(r_2) = o(q^2)$.

(i) If $o(p^3) > o(q^2) = 6 \cdot o(r_1)$, then $o(B) = o(p^3 + q^2) = o(q^2)$

$= 6 \cdot o(r_1) = 2 \cdot o(2r_1^2 + r_2) + o(r_1^2 - 4r_2) \geq 4 \cdot o(r_1) + 2 \cdot o(r_1)$
$= 6 \cdot o(r_1)$. So $o(2r_1^2 + r_2) = 2 \cdot o(r_1) = o(r_1^2 - 4r_2)$.

(ii) If $o(B) = o(p^3) = o(q^2)$, then by the same argument as case (i), $o(2r_1^2 + r_2) = o(r_1^2 - 4r_2)$.

Now if $o(B) > o(p^3) = o(q^2)$ then either $o(B)$ = even or $o(B)$ = odd. We know that either $o(2r_1^2 + r_2) > 2 \cdot o(r_1) = o(r_2)$ or $o(r_1^2 - 4r_2) > 2 \cdot o(r_1) = o(r_2)$, but not both, otherwise $o(p) = o(-r_1^2 + r_2) = o[2r_1^2 + r_2 - (r_1^2 - 4r_2)] > 2 \cdot o(r_1) = o(r_2)$ implies $o(p^3) > o(q^2)$. If $o(B)$ = odd, then $o(B) = 2 \cdot o(2r_1^2 + r_2) + o(r_1^2 - 4r_2)$ implies $o(r_1^2 - 4r_2)$ = odd. So $o(r_1^2 - 4r_2) > 2 \cdot o(r_1) = o(r_2)$. Therefore, if $o(B)$ = odd, $o(r_1^2 - 4r_2) > o(2r_1^2 + r_2) = 2 \cdot o(r_1)$ $= o(r_2)$.

<u>Proposition 4.2</u> Let V satisfy (4.1). Suppose that we have completed t of the n-steps for a resolution by (2.2). Let $V^{(t)}$ be defined by $\{f_t = z^3 + 3v^\lambda u^\alpha p_t(u,v)z + 2v^\mu u^\beta q_t(u,v) = 0\}$ near $(u,v,z) = (0,0,0)$ where $v \nmid p_t$, $v \nmid q_t$, $u \nmid p_t$ and $u \nmid q_t$. Let $E_t = \{v = 0\}$ be an exceptional curve which appears in $V^{(t)}$. Assume that $o(B) = 3\lambda < 2\mu$ along E_t or $o(B) = 2\mu < 3\lambda$ along E_t and that E_t intersects irreducible curves of the proper transform of B which vanish at $(u,v) = (0,0)$. Then f_t is irreducible in $\mathbb{C}\langle u,v\rangle[z]$. Observe that irreducibility in $\mathbb{C}\langle u,v\rangle[z]$ is the same as irreducibility in $\mathbb{C}\langle u,v,z\rangle$ (see [G & R, Lemma 5, p. 71]). Therefore (0,0,0) is an irreducible singular point of $V^{(t)}$.

Proof First we assume that $o(B) = 3\lambda < 2\mu$ along E_t. To prove this, let us divide it into the following three cases:

(i) $3\alpha = 2\beta$ which may be equal to zero

(ii) $3\alpha < 2\beta$

(iii) $2\beta < 3\alpha$

(i) Suppose that f_t is reducible in $\mathbb{C}\langle u,v\rangle[z]$. We write f_t as
$f_t = (z - r_1)\cdot(z^2 + r_1 z + r_2) = z^3 + (-r_1^2 + r_2)z - r_1 r_2$
where r_1 and r_2 are holomorphic near (0,0). Then in terms of this expression, the z-discriminant of f_t is
$(2r_1^2 + r_2)^2\cdot(r_1^2 - 4r_2) = -108v^{3\lambda}\cdot u^{3\alpha}[p_t^3 + v^{2\mu-3\lambda}q_t^2]$. By Lemma 4.1, $2r_1^2 + r_2 = v^\lambda u^\alpha h$ and $r_1^2 - 4r_2 = v^\lambda u^\alpha [p_t^3 + v^{2\mu-3\lambda}q_t^2]k$ where h and k are units near (0,0) because $p_t^3 + v^{2\mu-3\lambda}q_t^2$ is a product of distinct prime factors up to a unit near (0,0).
So $3v^\lambda u^\alpha p_t(u,v) = -r_1^2 + r_2 = -\frac{1}{3}[(2r_1^2 + r_2) + (r_1^2 - 4r_2)]$
$= -\frac{1}{3}[v^\lambda u^\alpha h + v^\lambda u^\alpha (p_t^3 + v^{2\mu-3\lambda}q_t^2)k]$
$= -\frac{1}{3}\, v^\lambda\cdot u^\alpha [h + (p_t^3 + v^{2\mu-3\lambda}q_t^2)k]$. Then we would get
$3p_t = -\frac{1}{3}[h + (p_t^3 + v^{2\mu-3\lambda}q_t^2)k]$ and $p_t(0,0) \neq 0$. By assumption $p_t^3 + v^{2\mu-3\lambda}q_t^2$ vanishes at (0,0) and so $p_t(0,0) = 0$. It is absurd.

(ii) Let $3\alpha < 2\beta$. Suppose that f_t is reducible in $\mathbb{C}\langle u,v\rangle[z]$. f_t can be written as
$f_t = (z - r_1)\cdot(z^2 + r_1 z + r_2) = z^3 + (-r_1^2 + r_2)z - r_1 r_2$
where r_1 and r_2 are holomorphic near (0,0). Then the

z-discriminant of f_t is $(2r_1^2 + r_2)^2 \cdot (r_1^2 - 4r_2)$
$= -108v^{3\lambda} \cdot u^{3\alpha}[p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2]$. By Lemma 4.1,
$2r_1^2 + r_2 = v^\lambda u^\alpha h$ and $r_1^2 - 4r_2 = v^\lambda u^\alpha [p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2]k$
where h and k are units near (0,0) because $p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2$ is a product of distinct prime factors up to a unit near (0,0). So $3v^\lambda u^\alpha p_t(u,v) = -r_1^2 + r_2 = -\frac{1}{3}[(2r_1^2 + r_2) + (r_1^2 - 4r_2)]$
$= -\frac{1}{3}[v^\lambda u^\alpha h + v^\lambda u^\alpha (p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2)k]$
$= -\frac{1}{3} v^\lambda u^\alpha [h + (p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2)k]$. Then we would get
$3p_t = -\frac{1}{3}[h + (p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2)k]$ and then $p_t(0,0) \neq 0$.
But by assumption $p_t^3 + v^{2\mu-3\lambda}u^{2\beta-3\alpha}q_t^2$ vanishes at (0,0).
Thus $p_t(0,0) = 0$. It is impossible.

(iii) Let $2\beta < 3\alpha$. If $2\beta \not\equiv 0 \pmod 3$, then it is trivial by Corollary 2.5. So we may assume that $2\beta \equiv 0 \pmod 3$.
Suppose that f_t is reducible in $\mathbb{C}\langle u,v\rangle[z]$. f_t can be written as $f_t = (z - r_1)\cdot(z^2 + r_1 z + r_2)$ where r_1 and r_2 are holomorphic near (0,0). Then the z-discriminant of f_t is
$(2r_1^2 + r_2)^2 \cdot (r_1^2 - 4r_2) = -108v^{3\lambda} \cdot u^{2\beta}[u^{3\alpha-2\beta}p_t^3 + v^{2\mu-3\lambda}q_t^2]$.
Since $2\beta \equiv 0 \pmod 3$, let $\beta = 3\beta'$. Then by Lemma 4.1,
$2r_1^2 + r_2 = v^\lambda u^{2\beta'}h$ and $r_1^2 - 4r_2 = v^\lambda u^{2\beta'}[u^{3\alpha-6\beta'}p_t^3 + v^{2\mu-3\lambda}q_t^2]k$
where h and k are units near (0,0). So

$3v^\lambda u^\alpha p_t(u,v) = -r_1^2 + r_2 = -\frac{1}{3}[(2r_1^2 + r_2) + (r_1^2 - 4r_2)]$
$= -\frac{1}{3}[v^\lambda u^{2\beta'}h + v^\lambda u^{2\beta'}(u^{3\alpha-6\beta'}p_t^3 + v^{2\mu-3\lambda}q_t^2)k]$
$= -\frac{1}{3} v^\lambda u^{2\beta'}[h + (u^{3\alpha-6\beta'}p_t^3 + v^{2\mu-3\lambda}q_t^2)k]$.

Thus $3\cdot u^{\alpha-2\beta'} p_t = -\frac{1}{3}[h + (u^{3\alpha-6\beta'} p_t^3 + v^{2\mu-3\lambda} q_t^2)k]$.

This is a contradiction because $u^{\alpha-2\beta'} p_t$ vanishes at (0,0) but the right side of this equation does not vanish at (0,0).

Next, if $o(B) = 2\mu < 3\lambda$ along E_t, similarly we can prove it.

Corollary 4.3. Let V and $V^{(t)}$ be defined as in Proposition 4.2. Recall that $\omega_t : N^{(t)} \to V^{(t)}$ is the normalization of $V^{(t)}$. Let $E_t = \{v = 0\}$ and $E_s = \{u = 0\}$ be exceptional curves which appear in $V^{(t)}$. Assume that $o(B) = \text{odd} = 3\lambda < 2\mu$ along E_t and $o(B) = \text{odd} = 3\alpha < 2\beta$ along E_s. Suppose that $E_t \cap E_s \neq \phi$ and the proper transform of B does not vanish at $(u,v) = (0,0)$. Then $V^{(t)}$ is reducible near (0,0,0). So $N^{(t)}$ has a double point singularity at a point of $\rho_t^{-1}(0,0,0)$.

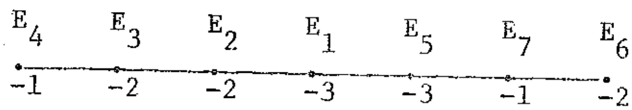
Proof Recall that the z-discriminant of the local defining equation f_t for $V^{(t)}$ is $-108v^{3\lambda}\cdot u^{3\alpha}\cdot h$ where h is a unit near (0,0) since $3\lambda < 2\mu$ and $3\alpha < 2\beta$. Resolve $V^{(t)}$ over (0,0) by a resolution process by (2.2). Since 3λ and 3α are odd, after just one blowing-up at $(u,v) = (0,0)$, the new exceptional curve E is not part of the branch locus of ρ. The corresponding weighted dual graph for a resolution by (2.2) of $V^{(t)}$ near $(u,v) = (0,0)$ is $A_1 \cup A_2$ where $\rho^{-1}(E) = A_1 \cup A_2$, $A_1\cdot A_1 = -2$ and $A_2\cdot A_2 = -1$. Note that $A_1 \cup A_2$ is disconnected. Thus we proved that $V^{(t)}$ is reducible near (0,0,0) and that $N^{(t)}$ has a double point singularity at a point of $\omega_t^{-1}(0,0,0)$.

Examples of Corollary 4.3

Let $V = \{z^3 + 3x(x + 2y)^2 z + 2(x^9 + y^9) = 0\}$.

Then $B = \{-108[x^3(x + 2y)^6 + (x^9 + y^9)^2] = 0\}$.

Note that V has a normal triple point singularity at (0,0,0). Then after 7-steps, we may assume that



$(B) = 9E_1 + 12E_2 + 15E_3 + 18E_4 + 15E_5 + 18E_6 + 36E_7 + W^{(7)}$ and $W^{(7)}$ meets E_4 and E_7 in three points respectively. Since $\rho^{(7)}$ are 2 - 1 over E_1 and E_5 and $E_1 \cap E_5 \neq \phi$, we have the situation mentioned in Corollary 4.3.

Proposition 4.4 Let V satisfy the condition (4.1). Suppose that we have completed t of the n-steps needed for a resolution by (2.2). Let $V^{(t)}$ be defined by $\{f_t = z^3 + 3v^\lambda p_t(u,v)z + 2v^\mu q_t(u,v) = 0\}$ near $(u,v,z) = (0,0,0)$ where $v \nmid p_t$ and $v \nmid q_t$. Let $E_t = \{v = 0\}$ be an exceptional curve which appears first in $V^{(t)}$. Assume that $o(B) = 3\lambda = 2\mu$ along E_t. Note that the z-discriminant of f_t is $-108v^{3\lambda}b = -108v^{3\lambda}(p_t^3 + q_t^2)$ where $b = b(u,v) = p_t^3 + q_t^2$. Suppose that irreducible curves of the proper transform of B just after t-steps vanish at a point in E_t, say (0,0), or $b(0,0) = 0$. Then either $p_t(0,0) \neq 0 \neq q_t(0,0)$ or $p_t(0,0) = q_t(0,0) = 0$. If $p_t(0,0) \neq 0 \neq q_t(0,0)$, then only one irreducible curve of the proper transform vanishes at that point and so meets E_t with normal crossing. In this case after the n-steps, $\rho^{-1}(E_t)$ is connected. If $p_t(0,0) =$

$q_t(0,0) = 0$ then $(0,0,0)$ is an irreducible triple point of $V^{(t)}$. Whenever $b(u_0,0) = 0$ for some u_0 implies $p_t(u_0,0) = q_t(0,0) = 0$, then $\rho^{-1}(E_t)$ has globally three components. In other words, if there exists at least one irreducible curve of the proper transform of B which intersects only E_t transversely, then $\rho^{-1}(E_t)$ is connected otherwise $\rho^{-1}(E_t)$ is composed of globally three components.

Proof Recall that the z-discriminant of f_t is $-108v^{3\lambda}(p_t^3 + q_t^2)$. Suppose that the proper transform vanishes at some point $(0,0)$. Then clearly either $p_t(0,0) \neq 0 \neq q_t(0,0)$ or $p_t(0,0) = q_t(0,0) = 0$. If $p_t(0,0) \neq 0 \neq q_t(0,0)$, then there exists only one irreducible curve of the proper transform such that it meets E_t with normal crossing at that point otherwise it would contradict to (4.1). If $p_t(0,0) = q_t(0,0) = 0$, then similar arguments as in the proof of Proposition 4.2 show that $V^{(t)}$ is irreducible above $(0,0)$. Now suppose that there is no such s at which p_t and q_t vanish. Then after n-steps there exists F_i, $1 \leq i \leq k$ and G_j, $1 \leq j \leq \ell$ such that $o(B) = 3\lambda_i \leq 2\mu_i$ along F_i and $o(B) = 2\beta_j \leq 3\alpha_j$ along G_j where the F_i and the G_j are irreducible components of $E = \cup E_i$, $1 \leq i \leq n$ which intersects E_t, by (4.1). Then we claim that $o(B) = o(p^3) = o(q^2)$ along any component of E which intersects E_t. Also we may assume without changing the number of components of $\rho^{-1}(E_t)$ that if necessary, then by successive blow-ups there are no additional components of (p^3) or (q^2) at any point in E_t. Let $m = -E_t \cdot E_t$ after n-steps. Then since $(B)\cdot E_t = 0$, $3\lambda m = 2\mu m = \sum_{i=1}^{k} 3\lambda_i + \sum_{j=1}^{\ell} 2\beta_j$. But note that $3\lambda m = \sum_{i=1}^{k} 3\lambda_i + \sum_{j=1}^{\ell} 3\alpha_j$ and $2\mu m = \sum_{i=1}^{k} 2\mu_i + \sum_{j=1}^{\ell} 2\beta_j$ because

$(p^3)\cdot E_t = (q^2)\cdot E_t = 0$. Therefore, the above three equations show that $2\beta_j = 3\alpha_j$, $1 \leq j \leq \ell$ and $3\lambda_i = 2\mu_i$, $1 \leq i \leq k$. Thus $\rho^{-1}(E_t)$ has three components. So it is enough to consider the case that there is an irreducible curve of the proper transform which meets only E_t with normal crossings. Recall that $\rho_t : V^{(t)} \to M^{(t)}$, $\omega_t : N^{(t)} \to V^{(t)}$, the normalization of $V^{(t)}$ and $\rho^t = \rho_t \circ \omega_t$. Let $L^{(t)} = \{g_t = z^3 + 3p_t(u,v)z + 2q_t(u,v) = 0\}$ and let $\rho_t' : L^{(t)} \to M^{(t)}$ be defined by $\rho_t'(u,v,z) = (u,v)$. Clearly $L^{(t)}$ and $V^{(t)}$ have the same normalization $N^{(t)}$ since the fact that ρ_t and ρ_t' are proper implies that the induced map $\omega_t' : L^{(t)} \to V^{(t)}$ is proper, and biholomorphic over $V^{(t)} - \{v = 0\}$ and ω_t' is finite. Observe that the number of components of the regular set of $(\rho^t)^{-1}(E_t)$ and that of components of the regular set of $L^{(t)}(u,0,z) = \{g_t(u,0,z) = z^3 + 3p_t(u,0)z + 2q_t(u,0) = 0\}$ in $L^{(t)}$ are same since the singular set of $L^{(t)}$ is finite over E_t. Also $\rho^{-1}(E_t)$ is connected if and only if the regular set of $(\rho^t)^{-1}(E_t)$ is connected. If $L^{(t)}(u,0,z)$ is nonsingular everywhere, then $\rho^{-1}(E_t)$ must be connected since E_t can be blown down to an irreducible singular point of $V^{(i)}$ for some $i < t$ by Propositions 4.2 and 4.3. Suppose that $L^{(t)}(u,0,z)$ is singular at some point. To prove the connectedness of $\rho^{-1}(E_t)$ assume the contrary. Then the regular set of $L^{(t)}(u,0,z)$ would be disconnected. Note that $p_t(u,0)$ and $q_t(u,0)$ are polynomials in u recalling that $v^\lambda p_t(u,0) + v^\mu q_t(u,0)$ are the leading terms of $p_i(uv,v)$ and $q_i(uv,v)$ for some $i < t$ when we write $p_i(uv,v)$ and $q_i(uv,v)$ in terms of power series in v whose coefficients are polynomials in u, respectively.

So the local defining equation $g_t(u,0,z)$ for $L^{(t)}(u,0,z)$ would be reducible in $\mathbb{C}[u,z]$ where $\mathbb{C}[u,z]$ is the polynomial ring in z and u over $\mathbb{C}$. Let $(u_0,0) \in E_t$ and let o(—) be the order of zero of —— at $(u_0,0)$. If $o(p_t^3(u,0)) = o(q_t^2(u,0)) > 0$ at $(u_0,0)$ then we must have $o(p_t^3(u,0) + q_t^2(u,0)) = o(p_t^3(u,0)) = o(q_t^2(u,0))$ at $(u_0,0)$ otherwise it would contradicts to (4.1) as follows. Take $u_0 = 0$. We write $p_t = p_t(u,v)$ and $q_t = q_t(u,v)$ in $V^{(t)}$ as

$$p_t^3 = u^{6s}h + v^{\ell}A$$

$$q_t^2 = u^{6s}k + v^{m}B$$

where h and k are polynomials in u and units near u = 0, and A and B are holomorphic near v = 0 and $v \nmid A$ and $v \nmid B$ and s is a positive integer. Let ℓ_1 and m_1 be total orders of $v^{\ell}A$ and $v^{m}B$ at (0,0), respectively. If $6s < \ell_1$ and $6s < m_1$, then it leads to a contradiction to (4.1) by just one blowing up at (u,v) = (0,0). If either $6s \geq \ell_1$ or $6s \geq m_1$, then by successive blowing ups at (u,v) = (0,0) similarly we can find a contradiction to (4.1). Since $g_t(u,0,z)$ is reducible in $\mathbb{C}[u,z]$, it may be written as $(z - r_1)\cdot(z^2 + r_1 z + r_2)$ where r_1 and r_2 are polynomials in u. Note that the z-discriminant of $g_t(u,0,z)$ is $-108[p_t^3(u,0) + q_t^2(u,0)] = (2r_1^2 + r_2)^2\cdot(r_1^2 - 4r_2)$. Since E_t meets at most two exceptional curves in $V^{(t)}$, either the degree of $p_1^3(u,0)$ or the degree of $q_1^2(u,0)$ as polynomials in u are the same as the degree of $p_1^3(u,0) + q_1^2(u,0)$. So we may assume without loss of generality that the degree of $p_1^3(u,0)$ and the degree of $p_1^3(u,0) + q_1^2(u,0)$ are same. But by Lemma (4.1) and (4.1)

$$2r_1^2 + r_2 = a(u-u_1)^{\alpha_1} \cdots (u-u_m)^{\alpha_m}$$

$$r_1^2 - 4r_2 = b(u-u_1)^{\alpha_1} \cdots (u-u_m)^{\alpha_m} (u-s_1) \cdots (u-s_k)$$

where a, b and the s_i are constant and $\alpha_1,\ldots,\alpha_m$,m and k are integers and each $u-s_i = 0$ is an irreducible curve of the proper transform of B which intersects only E_t with normal crossing. Since $3p_t = -r_1^2 + r_2 = -\frac{1}{3}[(2r_1^2 + r_2) + (r_1^2 - 4r_2)]$ then the degree of $p_t = p_t(u,0)$ is $\alpha_1 + \ldots + \alpha_m + k$. But the degree of $p_t^3(u,0) + q_t^2(u,0)$ is the degree of $(2r_1^3 + r_2)^2 \cdot (r_1^2 - 4r_2)$, that is $3\alpha_1 + \ldots + 3\alpha_m + k$. Therefore $3(\alpha_1 + \ldots + \alpha_m + k)$ would be equal to $3\alpha_1 + \ldots + 3\alpha_m + k$. Hence we would get $k = 0$. It contradicts to the assumption that $k > 0$.

By Proposition 4.2 and Proposition 4.4, we can compute a resolution $r : N \to V$ directly. Let us recall that $E = \cup E_i$, $1 \le i \le n$. Then we summarize it below:

(i) If $o(B) = o(p^3) = o(q^2)$ along E_i, then $\rho^{-1}(E_i)$ has just one component or three components depending on whether some irreducible curves of the proper transform of B intersect E or not respectively.

(ii) If $o(B) = o(p^3) < o(q^2)$ along E_i, then either $o(p^3) \not\equiv 0$ (mod 2) or not. If $o(p^3) \not\equiv 0$ (mod 2), then $\rho^{-1}(E_i)$ consists of just two components. If $o(p^3) \equiv 0$ (mod 2), then $\rho^{-1}(E_i)$ consists of two or three components depending on whether part of the branch locus of ρ intersects E_i or not.

(iii) Let $o(B) = o(q^2) < o(p^3)$ along E_i. If $o(q^2) \not\equiv 0$ (mod 3), then

$\rho^{-1}(E_i)$ is just one component. If not, $\rho^{-1}(E_i)$ has one component or three components depending on whether part of the branch locus of ρ intersects E_i or not.

Corollary 4.5

(1) Let P_1 and P_2 be the singularities at (0,0,0) of $V_1 = \{(x,y,z) \mid z^3 + 2q_1(x,y) = 0\}$ and $V_2 = \{(x,y,z) \mid z^3 + 2q_2(x,y) = 0\}$ respectively. Let V_1 and V_2 be the normal analytic spaces. If q_1 and q_2 define equisingular plane curve singularities at (0,0), then P_1 and P_2 have homeomorphic resolutions by (2.2).

(2) Let P_1 and P_2 be the singularities at (0,0,0) of $V_1 = \{(x,y,z) \mid z^3 + 3p_1(x,y)z + 2q_1(x,y) = 0\}$ and $V_2 = \{(x,y,z) \mid z^3 + 3p_2(x,y)z + 2q_2(x,y) = 0\}$, respectively with $p_i(x,y) \not\equiv 0$, $i = 1,2$. Let V_1 and V_2 satisfy the condition (4.1). If $p_1^3 + q_1^2$ and $p_2^3 + q_2^3$ define equisingular plane curve singularities at (0,0), then P_1 and P_2 have homeomorphic resolutions by (2.2).

Proof By section 2, Proposition 4.2 and Proposition 4.4.

Corollary 4.6 Let V of Lemma 1.12 satisfy (4.1). Let $r : N \to V$ be a resolution by (2.2). Suppose that we have completed t of the n-steps needed for such a resolution. Recall that $B^{(t)}$ is the branch locus for $\rho^{(t)}: N^{(t)} \to M^{(t)}$ and $V^{(t)}$ in (2.2). Then $V^{(t)}$ is irreducible at any singular point of $B^{(t)}$ except possibly for the points $E_i \cap E_j$ where ρ is two to one over E_i and E_j and no other component of $B^{(t)}$ passes through $E_i \cap E_j$.

Proof By Propositions 4.2, 4.3 and 4.4.

Observe that if $V^{(t)}$ has an irreducible singular point P and $r : N \to V^{(t)}$ is a resolution by (2.2) near P then $r^{-1}(P)$ is connected and $X \cdot X = -3$ where X is a divisor of $r^*(m)$ and m is the maximal ideal of P.

Let V satisfy the condition (4.1). Now let us apply the results from Proposition 4.2 to Corollary 4.5 to Proposition 3.4. We use the same notations in Proposition 3.4.

(4.2) Case (A) of (I) in Proposition 3.4.

Let E_1 be the curve appearing at the initial quadratic transformation at Q. Let $\rho^{-1}(E_1) = A_1 \cup A_2$. Note that E_1 is the branch locus of ρ over which ρ is two to one. Recall that ρ': $N' \to M'$ and B' is the branch locus for ρ'. Then B' has only one singular point in E_1. If not, then by Propositions 4.2 and 4.4 $D \cdot A_1 \geq 2$. Recall that $D \cdot A_1 = (X + Z) \cdot A_1 = 1$ and it is impossible. Therefore there is only one component of $\cup A_i$, $i \neq 1,2$. In fact this component intersects both A_1 and A_2.

(4.3) Case (B2) of (I) in Proposition 3.4.

We may assume without loss of generality that $X \cdot A_1 = -1$, $X \cdot A_2 = 0$ and $X \cdot A_3 = -1$ where A_1 and A_3 have coefficients 1 and 2 in X respectively. Then note that $D \cdot A_2 = 1$ and $D \cdot A_1 = 0$. Let E_1 be the curve appearing at the initial quadratic transformation at Q. Let $\rho^{-1}(E_1) = A_1 \cup A_3$. Note that E_1 is the branch locus of ρ over which ρ is two to one. Recall that B' is the branch locus for $\rho' : N' \to M'$. Then B' has only one singular point in E_1. If not, then by Propositions 4.2 and 4.4 there would be another component of $\cup A_i$, $i \neq 1,3$ which meets A_1 and A_3 both. But note that $D \cdot A_1 = 0$. So it is impossible. Next, let E_2 be the next exceptional curve which results from blowing up at that point in E_1. Recall that $B^{(2)}$ is the branch locus for $\rho^{(2)}: N^{(2)} \to M^{(2)}$. Note that by (B2) of (I) of Proposition 3.4 E_2 is not part of the branch locus for ρ. If $B^{(2)}$ is singular at $E_1 \cap E_2$, then $D \cdot A_1 \geq 1$ by Propositions 4.2 and 4.4. Since $D \cdot A_1 = 0$, it is absurd. Thus we proved that $B^{(2)}$ is nonsingular at $E_1 \cap E_2$. After n-steps, $E_1 \cdot E_1 = -2$. Therefore by (II) of section 2 $A_1 \cdot A_1 = -2$ and $A_3 \cdot A_3 = -1$. Let us recall that Z_* is the fundamental cycle on a component of $\cup A_i$, $i \neq 1,3$. Note that there is only one component of $\cup A_i$, $i \neq 1,3$ and supp Z_* intersects A_1 and A_3 both. If $Z_* \cdot Z_* = -3$ and $Z_* \cdot A_2 = -3$, then $A_1 \cap A_2 \neq \phi$ and $A_2 \cap A_3 \neq \phi$ since E_2 follows E_1 and $E_1 \cap E_2 \neq \phi$. If $Z_* \cdot Z_* = -3$, $Z_* \cdot A_2 = -2$ and $Z_* \cdot A_t = -1$, then $A_1 \cap A_t \neq \phi$ and $A_3 \cap A_2 \neq \phi$ since E_2 follows E_1 and $E_1 \cap E_2 \neq \phi$.

(4.4) Case (B) of (II) in Proposition 3.4.

Note that A_1 and A_t follows A_2 at the same time and A_p follows A_1 and A_t. Let E_1 be the curve appearing at the initial quadratic transformation at Q. Then $\rho^{-1}(E_1) = A_2$. Let E_2 and E_3 be such that $\rho^{-1}(E_2) = A_1 \cup A_t$ and $\rho^{-1}(E_3) = A_p$.

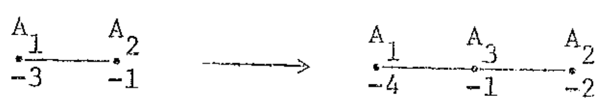
Recall that Z_* is the fundamental cycle on the connected component of $\cup A_i$, $i \neq 2$ which contains A_1, A_p and A_t. Note that $Z_* \cdot A_1 = Z_* \cdot A_p = -1$, $Z_* \cdot Z_* = -2$ and that X_* is the cycle on $\operatorname{supp} Z_*$ such that $X_* \cdot A_1 = X_* \cdot A_t = -1$ and that A_1 and A_t have coefficients 1 and 2 in X_* respectively. So after A_2 appears, we get the same situation on $\operatorname{supp} Z_*$ as in (4.3). Therefore $A_1 \cdot A_1 = -2$, $A_t \cdot A_t = -1$, $A_1 \cap A_p \neq \phi$ and $A_t \cap A_p \neq \phi$. Moreover, there is no connected component of $\cup A_i$, $i \neq 1,t$ which does not contain A_p.

<u>Theorem 4.7</u> Let $V = \{(x,y,z) \mid z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ be a normal two-dimensional analytic space with $P = (0,0,0)$ its only singularity. Let V satisfy (4.1). Let $r : N \to V$ be a resolution by (2.2). Let $\tilde{r} : \tilde{N} \to V$ be the minimal good resolution. Then N is obtained from $\tilde{N}$ by at most 5-time quadratic transformations at each s_j, $1 \leq j \leq \ell$ in $A = \tilde{r}^{-1}(P)$. Each $A_i - \cup A_k$, $k \neq i$ contains at most two s_j.

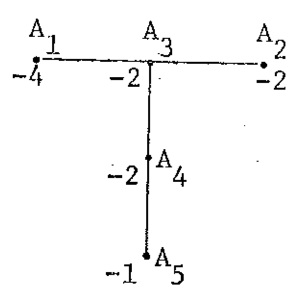
<u>Proof</u> Let $\pi : N \to \tilde{N}$ satisfy $r = \tilde{r} \circ \pi$. Proposition 3.1 and (*) tell what happens in each step of a resolution by (2.2) in terms of a resolution $r' : N' \to V$ with $(r')^*(m)$ locally principal satisfying $X' \cdot X' = -3$ where X' is the divisor of $(r')^*(m)$. Start with $r' = \tilde{r}$. Let Z be the fundamental cycle on N. Then Lemma 3.2, Corrolaries 4.3 and 4.6 show that it is enough to consider the case that $r^*(m)$ is not principal and the case mentioned in Corollary 4.3. $Z \cdot Z = -1$ or -2. We use the results and notations of Proposition 3.4.

(I) Let $Z \cdot Z = -2$.

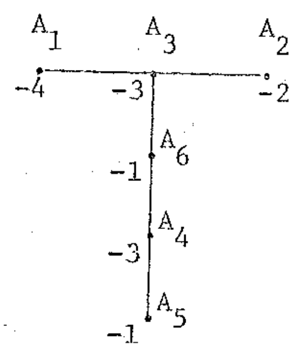
(A) There exists A_1 such that $Z \cdot A_1 = -2$ and A_1 has coefficient 1 in Z. If $r^*(m)$ is not principal, by (4.2) note that $Z = A_1$ and $A_1 \cdot A_1 = -2$. In case of Corollary 4.3, $X = A_1 + A_2$ where $\operatorname{supp} X = A_1 \cup A_2$ is not connected, $A_1 \cdot A_1 = -2$ and $A_2 \cdot A_2 = -1$. If this is not the case from Corollary 4.3, then by Proposition 3.6 there is an embedded point $s \in A_1$ which is blown up to a new exceptional curve A_2. $A_1 \cap A_2 \neq \phi$. By Proposition 3.4, E_1 is part of the branch locus of ρ where $\rho^{-1}(E_1) = A_1 \cup A_2$. So to separate A_1 and A_2, we need an additional blow-up at $A_1 \cap A_2$ as the following.



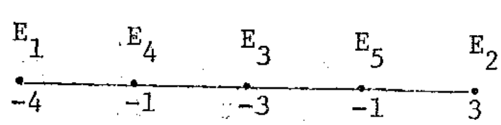
After A_1 and A_2 appear first in a resolution process by (2.2), the fundamental cycle Z_* of the connected component of $A_1 \cup A_2 \cup A_3$, $i \neq 1,2$ is A_3 with $A_3 \cdot A_3 = -1$. Again by Proposition 3.4 and Proposition 3.6 we need two time blow-ups at $s \in A_3 - \cup A_i$, $i \neq 1,2$. Then it becomes



Let X_* be the cycle on $\cup A_i$, $i \neq 1,2$ such that $X_* \cdot X_* = -3$ and $X_* \cdot A_j \leq 0$ for $j \neq 1,2$. Then we know that $X_* \cdot A_5 = -1$ and A_5 has coefficient 3 in X_*. Next, after A_5 appears in the second step of a resolution process by (2.2), the fundamental cycle Z_{**} of the connected component $A_3 \cup A_4$ is just $A_3 + A_4$ with $Z_* \cdot Z_* = Z_* \cdot A_3 + Z_* \cdot A_4 = -2$. So by Proposition 3.4 and Proposition 3.6 there is an embedded point s_1 in either $A_3 \cap A_4$ or $A_4 - A_3$. But E_1 is part of the branch locus. s_1 must be $A_3 \cap A_4$. Therefore blowing up at $A_3 \cap A_4$ we have the following graph:



Thus after A_6 follows A_5 in the third step, A_3 and A_4 follows A_6 respectively. Let E_2, E_3, E_4 and E_5 be such that $\rho^{-1}(E_2) = A_5$, $\rho^{-1}(E_3) = A_6$, $\rho^{-1}(E_4) = A_3$ and $\rho^{-1}(E_5) = A_4$. Note that ρ is 2 - 1 over E_1, 1 - 1 over E_2 and E_3, and E_4 and E_5 are not part of the branch locus. Since A_3 and A_4 are of genus 0, the resolution of corresponding branch locus B is:



Note that $B = 3E_1 + 4E_2 + 8E_3 + 12E_4 + 12E_5 + W^{(5)}$ and $W^{(5)}$ meets E_4 in one point. B will be found to be equisingular to $x^3 + y^4$ near (0,0). Therefore V might be a $\{z^3 + 3xy + 2y^2 = 0\}$. Note that in this case we did not use the condition (4.1). Thus in the above case we need five time quadratic transformations at $s \in A_1'$ from $\tilde{N}$ in order to get N.

(B) There exist $A_1 \neq A_2$ such that $Z \cdot A_1 = Z \cdot A_2 = -1$.

(B1) If $r^*(m)$ is not principal, then by Proposition 3.6 blowing up $\tilde{N}$ at $s = A_1 \cap A_2$, $A_3 = \pi^{-1}(s)$ is a new exceptional curve; A_3 appears in the first step of a resolution process by (2.2) where $\pi : N \to \tilde{N}$ is the blow-up at s. Then A_1 follows A_3 and A_2 follows A_3. So there is no subsequent embedded point that can appear in A_3. Also there is no embedded point in $A_1 - \cup A_i$, $i \neq 1$ and $A_2 - \cup A_i$, $i \neq 2$.

(B2) By Proposition 3.4 and (4.3) we may assume without loss of generality that there is no connected component of $\cup A_i$, $i \neq 1$ which does not contain A_2. If $r^*(m)$ is not principal, then by Proposition 3.6, let us blow up $\tilde{N}$ at $s \in A_2 - \cup A_i$, $i \neq 2$. Note that $A_1 \cdot A_1 = -2$. Let $A_3 = \pi^{-1}(s)$ where $\pi : N \to \tilde{N}$ is the blow-up at s. Then A_1 and A_3 appear in the first step of a resolution process by (2.2). No subsequent embedded point can appear in A_3. Observe that E_1 with $\rho^{-1}(E_1) = A_1 \cup A_3$ is part of the branch locus of ρ and $A_1 \cdot A_1 = 2A_3 \cdot A_3 = -2$. There is no embedded point in A_1. Let Z_* be the fundamental cycle on the connected component $\cup A_i$, $i \neq 1,3$. Then $Z_* \cdot Z_* = -3$ and either

$(Z_* \cdot A_2 = -2,\ Z_* \cdot A_t = -1)$ or $Z_* \cdot A_2 = -3$ by following notations in Proposition 3.4. If $Z_* \cdot A_2 = -2$ and $Z_* \cdot A_t = -1$, then it is clear that there is no more embedded point in $A_2 - \cup A_i$, $i \neq 2$ and there is no embedded point in $A_t - \cup A_i$, $i \neq t$. A_t and A_2 follow A_1 and A_3. If $Z_* \cdot A_2 = -3$, then there is no more embedded point in $A_2 - \cup A_i$, $i \neq 2$. A_2 follows A_1 and A_3 at the same time.

(C) There exists A_1 such that $Z \cdot A_1 = -2$ and A_1 has coefficient 2 in Z. If $r^*(m)$ is not principal, then by Proposition 3.6 blowing up at $s \in A_1 - \cup A_i$, $i \neq 1$, let $A_2 = \pi^{-1}(s)$ where $\pi : N \to \tilde{N}$ is the blow-up of $\tilde{N}$ at s. Then A_2 appears in the first step of a resolution process by (2.2). No subsequent embedded point can appear in A_2. Let us recall that X_* is the cycle on supp$(Z - 2D)$ such that $X_* \cdot A_i \leq 0$ for all $A_i \subset \text{supp}\, X_*$ and $X_* \cdot X_* = -3$. Then we have the following subcases. $X_* \cdot A_1 = 0$.

(C1) Let $A_s \neq A_1$ be such that $X_* \cdot A_s = -3$. Then A_s follows A_2. There is no embedded point in $A_s - \cup A_i$, $i \neq s$. Also there is no more embedded point in $A_1 - A_i$, $i \neq 1$ by case (C1) of (I) in Proposition 3.4.

(C2) Let $A_s \neq A_t$ be such that $X_* \cdot A_s = -2$ and $X_* \cdot A_t = -1$. Then A_s and A_t follow A_2. There is no embedded point in $A_s - \cup A_i$, $i \neq s$ and in $A_t - \cup A_i$, $i \neq t$. Also, there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$ by case (C2) of (I) in Proposition 3.4.

(C3) Let A_{s1}, A_{s2} and A_{s3} be distinct with $X_* \cdot A_{s1} = X_* \cdot A_{s2} = X_* \cdot A_{s3}$

$= -1$. Then A_{s1}, A_{s2} and A_{s3} follow A_2. There is no embedded point in $A_{s1} - \cup A_i$, $i \neq s1$, $A_{s2} - \cup A_i$, $i \neq s2$ and $A_{s3} - \cup A_i$, $i \neq s3$. Also, there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$ by case (C3) of (I) of Proposition 3.4.

(C4) Let A_s be such that $X_* \cdot A_s = -1$ with coefficient 3 in X_*. Then A_s follows A_2. If X_* is the fundamental cycle on supp$(Z - 2D)$ then there is no embedded point in $A_s - \cup A_i$, $i \neq s$. Also there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$ by case (C4) of (I) in Proposition 3.4. If X_* is not fundamental, then let us recall Z_*, the fundamental cycle on supp X_*. Note that $Z_* \cdot Z_* = -2$, A_1 has coefficient 1 in Z_* and $Z_* \cdot A_1 = 0$ or -1. Of course, there is no embedded point in $A_s - \cup A_i$, $i \neq 1$. By the case (I) in Proposition 3.4 there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$.

(C5) Let $A_s \neq A_t$ be such that $X_* \cdot A_s = X_* \cdot A_t = -1$ and A_s and A_t have coefficients 2 and 1 in X, respectively. By case (C5) of (I) in Proposition 3.4, X_* is the fundamental cycle on supp$(Z - 2D)$. A_s and A_t follows A_2. There is no embedded point in $A_s - \cup A_i$, $i \neq s$ and $A_t - \cup A_i$, $i \neq t$. By case (C5) of (I) in Proposition 3.4, there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$.

(II) Let $Z \cdot Z = -1$. Let $Y = \Sigma m_i A_i$ where m_i is the order to which functions g o r, $g \in m$, generically vanish on A_i. Then $Y \cdot Y = -2$ or $Y \cdot Y = -1$.

(i) Let $Y \cdot Y = -2$. Let A_p be such that $Y \cdot A_p = -1$ with coefficient 2 in Y by Proposition 3.4. By Proposition 3.6, there is an embedded point s in $A_p - \cup A_i$, $i \neq p$. Let $\pi : N \to \tilde{N}$ be the blow-up at s and $A_2 = \pi^{-1}(s)$. Then A_2 appears in the first step of a resolution process by (2.2). There is no subsequent embedded point in A_2. Let us recall that Z_* is the fundamental cycle on supp(2Z − D) by Proposition 3.4, (II). Then $Z_* \cdot Z_* = -2$ and $Z_* \cdot A_1 = Z_* \cdot A_p = -1$. By Proposition 3.4 and Proposition 3.6, we have two subcases.

<u>Case (A) of (II) of Proposition 3.4</u>: Then either $A_1 \cap A_p \neq \phi$ or $A_1 \cap A_p = \phi$. If $A_1 \cap A_p \neq \phi$, then blowing up at $A_1 \cap A_p$ by Proposition 3.6, A_t appears in the next step where $\pi : N_1 \to N$ is the blow-up at $A_1 \cap A_p$ and $\pi^{-1}(A_1 \cap A_p) = A_t$. There is no subsequent embedded point in $A_t - \cup A_i$, $i \neq t$. Also A_1 follows A_t and A_p follows A_t. Therefore there is no embedded point in $A_1 - \cup A_i$, $i \neq 1$ and $A_p - \cup A_i$, $i \neq p$. If $A_1 \cap A_p = \phi$, then there must be only one component of $\cup A_i$, $i \neq 2$ which intersects both A_1 and A_p by case (B1) of (I) in Proposition 3.4. That component contains A_t which is not a nonsingular rational curve with $A_t \cdot A_t = -1$. In this case, also there does not exist such an embedded point as in case of $A_1 \cap A_p = \phi$.

<u>Case (B) of (II) of Proposition 3.4</u>: By (4.4), note that $A_1 \cdot A_1 = -2$, $A_t \cdot A_t = -1$ and A_t is a nonsingular rational curve. So after blowing up at $s_1 \in A_p - \cup A_i$, $i \neq p$, let $A_t = \pi_1^{-1}(s_1)$ where $\pi_1 : N_1 \to N$ is the blow-up of N at s_1. Then A_1 and A_t follow A_2. So there is no subsequent embedded point in A_t

and there is no embedded point in A_1. A_p follows A_1 and A_t. Also $A_1 \cap A_p \neq \phi$ and $A_t \cap A_p \neq \phi$. Thus we showed that there is only two distinct embedded points in $A_p - \cup A_i$, $i \neq p$.

(ii) Let $Y \cdot Y = -1$. That is $Z = Y$. Let A_1 be such that $Z \cdot A_1 = -1$. By Proposition 3.6, there is an embedded point s in $A_1 - \cup A_i$, $i \neq 1$. Let $\pi : N \to \tilde{N}$ be the blow-up at s and $A_p = \pi^{-1}(s)$. Start with $Z_1 + A_p$ on N where $Z_1 = \pi^*(Z)$. Then we have the same situation as the case (i), because $(Z_1 + A_p) \cdot (Z_1 + A_p) = -2$, $(Z_1 + A_p) \cdot A_p = -1$ and A_p has coefficient 2 in $Z_1 + A_p$. Therefore we need three time quadratic transformations at $s \in A_1 - \cup A_i$ in order to get a resolution by (2.2). Moreover, there is no more embedded point in $A_1 - \cup A_i$, $i \neq 1$.

<u>Theorem 4.8</u> Let $r : N \to V$ be a resolution by (2.2) of a normal two dimensional triple point singularity P.

(1) Let $V = \{(x,y,z) \mid z^3 + 2q(x,y) = 0\}$ with $P = (0,0,0) \in V$. Then there is an algorithm to determine the equisingular type of the plane curve singularity (0,0) of $\{(x,y) \mid q(x,y) = 0\}$ from Γ, the topological type of the embedding of $A = r^{-1}(P)$ in N.

(2) Let $V = \{(x,y,z) \mid z^3 + 3p(x,y)z + 2q(x,y) = 0\}$ with $P = (0,0,0) \in V$. Let V satisfy (4.1). Then there is an algorithm to determine the equisingular type of the plane curve singularity (0,0) of $\{(x,y) \mid -108(p^3(x,y) + q^2(x,y)) = 0\}$ from Γ, the topological type of the embedding of $A = r^{-1}(P)$ in N.

Proof We shall describe the algorithm. It will suffice to identify $r^*(m)$ for P and for all subsequent singularities in $V^{(i)}$ for some i which appear in a resolution process by (2.2). These subsequent singularities are either (a) irreducible triple point singularities of $V^{(t)}$, for some $t < n$ or (b) singularities each of which is mapped by ρ^t, some $t < n$, to an intersection of only two exceptional curves which are two to one branch locus of ρ by Corollary 4.6. Suppose that we have completed t of the n-steps needed for a resolution by (2.2). $N^{(t)}$ is the obtained normal space and let us recall that $\rho^t = \rho_t \circ \omega_t : N^{(t)} \to M^{(t)}$ expresses $N^{(t)}$ as a three-fold branched covering of the manifold $M^{(t)}$. $M^{(t)}$ is obtained from $\mathbb{C}^2$ by a sequence of quadratic transformations. Choose subscripts for the C_i of cases (I), (II) and (III) of section 2 so that C_i first appear in $N^{(i)}$. Then the singularities of case (a) in $V^{(t)}$ have resolutions by (2.2) with exceptional sets given by connected components of $\cup A_j$, $A_j \not\subset \cup C_i$, $1 \leq i \leq t$, or each of the singularities of case (b) in $V^{(t)}$ has a resolution by (2.2) with exceptional sets given by the union of two disjoint irreducible curves, say, A_k, A_ℓ of $\cup A_j$. $A_j \not\subset \cup C_i$, $1 \leq i \leq t$ with $A_k \cdot A_k = -2$ and $A_\ell \cdot A_\ell = -1$ by Corollary 4.3. So, it is enough to consider the case (a). Let A' be such a connected component for a singularity $P^{(t)}$ in $N^{(t)}$. Then the initial step in that resolution of $P^{(t)}$ corresponds to a quadratic transformation in $M^{(t)}$ at a center $Q^{(t)}$. If $C_i \cap A' \neq \phi$, then $Q^{(t)} \in \rho^{(t)}(C_i)$ and conversely. This determines the topology of $M^{(t+1)}$ together with the case (b). The new exceptional curve in $M^{(t+1)}$ is denoted $E^{(t+1)}_{t+1}$.

We omit the super-script for $E_{t+1}^{(t+1)}$ and its proper transforms when no ambiguity in notation arises. So after n-steps, we know the nature of the exceptional set $E = \cup E_i$, $1 \leq i \leq n$, in $M^{(n)}$. We also keep track, as follows, of which E_i is part of the branch locus $B^{(n)}$ of $\rho^{(n)}$. Namely, after t steps, let $X = \Sigma m_j A_j$ be the divisor near A' of the pull-back of the maximal ideal of $P^{(t)}$. $X \cdot X = -3$. If there exists an A_k with $A_k \cdot X = -1$ and $m_k = 3$, then we are in case (I) of section 2. Blowing up at $Q^{(t)}$ and normalizing $V^{(t+1)}$ induced by this blowing-up gives an E_{t+1} over which ρ^{t+1} is one to one. If there exist $A_k \neq A_\ell$ with $A_k \cdot X = A_\ell \cdot X = -1$, $A_k \cdot A_k = 2A_\ell \cdot A_\ell$ and $m_k = 1$, $m_\ell = 2$, then we are in case (II) of section 2. Blowing at $Q^{(t)}$ and normalizing $V^{(t+1)}$ induced by this blowing up gives an E_{t+1} over which ρ^{t+1} is a two-fold branch cover. In other cases, we are in case (III) of section 2. E_{t+1} is not part of the branch locus. If there is an A_k such that $A_k \cdot X = -3$, then $C_{t+1} = A_k$. If there are A_k, A_ℓ and A_m such that $A_k \cdot X = A_\ell \cdot X = A_m \cdot X = -1$, then $C_{t+1} = A_k \cup A_\ell \cup A_m$. If there exist A_k and A_ℓ such that $X \cdot A_k = -1$ and $X \cdot A_\ell = -2$ then $C_t = A_k \cup A_\ell$ and $A_\ell \cdot A_\ell = 2A_k \cdot A_k$. The topological types of the C_i and how they intersect are known from Γ and the above paragraphs. A C_i above an E_i which is not part of $B^{(n)}$ is a three-fold branch cover of E_i with some known branch points at the E_j in $B^{(n)}$. The other branch points come from $W^{(n)}$, a proper transform on $M^{(n)}$ of $B = \{p^3(x,y) + q^2(x,y) = 0\}$. Then Γ determines how $W^{(n)}$, which is nonsingular, meets E with normal crossings. This determines the equisingular type of the plane curve singularity of

B at (0,0), as desired.

Thus there remains to find X, the divisor of $r^*(m)$ on N. If Z, the fundamental cycle, satisfies $Z \cdot Z = -3$, then $Z = X$ by Lemma 3.2. For $Z \cdot Z = -2$ or -1, then X satisfies the hypotheses of Proposition 3.4. We will follow the same notation of Proposition 3.4.

(I) If $Z \cdot Z = -2$, then there are three cases below:

(A) If there is A_1 such that $Z \cdot A_1 = -2$, then there is only one component C of $\cup A_i$, $i \neq 1$ by (4.2), which contains A_2. It is trivial to find X.

(B) Let A_1 and A_2 be such that $Z \cdot A_1 = Z \cdot A_2 = -1$. If $A_1 \cdot A_1 \neq -2$ and $A_2 \cdot A_2 \neq -2$, then there is only one component C of $\cup A_i$, $i \neq 1,2$ which intersects both A_1 and A_2 by case (B) of (I) in Proposition 3.4 and (4.3). So to find X is obvious. This is the case (B1) of (I) in Proposition 3.4. Now without loss of generality we may assume that $A_1 \cdot A_1 = 2A_3 \cdot A_3 = -2$ with $X \cdot A_1 = X \cdot A_3 = -1$. Then by case (B2) of (I) in Proposition 3.4 and (4.3), $A_2 \cap A_3 \neq \phi$ and $A_j \cap A_3 = \phi$ for all j, $j \neq 2,3$. So it is trivial. This is just the case (B2) of (I) of Proposition 3.4.

(C) If there exists A_1 such that $Z \cdot A_1 = -1$ and A_1 has coefficient 2 in Z, then by Propositions 3.4, 3.5, 3.6 and (4.4) we must find the correct component C of $\cup A_i$, $i \neq 1$ which contains A_2. Consider the following inductively defined sets S_k of subscripts of the A_i: $S_o = \phi$. With S_k defined, consider all fundamental cycles Z_{k+1} for connected components of UA_i, $i \notin S_k$. For each Z_{k+1} such that $Z_{k+1} \cdot Z_{k+1} = -1$, there is a unique A_{k+1} such that $A_{k+1} \cdot Z_{k+1} = -1$. Let T_{k+1} be the set of subscripts for such A_{k+1}. Let $S_{k+1} = S_k \cup T_{k+1}$. For sufficiently large k, $S_k = S_{k+1}$. Call this largest set S. Then by Proposition 3.1, 3.4, 3.5, 4.6 and by (4.4) S has $4k+3$ elements for some integer k. For each component C^j of UA_i, $i \neq 1$ with its fundamental cycle Z_j satisfying $Z^j \cdot Z^j = -1$, we may form the corresponding set S^j. But for each C^j with $A_2 \not\subset C^j$, S^j has 4ℓ elements for some integer ℓ by case (II) of Proposition 3.4 and Corollary 3.5 and (4.4) because $Z^j \cdot Z^j = -1$. Therefore the correct component C is that component whose S^j has $4m+3$ elements for some integer m.

(II) If $Z \cdot Z = -1$, then there is A_1 such that $Z \cdot A_1 = -1$. By Proposition 3.4, 3.5, 3.6 and (4.4) we may find the correct component C of UA_i, $i \neq 1$ which contains A_2 as follows. Consider the following inductively defined sets S_k of subscripts of the A_i: $S_o = \{1\}$. With S_k defined, consider all fundamental cycles Z_{k+1} for connected components of UA_i, $i \notin S_k$. For each Z_{k+1} such that $Z_{k+1} \cdot Z_{K+1} = -1$, there is a unique A_{k+1} such that $A_{k+1} \cdot Z_{k+1} = -1$. Let T_{k+1} be the set of subscripts for such A_{k+1}. Let $S_{k+1} = S_k \cup T_{k+1}$. For sufficiently large k, $S_k = S_{k+1}$. Call this largest set S. Then by Proposition 3.1, 3.4, 3.5, 4.6 and by (4.4), S has 4k elements for some integer k. For each component C^j of UA_i,

$i \neq 1$ with its fundamental cycle Z^j satisfying $Z^j \cdot Z^j = -1$, we may form the corresponding set S^j. But for each C^j with $A_2 \not\subset C^j$, S^j has 4ℓ elements for some integer ℓ by case (II) of Proposition 3.4 and Corollary 3.5 and (4.4) because $Z^j \cdot Z^j = -1$. Therefore the correct component C is that component whose S^j has $4m+3$ elements for some integer m.

Corollary 4.9

(1) Let P_1 and P_2 be the singularities at $(0,0,0)$ of $V_1 = \{(x,y,z) : z^3 + 2q_1(x,y) = 0\}$ and $V_2 = \{(x,y,z): z^3 + 2q_2(x,y) = (0,0)\}$ respectively. Let V_1 and V_2 be the normal analytic spaces. Then P_1 and P_2 have homeomorphic resolutions by (2.2) if and only if q_1 and q_2 have equisingular plane curve singularities at $(0,0)$.

(2) Let P_1 and P_2 be the singularities at $(0,0,0)$ of $V_1 = \{(x,y,z) : z^3 + 3p_1(x,y)z + 2q_1(x,y) = 0\}$ and $V_2 = \{(x,y,z) : z^3 + 3p_2(x,y)z + 2q_2(x,y) = 0\}$, respectively with $p_i(x,y) \not\equiv 0$, $i = 1,2$. Let V_1 and V_2 satisfy (4.1). Then P_1 and P_2 have homeomorphic resolutions by (2.2) if and only if $p_1^3 + q_1^2$ and $p_2^3 + q_2^2$ have equisingular plane curve singularities at $(0,0)$.

Proof By Corollary 4.5 and Theorem 4.8.

Corollary 4.10 Let $\tilde{\Gamma}$ be the topological type of the exceptional set for the minimal resolution of a normal two-dimensional singularity. Then there are only a finite number of equisingular types for plane curve singularities such that the corresponding two-dimensional triple point with condition (4.1) has a minimal resolution of the topological type of $\tilde{\Gamma}$.

REFERENCES

[Ab] S. Abhyanker, Resolution of singularities of embedded algebraic surfaces, New York, Academic Press, 1966.

[Ar] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129-136.

[B1] E. Brieskorn, Uber die Auflosung gewisser Singularitaten von Holomorphen Abbildungen. Math. Ann. 166(1966), 76-102.

[B2] E. Brieskorn, Rationale Singularitaten Komplexer Flachen, Inv. Math. 4, 366-358 (1967-68).

[B&C] S. Barnard and J. M. Child, Higher Algebra, Macmillian Company, 1959.

[D] P. Duval, On isolated singularities of surfaces which do not affect the condition of adjunction, Proc. Cambridge Philosophical Soc. 30 (1933-34), 453-491.

[G] R. Gunning, Lectures on complex analytic varieties, Finite analytic mappings, Princeton Univ. Press, Princeton, 1974.

[G&F] H. Grauert and K. Fritzche, Several complex variables, Springer-Verlag, 1976.

[G&R] R. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[Gr] H. Grauert, Uber Modifikationen and Exzeptionnnelle Analytiske Mengen, Math. Ann. 146 (1962), 331-368.

[Hr] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II, Ann. Math. 79 (1964), 109-326.

[Hz] F. Hirzebruch, Uber vierdimensionale Riemannsch Flachen mehrdeutiger analytischer Funktionen von zwei komplexen Veranderlichen, Math. Ann., 126 (1953), 1-22.

[K1] U, Karras, Dissertation, Bonn, 1973.

[K2] U. Karras, Eigenschaften der lokalen Ringe in zweidimensionalten Spitzen, Math, Ann., 215 (1975), 119-129.

[L1] H. Laufer, Normal two-dimensional singularities, Annals of Math. Studies, No. 71, Princeton Univ. Press, Princeton, 1971.

[L2] H. Laufer, On rational singularities, Amer. J. Math., 94 (1972), 597-608.

[L3] H. Laufer, Taut two-dimensional singularities, Math. Ann. 205 (1973), 131-164.

[L4] H. Laufer, On minimally elliptic singularities, Amer. J. Math., 99 (1977), 1257-1295.

[L5] H. Laufer, On normal two-dimensional double point singularities, Israel J. Math. Vol. 31, Nos. 3-4 (1978), 315-334.

[Li] J. Lipman, Rational singularities with application to algebraic surfaces and unique factorization, Publ. Math. IHES n° 36 (1969), 195-279.

[M] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. IHES n° 9 (1961), 5-22.

[M&O] J. Milnor and P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, Topology 9 (1970), 385-393, MR 45 #2757.

[S] J-P. Serre, Algèbre locale-multiplicités, Springer-Verlag, Berlin (1965).

[Sa] K. Saito, Einfach-elliptische Singularitaten, Invent. Math. 23 (1974), 289-325.

[T1] G. Tyurina, Absolutely isolatedness of rational singularities and triple rational points, Funtional analysis and its applications, 2 (1968), 324-332.

[T2] G. Tyurina, The rigidity of rationally contractible curves on a surface (Russian) IZV. Akad. Nauk. SSSR Ser mat 32, 943-970, 1968.

[W1] P. Wagreich, Elliptic singularities of surfaces, Amer. J. Math. 92 (1970), 419-454.

[W2] P. Wagreich, Singularities of complex surfaces with solvable local fundamental group, Topology II (1972), 51-72.

[Z1] O. Zariski, A simplified proof of resolution of singularities of algebraic surfaces, Ann. Math. 43 (1942), 583-593.

[Z2] O. Zariski, Studies in equisingularity I, Equivalent singularities of plane algebroid curves, Amer. J. Math. 87 (1965), 507-536.