MANIFOLDS WITH ALMOST EQUAL DIAMETER AND INJECTIVITY RADIUS

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i) If there exists a point p in M with $i_p/d_p > 1-\varepsilon(C)$, then $\pi_1(M)$ is trivial or $\mathbb{Z}_2$;

ii) If $\pi_1(M)=\mathbb{Z}_2$, and there exists a point p in M such that $i_M/d_p > 1-\varepsilon(C)$, then M^n has the homotopy type of $\mathbb{R}P^n$;

iii) If for some p in M, $i_M/d_p > 1-\varepsilon(C)$, and the exponential map from p is of maximal rank on a closed ball of radius d_p about 0 in TM_p, then the universal cover of M is homeomorphic to S^n, and $\pi_1(M)=\mathbb{Z}_2$. Moreover, for $n \leqq 4$, M^n is homeomorphic to $\mathbb{R}P^n$. Also , it can be shown that the cut locus of p is a stratified set which has strata of smooth submanifolds of various dimensions, for all n. In this case, i_M is bounded from below in terms of curvature. For a smaller $\varepsilon(C)$, the cut locus becomes a smooth submanifold of codimension 1.

There are examples showing that the curvature condition of i) can not be removed.

The $\varepsilon(C)$'s above are different in each case, and their scale is approximately between 1/10 and 1/20.

To Robin

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INTRODUCTION

In this paper, we will give some constraints on the topology of compact, connected Riemannian manifolds whose injectivity radii and diameters are close to each other, in terms of the sectional curvature. For the notation and definitions, we refer to Chapter I.

The case of the spherical cut locus of a point p in M and also the stronger case of the equality of diameter and injectivity radius of M, i.e. $i_p=d_p$ and $i_M=d_M$, have been studied by various authors.

F.W. Warner [Wa] has shown that if there exists a point p in a compact and simply connected Riemannian manifold M for which each point of the spherical conjugate locus $\tilde{Q}_p$ in TM_p is regular, then that has the same multiplicity as conjugate points which is $\geqq 1$, and M is homeomorphic to a sphere or has the integral cohomology ring of one of compact irreducible symmetric spaces of rank 1.

Nakagawa and Shiohama [NS-1], [NS-2], have studied the spherical cut locus for the cases of $K_M \leqq 1$ and $0<k \leqq K_M \leqq K=1$, and obtained that M should be either $\mathbb{R}P^n$ or S^n in some sense, or have a cohomology ring with one generator. For precise

statements, see Chapter I.

It was observed by Omori [O] that if the metric of M is real analytic and if N is a real analytic submanifold whose cutlocus N' has constant distance from N, then N' is a real analytic submanifold of M, and M has a decomposition $M=D_N \cup_\phi D_{N'}$, where D_N and $D_{N'}$ are normal disc bundles of N, N', respectively.

In Besse [Bs], it is claimed that a point $p \in M$, where M is C^∞, has a spherical cut locus if and only if M is a pointed Blaschke manifold at p. There is an extensive theory for Blaschke manifolds (see [Bs]). Especially, the Bott-Samelson Theorem gives topological information about M, similar to the Nakagawa-Shiohama results (see Chapter 7 of [Bs]).

Berger [Bs] has shown that if M is S^n or $\mathbb{R}P^n$ and a Blaschke manifold, i.e. $i_M=d_M$, then M is actually isometric to S^n or $\mathbb{R}P^n$. The Blaschke Conjecture states that any Blaschke manifold is isometric to one of S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, or $\mathbb{C}aP^2$, with their canonical metrics, up to a constant factor.

The theorems above show that the condition $i_p=d_p$ for some point p in a compact Riemannian manifold is a very rigid restriction. Then, a natural question to ask is "What happens if we allow some flexibility in this condition?" This cannot be done arbitrarily, since the example 2 of Chapter II, Section 5, shows that for any given ε_0, any compact Riemannian manifold has a new metric on it so that i_p and d_p are ε_0-close. Hence, other conditions, such as on curvature, may be needed. Also, the theorems above do not take this case into consideration and do not seem to generalize easily in this direction, because the nature of the proofs depends very much on the rigidity, with the exception of the Samelson map (see Chapter II, Section 3).

One important result in a similar direction is due to Weinstein [Wa], [Bs]: If M can be written as $D \cup_a E$, where D is the n-dimensional closed ball, E is a C^∞ closed k-disc bundle over a (n-k)-dimensional compact C^∞ manifold, with boundary ∂E diffeomorphic to S^{n-1}, and $a:\partial D \to \partial E$ an attaching diffeomorphism; then there exists a new Riemannian metric on M such that the center of D has a spherical cut locus. On the other hand, given a manifold with i_p close to d_p; it appears that to show that some neighborhood of C_p has the structure of a smooth disc bundle over some smooth submanifold is very difficult or perhaps impossible.

A problem which makes this situation i_M close to d_M interesting and also illustrates its complexity explicitly is the following: To find quantitative topological restrictions on even dimensional manifolds with $1 \leqq K_M \leqq 4+\varepsilon$, for some $\varepsilon>0$. Grove and Shiohama [GS] have shown that if $d_M>\pi/2$ then M is homeomorphic to S^n. Gromoll and Grove [GG] extended this result: If $d_M=\pi/2$, then M is homeomorphic to S^n, or isometric to a symmetric space of rank 1. By Klingenberg's Lemma, we have $i_M \geqq \pi(4+\varepsilon)^{-1/2}$. The case of $\pi(4+\varepsilon)^{-1/2} \leqq i_M \leqq d_M < \pi/2$ is recently resolved by Berger [B-3]: "There exists a real number $\delta<1/4$ such that any simply connected, even dimensional compact Riemannian manifold M with $0<\delta \leqq K_M \leqq 1$, is necessarily homeomorphic to S^n or diffeomorphic to a symmetric space of rank 1."

The primary goal of this paper is to construct some universal constants such that if i_p or i_M is close to d_p or d_M in terms of these constants, then there will be constraints on the topology of the manifold. These universal constants depend only on the lower bound of the sectional curvature, and in some cases also on the dimension of M. Although different in each case, their general scale is not too small by comparison.

In Chapter I, we list the basic definitions, notations, and theorems that we will use.

Chapter II contains the proofs of the main theorems, corollaries, and examples. An elementary description of the universal cover is given in Section 1. In Section 2, we present and prove Theorem I, which gives the restriction on the fundamental group if i_p is close to d_p, in terms of a universal constant which depends on the lower bound of the sectional curvature. Section 3 deals with Theorem 2, which classifies the non-simply connected case. The case of the conjugate locus Q_p being bounded away from the cut locus C_p of a point p of M is investigated in Section 4. We show that M cannot be simply connected in this case, and give some stratification structure to the cut locus, and conditions which make C_p a codimension 1 submanifold, so that we can apply Weinstein's result. Finally, we present some examples, counterexamples, and immediate consequences of the main theorems in Section 5.

The Theorems 1-5B generalize some of the results of Shiohama and Nakagawa, especially in the non-simply connected case.

Although Berger [B-3] has resolved the problem of almost-1/4-pinched manifolds; the problem of finding more topological restrictions on the simply connected manifolds whose injectivity radius is close to the diameter in terms of the lower bound of the sectional curvature or any other geometric quantities, still needs to be investigated.

CHAPTER I

NOTATION AND PRELIMINARIES

In this chapter, the basic definitions, notations, and the theorems which will be used in Chapter II are given.

For the basic notions of manifolds and Riemannian Geometry, we refer to Cheeger and Ebin [CE], Gromoll, Klingenberg, and Meyer [GKM], and Kobayashi and Nomizu [KN].

In this text, M^n always denotes a compact, smooth, connected, n-dimensional Riemannian manifold without boundary. TM, UM, TM_p, and UM_p denote the tangent bundle of M, the unit sphere bundle of the tangent bundle, the tangent space at p, and the unit vectors in TM_p with respect to the Riemannian metric $< , >_p$, respectively.

For any smooth map f: $M_1 \to M_2$, where M_i are smooth manifolds, for i=1,2, f_* denotes the differential (Jacobian) of f, i.e. the induced map $f_*: TM_1 \to TM_2$.

Unless it is specified, all coordinate systems around any point of M are normal, and all geodesics are parametrized by their arclength; that is, the velocity vectors are unit vectors. If $\gamma(t)$ is said to be any geodesic from p to q, then it is assumed that $\gamma: [0,\ell] \to M$ such that $\gamma(0)=p$, $\gamma(\ell)=q$, and the length of γ, which is denoted by $\ell(\gamma)$ is to be ℓ. If γ is also said to be minimal, then $d(p,q)=\ell$, where $d(p,q)$ denotes the distance between p and q. $\gamma'(t)$ denotes the velocity vector of γ at $\gamma(t)$. Similarly, if $r=\gamma(t_r)$ and t_r is unique, then $\gamma'(r)$ also denotes $\gamma'(t_r)$.

Let v_1,v_2 be non-zero vectors in TM_p for some $p\in M$. The angle $\measuredangle(v_1,v_2)$ between v_1 and v_2 is always measured to be between 0 and π, and it is given by $\cos(\measuredangle(v_1,v_2))\cdot\|v_1\|\cdot\|v_2\|=\langle v_1,v_2\rangle_p$.

$d_M:M\times M\to[0,\infty)$ is the distance function. The diameter of M which is the maximum value of the function $d_M(.,.)$ will also be denoted by d_M or $d(M)$. If there is no chance of ambiguity, M will be suppressed in $d_M(.,.)$ and $d(p,q)$ will be written instead of $d_M(p,q)$, for $p,q\in M$.

<u>DEFINITION 1.</u> For any subset X of M, the closure, the interior, and the boundary of X will be denoted by $\overline{X}$, int(X), and ∂X, respectively. Let $\exp_p:TM_p\to M$ be the <u>exponential map</u>. For any $p\in M$ and $v\in UM_p$, the <u>cut value in the direction of v</u> $c_p(v)$ is to be $\mathrm{Max}\{\lambda\in\mathbb{R}|\ \lambda>0,\ d(p,\exp_p(\lambda v))=\lambda\}$ and the <u>fundamental region</u> A_p to be $\{v\in TM_p|\ d(p,\exp_p(v))=\|v\|\}$. The <u>tangential cut locus</u> of p, $\tilde{C}_p$, is defined to be ∂A_p and the <u>cut locus</u> of p, C_p, be $\exp_p\tilde{C}_p$.

One can show that $c_p(v)$ depends on p and v continuously, and $c_p(v)>0$ is finite for all $v\in UM_p$, since M is compact. Hence, $\partial A_p=\{c_p(v)\cdot v|\ v\in UM_p\}$, and it is homeomorphic to S^{n-1}. See [GKM].

DEFINITION 2. Let $p\varepsilon M$ be given. The injectivity radius at p is defined to be $\mathrm{Min}\{c_p(v) \mid v\varepsilon UM_p\}$ and is denoted by i_p. $d_p=\mathrm{Max}\{c_p(v) \mid v\varepsilon UM_p\}$ is in fact the distance to the furthest point from p. Let i_M and d_M be $\mathrm{Min}\{i_p \mid p\varepsilon M\}$ and $\mathrm{Max}\{d_p \mid p\varepsilon M\}$, respectively. i_M and d_M are called the injectivity radius of M and the diameter of M.

This definition of the diameter is equivalent to the previous definition. If there is more than one metric on M, then $i_p(M,g)$, $d_p(M,g)$, $d(M,g)$ and $i(M,g)$ will be used to indicate dependence on some metric g on M.

DEFINITION 3. For any metric space X, $B_r(x_0,X)$ and $\overline{B}_r(x_0,X)$ for some $x_0\varepsilon X$, denote the balls $\{x\varepsilon X \mid d_X(x,x_0)<r\}$ and $\{x\varepsilon X \mid d_X(x,x_0)\leq r\}$, respectively.

Let $\tilde{M}$ be the universal cover of M and $\rho:\tilde{M}\to M$ be the natural projection map. Since ρ is a local homeomorphism, it induces a smooth Riemannian manifold structure on $\tilde{M}$ by pulling back the strucutre on M locally. With this natural structure on $\tilde{M}$, ρ becomes a local isometry and $\forall p'\varepsilon\tilde{M}$, $\forall v'\varepsilon T\tilde{M}_{p'}$, $\forall t\varepsilon\mathbb{R}$, $\rho(\exp_{p'}(tv'))=\exp_{\rho(p')}(t\rho_*(v'))$. In this paper, whenever the universal cover $\tilde{M}$ of M is used, this natural Riemannian structure will always be considered.

DEFINITION 4. Let $p\varepsilon M$. The first tangential conjugate locus $\tilde{Q}_p$ of p is defined to be:

$$\left\{ v \in TM_p \,\middle|\, \begin{array}{l} (\exp_p)_*(tv): T(TM_p)_{tv} \to TM_{\exp_p(tv)} \text{ is maximal rank for} \\ 0 \leq t < 1 \text{ and not maximal rank for } t=1. \end{array} \right\}$$

The <u>first conjugate locus</u> Q_p <u>of</u> <u>p</u> is defined to be $\exp_p(\tilde{Q}_p)$.

Let K_M denote the sectional curvature of the Riemannian connection on M, which is torsion free.

<u>DEFINITION 5.</u> For any $C \in \mathbb{R}$, M_C^2 denotes the simply connected two-dimensional, complete Riemannian manifold of constant sectional curvature C, i.e. a space form which is unique up to isometry. For example, see [CE].

The theorem below was first proved by V.A. Toponogov [T-1], [T-2]. For other proofs and a complete treatment of the subject, see [CE,pg. 43], [GKM, pg. 184]. The following form of the theorem and definitions appear in [CE,pg. 43]. All indices below are taken modulo 3.

<u>DEFINITION 6.</u> A <u>geodesic triangle</u> in the Riemannian manifold M is a set of three geodesic segments parametrized by arclength $(\gamma_1, \gamma_2, \gamma_3)$ of lengths ℓ_1, ℓ_2, ℓ_3 such that $\gamma_i(\ell_i) = \gamma_{i+1}(0)$ and $\ell_i + \ell_{i+1} \geq \ell_{i+2}$. Set $\alpha_i = \measuredangle(-\gamma_{i+1}'(\ell_{i+1}), \gamma_{i+2}'(0))$, the angle between $-\gamma_{i+1}'(\ell_{i+1})$ and $\gamma_{i+2}'(0)$, $0 \leq \alpha_i \leq \pi$.

<u>THEOREM (Toponogov).</u> Let M be a complete manifold with $K_M \geq C$.

a) Let $(\gamma_1, \gamma_2, \gamma_3)$ determine a geodesic triangle in M. Suppose γ_1, γ_3 are minimal and if $C > 0$, suppose $\ell(\gamma_2) \leq \pi \cdot C^{-1/2}$. Then in M_C^2, there exists a geodesic triangle $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$ such that

$\ell(\gamma_i)=\ell(\bar{\gamma}_i)$ and $\bar{\alpha}_1\leqq\alpha_1, \bar{\alpha}_3\leqq\alpha_3$. Except in the case $C>0$ and $\ell(\gamma_i)=\pi\cdot C^{-1/2}$ for some i, the triangle in M_C^2 is uniquely determined.

b) Let γ_1,γ_2 be geodesic segments in M such that $\gamma_1(\ell_1)=\gamma_2(0)$ and $\measuredangle(-\gamma_1'(\ell_1),\gamma_2'(0))=\alpha$. We call such a configuration a hinge L and denote it by $(\gamma_1,\gamma_2,\alpha)$. Let γ_1 be minimal, and if $C>0$, $\ell(\gamma_2)\leqq\pi\cdot C^{-1/2}$. Let $\bar{\gamma}_1,\bar{\gamma}_2\subseteq M_C^2$ be such that $\bar{\gamma}_1(\ell_1)=\bar{\gamma}_2(0)$, $\ell(\gamma_i)=\ell(\bar{\gamma}_i)=\ell_i$ and $(-\bar{\gamma}_1'(\ell_1),\bar{\gamma}_2'(0))=\alpha$. Then $d_M(\gamma_1(0),\gamma_2(\ell_2))\leqq d_{M_C^2}(\bar{\gamma}_1(0),\bar{\gamma}_2(\ell_2))$.

Toponogov's Theorem guarantees the existence of the triangles in M_C^2, as long as the lengths of the sides satisfy the triangle inequalities. In this text, unless otherwise specified, the geodesic triangles in M have sides given by minimal geodesics, so the triangle inequalities are automatically satisfied. Also, all comparisons are done with M_C^2 for $C\leqq 0$. Hence, the triangles are uniquely determined up to congruences of M_C^2. At the same time, the angles $\bar{\alpha}_i$ are known, if the side lengths ℓ_i are known, by the laws of cosine, [GKM,pg. 195]:

if $C=0$: $\ell_i^2=\ell_{i+1}^2+\ell_{i+2}^2-2\ell_{i+1}\cdot\ell_{i+2}\cdot\cos\bar{\alpha}_i$;

if $C<0$:

$\cosh(\kappa\cdot\ell_i)=\cosh(\kappa\cdot\ell_{i+1})\cdot\cosh(\kappa\cdot\ell_{i+2})-\sinh(\kappa\cdot\ell_{i+1})\sinh(\kappa\cdot\ell_{i+2})\cos\bar{\alpha}_i$,

where $\kappa=(-C)^{1/2}$.

THEOREM (Rauch [R]). Let M_1, M_2 be Riemannian manifolds of the same dimension, $\gamma_1:[0,\ell]\to M_1$, $\gamma_2:[0,\ell]\to M_2$ be normal geodesics, Y_1, Y_2 be Jacobi fields along γ_1, γ_2 with $Y_1(0)=Y_2(0)=0$ and $\langle Y_1', \gamma_1'\rangle|_{t=0}=\langle Y_2', \gamma_2'\rangle|_{t=0}=0$ and $\|Y_1'(0)\|=\|Y_2'(0)\|$. Let γ_2 have no conjugate points on $(0,\ell)$. If the sectional curvature of M_1 along γ_1 is smaller than or equal to the sectional curvature of M_2 along γ_2; i.e. $K_{M_1}(\sigma_{1,t})\leq K_{M_2}(\sigma_{2,t})$ for any two-plane $\sigma_{i,t}$ in $T(M_i)_{\gamma_i(t)}$, (in fact, it is sufficient to have the inequality to be true for the two-planes generated by $Y_i(t)$ and $\gamma_i'(t)$, $i=1,2$) for all $t\in[0,\ell]$. Then $\|Y_1(t)\|\geq\|Y_2(t)\|$. In fact, for $t>0$,

$$\frac{d}{dt}\left(\|Y_1\|^2/\|Y_2\|^2\right)\geq 0.$$

PROOF. See [GKM, pg. 181], [CE, pg. 28].

LEMMA (Berger [B-2]). Let M be a complete Riemannian manifold. Let $p,q\in M$ be such that $d_M(p,q)=d_p$. For any vector $v\in TM_q$, there exists a minimal geodesic γ from q to p such that $\measuredangle(\gamma'(0),v)\leq\pi/2$.

PROOF. See [GKM, pg. 257] or [CE, pg. 106].

DEFINITION 7. A subset S of a Riemannian manifold M is called strongly convex if for any $q_1,q_2\in\bar{S}$, there exists a unique minimal geodesic γ_{q_1,q_2} from q_1 to q_2 such that $\gamma_{q_1,q_2}:[0,d(q_1,q_2)]\to M$, $\gamma_{q_1,q_2}(0)=q_1$, $\gamma_{q_1,q_2}(d(q_1,q_2))=q_2$ and $\gamma_{q_1,q_2}((0,d(q_1,q_2)))\subseteq S$.

THEOREM (Whitehead [Wh]). If $r<(1/2)\cdot\mathrm{Min}(\pi\cdot K^{-1/2}, i_M)$, then $B_r(p,M)$ is strongly convex, where $K=\mathrm{Max}(K_M)$. (If $K\leq 0$, consider ∞

instead of $\pi/K^{1/2}$.)

PROOF. See [CE,pg. 103].

LEMMA A. Let $p,q\varepsilon M$ be such that $d(p,q)=i_p$. Then either there is a minimal geodesic from p to q along which p is conjugate to q, or there are precisely two minimal geodesics γ_1,γ_2 from p to q such that $\gamma_1'(q)=-\gamma_2'(q)$.

PROOF. See [CE,pg. 95].

LEMMA (Klingenberg). If $K_1 \geqq K_M \geqq K_0 > 0$, then $i_M \geqq \mathrm{Min}(\pi/K_1^{1/2}, \ell/2)$, where ℓ is the length of the shortest smooth closed geodesic in M. If M is also an even dimensional, oriented manifold, then $i_M \geqq \pi/K_1^{1/2}$.

PROOF. See [GKM,pg.227], [CE,pg.96,98].

DEFINITION 8. Let $p\varepsilon M$. p is said to have a spherical cut locus if $i_p=d_p$.

DEFINITION 9. Let $p\varepsilon M$ and $q\varepsilon C_p$. The link from p to q is defined to be $\Lambda(p,q)=\{v\varepsilon UM_q \mid \exp_q(d(p,q)\cdot v)=p\}$.

DEFINITION 10. A compact Riemannian manifold M is called a pointed Blaschke manifold at p, for some $p\varepsilon M$, if $\forall q\varepsilon C_p$, $\Lambda(p,q)$ is the intersection of UM_q with a subspace of TM_q. M is called a Blaschke manifold if it is a pointed Blaschke manifold at p, for all $p\varepsilon M$.

THEOREM (Nakagawa-Shiohama [NS-1]). Let M be a compact, connected Riemannian manifold with $K_M \leq 1$ such that there exists $p \in M$ with $\ell = i_p = d_p$. Then

i) $\ell \geq \pi/2$;

ii) if $\ell = \pi/2$, then M is isometric to $\mathbb{R}P^n$ with constant sectional curvature 1;

iii) if $\pi/2 < \ell < \pi$, then M has the same cohomology groups as that of $\mathbb{R}P^n$ and $\tilde{M}^n$ is homeomorphic to S^n. Hence, if M is simply connected, then $\ell \geq \pi$.

iv) if $C_p \nsubseteq Q_p$, then $\tilde{C}_p \cap \tilde{Q}_p = \phi$ and hence, M has the same cohomology groups as that of $\mathbb{R}P^n$, and $\tilde{M}$ is homeomorphic to S^n.

THEOREM (Nakagawa-Shiohama [NS-2]). Let M be an n-dimensional, connected, compact C^∞- manifold. Assume that there exists $p \in M$ such that $d(p,q) = \ell$ for all $q \in C_p$, where $\ell = \pi \cdot (\mathrm{Max}(K_M))^{-1/2}$. Then every geodesic segment starting from p with length 2ℓ is a geodesic loop at p, and we have, for any point $q \in Q_p$, the multiplicity of p and q as a conjugate pair is constant λ, where $\lambda = 0,1,3,7,n-1$. Moreover,

i) If $\pi_1(M) \neq 0$, then M has the same cohomology groups as that of $\mathbb{R}P^n$, and $\tilde{M}^n$ is homeomorphic to S^n, where $\lambda = 0$ holds.

ii) If $\pi_1(M) = 0$, then the integral cohomology ring $H^*(M,\mathbb{Z})$ is a truncated polynomial ring generated by one element (in $H^{\lambda+1}(M,\mathbb{Z})$). In particular, if $\lambda = n-1$, then M is isometric to a sphere of constant sectional curvature $\mathrm{Max}(K_M)$.

CHAPTER II

SECTION 1. A DESCRIPTION OF THE UNIVERSAL COVER $\tilde{M}$.

The following description is elementary, and it gives a proper perspective of the universal cover, which is used in the proofs of Theorems 1 and 2.

Let M be any non-simply connected, compact Riemannian manifold, $\tilde{M}$ be its Riemannian universal cover, and $\rho:\tilde{M}\to M$ be the natural Riemannian projection map which is a local isometry. There is a natural one-to-one correspondence between $\pi_1(M)$ and the deck transformations of $\tilde{M}$. For $[\gamma_i]\in\pi_1(M)$, let $\gamma_i:\tilde{M}\to\tilde{M}$ also represent the corresponding deck transformation.

Let $U=M-C_p$, for some $p\in M$. It is known that U is homeomorphic to an open ball and is dense in M. [CE], [GKM].

Fix $p\in M$ and $p_0\in\tilde{M}$ such that $\rho(p_0)=p$, and set $p_i=\gamma_i(p_0)$, for $[\gamma_i]\in\pi_1(M)$. Since U is contractible, there exists a unique open connected set U_i in $\tilde{M}$ such that $p_i\in U_i$ and $\rho|U_i:U_i\to U$ is a homeomorphism. Thus, we have $U_i\cap U_j=\phi$, if $[\gamma_i]\neq[\gamma_j]$ in $\pi_1(M)$, and $\tilde{M}=\bigcup_{[\gamma_i]\in\pi_1(M)}\overline{U}_i$. To see that, if $U_i\cap U_j\neq\phi$, then we would obtain a continuous curve from p_i to p_j in $U_i\cup U_j$, whose image under ρ lies in U, and hence, a curve representing a non-trivial element of $\pi_1(M)$ would be lying in a contractible set, which is a contradiction. Second part follows from $\rho(\overline{U}_0)=M$.

For U_i, U_j such that $\overline{U}_i \cap \overline{U}_j \neq \phi$, let q_{ij} be any point $\overline{U}_i \cap \overline{U}_j$ and θ_i, θ_j be minimal geodesic segments such that $\theta_i(0)=p_i$, $\theta_i(d(p_i,q_{ij}))=q_{ij}$, $\theta_j(0)=q_{ij}$ and $\theta_j(d(q_{ij}, p_j))=p_j$. Define

$$\gamma_{ij}(t)=\begin{cases}\theta_i(t), & \text{if } 0 \leqq t \leqq d(p_i,q_{ij}) \\ \theta_j(t-d(p_i,q_{ij})), & \text{if } d(p_i,q_{ij})<t\leqq d(p_i,q_{ij})+d(q_{ij},p_j).\end{cases}$$

Obviously, γ_{ij} represents $[\gamma_j]\cdot[\gamma_i]^{-1} \epsilon \pi_1(M)$. For any $[\gamma_{i_0}] \epsilon \pi_1(M)$, consider any minimal geodesic γ from p_0 to p_{i_0}. The set $I=\{[\gamma_j] \epsilon \pi_1(M) \mid \overline{U}_j \cap Im(\gamma) \neq \phi\}$ is a finite set, because all such $\overline{U}_j$'s lie in $2d_M$ neighborhood of γ whose length is finite, and all $\overline{U}_j$ have the same volume. $Im(\gamma)$ is a connected set, so we can find a sequence $n_0, n_1, \ldots, n_k$, with $[\gamma_{n_k}] \epsilon I$, such that $n_0=0$, $n_k=i_0$ and $\overline{U}_{n_\ell} \cap \overline{U}_{n_{\ell+1}} \neq \phi$, for $\ell=1,\ldots,k-1$. If $\gamma_{n_\ell n_{\ell+1}}$ is constructed as above, then clearly the union of these curves $\gamma_{n_\ell n_{\ell+1}}$, $\ell=1,\ldots,k-1$; is a continuous curve from p_0 to p_{i_0} and hence, its image under ρ represents $[\gamma_{i_0}] \epsilon \pi_1(M)$. Let $\Theta=\rho(\{\gamma_{ij} \mid \overline{U}_i \cap \overline{U}_j \neq \phi\})$. Θ is a set of loops at p, and it generates $\pi_1(M)$. Θ is a set that contains some curves which are the union of two minimal geodesic segments: the first one is from p to a point in C_p and the second is from the end point of the first, back to p along possibly another minimal geodesic.

It follows that $\Lambda=\{[\gamma_j]\in\pi_1(M) \mid \overline{U}_j\cap\overline{U}_0\neq\phi\}$ is a set of generators for $\pi_1(M)$.

$\exp_p:\mathrm{int}(A_p)\to U$ is a homeomorphism, where $A_p \subsetneq TM_p$. [CE], [GKM]. Let ρ_* be the induced map on the tangent spaces. Since ρ is a local isometry, the following diagram is commutative:

$$\begin{array}{ccc}
(\rho_*(p_0))^{-1}(\mathrm{int}(A_p)) & \xrightarrow{\exp_{p_0}} & U_0 \\
\simeq \Big\downarrow \rho_* & & \Big\downarrow \rho \\
\mathrm{int}(A_p) & \xrightarrow{\exp_p} & U
\end{array}$$

Since A_p is compact, $\exp_{p_0}((\rho_*(p_0))^{-1}(A_p))$ is compact. $\mathrm{Int}(A_p)$ is dense in A_p, so $\exp_{p_0}((\rho_*(p_0))^{-1}(\mathrm{int}(A_p)))= U_0$ is dense in $\exp_{p_0}((\rho_*(p_0))^{-1}(A_p))$. Therefore, $\overline{U}_0=\exp_{p_0}((\rho_*(p_0))^{-1}(A_p))$. Since $\exp_{p_0}$ is a homeomorphism on $(\rho_*(p_0))^{-1}(\mathrm{int}(A_p))$, it takes interior points to interior points. Hence,

$$\partial U_0=\overline{U}_0-U_0 \subsetneq \exp_{p_0}((\rho_*(p_0))^{-1}(\partial A_p)).$$

$\exp_p(\partial A_p)=C_p$ and $C_p\cap U=\phi$, so $\exp_{p_0}((\rho_*(p_0))^{-1}(\partial A_p))\cap U_0=\phi$. Therefore, $\exp_{p_0}((\rho_*(p_0))^{-1}(\partial A_p))\subsetneq\partial U_0$, and hence, $\exp_{p_0}((\rho_*(p_0))^{-1}(\partial A_p))=\partial U_0$. It is known that for a compact

Riemannian manifold M^n, ∂A_p is homeomorphic to S^{n-1}. Hence, ∂U_0 is an image of a connected set under a continuous map $\exp_{p_0} \circ (\rho_*(p_0))^{-1}$. We have now proved that:

LEMMA 1. ∂U_0 is connected.

We have $\rho(\partial U_i)=C_p$, for all i. $\forall q \in \partial U_i = \exp_{p_i}((\rho_*(p_i))^{-1}(\partial A_p))$, that is, there is a $v \in \partial A_p$ such that $q=\exp_{p_i}((\rho_*(p_i))^{-1}(v))$ and $\exp_{p_i}(t\cdot((\rho_*(p_i))^{-1}(v)))$ is a minimal geodesic from p_i to q. So, $i_p \leq d(q,p_i) \leq c_p(v/\|v\|) \leq d_p$.

REMARK. If any statement is true for p_i or U_i, for some i, then the analogue is true for all i, by using an appropriate deck transformation which is an isometry of $\tilde{M}$.

Some facts about U_i that we will use in the following without referring: For U_i, U_j disjoint, for any point q in $\overline{U}_i \cap \overline{U}_j$, every neighborhood of q intersects with U_i and U_j, so $\overline{U}_i \cap \overline{U}_j \subseteq \partial U_i$. Obviously, $\partial U_i = \overline{U}_i - U_i$ and $\partial \overline{U}_i = \overline{U}_i - \text{int}(\overline{U}_i)$. Since U_i is open, we have $U_i \subseteq \text{int}(\overline{U}_i)$; therefore, $\partial \overline{U}_i \subseteq \partial U_i$. U_i is dense in $\overline{U}_i$, so U_i is dense in $\text{int}(\overline{U}_i)$. $U_i \cap U_j = \phi$ implies that $(\text{int}(\overline{U}_i)) \cap U_j = \phi$. Moreover, $(\text{int}(\overline{U}_i)) \cap (\text{int}(\overline{U}_j)) = \phi$. Finally, $\overline{U}_i \cap \overline{U}_j = \partial \overline{U}_i \cap \partial \overline{U}_j = \partial U_i \cap \partial U_j$, and hence, $\overline{U}_i \cap \overline{U}_j \subseteq \partial \overline{U}_i$.

SECTION 2. THE FUNDAMENTAL GROUP.

The main result of this section is:

THEOREM 1. Given $C \in \mathbb{R}$, there exists a universal constant $\varepsilon_1(C)$ depending only on C, such that: For any compact Riemannian manifold M^n, $n \geq 2$; if

i) $d_M^2 \cdot K_M \geq C$, and

ii) there exists a point p in M, such that $i_p/d_p > 1-\varepsilon_1(C)$;

then $\pi_1(M)=1$ or $\mathbb{Z}_2$.

For the proof of Theorem 1, we need some preliminary lemmas.

LEMMA 2. For $C \leq 0$. Let two geodesic triangles in M_C^2 be given with sides of lengths A_1, B_1, C_1 and A_2, B_2, C_2, respectively. Let $\alpha_i, \beta_i, \gamma_i$ be the angles between the sides of length B_i, C_i; A_i, C_i; and A_i, B_i, respectively, for $i=1,2$.

a) If $A_1=A_2$, $C_1=C_2$ and $B_1<B_2$, then $\beta_1<\beta_2$.

b) If $A_1>A_2$, $C_1=C_2$, $B_1=B_2$ and $\beta_1>\pi/2$, then $\beta_1<\beta_2$.

PROOF. Case for $C<0$: By multiplying the metric with $(-C)^{1/2}$, we can reduce the problem to the case $C=-1$. The cosine theorem for the hyperbolic space [GKM,pg.195] states that:

$$\cos\beta = \frac{(\cosh A)\cdot(\cosh C) - \cosh B}{(\sinh A)\cdot(\sinh C)}. \qquad (1)$$

a) is obvious from (1).

b) Let $B_0=B_1=B_2$, $C_0=C_1=C_2$ and $A(t)=t\cdot A_2+(1-t)\cdot A_1$, for $t\varepsilon[0,1]$. Define

$$f(t) = \frac{(\cosh A(t))\cdot(\cosh C_0)-(\cosh B_0)}{(\sinh A(t))\cdot(\sinh C_0)} .$$

f is a smooth function, because A_1, A_2,C_0 are all positive.

A straightforward calculation shows that:

$$f'(t)=A'(t)\cdot \frac{(\sinh C_0)\cdot((\cosh A(t))\cdot(\cosh B_0)-(\cosh C_0))}{((\sinh A(t))\cdot(\sinh C_0))^2} .$$

On the other hand:

$$\begin{aligned}(\cosh A(t))\cdot(\cosh B_0)-(\cosh C_0) &\geqq (\cosh A(0))\cdot(\cosh B_0)-(\cosh C_0)\\ &=(\cos\gamma_1)\cdot(\sinh A_1)\cdot(\sinh B_1)\\ &> 0,\end{aligned}$$

because $\beta_1>\pi/2$ and by Gauss-Bonnet Theorem the sum of the internal angles of a geodesic triangle in M_C^2 is $\leqq\pi$,for $C<0$, and hence, $\gamma_1<\pi/2$. Therefore, $f'(t)$ has the same sign of $A'(t)=A_2-A_1<0$.

$\therefore \cos\beta_2=f(1)<f(0)=\cos\beta_1$.

$\therefore \beta_2>\beta_1$.

Case for C=0: It follows from $\cos\beta=(A^2+C^2-B^2)/2AC$ by a similar argument. Lemma 2 QED.

LEMMA 3. Let $x_1, x_2, \ldots, x_k$ be distinct unit vectors in $\mathbb{R}^N$, with the standard inner product, such that $\measuredangle(x_i, x_j) > \arccos(-1/n)$, for $x_i \neq x_j$. Then $k<n+1$.

PROOF. $\measuredangle(x_i, x_j) > \arccos(-1/n) \Leftrightarrow \langle x_i, x_j\rangle < -1/n$.

$$0 \leq \left\| \sum_{i=1}^{k} x_i \right\|^2 = \left\langle \sum_{i=1}^{k} x_i, \sum_{j=1}^{k} x_j \right\rangle = \sum_{i=1}^{k} \sum_{j=1}^{k} \langle x_i, x_j\rangle$$

$$= \sum_{i=1}^{k} \Big(\Big(\sum_{\substack{j=1 \\ j\neq i}}^{k} \langle x_i, x_j\rangle \Big) + 1 \Big) < \sum_{i=1}^{k} \big((k-1)(-1/n) + 1 \big).$$

$\Rightarrow 0 < 1-(k-1)/n$, and hence, $k<n+1$. Lemma 3 QED.

First, we prove Theorem 1 by using Lemma 4, second, we give some facts and prove Lemma 5 and finally, we prove Lemma 4 by using Lemma 5.

PROOF (Theorem 1). $\varepsilon_1(C)$ is constructed as follows: Given $C \in \mathbb{R}$. Case for $C \leq 0$. Let $x \in [0,1)$. Consider the following two geodesic triangles in M_C^2. The first one has sides of length $1+x, 1+x$, and 2. Let $\beta_1(x)$ be the angle between the sides of length $1+x$. The second one has sides of length $1+x$, $1+3x$, and 2. Let $\beta_2(x)$ be the angle between the sides of length $1+x$ and $1+3x$. See Figures 1 and 2. $\beta_1(x)$ and $\beta_2(x)$ are strictly decreasing continuous functions of x, whenever each is $\geq \pi/2$; by the Laws of Cosine and by applying

Lemma 2b twice, by changing one side at a time. $\beta_2(1)=0$ and if $\beta_2(x)<\pi/2$, then $2\beta_2(x)+\beta_1(x)<\pi+\beta_1(x)<2\pi$. $\beta_1(0)=\beta_2(0)=\pi$. Therefore, there exists a unique $x_0(C)$ such that $\beta_1(x_0(C))+2\beta_2(x_0(C))=2\pi$. If $\beta_2(x)\geqq\pi/2$ and $x\neq 0$, by Lemma 2b we will obtain that $\beta_1(x)>\beta_2(x)$ and hence, $\beta_1(x_0(C))>2\pi/3$.

Let q_1, q_2 and q_3 be points in M_C^2 with $d(q_1,q_2)=d(q_1,q_3)=d(q_2,q_3)=1$ and γ_1 be the minimal geodesic from q_2 to q_3, γ_1 be defined after passing through q_3. Let q_4 be $\gamma_1(1+2x_0)$, and γ_2, γ_3 be the minimal geodesic segments from q_4 and q_3 to q_1, respectively. If $\alpha_1=\measuredangle(-\gamma_1'(q_4),\gamma_2'(q_4))$, then define $\varepsilon_1'(C)=\mathrm{Min}(x_0, \beta_1^{-1}(\pi-\alpha_1))$ and $\varepsilon_1(C)=1-(1+\varepsilon_1'(C))^{-1}$. Also set $\alpha(C)=\beta_1(\varepsilon_1'(C))=\mathrm{Max}(\pi-\alpha_1,\beta_1(x_0))$. See Figure 3.

<u>Case for C>0.</u> Set $\varepsilon_1(C)=\varepsilon_1(0)$.

Lemma 6 shows that $x_0(C)<1/10$, for all $C\in\mathbb{R}$.

Let M^n and $p\in M^n$ be as in the hypothesis. Multiply the metric with $1/i_p$ and normalize it so that with this new metric the hypothesis becomes:

i) $K_M \geqq \mathrm{Min}(C,0)$,

ii) $1=i_p \leqq d_p < 1+\varepsilon_1'(C)$.

For i), if $\mathrm{Min}(K_M)\leqq 0$, then $K_M \geqq \mathrm{Min}(K_M)=i_p^2\cdot\mathrm{Min}(K_M)\geqq d_M^2\cdot\mathrm{Min}(K_M)\geqq C$. If $\mathrm{Min}(K_M)>0$, then $K_M\geqq 0$. $i_p/d_p=1/d_p>1-\varepsilon_1(C)=(1+\varepsilon_1'(C))^{-1}$.

<u>REMARK 1.</u> The way we choose $\varepsilon_1(C)$, for $C>0$, enables us to deal with the change of the lower bound of the curvature for a positively

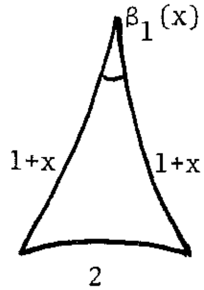


Figure 1. In M_C^2.

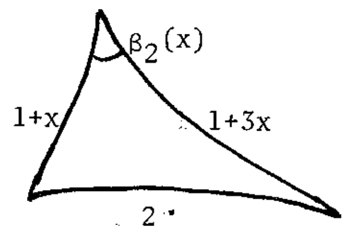


Figure 2. In M_C^2.

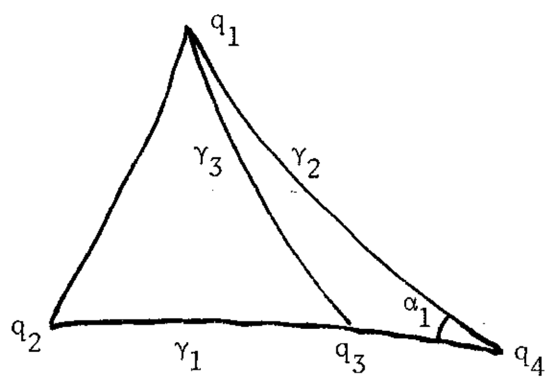


Figure 3. In M_C^2.

Figure 4. In $\tilde{M}$.

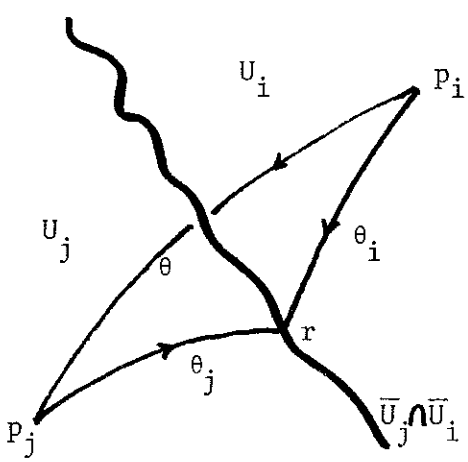
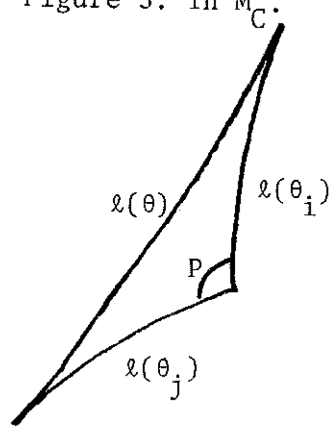


Figure 5. In M_C^2.



curved manifold when we normalize the metric. So, for any positively curved manifold, we will take C to be 0. In fact, it is obvious from the proof that the hypothesis i) can be replaced by $i_p^2 \cdot K_M \geq C$. If this is done, then $\varepsilon_1(C)$ can be made bigger for $C>0$.

Let $U=M-C_p$, and construct U_i in $\tilde{M}$ as in Section 1. Fix p_0 and U_0. Recall that there is a one-to-one correspondence between $p_i=\gamma_i(p_0)$ and $[\gamma_i]\varepsilon\pi_1(M)$. If $\overline{U}_0=\tilde{M}$, then $\pi_1(M)=1$ and there is nothing to prove. If $\tilde{M}\neq\overline{U}_0$, then pick any $q\ \varepsilon\tilde{M}-\overline{U}_0$ and let $f:[0,1]\to\tilde{M}$ be any continuous path from q to p_0. f exists, since $\tilde{M}$ is path-connected. As in Section 1, $\{[\gamma_j]\varepsilon\pi_1(M) \mid \overline{U}_j\cap f(I)\neq\phi\}$ is a finite set, where $I=[0,1]$; and hence, both $f^{-1}(\overline{U}_0)$ and $f^{-1}(\bigcup_{j\neq 0}\overline{U}_j)$ are closed. Therefore, there exists q' in $\overline{U}_0\cap(\bigcup_{j\neq 0}\overline{U}_j)$; because $I=f^{-1}(\overline{U}_0)\cup f^{-1}(\bigcup_{j\neq 0}\overline{U}_j)$ and each is non-empty. Choose any of U_j with $\overline{U}_j\cap\overline{U}_0\neq\phi$ and $U_j\neq U_0$; call it U_1. If $\overline{U}_1\cup\overline{U}_0=\tilde{M}$, then $\pi_1(M)=\mathbb{Z}_2$, and there is nothing more to prove.

Suppose $\overline{U}_1\cup\overline{U}_0\neq\tilde{M}$. Let $q\varepsilon\tilde{M}-(\overline{U}_0\cup\overline{U}_1)$ be any element. By a similar proof as above, it can be shown that there exists $q'\varepsilon(\overline{U}_1\cup\overline{U}_0)\cap(\bigcup_{j\neq 0,1}\overline{U}_j)$. If $q'\varepsilon\overline{U}_0$, then there exists U_2 such that $U_2\neq U_0$, $U_2\neq U_1$, and $\overline{U}_2\cap\overline{U}_0\neq\phi$. In the case of $q'\varepsilon\overline{U}_1-\overline{U}_0$, there is a U_{j_0} with $U_{j_0}\neq U_0$, $U_{j_0}\neq U_1$ and $\overline{U}_{j_0}\cap\overline{U}_1\neq\phi$. So there are at least two other U_i intersecting with U_1, namely U_0 and U_{j_0}. By taking an appropriate deck transformation (i.e. γ_1^{-1}), there are at least two other U_i intersecting with U_0. In either case, U_2 exists with $U_2 \neq U_1$, $U_2 \neq U_0$, $\overline{U}_2 \cap \overline{U}_0 \neq \phi$ and previously we had $U_1 \neq U_0$, $\overline{U}_1 \cap \overline{U}_0 \neq \phi$. If this is the case, define

$F: \partial U_0 \to \mathbb{R}$ by $F(q) = d_{\tilde{M}}(q, \overline{U}_1)$.

<u>LEMMA 4.</u> With the hypothesis of Theorem 1, and if F is as above, then, a) There does not exist $q \varepsilon \partial U_0$ such that $F(q) = 3x_0(C)$;

b) For any $q \varepsilon \overline{U}_0 \cap \overline{U}_2 \subseteq \partial U_0$, $F(q) \geqq 1/2 - x_0(C)$.

Now we can prove Theorem 1, by using Lemma 4:

F is defined by restricting the distance function to ∂U_0, so it is continuous. By Lemma 1, ∂U_0 is connected. Therefore, $F(\partial U_0)$ is a connected subset of $\mathbb{R}$. For any $q \varepsilon \overline{U}_0 \cap \overline{U}_1 \subseteq \partial U_0$, $F(q) = 0$, obviously. By Lemma 4, for any $q \varepsilon \overline{U}_0 \cap \overline{U}_2$, $F(q) \geqq 1/2 - x_0(C)$ and there does not exist any q such that $F(q) = 3x_0(C)$. This is a contradiction, since $0 < x_0(C) \leqq 1/10$ and $\overline{U}_2 \cap \overline{U}_0$ and $\overline{U}_1 \cap \overline{U}_0$ are non-empty. Therefore, such U_2 as above does not exist. Consequently, $\tilde{M} = \overline{U}_0 \cup \overline{U}_1$ or $\tilde{M} = \overline{U}_0$, that is $\pi_1(M) = \mathbb{Z}_2$ or 1. Theorem 1 QED.

To prove Lemma 4, we need the following facts and Lemma 5.

<u>CLAIM 1.</u> If $p_i \neq p_j$, then $d_{\tilde{M}}(p_i, p_j) \geqq 2$.

To see this, consider any minimal geodesic from p_i to p_j.

Its image under ρ is a geodesic loop at p in M. So its length is bigger than $2i_p=2$. See Klingenberg's Lemma. [CE], [GKM].

CLAIM 2. Let U_i, U_j be such that $U_i \cap U_j = \phi$ and $\bar{U}_i \cap \bar{U}_j \neq \phi$, and r be in $\bar{U}_i \cap \bar{U}_j$; θ_i, θ_j be minimal geodesics from p_i and p_j to r, respectively. Then $\sphericalangle(\theta_i'(r), \theta_j'(r)) > \beta_1(\varepsilon_1') > 2\pi/3$.

To prove that: Let θ be any minimal geodesic from p_i to p_j. Consider a geodesic triangle with sides of length $\ell(\theta_i)$, $\ell(\theta_j)$, and $\ell(\theta)$ in M_C^2 and P be the angle between the sides of length $\ell(\theta_i)$ and $\ell(\theta_j)$. See Figures 4 and 5. We have:

$$1 \leqq \ell(\theta_i) \leqq d_p < 1+\varepsilon_1'$$

$$1 \leqq \ell(\theta_j) \leqq d_p < 1+\varepsilon_1'$$

$$2 \leqq \ell(\theta).$$

Consider a geodesic triangle with sides of length $1+\varepsilon_1'$, $1+\varepsilon_1'$, and 2 in M_C^2, that is a triangle with $x=\varepsilon_1'$ in Figure 1. Obviously, the angle between the sides of length $1+\varepsilon_1'$ is $\beta_1(\varepsilon_1')$. To compare $\sphericalangle P$ and $\beta_1(\varepsilon_1')$, apply Lemma 2 to the triangles above with sides of length $\ell(\theta_i), \ell(\theta_j), \ell(\theta)$ and $1+\varepsilon_1'$, $1+\varepsilon_1'$, 2 in M_C^2, successively three times by changing one side length at a time. It follows that $\sphericalangle P > \beta_1(\varepsilon_1') > 2\pi/3$. Now apply Toponogov's Theorem to the geodesic triangle in $\tilde{M}$ with vertices p_i, p_j, r and sides given by the minimal geodesics θ_i, θ_j and θ, and obtain that $\sphericalangle(\theta_i'(r), \theta_j'(r)) \geqq \sphericalangle P$. Consequently, $\sphericalangle(\theta_i'(r), \theta_j'(r)) > \beta_1(\varepsilon_1') > 2\pi/3$.

CLAIM 3. If U_i, U_j, U_k are distinct, then $\overline{U}_i \cap \overline{U}_j \cap \overline{U}_k = \phi$.

Suppose the contrary. Let $r \in \overline{U}_i \cap \overline{U}_j \cap \overline{U}_k$ be any element, and $\theta_i, \theta_j, \theta_k$ be any minimal geodesics from r to p_i, p_j, p_k, respectively. By Claim 2, $\measuredangle(\theta_i'(r), \theta_j'(r)), \measuredangle(\theta_i'(r), \theta_k'(r)), \measuredangle(\theta_j'(r), \theta_k'(r))$; all are $> \beta_1(\varepsilon_1') \geqq 2\pi/3 = \arccos(-1/2)$. This contradicts Lemma 3.

REMARK 2. $\partial\overline{U}_0$ is not necessarily connected. If it is connected, then Claim 3 would be enough to prove Theorem 1, without Lemma 4.

Let $q \in \tilde{M} - \overline{U}_i$, for some i, and θ be any minimal geodesic from p_i to q. Define $t_r =: \mathrm{Max}\{ t \mid t \geqq 0, \theta(t) \in \overline{U}_i \}$. Also set $r = \theta(t_r)$.

CLAIM 4. $t_r = c_p(\rho_*(\theta'(p_i)))$, that is, $\exp_{p_i}(t \cdot \theta'(p_i)) = \theta(t)$ as a radial geodesic, it reaches ∂U_i at r; it does not stay in ∂U_i after r, and leaves $\overline{U}_i$, does not intersect $\overline{U}_i$ again before it reaches q. Equivalently, $\{r\} = (\mathrm{Im}(\theta)) \cap \partial\overline{U}_i = (\mathrm{Im}(\theta)) \cap \partial U_i$.

To prove this: Since $t_r = \mathrm{Max}\{t \mid 0 \leqq t, \theta(t) \in \overline{U}_i \}$, we have $r \in \partial\overline{U}_i \subseteq \partial U_i = \exp_{p_i}(\rho_*^{-1}(\partial A_p))$. Therefore, there exists v in $\rho_*^{-1}(\partial A_p)$ such that $r = \exp_{p_i} v$. $\exp_p(t \cdot \rho_*(v/\|v\|))$ is a minimal geodesic from p to its image points, for $t \leqq \|v\|$, so its lift $\exp_{p_i}(t \cdot v/\|v\|)$ to $\tilde{M}$ has the same property, from p_i. On the other hand, θ is a minimal geodesic from p_i to q, so it is the only minimal geodesic from p_i to any point ($\neq q$) on θ. $q \neq r$, therefore, $v/\|v\| = \theta'(p_i)$. But, $\exp_p(t \cdot \rho_*(v/\|v\|))$ lies in U, for

$t<\|v\|=c_p(\rho_*(v/\|v\|))$, so $\exp_{p_i}(t\cdot v/\|v\|)$ lies in U_i for $t<c_p(\rho_*(\theta'(p_i)))$ and $r=\exp_{p_i} v=\theta(\|v\|)$. Finally, $\|v\|=t_r=c_p(\rho_*(\theta'(p_i)))$. This proves Claim 4.

Let $q\in\tilde{M}-\overline{U}_i$ be any element, and θ be any minimal geodesic from p_i to q, also θ be defined after passing through q. Let r be the unique element in $\partial\overline{U}_i\cap\{\theta(t)\mid 0\leqq t\leqq d(p_i,q)\}$. By Claim 3, there is a unique j_0, with $U_{j_0}\neq U_i$, such that $r\in\overline{U}_i\cap\overline{U}_{j_0}$.

LEMMA 5. a) $\{\theta(t_r+t)\mid 0<t\leqq \mathrm{Min}(2x_0,d_{\tilde{M}}(q,r))\}\subseteq \mathrm{int}(\overline{U}_{j_0})$,

b) If $d_{\tilde{M}}(q,r)>2x_0$, then

$\{\theta(t_r+t)\mid 2x_0<t\leqq \mathrm{Min}(1/2,\ d_{\tilde{M}}(q,r))\}\subseteq U_{j_0}$.

PROOF (Lemma 5).

a) Suppose the contrary:

$\{\theta(t_r+t)\mid 0<t\leqq \mathrm{Min}(2x_0,d_{\tilde{M}}(q,r))\}\nsubseteq \mathrm{int}(\overline{U}_{j_0})$. Then, there exists t_0 with $0<t_0\leqq \mathrm{Min}(2x_0,d_{\tilde{M}}(q,r))$, such that $\theta(t_r+t_0)\in\partial\overline{U}_{j_0}$. So, there exists j_1 with $U_{j_1}\neq U_{j_0}$, such that $\theta(t_r+t_0)\in\overline{U}_{j_0}\cap\overline{U}_{j_1}$. Since, $\{r\}=\partial\overline{U}_i\cap\{\theta(t)\mid 0\leqq t\leqq d(p_i,q)\}$, we have $U_{j_1}\neq U_i$. Let θ_0,θ_1 be minimal geodesics from $\theta(t_r+t_0)$ to p_{j_0} and p_{j_1}, respectively. Let θ_2 be any minimal geodesic from p_i to p_{j_0}. See Figure 6. Consider the geodesic triangle in $\tilde{M}$ with vertices p_i,p_j, and $\theta(t_r+t_0)$, and the sides given by the minimal geodesics:

θ from p_i to $\theta(t_r+t_0)$,

θ_0 from $\theta(t_r+t_0)$ to p_{j_0},

θ_2 from p_i to p_{j_0}.

Now, consider a geodesic triangle in M_C^2 with sides of length t_r+t_0, $\ell(\theta_0)$ and $\ell(\theta_2)$; such a triangle exists, since triangle inequalities are satisfied. Figure 7. Let R be the angle between the sides of length t_r+t_0 and $\ell(\theta_0)$. Therefore, Toponogov's Theorem is applicable to these triangles and

$\sphericalangle(-\theta'(t_r+t_0),\theta_0'(0)) \geqq R$.

$1 \leqq \ell(\theta_0) < 1+x_0$, $\quad (\theta(t_r+t_0)\ \varepsilon\partial U_{j_0})$

$1 \leqq t_r+t_0 < 1+3x_0$, $\quad (r\varepsilon\partial U_i)$

$2 \leqq \ell(\theta_2)$. $\quad$ (Claim 1)

Similar to the proof of Claim 2, consider a geodesic triangle in M_C^2 with sides of length $1+x_0$, $1+3x_0$, and 2, that is a triangle in Figure 2, for $x=x_0$. The angle between the sides of length $1+x_0$ and $1+3x_0$ is $\beta_2(x_0)$. To compare R and $\beta_2(x_0)$, we apply Lemma 2 to these triangles with side lengths $\ell(\theta_0),\ell(\theta_2)$, t_r+t_0 and $1+x_0$, 2, $1+3x_0$ in M_C^2, successively three times by changing only one side length at a time. Recall that $\beta_2(x_0)>\pi/2$, to apply Lemma 2. It follows that $R>\beta_2(x_0)$. Consequently, $\sphericalangle(-\theta'(t_r+t_0),\theta_0'(0))>\beta_2(x_0)$. Similarly, $\sphericalangle(-\theta'(t_r+t_0),\theta_1'(0))>\beta_2(x_0)$. By Claim 2, $\sphericalangle(\theta_0'(0),\theta_1'(0))>\beta_1(\varepsilon_1') \geqq \beta_1(x_0)$, since $\theta(t_r+t_0)\varepsilon\overline{U}_{j_0}\cap\overline{U}_{j_1}$. Therefore, we have:

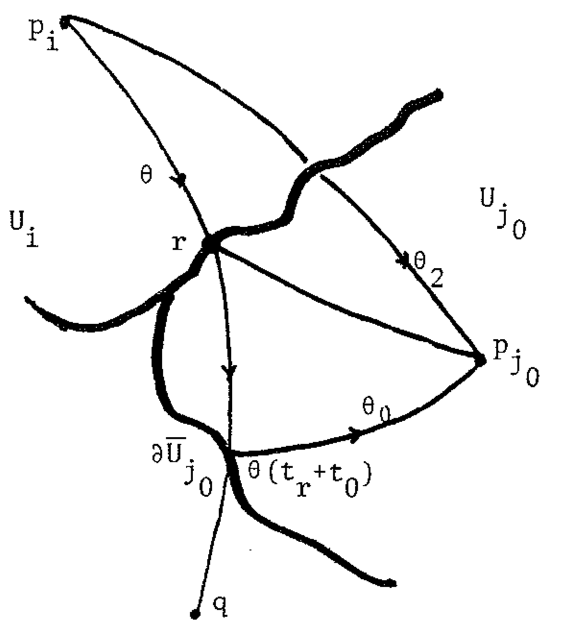


Figure 6. In $\tilde{M}$.

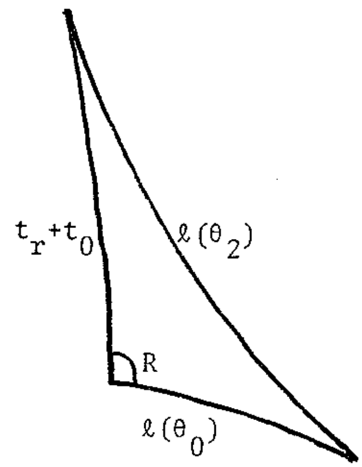


Figure 7. In M_C^2.

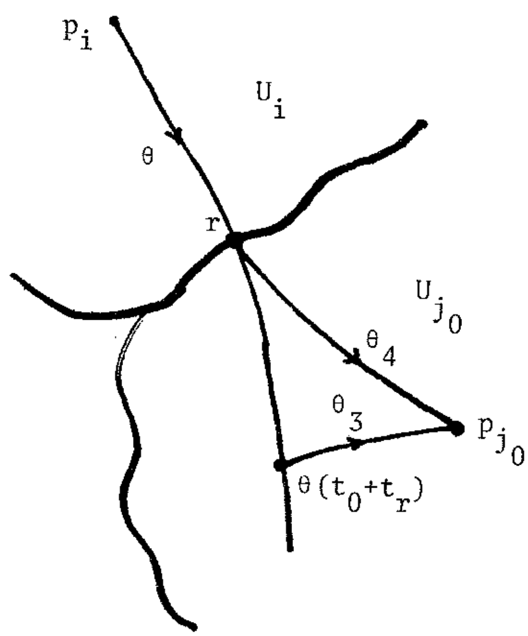


Figure 8. In $\tilde{M}$.

$$\sphericalangle(-\theta'(t_r+t_0),\theta_0'(0))+\sphericalangle(-\theta'(t_r+t_0),\theta_1'(0))+\sphericalangle(\theta_0'(0),\theta_1'(0))$$
$$> 2\beta_2(x_0)+\beta_1(x_0)=2\pi.$$

Obviously, $-\theta'(t_r+t_0),\theta_0'(0),\theta_1'(0)$ are distinct, since U_{j_0}, U_{j_1}, U_i are distinct. For any distinct three vectors in $\mathbb{R}^n$, they lie in some 3-dimensional subspace;and in $\mathbb{R}^3$, the sum of all of the angles between any two of three vectors is $\leqq 2\pi$. This is a contradiction. Therefore, for all t with $0<t\leqq \mathrm{Min}(2x_0,d(q,r))$, $\theta(t_r+t)\in \mathrm{int}(\overline{U}_{j_0})$ holds.

b) Given any $t_0\in\mathbb{R}$ such that $2x_0<t_0\leqq \mathrm{Min}(1/2,d_{\tilde{M}}(r,q))$. Let θ_3,θ_4 be any minimal geodesics from $\theta(t_0+t_r)$ to p_{j_0} and from r to p_{j_0}, respectively. $1\leqq \ell(\theta_4)<1+\varepsilon_1'$ and $\sphericalangle(\theta_4'(0),-\theta'(t_r))>\beta_1(\varepsilon_1')=\alpha$, by Claim 2. See Figure 8.

Let $q_1,q_2,q_3,q_4,\gamma_1,\gamma_2$ and γ_3 be as in the construction of $\varepsilon_1(C)$, in M_C^2. Figures 3 and 9. Since the triangle with vertices q_1,q_2, and q_3 is an equilateral triangle in M_C^2, $C\leqq 0$, by Toponogov's Theorem: $\sphericalangle(-\gamma_1'(q_3),\gamma_3'(q_3))\leqq\pi/3$. By Toponogov's Theorem and the Law of Cosines in the flat case:

$$d(q_1,q_4)^2 \geqq d(q_3,q_4)^2+d(q_1,q_3)^2-2d(q_3,q_4)\cdot d(q_1,q_3)\langle\gamma_3'(q_3),\gamma_1'(q_3)\rangle$$
$$\geqq(2x_0)^2+1-4x_0\cos(2\pi/3)$$
$$=1+2x_0+4x_0^2 >(1+x_0)^2.$$

Let q_5 be a point on γ_1 between q_3 and q_4 with $d(q_1,q_5)=\ell(\theta_4)$. q_5 exists by the continuity of the distance function and

$d(q_1,q_3)=1\leqq\ell(\theta_4)<1+\varepsilon_1'\leqq 1+x_0\leqq d(q_1,q_4)$. q_5 is unique, because, for $C\leqq 0$, every metric ball is strongly convex in M_C^2. Let γ_4 be the minimal geodesic from q_5 to q_1. If $q_5=\gamma_1(t_1)$, then set $q_6=\gamma_1(t_1-t_0)$. $2x_0<t_0\leqq\text{Min}(1/2,\ d_{\tilde{M}}(q,r))$ and $1\leqq t_1\leqq 1+2x_0$ imply that $1/2\leqq t_1-t_0<1$. Consider any geodesic triangle in M_C^2 with sides of length $t_0,\ell(\theta_3)$, and $\ell(\theta_4)$; and let Q be the angle between the sides of length t_0 and $\ell(\theta_4)$. Figure 10. Such a triangle exists, since those lengths satisfy triangle inequalities. Also consider the geodesic triangle in M_C^2 with vertices q_1,q_5,q_6 given as above. Figure 11.

For $C\leqq 0$, every metric ball is strongly convex in M_C^2, so $d(q_1,q_6)<1$. Now suppose that $\ell(\theta_3)\geqq 1$. Consider the geodesic triangles in M_C^2 mentioned above. Since two of their side lengths are the same, and the third ones are $\ell(\theta_3)$ and $d(q_1,q_6)$ and $\ell(\theta_3)\geqq 1>d(q_1,q_6)$, by Lemma 2 we conclude that $Q>\measuredangle(-\gamma_1'(q_5),\gamma_4'(q_5))$.

By Toponogov's Theorem applied to the geodesic triangle in $\tilde{M}$ with vertices r,p_{j_0}, and $\theta(t_0+t_r)$ with sides given by the minimal geodesics θ,θ_3 and θ_4, we will obtain that: $Q\leqq\measuredangle(\theta'(t_r),\theta_4'(0))$. By Claim 2: $\measuredangle(\theta_4'(0),-\theta'(t_r))>\alpha$. Combining all of the above, we obtain that $\measuredangle(\gamma_1'(q_5),\gamma_4'(q_5))>\alpha$. On the other hand, $\measuredangle(-\gamma_1'(q_4),\gamma_2'(q_4))=\alpha_1\geqq\pi-\alpha$, because $\alpha=\text{Max}(\pi-\alpha_1,\ \beta_1(x_0))$. Therefore, $\measuredangle(-\gamma_1'(q_4),\gamma_2'(q_4))+\measuredangle(\gamma_1'(q_5),\gamma_4'(q_5))>\pi$.

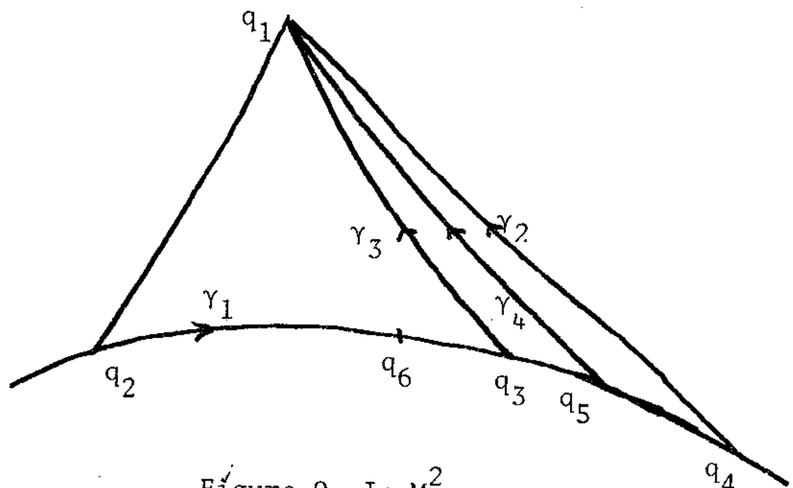


Figure 9. In M_C^2.

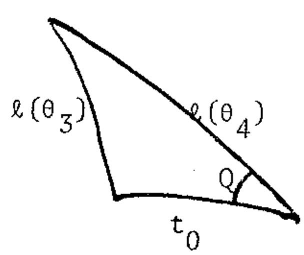


Figure 10. In M_C^2.

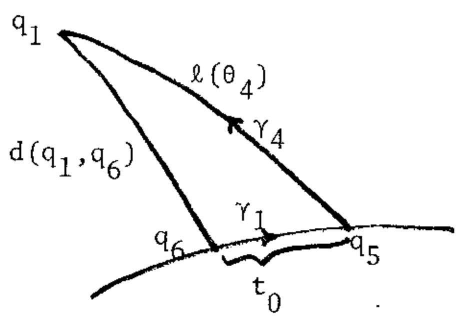


Figure 11. In M_C^2.

By Gauss-Bonnet Theorem, this can not happen for a geodesic triangle in M_C^2, for C non-positive. Hence, $\ell(\theta_3)<1$; and by the minimality of θ_3, we conclude that $d(p_{j_0},\theta(t_r+t_0))=\ell(\theta_3)<1=i_p$. t_0 was fixed, but arbitrary, so $\{\theta(t_r+t) \mid 2x_0<t\leqq \text{Min}(1/2,\ d_{\tilde{M}}(r,q))\}\subseteq U_{j_0}$. Lemma 5 QED.

PROOF (Lemma 4).

a) Suppose there exists $q\varepsilon\partial U_0$ such that $d_{\tilde{M}}(q,\bar{U}_1)=3x_0$. Let θ be any minimal geodesic from p_1 to q. Let r be the unique element in $\text{Im}(\theta)\cap\partial\bar{U}_1$. $d(q,r)\geqq 3x_0$. Suppose $d(q,r)>4x_0$, then $d(p_1,q)=d(q,r)+d(r,p_1)>1+4x_0$. $\forall r'\varepsilon\partial\bar{U}_1$, $d(r',p_1)<1+x_0$; so $d(q,r')>1+4x_0-(1+x_0)=3x_0$ and this is a contradiction. Hence, $3x_0\leqq d(q,r)\leqq 4x_0<1/2$. By Lemma 5b, $q\varepsilon U_{j_0}$, for some j_0. $q\varepsilon\partial U_0$, so every neighborhood of q intersects with U_0. $U_{j_0}\cap U_0\neq\phi$. Therefore, $U_{j_0}=U_0$. U_0 is open, so $\partial U_0=\bar{U}_0-U_0$, $\partial U_0\cap U_0=\phi$ and $q\varepsilon U_{j_0}=U_0$ and $q\varepsilon\partial U_0$. This is a contradiction.

b) Let $q\varepsilon\bar{U}_2\cap\bar{U}_0$ be any element, and θ be a minimal geodesic from p_1 to q; and r be the unique element in $\partial\bar{U}_1\cap\text{Im}(\theta)$. $r\neq q$, by Claim 3. Let $r\varepsilon\ \partial\bar{U}_{i_0}$, for some i_0. By Lemma 5, $\theta(t_r+t)\varepsilon\text{int}(\bar{U}_{i_0})$ for $0<t\leqq\text{Min}(1/2,\ d_{\tilde{M}}(r,q))$. Suppose that $q\varepsilon\text{int}(\bar{U}_{i_0})$, then:

$$
\begin{aligned}
q\varepsilon\overline{U}_2\cap\overline{U}_0 &\Rightarrow q\varepsilon\overline{U}_2\\
&\Rightarrow q\varepsilon\overline{U}_2\cap\mathrm{int}(\overline{U}_{i_0})\\
&\Rightarrow U_2\cap\mathrm{int}(\overline{U}_{i_0})\neq\phi \text{ (Since } \mathrm{int}(\overline{U}_{i_0}) \text{ is open)}\\
&\Rightarrow U_2\cap\overline{U}_{i_0}\neq\phi\\
&\Rightarrow U_2\cap U_{i_0}\neq\phi \text{ (Since } U_2 \text{ is open)}\\
&\Rightarrow U_2=U_{i_0}.
\end{aligned}
$$

Similarly, $U_0=U_{i_0}$. But, $U_2\neq U_0$, which gives a contradiction. Therefore, $q\notin\mathrm{int}(\overline{U}_{i_0})$. Finally, $d_{\tilde{M}}(r,q)>1/2$. θ is a minimal geodesic:

$$d_{\tilde{M}}(q,p_1)=d_{\tilde{M}}(q,r)+d_{\tilde{M}}(r,p_1)\geq 1/2+1=3/2.$$

Since $\overline{U}_1$ is compact, there exists a $q_0\varepsilon\partial\overline{U}_1$ such that $d_{\tilde{M}}(q,q_0)=d_{\tilde{M}}(q,\overline{U}_1)$. Hence,

$d_{\tilde{M}}(q,q_0)\geq d_{\tilde{M}}(q,p_1)-d_{\tilde{M}}(p_1,q_0)\geq 3/2-d_p\geq 3/2-(1+\varepsilon_1')\geq 1/2-x_0$.

Lemma 4 QED. This concludes the proof of Theorem 1.

<u>REMARK 3.</u>a) In Lemma 5: 1/2 can be replaced by any δ with $1/2\leq\delta<1$; b) We do not need the minimality of θ between r and q to apply Toponogov's Theorem in Lemma 5b. All we need is the triangle inequality to be satisfied; so $\mathrm{Min}(1/2, d_{\tilde{M}}(r,q))$ can be replaced by any δ as above.

We will need the following fact in the proof of Theorem 2:

PROPOSITION B. If the hypothesis of Theorem 1 is satisfied and $\overline{U}_i \cap \overline{U}_j \neq \phi$, then $\forall q \in \partial U_j$, $d_{\tilde{M}}(q, \overline{U}_i) \leqq 2x_0$.

PROOF. It follows from the proof of Lemma 4, Theorem 1 by the connectedness of ∂U_j and taking $2x_0+\delta$ instead of $3x_0$ in Lemma 4a, for any $\delta>0$. Also, Lemma 4a does not use the existence of U_2, it only depends on the existence of U_0 and U_1, so it is applicable to this case; since the contradiction of the proof of Theorem 1 comes from the existence of U_2 only.

LEMMA 6. If $C_1 \leqq C_2$, then $x_0(C_1) \leqq x_0(C_2)$. $x_0(0)<1/10$.

PROOF. Let $\beta_i(x,C)$ denote $\beta_i(x)$ in M_C^2, $i=1,2$, in the construction of $\varepsilon_1(C)$ of Theorem 1. By Toponogov's Theorem:
$\beta_i(x,C_2) \geqq \beta_i(x,C_1)$, for $i= 1,2$; and hence, $x_0(C_2) \geqq x_0(C_1)$.

Fix $C=0$. An elementary calculation shows that:
$\beta_1(1/10, 0)=\arccos(-79/121)$ and $\beta_2(1/10, 0)=\arccos(-5/13)$; hence, $\sin(\beta_1(1/10, 0)+2\beta_2(1/10, 0))<0$. Since $\beta_i<\pi$, we have $2\beta_2(1/10, 0)+\beta_1(1/10, 0)<3\pi$. Consequently, $2\beta_2(1/10, 0)+\beta_1(1/10, 0)<2\pi$. Therefore, $x_0(0)<1/10$.

REMARK 4. a) Example 2 of Section 5 shows that $\lim_{C\to-\infty} \varepsilon_1(C)=0$.

b) For $C=0$; if we solve the system of equations:

$\cos\beta_1 = 1 - 2/(1+x)^2$,

$\cos\beta_2 = (5x-1)/(3x+1)$,

$2\beta_2+\beta_1=2\pi$;

then we will obtain that $x_0(0)$ is a solution of $16x^4+16x^3-7x^2-10x+1=0$. It is obvious that the derivative of this polynomial is negative for $0<x<1/10$. So, there is only one root in $[0,1/10]$. An approximate solution is $x_0(0)\simeq 0.095$. Since $\pi-\alpha_1(0)<\beta_1(x_0(0))$, we have $\varepsilon_1'(0)= x_0(0)$; and $\varepsilon_1(0)\simeq 0.087$.

SECTION 3. THE NON-SIMPLY CONNECTED CASE.

Theorem 1 of Section 2 shows that a compact Riemannian manifold M^n with i_p close to d_p (for some $p \in M$), in terms of the lower bound of the sectional curvature, has fundamental group $\mathbb{Z}_2$ or trivial. In this section, more restrictions on the topology of these manifolds with the fundamental group $\mathbb{Z}_2$ will be obtained by imposing a slightly stronger hypothesis.

THEOREM 2. Given $C \in \mathbb{R}$, there exists a universal constant $\varepsilon_2(C)$, only depending on C, such that: For any compact Riemannian manifold M^n, $n \geqq 2$, if

i) $d_M^2 \cdot K_M \geqq C$,

ii) There exists a point p in M such that $i_M/d_p > 1-\varepsilon_2(C)$, and

iii) $\pi_1(M) = \mathbb{Z}_2$;

then, a) M^n is oriented if and only if n is odd,

b) $H^*(M^n, \mathbb{Z}) \cong H^*(\mathbb{R}P^n, \mathbb{Z})$, the isomorphism is naturally induced by a map from $\mathbb{R}P^n$ to M^n. Moreover, M^n has the homotopy type of $\mathbb{R}P^n$.

PROOF. We construct $\varepsilon_2(C)$ as follows: Given $C \in \mathbb{R}$.

Case for $C \leq 0$. Let $x \in [0, 1/4)$. Consider the following two geodesic triangles in M_C^2. The first one has sides of length

1, 1, 1-4x, and the second one has sides of length 1, 1, 2-4x. Figures 12 and 13. Let $\beta_3(x)$ and $\beta_4(x)$ be the angles between the sides of length 1 in the first and second triangles, respectively. $\beta_3(x)$ and $\beta_4(x)$ are strictly decreasing continuous functions of x. By a similar argument as in Theorem 1, using Lemma 2. $\beta_3(0)+\beta_4(0)>\beta_4(0)=\pi$, and $\beta_3(1/4)+\beta_4(1/4)=\beta_4(1/4)<\pi$. Therefore, there exists $x_1\varepsilon(0, 1/4)$ such that $\beta_3(x_1(C))+\beta_4(x_1(C))=\pi$. Let $x_2(C)=\text{Min}(x_1(C), x_0(C))$, where $x_0(C)$ is as in Theorem 1. Let q_1, q_2 and q_3 be points of M_C^2 with $d(q_1,q_2)=d(q_1,q_3)=d(q_2,q_3)=1$, γ_1 be the minimal geodesic from q_2 to q_3 and γ_1 be defined beyond q_3, as in the construction of $\varepsilon_1(C)$ of Theorem 1. Set $q_4'=\gamma_1(1+2x_2(C))$, and let γ_2 be the minimal geodesic from q_4' to q_1. If $\alpha_2=\measuredangle(-\gamma_1'(q_4'),\gamma_2'(q_4'))$, then define $\varepsilon_2'(C)=\text{Min}(x_2(C), \beta_1^{-1}(\pi-\alpha_2))$, and $\varepsilon_2(C)=1-(1+\varepsilon_2'(C))^{-1}$. Figure 14.

Case for C>0. Set $\varepsilon_2(C)=\varepsilon_2(0)$.

Let M^n and $p\varepsilon M^n$ be as in the hypothesis. We normalize the metric by multiplying by $1/i_M$. With the new metric, the hypothesis becomes:

i) If $\text{Min}(K_M)\leqq 0$, then $K_M\geqq\text{Min}(K_M)=i_M^2\cdot\text{Min}(K_M)\geqq d_M^2\cdot\text{Min}(K_M)\geqq C$.

If $K_M>0$, then $K_M\geqq 0$, obviously. In short, $K_M\geqq\text{Min}(C,0)$.

ii) $i_M/d_p>1-\varepsilon_2(C)$, or equivalently, $i_M=1$ and $1=i_M\leqq d_p<1+\varepsilon_2'(C)$,

iii) $\pi_1(M)=\mathbb{Z}_2$. Recall Remark 1 of Section 2.

Let $U=M-C_p$, and construct U_0 and U_1 in $\tilde{M}$ as in Section 1. We have $p_i \varepsilon U_i$, $\rho(p_i)=p$, for $i=1,2$, $U_0 \cap U_1=\phi$ and $\overline{U}_0 \cup \overline{U}_1=\tilde{M}$.

To prove Theorem 2, we need Lemmas 7 and 8.

<u>LEMMA 7.</u> For any $w \varepsilon UM_p$, $d_M(\exp_p w, \exp_p -w)<1=i_M$; if the hypothesis of Theorem 2 holds true.

<u>PROOF of Lemma 7.</u> Given any $v \varepsilon U\tilde{M}_{p_0}$, let $q(v)=\exp_{p_0} v$ and $r(v)=\exp_{p_0}(c_p(\rho_*(v))\cdot v)$. $d_{\tilde{M}}(q(v),r(v)) \leqq c_p(\rho_*(v))-1 \leqq d_p-i_p$, and obviously by the hypothesis: $d_p-i_p<\varepsilon_2'(C) \leqq x_2(C)$. Since $d_p<1+\varepsilon_2'(C) \leqq 1+x_2$, $\varepsilon_2'(C)=\mathrm{Min}(x_2(C), \beta_1^{-1}(\pi-\alpha_2))$, $x_2(C) \leqq x_0(C)$ and α_2 is constructed in a similar way to α_1; x_0 can be replaced by x_2 in the proof of Lemmas 4a and 5b and therefore in Proposition B, with the hypothesis of Theorem 2. So, $d_{\tilde{M}}(r(v),\overline{U}_1) \leqq 2x_2(C)$, since $r(v) \varepsilon \exp_{p_0}((\rho_*(p_0))^{-1}(\partial A_p))=\partial U_0$. $\overline{U}_1$ is compact, so there exists $s(v) \varepsilon \partial \overline{U}_1$ such that $d_{\tilde{M}}(s(v),r(v))=d_{\tilde{M}}(r(v),\overline{U}_1)$.

$s(v) \varepsilon \partial \overline{U}_1 \subseteq \partial U_1=\exp_{p_1}((\rho_*(p_1))^{-1}(\partial A_p))$; hence, there exists $v' \varepsilon U\tilde{M}_{p_1}$ such that $s(v)=\exp_{p_1}(c_p(\rho_*(v'))\cdot v')$. Obviously v' depends only on v.

$$d_{\tilde{M}}(\exp_{p_0} v, \exp_{p_1} v') \leqq d_{\tilde{M}}(\exp_{p_0} v, r(v))+d_{\tilde{M}}(r(v),s(v))+d_{\tilde{M}}(s(v),\exp_{p_1} v')$$
$$<x_2+2x_2+x_2=4x_2(C).$$

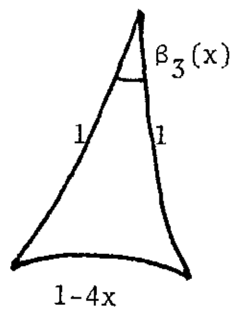


Figure 12. In M_C^2.

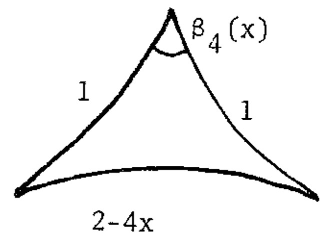


Figure 13. In M_C^2.

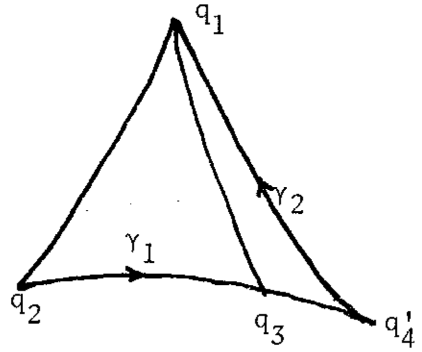


Figure 14. In M_C^2, where $q_2=\gamma_1(0)$, $q_3=\gamma_1(1)$, and $q_4'=\gamma_1(1+2x_2(C))$.

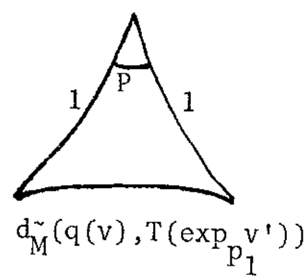


Figure 16. In M_C^2.

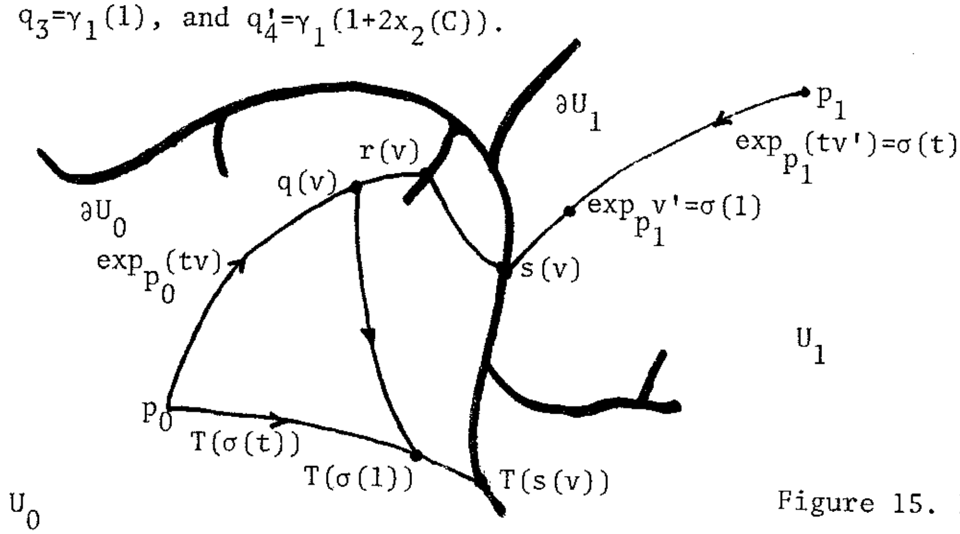


Figure 15. In $\tilde{M}$.

Let T be the non-trivial deck transformation on $\tilde{M}$, i.e. $\rho(T(m))=\rho(m)$, $T(m)\neq m$, $T^2(m)=m$, for all $m\in\tilde{M}$. Recall that T is an isometry, since ρ is a local isometry as a covering map. $d_{\tilde{M}}(m,T(m))\geqq 2i_M=2$ for all $m\in\tilde{M}$. Proof of this fact is the same as Claim 1 of Theorem 1. (Klingenberg's Lemma) Therefore:

$$d_{\tilde{M}}(q(v),T(\exp_{p_1}v'))\geqq d_{\tilde{M}}(\exp_{p_1}v',T(\exp_{p_1}v'))-d_{\tilde{M}}(q(v),\exp_{p_1}v')$$
$$> 2-4x_2(C)\geqq 2-4x_1(C).$$

Let $\sigma(t)=\exp_{p_1}(tv')$. Consider the geodesic triangles in $\tilde{M}$ with vertices p_0, $q(v)$, and $T(\exp_{p_1}v')$ and sides are given by the minimal geodesics $\exp_{p_0}(tv)$, $0\leqq t\leqq 1$; $T(\exp_{p_1}(tv'))=T(\sigma(t))$, $0\leqq t\leqq 1$; and any minimal geodesic from $q(v)$ to $T(\sigma(1))=T(\exp_{p_1}(v'))$. Figure 15. We have:

$$d_{\tilde{M}}(q(v),p_0)=1,$$
$$d_{\tilde{M}}(p_0,T(\sigma(1)))=d_{\tilde{M}}(T(p_0),\sigma(1))=d_{\tilde{M}}(p_1,\sigma(1))=1,$$
$$d_{\tilde{M}}(q(v),T(\sigma(1)))>2-4x_1(C).$$

Take any geodesic triangle in M_C^2 with side lengths 1, 1, and $d_{\tilde{M}}(q(v),T(\sigma(1)))$. Figure 16. Let P be the angle between the sides of length 1. By Toponogov's Theorem, $\measuredangle(v,T_*(v'))\geqq P$, since $T(\sigma(t))=T(\exp_{p_1}(tv'))=\exp_{p_0}(t\cdot T_*(v'))$. On the other hand, by Lemma 2, $P>\beta_4(x_1(C))$; since $d_{\tilde{M}}(q(v),T(\sigma(1)))>2-4x_1(C)$ and Figures 13 and 16.

Therefore, $\sphericalangle(v,T_*(v'))>\beta_4(x_1(C))$.

$\sphericalangle(-v,T_*(v'))<\pi-\beta_4(x_1)=\beta_3(x_1)$. Consider the geodesic hinge in $\tilde{M}$ with vertices $\exp_{p_0}-v$, p_0 and $T(\exp_{p_1}v')$, with minimal geodesics $\exp_{p_0}-tv$ from p_0 to $\exp_{p_0}-v$ and $T(\exp_{p_1}tv')$ from p_0 to $T(\exp_{p_1}v')$. Also consider the geodesic triangle with side lengths 1, 1, and $1-4x_1(C)$ in M_C^2, Figure 12. Apply Toponogov's Theorem and Lemma 2, in a similar fashion as above to obtain $d_{\tilde{M}}(\exp_{p_0}-v, T(\exp_{p_1}v'))<1-4x_1(C)$, by taking a hinge in M_C^2 of two minimal geodesics of length 1,beginning from the same point with an angle between them $\sphericalangle(-v,T_*(v'))$.

Let $w\varepsilon UM_p$ be any element. There exists a unique $v\varepsilon U\tilde{M}_{p_0}$ such that $\rho_*(v)=w$. Choose v' as above depending on v and hence, on w. Since, $\forall m_1,m_2\varepsilon\tilde{M}$, $d_{\tilde{M}}(m_1,m_2)\geqq d_M(\rho(m_1),\rho(m_2))$, $\rho\circ T=\rho$, ρ is a local isometry and ρ commutes with exp by the diagram in Section 1; we have:

$$d_M(\exp_p w, \exp_p -w)\leqq d_M(\exp_p -w,\rho(T(\exp_{p_1}v')))+d_M(\rho(\exp_{p_1}v'),\exp_p w)$$
$$\leqq d_{\tilde{M}}(\exp_{p_0}((\rho_*(p_0))^{-1}(-w)),T(\exp_{p_1}v'))+d_{\tilde{M}}(\exp_{p_1}v',\exp_{p_0}((\rho_*(p_0))^{-1}w))$$
$$=d_{\tilde{M}}(\exp_{p_0}-v,T(\exp_{p_1}v'))+d_{\tilde{M}}(\exp_{p_1}v', \exp_{p_0}v)$$
$$<1-4x_1(C)+4x_2(C)\leqq 1=i_M.$$

Therefore, $d_M(\exp_p w, \exp_p -w)<1$ and this does not depend on the choice of v'. w was arbitrary, so it is true for all $w \in UM_p$. Lemma 7 QED.

From the proof above, we also conclude that:

LEMMA 7B. $\forall v \in U\tilde{M}_{p_0}$, $d_{\tilde{M}}(T(\exp_{p_0} -v), \exp_{p_0} v) < 1 = i_M \leq i_{\tilde{M}}$.

PROOF.

$$d_{\tilde{M}}(T(\exp_{p_0} -v), \exp_{p_0} v) \leq d_{\tilde{M}}(T(\exp_{p_0} -v), \exp_{p_1} v') + d_{\tilde{M}}(\exp_{p_1} v', \exp_{p_0} v)$$
$$= d_{\tilde{M}}(\exp_{p_0} -v, T(\exp_{p_1} v')) + d_{\tilde{M}}(\exp_{p_1} v', \exp_{p_0} v) < 1$$

from the proof of Lemma 7.QED.

LEMMA 8. There exists a continuous function $f: \mathbb{R}P^n \to M$ such that $f: f^{-1}(B_r(p,M)) \to B_r(p,M)$ is a diffeomorphism, for some $r>0$, and $f(B_{r'}(a, \mathbb{R}P^n)) = B_{r'}(p,M)$ for all $r' \leq r$ and for some $a \in \mathbb{R}P^n$; if the hypothesis of Theorem 2 holds true.

PROOF. Given any $w \in UM_p$, there exists a unique minimal geodesic θ_w from $\exp_p w$ to $\exp_p -w$, since $d_M(\exp_p w, \exp_p -w) < i_M$. $\ell(\theta_w) = d_M(\exp_p w, \exp_p -w) = \ell(\theta_{-w})$. Fix a point $a \in \mathbb{R}P^n(1)$ and ψ be an isometry of $T\mathbb{R}P^n_a$ onto TM^n_p.

$$\begin{array}{ccc} \overline{B}_{\pi/2}(0,T\mathbb{R}P^n_a) & \xrightarrow{\psi} & \overline{B}_{\pi/2}(0,TM_p) \\ \downarrow \exp_a & & \downarrow h \\ \mathbb{R}P^n & \xrightarrow{f} & M^n \end{array}$$ where

$$h(y)=\begin{cases} \exp_p y & \text{if } \|y\|\leq 1; \\ \theta_{y/\|y\|}\left(\frac{\|y\|-1}{\pi-2}\cdot \ell(\theta_{y/\|y\|})\right), & \text{if } 1<\|y\|\leq \pi/2. \end{cases}$$

By symmetry, $\theta_w(t)=\theta_{-w}(\ell(\theta_w)-t)$ and hence, $\theta_w(\ell(\theta_w)/2)=\theta_{-w}(\ell(\theta_w)/2)$. If $w_1,w_2\varepsilon T\mathbb{R}P^n_a$ with $\pi/2=\|w_1\|=\|w_2\|$; then ($\exp_a w_1=\exp_a w_2$ if and only if $w_1=\pm w_2$). Let $w\varepsilon T\mathbb{R}P^n_a$ such that $\|w\|=\pi/2$.

$$h(\psi(w))=\theta_{\psi(w)/\|\psi(w)\|}\left(\frac{\pi/2\ -1}{\pi\ -2}\cdot \ell(\theta_{\psi(w)/\|\psi(w)\|})\right)$$

$$=\theta_{-\psi(w)/\|\psi(w)\|}(\ell(\theta_{\psi(w)/\|\psi(w)\|})/2)=h(\psi(-w)).$$

Since $\exp_a$ is one-to-one on the interior of $\overline{B}_{\pi/2}(0,T\mathbb{R}P^n_a)$; by above, there exists a unique well-defined function $f:\mathbb{R}P^n\to M$ which makes the diagram above commutative.

CLAIM: f is continuous.

Continuity of f on $\exp_a(B_1(0,T\mathbb{R}P^n_a))$ is obvious, because it is defined by diffeomorphisms on the interior.

Let $t_n\varepsilon[0,1]$, $n\varepsilon\mathbb{N}$, such that $t_n\to t_0$ as $n\to\infty$; and $w_n\varepsilon UM_p$, $n\varepsilon\mathbb{N}$, such that $w_n\to w_0$ as $n\to\infty$. Consider the sequence $q_n=\theta_{w_n}(t_n\cdot\ell(\theta_{w_n}))$. By compactness of M, this sequence has a convergent subsequence. Take any convergent subsequence q_{n_k}, let $q_{n_k} \to q$. By the continuity of the exponential map and the distance function and $d_M(q_{n_k}, \exp_p w_{n_k}) = t_{n_k} \cdot \ell(\theta_{w_{n_k}})$, we obtain that $\exp_p w_{n_k} \to \exp_p w_0$, $d_M(q_{n_k}, \exp_p w_{n_k}) \to d_M(q, \exp_p w_0)$ and $t_n \cdot \ell(\theta_{w_n}) \to t_0 \cdot \ell(\theta_{w_0})$. Therefore, $d_M(q, \exp_p w_0) = t_0 \cdot \ell(\theta_{w_0})$. By the same argument: $d_M(q, \exp_p -w_0) = (1-t_0) \cdot \ell(\theta_{w_0})$. There is exactly one point $q \in M$ with these properties, otherwise we would obtain two curves from $\exp_p w_0$ to $\exp_p -w_0$ of length $\ell(\theta_{w_0}) = d_M(\exp_p w_0, \exp_p -w_0)$; and this is not the case. So,

$q = \theta_{w_0}(t_0 \cdot \ell(\theta_{w_0}))$. q_{n_k} was any convergent subsequence of q_n; by compactness of M, we have $\theta_{w_n}(t_n \cdot \ell(\theta_{w_n})) \to \theta_{w_0}(t_0 \cdot \ell(\theta_{w_0}))$. Any convergence sequence in $\mathbb{R}P^n - \exp_a(B_1(0, T\mathbb{R}P^n_a))$ can be written of the form $\exp_a w_n(1+t_n(\pi-2))$ where w_n, t_n as above, for n large. $\theta_w(0) = \exp_p w$ and $\theta_w(t \cdot \ell(\theta_w)) = \theta_{-w}((1-t) \cdot \ell(\theta_w))$, for $0 \leqq t \leqq 1$. Now the continuity of f follows easily.

Although f is continuous, it may not be smooth. But, it is smooth everywhere except on $\exp_a(\partial B_1(0, T\mathbb{R}P^n_a))$. See above.

$\ell(\theta_w) < 1$ and $d_M(p, \exp_p w) = d_M(p, \exp_p -w) = 1$, so θ_w never passes through p. Let $r \in \mathbb{R}$ be such that

$$1 \geqq \operatorname*{Min}_{w \in UM_p} \left(\operatorname*{Min}_{0 \leqq t \leqq \ell(\theta_w)} d_M(p, \theta_w(t)) \right) = 2r > 0.$$

Therefore, $f^{-1}(B_r(p,M))=B_r(a,\mathbb{R}P^n)$ and on this set f is defined by non-singular one-to-one exponential maps, so it is a diffeomorphism onto $B_r(p,M)$. The second part of the conclusion is obvious from the construction of f. Lemma 8 QED.

We complete the proof of Theorem 2 as follows:

By Lemma 7B; $\forall v\varepsilon U\tilde{M}_{p_0}$, $d_{\tilde{M}}(\exp_{p_0}v,\ T(\exp_{p_0}-v))<1=i_M\leq i_{\tilde{M}}$.

Let $\tilde{\theta}_v$ be the unique minimal geodesic from $\exp_{p_0}v$ to $T(\exp_{p_0}-v)$. $\rho(\tilde{\theta}_v)$ is a geodesic from $\rho(\exp_{p_0}v)=\exp_p(\rho_*(v))$ to $\rho(T(\exp_p-v))=\exp_p(-\rho_*(v))$, whose length is $<1=i_M$. Therefore $\rho(\tilde{\theta}_v)=\theta_{\rho_*(v)}$. Define

$$\gamma_v(t)=\begin{cases}\exp_{p_0}tv & \text{if } 0\leq t\leq 1,\\ \tilde{\theta}_v\left((t-1)\cdot\frac{\ell(\tilde{\theta}_v)}{\pi-2}\right) & \text{if } 1<t\leq\pi-1,\\ T(\exp_{p_0}(-v(\pi-t))) & \text{if } \pi-1<t\leq\pi.\end{cases}$$

Clearly, $\gamma_v(t)$ is a continuous curve from p_0 to p_1. Hence, it represents the non-trivial element of $\pi_1(M)$. On the other hand,

$$f(\exp_a(t(\psi^{-1}(\rho_*(v)))))=\begin{cases}\exp_p(t\rho_*(v)) & \text{if } t\varepsilon[0,1],\\ \theta_{\rho_*(v)}\left((t-1)\cdot\frac{\ell(\theta_{\rho_*(v)})}{\pi-2}\right) & \text{if } t\varepsilon(1,\pi-1],\\ \exp_p(-\rho_*(v))(\pi-t) & \text{if } t\varepsilon(\pi-1,\pi].\end{cases}$$

Clearly, $f(\exp_a(t(\psi^{-1}(\rho_*(v)))))=\rho(\gamma_v(t))$.

Recall that $\exp_a tv = \exp_a(t-\pi)v$ for all $v \in U\mathbb{R}P^n_a$, for the above equality. Hence, $f_*: \pi_1(\mathbb{R}P^n) \to \pi_1(M)$ is an isomorphism. By Lemma 8, $f_*: H_n(\overline{B}_{r/2}(a, \mathbb{R}P^n), \overline{B}_{r/2}(a, \mathbb{R}P^n) - a) \to$

$$H_n(\overline{B}_{r/2}(p,M), \overline{B}_{r/2}(p,M)-p) \text{ is an}$$

isomorphism;and consequently,

$f_*: H_n(\mathbb{R}P^n, \mathbb{R}P^n - a) \to H_n(M, M-p)$ is an isomorphism by excision. In other owrds, f has local degree ± 1, with $\mathbb{Z}$-coefficients. M^n or $\mathbb{R}P^n$ may not be $\mathbb{Z}$-orientable, but if we use $\mathbb{Z}_2$ coefficients:

$$\begin{array}{ccc} \mathbb{Z}_2 \cong H_n(\mathbb{R}P^n, \mathbb{R}P^n - a; \mathbb{Z}_2) & \xrightarrow[f_*]{\cong} & H_n(M, M-p; \mathbb{Z}_2) \cong \mathbb{Z}_2 \\ \cong \updownarrow & & \updownarrow \cong \\ \mathbb{Z}_2 \cong H_n(\mathbb{R}P^n, \mathbb{Z}_2) & \xrightarrow[f_*]{} & H_n(M, \mathbb{Z}_2) \cong \mathbb{Z}_2 \end{array}$$

By commutativity, f induces an isomorphism of $H_n(\mathbb{R}P^n, \mathbb{Z}_2)$ onto $H_n(M, \mathbb{Z}_2)$, and hence, it induces isomorphisms on H^0, H_0, H^n levels with $\mathbb{Z}_2$ coefficients, by the duality of H^0 and H_0, and Poincaré duality.

Hence, $f^*: H^n(M, \mathbb{Z}_2) \to H^n(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$ is an isomorphism. We follow [SA] and [B-1,pg.135-141] in the following, to obtain the cohomology ring of M. Although the results of Samelson are obtained under different hypotheses, he uses only the existence of a continuous function from $\mathbb{R}P^n$ to M of local degree ± 1, and the rest of his arguments do not use any other assumption. Those proofs are purely algebraic topological, so they are applicable to our case.

CLAIM: $f^*: H^*(M, \mathbb{Z}_2) \to H^*(\mathbb{R}P^n, \mathbb{Z}_2)$ is an isomorphism.

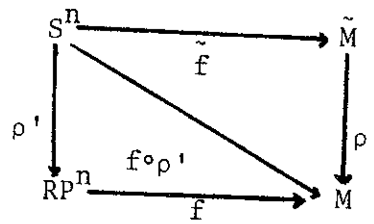
To prove this, given any $e \in H^*(M, \mathbb{Z}_2), e \neq 0$, there exists $e' \in H^*(M, \mathbb{Z}_2)$ such that $e \cup e' = [M]$ where $[M]$ is the generator of $H^n(M, \mathbb{Z}_2)$, by Poincaré duality and the duality of H_k and H^k with field coefficients. $f^*(e \cup e') = f^*(e) \cup f^*(e') = f^*([M]) = [\mathbb{R}P^n] \neq 0$, thus f^* is injective. Hence, $H^*(M, \mathbb{Z}_2)$ is isomorphic to a subring of $H^*(\mathbb{R}P^n, \mathbb{Z}_2)$ which is a truncated polynomial algebra with one generator in $H^1(\mathbb{R}P^n, \mathbb{Z}_2)$. $\pi_1(M) = \mathbb{Z}_2 = H_1(M, \mathbb{Z}_2) = H^1(M, \mathbb{Z}_2)$. This completes the argument for the claim.

By [SA], Proposition C: M^n is oriented if and only if n is odd.

Whenever n is odd, both M^n and $\mathbb{R}P^n$ are $\mathbb{Z}$-orientable, and f_* has local and global degree ± 1 with $\mathbb{Z}$-coefficients. The proof above does not work, since we do not have field coefficients; but $f^*: H^*(M, \mathbb{Z}) \to H^*(\mathbb{R}P^n, \mathbb{Z})$ is still injective, see [Bw], pg. 8, Theorem I.2.5.

By [SA], Theorems D,E: $f^*: H^*(M, \mathbb{Z}) \to H^*(\mathbb{R}P^n, \mathbb{Z})$ is an isomorphism, for n odd or even.

By a similar argument as in Samelson's proof, a stronger conclusion can be obtained as follows: There exists a unique function $\tilde{f}: S^n \to \tilde{M}$ which makes the following diagram commutative:



Let $a' \varepsilon S^n$ such that $\rho'(a')=a$, and $w \varepsilon US^n_{a'}$. Define $\zeta(w)$ to be $((\rho_*(p_0))^{-1} \circ \psi \circ \rho'_*)(w)$. In fact, a straightforward calculation shows that $(f\circ\rho')(\exp_{a'} tw)=f(\exp_a t\rho'_*(w))=\rho(\gamma_{\zeta(w)}(t))$, for $0 \leq t \leq \pi$. Therefore, $\tilde{f}(\exp_{a'} tw)=\gamma_{\zeta(w)}(t)$ which is also equal to:

$$\begin{cases} \exp_{p_0} t\zeta(w) & \text{if } 0 \leq t \leq 1, \\ \tilde{\theta}_{\zeta(w)}\left((t-1)\cdot\frac{\ell(\tilde{\theta}_{\zeta(w)})}{\pi-2}\right) & \text{if } 1 < t \leq \pi-1, \\ T(\exp_{p_0}(\pi-t)\cdot(-\zeta(w))) & \text{if } \pi-1 < t \leq \pi; \text{for all } t \varepsilon [0,\pi]. \end{cases}$$

Hence, $\tilde{f}^{-1}(B_r(p_0,\tilde{M}))=B_r(a',S^n)$ and $\tilde{f}:B_r(a',S^n)\rightarrow B_r(p_0,\tilde{M})$ is a diffeomorphism. $\tilde{f}$ has local degree ± 1 on this open set, that gives $\tilde{f}_*:H_n(S^n,S^n-a') \xrightarrow{\cong} H_n(\tilde{M},\tilde{M}-p_0)$. $\tilde{M}$ and S^n are both oriented, so $\tilde{f}_*:H_n(S^n, Z) \xrightarrow{\cong} H_n(\tilde{M}, Z)$. By [Bw],pg.8,Theorem I.2.5; $\tilde{f}^*:H^*(\tilde{M})\rightarrow H^*(S^n)$ is injective and hence, an isomorphism. By Whitehead's Theorem, $\tilde{f}$ induces isomorphisms for all homotopy groups, so $\tilde{M}$ is a homotopy sphere. By [LM],pg.43, M has the homotopy type of $\mathbb{R}P^n$, since the $\mathbb{Z}_2$ action on $\tilde{M}$ which yields M as a quotient is a smooth action. Theorem 2 QED.

An elementary calculation for $C=0$ shows that: $x_1(0)=(3-7^{1/2})/8$; so $x_2(0)=x_1(0)$ and $\alpha_2(0)>\alpha_1(0)$. Hence, $\beta_1^{-1}(\pi-\alpha_2(0))>\beta_1^{-1}(\pi-\alpha_1(0))$. Finally, $\varepsilon_2'(0)=x_2(0)$ and $\varepsilon_2(0)=(13-4\cdot 7^{1/2})/57\simeq 0.04$.

SECTION 4. A SPECIAL CASE: CONJUGATE LOCUS BOUNDED AWAY FROM THE CUT LOCUS.

In this section, the case in which the first conjugate locus of a particular point is bounded away from the cut locus of the same point will be investigated. The main results are Theorems 3, 4, and 5B. Before proving these theorems, some preliminary results are needed.

LEMMA 9 ([GKM],pg 198). Let M be a complete Riemannian manifold, $p\in M$ and $\exp_p: B_R(0,TM_p)\to M$ be of maximal rank. Given $v,w\in B_R(0,TM_p)$ such that $v\neq w$, and $\exp_p v=\exp_p w=:r\in M$. For $t_0\in[0,1]$ fixed, let $q=\exp_p t_0 v$, $c_0:[0,1]\to M$ be the geodesic given by $c_0(t)=\exp_p tt_0 v$, from p to q, and $c_1:[0,1]\to M$ be the broken geodesic given by

$$c_1(t)=\begin{cases}\exp_p(2tw) & \text{if } 0\leqq t\leqq 1/2,\\ \exp_p((1-(2t-1)(1-t_0)v) & \text{if } 1/2< t\leqq 1.\end{cases}$$

For any homotopy $H:[0,1]\times[0,1]\to M$ between c_0 and c_1, fixing the end points, i.e. $H(0,t)=c_0(t)$ and $H(1,t)=c_1(t)$, for all $t\in[0,1]$, and $H(s,0)=p$, $H(s,1)=q$, for all $s\in[0,1]$, then there exists $s_0\in[0,1]$ so that $\ell(c_0)+\ell(H(s_0,t))\geqq 2R$.

PROOF. See [GKM],pg.198-199.

<u>LEMMA 10.</u> For all $C\varepsilon\mathbb{R}$, for all $\alpha\varepsilon(0,\pi)$, there exists $\varepsilon=\varepsilon(\alpha,C)>0$ such that: for any compact Riemannian manifold M^n with $K_M\cdot i_M^2\geq C$, and if there exists a point $p\varepsilon M$ with

i) $i_M/d_p> 1-\varepsilon(\alpha,C)$ and

ii) $\exp_p:\overline{B}_{d_p}(0,TM_p)\to M$ is of maximal rank;

then, for any $q\varepsilon C_p$ and for any two distinct minimal geodesics γ_1,γ_2 from p to q, we have $\measuredangle(\gamma_1'(q),\gamma_2'(q))>\alpha$.

<u>PROOF.</u> $\varepsilon(\alpha,C)$ will be constructed as follows: Given $C\varepsilon\mathbb{R}$ and $\alpha\varepsilon(0,\pi)$. Case for $C\leq 0$: Let $x\varepsilon[0,\infty)$. Consider a geodesic triangle in M_C^2 with sides of length 1, x+ 1/2, and x+ 1/2. Figure 17. Let $\beta_5(x)$ be the angle between the sides of length x+ 1/2. By Lemma 2, $\beta_5(x)$ is a strictly decreasing function of x, for all $x\varepsilon[0,\infty)$. In fact, $\lim_{x\to\infty}\beta_5(x)=0$. Obviously $\beta_5(0)=\pi$. Define $\varepsilon'(\alpha,C)=\beta_5^{-1}(\alpha)$ in M_C^2, and $\varepsilon(\alpha,C)=1-(1+\varepsilon'(\alpha,C))^{-1}$. For $C>0$: define $\varepsilon(\alpha,C):=\varepsilon(\alpha,0)$.

Similar to the proofs of Theorems 1 and 2, multiply the metric with $1/i_M$, and with the new metric, the hypothesis becomes $K_M\geq \mathrm{Min}(C,0)$, $1=i_M\leq d_p\leq 1+\varepsilon'(\alpha,C)$; the other conditions remain unchanged. Now let γ_1 and γ_2 be as in the hypothesis, and $\ell_0=d_M(p,q)$. Define $f:[0,\ell_0]\to\mathbb{R}$ by $f(s)=d_M(\gamma_1(s),\gamma_2(s))$. f is continuous;and $f(s)>0$, for $s\varepsilon(0,\ell_0)$.

Suppose that $f(s)<i_M=1$, for all $s\varepsilon[0,\ell_0]$. For any fixed

$s\in[0,\ell_0]$, let $\theta_s(t)$ be the unique minimal geodesic from $\gamma_1(s)$ to $\gamma_2(s)$. See Figure 18. $\theta_s(t)$ depends on s continuously, i.e. $\lim_{s\to s_0}\theta_s(t)=\theta_{s_0}(t)$. Proof of this fact is similar to the continuity of the function f of Lemma 8. In short, we can say that the minimal geodesics $\theta_s(t)$ from $\gamma_1(s)$ to $\gamma_2(s)$ have a convergent subsequence and any convergent subsequence converges to a minimal geodesic from $\gamma_1(s_0)$ to $\gamma_2(s_0)$, but there is only one such minimal geodesic, namely $\theta_{s_0}(t)$. So $\lim_{s\to s_0}\theta_s(t)=\theta_{s_0}(t)$. By definition, $d_p\geq\ell_0$.

$f(s)=d(\gamma_1(s),\gamma_2(s))\leq d(\gamma_1(s),p)+d(\gamma_2(s),p)=2s$. If $s>0$, then $f(s)<3s$. Similarly, $f(s)\leq d(\gamma_1(s),q)+d(\gamma_2(s),q)=2(\ell_0-s)$. Let $v=\gamma_1'(0)\cdot\ell_0$, $w=\gamma_2'(0)\cdot\ell_0$, $t_0=1/2$ and $c_0(t)=\exp_p(tv/2)=\gamma_1(\ell_0 t/2)$,

$$c_1(t)=\begin{cases}\exp_p 2tw & \text{if } 0\leq t\leq 1/2,\\ \exp_p(3/2-t)v & \text{if } 1/2\leq t\leq 1.\end{cases}$$

Obviously, $\exp_p w=\exp_p v=q$.

Set $I=[0,1]$ and define a homotopy $G:I\times I\to M$ as follows:

$$G(s,t)=\begin{cases}\gamma_2(3ts\ell_0) & \text{if } 0\leq t\leq 1/3,\\ \theta_{s\ell_0}((2-3t)\cdot f(\ell_0 s)) & \text{if } 1/3\leq t\leq 2/3,\\ \gamma_1(\ell_0((3/2-3s)t+3s-1)) & \text{if } 2/3\leq t\leq 1.\end{cases}$$

Continuity of G follows from the continuity of γ_1 and γ_2, and the continuous dependence of $\theta_s(t)$ on s. See Figures 19 and 20. Clearly, $\gamma_2(s\ell_0)=\theta_{s\ell_0}(f(\ell_0 s))$ and $\theta_{s\ell_0}(0)=\gamma_1(s\ell_0)$.

$$G(0,t)=\begin{cases} p & \text{if } 0\leq t\leq 2/3, \\ \gamma_1((3t/2 -1)\ell_0) & \text{if } 2/3\leq t\leq 1. \end{cases} = c_0(h_0(t)) \text{ and}$$

$$G(1,t)=\begin{cases} \gamma_2(3t\ell_0) & \text{if } 0\leq t\leq 1/3, \\ q & \text{if } 1/3\leq t\leq 2/3, \\ \gamma_1((-3t/2 +2)\ell_0) & \text{if } 2/3\leq t\leq 1. \end{cases} = c_1(h_1(t)), \text{ where}$$

$$h_0(t)=\begin{cases} 0 & \text{if } 0\leq t\leq 2/3, \\ 3t-2 & \text{if } 2/3\leq t\leq 1. \end{cases} \quad \text{and}$$

$$h_1(t)=\begin{cases} 3t/2 & \text{if } 0\leq t\leq 1/3, \\ 1/2 & \text{if } 1/3\leq t\leq 2/3, \\ 3t/2 -1/2 & \text{if } 2/3\leq t\leq 1. \end{cases}$$

Trivially, h_0 and h_1 are continuous.

Define $H: I\times I\to M$ by:

$$H(s,t)=\begin{cases} c_0(3st+(1-3s)h_0(t)) & \text{if } 0\leq s\leq 1/3, \\ G(3s-1,t) & \text{if } 1/3\leq s\leq 2/3, \\ c_1((3s-2)t+(3-3s)h_1(t)) & \text{if } 2/3\leq s\leq 1. \end{cases}$$

H is clearly continuous by the above. $H(s,0)=p$, $H(s,1)=\exp_p v/2= \gamma_1(\ell_0/2)$, for all $s\varepsilon I$; and $H(0,t)=c_0(t)$, $H(1,t)=c_1(t)$, for all $t\varepsilon I$. $\exp_p:\overline{B}_{d_p}(0,TM_p)\to M$ is of full rank, therefore there exists $\delta>0$ such that $\exp_p:B_{d_p+\delta}(0,TM_p)\to M$ is still of maximal rank, because being singular is a closed condition. Hence, Lemma 9 is applicable and there exists $s_0\varepsilon[0,1]$ such that $\ell(H(s_0,t))+\ell(c_0)\geq 2(d_p+\delta)$.

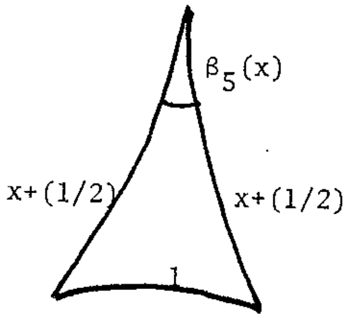


Figure 17. In M^2_C.

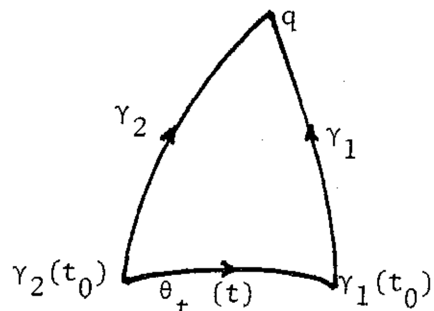


Figure 21. In M.

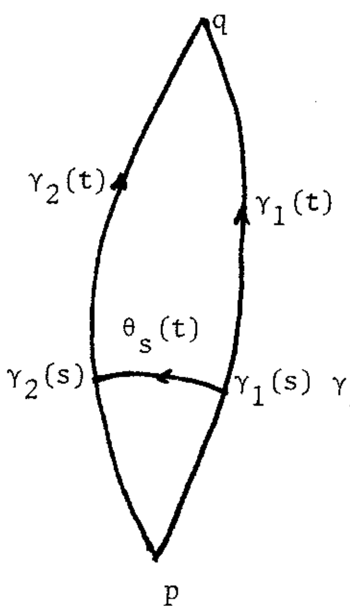


Figure 18. In M.

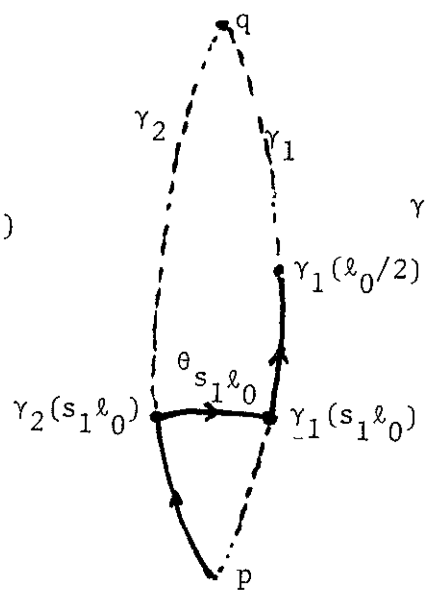


Figure 19.
For fixed $0 \leq s_1 \leq 1/2$,
Graph of $G(s_1,t)$ in M.

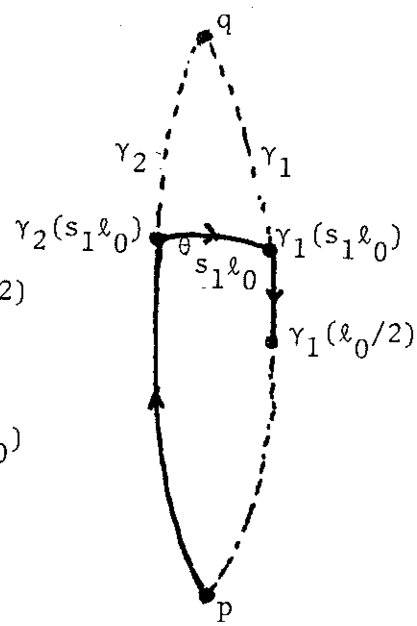


Figure 20.
For fixed
$1/2 \leq s_1 \leq 1$, Graph of
$G(s_1,t)$ in M.

If $0 \leqq s_0 \leqq 1/3$, then $\ell(H(s_0,t))=\ell(c_0(t))$. If $2/3 \leqq s_0 \leqq 1$, then $\ell(H(s_0,t))=\ell(c_1(t))$. $2\ell(c_0)=\ell_0<2(d_p+\delta)$.
$\ell(c_0)+\ell(c_1)=2\ell_0 \leqq 2d_p<2(d_p+\delta)$. Therefore, $s_0 \varepsilon [1/3, 2/3]$. Set $s_1=3s_0-1$. For any $s_1 \varepsilon [0,1]$, $G(s_1,t)$ is a union of broken geodesics with parametizations different from arc length: from p to $\gamma_2(s_1\ell_0)$ along γ_2, from $\gamma_2(s_1\ell_0)$ to $\gamma_1(s_1\ell_0)$ along $\theta_{\ell_0 s_1}$ with opposite direction, and from $\gamma_1(s_1\ell_0)$ to $\gamma_1(\ell_0/2)$ along γ_1 with the same direction or opposite direction depending on whether $s_1 \varepsilon [0, 1/2]$ or $s_1 \varepsilon [1/2 ,1]$. See Figures 19 and 20. Since γ_1, γ_2, and θ_s are minimal geodesics between those points:

$$\ell(G(s_1,t))=d(p,\gamma_2(s_1\ell_0))+d(\gamma_2(s_1\ell_0),\gamma_1(s_1\ell_0))+d(\gamma_1(s_1\ell_0),\gamma_1(\ell_0/2))$$
$$=s_1\ell_0+f(s_1\ell_0)+|s_1 -1/2|\ell_0.$$

If $0 \leqq s_1 \leqq 1/2$, then
$$\ell(G(s_1,t))+\ell(c_0) \leqq s_1\ell_0+f(s_1\ell_0)+(1/2 -s_1)\ell_0+\ell_0/2$$
$$\leqq 1+\ell_0< 2(d_p+\delta).$$

If $1/2 \leqq s_1 \leqq 1$, then
$$\ell(G(s_1,t))+\ell(c_0) \leqq s_1\ell_0+f(s_1\ell_0)+(s_1- 1/2)\ell_0+\ell_0/2$$
$$\leqq 2s_1\ell_0+0+2(\ell_0-\ell_0 s_1)$$
$$=2\ell_0< 2(d_p+\delta),$$

since $f(s) \leqq 2(\ell_0-s)$. $H(s_0,t)=G(s_1,t)$, and previously we had obtained that $\ell(H(s_o,t))+\ell(c_0) \geqq 2(d_p+\delta)$ by using Lemma 9. This is a contradiction. Therefore, such H does not exist. Consequently, there exists $t_0 \varepsilon [0,\ell_0]$ such that $f(t_0)=1=i_M$.

By the triangle inequality, $1/2 \leqq t_0 \leqq \ell_0 -1/2$. Now consider the geodesic triangle in M determined by:

$\gamma_1(t)$, for $t_0 \leq t \leq \ell_0$, from $\gamma_1(t_0)$ to q,

$\gamma_2(t)$, for $t_0 \leq t \leq \ell_0$, from $\gamma_2(t_0)$ to q,

$\theta_{t_0}(t)$, for $0 \leq t \leq f(t_0)=1$, from $\gamma_1(t_0)$ to $\gamma_2(t_0)$. Figure 21.

$\ell_0 - t_0 \leq \ell_0 - 1/2 \leq d_p - 1/2 < 1+\varepsilon'(\alpha,C) - 1/2 = 1/2 + \varepsilon'(\alpha,C)$. Now consider a geodesic triangle in M_C^2 with side lengths 1, $\ell_0 - t_0$, and $\ell_0 - t_0$. By Toponogov's Theorem and the construction of $\beta_5(x)$, and Lemma 2, we obtain that $\measuredangle(\gamma_1'(q),\gamma_2'(q)) \geq \beta_5(\ell_0 - t_0 - 1/2) > \beta_5(\varepsilon'(\alpha,C)) = \alpha$. Lemma 10 QED.

Straightforward calculations show that:
$\varepsilon(\alpha,0) = (1-\sin(\alpha/2))/(1+\sin(\alpha/2))$, and if $C<0$, then
$\varepsilon(\alpha,C) = -1+(\ln((e^{\kappa}-1+(e^{2\kappa}+1-2e^{\kappa}\cos\alpha)^{1/2})/2\sin(\alpha/2))/\kappa)$, where $\kappa = (-C)^{1/2}$.

<u>THEOREM 3.</u> For any given $C \in \mathbb{R}$, there exists $\varepsilon_3(C)$ such that: for any compact Riemannian manifold M^n, $n \geq 2$, if

i) $d_M^2 \cdot K_M \geq C$,

ii) there exists a point p in M such that $i_M/d_p > 1-\varepsilon_3(C)$, and

iii) for the same point p, $\exp_p : \overline{B}_{d_p}(0,TM_p) \to M$ is of maximal rank; then $\tilde{M}$ is homeomorphic to S^n and $\pi_1(M) = \mathbb{Z}_2$.

If $n \leq 4$, then M is homeomorphic to $\mathbb{R}P^n$.

For the proof of Theorem 3, we need the following:

THEOREM (Sugahara, [Su]). For any compact Riemannian manifold M^n, if there exists a point $p \in M$ such that the first conjugate locus of p is disjoint from the cut locus of p and the number of minimal geodesics from p to any point on its cut locus is 2, then $\pi_1(M) = \mathbb{Z}_2$ and $\tilde{M}$ is homeomorphic to S^n.

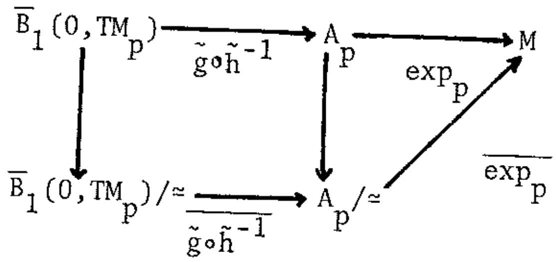
THEOREM (Livesay, [L]). If $T: S^3 \to S^3$ is any fixed point free continuous involution, i.e. $T^2x = x$, $Tx \neq x$ for all $x \in S^3$; then there exists a homeomorphism $h: S^3 \to S^3$ such that $(hTh^{-1})(x) = Ax$, where A is the antipodal map.

PROOF. (Theorem 3) Take $\varepsilon_3(C) = \varepsilon(2\pi/3, C)$. $d_M^2 \cdot K_M \geqq C$ implies that $i_M^2 \cdot K_M \geqq \mathrm{Min}(C,0)$. By Lemma 10, for any $q \in C_p$, if γ_1, γ_2 are any two minimal geodesics from p to q, then $\measuredangle(\gamma_1'(q), \gamma_2'(q)) > 2\pi/3 = \arccos(-1/2)$. There are at most two geodesics from p to q by Lemma 3. Since q is not conjugate to p along any minimal geodesic, there are at least 2 such geodesics. (For example, see [CE, pg.93]) Therefore, the hypothesis of Sugahara's Theorem B is satisfied. Hence, $\tilde{M}$ is homeomorphic to S^n and $\pi_1(M) = \mathbb{Z}_2$.

If M^n is as above, define $\psi: \partial A_p \to \partial A_p$ by $\psi(v) = w$ if and only if $\exp_p v = \exp_p w$ and $v \neq w$. Obviously, $\psi^2 = \text{identity}$, and ∂A_p is homeomorphic to S^{n-1}. [CE], [GKM].

Define $\tilde{g}: \bar{B}_1(0, TM_p) \to A_p$ by $\tilde{g}(u) = u \cdot c_p(u/\|u\|)$, if $u \neq 0$; and

$\tilde{g}(0)=0$. $\tilde{g}$ is a homeomorphism. Let $g=\tilde{g}|_{UM_p}$. The map $g^{-1}\circ\psi\circ g:UM_p\to UM_p$ is a continuous fixed point free involution. If $n=4$, by Livesay's Theorem, there exists $h:UM_p\to UM_p$ such that $(h\circ g^{-1}\circ\psi\circ g\circ h^{-1})(x)=-x$, that is $\psi((g\circ h^{-1})(x))=(g\circ h^{-1})(-x)$, for all $x\in UM_p$. Now define $\tilde{h}:\overline{B}_1(0,TM_p)\to\overline{B}_1(0,TM_p)$ by $\tilde{h}(0)=0$ and if $u\neq 0$, $\tilde{h}(u)=\|u\|\cdot h(u/\|u\|)$. Also define the following two equivalence relations: For $x,y\in\overline{B}_1(0, TM_p)$; $x\simeq y$ if and only if $x=-y$, and $x,y\in UM_p$. For $u,v\in A_p$; $u\simeq v$ if and only if $u,v\in\partial A_p$ and $u=\psi(v)$. The following diagram commutes:



$\overline{exp_p}$ and $\overline{\tilde{g}\circ\tilde{h}^{-1}}$ are well defined and one-to-one, since $exp_p w=exp_p\psi(w)$ and $\psi((g\circ h^{-1})(x))=(g\circ h^{-1})(-x)$. They are continuous since exp_p and $\tilde{g}\circ\tilde{h}^{-1}$ are continuous. Hence, they are homeomorphisms. $\overline{B}_1(0,TM_p)/\simeq$ is obviously homeomorphic to $\mathbb{R}P^4$.

If $n=3$, Livesay's Theorem directly implies that any free continuous action of $\mathbb{Z}_2$ on S^3 gives a quotient homeomorphic to $\mathbb{R}P^3$. If $n=2$, a similar proof to $n=4$ case can be given by an elementary version of Livesay's Theorem on S^1. Theorem 3 QED.

<u>NOTATION.</u> $\sigma_m=\arccos(-1/m)$.

THEOREM 4. For any compact Riemannian manifold M^n, $n \geqq 2$, if

i) $d_M^2 \cdot K_M \geqq C$, and

ii) there exists a point p in M such that

$i_M/d_p > 1-\varepsilon(\sigma_4, C)$ and $\exp_p : \bar{B}_{d_p}(0, TM_p) \to M$ is of maximal rank; then $C_p = V_1 \cup V_2 \cup V_3$, where V_i are disjoint smooth submanifolds of codimension i, open in their dimensions. If n=2, then $V_3 = \phi$. If σ_4 is replaced by σ_3, then $V_3 = \phi$. If σ_4 is replaced by σ_2, then $V_3 = V_2 = \phi$ and hence, C_p is a smooth submanifold of dimension n-1, and by the theorem of Weinstein below, there exists a new Riemannian metric on M such that M becomes a non-simply connected, pointed Blaschke manifold.

THEOREM (Weinstein, [Wa], [BS].) If M can be written as $D \cup_a E$, where D is the n-dimensional closed ball, E a C^∞ closed disc bundle over a (n-k)-dimensional compact C^∞ manifold, with boundary ∂E diffeomorphic to S^{n-1} and $a: \partial D \to \partial E$ an attaching diffeomorphism; then there exists a new Riemannian metric on M, such that M becomes a Blaschke manifold at p which is the center of D.

PROOF (Theorem 4). We define $N_p : C_p \to \mathbb{Z}$, by for any $q \in C_p$, $N_p(q)$ to be the number of distinct minimal geodesics from p to q. Since $\exp_p | \bar{B}_{d_p}(0, TM_p)$ is maximal rank, for δ sufficiently small $\exp_p | B_{d_p+\delta}(0, TM_p)$ is still maximal rank. Therefore, for all $q \in C_p$, q is not conjugate to p along any minimal geodesic, and there are finitely many minimal geodesics from p to q, this number is $\geqq 2$. [CE,pg.93],[Su]. Take $V_i = N_p^{-1}(i+1)$. Clearly, $C_p = \bigcup_{i=1}^{\infty} V_i$, $V_i \cap V_j = \phi$, if $i \neq j$.

Let $q \in C_p$ be any fixed element, and $\gamma_1, \ldots, \gamma_k$ be the all of the distinct minimal geodesics from p to q, i.e. $N_p(q) = k$. $\measuredangle(\gamma_i'(q), \gamma_j'(q)) > \sigma_4 = \arccos(-1/4)$, if $i \neq j$; by Lemma 10; and $2 \leqq k \leqq 4$, by Lemma 3. Therefore, $V_i = \phi$, if $i \geqq 5$. Obviously, by the above, if σ_4 is replaced by σ_3 or σ_2, then furthermore, $V_3 = \phi$ or $V_2 = V_3 = \phi$, respectively.

Set $\ell = d(p,q)$. $\exp_p | B_{d_p + \delta}$ is a local diffeomorphism, so there exist $U \subseteq TM_p$, $U_q \subseteq M$ both open sets, such that $0 \in U$, $q \in U_q$, $p \notin U_q$, $U_i = \ell \cdot \gamma_i'(0) + U$; all U_i are disjoint and $\exp_p|_{U_i} : U_i \to U_q$ is a diffeomorphism for all i, $1 \leqq i \leqq k$. Let $f_i = (\exp_p | U_i)^{-1} : U_q \to U_i$ and define $H_{ij}(q) = \{x \in U_q | \, \|f_i(x)\| = \|f_j(x)\|\}$. $\|f_i(x)\| : U_q \to \mathbb{R}$ is a smooth function of x, since $f_i(x)$ is smooth and $p \notin U_q$ implies that $0 \notin U_i$, for $i = 1, \ldots, k$. $(\text{grad} \|f_i(x)\|)(q) = \gamma_i'(q)$ by Gauss Lemma and the way $\exp_p$ is defined. [CE],[GKM]. $(\text{grad}(\|f_i(x)\| - \|f_j(x)\|))(q) = \gamma_i'(q) - \gamma_j'(q) \neq 0$, if $i \neq j$, i.e. the function $\|f_i(x)\| - \|f_j(x)\|$ is regular at q. Therefore, there exists an open set $U_q' \subseteq U_q$ such that $\|f_i(x)\| - \|f_j(x)\|$ is regular on U_q'. $H_{ij}(q) \cap U_q' = \{x \in U_q' | \, \|f_i(x)\| - \|f_j(x)\| = \|f_i(q)\| - \|f_j(q)\| = 0\}$ is locally a smooth submanifold of M of dimension n-1, it contains q and is open in its dimension, by Implicit Function Theorem. Furthermore, $\gamma_i'(q)-\gamma_j'(q)$ is orthogonal to $T(H_{ij}(q))_q$ which is a hyperplane.

We will prove the following Lemma later; it is needed to complete the proof of Theorem 4.

LEMMA 11. Let $w_i \in \mathbb{R}^n$, $i=1,\ldots,k\leq 4$, such that $\|w_i\|=1$, $<w_i,w_j><0$, for $i\neq j$. Then $w_1-w_k,\ldots,w_{k-1}-w_k$ are linearly independent.

Now we continue with the proof of Theorem 4. If we set $w_i=\gamma_i'(q)$, then $\gamma_i'(q)-\gamma_k'(q)$'s, $i=1,\ldots,k-1$, form a linearly independent set, and hence, the set of $H_{ik}(q)$'s is transversal at q. Therefore, there exists U_q'', an open neighborhood of q such that $U_q''\subseteq U_q'$ and $H(q)=U_q''\cap\bigcap_{i=1}^{k-1}H_{ik}(q)$ is an n-k+1 dimensional submanifold of M locally, open in its dimension, containing q. Obviously, if n=2, then $k\leq 3$.

CLAIM. There exists an open neighborhood U_q''' of q such that $U_q'''\subseteq U_q''$ and $U_q'''\cap H(q)=U_q'''\cap V_{k-1}\subseteq C_p$.

First we prove the existence of U_q''' with $U_q'''\cap H(q)\subseteq U_q'''\cap V_{k-1}$. Suppose that there does not exist any such U_q''', i.e. there exist $q_n\in H(q)\cap U_q''$, $n\in\mathbb{N}$ such that $q_n\to q$ and $q_n\notin V_{k-1}$, $q\in H(q)\cap V_{k-1}$. q_n are in $H(q)\subseteq\bigcap_{i=1}^{k-1}H_{ik}(q)=\{x\in U_q \mid \|f_i(x)\|=\|f_k(x)\|, \forall i\}$. Hence, $\|f_i(q_n)\|=\|f_k(q_n)\|$, for i=1,...,k-1. Define $\theta_{n,i}(t)=\exp_p(t\cdot f_i(q_n)/\|f_i(q_n)\|)$, $0\leq t\leq f_i(q_n)$, geodesics from p to q_n. $\theta_{n,i}$ have the same length and all are distinct, for a fixed n. If $\|f_k(q_n)\|=d(p,q_n)$ and there is no other geodesic different from $\theta_{n,i}$'s from p to q_n, then $q_n \in V_{k-1}$. If $q_n \notin V_{k-1}$, then there exists a minimal geodesic $\psi_n(t)$ from p to q_n which is distinct from all $\theta_{n,i}$'s, i=1,...,k. Since $q_n \to q$, $\psi_n(t)$ has a convergent subsequence ψ_{n_m} converging to a minimal geodesic from p to q, i.e. $\gamma_{i_0}(t)$ for some $1\leq i_0 \leq k$. In this case $\theta_{n_m,i_0}(t)$ and $\psi_{n_m}(t)$ are distinct geodesics from p to q_{n_m} and both converge to $\gamma_{i_0}(t)$, as geodesics. $\exp_p|B_{d_p+\delta}(0,TM_p)$ is of maximal rank, so we conclude that:

$\psi'_{n_m}(0)\cdot d(p,q_{n_m})\to f_{i_0}(q)$; $f_{i_0}(q_n)\to f_{i_0}(q)$ in TM_p and

$\psi'_{n_m}(0)\cdot d(p,q_{n_m})\neq f_{i_0}(q_{n_m})$, since $\psi_{n_m}(t)$ and $\theta_{n_m,i_0}(t)$ are distinct geodesics from p to q_{n_m}. $\exp_p\psi'_n(0)\cdot d(p,q_n)=\psi_n(d(p,q_n))=q_n=\exp_p f_{i_0}(q_n)$. This contradicts the fact that $\exp_p|U_{i_0}$ is a diffeomorphism. Therefore, such $\psi_n(t)$ do not exist. So,in fact, $q_n \in V_{k-1}$, for all n large, and we conclude that there exists U'''_q open, with $U'''_q \cap H(q) \subseteq U'''_q \cap V_{k-1}$.

Second, we need to show the existence of an open set U'''_q with $U'''_q \cap V_{k-1} \subseteq U'''_q \cap H(q)$. Suppose that it does not exist, i.e.

there exist $q_n \varepsilon (V_{k-1} - H(q)) \cap U_q''$, $n \varepsilon \mathbb{N}$, and $q_n \to q$. $q_n \varepsilon V_{k-1}$, so there are k distinct minimal geodesics $\theta_{n,i}$ from p to q_n. By Lemma 10: $<\theta'_{n,i}(0), \theta'_{n,j}(0)> < -1/4$, for $i \neq j$. Therefore, the limit set of these geodesics contains at least k distinct minimal geodesics from p to q. They have to be $\gamma_1, \ldots, \gamma_k$. For sufficiently large n, by rearranging i indices for fixed n's, and by taking convergent subsequences, we have $\theta_{n_m,i}(t) \to \gamma_i(t)$, as $m \to \infty$, as curves. $\theta'_{n_m,i}(0) \to \gamma'_i(0)$; $\theta'_{n_m,i}(0) \cdot d(p,q_{n_m}) \to \gamma'_i(0) \cdot d(p,q) = f_i(q)$. For sufficiently large m, $\theta'_{n_m,i}(0) \cdot d(p,q_{n_m}) \varepsilon U_i$ and hence, $d(p,q_{n_m}) = \| \theta'_{n_m,i}(0) \cdot d(p,q_{n_m}) \| = \| f_i(q_{n_m}) \|$. So, $\| f_i(q_{n_m}) \| = \| f_j(q_{n_m}) \|$, for all $1 \leq i < j \leq k$, and for sufficiently large m, $q_{n_m} \varepsilon H(q)$. This is a contradiction. Finally, the claim holds to be true, by the existence of U_q''', an open set with $U_q''' \cap H(q) = U_q''' \cap V_{k-1}$.

For the argument above, q was fixed, but arbitrarily. For any $q \varepsilon V_{k-1}$, $H(q) \cap U_q''' \subseteq V_{k-1}$ and $H(q) \cap U_q'''$ is an open piece of an n-k+1 dimensional smooth submanifold of M. This shows that V_{k-1} is an n-k+1 dimensional smooth submanifold of M, which is open in its dimension. If $q \varepsilon \overline{V}_i$, i.e. there exist $q_n \varepsilon V_i$, $n \varepsilon N$, $q_n \to q$ as $n \to \infty$, then there are exactly i+1 distinct minimal geodesics from p to q_n, and they have a limit set of at least i+1 minimal geodesics from p to q, as above, all are distinct. But, there may be other minimal geodesics from p to q, that implies that $q \in V_{i+m}$, for some $m \geq 0$. Therefore, $\overline{V}_i - V_i \subseteq \bigcup_{j>i} V_j$. By a theorem of Sugahara, [Su], V_1 is an open and dense subset of C_p. $\partial V_1 = \overline{V}_1 - V_1 = C_p - V_1 = V_2 \cup V_3$. We only have $\partial V_2 \subseteq V_3$, since $V_2 \cup V_3$ is not necessarily connected, V_2 is not necessarily dense in $V_2 \cup V_3$.

If σ_4 is replaced by σ_2, then by Lemmas 3 and 10, we have $C_p = V_1$, which is an n-1 dimensional compact smooth submanifold of M. V_1 is locally defined by $H(q) \cap U_q''' = \{x \in U_q''' \mid \|f_1(x)\| = \|f_2(x)\|\}$, a level set of a regular smooth function locally and $\|f_1(x)\|$ is a smooth function on U_q'''. Therefore, $\|f_1(x)\| = d(p,x)$ is a smooth function on $C_p = V_1$. Hence, $c_p(v): UM_p \to \mathbb{R}$ is smooth; and for any δ', $i_M > \delta' > 0$, $V_{\delta'} = \{\exp_p tv \mid v \in UM_p,\ 0 \leq t \leq c_p(v) - \delta'\}$ is diffeomorphic to D^n and $\partial D^n = S^{n-1}$ is diffeomorphic to $\partial V_{\delta'}$. Since $\exp_p$ is of maximal rank on $B_{d_p+\delta'}$ and C_p is smooth submanifold, $(\exp_p tv)'|_{t=c_p(v)}$ depends on $\exp_p c_p(v) \cdot v$ smoothly, for $v \in UM_p$. Hence, $M - V_{\delta'}$ is diffeomorphic to a smooth 1-disc bundle over V_1. So, Weinstein's Theorem is applicable. Theorem 4 QED.

<u>PROOF (Lemma 11).</u> Obviously, $w_i \neq w_j$ if $i \neq j$.

<u>Case for k=2:</u> Obvious, since $w_1 - w_2 \neq 0$.

<u>Case for k=3:</u> Suppose that there exists $c \in \mathbb{R}$ such that $w_1 - w_3 = c(w_2 - w_3)$. $\langle w_1 + w_2, w_3 \rangle = \langle w_1, w_3 \rangle + \langle w_2, w_3 \rangle < 0$. So, $w_1 + w_2 \neq 0$, and hence, $\langle w_1, w_2 \rangle > -1$.

$c\|w_2-w_3\|^2=<w_1-w_3,w_2-w_3>=\|w_3\|^2-<w_2,w_3>-<w_1,w_3>+<w_1,w_2>$

$\geq 1+<w_1,w_2> >0.$

Thus, $c>0$. By the symmetry, we may assume that $0<c\leq 1$. If $c=1$, then $w_2=w_1$, which is not the case. If $0<c<1$, then $1>(1-c)^2=\|(1-c)w_3\|^2=\|w_1-cw_2\|^2=1+c^2-2c<w_1,w_2> \quad >1+c^2>1$. This gives a contradiction. Hence w_1-w_3, w_2-w_3 which are both non-zero, are linearly independent.

Case for k=4: First of all, we observe that there do not exist four distinct non-zero vectors in $\mathbb{R}^2$ such that all angles between any two are $>\pi/2$. Hence, if $<w_i,w_j> <0$ for $1\leq i<j\leq 4$, then $\dim(\mathrm{span}(w_1,w_2,w_3,w_4))\geq 3$. Let $W=\mathrm{span}(w_1-w_4,w_2-w_4,w_3-w_4)$. By the case for k=3, $\dim W\geq 2$. Suppose that $\dim W=2$. If $w_4\in W$, then obviously, $w_1,w_2,w_3,w_4\in W$. This is not the case, so $w_4\notin W$. Let W_1 be the three dimensional subspace spanned by W and w_4. Since $w_4\notin W$, there exists a unique vector $w\in W_1$ such that $\|w\|=1$, $w\perp W$ and $<w_4,w>=c>0$. $<w,w_i-cw>=<w,w_i-w_4+w_4-cw>=<w,w_i-w_4>+<w,w_4>-<w,cw>=0+c-c=0$, for $i=1,2,3,4$. Hence, $w_i-cw\in W$, for all i. Obviously, w_i-cw are all distinct. $<w_i-cw,w_j-cw>=<w_i,w_j>-c<w,w_i>-c<w,w_j>+c^2<w,w>$ $=<w_i,w_j>-c<w,w_i-w_4+w_4>-c<w,w_j>+c^2=<w_i,w_j>-c^2$. Since $\|w_i-cw\|^2=1-c^2$, and w_i-cw are distinct; we have $0<c<1$. So, w_i-cw are all non-zero. If $i\neq j$, then $<w_i-cw,w_j-cw>=<w_i,w_j>-c^2<0$. Therefore, we obtain four non-zero vectors in W which is two dimensional, such that all angles between any two are $>\pi/2$.

This gives a contradiction. Consequently, $\dim W=3$. This proves the case for $k=4$ and hence, Lemma 11.

REMARK. Lemma 11 nay be extended for $n+1>k>4$, if one could prove that there do not exist $w_1,\ldots,w_k$, unit vectors in $\mathbb{R}^{k-2}$ with $\langle w_i,w_j\rangle <-1/k$, for $i\neq j$. If this is done, then we can let $V_4,\ldots,V_k$ be non-empty submanifolds and replace σ_4 by σ_k in Theorem 4.

THEOREM 5. For any compact Riemannian manifold M^n, $n\geqq 2$, if

i) $d_M^2\cdot K_M \geqq C$, and

ii) there exists a point $p\varepsilon M$ such that $i_M/d_p >1-\varepsilon(\sigma_{n'},C)$, where $n'=\mathrm{Min}(n,4)$;

Then, $d_p\geqq\pi/(2K^{1/2})$, where $K=\mathrm{Max}(K_M)$, and hence, $K>0$.

PROOF. Let $q\varepsilon M$ be such that $d(p,q)=d_p$ and suppose that $d_p<\pi/(2K^{1/2})$. We will use the notation and construction in Theorem 4. By Rauch Comparison, $\exp_p|\overline{B}_{d_p}(0,TM_p)$ is of maximal rank. Hence, by Lemmas 3 and 10, there are at most n' geodesics from p to q. $q\varepsilon V_k$ for some $k<n'=\mathrm{Min}(n,4)$. $\dim(V_k)\geqq n-n'+1=\mathrm{Max}(1,n-3)\geqq 1$. V_k is at least a one dimensional submanifold of M, lying in C_p. There exists a smooth curve $\theta(t)$ defined for $t\varepsilon(-\varepsilon,\varepsilon)$, for ε small enough, such that $\theta(t)\varepsilon V_k\cap U_q'''$, $\theta(0)=q$.

$\theta(t)\varepsilon H(q)\cap U_q'''$. If f_i are constructed as in Theorem 4, then $\forall r\varepsilon H(q)\cap U_q'''=V_k\cap U_q'''\subseteq C_p$, we have $d(p,r)=\|f_i(r)\|$. H(q) was obtained by the intersection of the smooth hypersurfaces which are the level sets of the functions $\|f_i(x)\|-\|f_j(x)\|$ locally. Since $f_i(x)$ are smooth functions and $\|f_i(x)\|-\|f_j(x)\|$ are regular on $H(q)\cap U_q'''$; $f_i(x)$ restricted to $H(q)\cap U_q'''$, which is a smooth submanifold and a subset of C_p, is still a smooth function. Consequently, $d(p,\theta(t))>0$, is a smooth function of t. See the last part of the proof of Theorem 4. Let $\gamma_i(t)$, $i=1,\ldots, k+1$, be all of the distinct minimal geodesics from p to q. Define $F_i:(-\varepsilon,\varepsilon)\times[0,1]\to M$ by $F_i(s,t)=\exp_p t\cdot f_i(\theta(s))$. Since γ_i are minimal geodesics and $\exp_p$ is regular at $f_i(q)$, F_i is one-to-one, for small ε and $t>0$. Let $S_i=F_{i*}(\partial/\partial s)$ and $T_i=F_{i*}(\partial/\partial t)$, where $F_{i*}:T((-\varepsilon,\varepsilon)\times[0,1])\to TM$. Let $\ell_i(s)=\int_0^1\|T_i(s,t)\|dt=d(p,\theta(s))$, for $s\varepsilon(-\varepsilon,\varepsilon)$. Let S_i and T_i also denote $S_i(0,t)$ and $T_i(0,t)$, respectively. By the first variation formula: $d/ds(\ell_i(s))|_{s=0}=$
$(1/\ell_i(0))(<S_i,T_i>|_p^q-\int_0^1<S_i(0,t),\nabla_{T_i}T_i(0,t)>dt)=(1/\ell_i(0))(<S_i,T_i>_q)=0$, since $d(p,\theta(s))$ is a smooth function of s, T_i are tangent vectors of a geodesic and q is at the maximal distance. By a similar argument, $d^2/ds^2(\ell_i(s))|_{s=0}\leq 0$. By the second variation formula, [CE], pg.20, $d^2/ds^2(\ell_i(s))|_{s=0}=(<\nabla_{S_i}S_i,T_i>+<\nabla_{T_i}S_i, S_i>)|_q$, since the variation is through geodesics, i.e. S_i is a Jacobi field along γ_i and

$\langle S_i,T_i\rangle|_q=\langle S_i,T_i\rangle|_p=0$. Let X be $\nabla_{\theta'(t)}\theta'(t)|_{t=0}$. $(\nabla_{S_i}S_i)(q)=X$, for all i, since $S_i(q)=\theta'(0)$ and $F_i(s,1)=\theta(s)$ for $s\varepsilon(-\varepsilon,\varepsilon)$. If $X\neq 0$, then by Berger's Lemma, Chapter 1, there exists T_{i_0} such that $\langle X,T_{i_0}(q)\rangle \geqq 0$. If X=0, then such T_{i_0} exists obviously. Therefore, $\langle\nabla_{S_{i_0}}S_{i_0}, T_{i_0}\rangle|_q \geqq 0$ and hence, $\langle\nabla_{T_{i_0}}S_{i_0}, S_{i_0}\rangle|_q \leqq 0$. $\langle\nabla_{T_{i_0}}S_{i_0}, S_{i_0}\rangle=(1/2)T_{i_0}(\|S_{i_0}\|^2)|_q \leqq 0$, and S_{i_0} is a Jacobi field. If K>0, then let $S^n(K^{-1/2})$ be the standard sphere of constant curvature K. In fact, we will show that K>0, below. Let $p_0\varepsilon S^n=S^n(K^{-1/2})$ be any point, ψ be any geodesic from p_0 and E(t) be any parallel unit vector field along $\psi(t)$, so $(\sin t\cdot K^{1/2})E(t)$ is a Jacobi field along $\psi(t)$. Now apply Rauch Comparison Theorem 1, Chapter 1, to S_{i_0} along γ_{i_0} in M and $c(\sin t\cdot K^{1/2})E(t)$ along $\psi(t)$ in S^n, where $c=\|S_i'(0)\|/K^{1/2}$. For t>0, we will get

$$0\leqq d/dt(\|S_{i_0}\|^2/(c^2\cdot\sin^2 t\cdot K^{1/2})) \text{ and}$$

$$d/dt(\|S_{i_0}\|^2)|_q=(1/d(p,q))\cdot(T_{i_0}(\|S_{i_0}\|^2)|_q) \leqq 0.$$

Hence, $2(K^{1/2})(\sin t\cdot K^{1/2})(\cos t\cdot K^{1/2})|_{t=d(p,q)} \leqq 0$, and therefore $d(p,q) \geqq \pi/(2K^{1/2})$. If K were non-positive, then we would replace K above by $1/m^2$ and by the same proof, we would obtain that $d(p,q)\geqq \pi m/2$, for any $m\varepsilon N^+$. d(p,q) is finite, so K>0. We had assumed that $d_p<\pi/(2K^{1/2})$, in the beginning and then obtained that

$d_p \geq \pi/(2K^{1/2})$, which is a contradiction. So we should have $d_p \geq \pi/(2K^{1/2})$ in the beginning. Theorem 5 QED.

THEOREM 5B. Theorem 5 still holds, if $\sigma_{n'}$ is replaced by σ_n.

Theorem 5B follows from Lemma 12, below, which was known to J.Cheeger and D. Gromoll. In fact, Theorems 5 and 5B were known to Jeff Cheeger.

LEMMA 12 (Cheeger-Gromoll). For any compact Riemannian manifold M^n, if $d_p < \pi/(2K^{1/2})$ for some $p \varepsilon M$, where $K = \mathrm{Max}(K_M)$, and $d_p = d(p,q)$, for some $q \varepsilon C_p$; then there are at least n+1 distinct minimal geodesics from p to q. (For $K \leq 0$, we again mean ∞ instead of $K^{-1/2}$.)

PROOF (Theorem 5B). By replacing $\sigma_{n'}$ by σ_n and supposing that $d_p < \pi/(2K^{1/2})$, we would obtain that there are at most n geodesics from p to q by Lemmas 3 and 10. This contradicts Lemma 12. Hence, $d_p \geq \pi/(2K^{1/2})$.

PROOF (Lemma 12). Let $\gamma_1, \ldots, \gamma_k$ be all distinct minimal geodesics from q to p. Suppose $k \leq n$. There exists $v_0 \varepsilon TM_q$ such that $\langle v_0, \gamma_i'(0)\rangle \leq 0$, for all $i = 1, 2, \ldots, k$. For the existence of v_0: $\gamma_1'(0), \ldots, \gamma_{k-1}'(0)$ span at most an n-1 dimensional subspace of TM_q.

The orthogonal complement of this subspace is at least one dimensional, and contains at least two vectors in opposite directions, one of which makes an angle $\geqq\pi/2$ with $\gamma_k'(0)$. Let $\theta(t)=\exp_q(v_o t)$, for $t\geqq 0$. $d_p<\pi/(2K^{1/2})$, so $\exp_p|\bar{B}_{d_p}(0,TM_p)$ is non-singular. Construct f_i around q as in Theorem 4. If $<v_o,\gamma_i'(0)> <0$, then obviously $\|f_i(\theta(t))\|$ is strictly increasing at t=0. If $<v_0,\gamma_i'(0)>=0$, then $\|f_i(\theta(t))\|$ is still strictly increasing. To observe that, consider the pull back metric of M on TM_p by $\exp_p|B_{d_p+\delta}(0,TM_p)$ which is a local diffeomorphism. With this new metric, the metric ball of radius d_p in TM_p is strictly convex by Whitehead's Lemma, and hence, $\|f_i(\theta(t))\|$ is strictly increasing at t=0. For large $n\in\mathbb{N}$, let $q_n=\exp_q v_0/n$ and θ_n be any minimal geodesic from p to q_n. $q_n\to q$, therefore θ_n has a convergent subsequence $\theta_{n_m}(t)\to\theta_0(t)$, as $m\to\infty$, and hence, θ_0 is a minimal geodesic from p to q. $\Rightarrow\theta_0=\gamma_{i_0}$, for some i_0 with $1\leqq i_0\leqq k$. For large $m\in\mathbb{N}$, $\nu_m(t)=\exp_p tf_{i_0}(q_{n_m})$ is not minimal, since $\ell(\nu_m)=\|f_{i_0}(q_{n_m})\|>\|f_{i_0}(q)\|=d(p,q)\geqq d(p,q_{n_m})$. So, we have: $f_{i_0}(q_{n_m})\to f_{i_0}(q)$, $\theta_{n_m}'(0)\cdot\ell(\theta_{n_m})\to f_{i_0}(q)$, $f_{i_0}(q_{n_m})\neq\theta_{n_m}'(0)\cdot\ell(\theta_{n_m})$, and $\exp_p f_{i_0}(q_{n_m})= \exp_p\theta_{n_m}'(0)\cdot\ell(\theta_{n_m})= q_{n_m}\to q$. This is a contradiction, since $\exp_p$ is a local diffeomorphism around $f_{i_0}(q)$. Consequently, $k\geqq n+1$. Lemma 12 QED.

SECTION 5. SOME EXAMPLES AND IMMEDIATE COROLLARIES.

S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ with their standard metrics (normalized with diameter equal to π except for $\mathbb{R}P^n$, and $d_{\mathbb{R}P^n} = \pi/2$) have equal diameter and injectivity radius, so they are obvious examples for Theorem 1 and $\mathbb{R}P^n$ is for all Theorems 1-5B.

EXAMPLE 1. Let M be either of the above examples and $g(t)$ be a C^2, 1-parameter family of Riemannian metrics on M such that $g(0)$ is the standard metric. Since the diameter and injectivity radius depend on the metrics continuously [E], and $g(0)$ has positive sectional curvature, there exists a neighborhood $(-\delta,\delta)$ of 0 such that for every $t\varepsilon(-\delta,\delta)$, $i_p(g(t))/d_p(g(t)) > 1-\varepsilon_1(0)$. Those are non-trivial examples for Theorem 1.

EXAMPLE 2. Let M be any compact Riemannian manifold with metric g_0, $p\varepsilon M$ and $\varepsilon_0 > 0$ be given. There exists a Riemannian metric g_1 on M such that $i_p(g_1)/d_p(g_1) > 1-\varepsilon_0$.

We construct g_1 as follows. Let r be small enough so that $\exp_p | B_r(0,TM_p)$ is a diffeomorphism from $B_r(0,TM_p)$ onto $B_r(p,M)$. There exists a smooth function $\psi: M \to [0,1]$ such that $\mathrm{supp}(\psi) \subseteq B_r(p,M)$

and $\psi|_{B_{r(1-\varepsilon_0/2)}(p,M)}\equiv 1$. Let d denote the diameter of (M,g_0).

Define $g_1=(1+2d\psi/\varepsilon_0 r)g_0$. Then, we have $i_p(g_1)\geqq(1-\varepsilon_0/2)r\cdot(2d/\varepsilon_0 r)$ and $d_p\leqq 2dr/\varepsilon_0 r+d$. Therefore, $i_p(g_1)/d_p(g_1)\geqq(2-\varepsilon_0)/(2+\varepsilon_0)>1-\varepsilon_0$.

REMARKS. 1) Example 2 shows that the curvature condition of Theorem 1 can not be removed. It may be possible that it can be weakened or replaced by using other geometric quantities.

2) $\lim_{C\to-\infty}\varepsilon_1(C)=0$. We observe this as follows: $\varepsilon_1(C)$ is decreasing, if C is decreasing, by Toponogov's Theorem and the construction of $\varepsilon_1(C)$. If $\lim_{C\to-\infty}\varepsilon_1(C)$ were equal to $\varepsilon_0>0$, then example 2 would give a counterexample to Theorem 1. Hence, $\lim_{C\to-\infty}\varepsilon_1(C)=0$. In fact, if one could give a stronger theorem with better $\varepsilon_1(C)$, then still $\inf_C \varepsilon_1(C)=0$ holds.

3) In example 2, possibly the sectional curvature of g_1 is large, both positively and negatively on the set $B_{2d/\varepsilon_0}(p,M;g_1)-B_{(1-\varepsilon_0/2)2d/\varepsilon_0}(p,M;g_1)$. In fact, Theorem 1 implies that the sectional curvature should become smaller than C_0, where $\varepsilon_1(C_0)=\varepsilon_0$, if $\mathrm{order}(\pi_1(M))\geqq 3$.

EXAMPLE 3. Consider the lattice $L=\mathbb{Z}e_1+\mathbb{Z}e_2$ in $\mathbb{R}^2$ given by $e_1=(1,0)$ and $e_2=(1/2,3^{1/2}/2)$. $T^2=\mathbb{R}^2/L$ is a flat hexagonal torus. Let $\rho:\mathbb{R}^2\to\mathbb{R}^2/L$ be the natural projection map. A fundamental domain can be chosen as a hexagonal region H with vertices: $v_1=e_1/3+e_2/3$, $v_2=-e_1/3+2e_2/3$, $v_3=-2e_1/3+e_2/3$, $v_4=-e_1/3-e_2/3=-v_1$, $v_5=-v_2$, and $v_6=-v_3$ and the sides to be the line segments joining v_i to v_{i+1}, mod 6. See Figure 22. In fact, if we consider $\mathbb{R}^2$ to be $T(T^2)_{(0,0)}$ and ρ be $\exp_{(0,0)}$, then $A_{(0,0)}=H$ and the tangential cut locus $\tilde{C}_{\rho(0,0)}$ of $\rho(0,0)$ is ∂H. Hence, $i_{T^2}=1/2$, $d_{T^2}=3^{-1/2}$ and $i_T/d_T=3^{1/2}/2$, since i_T and d_T are independent from the choice of the points in this example.

Example 3 shows that $\varepsilon(0)$'s of Theorems 1,2, and 3 can not be made better than $1-(3^{1/2}/2)$, even with another method.

In T^2 of Example 3, let $p,q_0,q_1\varepsilon T^2$ be such that $p=\rho(0,0)$, $q_1=\rho(v_1)=\rho(v_3)=\rho(v_5)$ and $q_0=\rho(v_2)=\rho(v_4)=\rho(v_6)$. $d(p,q_1)=d(p,q_0)=d_p=3^{-1/2}$. $C_p=\rho(\partial H)$ is the union of three distinct minimal geodesics from q_0 to q_1. In fact, there are exactly three distinct minimal geodesics from p to q_0 and q_1, each. With the notation of Theorem 4; $V_2=\{q_0,q_1\}$ and $V_1=\rho(\partial H)-\{q_0,q_1\}$. Although $i_T/d_T=3^{1/2}/2$ $<2\cdot 6^{1/2}-4=1-\varepsilon(\sigma_3,0)$, T^2 perfectly describes an example of C_p for the situation of Theorem 4 for $V_2\neq\phi$; but still it is not an example for Theorem 4.

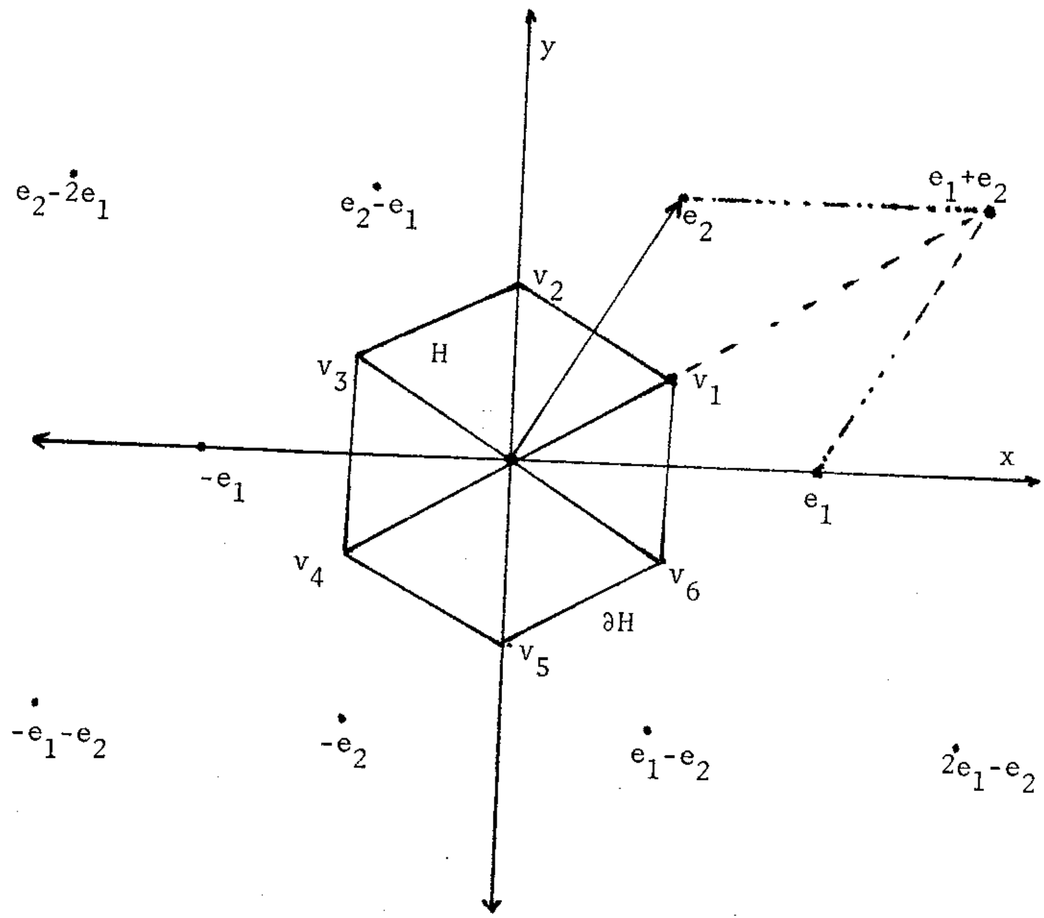


Figure 22. In $\mathbb{R}^2$.

Some immediate corollaries of Theorems 1-5B.

For any $C\in\mathbb{R}$, there is a universal constant $\varepsilon(C)$ such that for any compact Riemannian manifold M^n with $K_1 \geq K_M \geq K_0$ and diameter d, the following corollaries hold, ($\varepsilon(C)$ can be made better, if different ones are used in each case, sometimes depending on the dimension, as in the Theorems. Otherwise, we consider the smallest of all of them, which is positive):

COROLLARY 1. If $i_M/d_M > 1-\varepsilon(d^2K_0)$, then $\pi_1(M)=1$ or $\mathbb{Z}_2$.

COROLLARY 2. If $i_M/d_M > 1-\varepsilon(d^2K_0)$ and M is not simply connected, then M^n has the homotopy type of $\mathbb{R}P^n$.

Corollaries 1 and 2 follow from Theorems 1 and 2, with the fact that $i_M \leq i_p \leq d_p \leq d_M$ and hence, $i_p/d_p \geq i_M/d_M$.

COROLLARY 3. If $K_1 \leq 0$, then $i_p \leq d_p(1-\varepsilon(d^2K_0))$ and hence, for any flat manifold, $i_p \leq 0.914 \cdot d_p$.

This follows from Theorem 1 and the fact that any compact Riemannian manifold whose sectional curvature is non-positive, has infinite fundamental group, since its universal cover is diffeomorphic to $\mathbb{R}^n$.

COROLLARY 4. If $K_1>0$ and $d_p \leq \pi/(2K_1^{1/2})$, then $i_M \leq d_p(1-\varepsilon(d^2K_0))$.

This follows from Theorems 4 and 5.

COROLLARY 5. If $d_p<\pi/K_1^{1/2}$ and $i_M>d_p(1-\varepsilon(d^2K_0))$, then

i) $d_p \geq \pi/(2K_1^{1/2})$,

ii) $\tilde{M}$ is homeomorphic to S^n,

iii) $\pi_1(M)=\mathbb{Z}_2$.

Hence, M^n has the homotopy type of $\mathbb{R}P^n$, and if $n\leq 4$, it is homeomorphic to $\mathbb{R}P^n$. C_p is a n-1 dimensional submanifold which has the homotopy type of $\mathbb{R}P^{n-1}$. Thus, if we have a simply connected compact Riemannian manifold with $i_M/d_p > 1-\varepsilon(d^2K_0)$, then $d_p \geq \pi/K_1^{1/2}$.

This follows from Theorems 3,4, and 5.

COROLLARY 6. Let $g(t)$ be a C^2 one parameter family of Riemannian metrics on $\mathbb{R}P^n$, $t\varepsilon(-\delta,\delta)$ such that $g(0)$ is the standard metric on $\mathbb{R}P^n$ of constant curvature 1. Then there exists $\delta_1>0$ such that, for all $t\varepsilon(-\delta_1,\delta_1)$, cut locus of any point of $\mathbb{R}P^n$ with the metric $g(t)$ is an n-1 dimensional submanifold.

For the proof of this, see Example 1 and Theorem 4.

REMARK. Obviously, the cases of $i_p=d_p$ and $i_M=d_M$ are included in Theorems 1-5B and above Corollaries, whenever it is appropriate.

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