997-69 (2)

THE DIRAC OPERATOR ON SPACES WITH CONICAL SINGULARITIES

A Dissertation presented

by

Arthur Weichung Chou

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

State University of New York

at

Stony Brook

August 1982

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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Dean of the Graduate School

Abstract of the Dissertation

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The Dirac operator on compact spaces with conical singularities is studied via the separation of variables formula and the functional calculus of the Dirac Laplacian on the cone. We prove a Bochner type vanishing theorem which gives topological obstructions to the existence of non-negative scalar curvature $\kappa \geq 0$ in the singular case. We also obtain an index formula relating the index of the Dirac operator to the $\hat{A}$-genus and Eta-invariant similar to that of Atiyah-Patodi-Singer.

In an appendix, we study manifolds with boundary with non-negative scalar curvature $\kappa \geq 0$, and obtain several new results on constructing complete metrics with $\kappa \geq 0$ on them.

To my parents.

Acknowledgements

I wish to express my deepest gratitude to my advisor, Professor Jeff Cheeger, for his guidance and encouragement. Without his inspiration and generous help, this work would hardly be possible.

My sincere thanks also go to Professors Detlef Gromoll and Blaine Lawson for many helpful conversations.

Finally, I would like to thank the members of the Stony Brook Mathematics Department for their help during my graduate years.

Table of Contents

Abstract iii

Acknowledgements vi

0. Introduction 1

1. Preliminaries 15

2. The Separation of Variables Formula and Eigenspinors of the Dirac Operator on Cones 22

3. The Self-adjointness of the Dirac Operator . 40

4. Vanishing Theorems 49

Appendix to 4. Deformations of the Metric near a Boundary and Positive Scalar Curvature 53

5. The Index Theorems 68

References 92

0. INTRODUCTION

In recent years, analytic tools (e.g. the elliptic operator theory) have become increasingly important in studying the topology of compact smooth manifolds. This trend can be traced back at least to the Hodge theorem which states that the space of harmonic i-forms on a compact manifold M is isomorphic to the real cohomology group $H^i(M, \mathbb{R})$ (see [R]). Other examples are Bochner's method and the Atiyah-Singer index theorem.

Bochner's method is to obtain a local formula expressing a geometric (elliptic) operator as the sum of a "rough Laplacian," which is non-negative on closed manifolds, and a curvature term. Thus if the curvature term is positive, the kernel of this operator vanishes. Results of this type are called "vanishing theorem." In the case of the Laplacian Δ on i-forms over closed manifolds, the local formula says $\Delta = \nabla^* \circ \nabla + \mathcal{R}$ where ∇ is the (induced) connection and $\mathcal{R}$ is a curvature piece (see [R],[LM]). Thus $\mathcal{R} > 0$ implies Kernel $(\Delta) = \{0\}$, i.e., $H^i(M,\mathbb{R})$ vanishes. For the Dirac operator D on spinors, we have the Lichnerowicz-Bochner formula

$$(0.1) \qquad D^2 = \nabla^* \circ \nabla + \tfrac{1}{4}\kappa,$$

where ∇ is the connection on the bundle of spinors and κ is the scalar curvature. Again for closed manifolds, $\kappa > 0$ yields Kernel $(D^2) = \{0\}$, i.e., there are no harmonic spinors (see [Li] [LM]).

The Atiyah-Singer index theorem ([As] [ABP]) tells us that, on a closed manifold, the index of an elliptic operator L, i.e., dim(Kernel of L) - dim(Cokernel of L), can be expressed as a linear combination of Pontrjagin numbers, which are topological invariants. For the case of the Dirac operator D on an even dimensional closed spin manifold M, Lichnerowicz ([Li]) obtained from the index theorem that

(0.2) $\quad \text{Index}(D) = \hat{A}(M)$; the $\hat{A}$-genus of M.

Combined with the vanishing theorem (0.1), it shows that $\kappa > 0$ implies $\hat{A}(M) = 0$. This provides us with a topological obstruction to the existence of a metric on M with $\kappa > 0$. (Gromov and Lawson have greatly extended the scope of this method by considering spinors with coefficients in suitable bundles, and they also developed a parallel theory for the Dirac operator on <u>complete manifolds</u> (see [GL1] [GL3]).)

An extension of this kind of theory to <u>singular spaces</u> for the Laplacian Δ was developed by Jeff Cheeger in [C1][C2][C3] (see also [CT][CGM]). In this thesis, we shall establish the corresponding theory for the Dirac operator by use of his ideas. Let us now recall some definitions and basic ideas from [C1][C2][C3].

Let N^m be a closed riemannian manifold of dimension m with metric g. By the cone C(N) on N, we mean the space (0.00) x N equipped with the metric $dr \otimes dr + r^2 g$ where $r \in (0,\infty)$. Set

$$C_{o,u}(N) = \{(r,x) \in C(N),\ 0<r\leq u\} \text{ and}$$

$$\overline{C_{o,u}(N)} = \{(r,x) \in C(N),\ 0\leq r\leq u\}.$$

(0.3). <u>Definition</u>. X^{m+1} is called a <u>space with conical singularities</u> if there exists $P_j \in X^{m+1}$, $j=1,2,\ldots,k$ such that $X^{m+1} \setminus \bigcup_{j=1}^{k} \{P_j\}$ is a smooth riemannian manifold and each P_j has a neighborhood U_j such that $U_j \setminus \{P_i\}$ is isometric to $C_{o,u_j}(N_j^m)$ for some U_j and N_j^m.

Without loss of generality, we assume that k = 1 and $u_j \geq 1$. We write $X^{m+1} = \overline{C_{0,1}(N^m)} \cup M^{m+1}$ where $N = \partial M$ and the union is along the boundary. Of course, X^{m+1} is not a manifold in general. For the purpose of this paper, we also assume that $X^{m+1}\{P\}$ is a spin manifold and N^m has the induced spin structure (see section 1 for definitions). Notice that $C(S_1^m) = \mathbb{R}^{m+1}$

where S_1^m is the standard sphere of radius one. This is the Euclidean space in polar coordinates.

By definition, analysis on a space with conical singularities X means analysis on the smooth part $X \setminus \{P\}$ of X. Since this manifold is incomplete, the situation is quite different from that of a compact or complete manifold. For example, the elliptic operators (e.g. the Laplacian and the Dirac operator) are no longer essentially self-adjoint. Thus we have to choose particular self-adjoint extension. See [c1][C2] for the case of the Laplacian.

Because the local analysis on M is well-understood, we first restrict our attention to the cone part, $C_{0,1}(N)$. To do analysis on the cone, we observe that by using the separation of variables technique, we can reduce the local analysis on the cone to the global analysis on the cross section N. In fact, if we restrict a function g(r,x) on C(N) to $\{r\} \times N$, then by the standard theory of the eigenfunction expansion of the elliptic operator L on compact manifolds, we can write

$$g(r,x) = \Sigma g_i(r)\phi_i(x); \tag{0.4}$$

$\{\phi_i\}$ are the eigenfunctions of $\tilde{L}_r = L|_{\{r\}\times N}$ which can be identified with those of $\tilde{L} = L|_{\{1\}\times N}$ by (parallel) translation along the radial geodesic $\mathbb{R} \times \{x\}$. The convergence of (0.4) is in the L^2 sense; moreover, if g is smooth, then a standard argument shows that the convergence is uniform on each compact subset away from the singularity at p. Note that in the case of $C(S_1^m) = \mathbb{R}^{m+1}$, this is nothing but the usual Fourier series expansion.

On functions of the type $g(r)\phi(x)$, the action of L will give us singular Sturm-Liouville ordinary differential equations (see (2.6)), and hence we can solve for the eigenfunctions of L explicitly on the cone. As we will see, there are limit circle cases (see [St]) of singular Sturm-Liouville equations corresponding to the small eigenvalues of $\tilde{L}$ on $\{1\} \times N$. This is the reason why L fails to be self-adjoint without any condition on the behavior of the function near the singularity $r = 0$.

In section 2, we derive the separation of variables formula, write down the eigenspinors for the Dirac operator D on the cone, and construct counterexamples to the self-adjointness of D. Then, in section 3, we obtain the criterion for the self-adjointness of D from an a priori estimate. The main result is the following (compare [C2]). Let D_0 denote the Dirac operator on the space of smooth spinors with compact support on $X = C_{0,1}(N^m) \cup M$, and $\tilde{D}$ denote the Dirac operator on $\{1\} \times N = N$. Then we have

<u>Theorem</u> (3.2). The Dirac operator D_0 is essentially self-dajoint if and only if there is no eigenvalue μ_j of $\tilde{D}$ such that $|\mu_j| < \frac{1}{2}$.

Let D denote the Dirac operator with domain $\{\phi | \phi \in L^2 \cap C^\infty$ and $D\phi \in L^2\}$, and $\overline{D}$ denote its L^2 closure. It follows that

$$(0.5) \qquad \overline{D}_0 = \overline{D} \text{ and } \overline{D}_0^* \overline{D}_0 = \overline{D}^* \overline{D} \text{ if } |\mu_j| \geq \frac{1}{2}.$$

Let us write $\Delta_D = \overline{D}_0^* \cdot \overline{D}_0$ and $\Delta_N = \overline{D}^* \cdot \overline{D}$, which correspond to the generalized Dirichlet and Newmann condition respectively. Both Δ_D and Δ_N are self-adjoint extensions of $\Delta_0 = (D_0)^2$. We should mention that even if D_0 is essentially self-adjoint, Δ_0 may still not be essentially self-dajoint because of the limit circle phenomenon (see [St][Cl]).

In section 4, we show that if $C_{0,1}(N)$ has non-negative scalar curvature, then the condition $|\mu_j| \geq \frac{1}{2}$ is automatically satisfied. This, together with (0.1), gives us the vanishing theorem for spaces with conical singularities.

<u>Theorem</u> (4.2). If $X^{m+1} = C_{0,1}(N^m) \cup M$ has scalar curvature $\kappa \geq 0$ and $\kappa > 0$ somewhere, then

$$\mathrm{Kernel}(\overline{D}) = \{0\}.$$

This vanishing theorem has possible generalization to PL-manifolds or Pseudomanifolds. They are spaces which can be built up inductively by spaces with conical singularities (see [C2][C3]). This theorem can also be construed as giving necessary conditions for a manifold with boundary to admit a metric with scalar curvature $\kappa \geq 0$ for which the metric near the boundary is conical. In section 5, we combine this theorem with our index formulas to obtain topological conclusions.

A general discussion of manifolds with boundary is given in the appendix to section 4. By use of certain deformation techniques, we obtain the following theorems. Let M be a manifold with boundary N^m. Let κ denote the scalar curvature of M, and κ_r denote the scalar curvature of the hypersurface at distance r to the boundary.

Theorem [A.2]. Suppose that

(i) $\kappa \geq 0$

(ii) $\kappa_r \geq 0$ for $\forall$ $r \in [0,\epsilon]$

(iii) $H_r \geq 0$ for $\forall$ $r \in [0,\epsilon]$

where H_r is the mean curvature w.r.t the exterior normal. Then in this neighborhood $[0,\epsilon] \times N$ the metric can be deformed to a complete metric, which ends with the product metric $\mathbb{R} \times N$ near infinity, with non-negative scalar curvature.

This is similar to Theorem 5.7 in [GL1].

Theorem [A.12]. Suppose that dim $N = m \geq 2$ and

(i) the tubular neighborhood of N in M is normalized to be of width 1 and $\kappa > \frac{16m}{m+1}$ on it.

(ii) $\kappa_r \geq 0$ (>0) on $\{r\} \times N$, $r \in [0,1]$.

Then the metric can be deformed to a complete metric, which ends with the product metric $\mathbb{R} \times N$ near infinity, with scalar curvature $\kappa' \geq 0$ (>0).

The discussion of the appendix shows that our vanishing theorem could also be obtained from the work of Gromov and Lawson [GL3] except for the case where $K \equiv 0$ on a conical neighborhood of the boundary. This case is not covered by their method.

The next step is to study index formulas. We follow the same procedure as in [C1][C3], which is based on the functional calculus of the Laplacian on the cone and the heat equation method of deriving the index formula (see [ABP]). Let us briefly recall the ideas in [C3] as follows:

1. Using the technique of separation of variables, i.e. the eigenfunction expansion (0.4), and the Hankel transform, we can obtain a spectral representation of the Laplacian Δ on the cone $C(N)$ such that the action of Δ is carried into multiplication of λ^2; moreover, according to the Hankel inversion formula (5.5), the following formal representation for the kernal $f(\Delta)$ on $C(N) \times C(N) = \{(r_1, x_1, r_2, x_2)\}$ holds.

$$(0.6) \quad f(\Delta) = (r_1 r_2)^c \sum_j \left(\int_0^\infty f(\lambda^2) J_{\nu_j}(\lambda r_1) J_{\nu_j}(\lambda r_2) \lambda d\lambda \right) \phi_j(x_1) \otimes \phi_j(x_2)$$

where $c = \frac{1-m}{2}$, ϕ_j's are the eigenfunctions of $\tilde{\Delta}$ (the Laplacian on N) with eigenvalue μ_j, and J_{ν_j} is the Bessel function of order $\nu_j = \sqrt{c^2+\mu_j}$.

Thus we can regard (0.5) as the sum of series consisting of a family of functions of $\tilde{\Delta}$ on N parametrized by (r_1, r_2), in the distribution sense. Notice that in the case of $C(S_1^m) = \mathbb{R}^{m+1}$, the Hankel transform is nothing but the Fourier transform in polar coordinates and (0.5) is just the representation of the kernel $f(\Delta)$ via the Fourier inversion formula.

2. Making use of the classical integral formulas of Bessel functions, we can explicitly integrate (0.5) out for certain functions f; e.g., the heat kernel $f(\Delta) = e^{-t\Delta}$ and the Zeta function $f(\Delta) = \Gamma(s)\Delta^{-s}$ (see (5.9)(5.11)).

These explicit expressions, together with the property of conformal homogeneity of the cone, enable us to compute the asymptotic expansion of the trace of the heat kernel on the cone in terms of functions of $\tilde{\Delta}$ on N^m.

3. It follows from Duhamel's principle (see [C] [C3]) that a parametrix for the heat kernel on $X = C_{0,1}(N) \cup M$ can be gotten by gluing together the heat kernel on M with the one on $C_{0,1}(N^m)$. Then, from the behavior of the heat kernel near the singularity $r = 0$, we can conclude that the heat kernel on X is

of trace class. This gives that the Green's operator is compact and hence Frodholm theory can be applied to obtain the standard global results as in the non-singular case; e.g., the existence of a complete orthonormal basis of L^2 consisting of eigenfunctions (-forms) of Δ with discrete eigenvalues $0 \leq \lambda_0 \leq \lambda_1, \ldots, \to \infty$.

4. In order to compute the index of the geometric operator on X, we apply the heat equation method which says that the index is equal to the constant term in the asymptotic expansion of the trace of a certain modified heat kernel on X^{m+1}. As can be easily seen from the previous discussion, this constant term must consist of two separate terms; one is from the manifold M, and the other from the singular part $C_{0,1}(N)$. The first contribution is the integral of the same characteristic form over M as in the non-singular case, by Patodi-Gilkey theorem (see [ABP]). The second contribution from the singularity can be shown to be an Eta-invariant of the manifold N (see [APS]).

In section 5, we carry out the above program for the Dirac operator, and the index formula that we obtain is the following. Suppose that $X^{m+1} \setminus \{p\}$ is an even dimensional spin manifold. Let $\overline{D}_0$ and $\overline{D}$ denote the Dirac operators as in (0.5). Let $\overline{D}_0^+$ and $\overline{D}^+$ be the operators restricted to the (+)-spinors corresponding to the (+)-spin representation. Then

Theorem (5.23).

$$(1)\quad \mathrm{Index}(\overline{D}_0^+) = \int_X \hat{A}(p) + \frac{\eta(0)-h}{2} - \sum_{0<\mu_j<\frac{1}{2}} \dim(E_{\mu_j})$$

$$(2)\quad \mathrm{Index}(\overline{D}^+) = \int_X \hat{A}(p) + \frac{\eta(0)+h}{2} + \sum_{-\frac{1}{2}<\mu_j<0} \dim(E_{\mu_j})$$

where $\hat{A}$ is the Hirzebruch $\hat{A}$-polynomial of Pontrjagin forms $\{P_i\}$, $\eta(s) = \sum_{\mu_j\neq 0} (\mathrm{sign}\,\mu_j)|\mu_j|^{-s}$ the Eta function, $h = \dim \mathrm{Kernel}(\tilde{D})$, and E_{μ_j} is the eigenspace of $\tilde{D}$ with eigenvalue μ_j.

In case that $\overline{D}$ is self-adjoint, i.e., $|\mu_j| \geq \frac{1}{2}$, we have $\mathrm{Index}(\overline{D}_0^+) = \mathrm{Index}(\overline{D}^+) = \int_X \hat{A}(p) + \frac{\eta(0)}{2}$.

Combined with the vanishing theorem (4.3), this gives

Theorem (5.24). Suppose that the scalar curvature κ of $X = C_{0,1}(N)\cup M$ satisfies $\kappa \geq 0$ and $\kappa > 0$ somewhere. Then

$$\mathrm{Index}(\overline{D}^{+}) = 0 = \int_X \hat{A}(p) + \frac{\eta(0)}{2} .$$

Although our index formula is essentially the same as that of Atiyah-Patodi-Singer [APS], we would like to emphasize that it is the natural index formula for spaces with conical singularities. Moreover, it gives a topological obstruction to the existence of the metrics with $\kappa \geq 0$ on these spaces.

We conclude our thesis by noting that all the results we have obtained for spinors immediately generalize to spinors with coefficients in a bundle E, i.e., to sections of the twisted bundle of spinors $S(X) \otimes E$, if the connection of E is flat in a neighborhood of the singularity. The following vanishing theorem is an easy consequence. Set $\mathcal{R}_0(\sigma \otimes e) = \frac{1}{2}\sum_{j,k} (e_j e_k \sigma) \otimes R^E_{e_j e_k} e$ as in (1.11) and (1.12).

<u>Theorem</u> (5.28). Suppose that on $X = C_{0,1}(N) \cup M^{2k}$ $\kappa \geq 4\|\mathcal{R}_0\|$ and $\kappa > 4\|\mathcal{R}_0\|$ somewhere. Then

$$\int_X \hat{\mathrm{Ch}}(E) \cdot \hat{A} = 0$$

where $\hat{\mathrm{Ch}}(E)$ is the reduced Chern character, $\hat{\mathrm{Ch}}(E) = \mathrm{Ch}(E) = \mathrm{Ch}_1(E) + \mathrm{Ch}_2(E) + \cdots\cdots$.

We can also define the notion of enlargeability as in [GL2] and obtain a similar result for singular spaces.

Theorem (5.30). Suppose that in $X = C_{0,1}(N) \cup M^{2k}$, M is of dimension 2k and the interior of M is enlargeable. Then there exists no metric, which is conical near the singularity at the cone tip, with scalar curvature such that $\kappa \geq 0$ and $\kappa > 0$ on the interior of M.

1. PRELIMINARIES

In this section, we shall briefly recall some basic facts about spin manifolds and the Dirac operator. The general references are [ABS][GL1][GL3][LM][M].

An orientable manifold X is called a spin manifold if its second Stiefel-Whitney class $W_2(X)$ is zero. Suppose that X is equipped with a riemannian metric and let $P_{SO_n}(X)$ be the Bundle of oriented orthonormal tangent frames. Let $Spin_n$ denote the spin group, which is the universal 2-fold covering of SO_n for $n \geq 3$. A spin structure on X is a principle $Spin_n$-bundle $P_{Spin_n}(X)$ together with a $Spin_n$-equivariant map $\xi : P_{Spin_n}(X) \to P_{SO_n}(X)$ which commutes with the projection maps onto X. The condition $W_2(X) = 0$ is equivalent to the existence of a spin structure. In fact, using Cech Cohomology, we can easily see that the topological obstruction cocycle to the globalization of the local covering map:

(1.1) $$P_{Spin_n}(X)\Big|_U \cong Spin_n \times U \to P_{SO_n}(X)\Big|_U \cong SO_n \times U,$$

where U is a small neighborhood, is exactly $W_2(X) \in H^2(X,\mathbb{Z})$.

Let Cl_n denote the Clifford algebra of R^n with its standard inner product. In this thesis, we will only consider the complexified Clifford algebra $\mathbb{C}l_n = Cl \otimes_R \mathbb{C}$. The (complex) Spin representation of the Spin group $Spin_n$ is, by definition, the restriction of the algebra representation ρ of $\mathbb{C}l_n$ to $Spin_n \subset \mathbb{C}l_n$. The Spin group $Spin_n$ has only one irreducible representation if n is odd, and two irreducible representations $\Delta^{\pm}$ if n is even. These two irreducible representations $\Delta^{\pm}$ of $Spin_{2k}$ comes from the irreducible representation Δ of $\mathbb{C}l_{2k}$;

(1.2) $\Delta : \mathbb{C}l_{2k} \overset{\cong}{\to} End(\mathbb{C}^{2^k})$: the group of endomorphisms on $\mathbb{C}^{2^k}$.

When restricted to $Spin_{2k}$, Δ breaks into two irreducible ones $\Delta^{\pm}$ corresponding to the $(\pm)$-eigenspace of multiplication by the Volume form $w = i^k e_1 e_2, \ldots, e_{2k}$, where $\{e_1, \ldots, e_{2k}\}$ is the orthonormal basis of R^{2k}.

Suppose now that X is a spin manifold of dimension n and $P_{Spin_n}(X) \to P_{SO_n}(X)$ is a spin structure on X. Then from the spin representation ρ of $Spin_n$, we have the associated (complex) vector bundle

(1.3) $$S(X) = P_{Spin_n}(X) \times_{\rho} V$$

where V is the representation space of ρ. This is called the bundle of (complex) spinors. If $n = 2k$ is even, then (1.3) breaks into two pieces:

(1.4) $$\begin{aligned} S(X) &= P_{Spin_n}(X) \times_{\Delta} V \\ &= P_{Spin_n}(X) \times_{\Delta^+} V^+ \oplus P_{Spin_n}(X) \times_{\Delta^-} V^- \\ &= S^+(X) \oplus S^-(X) \end{aligned}$$

The sections of $S(X)$, $\Gamma(S)$, are called spinors, and the sections of $S^+(S^-)$ are called (+)-spinors ((-)-spinors). Let us denote the associate principle SO-bundle of $S(X)$ by $P_{S(X)}$. Then a local section $e = \{e_1,\ldots,e_n\}$ of $P_{SO_n}(X)$ can be lifted up to $P_{Spin_n}(X)$ via (1.1) and then imbedded into $P_{S(X)}$ as a local section $\phi = \{\phi_1,\ldots,\phi_N\}$. This section ϕ is a local orthonormal basis of the bundle $S(X)$ and will be called the spinor basis.

Let $\mathbb{C}l(X)$ denote the associated bundle of Clifford algebras. This is the bundle over X whose fiber at each point x is the (complex) Clifford algebra of the tangent space $T_x(X)$ with its given metric. This bundle carries a natural unitary connection ∇, induced from the principle SO_n-bundle, and characterized by the fact that ∇ acts as derivation on the algebra of sections: $\Gamma(\mathbb{C}l(X))$, i.e.,

$$\nabla(\alpha\cdot\beta) = (\nabla\alpha)\cdot\beta + \alpha\cdot(\nabla\beta)$$

for all $\alpha,\beta \in \Gamma(\mathbb{C}l(X))$, where "$\cdot$" is the Clifford multiplication.

We can easily see that $S(X)$ is a bundle of modules over $\mathbb{C}l(X)$, i.e., there is a module multiplication

(1.5) "$\cdot$": $\Gamma(\mathbb{C}l) \times \Gamma(S) \to \Gamma(S)$ defined by

$$(\alpha\cdot\phi)(x) = \rho(\alpha(x))(\phi(x))$$

for all $\alpha \in \Gamma(\mathbb{C}l)$ and all $\phi \in \Gamma(S)$, where $\phi(x) \in V_x$ and $\rho(\alpha(x)) \in \mathrm{End}(V_x)$ (see (1.4)).

Lifting the Riemannian connection on $P_{SO_n}(X)$ to $P_{Spin_n}(X)$ via the Lie algebra isomorphism: $(Spin_n)_* \cong (SO_n)_*$ determines an associated connection ∇^S on $S(X)$, whose action on the spinor basis $\phi = \{\phi_1,\ldots,\phi_N\}$ can be described as follows. Let $e = \{e_1,\ldots,e_n\}$ be a local section of $P_{SO_n}(X)$ and ∇^T be the Riemannian connection on the tangent bundle $T(X)$. Suppose that $\{\omega_{ij}\}$ are the 1-form defined by $\nabla^T e_i = \sum_{j=1}^{n} \omega_{ij} e_j$ for $i = 1,2,\ldots,n$.

Then

$$(1.6) \qquad \nabla^S \phi_\ell = \frac{1}{2} \sum_{i<j} \omega_{ij} e_i e_j \cdot \phi_\ell \quad \text{for } \ell = 1,2,\ldots,N,$$

where "," is the module multiplication in (1.5).

It can also be shown that ∇^S acts as derivation w.r.t. the module multiplication, i.e.,

$$(1.7) \qquad \nabla^S(\alpha\cdot\phi) = (\nabla\alpha)\cdot\phi + \alpha\cdot(\nabla^S\phi)$$

for all $\alpha \in \Gamma(\mathbb{C}l)$ and all $\phi \in \Gamma(S)$. From now on, we will drop the superscripts and simply use ∇ to denote various connections if no ambiguity occurs.

The <u>Dirac</u> <u>operator</u> $D: C^\infty(S) \to C^\infty(S)$ is defined by

$$(1.8) \qquad D\phi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\phi$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal basis on X and $\phi \in C^\infty(S)$. This is an invariantly defined first order elliptic differential operator with symbol $\sigma_\xi(D) = \xi\cdot$, for $\xi \in T^*(X)$. Notice that in case of an even dimensional manifold, $D: C^\infty(S^\pm) \to C^\infty(S^\pm)$ and we shall denote $D|_{S^\pm}$ by $D^\pm$.

Under the normal coordinate, we have

$D^2\phi = \sum_{i,j} e_i \cdot e_j \cdot \nabla_{e_i,e_j}\phi$ where $\nabla_{v,w} = \nabla_v\nabla_w - \nabla_{\nabla_v w}$,

and the symbol of D^2 is $\sigma_\xi(D^2) = -\|\xi\|^2$ for $\xi \in T^*(X)$.
Define

$$(1.9) \qquad -\nabla^2\phi = -\sum_{j=1}^{n} \nabla_{e_j,e_j}\phi \text{ for } \phi \in C^\infty(S).$$

Then the following relation holds poinwisely (cf. [LM][Li]).

(1.10) Theorem (Lichnerowicz-Bochner-Weitzenböck formula]

$$D^2\phi = -\nabla^2\phi + \frac{1}{4}\kappa\cdot\phi$$

where κ is the scalar curvature of X.

We can extend above notions to more general classes of bundles. Suppose that E is any hermitian vector bundle over X with a unitary connection ∇^E. Consider the bundle $S(X) \otimes E$ with the canonical tensor product connection ∇. This is again a bundle of modules over X satisfying (1.7). We can still define the (generalized) Dirac operator D and $-\nabla^2$ by using this new connection ∇ as in (1.8) and (1.9). Thus the corresponding formula to (1.10) is

(1.11) $$D^2 = -\nabla^2 + \frac{1}{4}\kappa + R_0$$

where

(1.12) $$R_0(\phi \otimes \sigma) = \frac{1}{2} \sum_{i,j} (e_i \cdot e_j \cdot \phi) \otimes R^E_{e_i, e_j}(\sigma)$$

for all $\phi \in S(X)$ and all $\sigma \in \Gamma(E)$, and R^E denotes the curvature tensor of the connection ∇^E.

If X is a closed manifold, then D is self-adjoint, and $-\nabla^2 = \nabla^* \nabla$ is self-dajoint and non-negative. Thus (1.11) gives (see [GL1])

(1.13) Theorem. If $\kappa > 4\|R_0\|$, where $\| \ \|$ is the operator norm, then Kernel (D) = {0}.

2. THE SEPARATION OF VARIABLES FORMULA AND EIGENSPINORS OF THE DIRAC OPERATOR ON THE CONE

In this section, we are going to derive the separation of variables formula for the Dirac operator D and D^2 on cones. We will also write down the eigenspinors of D and D^2 explicitly, and discuss the domains of closed extensions of them.

Recall that a cone on N, C(N), is a space $(0,\infty) \times N$ with the metric of $dr \otimes dr + r^2\tilde{g}$ where $\tilde{g}$ is the metric on N. We assume that N^m is a closed manifold of dimension m, and C(N) is a spin manifold. Let N be endowed with the induced spin structure from C(N), i.e., the principle $Spin_m$-bundle $P_{Spin_m}(N)$ on N is the reduction of $P_{Spin_{m+1}}(C(N))$ via the inclusion maps on N^m:

$$
(2.1) \qquad \begin{array}{ccc} Spin_{m+1} & \rightarrow & SO_{m+1} \\ \cup & & \cup \\ Spin_m & \rightarrow & SO_m \end{array}
$$

Let $\frac{\partial}{\partial r}$ denote the unit tangent vector to the radial geodesic $(0,\infty) \times \{x\}$ for some $x \in N$. Then the inclusion maps in (1.1) are just "adding $\frac{\partial}{\partial r}$ to the (oriented) orthonormal frames $(\tilde{e}_1,\ldots,\tilde{e}_m)$ on N to form the (oriented) orthonormal frames $(\frac{\partial}{\partial r},\tilde{e}_1,\ldots,\tilde{e}_m)$ on C(N)." This also defines the orientations on C(N). By taking the induced spin representation of $Spin_m$ from $Spin_{m+1}$, the bundle of spinors S(N) over N can be canonically imbedded into the bundle of spinors S(C(N)) over C(N). Let us denote everything intrinsic to N by a tilda "~" and the parallel translation along the radial geodesics by a bar "—". Thus we can imbed a section $\tilde{\sigma}$ of S(N) into S(C(N)) and then extend it to $\bar{\tilde{\sigma}}$ on C(N), with the property that $\nabla_{\frac{\partial}{\partial r}} \bar{\tilde{\sigma}} = 0$.

We now derive the separation of variables formula for a more general kind of metric on $(0,\infty) \times N$: $g = dr \otimes dr + h^2(r)\tilde{g}$ for some $h > 0$.

(2.2) <u>Lemma</u>. Let $\{\tilde{e}_i, i=1,2,\ldots,m\}$ be a local orthonormal basis on N. Then $\{\frac{\partial}{\partial r}, E_i=\bar{\tilde{e}}_i, i=1,2,\ldots,m\}$ is a local orthonormal basis on C(N), and

$$\langle \nabla_{E_i} E_j, E_k \rangle = \frac{1}{h} \langle \tilde{\nabla}_{\tilde{e}_i} \tilde{e}_j, \tilde{e}_k \rangle_{\tilde{g}}$$

$$\langle \nabla_{E_i} E_j, \frac{\partial}{\partial r} \rangle = - \frac{h'}{h} \delta_{ij}$$ where $h' = \frac{\partial}{\partial r} h$,

$$\langle \nabla_{E_i} \frac{\partial}{\partial r}, E_j \rangle = \frac{h'}{h} \delta_{ij}$$ ∇ and $\langle\ ,\ \rangle$ are the connection and inner product of g.

$$\langle \nabla_{E_i} \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = 0$$

Proof: This follows in a straightforward fashion from the following formula for the riemannian connection ∇ (see [CE]):

$$2 \langle \nabla_x y, z \rangle = x\langle y,z \rangle + y\langle x,z \rangle - z\langle x,y \rangle + \langle [x,y],z \rangle - \langle [x,z],y \rangle - \langle [y,z],x \rangle.$$ Q.E.D.

Assume that $\{\sigma_i\}$ is a local spinor basis (see section 1) of S(N). Because S(N) is induced from S(C(N)), $\{\overline{\sigma}_i\}$ is a local spinor basis of S(C(N)), and we have

2.3 Lemma 1) $\nabla_{\frac{\partial}{\partial r}} \overline{\phi} = 0$, $\nabla_{\frac{\partial}{\partial r}} (\frac{\partial}{\partial r} \cdot \overline{\phi}) = 0$ for $\forall$ spinor ϕ on N,

where "." is the module multiplication.

2) $$\nabla_{E_i} \overline{\sigma}_j = \frac{1}{2} \frac{h'}{h} \frac{\partial}{\partial r} \cdot E_i \cdot \overline{\sigma}_j + \frac{1}{h} \overline{\tilde{\nabla}_{\tilde{e}_i} \sigma_j}$$

3) $\nabla_{E_i}(\frac{\partial}{\partial r} \cdot \bar{\sigma}_j) = \frac{1}{2}\frac{h'}{h} E_i \cdot \bar{\sigma}_j + \frac{1}{h}\frac{\partial}{\partial r} \cdot \overline{\tilde{\nabla}_{\tilde{e}_i}\sigma_j}$

Proof: 1) follows from the definition of the parallel translation and (1.7).

2) We assume that $E_0 = \frac{\partial}{\partial r}$. Then by (1.6)

$$\nabla_{E_i}\bar{\sigma}_j = \frac{1}{2}\sum_{\substack{k<\ell \\ k,\ell=0}}^{m} \omega_{k\ell}(E_i)E_kE\,\bar{\sigma}_j$$

$$= \frac{1}{2}\sum_{\ell=1}^{m}\omega_{o\ell}(E_i)E_oE_\ell\bar{\sigma}_j + \frac{1}{2}\sum_{\substack{k=1 \\ k<\ell}}^{m}\omega_{k\ell}(E_i)E_kE_\ell\bar{\sigma}_j$$

$$= \frac{1}{2}\frac{h'}{h}\frac{\partial}{\partial r}E_i\bar{\sigma}_j + \frac{1}{2}\sum_{\substack{k=1 \\ k<\ell}}^{m}\tilde{\omega}_{k\ell}(E_i)E_kE_\ell\bar{\sigma}_j$$

$$= \frac{1}{2}\frac{h'}{h}\frac{\partial}{\partial r}E_i\bar{\sigma}_j + \frac{1}{h}\overline{\tilde{\nabla}_{\tilde{e}_i}\sigma_j},$$

since $\omega_{k\ell}(\tilde{e}_i) = \tilde{\omega}_{k\ell}(\tilde{e}_i)$ for $k,\ell \geq 1$ by Lemma (2.2), where $\omega_{k\ell}$ and $\tilde{\omega}_{k\ell}$ are the connection 1-forms w.r.t $\{E_i\}$ on C(N) and $\{\tilde{e}_i\}$ on N respectively, and $E_i \cdot \bar{\sigma} = \overline{\tilde{e}_i \cdot \sigma}$. By using (1.7), 3) follows similarly. Q.E.D.

Notice that we use the same notation ∇ for different connections whenever no ambiguity occurs.

(2.4) Proposition $\nabla_{E_i}\bar{\phi} = \frac{1}{2}\frac{h'}{h}\frac{\partial}{\partial r}E_i\bar{\phi} + \frac{1}{h}\overline{\tilde{\nabla}_{\tilde{e}_i}\phi}$.

$$\nabla_{E_i}(\frac{\partial}{\partial r}\cdot\bar{\phi}) = \frac{1}{2}\frac{h'}{h}E_i\bar{\phi} + \frac{1}{h}\frac{\partial}{\partial r}\overline{\tilde{\nabla}_{\tilde{e}_i}\phi},$$

for any spinor ϕ on N^m.

Proof: Write $\phi = \Sigma a_i\sigma_i$ where $\{\sigma_i\}$ is a spinor basis of $S(N)$ and apply (2.3). Q.E.D.

The separation of variables formulas for the Dirac operator D and D^2 are the following:

(2.5) Proposition Let $\theta = f(r)\bar{\phi} + g(r)\frac{\partial}{\partial r}\bar{\omega}$, where $\bar{\phi},\bar{\omega}$ are the parallel translations of the spinors ϕ,ω on $N^m = \{1\} \times N^m$ to $C(N^m)$. Then

$$D\theta = (f' + \frac{m}{2}\frac{h'}{h}f)\frac{\partial}{\partial r}\bar{\phi} + \frac{1}{h}f\,\overline{\tilde{D}\phi}$$
$$- (g' + \frac{m}{2}\frac{h'}{h}g)\bar{\omega} - \frac{1}{h}g\frac{\partial}{\partial r}\overline{\tilde{D}\omega} .$$

$$-D^2\theta = \{-f'' - m(\frac{h'}{h})f' - [\frac{m}{2}(\frac{h'}{h})' + (\frac{m}{2}\frac{h'}{h})^2]f\}\bar{\phi}$$
$$+ \{-\frac{h'}{h^2}\}f\frac{\partial}{\partial r}\overline{\tilde{D}\phi} + \{\frac{1}{h^2}f\}\overline{\tilde{D}^2\phi}$$
$$+ \{-g'' - m(\frac{h'}{h})g' - [\frac{m}{2}(\frac{h'}{h})' + (\frac{m}{2}\frac{h'}{h})^2]g\}\frac{\partial}{\partial r}\bar{\omega}$$
$$+ \{-\frac{h'}{h^2}g\}\overline{\tilde{D}\omega} + \{\frac{1}{h^2}g\}\frac{\partial}{\partial r}\overline{\tilde{D}^2\omega},$$

where $\tilde{D}$ is the Dirac operator on N^m.

Proof: This is just a straightforward computation.

$$\theta = f(r)\overline{\phi} + g(r)\frac{\partial}{\partial r}\overline{\omega}$$

$$D\theta = \frac{\partial}{\partial r}\nabla_{\frac{\partial}{\partial r}}\theta + \sum_{i=1}^{m} E_i \cdot \nabla_{E_i}\theta$$

$$= \frac{\partial}{\partial r}\cdot\{f'(r)\overline{\phi}+g'(r)\frac{\partial}{\partial r}\overline{\omega}\} + \sum_{i=1}^{m} E_i \cdot \{f(r)\nabla_{E_i}\overline{\phi}+g(r)\nabla_{E_i}\frac{\partial}{\partial r}\overline{\omega}\}$$

$$= f'(r)\frac{\partial}{\partial r}\overline{\phi} - g'(r)\overline{\omega} + \frac{f(r)}{h}\overline{\tilde{D}\phi} + \frac{m}{2}\frac{h'}{h}\frac{\partial}{\partial r}f(r)\overline{\phi}$$

$$+ \frac{g(r)}{h}(-\frac{\partial}{\partial r})\overline{\tilde{D}\omega} - \frac{m}{2}\frac{h'}{h}g(r)\overline{\omega}$$

$$= [f'(r) + \frac{m}{2}\frac{h'}{h}f(r)]\frac{\partial}{\partial r}\overline{\phi} + \frac{f(r)}{h}\overline{\tilde{D}\phi}$$

$$- [g'(r) + \frac{m}{2}\frac{h'}{h}g(r)]\overline{\omega} - \frac{1}{h}g(r)\frac{\partial}{\partial r}\overline{\tilde{D}\omega}\ .$$

$$D(D\theta) = \frac{\partial}{\partial r}\nabla_{\frac{\partial}{\partial r}}(D\theta) + \sum_{i=1}^{m} E_i \cdot \nabla_{E_i}(D\theta)$$

$$\nabla_{\frac{\partial}{\partial r}}(D\theta) = [f'' + \frac{m}{2}(\frac{h'}{h})'f + \frac{m}{2}\frac{h'}{h}f']\frac{\partial}{\partial r}\overline{\phi} - \frac{h'}{h^2}f\,\overline{\tilde{D}\phi} + \frac{1}{h}f'\overline{\tilde{D}\phi}$$

$$- [g'' + \frac{m}{2}(\frac{h'}{h})'g + \frac{m}{2}\frac{h'}{h}g']\overline{\omega} + \frac{h'}{h^2}g\frac{\partial}{\partial r}\overline{\tilde{D}\omega} - \frac{1}{h}g'\frac{\partial}{\partial r}\overline{\tilde{D}\omega}$$

$$\nabla_{E_i}(D\ ) = (f' + \frac{m}{2}\frac{h'}{h} f)[\frac{1}{2}\frac{h'}{h} E_i\overline{\phi} + \frac{1}{h}\frac{\partial}{\partial r}\overline{\tilde{\nabla}_{e_i}\phi}]$$

$$+ (\frac{1}{h} f)[\frac{1}{2}\frac{h'}{h}\frac{\partial}{\partial r} E_i\ \overline{\tilde{D}\phi} + \frac{1}{h}\overline{\tilde{\nabla}_{e_i}(\tilde{D}\phi)}\ ]$$

$$- (g' + \frac{m}{2}\frac{h'}{h} g)[\frac{1}{2}\frac{h'}{h}\frac{\partial}{\partial r} E_i\overline{\omega} + \frac{1}{h}\overline{\tilde{\nabla}_{e_i}\omega}\ ]$$

$$- (\frac{1}{h} g)[\frac{1}{2}\frac{h'}{h} E_i\overline{\tilde{D}\omega} + \frac{1}{h}\frac{\partial}{\partial r}\overline{\tilde{\nabla}_{e_i}(\tilde{D}\omega)}\ ]$$

Multiplying out and collecting terms, we obtain the desired formula. Q.E.D.

Now we specialize to the metric $h(r) = r$. In order to treat spinors of the type $f(r)\overline{\phi} + g(r)\frac{\partial}{\partial r}\overline{\omega}$, it suffices to consider spinors of the type $f(r)(\overline{\phi}\pm\frac{\partial}{\partial r}\overline{\phi})$, since

$$f\overline{\phi} + g\frac{\partial}{\partial r}\overline{\omega} = \frac{f}{2}(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) + \frac{f}{2}(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi}) + \frac{g}{2}(\overline{\omega} + \frac{\partial}{\partial r}\overline{\omega})$$
$$- \frac{g}{2}(\overline{\omega} - \frac{\partial}{\partial r}\overline{\omega}) \quad .$$

Let ϕ be an eigenspinor of $\tilde{D}$ on N^m with eigenvalue μ, i.e., $\tilde{D}\phi = \mu\phi$, and $\theta = f(r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$ be an eigenspinor of D^2 with eigenvalue $\lambda^2 \neq 0$. Then (2.5) becomes

$$(2.6) \qquad D^2\theta = \{-f'' - \frac{m}{r} f' + [\mu^2 - \mu - \frac{(m^2-2m)}{4}]\frac{1}{r^2} f\}(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$$

$$= \lambda^2 f(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}), \text{ hence}$$

$$(2.7) \qquad + f'' + \frac{m}{r} f' + \{\lambda^2 - [\mu^2 - \mu - \frac{(m^2-2m)}{4}]\frac{1}{r^2}\} f = 0,$$

and the solutions are $f(r) = \gamma^c J_{\pm\nu^+}(\lambda r)$, where $c = \frac{-m+1}{2}$, $\nu^+ = \frac{|2\mu-1|}{2}$, and J_ν is the Bessel function of order ν. (see [In]).

If $\theta = f(r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$ is an eigenspinor of eigenvalue $\lambda^2 \neq 0$, then

$$(2.8) \qquad D^2\theta = \{-f'' - \frac{m}{r} f' + [\mu^2 + \mu - \frac{(m^2-2m)}{4}]\frac{1}{r^2} f\} (\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$$

$$= \lambda^2 f(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi}), \text{ hence}$$

$$(2.9) \qquad + f'' + \frac{m}{r} f' + \{\lambda^2 - [\mu^2 - \mu - \frac{(m^2-2m)}{4}]\frac{1}{r^2}\} f = 0,$$

and the solutions are $f(r) = r^c J_{\pm\nu^-}(\lambda r)$, where $c = \frac{-m+1}{2}$, $\nu^- = \frac{|2\mu+1|}{2}$, and J_ν is as before.

Thus we have the following four types of eigenspinors with eigenvalue $\lambda^2 \neq 0$ of D^2:

$$(2.10) \qquad r^c J_{\nu^+}(\lambda r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$$

$$(2.11) \qquad r^c J_{-\nu^+}(\lambda r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) \qquad \text{where } c = \frac{1-m}{2}$$

(2.12) $\quad r^c J_{\nu^-}(\lambda r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$ and $\nu^{\pm} = \frac{|2\mu \mp 1|}{2}$.

(2.13) $\quad r^c J_{-\nu^-}(\lambda r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$

The corresponding harmonic spinors, i.e., $\lambda^2 = 0$, are the following:

(2.14) $\quad r^{c+\nu^+}(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$

(2.15) $\quad r^{c-\nu^+}(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$

(2.16) $\quad r^{c+\nu^-}(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$

(2.17) $\quad r^{c-\nu^-}(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$

Henceforth we shall call (2.10) (2.12) (2.14) (2.16) positive solutions, and (2.11) (2.13) (2.15) (2.17) negative solutions. If $\nu^{\mp}$ is an integer, we have to introduce logarithmic negative solutions. This can happen only when $\mu = \pm \frac{1}{2}$ and $\nu^{\pm} = 0$, and the negative solutions should be, in stead of (2.11) (2.13),

(2.11)' $\quad r^c Y_0(\lambda r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$

(2.13)' $\quad r^c Y_0(\lambda r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$

where Y_0 is the Bessel function of the second kind of order 0.

Notice that $Y_0(\lambda r) \sim \frac{2}{\pi}\log\frac{\lambda r}{2}$ as $r \to 0$ (see [Le]). Similarly (2.15) and (2.17) should be replaced by

$$(2.15)' \qquad r^c \log r \left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$$

$$(2.17)' \qquad r^c \log r \left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$$

in this case. (2.14)-(2.17) can also be written in the following way:

(2.18)

(2.14)	(a)	$r^{-\frac{m}{2}+\mu}\left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu \geq \frac{1}{2}$
	(b)	$r^{-\frac{m}{2}-\mu+1}\left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu < \frac{1}{2}$
(2.15)	(a)	$r^{-\frac{m}{2}-\mu+1}\left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu > \frac{1}{2}$
	(b)	$r^{-\frac{m}{2}+\mu}\left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu < \frac{1}{2}$
	(c)	$r^c \log r\left(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu = \frac{1}{2}$
(2.16)	(a)	$r^{-\frac{m}{2}+\mu+1}\left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu > -\frac{1}{2}$
	(b)	$r^{-\frac{m}{2}-\mu}\left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu \leq -\frac{1}{2}$
(2.17)	(a)	$r^{-\frac{m}{2}-\mu}\left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu > -\frac{1}{2}$
	(b)	$r^{-\frac{m}{2}+\mu+1}\left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu < -\frac{1}{2}$
	(c)	$r^c \log r\left(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}\right)$	if $\mu = -\frac{1}{2}$

Set $\nu_1 = \frac{2\mu+1}{2}$ and $\nu_2 = \frac{2\mu-1}{2}$.

Then the eigenspinors of D^2 can be classified as follows:

(2.19) (1) $r^{c+\nu_1}(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}) = r^{-\frac{m}{2}+\mu+1}(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$ (I) $r^c J_{\nu_1}(\lambda r)(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$

(2) $r^{c-\nu_1}(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi}) = r^{-\frac{m}{2}-\mu}(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$ (II) $r^c J_{-\nu_1}(\lambda r)(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$

(2)' $r^c \log r(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$ (II)' $r^c Y_0(\lambda r)(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$

(3) $r^{c+\nu_2}(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}) = r^{-\frac{m}{2}+\mu}(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$ (III) $r^c J_{\nu_2}(\lambda r)(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$

(4) $r^{c-\nu_2}(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi}) = r^{-\frac{m}{2}-\mu+1}(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$ (IV) $r^c J_{-\nu_2}(\lambda r)(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$

(4') $r^c \log r(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$ (IV)' $r^c Y_0(\lambda r)(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$

From (2.5), we have

(2.20)
$$D[f(\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})] = [-f' - \frac{(\frac{m}{2}-\mu)}{r} f](\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})$$
$$D[f(\bar{\phi} - \frac{\partial}{\partial r}\bar{\phi})] = [f' + \frac{(\frac{m}{2}+\mu)}{r} f](\bar{\phi} + \frac{\partial}{\partial r}\bar{\phi})$$

Apply this to (2.19), we get

(2.21) $$D(1) = 2\nu_1(3)$$
$$D(2) = 0, \; D(2)' = (3)$$
$$D(3) = 0$$
$$D(4) = 2\nu_2(2), \; D(4)' = -(2)$$

Hence (2)(3) are the only types of spinors in the kernel of D. Also, using the identity (see [Le])

$$J'_\nu(z) - \frac{\nu}{z}J\ (z) = -J_{\nu+1}(z) \text{ and}$$

$$J'_\nu(z) + \frac{\nu}{z}J_\nu(z) = J_{\nu-1}(z), \text{ we have}$$

(2.22) $$D(I) = \lambda(III) \qquad D(III) = \lambda(I)$$
$$D(II) = -\lambda(IV) \qquad D(IV) = -\lambda(II)$$

Thus we have the following two types of eigenspinors of D with eigenvalue $\lambda \neq 0$:

$$(I) + (III): \; r^c J_{\nu_1}(\lambda r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi}) + r^c J_{\nu_2}(\lambda r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$$

$$(I) - (IV): \; r^c J_{-\nu_1}(\lambda r)(\overline{\phi} - \frac{\partial}{\partial r}\overline{\ }) + r^c J_{-\nu_2}(\lambda r)(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) \; .$$

Next we are going to determine those eigenspinors of D^2 which are square-integrable when restricted to the finite cone $C_{0,1}(N)$. Let the domain of D be

(2.23) $\quad \mathrm{dom}(D) = \{\phi \mid \phi \text{ is } C^\infty,\ \phi \in L^2, \text{ and } D\phi \in L^2 \text{ on the finite cone } C_{0,1}(N)\}.$

Since $r^a(\bar{\phi} \pm \frac{\partial}{\partial r}\bar{\phi}) \in L^2(C_{0,1}(N)) \Longleftrightarrow a > \frac{-1-m}{2}$, it follows from (2.21) that in (2.19), we have

(2.24):(2.19) (1), (I) $\in \mathrm{dom}(D) \Longleftrightarrow \mu > -\frac{1}{2}$

(2), (II) $\in \mathrm{dom}(D) \Longleftrightarrow \mu < \frac{1}{2}$

(2)'(II)' $\notin \mathrm{dom}(D)$

(3), (III) $\in \mathrm{dom}(D) \Longleftrightarrow \mu > -\frac{1}{2}$

(4),(IV) $\in \mathrm{dom}(D) \Longleftrightarrow \mu < \frac{1}{2}$

(4)'(IV)' $\notin \mathrm{dom}(D)$

(2.25) Proposition In the four types of eigenspinors of D^2: (2.10)-(2.13) and (2.14)-(2.17), the positive solutions are always in the domain of D. The negative solutions, corresponding to the small eigenvalues μ of $\tilde{D}$ such that $|\mu| < \frac{1}{2}$, are also in the domain of D.

Proof: We need only examine (2.18) using (2.19)(2.14).

(2.15)(b) $\in \mathrm{dom}(D) \Longleftrightarrow -\frac{1}{2} < \mu < \frac{1}{2}$

(2.17)(a) $\in \mathrm{dom}(D) \Longleftrightarrow -\frac{1}{2} < \mu < \frac{1}{2}$. Q.E.D.

We shall see that the negative solutions in the domain of D are exactly the spinors which prevent D from being self-adjoint.

Define D_0 to be the Dirac operator restricted to the spinors with compact support on $X^{m+1} = C_{0,1}(N^m) \cup M$. Let $\overline{D}_0$ and $\overline{D}$ denote the closures of these operators on X. Using the idea of Friedrichs mollifier (see [C2][Fr][Ga]), one can show that

(2.26) $\overline{D}_0^* = \overline{D}$, where "$*$" denote the adjoint.

Therefore $\overline{D}^* = \overline{D}$ is equivalent to $\overline{D} \subset \overline{D}^*$, i.e.,

(2.27) $\langle \overline{D}\alpha, \beta \rangle = \langle \alpha, \overline{D}\beta \rangle$ for all $\alpha, \beta \in \mathrm{dom}(\overline{D})$.

To show that this is not in general true, we first establish the following formula.

(2.28) <u>Proposition</u> (Integration by Parts Formula)

Let Y^m be a compact spin manifold with boundary ∂Y, and $\langle \, , \, \rangle$ denote the inner product on the space of L^2 spinors. Suppose that α and β are smooth spinors. Then

$$\langle D\alpha,\beta\rangle = \langle\alpha,D\beta\rangle - \int_{\partial Y}\langle\alpha,N\cdot\beta\rangle_x$$

where $\langle\ ,\ \rangle_x$ is the pointwise inner product on the fiber at x, and N is the unit outer normal at the boundary.

<u>Proof</u>: $\langle D\alpha,\beta\rangle = \int_Y \langle D\alpha,\beta\rangle_x$

$\langle D\alpha,\beta\rangle_x = \sum_{i=1}^{m}\langle e_i\cdot\nabla_{e_i}\alpha,\beta\rangle_x$ where $\{e_i\}$ is the orthonormal basis of the tangent bundle,

$$= \sum_{i=1}^{m}\langle -\nabla_{e_i}\alpha, e_i\cdot\beta\rangle_x$$

$$= \sum_{i=1}^{m}(e_i\langle -\alpha,e_i\cdot\beta\rangle_x - \langle -\alpha,\nabla_{e_i}(e_i\cdot\beta)\rangle_x)$$

$$= -\sum_{i=1}^{m} e_i\langle -\alpha,e_i\cdot\beta\rangle_x + \sum_{i=1}^{m}\langle\alpha,(\nabla_{e_i}e_i)\cdot\beta\rangle_x + \langle\alpha,D\beta\rangle_x$$

$$= -(\mathrm{div}\ V)_x + \langle\alpha,D\beta\rangle_x$$

where V is the vector field on Y defined by

$$(V,W) = \langle\alpha,W\cdot\beta\rangle \text{ for } \forall \text{ vector field } W,$$

and $\mathrm{div}\ V = \sum_{i=1}^{m}(\nabla_{e_i}V,e_i)$ is the divergence of V.

Hence
$$\langle D\alpha,\beta\rangle = \langle\alpha,D\beta\rangle - \int_Y \operatorname{div} V$$
$$= \langle\alpha,D\beta\rangle - \int_Y (V,N)$$
$$= \langle\alpha,D\beta\rangle - \int_{\partial Y} \langle\alpha,N\cdot\beta\rangle. \qquad \text{Q.E.D.}$$

It follows that on $X = C_{0,1}(N) \cup M$,

$$\langle \overline{D}\alpha,\beta\rangle = \langle\alpha,\overline{D}\beta\rangle$$

if α or β has compact support. In order to show that $\overline{D}$ is self-adjoint, we have to prove that this is true for all $\alpha,\beta \in \operatorname{dom}(\overline{D})$, i.e., the stoke's theorem holds for $\overline{D}$. Unfortunately this is not the case in general because of the negative solutions in the domain of $\overline{D}$. The following example illustrates this situation.

(2.29) Example. Let $X = C_{0,1}(N^m) \cup M$. Choose two negative solutions from (2.18), say,

(2.15)(b) $\theta_1 = r^{-\frac{m}{2}+\mu}(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi})$

(2.17)(a) $\theta_2 = r^{-\frac{m}{2}-\mu}(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$

on $C_{0,1}(N)$ with $-\frac{1}{2} < \mu < \frac{1}{2}$. We can extend them to X by multiplying a cut-off function $f: f(r) \equiv 1$ if $r < 1-\epsilon$ and $f(r) \equiv 0$ if $r \geq 1$ for some small number ϵ. Then $\theta_1, \theta_2 \in \mathrm{dom}(\overline{D})$. Let $X_\epsilon = C_{\epsilon,1}(N) \cup M$, then

$$(2.30) \qquad \langle D\theta_1, \theta_2 \rangle = \int_X \langle D\theta_1, \theta_2 \rangle_x = \lim_{\epsilon \to 0} \int_{X_\epsilon} \langle D\theta_1, \theta_2 \rangle_x$$

$$= \lim_{\epsilon \to 0} \left(\int_{X_\epsilon} \langle \theta_1, D\theta_2 \rangle + \int_{\partial X_\epsilon} \langle \theta_1, N \cdot \theta_2 \rangle \right)$$

$$= \langle \theta_1, D\theta_2 \rangle + \lim_{\epsilon \to 0} \int_{\partial X_\epsilon} \langle \theta_1, N \cdot \theta_2 \rangle \text{ by (2.27).}$$

$$(2.31) \qquad \lim_{\epsilon \to 0} \int_{\partial X_\epsilon} \langle \theta_1, N \cdot \theta_2 \rangle = \lim_{\epsilon \to 0} \int_{\{\epsilon\} x N} \langle \theta_1, (-\tfrac{\partial}{\partial r}) \cdot \theta_2 \rangle$$

$$= -\lim_{\epsilon \to 0} \int_{\{\epsilon\} x_N} 2 \|\overline{\phi}\|^2 \epsilon^{-m} \cdot \epsilon^m dV$$

$\neq 0$, where dV is the volume element on $\{1\} x_N{}^m$.

Thus $\langle D\theta_1, \theta_2 \rangle \neq \langle \theta_1, D\theta_2 \rangle$ on X.

We can conclude that, in general, $\overline{D}_0 \neq \overline{D}$ and the two self-adjoint extensions of $(D_0)^2: \Delta_D = \overline{D}_0^* \overline{D}_0$ and $\Delta_N = \overline{D}^* \overline{D}$ are not the same.

(2.32) Remark. Since $\langle \overline{D}_0\alpha, \beta\rangle = \langle \alpha, \overline{D}\beta\rangle$ for all $\alpha \in \mathrm{dom}(\overline{D}_0)$ and $\beta \in \mathrm{dom}(\overline{D})$, it can be easily shown that the negative solutions are not in the domain of $\overline{D}_0$ by use of the same argument as in the previous example.

3. THE SELF-ADJOINTNESS OF THE DIRAC OPERATOR

In view of (2.29) and (2.32), it is natural to conjecture that $\overline{D}$ (or $\overline{D}_0$) is self-adjoint provided that there is no eigenvalue μ_j of $\tilde{D}$ on N^m such that $|\mu_j| < \frac{1}{2}$. In fact, if we take for granted that the eigenvalues of $\overline{D}^*\overline{D}$ on X are discrete and the usual Fredholm theory holds for $\overline{D}^*\overline{D}$ on X, which will be established in section 5, then it is easy to see that if $|\mu_j| \geq \frac{1}{2}$, then $\langle \overline{D}\alpha, \beta \rangle = \langle \alpha, \overline{D}\beta \rangle$ holds for all eigenspinors α, β of $\overline{D}^*\overline{D}$, and hence for all $\alpha, \beta \in \mathrm{dom}(\overline{D})$. Therefore the conjecture is proved.

In spite of this, we prefer to show directly that the boundary term (2.31) goes to zero as the boundary approaches the cone tip and hence (2.27) holds, if $|\mu_j| \geq \frac{1}{2}$. To this end we first obtain the following pointwise a priori estimate (compare [C2]). The idea is to construct D^{-1} via the separation of variables formula, and then use Schwarz inequality.

(3.1) Proposition. Suppose that $\alpha = f(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) + g(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi}) = \alpha^+ + \alpha^- \in \mathrm{dom}(D)$ (see (2.23)), where ϕ is an eigenspinor of $\tilde{D}$ on N with eigenvalue μ. Let us denote the L^2-norm on $C_{0,1}(N)$ by "$\| \;\|$" and the L^2-norm on N by $\| \;\|_N$. Assume that $\|\phi\|_N = 1$. Then

$$|f(r)| \leq \begin{cases} K\|\alpha^+\|(2r)^{-\frac{m}{2}+\mu} + \frac{1}{2}\|D\alpha^+\| \left|\frac{1-r^{-2\mu+1}}{-2\mu+1}\right|^{\frac{1}{2}} r^{-\frac{m}{2}+\mu} & \text{if } \mu \neq \frac{1}{2}. \\ K\|\alpha^+\|(2r)^{-\frac{m}{2}+\frac{1}{2}} + \frac{1}{2}\|D\alpha^+\| \; |\log r|^{\frac{1}{2}} r^{-\frac{m}{2}+\frac{1}{2}} & \text{if } \mu = \frac{1}{2}. \end{cases}$$

where K is a constant depending only on m. Moreover, if $\mu \leq -\frac{1}{2}$, then $D\alpha^+ \neq 0$, and

$$|f(r)| \leq \frac{1}{2}\|D\alpha^+\| \; r^{-\frac{m}{2}+\frac{1}{2}} / \sqrt{-2\mu+1} .$$

Similarly,

$$|g(r)| \leq \begin{cases} K\|\alpha^-\|(2r)^{-\frac{m}{2}-\mu} + \frac{1}{2}\|D\alpha^-\| \left|\frac{1-r^{2\mu+1}}{2\mu+1}\right|^{\frac{1}{2}} r^{-\frac{m}{2}-\mu} & \text{if } \mu \neq -\frac{1}{2}. \\ K\|\alpha^-\|(2r)^{-\frac{m}{2}+\frac{1}{2}} + \frac{1}{2}\|D\alpha^-\| \; |\log r|^{\frac{1}{2}} r^{-\frac{m}{2}+\frac{1}{2}} & \text{if } \mu = -\frac{1}{2}. \end{cases}$$

If $\mu \geq \frac{1}{2}$, then $D\alpha^- \neq 0$, and

$$|g(r)| \leq \frac{1}{2}\|D\alpha^-\| \, r^{-\frac{m}{2}+\frac{1}{2}}/\sqrt{2\mu+1} .$$

<u>Proof</u>: Notice that, according to (2.19)(2.24), $D\alpha^+ = 0 \Rightarrow \mu > -\frac{1}{2}$, and $D\alpha^- = 0 \Rightarrow \mu < \frac{1}{2}$, since $\alpha \in \mathrm{dom}(D)$. The fact that $\alpha \in \mathrm{dom}(D)$ gives

$$\int_0^1 (f^2+g^2)r^m dr < \infty \qquad \text{and, by (2.20),}$$

$$\int_0^1 \{[f'+(\tfrac{m}{2}-\mu)\tfrac{f}{r}]^2 + [g' + (\tfrac{m}{2}+\mu)\tfrac{g}{r}]^2\}r^m dr < \infty .$$

To obtain estimates for f, we first observe that

$$\frac{\partial}{\partial r}(r^{\frac{m}{2}-\mu}f) = r^{\frac{m}{2}-\mu}[f' + (\tfrac{m}{2}-\mu)\tfrac{f}{r}], \text{ and hence}$$

$$r^{\frac{m}{2}-\mu}f = -\int_r^a s^{\frac{m}{2}-\mu}[f' + (\tfrac{m}{2}-\mu)\tfrac{f}{s}]ds + a^{\frac{m}{2}-\mu}f(a),$$

for any $a \in [\frac{1}{2},1]$.

By the Schwarz inequality with the measure $r^m dr$

$$|r^{\frac{m}{2}-\mu}f(r)| \leq \left(\int_r^1 s^{-m-2\mu}s^m ds\right)^{\frac{1}{2}}\left(\int_r^1 [f'+(\tfrac{m}{2}-\mu)\tfrac{f}{s}]^2 s^m ds\right)^{\frac{1}{2}}$$

$$+ a^{\frac{m}{2}-\mu}|f(a)|$$

$$= a^{\frac{m}{2}-\mu}|f(a)| + \left|\frac{1-r^{-2\mu+1}}{-2\mu+1}\right|^{\frac{1}{2}}(\tfrac{1}{2}\|D\alpha^{+}\|), \quad \text{if } -2\mu+1 \neq 0$$

and $|r^{\frac{m}{2}-\mu} f(r)| \leq a^{\frac{m}{2}-} |f(a)| + |\log r|^{\frac{1}{2}}(\frac{1}{2}\|D\alpha^{+}\|)$ if $-2\mu+1 = 0$.

Using the fact that $\exists\, a \in [\frac{1}{2},1]$ and a constant K depending only on m such that $|f(a)| \leq K\int_0^1 |f|^2 r^m dr$, we obtain the desired inequality.

Similarly the inequality for g follows.

Note that $\int_0^r r^{\frac{m}{2}-\mu}(f' + (\frac{m}{2} - \mu)\frac{f}{r})dr$

$= \int_0^r \frac{\partial}{\partial r}(r^{\frac{m}{2}-\mu} f)dr = r^{\frac{m}{2}-\mu} f(r) - \lim_{r\to 0} r^{\frac{m}{2}-\mu} f(r)$. Since

$\int_0^1 |f|^2 r^m dr < \infty$, we can find $r_i \to 0$ such that

$|f^2(r_i)r_i^m| = o(\frac{1}{r_i})$, i.e., $|f(r_i)| = o\left(\frac{1}{r_i^{\frac{m+1}{2}}}\right)$. (see [C2].)

Therefore, if $\mu < \frac{1}{2}$, then the above limit exists and the integral converges. Moreover,

$$\lim_{r\to 0} r^{\frac{m}{2}-\mu} f(r) = \lim_{r_i\to 0} r_i^{\frac{m}{2}-\mu} f(r_i) = \lim_{r_i\to 0} r_i^{\frac{m}{2}-\mu} o\left(r_i^{-\frac{m+1}{2}}\right) = 0$$

provided that $\mu \leq -\frac{1}{2}$.

Hence we obtain that if $\mu \leq -\frac{1}{2}$,

$$|r^{\frac{m}{2}-\mu} f(r)| = |\int_0^r \frac{\partial}{\partial r}(r^{\frac{m}{2}-\mu} f)|$$

$$= |\int_0^r s^{-\frac{m}{2}-\mu}(f' + (\frac{m}{2}-\mu)\frac{f}{s}) s^m ds|$$

$$\leq |\int_0^r (s^{-m-2\mu}) s^m ds|^{\frac{1}{2}} |\int_0^r [f' + (\frac{m}{2}-\mu)\frac{f}{s}]^2 s^m ds|^{\frac{1}{2}}$$

$$= \left(\frac{r^{-2\mu+1}}{-2\mu+1}\right)^{\frac{1}{2}} \frac{1}{2}||D\alpha^+||, \text{ i.e.,}$$

$$|f(r)| \leq r^{-\frac{m}{2}+\frac{1}{2}}/\sqrt{-2\mu+1} \cdot \frac{1}{2}||D\alpha^+|| .$$

Similarly, we have

$$|g(r)| \leq r^{-\frac{m}{2}+\frac{1}{2}}/\sqrt{-2\mu+1} \cdot \frac{1}{2}||D\alpha^-|| \text{ if } \mu \geq \frac{1}{2}. \quad \text{Q.E.D.}$$

(3.2) Theorem. Suppose that $X^{m+1} = \overline{C_{0,1}(N^m)} \cup M$ is a compact space with a conical singularity. Let D, $\tilde{D}$ denote the Dirac operators on X^{m+1}, N^m respectively, and $\text{dom } D = \{\phi | \phi \in C^\infty \cap L^2, D\phi \in L^2\}$. Then $\overline{D}$ is self-adjoint if and only if there is no eigenvalue of $\tilde{D}$ whose absolute value is strictly less than $\frac{1}{2}$.

<u>Proof</u>. To show that $\overline{D}$ is self-adjoint, it suffices to show that the boundary term in (2.30)(2.31) goes to zero. We need only to check this for spinors of the form $f(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) + g(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi})$ where ϕ is an eigenspinor of $\tilde{D}$ with eigenvalue μ, and $\|\phi\|_N = 1$. Let

$$\theta_1 = f_1(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) + g_1(\overline{\phi} - \frac{\partial}{\partial r}\overline{\phi}) = \theta_1^+ + \theta_1^-$$

$$\theta_2 = f_2(\overline{\phi} + \frac{\partial}{\partial r}\overline{\phi}) + g_2(\overline{\Phi} - \frac{\partial}{\partial r}\overline{\phi}) = \theta_2^+ + \theta_2^-$$

Assume that $\theta_1, \theta_2 \in \mathrm{dom}(D)$ and $|\mu| \geq \frac{1}{2}$. For the boundary term $\int_{\{r\}\times N^m} \langle \theta_1, \frac{\partial}{\partial r} \cdot \theta_2 \rangle = 2(f_2 g_1 - f_1 g_2) r^m$ (see (2.31)), we have two cases:

(i) $\mu \leq -\frac{1}{2}$. Then $|f_1(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$ (since $\mu \leq -\frac{1}{2} \Rightarrow D(\theta^+) \neq 0$)

$$|g_1(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$$

$|f_2(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$ (since $\mu \leq -\frac{1}{2} \Rightarrow D(\theta^+) \neq 0$)

$$|g_2(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$$

(ii) $\mu \geq \frac{1}{2}$. Then $|f_1(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$

$|g_1(r)| \leq c r^{-\frac{m}{2}+\frac{1}{2}}$ (since $\mu \geq \frac{1}{2} \Rightarrow D(\theta_1^-) \neq 0$)

$$|f_2(r)| \le cr^{-\frac{m}{2}+\frac{1}{2}}$$

$$|g_2(r)| \le cr^{-\frac{m}{2}+\frac{1}{2}} \quad (\text{since } \mu \ge \tfrac{1}{2} \Rightarrow D(\theta_2^-)\neq 0)$$

This shows anyhow $|f_1g_2-f_2g_1|r^m \to 0$ as $r \to 0$. Q.E.D.

(3.3) Remark. (1) This theorem says that D_0 is essentially self-adjoint (see [RS]) if and only if $|\mu_j| \ge \frac{1}{2}$ for all eigenvalue μ_j of D on N^m.

(2) If the condition $|\mu_j| \ge \frac{1}{2}$ is not satisfied, we can still obtain self-adjoint extensions of D_0 by imposing conditions on the domain of D, which corresponds to the boundary conditions in case of a manifold with boundary. Notice that in our case the boundary is crashed to a point: the singularity at the cone tip. We therefore call the conditions, which are imposed on the domain of D to make it self-dajoint, the ideal boundary conditions (see [Ga][C1][C2]). For example, suppose that $E = \bigoplus_{|\mu_j|\le\frac{1}{2}} E_{\mu_j}$ where E_{μ_j} is the eigenspace of $\tilde{D}$ with eigenvalue μ_j. Let $\theta|_E = \Sigma\, f_i^+(r)(\bar{\phi}_i + \frac{\partial}{\partial r}\bar{\phi}_i) + f_i^-(\bar{\phi}_i - \frac{\partial}{\partial r}\bar{\phi}_i)$, where ϕ_i's are the eigenspinors of $\tilde{D}$ with eigenvalues μ_j such that $|\mu_j| \le \frac{1}{2}$, denote the projection of θ to the part consisting of eigenspinors in E only. Define D_b to be D restricted to the space:

(3.4) $\{\theta \mid \theta \in L^2$, and $\theta|_E$ satisfies $|f_i^{\pm}| = o(r^{-\frac{m}{2}})\}$.

Then the proof of the previous theorem (3.2) tells us that the boundary term also goes to zero under this ideal boundary condition (3.4), i.e., $\overline{D}_b \subset \overline{D}_b^*$. $\overline{D}_o \subset \overline{D}_b$ implies $\overline{D}_b^* \subset \overline{D}_o^* = \overline{D}$. It follows that $\overline{D}_b = D_b^*$ on the space (3.4).

It is natural to ask for geometric conditions on N^m which guarantees that $|\mu_j| \geq \frac{1}{2}$. In fact, if the scalar curvature $\tilde{\kappa}$ of N^m is positive and $k_o = \min \tilde{\kappa}$, then, using a slight modification of the Lichnerowicz-Bochner-Weitzenböck formula (1.10), we can show the following.

(3.5) <u>Lemma</u>. $|\mu_j| \geq \frac{1}{2}\sqrt{\frac{m}{m-1} k_o}$ where $m = \dim N \geq 2$.

<u>Proof</u>. See [Li] and [F].

Let κ denote the scalar curvature of C(N). It follows from the straightforward computation (see (A.8) in the appendix to section 4) that $\kappa = \frac{1}{r^2}[\tilde{\kappa}-m(m-1)]$. Therefore, if $\kappa \geq 0$, then $\tilde{\kappa} \geq m(m-1)$. It follows from (3.5) that $|\mu| \geq \frac{1}{2}m$, and hence

(3.6) Theorem. Let $X^{m+1} = C_{0,1}(N^m) \cup M$ and $m \geq 2$. Then $\overline{D}$ is self-adjoint if the scalar curvature of $C_{0,1}(N^m)$ is non-negative.

4. VANISHING THEOREMS

We shall prove vanishing theorems for singular spaces analogous to the one proved by Lichnerowicz in [Li] for smooth manifolds. Let us begin with some general facts for arbitrary manifolds.

(4.1) Lemma. Let Y be an arbitrary spin manifold. Assume that the scalar curvature κ of Y satisfies $\kappa \geq -k$ for some positive constant k. Then we have

$$\|D\theta\|^2 \geq \|\nabla\theta\|^2 + \frac{1}{4}\int_Y \kappa\|\theta\|_x^2 \text{ for all } \theta \in \mathrm{dom}(\bar{D}_0)$$

where ∇ is the connection, $\|\ \|$ denotes the global L^2-norm on Y, and $\|\ \|_x$ the pointwise norm at x.

Proof. Integrating the Lichnerowicz formula (1.10)

$$D^2\phi = -\nabla^2\phi + \frac{1}{4}\kappa\phi$$

for smooth ϕ with compact support over Y, we immediately see that

$$\|D\theta\|^2 = \|\nabla\theta\|^2 + \frac{1}{4}\int_Y \kappa\|\theta\|_x^2$$

$$\geq \|\nabla\theta\| - \frac{1}{4}k\|\theta\| \qquad \text{for all } \theta \in \operatorname{dom}(D_0).$$

Now let $\theta \in \operatorname{dom}(\overline{D}_0)$. By definition we can find $\theta_i \in \operatorname{dom}(D_0)$ such that $\theta_i \to \theta$ and $D\theta_i \to D\theta$ as $i \to \infty$, in the L^2 sense. Since $\|D\theta_i\|^2 = \|\nabla\theta_i\|^2 + \frac{1}{4}\int_Y \kappa\|\theta_i\|_x^2$ $\geq \|\nabla\theta_i\|^2 - \frac{1}{4}k\|\theta_i\|$, if we define $\|\phi\|_D = \|D\theta\| + (\frac{1}{4}k+1)\|\theta\|$ and $\|\theta\|_\nabla = \|\nabla\phi\| + \|\phi\|$, then clearly $\| \ \|_D \geq \| \ \|_\nabla$. Therefore θ_i is a cauchy sequence in the norm $\| \ \|_\nabla$ and hence converges to θ' for some θ'. But $\| \ \|_\nabla \geq \| \ \|$, so we must have $\theta' = \theta$, i.e., $\nabla\theta_i \to \nabla\theta$ in the L^2-norm $\| \ \|$. Thus

$$\|D\theta\|^2 = \lim_{i\to\infty} \|D\theta_i\|^2 = \lim_{i\to\infty}\left(\|\nabla\theta_i\|^2 + \frac{1}{4}\int_Y \kappa\|\theta_i\|_x^2\right)$$

$$= \left(\lim_{i\to\infty} \|\nabla\theta_i\|^2\right) + \lim_{i\to\infty} \frac{1}{4}\int_Y \kappa\|\theta_i\|_x^2$$

$$= \|\nabla\theta\|^2 + \frac{1}{4}\lim_{i\to\infty}\int_Y \kappa\|\theta_i\|_x^2.$$

An application of Fatou's lemma yields

$$\|D\theta\|^2 \geq \|\nabla\theta\|^2 + \frac{1}{4}\int_Y \kappa\|\theta\|_x^2 . \qquad \text{Q.E.D.}$$

(4.2) Theorem. Let Y be an arbitrary spin manifold with scalar curvature κ. Assume that $\kappa \geq 0$ and $\kappa > 0$ somewhere. Then

$$\mathrm{Kernel}(\overline{D}_0) = \{0\}$$

Proof. Let $\theta \in \mathrm{kernel}(\overline{D}_0)$. Then (4.1) gives

$$0 = \|D\theta\|^2 \geq \|\nabla\theta\|^2 + \frac{1}{4}\int_Y \kappa \|\theta\|_x^2$$

Since $\kappa > 0$ on an open subset U, we have $\theta \equiv 0$ on U and $\nabla\theta \equiv 0$ on Y. This implies $\theta \equiv 0$ on Y. Q.E.D.

Now let $X^{m+1} = C_{0,1}(N^m) \cup M$. Then $\kappa \geq 0$ guarantees that $\overline{D}$ is self-adjoint and $\overline{D}_0 = \overline{D}$ by (3.6). Thus we have

(4.3) Theorem. Suppose that on X^{m+1} $(m \geq 2)$ $\kappa \geq 0$ and $\kappa > 0$ somewhere. Then $\mathrm{Kernel}(\overline{D}^*\overline{D}) = \mathrm{Kernel}(\overline{D}) = \{0\}$, i.e., there exists no square integrable harmonic spinors on X.

(4.4) Remark. (1) This theorem can also be proved directly by showing that $\langle -\nabla^2\theta, \theta\rangle \geq 0$ for all θ in the kernel of $\overline{D}$ under the condition that $\kappa \geq 0$.

(2) The above theorem can be construed as giving necessary conditions for a manifold with boundary to admit a metric with non-negative scalar curvature for which the metric near the boundary is conical, i.e., like the exterior of a cone. In the appendix to this section we will give a general discussion of manifolds with boundary.

(4.2) and (4.3) can be easily generalized to the generalized Dirac operator on the "twisted" bundle of spinors $S(X) \otimes E$, i.e., on spinors with coefficients in the bundle E (see section 1), if the connection on E is flat in a neighborhood of the singularity at the cone tip {P}. For this situation we only need to replace a spinor by n-tuple of spinors (n=dim E) in our previous calculation and everything else stays the same. Under these assumptions we thus have

(4.5) Theorem. Suppose that on X^{m+1} $(m\geq 2)$, $\kappa > 4\|R_0\|$ then $\overline{D}$ is self-adjoint and $\mathrm{Ker}(\overline{D}) = \{0\}$ for the generalized Dirac operator $\overline{D}$.

Proof. Since E is flat near the cone tip, we can deduce from (1.12) that $R_0 \geq -k$ for some positive k. Therefore the same arguments as in the proofs of (4.1) (4.2) give us the desired result.

Appendix to Section 4. Deformations of the Metric near a Boundary and Positive Scalar Curvature.

§ Introduction

On a manifold with boundary, a differential operator, which is self-adjoint on a closed manifold, is no longer self-adjoint without a suitable boundary condition. In certain situations, we don't even know how to choose boundary conditions which make the problem meaningful. One way of avoiding this difficulty is to trivialize the Geometry near the boundary and treat the space as part of an ambient space formed by attaching a cylinder (or a cone) to the boundary. We then study the operator on the ambient space in which no boundary is present. This is a basic point in the proof of the geometric index theorem for manifolds with boundary (see [APS]).

In this appendix we will study two types of deformation of the metric near the boundary and their influence on the scalar curvature. We then obtain sufficient conditions under which the scalar curvature remains positive after the deformation. It follows from (A.8) that if the cylinder $N \times \mathbb{R}$ has scalar curvature $\kappa > 0$, then we can deform the metric slowly to a cone and keep the scalar curvature positive.

Conversely, if the cone has $\kappa > 0$, then, using the bending technique of Gromov and Lawson (see below and [GL2]), we can open up the cone tip to form a cylinder and maintain positive scalar curvature. Therefore, under this circumstance, attaching a cone to the boundary is equivalent to attaching a cylinder. But our vanishing theorem, when construed as for manifolds with boundary, emphasizes the case that $\kappa \equiv 0$ on the conical neighborhood of the boundary, for which no such deformation can be applied.

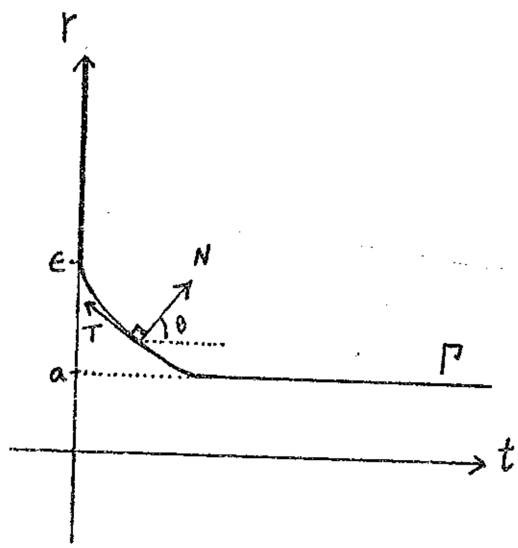
§2. The Bending Technique.

Let M^{m+1} be a manifold with boundary N^m. We can find an ϵ-neighborhood of N that is diffeomorphic to $[0,\epsilon] \times N = N_\epsilon$. Let r denote the geodesir distance to the boundary and $g(r,x)$ be the metric on the hypersurface $\{r\} \times N$ at distance r to the boundary. It is easy to see that the metric on this tubular neighborhood can be written as $dr \otimes dr + g(r,x)$.

The first method of deformation is an adaption of the "bending technique" due to Gromov and Lawson [GL2] for the connected sum of two manifolds. Let us define a hypersurface M' in $M \times \mathbb{R}$ with the product metric by

$M' = \{(y,t) \mid (y,t) \in M \times \mathbb{R}$ and $(\|y\|,t) \in \Gamma\}$

where $\|y\|$ denotes the geodesic distance from y to the boundary N, i.e. $\|y\| = r$, and Γ is the curve in the (r,t) plane as described below.



Γ starts from the r-Axis and finishes with a horizontal line to the t-Axis at $r = a$.

Notice that only the part N_ϵ is bent. The induced metric on M' from $M \times \mathbb{R}$ extends the metric on $M \setminus N_\epsilon$ smoothly and ends with the product metric on $(\{a\} \times N) \times \mathbb{R}$. To study the change of the curvature under this bending, we begin with the following observations. (See [GL2])

(i) In N_ϵ, let $\frac{\partial}{\partial r}$ denote the tangent vector to the geodesic $[0,\epsilon] \times \{x\} = \ell$. Let N be the unit normal to M' and $\overline{\nabla}$ be the connection on $M \times R$. Define the principle curvatures to be the eigenvalues of the operator $\overline{S}(X) = \overline{\nabla}_X N$.

Then $\gamma_\ell = (\ell \times \mathbb{R}) \cap M'$ is a principle curve on M', i.e., the tangent vector T to γ_ℓ is a principle direction of the second fundamental form $\overline{S}$, and the associated principle curvature at a point corresponding to $(r,t) \in \Gamma$ is exactly $-k$, where k is the curvature of Γ at that point.

(ii) Let θ be the angle between N and the t-direction and $\{e_1, \ldots, e_m\}$ be the orthonormal basis of principle vectors for the operator $S(X) = \nabla_X \frac{\partial}{\partial r}$ on the hypersurface $\{r\} \times N$ with principle curvatures λ_i, $i=1,2,\ldots,m$. Then $\{e_1, \ldots, e_m\}$ are the principle vectors for $\overline{S}$ on M' with principle curvatures $\overline{\lambda}_i = \lambda_i \sin\theta$, since

$$N = \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t} \quad \text{and}$$

$$\overline{\nabla}_{e_i} N = \overline{\nabla}_{e_i}(\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t}) = \sin\theta\overline{\nabla}_{e_i}\frac{\partial}{\partial r}$$

$$= \sin\theta\,\nabla_{e_i}\frac{\partial}{\partial r} = (\lambda_i \sin\theta)e_i.$$

(iii) Let K_{ij}, $\overline{K}_{ij}$, and K'_{ij} denote the sectional curvature of M, $M \times \mathbb{R}$, and M' respectively, corresponding to the plane $e_i \wedge e_j$. Set $e_0 = T = \cos\theta(\frac{\partial}{\partial r}) + \sin\theta(\frac{\partial}{\partial t})$. By the Gauss curvature equation,

$$K'_{ij} = \overline{K}_{ij} + \overline{\lambda}_i\overline{\lambda}_j \text{ where } \overline{\lambda}_0 = -k$$

$$\text{and } \overline{\lambda}_j = \lambda_j \sin\theta \text{ for } j \geq 1.$$

Since $M \times \mathbb{R}$ has the product metric, one sees that

$$\overline{K}_{oj} = K_{\frac{\partial}{\partial r},j} \cos^2\theta$$

$$\overline{K}_{ij} = K_{ij} \text{ for } i,j \geq 1.$$

Hence $$K'_{oj} = K_{\frac{\partial}{\partial r},j} \cos^2\theta - k\lambda_j \sin\theta$$

$$K'_{ij} = K_{ij} + \lambda_i\lambda_j \sin^2\theta \text{ for } i,j \geq 1.$$

It follows that the scalar curvature κ' of M' is given by

$$\kappa' = \sum_{\substack{i,j=0\\ i\neq j}}^{m} K'_{ij} = \sum_{\substack{i,j=1\\ i\neq j}}^{m} (K_{ij} + \lambda_i\lambda_j \sin^2\theta)$$

$$+ 2 \sum_{j=1}^{m} K_{\frac{\partial}{\partial r},j} \cos^2\theta - 2k \sum_{j=1}^{m} \lambda_j \sin\theta$$

$$= \kappa - 2 \operatorname{Ric}(\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial r}) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_i\lambda_j \sin^2\theta$$

$$+ 2 \operatorname{Ric}(\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial r}) \cos^2\theta - 2k \sum_{j=1}^{m} \lambda_j \sin\theta$$

$$= \kappa - 2 \operatorname{Ric}(\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial r}) \sin^2\theta + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_i\lambda_j \sin^2\theta$$

$$- 2k \sum_{j=1}^{m} \lambda_j \sin\theta$$

where κ and Ric denote the scalar and Ricci curvatures of M respectively.

Let us denote the scalar curvature of the hypersurface {r} x N in M by κ_r. Using the formula that $\kappa_r = \kappa - 2\,\mathrm{Ric}(\frac{\partial}{\partial r},\frac{\partial}{\partial r}) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_i\lambda_j$, we deduce

$$\text{(A.1)} \qquad \kappa' = \kappa + (\kappa_r - \kappa - \Sigma\lambda_i\lambda_j)\sin^2\theta + \Sigma\lambda_i\lambda_j\sin^2\theta - 2k(\Sigma\lambda_j)\sin\theta$$

$$= \kappa\cos^2\theta + \kappa_r\sin^2\theta - 2k(\mathrm{TrS})\sin\theta$$

where TrS is the trace of S.

It follows that

(A.2) Theorem. Let M^{m+1} be a manifold N^m. Suppose that

(i) $\kappa \geq 0$.

(ii) $\mathrm{TrS} \leq 0$, i.e., the trace of the second fundamental form points inward.

(iii) $\kappa_r \geq 0$ for $r \in [0,\epsilon]$.

Then in this tubular neighborhood $[0,\epsilon]$ x N the metric can be deformed to a complete metric, which ends with the product metric N x $\mathbb{R}$ near infinity, with scalar curvature $\kappa' \geq 0$. Moreover, if $\kappa > 0$, $\mathrm{Tr}S \leq 0$, and $\kappa_r > 0$, then $\kappa' > 0$.

§3. The Deformation Technique.

Let $\bar{g} = dr^2 + g(r,x)$ be the metric near the boundary N^m. We shall consider the deformation of the type $\phi^2(r)dr^2 + f^2(r)g(r,x)$. From now on we assume that $\dim N = m \geq 2$.

First let $g = dr^2 + f^2(r)g(r,x)$ and ∇ denote its connection. Let $\bar{\nabla}$ denote the connection of the original metric $\bar{g}$, $\bar{g} = \langle\,,\rangle_0$, and $g = \langle\,,\rangle$. Now if $\{\frac{\partial}{\partial r}, \bar{e}_i\}$ is an orthonormal basis of $\bar{g}$, then $\{\frac{\partial}{\partial r}, e_i = \bar{e}_i/f\}$ is an orthonormal basis of g. A straightforward computation gives

(2.3) <u>Lemma</u>. (1) $\nabla_{\frac{\partial}{\partial r}} e_j = \frac{1}{f}\bar{\nabla}_{\frac{\partial}{\partial r}}\bar{e}_j = \frac{f'}{f}e_j + \bar{\nabla}_{\frac{\partial}{\partial r}} e_j$

$$(2)\quad \nabla_{e_i} e_j = -\frac{f'}{f}\delta_{ij}\frac{\partial}{\partial r} + (f^2-1)\left\langle\bar{\nabla}_{e_i} e_j, \frac{\partial}{\partial r}\right\rangle\frac{\partial}{\partial r} + \bar{\nabla}_{e_i} e_j$$

$$= -\frac{f'}{f}\delta_{ij}\frac{\partial}{\partial r} - \left(1-\frac{1}{f^2}\right)\left\langle\bar{S}(e_i), e_j\right\rangle\frac{\partial}{\partial r} + \bar{\nabla}_{e_i} e_j$$

where $\bar{S}(e_i) = \bar{\nabla}_{e_i}\frac{\partial}{\partial r}$.

(3) $\nabla_{e_i}\frac{\partial}{\partial r} = \frac{f'}{f} e_i + \bar{\nabla}_{e_i} \frac{\partial}{\partial r}$.

Set $A(\bar{e}_i) = \langle \bar{\nabla}_{\bar{e}_i}\frac{\partial}{\partial r}, \bar{e}_i \rangle_o$. Then $A(\bar{e}_i) = \langle \bar{S}(\bar{e}_i), \bar{e}_i \rangle_o$ $= \langle \bar{S}(e_i), e_i \rangle$, and we have

(A.4) <u>Lemma</u> (1) $\langle \nabla_{e_i}\nabla_{e_j}e_j, e_i \rangle = -(\frac{f'}{f})^2 - \frac{f'}{f}[A(\bar{e}_i)+A(\bar{e}_j)]$

$$+ (\frac{1}{f^2} -1)A(\bar{e}_i)A(\bar{e}_j) + \langle \bar{\nabla}_{e_i}\bar{\nabla}_{e_j}e_j, e_i \rangle$$

(2) $\langle \nabla_{e_j}\nabla_{e_i}e_j, e_i \rangle = (\frac{1}{f^2} -1)(\langle \bar{S}(e_i), e_j \rangle)^2$

$$+ \langle \bar{\nabla}_{e_j}\bar{\nabla}_{e_i}e_j, e_i \rangle$$

(3) $\langle \nabla_{[e_i, e_j]}e_j, e_i \rangle = \langle \bar{\nabla}_{[e_i, e_j]}e_j, e_i \rangle$

Combining these three formulas, we deduce

(A.5) <u>Proposition</u>. The sectional curvatures of $g = dr^2 + f^2(r)g(r,x)$ for the plane sections $e_i \wedge e_j$ are given by

$$K(e_i \wedge e_j) = -(\frac{f'}{f})^2 - (\frac{f'}{f})[A(\bar{e}_i)+A(\bar{e}_j)] + (\frac{1}{f^2} -1)A(\bar{e}_i)A(\bar{e}_j)$$

$$- (\frac{1}{f^2} -1) \langle \bar{S}(e_i), e_j \rangle^2 + \frac{1}{f^2} \bar{K}(\bar{e}_i \wedge \bar{e}_j)$$

(A.6) Lemma. (1) $\langle \nabla_{e_i} \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, e_i \rangle = 0$

(2) $$\langle \nabla_{\frac{\partial}{\partial r}} \nabla_{e_i} \frac{\partial}{\partial r}, e_i \rangle = \frac{f''f-(f')^2}{f^2} + \frac{f'}{f} A(\bar{e}_i) + \langle \bar{\nabla}_{\frac{\partial}{\partial r}} \bar{\nabla}_{e_i} \frac{\partial}{\partial r}, e_i \rangle$$

(3) $$\langle \nabla_{[e_i, \frac{\partial}{\partial r}]} \frac{\partial}{\partial r}, e_i \rangle = (\frac{f'}{f})^2 + \frac{f'}{f} A(\bar{e}_i) + \langle \bar{\nabla}_{[e_i \cdot \frac{\partial}{\partial r}]} \frac{\partial}{\partial r}, e_i \rangle$$

These three formulas give us

(A.7) Proposition. The sectional curvatures of $g = dr^2 + f^2(r)g(r,x)$ corresponding to the plane sections $e_i \wedge \frac{\partial}{\partial r}$ are given by

$$K(e_i \wedge \frac{\partial}{\partial r}) = -\frac{f''}{f} - 2\frac{f'}{f} A(\bar{e}_i) + \bar{K}(\bar{e}_i \wedge \frac{\partial}{\partial r}) .$$

Choose $\{\bar{e}_i\}$ to be an orthonormal set of principle vectors with principle curvatures $\bar{\lambda}_i$, i.e.,

$\bar{S}(\bar{e}_i) = \bar{\nabla}_{\bar{e}_i} \frac{\partial}{\partial r} = \bar{\lambda}_i \bar{e}_i$. Then $A(\bar{e}_i) = \bar{\lambda}_i$ and $\langle \bar{S}(e_i), e_j \rangle = 0$ for $i \neq j$. Therefore we have

$$K(e_i \wedge e_j) = -(\frac{f'}{r})^2 - (\frac{f'}{f})(\bar{\lambda}_i + \bar{\lambda}_j) + (\frac{1}{f^2} - 1)\bar{\lambda}_i \bar{\lambda}_j + \frac{1}{f^2} \bar{K}(\bar{e}_i \wedge \bar{e}_j)$$

$$K(e_i \wedge e_o) = -\frac{f''}{f} - 2\frac{f'}{f}\bar{\lambda}_i + \bar{K}(\bar{e}_i \wedge e_o) \text{ where } e_o = \frac{\partial}{\partial r}.$$

It follows that for $i \geq 1$

$$\begin{aligned} Ric(e_i) = Ric(e_i, e_i) &= \sum_{\substack{j=0 \\ j\neq i}}^{m} K(e_i \wedge e_j) \\ &= -\frac{f''}{f} - (m-1)(\frac{f'}{f})^2 - (\frac{f'}{f})Tr\bar{S} - m(\frac{f'}{f})\bar{\lambda}_i \\ &\quad + (\frac{1}{f^2} - 1)\bar{\lambda}_i (Tr\bar{S} - \bar{\lambda}_i) \\ &\quad + (1 - \frac{1}{f^2})\bar{K}(\frac{\partial}{\partial r} \wedge e_i) + \frac{1}{f^2} \overline{Ric}(\bar{e}_i) \end{aligned}$$

$$Ric(\frac{\partial}{\partial r}) = -m(\frac{f''}{f}) - 2(\frac{f'}{f})Tr\bar{S} + \overline{Ric}(\frac{\partial}{\partial r}).$$

From this and the fact that $\bar{\kappa}_r = \bar{\kappa} - 2\overline{Ric}(\frac{\partial}{\partial r}) + (\sum_{\substack{i\neq j \\ i,j=1}}^{m} \bar{\lambda}_i \bar{\lambda}_j)$, we deduce that

(A.8) <u>Proposition</u>. Suppose that $\bar{g} = dr^2 + g(r,x)$ and $g = dr^2 + f^2(r)g(r,x)$ have scalar curvatures $\bar{\kappa}$ and κ respectively. Then

$$\kappa = -m(m-1)\left(\frac{f'}{f}\right)^2 - 2m\frac{f''}{f} - (2m+2)\frac{f'}{f}\,\mathrm{Tr}\bar{S}$$

$$+ \left(\frac{1}{f^2} - 1\right)\bar{\kappa}_r + \bar{\kappa}$$

where $\mathrm{Tr}\bar{S}$ is the trace of $\bar{S} = \bar{\nabla}\frac{\partial}{\partial r}$, and κ_r is the scalar curvature of the hypersurface $\{r\} \times N$ at distance r to the boundary.

Similar computations give us the following formulus for the deformation $\phi^2(r)dr^2 + g(r,x)$ (see also [Ing]).

(A.9) <u>Proposition</u>. Let $g' = \phi^2(r)dr^2 + g(r,x)$ and K' and Ric' denote the <u>sectional</u> and <u>Ricci</u> curvature of g'. Suppose that $\{\bar{e}_i\}$ is an orthonormal set of principle vectors of $\bar{S} = \bar{\nabla}\frac{\partial}{\partial r}$ with principle curvatures $\bar{\lambda}_i$, and $\frac{\partial}{\partial s} = \frac{1}{\phi}\frac{\partial}{\partial r}$. Then

$$\mathrm{Ric}'\left(\frac{\partial}{\partial s}\right) = \frac{1}{\phi^2}\,\overline{\mathrm{Ric}}\left(\frac{\partial}{\partial r}\right) + \frac{\phi'}{\phi^3}\,\mathrm{Tr}\bar{S}$$

$$\mathrm{Ric}'(\bar{e}_i) = \overline{\mathrm{Ric}}(\bar{e}_i) + \left[\frac{\phi'}{\phi^3} + \left(1-\frac{1}{\phi^2}\right)\mathrm{Tr}\bar{S}\right]\bar{\lambda}_i$$

$$+ \left(\frac{1}{\phi^2} - 1\right)\left[\bar{K}\left(e_i \wedge \frac{\partial}{\partial r}\right) + \bar{\lambda}_i^2\right]$$

$$= \overline{\mathrm{Ric}}(\bar{e}_i) + \frac{\phi'}{\phi^3}\,\bar{\lambda}_i + \left(1-\frac{1}{\phi^2}\right)\left[\overline{\mathrm{Ric}}_r(\bar{e}_i) - \overline{\mathrm{Ric}}(\bar{e}_i)\right]$$

where $\overline{\mathrm{Ric}}_r$ is the Ricci curvature of the hypersurface $\{r\} \times N$ at distance r to the boundary.

(A.10) Proposition. Suppose that $\bar{g} = dr^2 + g(r,x)$ and $g' = \phi^2(r)dr^2 + g(r,x)$ has scalar curvatures $\bar{\kappa}$ and κ' respectively. Then

$$\kappa' = 2\frac{\phi'}{\phi^3}\,\mathrm{Tr}\bar{S} + \frac{1}{\phi^2}\bar{\kappa} + \left(1-\frac{1}{\phi^2}\right)\bar{\kappa}_r$$

where $\bar{\kappa}_r$ is the scalar curvature of $\{r\} \times N$.

Combine (A.8) and (A.10), we have the following formula for the scalar curvature κ' of $g' = \phi^2(r)dr^2 + f^2(r)g(r,x)$ in terms of $\bar{\kappa}$, $\bar{\kappa}_r$ and $\mathrm{Tr}\bar{S}$ of $\bar{g} = dr^2 + g(r,x)$.

$$\text{(A.11)} \quad \kappa' = \frac{-m(m+1)}{\phi^2}\left(\frac{f'}{f}\right)^2 - 2m\frac{f''}{f}\frac{1}{\phi^2} + 2m\frac{\phi'}{\phi^3}\frac{f'}{f}$$

$$+ \mathrm{Tr}\bar{S}\left(\frac{2\phi'}{\phi^3} - \frac{(2m+2)}{\phi^2}\frac{f'}{f}\right)$$

$$+ \left(\frac{1}{f^2} - \frac{1}{\phi^2}\right)\bar{\kappa}_r + \frac{1}{\phi^2}\bar{\kappa}.$$

(A.12) Theorem. Let M be a manifold with boundary N^m and $m = \dim N \geq 2$. Suppose that the tubular neighborhood of N is normalized to be of width 1, i.e., $[0,1] \times N$. If $\bar{\kappa} \geq 0$, $\bar{\kappa} > {}^{16m}/_{m+1}$ on $[0,1] \times N$, and $\bar{\kappa}_r \geq 0$, then the metric can be deformed to a complete metric, which ends with the product metric $N \times \mathbb{R}$ near infinity, with scalar curvature $\kappa' \geq 0$.

(A.13) <u>Remark</u>. (1) For $m \geq 3$, since $m(m+1) \geq \frac{16m}{m+1}$, the condition "$\bar{\kappa} > \frac{16m}{m+1}$" can be replaced by "$\bar{\kappa} \geq$ the scalar curvature of the standard sphere S^{m+1} of the same dimension."

(2) $\kappa' > 0$ provided that $\bar{\kappa} > 0$ and $\bar{\kappa}_r > 0$.

<u>Proof of the theorem</u>. In (A.11), set $\phi = f^{m+1}$.

Then $\frac{2\phi'}{\phi^3} - \frac{(2m+2)}{\phi^2}\frac{f'}{f} = 0$, i.e., the term involving $Tr\bar{S}$ vanishes. Now we want to find a positive smooth function f such that (1) $f(r) \equiv 0$ for $r \geq 1$.

(2) $\lim_{r\to 0} f(r) = \infty$ and $\int_1^r \phi = \int_1^r f^{m+1}$ diverges.

(3) $$\frac{-m(m+1)}{\phi^2}\left(\frac{f'}{f}\right)^2 - 2m\frac{f''}{f}\frac{1}{\phi^2} + 2m\frac{\phi'}{\phi^3}\frac{f'}{f} + \frac{1}{\phi^2}\bar{\kappa} \geq 0.$$

i.e., $\frac{1}{f^{2m+6}}\left(\kappa + (m^2+3m)\left(\frac{f'}{f}\right)^2 - 2m\left(\frac{f''}{f}\right)\right) \geq 0$ for $0 < r \leq 1$.

Let k = the minimum of $\bar{\kappa}$ on $[0,1] \times N$ and $\frac{f'}{f} = g$. Then it suffices to find a function g satisfying

(a) $g(r) \equiv 0$ for $r \geq 1$

(b) $\lim_{r\to 0} g(r) = -\infty$

(c) $k + (m^2+m)g^2 - 2mg' \geq 0$.

The condition (b) will guarantee us a function f increases rapidly when r tends to zero and hence satisfies (2).

Let $g = -\frac{c}{r} + c$ $(c>0)$. Then

$$k + (m^2+m)g^2 - 2mg' = \frac{1}{r^2}\{r^2(k+c^2(m^2+m)) - 2(m^2+m)cr + (m^2+m)c^2 - 2mc\}$$

$$= \frac{1}{r^2} h(r).$$

$h(0) = (m^2+m)c^2 - 2mc \geq 0$ iff $c \geq \frac{2}{m+1}$, and

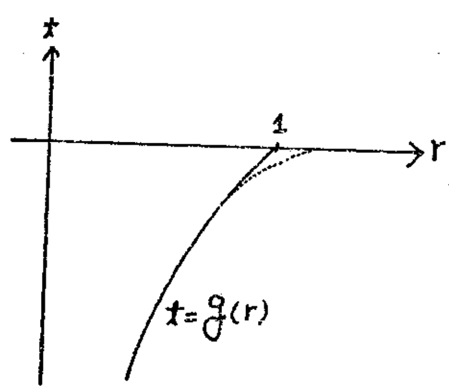
$h(1) = k - 2mc \geq 0$ iff $k \geq 2mc$.

The discriminant of $h(r) \leq 0$ iff $k \geq \frac{2c^2(m^2+m)}{(m+1)c-2} = k_0$.

$k_0 = \frac{16m}{m+1}$ is the smallest number if we choose $c = \frac{4}{m+1}$.

This function $g = -\frac{c}{r} + c$ $(c=\frac{4}{m+1})$ satisfies (a)(b)(c), except that it is not smooth at $r = 1$.

If we assume $k > \frac{16m}{m+1}$, then we can smooth the corner out without affecting the inequality (c).



Thus $f = e^{\int_1^r g} = e^{cr-c}r^{-c}$ $(c=\frac{4}{m+1})$ is the function we are seeking for.

It is easy to see that the trace of $S = \nabla_{\frac{\partial}{\partial s}}$ in this new metric is negative near infinity; therefore, using theorem (A.2), we can again deform the metric near infinity to the product metric on $N \times \mathbb{R}$. Q.E.D.

5. THE INDEX THEOREMS

In this section, we will set up the framework of the heat equation method and derive the index formulus for the Dirac operator. We shall omit the proofs of most of the standard facts here because they are just trivial modification of the proofs given in [C3][CT]. These papers are our main references for this section.

Let $X^{m+1} = C_{0,1}(N^m) \cup M$ be a space with conical singularity. We first assume that the Dirac operator $\overline{D}$ on X is self-adjoint, i.e., the eigenvalues of $\tilde{D}$, the Dirac operator on N, are greater than or equal to $\frac{1}{2}$. Let $\Delta = \overline{D}^*\overline{D} = \overline{D}_0^*\overline{D}_0$ be the Dirac Laplacian. The following facts can be proved by exactly the same arguments as in [C3].

(1) The heat kernel of the Dirac Laplacian $E(u,v,t) = e^{-t\Delta}$ on X is of trace class, i.e.,

$$\int_X E(v,v,t) < \infty \tag{5.1}$$

This implies that the Green's operator is compact; therefore, the Fredholm theory can be applied to show that the eigenvalues of Δ on X are discrete and that the L^2-completeness of the eigenspinors of Δ holds.

(2) Let E_1 and E_2 denote the heat kernel on $C(N)$ and M respectively. Then for all N,

$$(5.2) \qquad \left| \int_{C_{0,1}(N)} E(u,u,t) - \int_{C_{0,1}(N)} E_1(u,u,t) \right| < K_N t^N,$$

and hence

$$\int_X E(v,v,t) = \int_{C_{0,1}(N)} E_1(u,u,t) + \int_M E_2(y,y,t) + 0(t^N).$$

Thus the study of the trace of the heat kernel, $\mathrm{tr}E(t)$, is reduced to studying $\int_{C_{0,1}(N)} E_1(u,u,t)$ where E_1 is the heat kernel of the Dirac Laplacian on the cone $C(N)$. Using the conformal homogeneity of $C(N)$, one can again reduce the explicit calculation of the coefficients of the asymptotic expansion of $\int_{C_{0,1}(N)} E_1(u,u,t)$ to the calculation of the pointwise coefficients of $\mathrm{tr}E_1(t)$ at $r = 1$ and to the calculation of certain global spectral invariant $\psi(N)$ of N, which is the contribution from the singularity at the cone tip $\{P\}$.

(3) Let the pointwise asymptotic expansion of

$E_1((r,x),(r,x),t)$ be $\Sigma a_{j/2}(r,x)t^{-\frac{m+1}{2}+\frac{j}{2}}$ where $(r,x) = u$ is the (polar) coordinate on the cone $C(N)$. Define $a_{j/2}(1) = \int_N a_{j/2}(1,x)$, and $\mu_k(t) = \int_N [trE_1(1,x,t) - \sum_{j=0}^{k} a_{j/2}(1,x)t^{-\frac{m+1}{2}+\frac{j}{2}}]$. Let p.f.$\int_X \bar{a}_{j/2}(v,t)$ denote the finite part of the (divergent) integral $\int_X \bar{a}_{j/2}(v,t) = \lim_{\epsilon\to 0} \int_{X_\epsilon} \bar{a}_{j/2}(v,t)$ where $\bar{a}_{j/2}(v,t)$ are the coefficients of the pointwise asymptotic expansion of $E(v,v,t)$ on X, and $X_\epsilon = X \setminus C_{0,\epsilon}(N)$. (See [C3]), i.e.,

$$\text{p.f.}\int_X \bar{a}_{j/2} = \begin{cases} \int_M \bar{a}_{j/2}(y) + \frac{1}{(m+1)-j}\int_N a_{j/2}(1,x), & \text{if } j\neq m+1. \\ \int_M \bar{a}_{j/2}(y), & \text{if } j=m+1 \end{cases}$$

Then we have

$$(5.3)\quad trE(t) = \int_X E(v,v,t) \sim \sum_{j=0}^{k} (\text{p.f.}\int_X \bar{a}_{j/2})t^{-\frac{m+1}{2}+\frac{j}{2}}$$

$$- \frac{1}{2}a_{\frac{m+1}{2}}(1)\log t + \psi(N),$$

where $\psi(N) = \frac{1}{2}\{\int_{0\ N}^{\infty} t^{-1}\mathrm{tr}E_1(1,x)\,dt + \int_0^1 t^{-1}\mu_k(t)\,dt$

$$+ \sum_{\substack{j\neq m+1\\ j=0}}^{k} a_{j/2}(1)/_{-\frac{m+1}{2}+\frac{j}{2}}\ \}$$

It can be seen that this contribution $2\psi(N)$ is formally the constant term in the Laurent expansion at $s = 0$ of $\Gamma(s)\zeta(s)$ at $r = 1$ if the relation between the heat kernel and the zeta function holds, i.e.,

$$(5.4) \qquad \Gamma(s)\zeta(s) = \int_0^{\infty} t^{s-1}e^{-t\Delta}dt,$$

as in the compact case. Indeed, this relation will follow from the explicit calculation of the functional calculus of the (Dirac) Laplacian Δ on the cone $c(N)$ (see [C3]).

We now outline the theory of the functional calculus of the Laplacian on cones. For details, see [C3] and [CT]. Recall that if g is a smooth function with compact support in $(0,\infty)$, its Hankel transform is defined by

$$H_\nu(g)(\lambda) = \int_0^\infty J_\nu(\lambda r)g(r)rdr$$ where J_ν is the

Bessel function of order ν and $\nu > -1$.

The Hankel inversion formula (see [EMOT] p. 73 and [W]) says that

(5.5) $$g(r) = H_\nu[H_\nu(g)](r) \quad \text{for } \nu > -1.$$

Moreover we have the Plancherel formula (see [Sn])

$$\int_0^\infty |g(r)|^2 rdr = \int_0^\infty |H_\nu(g)(\lambda)|^2 \lambda d\lambda.$$

Thus H can be extended to an isometry from $L^2((0,\infty),rdr)$ to $L^2((0,\infty),\lambda d\lambda)$. Define

$$\Delta_\mu^\pm g = - g'' - \frac{m}{r} g' + [\mu^2 \mp \mu - \frac{(m^2-2m)}{4}]\frac{1}{r^2} g,$$

the ordinary differential operator in (2.6). In view of (2.6) - (2.9), we have

(5.6) $$H_{\nu_j^\pm}(r^{-c}\Delta_{\mu_j}^\pm g) = \lambda^2 H_{\nu_j^\pm}(r^{-c}g)$$

where $c = \frac{1-m}{2}$ and $\nu_j^{\pm} = \frac{|2\mu_j \mp 1|}{2}$. Set $S_j^{\pm} = \frac{1}{\sqrt{2}}(\bar{\phi}_j \pm \frac{\partial}{\partial r} \cdot \bar{\phi}_j)$

where $\tilde{D}\phi_j = \mu_j\phi_j$ and $||\phi_j||_N = 1$. Then the map F defined by

$$(5.7)\quad \theta = \sum_j f_j^+(r)S_j^+ + f_j^-(r)S_j^- \mapsto \sum_j [\lambda^c H_{\nu_j^+}(r^{-c}f_j^+)(\lambda)S_j^+ + \lambda^c H_{\nu_j^-}(r^{-c}f_j^-)(\lambda)S_j^-]$$

provides an isometry of L^2 spinors on the (r,x) cone onto L^2 spinors on the (λ,y) cone such that

$$F(\Delta\theta) = \lambda^2 F(\theta) \quad \text{for all } \theta \in \text{dom}(\Delta)$$

i.e., a spectral representation of Δ while Δ is carried into multiplication by λ^2. Notice that without the assumption that $|\mu_j| \geq \frac{1}{2}$, we must include some $H_{-\nu_j^{\pm}}$ in (5.6)(5.7), corresponding to each self-adjoint extension of $(D_0)^2$. We will discuss this later.

Combining this spectral representation with the Hankel inversion formula, we find that, at least formally, the following relation holds for suitable f.

$$(5.8)\quad f(\Delta) = (r_1 r_2)^c \sum_j \int_0^\infty f(\lambda^2)\,[J_{\nu_j^+}(\lambda r_1) J_{\nu_j^+}(\lambda r_2) S_j^+(x_1) \otimes S_j^+(x_2) + J_{\nu_j^-}(\lambda r_1) J_{\nu_j^-}(\lambda r_2) S_j^-(x_1) \otimes S_j^-(x_2)] \cdot \lambda d\lambda .$$

Notice that in case of $C(S_1^m) = \mathbb{R}^{m+1}$ the Hankel transform is nothing but the Fourier transform in polar coordinates, and (5.8) is just the representation of the kernel $f(\Delta)$ via the Fourier inversion formula.

The right hand side should be interpreted as a convergent sum in the distribution sense. It defines families of functions of the Dirac operator $\tilde{D}$ on N^m, parametrized by r_1, r_2. This crucial observation allows us to bring in the functional calculus of $\tilde{D}$ on N and thereby "sum" the series. We refer to [C3] and [CT] for a rigorous justification of (5.8) and the details of following discussions of kernels.

For the purpose of calculating the coefficients of the asymptotic expansion of the heat kernel, we only need to consider the following two examples of f.

Example 1. (The Heat Kernel $e^{-t\Delta}$, i.e., $f(\lambda) = e^{-t\lambda}$)

Using Weber's second exponential integral ([W] p. 395), we have

$$(5.9)\qquad e^{-t\Delta} = (r_1r_2)^c \sum_j \int_0^\infty e^{-\lambda^2 t}[J_{\nu_j^+}(\lambda r_1)J_{\nu_j^+}(\lambda r_2)S_j^+ \otimes S_j^+$$

$$+ J_{\nu_j^-}(\lambda r_1)J_{\nu_j^-}(\lambda r_2)S_j^- \otimes S_j^-]\lambda d\lambda$$

$$= (r_1r_2)^c \sum_j \frac{1}{2t} e^{-\frac{(r_1^2+r_2^2)}{4t}} [I_{\nu_j^+}(\frac{r_1r_2}{2t})S_j^+ \otimes S_j^-$$

$$+ I_{\nu_j^-}(\frac{r_1r_2}{2t})S_j^- \otimes S_j^-]$$

where I_ν is the modified Bessel function of order ν.

The elliptic estimates, together with the asymptotic expansion of I_ν, show that (5.9) converge uniformly on compact subsets of $R^+ \times C(N) \times C(N)$.

Example 2. (the zeta function $\Gamma(s)\Delta^{-s}$, i.e., $f(\lambda) = \Gamma(s)\lambda^{-s}$)

This is given by the Weber-Schafheitlin integral ([W] p. 401).

$$(5.10)\quad \Gamma(s)\int_0^\infty \lambda^{-2s} J_\nu(\lambda r_1) J_\nu(\lambda r_2)\lambda d\lambda$$

$$= \frac{r_1^\nu r_2^{-\nu+2(s-1)}\Gamma(\nu-s+1)}{2^{2s-1}\Gamma(\nu+1)} F(1-s+\nu, s+1, \nu+1, r_1^2/r_2^2)$$

if $r_1 < r_2$ and $\nu > s-1 > -\frac{1}{2}$,

where F is the Hypergeometric function.

We are primarily interested in the trace of the corresponding kernel at $r_1 = r_2 = 1$. In this case (5.10) is reduced to $\frac{1}{2\sqrt{\pi}} \frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)} \Gamma(s-\frac{1}{2})$ (see [Le] p. 243), and the kernel is given by

$$(5.11)\qquad \Gamma(s)\zeta(s) = \Gamma(s)\Delta^{-s}\Big|_{r_1=r_2=1}$$

$$= \sum_j \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}} \left[\frac{\Gamma(\nu_j^+-s+1)}{\Gamma(\nu_j^++s)} S_j^+ \otimes S_j^+ + \frac{\Gamma(\nu_j^--s+1)}{\Gamma(\nu_j^-+s)} S_j^- \otimes S_j^-\right].$$

It follows from a careful examination of (5.9) and (5.11) that a similar relation to (5.4) holds, and we can compute the coefficients of the asymptotic expansion of the heat kernel in terms of the residues of the zeta function $\Gamma(s)\zeta(s)$ at simple poles; see [C3] for the precise statements and the proofs. Thus the contribution to the constant term from the singularity $\psi(N)$ is indeed the constant term of the Laurent expansion of $\Gamma(s)\zeta(s)$ at $s = 0$, i.e.,

$$\psi(N) = \frac{1}{2}\frac{d}{ds}\left[s\Gamma(s)\zeta(s)\right]\Big|_{s=0}. \tag{5.12}$$

To compute this term out explicitly, we need the following lemma from [C3]. Let B_ℓ denote the ℓ-th Bernoulli number and $C_\ell = (-1)^{\ell-1}\frac{B_\ell}{\ell}$.

(5.13) Lemma (1) $\frac{\Gamma(\nu-s)}{\Gamma(\nu+s)} \sim \nu^{-2s}\left(1+\frac{s}{\nu}+s\sum_{\ell=1}^{\infty} C_\ell \nu^{-2\ell}\right)$

$+ O(s^2)$ as $\nu \to \infty$

(2) $\frac{\Gamma(\nu-s+1)}{\Gamma(\nu+s)} \sim \nu^{1-2s}\left(1+s\sum_{\ell=1}^{\infty} C_\ell \nu^{-2\ell}\right)$

$+ O(s^2)$ as $\nu \to \infty$

Now we are ready to compute the index of D^+ on the even dimensional space $X^{m+1} = C_{0,1}(N^m) \cup M$. Assume that $m+1 = 2k$. Recall that the bundle of spinors S on X splits into $S^+ \oplus S^-$ where S^+ and S^- are the two irreducible bundles corresponding to the two irreducible spin-representations of Spin(2k). Since N^m has the induced spin structure, $S|_N$ splits into two non-isomorphic modules T^+ and T^- over $\mathbb{C}\ell(2k-1)$ corresponding to the (+) and (-) eigenspace of multiplication by the volume form $\omega_{2k-1} = i^k e_1 e_2, \ldots, e_{2k-1}$ on N^m respectively, where $i = \sqrt{-1}$ and $\{e_i\}$ is an orthonormal basis on N. Note that T^+ and T^- are isomorphic irreducible bundles of spinors coming from the two isomorphic spin representations of Spin(2k-1). Let us choose T^+ to be the irreducible bundle of spinors on N^m, i.e., $T^+ = S(N)$ in the notation of section 2. Then the embedding of S(N) into S(C(N)) is given by the inclusion $T^+ \subset S|_N = T^+ \oplus T^-$ and $S|_N = T^+ \oplus \frac{\partial}{\partial r} \cdot T^+$. To fix the orientation, we define the volume form on C(N) to be $\omega_{2k} = i^k \frac{\partial}{\partial r} e_1 e_2, \ldots, e_{2k-1}$. Then it is easy to check that if $\{\phi_i\}$ is a local spinor basis of $S(N) = T^+$, then $\{\frac{1}{\sqrt{2}}(\overline{\phi}_i + \frac{\partial}{\partial r}\, \overline{\phi}_i)\}$ and $\{\frac{1}{\sqrt{2}}(\overline{\phi}_i + \frac{\partial}{\partial r}\, \overline{\phi}_i)\}$ are the local orthonormal bases of S^+ and S^- on the cone $C_{0,1}(N)$ respectively.

Let $\overline{D}^{\pm}, \Delta^{\pm}$ denote $\overline{D}, \Delta$ restricted to $S^{\pm}$. Then the heat equation method of computing the index (see [APS]) gives

$$(5.14)\qquad \begin{aligned}\mathrm{Index}(\bar{D}^{+}) &= \dim \mathrm{Ker}(\bar{D}^{+}) - \dim \mathrm{Ker}(\bar{D}^{-})\\ &= \mathrm{tr}(e^{-t\Delta^{+}}) - \mathrm{tr}(e^{-t\Delta^{-}})\\ &= a_k^{+} - a_k^{-}\\ &= \text{the constant term } a_k \text{ in the}\\ &\quad\ \text{asymptotic expansion of}\\ &\quad\ \mathrm{tr}(\omega_{2k}\cdot e^{-t\Delta})\end{aligned}$$

where ω_{2k} is the volume form on X^{2k} and "$\cdot$" is the module multiplication to the first variable of the heat kernel $e^{-t\Delta} = E(u,v,t)$.

According to (5.3), one easily sees that a_k consists of two terms. The first term is the integral of the same local contribution from the interior as in the smooth case. By a theorem of Gilkey [ABP], this is the integral of the $\hat{A}$-polynomial of Pontrjagin forms P_i over M: $\int_M \hat{A}(P)$, which is equal to $\int_X \hat{A}(P)$ since $\hat{A}\Big|_{C_{0,1}(N)} \equiv 0$ (see [C3]).

We now compute the second contribution $\psi(N)$. From (5.11)(5.12), it follows that

$$(5.15)\qquad \psi(N) = \frac{1}{2}\int_N \frac{d}{ds}\,[s(\Gamma(s)\omega_{2k}\cdot\zeta(s))]\Big|_{s=0}$$

where $\Gamma(s)\omega_{2k}\cdot\zeta(s) = \sum_j \frac{\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}} \left[\frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} S_j^+ \otimes S_j^+\right.$

$$\left. - \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)} S_j^- \otimes S_j^-\right]$$

where $\nu_j^{\pm} = \frac{|2\mu_j \mp 1|}{2}$.

Using (5.13) and the assumption that $|\mu_j| \geq \frac{1}{2}$, we obtain

$$(5.16) \quad \psi(N) = -\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} s\cdot\{\Sigma(\nu_j^{+\,1-2s} + s\sum_{\ell=1}^{\infty} C_\ell \nu_j^{+\,-2\ell-2s+1})$$

$$- \Sigma(\nu_j^{-\,1-2s} + s\sum_{\ell=1}^{\infty} C_\ell \nu_j^{-\,-2\ell-2s+1})\}$$

$$= -\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} s\cdot\{\sum_{\mu_j\geq\frac{1}{2}} [(\mu_j-\tfrac{1}{2})^{1-2s} + s\sum_{\ell=1}^{\infty} C_\ell (\mu_j-\tfrac{1}{2})^{-2\ell-2s+1}]$$

$$+ \sum_{\mu_j\leq-\frac{1}{2}} [(\tfrac{1}{2}-\mu_j)^{1-2s} + s\sum_{\ell=1}^{\infty} C_\ell (\tfrac{1}{2}-\mu_j)^{-2\ell-2s+1}]$$

$$- \sum_{\mu_j\geq\frac{1}{2}} [(\mu_j+\tfrac{1}{2})^{1-2s} + s\sum_{\ell=1}^{\infty} C_\ell (\mu_j+\tfrac{1}{2})^{-2\ell-2s+1}]$$

$$- \sum_{\mu_j\leq-\frac{1}{2}} [(-\mu_j-\tfrac{1}{2})^{1-2s} + s\sum_{\ell=1}^{\infty} C_\ell (-\mu_j-\tfrac{1}{2})^{-2\ell-2s+1}]$$

An application of the binomial expansion gives

(5.17) $$(\mu-\tfrac{1}{2})^{1-2s} - (\mu+\tfrac{1}{2})^{1-2s} \sim -(1-2s)\mu^{-2s} + \sum_{k=1}^{\infty} a_k(s)s\mu^{-2s-2k}$$

where $a_k(s)$ is a polynomial in s.

Therefore $$\psi(N) = \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\{\sum_{\mu_j\geq\frac{1}{2}}[s(1-2s)\mu_j^{-2s}] - \sum_{\mu_j\leq-\frac{1}{2}}[s(1-2s)\mu_j^{-2s}]\}$$

$$= \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}[s\eta(s)] = \frac{\eta(0)}{2},$$ where

$$\eta(s) = \sum_{\mu_j\neq 0}(\text{sign }\mu_j)|\mu_j|^{-s}.$$

This is because $\eta(s)$ is holomorphic for $\mathrm{Re}(s) > -\frac{1}{2}$; see [APS] and [C3].

Thus we can summarize what we have obtained as follows.

(5.18) Theorem. Let $X^{2k} = C_{0,1}(N) \cup M$ be a space with a conical singularity. Assume that $|\mu_j| \geq \frac{1}{2}$ where $\{\mu_j\}$ are the eigenvalues of $\tilde{D}$ on N; i.e., $\overline{D}$ is self-adjoint. Then

$$\mathrm{Index}(\overline{D}^+) = \int_X \hat{A}(p) + \frac{\eta(0)}{2}$$

where $\hat{A}$ is the Hirzebruch $\hat{A}$-polynomial of Pontrjagin forms $P = \{P_i\}$.

Although this index formula is equivalent to the one obtained by Atiyah, Patodi, and Singer [APS] for manifolds with boundary, we wish to emphasize that (5.18) is the natural index formula for a class of compact singular spaces which are in themselves interesting geometric objects (compare [C1][C3]).

We now treat the general case without assuming $|\mu_j| \geq \frac{1}{2}$. Note that we already had three closed extensions of $D_0 : \overline{D}_0, \overline{D}_b$, and $\overline{D}$, where $\overline{D}_b$ is the self-adjoint extension under the ideal boundary condition (3.4). Before computing the indices of these operators, we first examine the domains of $\overline{D}_0^*\overline{D}_0$, $\overline{D}_b^*\overline{D}_b$, and $\overline{D}^*\overline{D}$.

(5.19) (1) The domain of $\overline{D}_0^*\overline{D}_0 = \Delta_D$ contains all the positive solutions (2.10)(2.11)(2.14)(2.16) since $\langle \overline{D}^*\alpha, \phi \rangle = \langle \alpha, \overline{D}\phi \rangle$ for all $\phi \in \mathrm{dom}(\overline{D})$ and all positive solutions α; moreover, no negative solution is in the $\mathrm{dom}(\overline{D}_0)$ or the $\mathrm{dom}(\Delta_D)$ by (2.32).

(2) For $\overline{D}_b^*\overline{D}_b = \Delta_b$, it follows from (2.19)(2.22) that the negative solution $r^{\frac{-m+1}{2}} J_{-\nu_j^+}$ corresponding

to $0 < \mu_j < \frac{1}{2}$ and $r^{\frac{-m+1}{2}} J_{-\nu_j^-}$ corresponding to $-\frac{1}{2} < \mu_j < 0$ are in the $\mathrm{dom}(\Delta_b)$, but the positive solutions $r^{\frac{-m+1}{2}} J_{\nu_j^+}$ for $0 \leq \mu_j < \frac{1}{2}$ and $r^{\frac{-m+1}{2}} J_{\nu_j^-}$ for $-\frac{1}{2} < \mu_j \leq 0$ are not.

(3) For $\overline{D}^*\overline{D} = \Delta_N$, it also follows from (2.19)(2.22) that the negative solutions $r^{\frac{-m+1}{2}} J_{-\nu_j^+}$ and $r^{\frac{-m+1}{2}} J_{-\nu_j^-}$ for $|\mu_j| < \frac{1}{2}$ are in the $\mathrm{dom}(\Delta_N)$, but the positive solutions $r^{\frac{-m+1}{2}} J_{\nu_j^+}$ and $r^{\frac{-m+1}{2}} J_{\nu_j^-}$ for $|\mu_j| < \frac{1}{2}$ are not.

For the functional calculus of the operators Δ_D, Δ_b, and Δ_N, we must include or exclude $J_{\pm\nu_j^\pm}$ in (5.7) - (5.11) according to (5.19). Applying heat equation method to compute the indices as in (5.14), we have

(5.20) $\quad \mathrm{Index}(\overline{D}_0^+) = \mathrm{tr}(e^{-t\Delta_D^+}) - \mathrm{tr}(e^{-t\Delta_N^-}).$

(5.21) $\quad \mathrm{Index}(\overline{D}_b^+) = \mathrm{tr}(e^{-t\Delta_b^+}) - \mathrm{tr}(e^{-t\Delta_b^-}).$

$$(5.22) \qquad \mathrm{Index}(\overline{D}^+) = \mathrm{tr}(e^{-t\Delta_N^+}) - \mathrm{tr}(e^{-t\Delta_D^-}).$$

Simple calculations similar to (5.15) - (5.17) with appropriate $\pm\nu_j^\pm$ for $|\mu_j| < \frac{1}{2}$ give us the following theorem.

(5.23) Theorem.

(1) $$\mathrm{Index}(\overline{D}_0^+) = \int_X \hat{A}(p) + \frac{\eta(0)-h}{2} - \sum_{0<\mu_j<\frac{1}{2}} \dim(E_{\mu_j})$$

where $h = \dim \mathrm{Ker}(\tilde{D})$, and E_{μ_j} is the eigenspace of $\tilde{D}$ with eigenvalue μ_j.

(2) $$\mathrm{Index}(\overline{D}_b^+) = \int_X \hat{A}(p) + \frac{\eta(0)}{2}$$

(3) $$\mathrm{Index}(\overline{D}^+) = \int_X \hat{A}(p) + \frac{\eta(0)+h}{2} + \sum_{-\frac{1}{2}<\mu_j<0} \dim(E_{\mu_j}).$$

Proof. (1) $\psi(N) = -\frac{1}{2} \frac{d}{ds}\Big|_{s=0} s.$

$$\left\{\sum_j \frac{\Gamma(\nu_j^+-s+1)}{\Gamma(\nu_j^++s)} - \left[\sum_{|\mu_j|<\frac{1}{2}} \frac{\Gamma(-\nu_j^--s+1)}{\Gamma(-\nu_j^-+s)} + \sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^--s+1)}{\Gamma(\nu_j^-+s)}\right]\right\}$$

$$= \frac{-1}{2} \frac{d}{ds}\Big|_{s=0} s.\{[\sum_{|\mu_j|<\frac{1}{2}} \frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} - \sum_{|\mu_j|<\frac{1}{2}} \frac{\Gamma(-\nu_j^- - s+1)}{\Gamma(-\nu_j^- + s)}]$$

$$+ [\sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} - \sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)}]\}$$

$$= \frac{\eta(0)-h}{2} - \sum_{0<\mu_j<\frac{1}{2}} \dim(E_{\mu_j})$$

(2) $\psi(N) = - \frac{1}{2} \frac{d}{ds}\Big|_{s=0} s.$

$$\{\sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} + \sum_{-\frac{1}{2}<\mu_j<0} \frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} + \sum_{0<\mu_j<\frac{1}{2}} \frac{\Gamma(-\nu_j^+ - s+1)}{\Gamma(-\nu_j^+ + s)}$$

$$- \sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)} - \sum_{0<\mu_j<\frac{1}{2}} \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)} - \sum_{-\frac{1}{2}<\mu_j<0} \frac{\Gamma(-\nu_j^- - s+1)}{\Gamma(-\nu_j^- + s)}$$

$$= \frac{1}{2}\eta(0)$$

$$(3)\quad \psi(N) = -\frac{1}{2}\frac{d}{ds}\Big|_{s=0} S\cdot$$

$$\Big\{ \sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^+ - s+1)}{\Gamma(\nu_j^+ + s)} + \sum_{|\mu_j|<\frac{1}{2}} \frac{\Gamma(-\nu_j^+ - s+1)}{\Gamma(-\nu_j^+ + s)}$$

$$- \sum_{|\mu_j|\geq\frac{1}{2}} \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)} - \sum_{|\mu_j|<\frac{1}{2}} \frac{\Gamma(\nu_j^- - s+1)}{\Gamma(\nu_j^- + s)} \Big\}$$

$$= \frac{\eta(0)+h}{2} + \sum_{-\frac{1}{2}<\mu_j<0} \dim(E_{\mu_j}). \text{ Q.E.D.}$$

Note that

$$\mathrm{Index}(\overline{D}^+) - \mathrm{Index}(\overline{D}_0^+)$$

$$= \sum_{|\mu_j|<\frac{1}{2}} \dim(E_{\mu_j})$$

is also a trivial consequence of (2.21) and the above (5.19)(1).

Combining (4.3) with (5.18), we thus have

(5.24) <u>Theorem</u>. Suppose that $\kappa \geq 0$ and $\kappa > 0$ somewhere. Then

$$\text{Index}(\overline{D}^{+}) = 0 = \int_X \hat{A}(p) + \frac{\eta(0)}{2}$$

(5.25) Remark. If we consider spinors with coefficients in a bundle E^k which is (locally) flat in a neighborhood of the singularity at the cone tip, then from [A5][ABP], it is easy to see that the index formula becomes

$$\text{Index}(\overline{D}^{+}) = \int_X \text{Ch}(E)\cdot\hat{A} + \frac{\eta(0)}{2} \text{ if } |\mu_j| \geq \frac{1}{2},$$

where Ch(E) is the Chern character of E; $\text{Ch}(E) = k + \text{Ch}_1(E) + \text{Ch}_2(E) + \cdots$. Notice that here both $\eta(0)$ and $\{\mu_j\}$ belong to the Dirac operator $\tilde{D}$ on spinors with coefficients in the bundle $E|_N$.

We now consider the relative index of generalized Dirac operators on spinor with coefficients in a hermitian bundle E^k, which is trivial and flat in a neighborhood of $\{p\}$; i.e. $E^k|_{C_{0,u}(N)} \cong \mathbb{C}^k \times C_{0,u}(N)$ and the curvature R^E satisfies $R^E|_{C_{0,u}(N)} \equiv 0$.

Let D_1 and D_2 denote the (generalized) Dirac operator on sections of the Bundles $S(X) \otimes E$ and $S(X) \otimes C^k$ respectively. Define the relative index of $\overline{D}_1$ and $\overline{D}_2$ (see [GL3]) to be

$$\text{(5.26)} \qquad \mathrm{Ind}(\overline{D}_1^+, \overline{D}_2^+) = \mathrm{Index}(\overline{D}_1^+) - \mathrm{Index}(\overline{D}_2^+).$$

Similarly,

$$\mathrm{Ind}(\overline{D}_{1,0}^+, \overline{D}_{2,0}^+) = \mathrm{Index}(\overline{D}_{1,0}^+) - \mathrm{Index}(\overline{D}_{2,0}^+) .$$

Then it follows from (5.23) and (5.25) that

$$\text{(5.27)} \qquad \mathrm{Ind}(\overline{D}_1^+, \overline{D}_2^+) = \mathrm{Ind}(\overline{D}_{1,0}^+, \overline{D}_{2,0}^+) = \int_X \hat{Ch}(E)\cdot\hat{A},$$

where $\hat{Ch}(E) = Ch_1(E) + Ch_2(E) + \cdots = Ch(E) - k$, the reduced Chern character.

The notion of relative index was first exploited by Gromov and Lawson in [GL3] for complete manifolds with uniformly positive curvature condition at infinity. Our notion (5.26) is called the <u>analytical</u> relative index in [GL3] and (5.27) actually shows that the analytical rel. index is equal to the topological rel. index (see [GL3]) by way of the (absolute) index formulas (5.23) in our case. Notice that we haven't assumed any positive curvature condition near the cone tip yet.

Now, in view of (4.5), we can deduce the following result from (5.27). Let $\sigma \in \Gamma(S(X))$ and $\sigma \in \Gamma(E)$. Put $\mathcal{R}_0(\sigma \otimes e) = \frac{1}{2} \sum_{j,k} (e_j e_k \sigma) \otimes R^E_{e_j e_k}(e)$ as in (1.11) and (1.12).

(5.28) Theorem. Suppose that on $X = C_{0,1}(N) \cup M^{2k}$

$\kappa \geq 4\| \mathcal{R}_0 \|$ and $\kappa > 4\| \mathcal{R}_0 \|$ somewhere.

Then

$$\int_X \hat{Ch}(E) \cdot \hat{A} = 0$$

This serves as a topological obstruction to the existence of such a metric. Notice that $\hat{Ch}(E)\Big|_{C_{0,1}(N)} \equiv 0$ since $R^E\Big|_{C_{0,u}(N)} \equiv 0$.

We can also define the notion of enlargeability and obtain similar results to those in [GL2].

(5.29) Definition. A riemannian n-manifold X^n (possibly incomplete) is said to be enlargeable if for $\forall\ \epsilon > 0$, there exists a finite spin covering manifold $\tilde{X} \to X$ and ϵ-contracting map $\tilde{X} \to S^n$, which is constant outside a compact subset and is of non-zero degree.

Here the assumption on the finiteness of the covering is essential for preserving the type of singularities after passing to the covering space. One easily sees that the same untwisting trick of Gromov and Lawson [GL2] can be applied to prove the following theorem.

(5.30) Theorem. Suppose that in $X = C_{0,1}(N) \cup M^{2k}$, M is even dimensional and the interior of M is enlargeable. Then there exists no metric, which is conical near the singularity at the cone tip, with scalar curvature κ such that $\kappa \geq 0$ and $\kappa > 0$ on the interior of M.

[Proof] We proceed exactly as in the proof of Theorem 3.1 in [GL2]. Assume that κ satisfies $\kappa \geq 0$ and $\kappa > 0$ and Y, the interior of M. Since Y is enlargeable, for any $\epsilon > 0$ there exist a finite spin cover $\tilde{Y} \to Y$ and an ϵ-contracting map of non-zero degree $f:\tilde{Y} \to S^{2k}$ which is constant outside a compact subset $Z \subset \tilde{Y}$. Choose a complex vector bundle E_0 over S^{2k} such that $C_k(E_0) \neq 0$ where C_k is the top dimensional Chern class. Let $\tilde{X}$ denote the space formed by attaching cones to $\tilde{Y}$ along it boundary. Notice that the metric near the boundary of $\tilde{Y}$ can be smoothly extended to an attached cone because $\tilde{Y}$ is a finite cover of Y and has the induced metric from Y. Then the map f can be extended to $\tilde{X}$ by sending the attached cones to the same constant point. Let $f^*E_0 = E$ be the pull back of E_0

by f with the induced connection. If ϵ is small enough, then $\kappa > 4\|R_0\|$ on Z and hence $\kappa \geq 4\|R_0\|$ on the whole $\tilde{X}$. It follows from (5.28) that

$$0 = \int_{\tilde{X}} \hat{Ch}(E) \cdot \hat{A} = \frac{1}{(m-1)!} \int_{\tilde{X}} f^* C_k(E_0)$$

$$= \frac{1}{(m-1)!} \deg(f) \int_{S^{2k}} C_k(E_0) \neq 0$$

This is a contradiction. Q.E.D.

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