Extension Aspects and New Examples of Positively Ricci Curved Manifolds

A Dissertation presented

by

Paul Mason Ingram, Jr.

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Paul Mason Ingram, Jr.

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We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

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The dissertation is accepted by the Graduate School.

Dean of the Graduate School

# Abstract of the Dissertation


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Positively curved compact Riemannian manifolds with boundary are studied from the viewpoint of extending the metric to a complete metric with positive Ricci curvature. Sufficient conditions on the boundary are given for manifolds bounded by a closed hypersurface in a sphere and other simple spaces.

Also, a large new class of compact manifolds with positive Ricci curvature is constructed, in a manner pertinent to the above extension problem.

To my parents.

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# INTRODUCTION

In the global structure theory for complete Riemannian manifolds with positive curvature, it is a basic question whether or not the condition of Ricci curvature Ric $\geq 0$ is less restrictive than the condition of sectional curvature $K \geq 0$. More precisely, does there exist a manifold M that admits a complete metric with Ric $\geq 0$, but no complete metric with $K \geq 0$? It is now generally expected that the answer is yes, but no example whatsoever is known up to now, for M compact or not. There are various reasons why such examples should be ample, which we will not discuss here in detail. We just mention that numerous interesting spaces even with Ric $> 0$ have been constructed more recently, for example among Brieskorn varieties (compact case, [H]), and among certain fiber bundles (compact and non-compact cases), whereas there are only comparatively few spaces known with $K \geq 0$. It should also be pointed out that certain manifolds with Ric $> 0$ do not admit $K > 0$, by fairly simple arguments involving the fundamental group.

Our work was primarily directed toward the construction of non-compact examples as described above. The only general result for complete manifolds with $K \geq 0$ is that necessarily they must be diffeomorphic to vector bundles over compact manifolds. We have been trying to construct complete non-compact manifolds with Ric > 0 which are topologically not vector bundles over a compact manifold and thus do not admit $K \geq 0$. We have obtained an abundance of likely candidates where the problem is reduced to the verification of a purely numerical condition that can be easily satisfied in some non-trivial cases (which are still bundles, however).

We feel it is only because of the somewhat complicated nature of this condition that we have no explicit examples as yet. In the course of our investigation we have also found a very large new class of compact manifolds with strictly positive Ricci curvature.

The main idea in our approach to the construction of complete manifolds with Ric > 0 is to modify the metric of a compact positively curved manifold $\overline{M}$ with boundary $\partial\overline{M}$ in the interior M near the boundary. This is essentially equivalent to the problem of extending $\overline{M}$ to a complete Riemannian manifold with Ric > 0, which is an interesting question in its own right. Of course the shape of the boundary will play a dominant role. The analogous problem for positive sectional curvature has a solution whenever $\partial\overline{M}$ is convex [K], but convexity would be a much too restrictive boundary condition in our more subtle situation. We will give a sufficient condition on $\partial\overline{M}$ (at least in case the interior metric is reasonably simple) under which the above extension problem has a solution. In particular, the mean curvature of the boundary, and more surprisingly the Ricci curvature of the boundary in the induced metric should be positive.

## 1. Deformations of Ricci curvature near a boundary

For basic facts in Riemannian geometry, we refer to [CE], [GKM], [KN].

In this chapter, $\overline{M}^{n+1}$ is a compact connected Riemannian manifold with smooth boundary $\partial\overline{M}$ and metric $\overline{g}$. The interior of $\overline{M}$ will be denoted by M.

The process referred to above for extending a compact manifold with boundary to a complete manifold with constraints on the Ricci curvature is based on a fairly natural deformation of the metric $\overline{g}$ in M near $\partial\overline{M}$, which was also used in [K], but turns out to be more delicate in our situation. The metric $\overline{g}$ induces a canonical function on M, the metric distance to the boundary $\partial\overline{M}$, which we denote by $t : M \to \mathbb{R}$. This function is smooth in $\mathcal{T}_\epsilon$, the one-sided tubular neighborhood of $\partial\overline{M}$ of radius $\epsilon$, for $\epsilon > 0$ sufficiently small. The change of metric occurs entirely in $\mathcal{T}_\epsilon$ and our attention will be restricted to this tube from now on.

The new metric g can be visualized as the metric induced on the graph of a suitable "warping" function $f : M \to \mathbb{R}$ from the product metric in $M \times \mathbb{R}$.

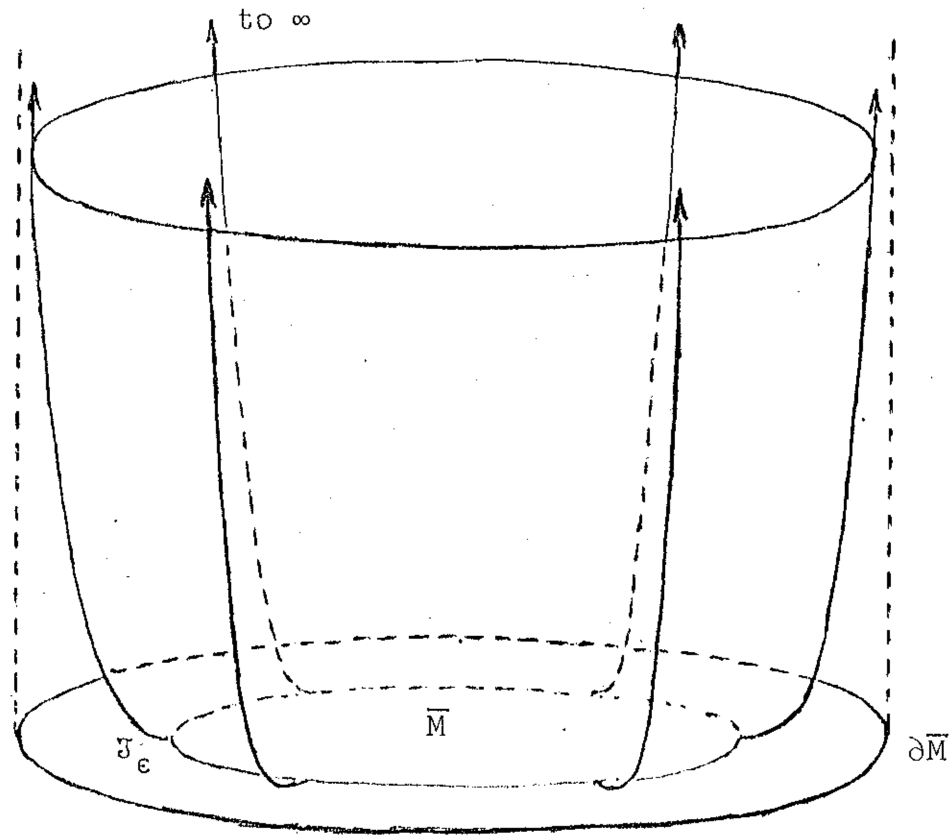
This function f satisfies the following conditions:

(i) $f \in C^\infty(M)$,

(ii) $f \equiv 0$ on $M \setminus \mathcal{T}_\epsilon$,

(iii) f depends only on t,

(iv) f is decreasing with t and $f \to \infty$ as $t \to 0$.

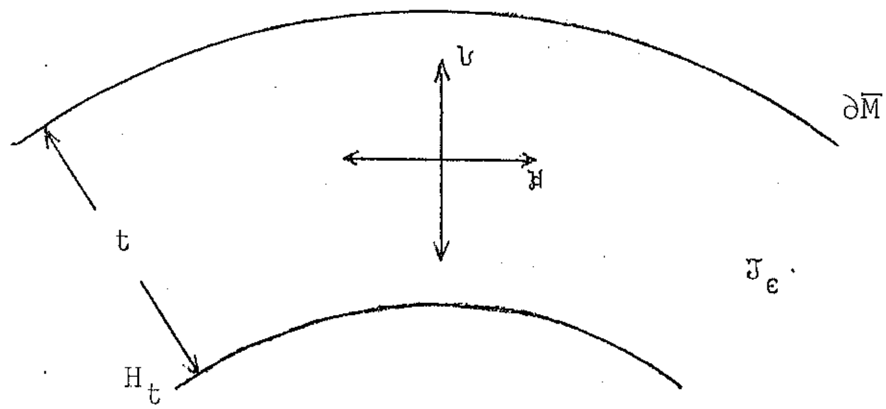



In order to make curvature estimates accessible, we use an alternate description of g in terms of the orthogonal splitting of the tangent bundle over $\mathcal{T}_\epsilon$ induced by t.

<u>Definition</u>. The <u>vertical distribution</u> $\mathcal{V}$ is the 1-dimensional distribution in $\mathcal{T}_\epsilon$ induced by $\overline{\nabla}t$ = grad t.

The <u>vertical geodesic</u> $v_x$ <u>based at</u> $x \in \partial\overline{M}$ is the integral curve of $\nabla t$ starting at x. (Clearly, $\overline{\nabla}t$ has unit length.)

The <u>horizontal distribution</u> $\mathcal{H}$ is the n-dimensional distribution in $\mathcal{T}_\epsilon$ of hyperplanes tangent to level sets of t. These level sets (or equidistant sets from the boundary) are smooth hypersurfaces in $\mathcal{T}_\epsilon$ which we will call <u>horizontal hyperfaces</u> and denote by $H_t$.




We can now make the formal definition of the new "warped" metric g.

$$g_{|x} = \begin{cases} \bar{g}_x, & x \in M \setminus \mathcal{T}_\epsilon, \\ \varphi^2(t(x))\,\bar{g}_{|\mathcal{V}_x} \oplus \bar{g}_{|\mathcal{H}_x}, & x \in \mathcal{T}_\epsilon, \end{cases}$$

where $\varphi : (0,\epsilon] \to \mathbb{R}$ satisfies the following conditions:

(1) $\varphi \in C^2$,

(2) $\varphi(\epsilon) = 1$, $\varphi^{(i)}(\epsilon) = 0$, $i = 1,2$ (for g to be $C^2$),

(3) $\varphi$ is decreasing,

(4) $\int_0^\epsilon \varphi(t)dt$ diverges, (for completeness of g).

<u>Remark</u>. Taking $\varphi^2 = 1+(f')^2$ gives the complete metric described above, in terms of the warping function f.

A function $\varphi$ satisfying 1) - 4) will be called admissible.

Having recast the definition of g in terms of $\mathcal{V}$, $\mathcal{H}$, and $\varphi$, the Ricci tensor for g, denoted Ric, can be computed in terms of the original Ricci tensor $\overline{\mathrm{Ric}}$, the function $\varphi$ and the second fundamental tensor $S_t$ of the horizontal hypersurfaces $H_t$, taken relative to the unit outward normal $T = -\overline{\nabla} t$ to $H_t$. We will often write S for $S_t$ if there is no danger of confusion. The original metric $\overline{g}$ will be denoted $\langle\ ,\ \rangle$.

Proposition 1.1. Let N be a vertical vector field, and X,Y be horizontal vector fields. Then:

(1) $\mathrm{Ric}(N,N) = \overline{\mathrm{Ric}}(N,N) + \frac{T\varphi}{\varphi}\|N\|^2 \operatorname{tr} S_t$,

(2) $\mathrm{Ric}(N,X) = \overline{\mathrm{Ric}}(N,X)$,

(3) $\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + [\frac{T\varphi}{\varphi^3} + (1-\frac{1}{\varphi^2})\operatorname{tr} S_t]\ \langle S_t X, Y\rangle$

$$+(\frac{1}{\varphi^2} - 1)\ [\langle \overline{R}(T,X)Y,T\rangle + \langle S_t^2 X, Y\rangle].$$

We proceed in three steps; first, the derivation of the Levi-Civita connection $\nabla$ for g; second, the derivation of the curvature tensor R; and finally, contraction of R to Ric.

The superscripts h and v to vector fields refer to horizontal and vertical components of these fields.

We will use the following obvious facts about horizontal and vertical fields:

(a) If X, Y horizontal, then [X, Y] is horizontal, since $\mathcal{H}$ is involutive.

(b) If $X \in \mathcal{H}$, then $X\varphi = 0$.

(c) $X \in \mathcal{H}$ implies $[X,T] \in \mathcal{H}$, because

$$<[X,T],T> = \frac{1}{2}X<T,T> - [T<X,T> - <X,\overline{\nabla}_T T>] = \frac{1}{2}X\varphi^2 = 0.$$

We first compute the Levi-Civita connection $\nabla$ for g. The metric g will also be written $< \, , \, >_0$.

<u>Lemma 1.2</u>: (1) $\nabla_N N = \overline{\nabla}_N N + (\frac{N\varphi}{\varphi})N$,

(2) $\nabla_X N = \overline{\nabla}_X N$,

(3) $\nabla_N X = \overline{\nabla}_N X$ is horizontal,

(4) $(\nabla_X Y)^h = (\overline{\nabla}_X Y)^h$

(5) $(\nabla_X Y)^v = \frac{1}{\varphi^2}(\overline{\nabla}_X Y)^v$.

$\Bigg\} \Rightarrow \nabla_X Y = \overline{\nabla}_X Y + (\frac{1}{\varphi^2} - 1)(\overline{\nabla}_X Y)^v$

<u>Proof of Lemma 1.2</u>: To prove the formulas (1) - (3), we compute the horizontal and vertical components of each left hand side using the implicit definition of the Levi-Civita connection in terms of the metric.

(1) It suffices to show $\nabla_T T = (\frac{T\varphi}{\varphi})T$, since $\overline{\nabla}_T T \equiv 0$.

<u>Vertical component</u>:

$$\begin{aligned} 2<\nabla_T T, T>_0 &= T<T,T>_0 \\ &= T(\varphi^2) \\ &= 2\varphi T\varphi \,. \end{aligned}$$

So $(\nabla_T T)^v = <\nabla_T T, \frac{T}{\varphi}>_0 \frac{T}{\varphi}$, since $\frac{T}{\varphi}$ is a unit vertical vector with respect to g

$= (\frac{T\varphi}{\varphi})T.$

Horizontal component:

$$2\langle \nabla_T T, X\rangle_0 = T\langle T,X\rangle_0 + T\langle X,T\rangle_0 - X\langle T,T\rangle_0$$

$$+ \langle X,[T,T]\rangle_0 - \langle T,[T,X]\rangle_0 + \langle T,[X,T]\rangle_0$$

$$= 0 \qquad \text{by (b) and (c).}$$

So $(\nabla_T T)^h = 0$, and $\nabla_T T = (\frac{T\varphi}{\varphi})T$.

(2) It suffices to compute $\nabla_X T$.

Vertical component:

$$\langle \nabla_X T, T\rangle_0 = \frac{1}{2}X\langle T,T\rangle_0 = 0, \qquad \text{by (b).}$$

So $(\nabla_X T)^v = 0 = (\overline{\nabla}_X T)^v$.

Horizontal component:

$$2\langle \nabla_X T, Y\rangle_0 = X\langle T,Y\rangle_0 + T\langle Y,X\rangle_0 - Y\langle X,T\rangle_0$$

$$+ \langle Y,[X,T]\rangle_0 - \langle X,[T,Y]\rangle_0 + \langle T,[Y,X]\rangle_0$$

$$= X\langle T,Y\rangle + T\langle Y,X\rangle - Y\langle X,T\rangle$$

$$+ \langle Y,[X,T]\rangle - \langle X,[T,Y]\rangle + \langle T,[Y,X]\rangle$$

$$= 2\langle \overline{\nabla}_X T, Y\rangle$$

$$= 2\langle \overline{\nabla}_X T, Y\rangle_0 .$$

So $(\nabla_X T)^h = (\overline{\nabla}_X T)^h$ and $\nabla_X T = \overline{\nabla}_X T$.

(3) follows from (2)

(4) $2\langle \nabla_X Y, Z\rangle_0 = X\langle Y,Z\rangle_0 + Y\langle Z,X\rangle_0 - Z\langle X,Y\rangle_0$

$+ \langle Z,[X,Y]\rangle_0 - \langle X,[Y,Z]\rangle_0 + \langle Y,[Z,X]\rangle_0$

$= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$

$+\langle Z,[X,Y]\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle$

$= 2\langle \overline{\nabla}_X Y, Z\rangle$

$= 2\langle \overline{\nabla}_X Y, Z\rangle_0$ .

So $(\nabla_X Y)^h = (\overline{\nabla}_X Y)^h$.

(5) $\langle \nabla_X Y, T\rangle_0 = X\langle Y,T\rangle_0 - \langle Y, \nabla_X T\rangle_0$

$= X\langle Y,T\rangle - \langle Y, \overline{\nabla}_X T\rangle$ by (2)

$= \langle \overline{\nabla}_X Y, T\rangle$ .

So $(\nabla_X Y)^v = \langle \nabla_X Y, \frac{T}{\varphi}\rangle_0 \frac{T}{\varphi}$

$= \frac{1}{\varphi^2}\langle \overline{\nabla}_X Y, T\rangle T$

$= \frac{1}{\varphi^2}(\overline{\nabla}_X Y)^v$.

Having computed the Levi-Civita connection for g, we next derive those terms of the curvature tensor R for g needed to compute Ric. The curvature tensor for $\overline{g}$ will be denoted by $\overline{R}$.

Lemma 1.3.

(1) $R(X,N)N = \overline{R}(X,N)N + \frac{T\varphi}{\varphi}\langle N,T\rangle^2 \cdot SX$,

(2) $R(X,Y)Z = \overline{R}(X,Y)Z$

$$+ \left(\frac{1}{\varphi^2} - 1\right)\{<\overline{R}(X,Y)Z,T>T + <SX,Z>SY - <SY,Z>SX\}.$$

(3) $R(N,X)Y = \overline{R}(N,X)Y + \{\frac{N\varphi}{\varphi^3}< SX,Y> + (\frac{1}{\varphi^2} - 1)<\overline{R}(N,X)Y,T>\}T$

Proof of lemma 1.3:

(1) It suffices to compute $R(X,T)T$.

$$R(X,T)T = \nabla_X\nabla_T T - \nabla_T\nabla_X T - \nabla_{[X,T]}T$$

$$= \nabla_X(\overline{\nabla}_T T + \frac{T\varphi}{\varphi}T) - \nabla_T(\overline{\nabla}_X T) - \overline{\nabla}_{[X,T]}T \qquad \text{by 1.2}$$

$$= \overline{\nabla}_X\overline{\nabla}_T T + X(\frac{T\varphi}{\varphi})T + \frac{T\varphi}{\varphi}\,\overline{\nabla}_X T$$

$$- \overline{\nabla}_T\overline{\nabla}_X T - \overline{\nabla}_{[X,T]}T \qquad \text{since } \overline{\nabla}_X T \in \mathcal{H}$$

$$= \overline{R}(X,T)T + \frac{T\varphi}{\varphi}\,S(X).$$

(2) $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

$$= \nabla_X((\overline{\nabla}_Y Z)^h + \frac{1}{\varphi^2}(\overline{\nabla}_Y Z)^v) - \nabla_Y((\overline{\nabla}_X Z)^h + \frac{1}{\varphi^2}(\overline{\nabla}_X Z)^v)$$

$$- [\overline{\nabla}_{[X,Y]}Z + (\frac{1}{\varphi^2} - 1)(\overline{\nabla}_{[X,Y]}Z)^v]$$

$$= \overline{\nabla}_X[(\overline{\nabla}_Y Z)^h] + (\frac{1}{\varphi^2} - 1)[\overline{\nabla}_X(\overline{\nabla}_Y Z)^h]^v + \frac{1}{\varphi^2}\overline{\nabla}_X[(\overline{\nabla}_Y Z)^v]$$

$$- \{\overline{\nabla}_Y[(\overline{\nabla}_X Z)^h] + (\frac{1}{\varphi^2} - 1)[\overline{\nabla}_Y(\overline{\nabla}_X Z)^h]^v + \frac{1}{\varphi^2}\overline{\nabla}_Y[(\overline{\nabla}_X Z)^v]\}$$

$$- \overline{\nabla}_{[X,Y]}Z - (\frac{1}{\varphi^2} - 1)(\overline{\nabla}_{[X,Y]}Z)^v.$$

The first line of this expression can be written as:

$$\overline{\nabla}_X((\overline{\nabla}_Y Z)^h + (\overline{\nabla}_Y Z)^v) + (\frac{1}{\varphi^2} - 1)\{\overline{\nabla}_X[(\overline{\nabla}_Y Z)^v] + [\overline{\nabla}_X(\overline{\nabla}_Y Z)^h]^v\}$$

$$= \overline{\nabla}_X \overline{\nabla}_Y Z + (\frac{1}{\varphi^2} - 1)\ \{\overline{\nabla}_X[(\overline{\nabla}_Y Z)^v] + [(\overline{\nabla}_X(\overline{\nabla}_Y Z)^h]^v\}$$

The second line can be written similarly by interchanging X and Y.

So $R(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$

$$+ (\frac{1}{\varphi^2} - 1)\ \{\overline{\nabla}_X[(\overline{\nabla}_Y Z)^v] + [\overline{\nabla}_X(\overline{\nabla}_Y Z)^h]^v - \overline{\nabla}_Y[(\overline{\nabla}_X Z)^v]$$

$$- [\overline{\nabla}_Y(\overline{\nabla}_X Z)^h]^v - (\overline{\nabla}_{[X,Y]} Z)^v\}$$

$$= \overline{R}(X,Y)Z + (\frac{1}{\varphi^2} - 1)\{A\}$$

To simplify A, rewrite the expression

$$\overline{\nabla}_X[(\overline{\nabla}_Y Z)^v] + [\overline{\nabla}_X(\overline{\nabla}_Y Z)^h]^v$$

$$= [\overline{\nabla}_X(\overline{\nabla}_Y Z)^v]^v + [\overline{\nabla}_X(\overline{\nabla}_Y Z)^v]^h + [\overline{\nabla}_X(\overline{\nabla}_Y Z - (\overline{\nabla}_Y Z)^v)]^v$$

$$= (\overline{\nabla}_X \overline{\nabla}_Y Z)^v + [\overline{\nabla}_X(\langle \overline{\nabla}_Y Z, T\rangle T)]^h$$

$$= (\overline{\nabla}_X \overline{\nabla}_Y Z)^v - \langle SY, Z\rangle SX$$

Substituting this in A yields:

$$A = (\overline{\nabla}_X \overline{\nabla}_Y Z)^v - \langle SY, Z\rangle SX$$

$$- (\overline{\nabla}_Y \overline{\nabla}_X Z)^V + \langle SX, Z\rangle SY$$

$$- (\overline{\nabla}_{[X,Y]} Z)^V$$

$$= (\overline{R}(X,Y)Z)^V + \langle SX, Z\rangle SY - \langle SY, Z\rangle SX.$$

Substitution of this expression for A gives the result.

(3) follows by taking horizontal and vertical components of (1) and (2).

We can finally compute Ric using Lemma 1.3.

Proof of Proposition 1.1:

(1) $\mathrm{Ric}(N,N) = \mathrm{tr}(X \mapsto R(X,N)N)$

$$= \mathrm{tr}_{\mathcal{H}}(X \mapsto \overline{R}(X,N)N + \frac{N\varphi}{\varphi}\langle N, T\rangle SX)$$

$$= \overline{\mathrm{Ric}}(N,N) + \frac{N\varphi}{\varphi}\langle N, T\rangle \mathrm{tr}\, S$$

(2) $\mathrm{Ric}(N,X) = \mathrm{tr}(Y \mapsto R(Y,N)X)$

$$= \mathrm{tr}_{\mathcal{H}}(Y \mapsto \overline{R}(Y,N)X + \text{vertical terms}) \quad \text{using 1.3(3)}$$

$$= \mathrm{tr}_{\mathcal{H}}(Y \mapsto \overline{R}(Y,N)X)$$

$$= \overline{\mathrm{Ric}}(N,X).$$

(3) Let $\{e_1 = T, e_2, \ldots e_{n+1}\}$ be an orthonormal basis of $T_xM$ with respect to $\overline{g}$.

Then $\mathrm{Ric}(X,Y) = \langle R(T,X)Y, T\rangle + \sum_{k=2}^{n+1} \langle R(e_k,X)Y, e_k\rangle$

$$= \langle \bar{R}(T,X)Y,T\rangle + \frac{T\varphi}{\varphi^3}\langle SX,Y\rangle + \left(\frac{1}{\varphi^2}-1\right)\langle \bar{R}(T,X)Y,T\rangle$$

$$+ \sum_{k=2}^{n+1}\{\langle \bar{R}(e_k,X)Y,e_k\rangle + \left(\frac{1}{\varphi^2}-1\right)[\langle Se_k,Y\rangle\langle SX,e_k\rangle$$

$$- \langle SX,Y\rangle\langle Se_k,e_k\rangle]\}$$

$$= \overline{Ric}(X,Y) + \frac{T\varphi}{\varphi^3}\langle SX,Y\rangle$$

$$+ \left(\frac{1}{\varphi^2}-1\right)\{\langle \bar{R}(T,X)Y,T\rangle + \langle S^2X,Y\rangle - trS\langle SX,Y\rangle\}.$$

This concludes the proof of Proposition 1.1 .

We can now find conditions on $\partial\bar{M}$ by means of 1.1 which guarantee that M has positive Ricci curvature with respect to g, at least "near infinity".

Proposition 1.4. Suppose $\partial\bar{M}$ has positive mean curvature relative to the outward unit normal field, and that the intrinsic Ricci curvature $Ric_0$ of $\partial\bar{M}$ is positive. Then there exists a tube $\mathfrak{T}_n$ of $\partial\bar{M}$ with $\mathfrak{T}_n \subset \mathfrak{T}_\epsilon$, and a function $\varphi$ such that Ric > on $\mathfrak{T}_n$.

Proof. Take $\varphi(t) = \frac{1}{t}$ in $\mathfrak{T}_{\frac{1}{2}\epsilon}$ , and let X be a unit horizontal vector at $x \in \mathfrak{T}_{\frac{1}{2}\epsilon}$ .

We will show the existence of a number n, $0 < n < \frac{1}{2}\epsilon$, with the following property:

(*) If $x \in \mathfrak{T}_n$ and $\alpha,\beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$, then

$$Ric(\alpha X+\beta\frac{T}{\varphi}) > 0.$$

To this, consider the expression in $(*)$ as a quadratic function in $\alpha$ and $\beta$.

$$\operatorname{Ric}(\alpha X+\beta\tfrac{T}{\varphi}) = \alpha^2 \operatorname{Ric} X + 2\alpha\beta \operatorname{Ric}(X,\tfrac{T}{\varphi}) + \beta^2 \operatorname{Ric}(\tfrac{T}{\varphi})$$

To establish $(*)$, it is enough to show that the discriminant

$$[\operatorname{Ric}(X,\tfrac{T}{\varphi})]^2 - (\operatorname{Ric}X)(\operatorname{Ric}\tfrac{T}{\varphi}) < 0 \text{ on } \mathfrak{T}_\eta \text{ for some } \eta \in (0,\tfrac{1}{2}\epsilon)$$

But in $\mathfrak{T}_{\frac{1}{2}\epsilon}$, letting $\operatorname{Ric}_t$ denote the intrinsic Ricci curvature of $H_t$

$$[\operatorname{Ric}(X,\tfrac{T}{\varphi})]^2 - (\operatorname{Ric}X)(\operatorname{Ric}\tfrac{T}{\varphi})$$

$$= -t \operatorname{tr}S_t \operatorname{Ric}_t X + O(t^2) \text{ by Proposition 1.1}$$

where $O(t^2)$ is a continuous function on $\mathfrak{T}_{\frac{1}{2}\epsilon}$ such that $\lim\limits_{t\to 0} \frac{O(t^2)}{t^2} < \infty$.

By hypothesis, $\operatorname{tr}S_0 > 0$ and $\operatorname{Ric}_0 > 0$ and $(*)$ follows.

At this point we remark that the existence of a metric on M of strictly positive Ricci curvature near infinity can be obtained more directly using only the assumption that $\operatorname{Ric}_0 > 0$, by regarding $\mathfrak{T}_\epsilon$ as the product $\partial\overline{M}\times(0,\infty)$ with $\operatorname{Ric}_{\partial\overline{M}} > 0$. Then $\operatorname{Ric}_{\partial\overline{M}\times\mathbb{R}} \geq 0$ in the product metric, and by changing the horizontal component of this metric by $\varphi(t) = t^{\frac{1}{2}}$ for example, Ric becomes strictly positive near infinity.

The point of our further discussion is to construct a a metric on M in such a way that M has positive Ricci curvature everywhere, a considerably more difficult problem which we deal with in the next chapter.

## 2. Global Extension of Positive Ricci Curvature

In this chapter we will assume that the Ricci curvature $\overline{Ric}$ of $\overline{M}$ is strictly positive everywhere in $\overline{M}$. According to 1.4, if the mean curvature $trS_0$ of $\partial\overline{M}$ is positive and the intrinsic Ricci curvature $Ric_0$ of $\partial\overline{M}$ is positive, then the original metric $\overline{g}$ can be deformed in $\mathcal{J}_\epsilon$ to a complete metric g with positive Ricci curvature near infinity. The obvious question which arises at this point is whether we can ensure that the Ricci curvature is positive throughout all of $\mathcal{J}_\epsilon$ and hence everywhere in M.

This does not seem to be possible (at least using our approach) without further restrictions, both on the boundary $\partial\overline{M}$ and the metric $\overline{g}$. For simplicity of presentation, we will assume that $\overline{g}$ has constant curvature 1, i.e. $\overline{M}^{n+1}$ is a submanifold of the euclidean sphere $S^{n+1}$. This assumption reduces the complexity of the analysis to some extent. The Ricci tensor splits along horizontal and vertical directions, and the problem reduces completely to $\partial\overline{M}$, but the final condition on $\partial\overline{M}$ is still fairly complicated.

The assumption that $\overline{g}$ has constant positive curvature is certainly not necessary, but in light of the potential examples of Chapter 3, this assumption does not seem severe. Theorem 2.2 holds as well for other "simple" metrics like those of ellipsoids of Chapter 3, but detailed estimates will be better in explicit examples.

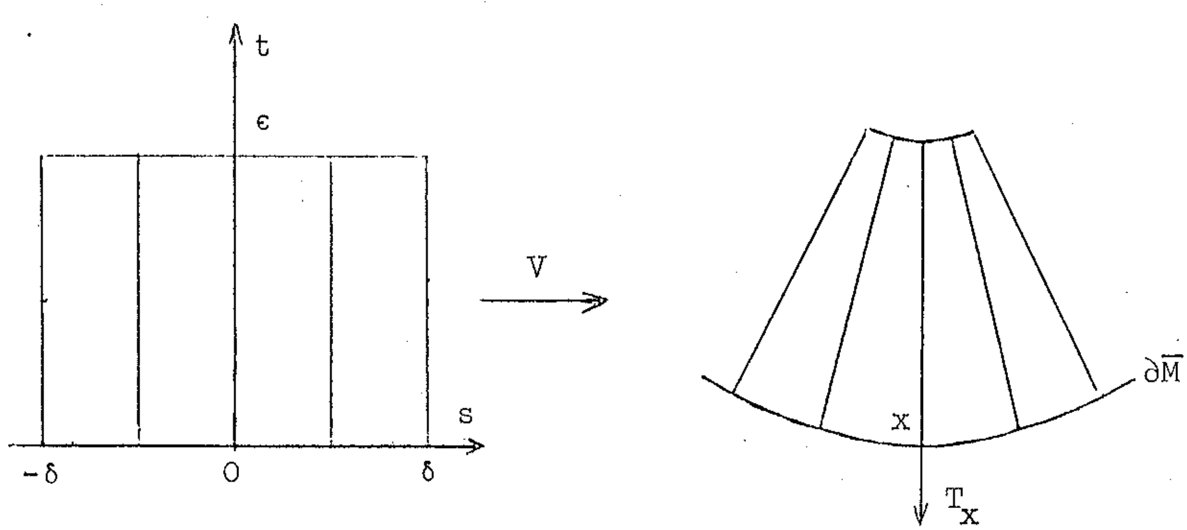
Recall that the horizontal hypersurfaces $H_t$ are the level sets of the metric distance function $t : \mathcal{T}_\epsilon \to \mathbb{R}$ to the boundary $\partial\overline{M}$, $T = -\overline{\nabla} t$ is the unit outward pointing normal vector field to $H_t$, and $S_t$ is the second fundamental tensor of $H_t$ taken with respect to T.

We begin the analysis of the formula for Ric in 1.1 by expressing $S_t$ in terms of $S_0$ and Jacobi fields along vertical geodesics.

Let $u \in T_x\partial\overline{M}$ and let $c : (-\delta, \delta) \to \partial\overline{M}$ be any curve with velocity u at 0; i.e. $c(0) = x$, $\dot{c}(0) = u$. At each $c(s) \in \partial\overline{M}$ we can shoot out the vertical geodesic based at c(s). This defines the variation V of nearby normals in direction u:

$$V: (-\delta, \delta) \times [0, \epsilon) \to M : (s, t) \mapsto \exp_{\partial\overline{M}}(-tT_{c(s)}),$$

where $\exp_{\partial\overline{M}}$ is the exponential map of the normal bundle of the boundary.



Let $X_t = V_{*|(o,t)}D_s$ be the variation field of V. Then X is a Jacobi field with initial values $X(0) = u$ and

$$X'(0) = (\nabla_{D_t} V_* D_s)|_{s,t=0} = (\nabla_{D_s} V_* D_t)|_{s,t=0}$$

$$= (\nabla_{D_s} -T)|_{s,t=0} = -SV_* D_s|_{s,t=0} = -S\dot{c}(0) = -Su .$$

The change of $S_t$ along a normal geodesic is now given by

Proposition 2.1. $S_t X_t = -X_t'$.

Proof.
$$S_t X_t = \nabla_{X_t} T = \nabla_{D_s}(-V_* D_t)|_{s=0}$$

$$= -\nabla_{D_t}(V_* D_s)|_{s=0} = -\nabla_{D_t} X_t = -X_t' .$$

Example. Suppose $\overline{M}$ has constant curvature $K \equiv 1$, and $x \in \partial\overline{M}$. Let u be a unit eigenvector of $S_x$ with corresponding eigenvalue $\lambda$, and let $U_t$ be the parallel field along the vertical geodesic $v_x$ based at $x \in \partial\overline{M}$ with $U(0) = u$.

Then $U_t$ is a unit eigenvector of $S_t$ with eigenvalue $\Lambda(t) = \tan(t + \tan^{-1}\lambda)$.

In fact, we can solve explicitly for the Jacobi field X of 2.1

$$X_t = (\cos t - \lambda \sin t)U_t,$$

so

$$S((\cos t - \lambda \sin t)U_t) = -X_t'$$

$$= (\sin t + \lambda \cos t)U_t .$$

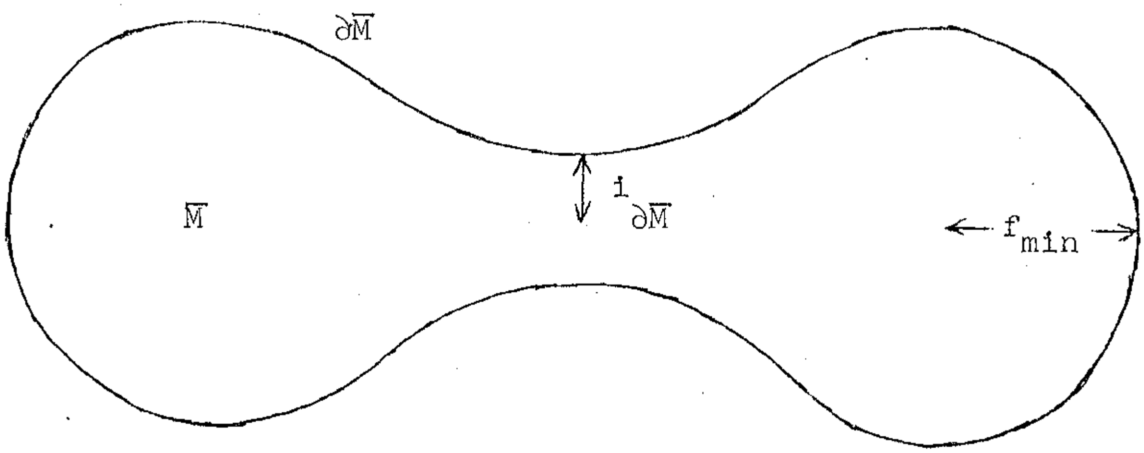
This implies that $U_t$ is an eigenvector of $S_t$ with eigenvalue

$$\Lambda(t) = \frac{\sin t + \lambda \cos t}{\cos t - \lambda \sin t}$$

$$= \tan(t + \tan^{-1}\lambda).$$

We now turn to estimating the largest radius $\epsilon$ of a tubular neighborhood $\mathcal{T}_\epsilon$ of the boundary $\partial\overline{M}$ in which the change of metric takes place. It will be necessary that $\epsilon$ is not too small. Since the exponential map $\exp_{\partial\overline{M}}$ along the boundary is a diffeomorphism on $\mathcal{T}_\epsilon$ for any $\epsilon$ less than the injectivity radius $i_{\partial\overline{M}}$ of $\exp_{\partial\overline{M}}$, it suffices to estimate $i_{\partial\overline{M}}$ from below in terms of boundary data.

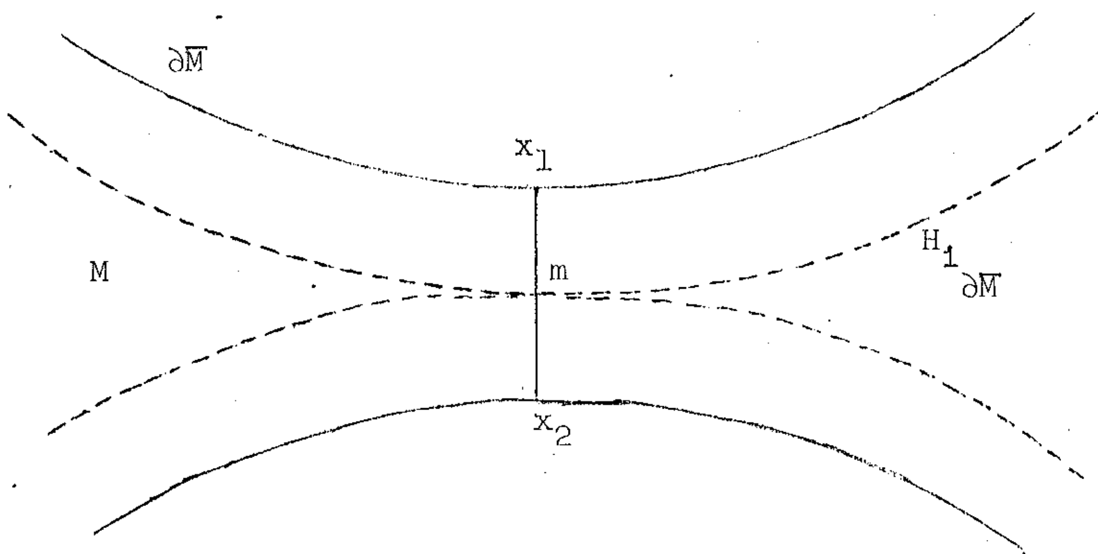
In general, $i_{\partial\overline{M}}$ may be less than the minimal focal radius $f_{min}$ of the boundary for global reasons.




However the conditions adopted in the previous chapter allow us to estimate $i_{\partial\overline{M}}$ in terms of boundary data as follows.

Theorem 2.2. If $\mathrm{Ric}_M > 0$ and $\mathrm{tr}\, S_{\partial\overline{M}} \geq 0$, then $i_{\partial\overline{M}} = f_{min}$.

Proof. Otherwise, $i_{\partial\overline{M}} < f_{min}$. Therefore, as t increases from 0 to $i_{\partial\overline{M}}$, the horizontal hypersurfaces $H_t$ are well-defined by hypothesis and sweep out a neighborhood of $\partial\overline{M}$. At $t = i_{\partial\overline{M}}$, $\exp_{\partial\overline{M}}$ fails to be injective, so there exists a "first intersection point" m and $x_1, x_2 \in \partial\overline{M}$ such that $\mathrm{dist}(x_1,m) = \mathrm{dist}(x_2,m) = i_{\partial\overline{M}}$. The hypersurfaces $H_t$ approach each other as $t \to i_{\partial\overline{M}}$, touching at m.




Let $v_{x_1}, v_{x_2} : [0, i_{\partial\overline{M}}] \to \overline{M}$ be the vertical geodesics based at $x_1$ and $x_2$. It follows by the Gauss Lemma that:

1. The piecewise smooth geodesic $\gamma : [0, 2i_{\partial\overline{M}}] \to \overline{M}$ defined as composition of $v_{x_1}$ and $-v_{x_2}$ is actually smooth at m;

2. $\gamma$ is a non-trivial locally minimising geodesic perpendicular to $\partial\overline{M}$ in the space of curves in $\overline{M}$ from $\partial\overline{M}$ to itself, since this is the first horizontal level of self-intersection.

On the other hand, by an "averaging" second variation argument [L], we have:

Lemma 2.3. If $Ric_{\overline{M}} > 0$ and $tr\, S \geq 0$, and $\gamma : [a,b] \to \overline{M}$ is a smooth geodesic perpendicular to $\partial\overline{M}$, then there exists a variation of $\gamma$ in the above space of curves, so that neighboring curves are strictly shorter than $\gamma$. This completes the proof of 2.2.

Remark. If $\overline{M}$ and $\partial\overline{M}$ satisfy the conditions of 2.2, then $\pi_1(\overline{M},\partial\overline{M}) = 0$, and $\pi_1(\partial\overline{M}) \to \pi_1(\overline{M})$ is surjective, cf. [L].

We now consider the case when $\overline{M}^{n+1}$ is a submanifold of the euclidean sphere $S^{n+1}$, with boundary $\partial\overline{M}$ of nonnegative mean curvature. (It follows then that $tr\, S_t > 0$ for $0 < t < \epsilon$.) The main result of this chapter is the derivation of a further condition on $\partial\overline{M}$, computable in terms of the second fundamental tensor S, which guarantees the existence in M of a warped metric g of positive Ricci curvature.

$$\text{Let } \lambda_{max} = \max_{x\in\partial\overline{M}} \max_{\|Y\|=1} \{\langle SY, Y\rangle \mid Y \in T_x\partial\overline{M}\}$$

be the maximum principal curvature of $\partial\overline{M}$, and let $\varepsilon = \tan^{-1} \frac{1}{\lambda_{max}}$ denote the injectivity radius of the boundary.

Theorem 2.4. Given $\mu > 0$, there exists a continuous real-valued function $F_\mu(S)$ defined on self-adjoint operators S on $\mathbb{R}^n$, with the following property:

$$\text{If } \lambda_{max} < \mu \text{ and } F_\mu(S_0) > 0$$

everywhere on $\partial\overline{M}$, then there exists an admissible warping function $\varphi : (0,\varepsilon] \to \mathbb{R}_+$ such that the metric g defined from $\varphi$ and $\overline{g}$ has strictly positive Ricci curvature throughout $\mathfrak{T}_\varepsilon$ and hence everywhere in M.

Remark. $F_\mu$ is defined explicitly in terms of $\lambda_{max}$, $\mu$, and the eigenvalues of S. The hypothesis is fairly sharp for $\mu > 1$.

Remark. If $\overline{M}$ has variable positive sectional curvature K, the same qualitative result holds, but F and $\varepsilon$ will also depend on the bounds for K. One has to modify some of the above arguments by using standard comparison techniques for Jacobi fields in order to get estimates for all data involving $S_t$. Also, as in the end of Chapter 1, the "cross terms" for the Ricci tensor will not necessarily vanish, so the discriminant must be estimated.

As we pointed out before, explicit bounds will be better in specific examples, so we will only carry out the arguments for constant curvature.

Proof of Theorem 2.4. At each $x \in \partial\overline{M}$, the second fundamental tensor $S_0$ has an orthonormal basis $\{U_1, \ldots, U_n\}$ of eigenvectors with corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Let $U_1, \ldots, U_n$ be parallel fields along the vertical geodesic $v_x$ based at $x$ with initial values $U_i(0) = u_i$. Then $U_i(t)$ is an eigenvector of $S_t$ with eigenvalue $\Lambda_i(t) = \tan(t + \tan^{-1}\lambda_i)$.

Furthermore, $\{U_1, \ldots, U_n, \frac{T}{\varphi} = U_{n+1}\}$ is an orthonormal basis with respect to g throughout $\mathcal{T}_\epsilon$, and are also eigenvectors of the new Ricci tensor Ric by 1.1 using again the fact that $\overline{g}$ has constant curvature.

The eigenvalue of Ric corresponding to $U_i$ is

$$\mathrm{Ric}_i \equiv \begin{cases} n - \Phi - \frac{1}{2}\Phi' \Lambda_i + \Phi \left[\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \Lambda_j \Lambda_i\right] & 1 \leq i \leq n, \\ \frac{1}{\varphi^2}\left(n - \frac{\varphi'}{\varphi} \operatorname{tr} S\right) & i = n+1, \end{cases}$$

where we have replaced the term $1 - \frac{1}{\varphi^2}$ in 1.1 (3) by $\Phi = 1 - \frac{1}{\varphi^2}$.

In order to define an admissible function $\varphi : (0, \epsilon] \to \mathbb{R}^+$, we will construct a function $\Phi : [0, \epsilon] \to [0, 1]$ with the following properties:

(1) $\Phi \in C^2$,

(2) $\Phi$ is decreasing,

(3) $\Phi(t) = 1 - t^2$ in a neighborhood of 0,

(4) $\Phi \equiv 0$ in a neighborhood of $\epsilon$,

(5) $\mathrm{Ric}_i \equiv n - \Phi - \frac{1}{2}\Phi' \Lambda_i + \Phi \left[\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \Lambda_j \Lambda_i\right] > 0$

on $(0, \epsilon]$ for all $1 \leq i \leq n$.

The warping function $\varphi$ will then be defined on $(0,\epsilon]$ as $\varphi = (1-\Phi)^{-\frac{1}{2}}$. The properties (1) - (4) above ensure that $\varphi$ is admissible according to the conditions in Chapter 1. Property (5) will imply that the Ricci curvatures $Ric_i$ in the metric warped by $\varphi$ are positive for $i=1,\ldots,n$. The fact that $\varphi$ will be decreasing and $tr\,S$ is increasing will imply that $Ric_{n+1}$ is positive.

So it remains to construct a function $\Phi$ subject to (1) - (5). At this point we observe that (5) can be replaced by the weaker

(5') Condition (5) holds for $i=1,n$.

In fact, the expression (5) can be rewritten as:

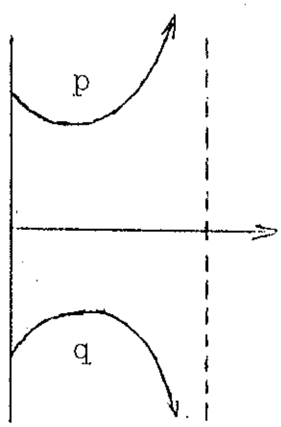
$$(6) \quad Ric_i = n - \Phi + [\Phi\, tr\,S - \tfrac{1}{2}\Phi'] \wedge_i - \Phi \wedge_i^2 .$$

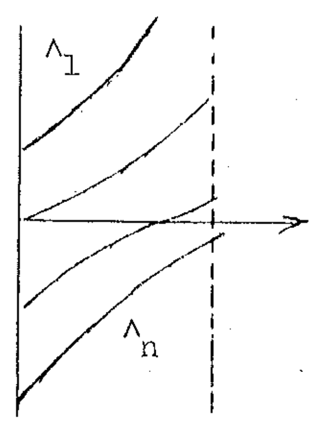
The quadratic formula implies that (6) is positive if and only if

$$(7) \quad p > \wedge_i > q \quad \text{where}$$

$$p = \frac{1}{2}\{tr\,S - \frac{1}{2}\frac{\Phi'}{\Phi} + \sqrt{(tr\,S - \frac{1}{2}\frac{\Phi'}{\Phi})^2 + 4(\frac{n-\Phi}{\Phi})}\} > 0,$$

$$q = \frac{1}{2}\{tr\,S - \frac{1}{2}\frac{\Phi'}{\Phi} - \sqrt{(tr\,S - \frac{1}{2}\frac{\Phi'}{\Phi})^2 + 4(\frac{n-\Phi}{\Phi})}\} < 0,$$



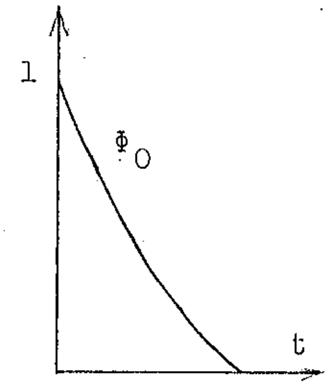



and where the inequalities follow from (1) - (4) for $\Phi$.

If (5') is true, then $p > \Lambda_1$ and $\Lambda_n > q$ by (7).

But $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n$, so $p > \Lambda_i > q$ for all i and (5) is true.

We now come to the construction of $\Phi$ satisfying (1) - (4) and (5'), and perform the construction in four steps.




Step 1. We first define a prototype $\Phi_0$ of $\Phi$ by

$$\Phi_0(t) = \begin{cases} -\mu \tan(t - \tan^{-1}\frac{1}{\mu}) & t \in [0, \tan^{-1}\frac{1}{\mu}], \\ 0 & t \in (\tan^{-1}\frac{1}{\mu}, \epsilon], \end{cases}$$

and observe that $\Phi_0$ is a function satisfying (2), and for which the minimum value of the expression for Ric in (5) can be estimated from below in terms of $\mu$ and $\lambda_1, \ldots, \lambda_n$, this yields the function F of the hypothesis.

In fact, $\lambda_{max} < \mu$ implies by elementary calculus that

$$(8) \quad \Phi_0 \Lambda_i \leq h(\lambda_i, \mu) \equiv \begin{cases} \lambda_i & \lambda_i > \frac{1}{\mu}, \\ \mu \tan^2[\frac{1}{2}(\tan^{-1}\lambda_i + \tan^{-1}\frac{1}{\mu})] & \frac{1}{\mu} \geq \lambda_i > -\frac{1}{\mu}, \\ 0 & -\frac{1}{\mu} \geq \lambda_i . \end{cases}$$

and the function h defined by the right hand side of (8) is continuous in $\lambda_i$.

Therefore, we have

$$\Phi_0 \Lambda_i \Lambda_1 \geq \begin{cases} h(\lambda_1,\mu)\lambda_i & \lambda_i < 0 , \\ 0 & \lambda_i \geq 0 , \end{cases}$$

and $\Phi_0 \sum_{i=2}^{n} \Lambda_i \Lambda_1 \geq \sum_{i=2}^{n} h(\lambda_1,\mu)k(\lambda_i)$ on $[0, \tan^{-1} \frac{1}{\mu}]$, where

$$k(\lambda_i) = \begin{cases} 0 & \lambda_i \geq 0 \\ \lambda_i & \lambda_i < 0 \end{cases} .$$

Also, if $\lambda_n < 0$, then by elementary properties of the tangent function,

$$\Phi_0 \Lambda_i \Lambda_n \geq h(\lambda_i,\mu)\lambda_n .$$

and
$$\Phi_0 \sum_{i=1}^{n-1} \Lambda_i \Lambda_n \geq \sum_{i=1}^{n-1} h(\lambda_i,\mu)\lambda_n ,$$

whereas $\lambda_n \geq 0$ implies

$$\Phi_0 \sum_{i=1}^{n-1} \Lambda_i \Lambda_n = \Phi_0(\operatorname{tr} S - \Lambda_n)\Lambda_n \geq 0$$

since $\operatorname{tr} S - \Lambda_n \geq 0$ and $\Lambda_n \geq 0$ in this case.

Together, we have

$$\Phi_0 \sum_{i=1}^{n-1} \Lambda_i \Lambda_n \geq \sum_{i=1}^{n-1} h(\lambda_i,\mu)k(\lambda_n) .$$

We can now estimate Ric for the prototype $\Phi_0$ :

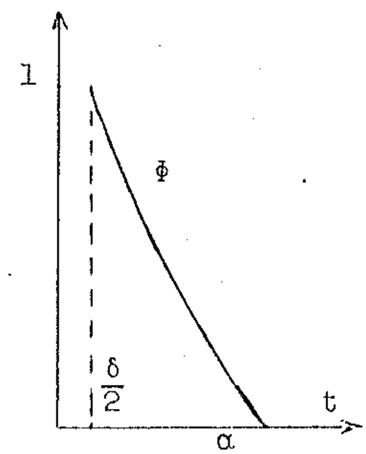
(9) $\mathrm{Ric}_1 \geq n - 1 + \sum_{i=2}^{n} h(\lambda_1,\mu)k(\lambda_i),$

(10) $\mathrm{Ric}_n \geq n - 1 + \frac{1}{2}\mu(1 + \frac{1}{\mu^2})k(\lambda_n) + \sum_{i=1}^{n-1} h(\lambda_i,\mu)k(\lambda_n)$

The functions $F_\mu^+$, $F_\mu^-$ defined as the right hand sides of (9) and (10) are continuous in the $\lambda_i$'s, and their minimum

(11) $F_\mu \equiv \min\{F_\mu^+, F_\mu^-\}$

defines a continuous function of the eigenvalues of S. We will take this as the definition of the $F_\mu$ in the hypothesis of 2.4, and then this hypothesis implies that (5') holds everywhere on $\partial\overline{M}$ for $\Phi_0$.




Step 2: In order to construct a function $\Phi$ which also satisfies (3), we shift the prototype $\Phi_0$ to the right a small amount $\frac{\delta}{2}$, where $\delta > 0$ is so small that

(12) $\Lambda_{max}(\frac{\delta}{2}) < \mu$,

(13) $F_\mu(S_{\frac{\delta}{2}}) > 0$, everywhere on $\partial\overline{M}$, and

(14) $\sum_{i=2}^{n} \Lambda_i\Lambda_1 - \sum_{i=2}^{n} h(\lambda_1,\mu)k(\lambda_i) > -\zeta$

(15) $\sum_{i=1}^{n-1} \Lambda_i\Lambda_n - \sum_{i=1}^{n-1} h(\lambda_i,\mu)k(\lambda_n) > -\zeta$

} in $[0,\delta]$, uniformly on $\partial\overline{M}$,

and where the number $\zeta > 0$ is defined as

(16) $\zeta = \frac{1}{2}\min_{\partial\overline{M}}\{F_\mu(S_0)\} > 0$ .

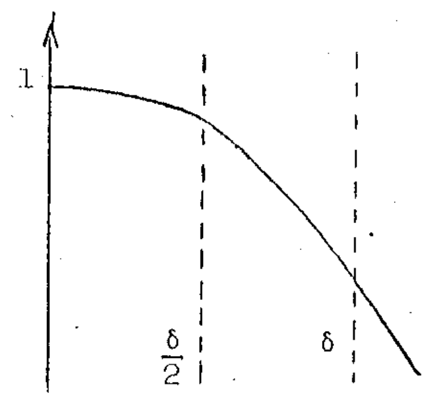
We now define $\Phi$ on $[\delta,\alpha]$, where $\alpha < \tan^{-1}\frac{1}{\mu} + \frac{\delta}{2}$ is a number specified in step 4.

$$(17)\quad \Phi(t) = \Phi_0(t - \tfrac{\delta}{2}) \qquad t \in [\delta,\alpha],$$

and observe that

$$\left.\begin{aligned} Ric_1 &\geq F_\mu(S_{\frac{\delta}{2}}) > 0 \\ Ric_n &\geq F_\mu(S_{\frac{\delta}{2}}) > 0 \end{aligned}\right\} \text{ on } [\delta,\alpha], \text{ by (13) and Step 1.}$$

$\therefore \Phi$ satisfies (1), (2), and (5') on $[\delta,\alpha]$.




Step 3: Define $\Phi$ on $[0,\frac{\delta}{2}]$ as $\Phi(t) = 1-t^2 \quad t \in [0,\frac{\delta}{2}]$ and define $\Phi$ on $(\frac{\delta}{2},\delta)$ as some $C^2$ decreasing function which joins $\Phi|[0,\delta]$ and $\Phi|[\delta,\alpha]$ smoothly and such that $\Phi' \geq -\mu(1+\frac{1}{\mu^2}) > \max_{[\delta,\alpha]} \Phi'$, which is clearly possible since $\Phi'(\frac{\delta}{2}) > \Phi'(\delta)$ and $\Phi(\frac{\delta}{2}) > \Phi(\delta)$.

Now $\Phi$ satisfies (1), (2) and (3).

Furthermore, (5') remains true because on $(0,\delta)$,

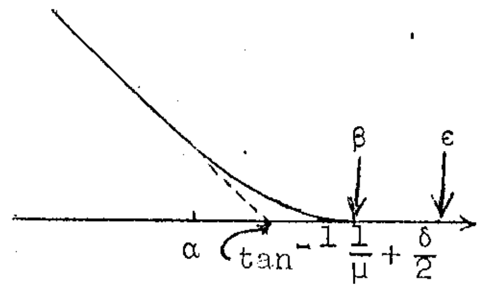
$$Ric_1 \geq n-1 + \sum_{i=2}^{n} h(\lambda_1,\mu)k(\lambda_i) - \zeta \qquad \text{by (14) and (5),}$$

$$> \tfrac{1}{2}F_\mu(S_0) \qquad \text{by (16),}$$

and $\mathrm{Ric}_n \geq n-1 + \mu(1+\frac{1}{\mu^2})k(\lambda_n) + \sum_{i=1}^{n-1} h(\lambda_i,\mu)k(\lambda_n) - \zeta$ by (15) and (5)

$> \frac{1}{2}F_\mu(S_0)$.

<u>Step 4</u>: It remains only to define $\Phi$ on $[\alpha,\epsilon]$ which satisfies (1) - (4) and (5').




Choose $\beta \in (\tan^{-1}\frac{1}{\mu}+\frac{\delta}{2},\epsilon)$ and find $A > 1$ so large that

(18) $\sum_{i=2}^{n} \Lambda_i\Lambda_1 \geq -A$, $\sum_{i=1}^{n-1} \Lambda_i\Lambda_n \geq -A$. } in $[0,\beta]$, uniformly over $\partial\overline{M}$.

Choose $\alpha \in (\delta, \tan^{-1}\frac{1}{\mu}+\frac{\delta}{2})$ so that

(19) $\Phi(\alpha) < \frac{\zeta}{A}$.

Finally, define $\Phi$ on $[\alpha,\epsilon]$ as some $C^2$ decreasing function smoothly joining $\Phi|_{[0,\alpha]}$ and the zero function on $[\beta,\epsilon]$ such that $\Phi' \geq -\mu(1+\frac{1}{\mu^2})$.

Then $\Phi$ satisfies (1) - (4).

As to (5') on $[\alpha,\epsilon]$,

$\mathrm{Ric}_1 \geq n-1 - \frac{\zeta}{A}\cdot A$ by (18) and (19),

$> \frac{1}{2}F_\mu(S_0)$, and

$$\mathrm{Ric}_n \geq n-1 + \frac{1}{2}\mu(1+\frac{1}{\mu^2})k(\lambda_n) - \frac{\zeta}{A} \cdot A$$

$$> \frac{1}{2}F_\mu(S_0).$$

This concludes the construction of $\Phi$ and the proof of 3.4.

Example. As a very simple illustration, we consider the case when $\overline{M}^{n+1}$ is a trivial disc bundle with boundary

$$\partial\overline{M} = S^p(r) \times S^q(s) \subset S^{n+1}(1), \; r^2 + s^2 = 1, \qquad p+q = n.$$

When $r = \sqrt{\frac{p}{p+q}}$ and $s = \sqrt{\frac{q}{p+q}}$, then $\partial\overline{M}$ is a minimal hypersurface in $S^{n+1}$ to which 2.4 applies.

The second fundamental tensor $S_0$ is

$$S_0 = (\sqrt{\frac{q}{p}})\,\mathrm{id}_{p\times p} \oplus (-\sqrt{\frac{p}{q}})\,\mathrm{id}_{q\times q}.$$

The condition of 2.4 becomes $q > \frac{1}{2} + \sqrt{\frac{n}{2} + \frac{1}{4}}$, and M has positive Ricci curvature in this case. In fact the same is true for any $q \geq 2$ by a choice of $\Phi$ more suitable to this example.

Actually, this can be proven more directly by virtue of the simple nature of the example. By giving $S^q$ its standard metric and $R^{p+1} \simeq \mathrm{int}\, D^{p+1}$ any complete metric of positive sectional curvature, then $M = R^{p+1} \times S^q$ has positive Ricci curvature in the product metric for $p \geq 1$, $q \geq 2$. If $q=1$, $\mathbb{R}^{p+1} \times S^1$ admits only $\mathrm{Ric} \geq 0$ and must always be a product, cf. [CG].

## 3. A New Class of Manifolds with Positive Ricci Curvature

In attempting to apply the results of Chapter 2, we were motivated to look for positively Ricci curved hypersurfaces in a sphere. Any compact hypersurface in a euclidean sphere is always the intersection of the sphere and a level set of a function smooth in the ambient euclidean space, and in this chapter we exhibit a large class of hypersurfaces arising from homogeneous functions in this way.

The Ricci curvature of such a hypersurface $M_0$ may not be positive in the metric induced from the sphere, but after a deformation of the sphere into an ellipsoid, the Ricci curvature of the new hypersurface $M_1$ of intersection becomes positive. $M_1$ is the orthogonal intersection of level sets, a fact which makes the Ricci curvature more easily computable.

Furthermore, a subclass of these hypersurfaces also has positive mean curvature. A compact connected hypersurface separates the ellipsoid into two components. Therefore, one of those components satisfies the conditions of 1.4. It seems most likely that Theorem 2.4 applies to many of the above examples, though at this point we have not carried out the (considerable) numerical analysis.

These hypersurfaces are of interest in their own right as new examples of compact manifolds of positive Ricci curvature. Our approach was suggested by a construction of such examples among Brieskorn varieties in the complex setting, cf. [H].

However, we work more generally with arbitrary homogeneous polynomials. In fact, our result says in particular, that in some "stable" sense, any real projective hypersurface admits positive Ricci curvature.

We first describe our family of hypersurfaces in the sphere and their deformation to hypersurfaces in ellipsoids.

<u>Notation</u>: For $i=1,2,\ldots,k$, let $f_i(x_i)$ be a homogeneous function on $\mathbb{R}^{n_i}$ of degree $d_i \geq 2$, where $x_i$ is the standard coordinate chart for $\mathbb{R}^{n_i}$. Set $n_1+\ldots+n_k = n$.

Consider the family $E_t$ of ellipsoids, $0 \leq t \leq 1$, defined as the zero set of $G_t : \mathbb{R}^{n+p+q} \to \mathbb{R}$ with

$$G_t(x_1,\ldots,x_k,y,z) = \alpha_1(t)|x_1|^2+\ldots+\alpha_k|x_k|^2 + |y|^2 + |z|^2 - r^2$$

where $\alpha_i(t) = 1 + t(\frac{2}{d_i} - 1)$, and $y,z$ are coordinates for $\mathbb{R}^p, \mathbb{R}^q$.

The assignment $t \mapsto G_t^{-1}(0) = E_t$ defines a smooth isotopy between the standard sphere $E_0 = S^{n+p+q-1}(r)$ of radius $r$ and the ellipsoid $E_1$ through ellipsoids $E_t$.

Now consider the function $F$ defined in terms of the homogeneous functions $f_1,\ldots,f_k$ by

$$F(x_1,\ldots,x_k,y,z) = f_1(x_1)+\ldots+f_k(x_k)+|y|^2-|z|^2.$$

The intersection of the zero levels of $F$ and $G_t$ defines a (possibly singular) hypersurface $M_t = M_t(f_1,\ldots,f_k;p,q)$ in the ellipsoid $E_t$. Our first observation is that if $f_i$ satisfies a mild restriction, then $M_t$ is actually nonsingular for all t, and $M_0$ is diffeomorphic to $M_1$.

Proposition 3.1: Suppose the origin in $\mathbb{R}^{n_i}$ is an isolated critical point for $f_i$, $1 \le i \le k$. Then for r sufficiently small, $M_t$ is a smooth hypersurface in $E_t$, $0 \le t \le 1$, and $M_0$ is diffeomorphic to $M_1$.

The proof is based on the following

Lemma 3.2. Suppose $f_i$ has no critical points in $0 < |x_i|^2 \le \frac{d_i}{2} r^2$. Then $\nabla F$ and $\nabla G_t$ are independent on $M_t$ for all $t \in [0,1]$.

Proof of 3.2. Otherwise, for some $a, b \in \mathbb{R}$ not both zero,

$$a\nabla F = b\nabla G_t \text{ at } (x_1,\ldots,x_k,y,z) \in M_t$$

$$\nabla F = (\nabla f_1,\ldots,\nabla f_k, 2y, -2z)$$

$$\nabla G_t = (2\alpha_1 x_1,\ldots,2\alpha_k x_k, 2y, 2z).$$

We get k+2 equations:

(1) $a\nabla f_i = b2\alpha_i x_i$, $\quad 1 \le i \le k$,

(2) $2ay = 2by$,

(3) $2az = -2bz$.

Furthermore, $(x_1,\ldots,x_k,y,z) \in M_t$ implies

(4) $f_1(x_1)+\ldots+f_k(x_k) + |y|^2 - |z|^2 = 0$

(5) $\alpha_1(t)|x_1|^2+\ldots+\alpha_k(t)|x_k|^2 + |y|^2 + |z|^2 = r^2$

We separate the argument into three cases:

Case 1: $y \neq 0$; Case 2: $z \neq 0$; Case 3: $y = 0$, $z = 0$.

Case 1: $y \neq 0 \Rightarrow a = b$ by (2).

Therefore, $z = 0$ by (3),

and $f_1(x_1)+\ldots+f_k(x_k) = -|y|^2$ by (4),

$< 0$ by hypothesis.

But $f_i(x_i) = \frac{1}{d_i}\langle \nabla f_i|x_i, x_i\rangle$ since $f_i$ homogeneous of degree $d_i$

$= \frac{1}{d_i} 2\alpha_i |x_i|^2$ by (1);

and $f_1(x_1)+\ldots+f_k(x_k) = \frac{2\alpha_1}{d_1}|x_1|^2+\ldots+\frac{2\alpha_k}{d_k}|x_k|^2$

$\geq 0$, contradiction.

Case 2: We can deal with this case in the same manner as for Case 1.

Case 3: $y = 0$, $z = 0$ implies

$f_1(x_1)+\ldots+f_k(x_k) = 0$ by (4)

and $\alpha_1|x_1|^2+\ldots+\alpha_k|x_k|^2 = r^2$ by (5).

Again, $af_i(x_i) = 2\frac{b}{d_i} \alpha_i |x_i|^2$, so $$0 = a\sum_{i=1}^{k} f_i(x_i) = 2b\sum_{i=1}^{k}\frac{\alpha_i}{d_i}|x_i|^2,$$

and this implies that $b = 0$ by (5).

Therefore $\nabla f_i = 0$ at $x_i$ by (1).

This contradicts the hypothesis that $f_i$ has no critical points in $0 < |x_i|^2 \leq \frac{d_i}{2} r^2$ for every i.

We can now prove Proposition 3.1:

Let $H : \mathbb{R}^{n+p+q} \times [0,1] \to \mathbb{R}^2$ be defined by

$H(x_1,\ldots,x_k,y,z,t) = (F(x_1,\ldots;x_k,y,z),\ G_t(x_1,\ldots,x_k,y,z))$.

First observe that (0,0) is a regular value for H, because $H_*$ can be represented relative to the standard basis for $\mathbb{R}^{n+p+q} \times [0,1]$ as the $2\times(n+p+q+1)$ - matrix

$$\begin{pmatrix} \bar{\nabla} F \\ \bar{\nabla} G \end{pmatrix} = \begin{pmatrix} \nabla F & 0 \\ \nabla G_t & \frac{\partial}{\partial t} G_t \end{pmatrix}$$

where $\bar{\nabla}$ is the gradient in $\mathbb{R}^{n+p+q} \times [0,1]$, and $\nabla$ is the gradient in $\mathbb{R}^{n+p+q}$.

It now follows by 3.2 that $H_*$ has maximal rank on $H^{-1}(0,0)$, so $H^{-1}(0,0)$ is a smooth submanifold of $\mathbb{R}^{n+p+q} \times [0,1]$ with boundary $M_0 \amalg M_1$.

Furthermore, the function $\pi : H^{-1}(0,0) \to \mathbb{R}$ defined by $\pi(x_1,\ldots,x_k,y,z,t) = t$ is nonsingular on $H^{-1}(0,0)$. In fact

$\overline{\nabla}\pi = (0,0,0,1)$ and so $\overline{\nabla}\pi \notin \text{span}\,\{\overline{\nabla}F, \overline{\nabla}G\}$.

We can conclude from this fact that $M_0$ is diffeomorphic to $M_1$.

The reason for working with $M_1$ is that this hypersurface in the ellipsoid $E_1$ is the orthogonal intersection of the hypersurfaces $F^{-1}(0)$ and $G_1^{-1}(0)$:

$$\nabla F = (\nabla f_1, \ldots, \nabla f_k, 2y, -2z)$$

$$\nabla G_1 = \left(\frac{4}{d_1} x_1, \ldots, \frac{4}{d_k} x_k, 2y, 2z\right)$$

$$\langle \nabla F, \nabla G_1 \rangle = \sum_{i=1}^{k} \frac{4}{d_i} \langle \nabla f_i, x_i \rangle + 4|y|^2 - 4|z|^2$$

$$= 4\, F(x,y,z) \equiv 0 \qquad \text{on } M_1.$$

This observation makes it possible to calculate the Ricci curvature for $M_1$ much more easily via the Gauss equations, which we now proceed to do.

Let Ric be the Ricci tensor of $M_1$, $S$ the second fundamental tensor of $M_1$ in $E_1$, and $K_{min}$ the minimum sectional curvature of $E_1$.

<u>Proposition 3.3.</u> For $v \in TM_1$ of unit length,

$$\text{Ric}\, v \geq (n+p+q-3)K_{min} - \frac{1}{\|\nabla F\|^2} \|H_F\| \left( \sum_{i=1}^{k} |\text{tr}\, H_{f_i}| + 2|p-q| + 3\|H_F\| \right)$$

<u>Proof</u>. The Gauss equation for $M_1 \hookrightarrow E_1$ is

(6) $\quad \mathrm{Ric}\, v = \overline{\mathrm{Ric}}_1\, v + \mathrm{tr}\, S \cdot \langle Sv, v\rangle - \|Sv\|^2$

where $\overline{\mathrm{Ric}}_1$ is the projection of the Ricci tensor $\overline{\mathrm{Ric}}$ of $E_1$ to $TM_1$.

We need to compute $S$ at $x \in M_1$.

Let $v \in T_x M_1 = (dF)_x^{-1}(0) \cap (dG_1)_x^{-1}(0)$

be a unit vector.

$$Sv = \left[\nabla_v \frac{\nabla F}{\|\nabla F\|}\right]^T$$

where $\nabla$ is the covariant derivative in $\mathbb{R}^{n+p+q}$ and $T$ denotes tangential projection onto TM. So, if as usual $H_F v = \nabla_v \nabla F$ is the (self-adjoint) Hessian tensor of $F$, then

(7) $\quad Sv = \frac{1}{\|\nabla F\|} H_F^T v$

and

$$\langle Sv, v\rangle = \frac{1}{\|\nabla F\|} \langle H_F^T v, v\rangle$$

$$= \frac{1}{\|\nabla F\|} \langle H_F v, v\rangle$$

implies

(8) $\quad |\langle Sv, v\rangle| \le \frac{1}{\|\nabla F\|} \|H_F\|,$

where the norm

(9) $\quad \|H_F\| = \max_{\|v\|=1} \{ |\langle H_F v, v\rangle| \,\big|\, v \in T_x M\}$

is the maximum of the eigenvalues of $H_F$ in absolute value. Observe that

(10) $H_F = H_{f_1} \oplus \dots \oplus H_{f_k} \oplus (2id_{p\times p}) \oplus (-2id_{q\times q}) : R^{n+p+q} \to \mathbb{R}$, and

(11) $\|H_F\| \le \max\{\|H_{f_1}\|, \dots, \|H_{f_k}\|, 2\}$.

From (6), we obtain

$$\operatorname{tr} S = \frac{1}{\|\nabla F\|} \operatorname{tr} H_F^T$$

$$= \frac{1}{\|\nabla F\|}\left[\operatorname{tr} H_F - \langle H_F \frac{\nabla F}{\|\nabla F\|}, \frac{\nabla F}{\|\nabla F\|}\rangle - \langle H_F \frac{\nabla G}{\|\nabla G\|}, \frac{\nabla G}{\|\nabla G\|}\rangle\right]$$

Therefore, using (9), (10),

(12) $|\operatorname{tr} S| \le \frac{1}{\|\nabla F\|}\left(\sum_{i=1}^{k} |\operatorname{tr} H_{f_i}| + 2|p-q| + 2\|H_F\|\right)$.

The proposition follows by estimating the terms in (6) using (8), (11) and (12).

The previous proposition allows us to estimate the Ricci curvature globally over $M_1$, giving the main result of this chapter.

<u>Theorem 3.4</u>. Given arbitrary homogeneous polynomials $f_1, \dots, f_k$ as above and any integer s, then there exists an integer $r \ge 0$ such that the hypersurface $M_1 = M_1(f_1, \dots, f_k; p, q)$ defined above has positive Ricci curvature everywhere when $p-q = s$, $p+q \ge r$.

Proof. We will show that the negative term in the lower bound for Ric of 3.3 can be bounded below, globally over $M_1$, by a number depending only on p-q and $f_1,\ldots,f_k$.

Let $B_i$ be the ball in $\mathbb{R}^{n_i}$ of radius $r\sqrt{\frac{d_i}{2}}$, and let $H_i = \sup_{B_i}\|H_{f_i}\|$, $K_i = \sup_{B_i}|\operatorname{tr} H_{f_i}|$. Then

$$\sup_{M_1}\|H_F\| \leq \max\{H_1,\ldots,H_k,2\} = H,$$

and

$$\|H_F\|\Big(\sum_{i=1}^{k}|\operatorname{tr} H_{f_i}| + 2|p-q| + 2\|H_F\|\Big)$$

$$\leq H\Big(\sum_{i=1}^{k} K_i + 2|p-q| + 2H\Big),$$

where the right hand side depends only on the difference p-q.

Also, $\frac{1}{\|\nabla F\|^2} \leq C(f_1,\ldots,f_k)$ a constant depending only on $f_1,\ldots,f_k$, but not on p and q. To see this, observe that $\|\nabla F\|^2 \geq \overline{C}(f_1,\ldots,f_k) > 0$ on $M_1 \cap \{y=0,\ z=0\}$, (where $\overline{C}$ depends only on $f_1,\ldots,f_k$), which follows from our assumptions on $f_1,\ldots,f_k$ and the identity

$$\|\nabla F\|^2 = \sum_{i=1}^{k}\|\nabla f_i\|^2 \text{ on } M_1 \cap \{y=0,z=0\}.$$

So there exists a number $\varepsilon > 0$ so small that

$$\|\nabla F\|^2 \geq \tfrac{1}{2}\overline{C} > 0 \text{ on } M_1 \cap \{|y|^2 + |z|^2 \leq \varepsilon\}.$$

Meanwhile, on $M_1 \cap \{|y|^2 + |z|^2 > \epsilon\}$,

$$\|\nabla F\|^2 = \sum_{i=1}^{n} \|\nabla f_k\|^2 + 4|y|^2 + 4|z|^2 \geq 4\epsilon.$$

Altogether, $\|\nabla F\|^2 \geq \min\{4\epsilon, \frac{1}{2}\overline{C}\} = L$ everywhere on $M_1$,

and
$$\mathrm{Ric} \geq (n+p+q-3)K_{min} - \frac{H}{L}\left(\sum_{i=1}^{k} K_i + 2|p-q| + 2H\right).$$

Since the negative term in this expression is bounded below on $M_1$ independently of $p+q$ when $p-q = s$ is fixed beforehand, the theorem follows.

Collary 3.5. Given $f_1, \ldots, f_k$ as above, there exist integers $s_0, r \geq 0$ such that $M_1 = M_1(f_1, \ldots, f_k; p, q)$ has positive mean curvature and positive Ricci curvature for $p-q = s_0$ and $p+q \geq r$. (The mean curvature is taken relative to the normal direction of $\nabla F$.)

Proof. We can bound the formula for $\mathrm{tr}\, S$ from below by

$$\mathrm{tr}\, S \geq \frac{1}{\|\nabla F\|}\{2(p-q) - \sum_{i=1}^{k} |\mathrm{tr}\, H_{f_i}| - 2\|H_F\|\}.$$

$$\geq \frac{1}{\|\nabla F\|}\{2(p-q) - (\sum_{i=1}^{k} K_i + 2H)\}.$$

By taking $s_0$ to be any integer such that $2s_0 - (\sum_{i=1}^{k} K_i + 2H) > 0$, and $r$ to be the integer specified in 3.4 corresponding to $s = s_0$, the corollary follows.

We conclude our discussion by raising two purely topological problems:

I) Analyse completely the topology of the manifolds $M_1(f_1,\ldots,f_k;p,q)$ in terms of the numbers p and q, and the real projective varieties defined by $f_1,\ldots,f_k$.

II) Decide when $M_1^{n+p+q-2}$ does not bound a disc bundle in the sphere $S^{n+p+q-1}$. This should be the case for most examples, and it would be particularly interesting in view of the extension problem for positive Ricci curvature (cf. also the introduction).

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