Recurrent Leaves in Foliations

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We, the disserattion committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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The thesis establishes a quasi-isometry property which must hold for a Riemannian manifold, if it is to appear as a minimal leaf in a compact foliated manifold.

First we prove a generalization to foliations of a result originally stated and proved by Birkhoff about flows. This is our Theorem 1; it states a leaf is minimal iff it is recurrent. The definition of recurrent leaf is a straightforward generalization of the classical notion of a recurrent orbit in a flow.

Theorem 1, together with some facts about the normal bundle to the foliation, is used to show Theorem 2, which states that a minimal leaf is "quasi-homogeneous". This last term is defined in the thesis.

Finally, examples are given of leaves which appear in foliations of all pairs (p,q) of dimension and codimension with $p > 1$, $q > 1$, such that the leaves in question appear in these foliations as non-minimal leaves, but they are not quasi-homogeneous.

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## Acknowledgements

I wish to thank my advisor Tony Phillips for steering me in a direction in which there are so many open questions that if one can't be answered, not many steps need be taken before encountering another. I am grateful to him for having pointed out to me at many turns, just what it was I was trying to say.

A foliated manifold is locally modelled on an affine space decomposed into parallel affine subspaces.

Definition. A p-dimensional class $C^r$ foliation $\mathcal{F}$ of an m-manifold $M^m$ is a decomposition of M into connected sets $L_\alpha$ in such a way that each point of M has a neighborhood U and a distinguished chart $x = (x^1, x^2, \ldots, x^m): U \longrightarrow R^m$, where the components of $L_\alpha \cap U$ are given by equations

$$x^{p+1} = \text{constant}, \ \ldots, \ x^m = \text{constant}.$$

Such a foliation is denoted $\mathcal{F} = \{\mathcal{L}_\alpha\}$. The codimension of $\mathcal{F}$ is the difference $q = m-p$. The connected sets $L_\alpha$ are called leaves.

It follows from the definition that the leaves are p-dimensional manifolds (possibly non-properly) embedded in M. This thesis deals only with foliations of compact M. If we choose a Riemannian metric $\hat{g}$ on $M^m$, then each leaf $L_\alpha$ as an embedded p-manifold will inherit a metric which we will denote $d_\alpha$, or simply d if the reference to the leaf is clear. Consider now a diffeomorphism $H: M \longrightarrow M'$. The foliation $\mathcal{F}$ on M will in a natural way induce another foliation $\mathcal{F}'$ on M' whose leaves $L_\alpha'$ are the H-images of the leaves $L_\alpha$ of $\mathcal{F}$. Since M is compact, it follows there will be global bounds on the factor by which $H_*$ can shrink or stretch a tangent vector of M:

$$k \leq \|H_* \vec{v}\| / \|\vec{v}\| \leq K$$

Such global bounds $(k,K)$ on shrinking or stretching of tangent vectors for a diffeomorphism are usually called dilation constants. Since the diffeomorphism between $L_\alpha$ and $L_\alpha'$ is just a restriction of $H$, the same dilation constants $(k,K)$ serve for $H|_L : L_\alpha \longrightarrow L_\alpha'$. A diffeomorphism between manifolds with global bounds on dilation is called a quasi-isometry. The above remarks show that, if $L_\alpha$ is a leaf of a foliation $\mathfrak{F}$ of a compact M, then it has a well-defined quasi-isometry type, independent of the metric $\hat{g}$ chosen on M.

If one considers the question, "What do leaves of a foliated compact manifold look like?" one might put the question more precisely: [D. Sullivan: Proc ICM 1974]

> Question 1. Which quasi-isometry types $[L^p]$ of p-dimensional Riemannian manifold admit realizations as leaves in some foliation of some compact M (depending on $[L^p]$)?

This question can be approached negatively by defining various notions of regularity and showing that "irregular" quasi-isometry types of p-manifolds cannot occur as leaves of foliated compact manifolds.

A manifold which appears as a leaf of such a foliation must of course be complete in its metric. In addition to this, there is a regularity that a manifold must have if it appears as a leaf in some compact foliated M. We can define a Riemannian p-manifold to be "plaque-like", provided it admits a cover by geodessically convex neighborhoods, $L = \bigcup_a N_a$, and there are diffeomorphisms $f_a\colon D^p \longrightarrow N_a$ from the standard disc in $R^p$ with its Euclidian metric onto each $N_a$ with their metrics inherited from L, and these diffeomorphisms $f_a$ all share a common pair (k,K) of dilation constants, in the sense that

$$k \leq \|f_a^* v\| / \|v\| \leq K \quad \text{for } v \in TD^p.$$

If $L^p$ occurs as a leaf in a foliation of some compact $M^m$, then L must be "plaque-like" with respect to the metric it inherits from M. This follows directly from the existence of a "plaque decomposition" of M into a finite number of "flow boxes". These last two terms will be defined later.

For an example of a non-plaque-like quasi-isometry type, consider a plane in $R^3$ with a sequence of smooth spikes going off to $\infty$, and becoming progressively longer and thinner. If we give this plane the metric it inherits from $R^3$, then it is not plaque-like.

So a leaf of a foliation must be plaque-like. The converse would say that if a p-manifold is plaque-like, then it must appear as a leaf in some compact foliated manifold. Does this hold? If not, what additional conditions must a p-manifold satisfy to ensure it appears as a leaf in some compact foliated manifold? This type of question, to the effect that "If a p-manifold is regular, must it occur?" will not be pursued here. Its answer would involve a construction, from the data of the regularity notion, of an appropriate compact foliated manifold in which $L^p$ occurs as a leaf. It is (as far as I know) quite possible that the assumption "L is plaque-like" might be enough to guarantee it appears, perhaps with very high codimension, as a leaf in some compact foliated manifold. [M. Gromov has a conjecture in this direction.]

Let P be a property which can be stated about a leaf L in a foliation $\mathfrak{F}$ of a compact M. This property P might depend only on the leaf L with its metric d, as is the case for example when P is the property "L has polynomial growth". On the other hand, the property P may depend on the way L is embedded in M. An example here is when P is the property "the closure of L in M is a minimal set of leaves". For such a property P we may ask the question:

> Question 2. Which quasi-isometry types of p-manifold $[L^p]$ admit realizations as leaves with the property P in some compact foliated M (depending on $[L^p]$)?

As with question 1 above, this question can be approached negatively by defining a regularity notion and showing that an irregular quasi-isometry type of p-manifold cannot occur as a leaf with property P in any foliated compact manifold whatsoever.

For the special case of class $C^2$, transversally oriented, codimension one foliations of compact manifolds, some answers to question 2 are known. Cantwell and Conlon[1] have given what amounts to an answer to question 2 where P is taken to be the property "L is a nowhere dense leaf of $\mathfrak{F}$ which has polynomial growth." Note that this property refers both to the metric on L and to the way L is embedded in M.

To describe their result we need the notion of the <u>class</u> of a leaf in a foliation. The leaves of class 0 are the compact leaves. A leaf is of class $k > 0$ if it is asymptotic only to leaves of class at most $k-1$ and to at least one leaf of class exactly $k-1$. To define the <u>asymptote</u> of a leaf L, choose an exhaustion $A_1 \subset A_2 \subset \ldots \subset L$ by compact sets; the asymptote is then $\bigcap_{i} \mathrm{closure}(L - A_i)$. One leaf is said to be <u>asymptotic</u> to another if the latter is in the asymptote of the former. Cantwell and Conlon show the following:

<u>Theorem</u> [C-C Thm. 1] The leaves of class k are exactly the nowhere dense leaves having polynomial growth of degree k.

The notion of class k still refers to the way L is embedded in M. In order to answer question two we must show that class k implies some property of the leaf L which depends only on its quasi-isometry type, and not on the ambient metric. Cantwell and Conlon do not explicitly define such a quasi-isometry property. However they do prove a theorem from which such a property easily follows. The theorem in question depends on the notion of an "infinite repetition", which we now describe.

Since the foliation is assumed to have codimension one, we can fix a transverse foliation $\mathfrak{X}$ of dimension one whose leaves are transverse to the leaves of $\mathfrak{F}$. We use a transverse orientation for $\mathfrak{F}$ to define a linear order in each leaf of $\mathfrak{X}$. Thus we can talk about subarcs $[x,y]$, $(x,y]$, etc. of such leaves.

Definition. Let $(x_0,x]$ be a subarc of a leaf of $\mathcal{X}$, and let $L_0$ be the leaf of $\mathcal{F}$ through $x_0$. If $(x_0,x] \cap L_0 = \emptyset$, we say that x projects (in the negative direction) to $L_0$ and we write this $p(x) = x_0 \in L_0$.

Let L and $L_0$ be leaves of $\mathcal{F}$ and let $B \subset L$ be a complete connected noncompact submanifold of dimension m-1 (= dimension L) with a single boundary component $N_0 = \partial B$ which is a compact connected manifold. Finally, suppose that each point of B projects to $L_0$ as above and remark that $p:B \longrightarrow L_0$ is locally a diffeomorphism.

Definition. We will say that B is an infinite repetition of $L_0$ (on the positive side) if the following conditions are satisfied.

a) $B = \bigcup B_i$ where each $B_i$ is a complete m-1 dimensional submanifold, $\partial B_i = N_i \cup N_{i+1}$ is a union of two compact components, and $\operatorname{int}(B_i) \cap \operatorname{int}(B_j) = \emptyset$ when $i \neq j$,

b) There is a compact connected m-2 dimensional manifold $N \subset L_0$ (called the juncture of the repetition) such that $p|N_i$ maps $N_i$ diffeomorphically onto N, $0 \leq i < \infty$,

c) For each $y \in L_0$ and each $i \geq 0$, $p^{-1}(y) \cap (B_i - N_{i+1})$ is a single point $y_i$,

d) For each $y \in L_0$, the sequence $p^{-1}(y) = \{y_i\}$ converges monotonically to y in $[y, y_0]$.

Theorem. [C-C Prop. 2] Let L be a leaf of $\mathcal{F}$ of class $k \geqslant 1$. Then L can be written

$$L = A \cup B^1 \cup \ldots \cup B^r$$

where A is a compact connected m-1 manifold with boundary components $N^1, \ldots, N^r$, and

1.) $A \cap B^j = N^j$, $1 \leqslant j \leqslant r$,
2.) $B^i \cap B^j = \emptyset$, $i \neq j$,
3.) Each $B^j$ is an infinite repition of a leaf $L^j$ of class at most k-1,
4.) For at least one value of j, $L^j$ is exactly of class k-1.

From [C-C Prp. 2] we will derive a quasi-isometry property, defined only via the metric on $L^{m-1}$, which a leaf must possess if it appears as a leaf of class k for some $k \geqslant 1$ in some transversally orientable $C^2$ codimension 1 foliation of a compact manifold. Note that, because of [C-C Thm 1], such a property will constitute a necessary condition for L to appear as a nowhere dense leaf of polynomial growth in some compact foliated manifold, giving an answer to question 2 in this context.

We treat here the case of class 1 leaves, which by [C-C Thm 1] is the same as nowhere dense leaves of linear growth. Now by [C-C Prop.2] such a leaf can be written

$$L = A \cup B^1 \cup \ldots \cup B^r$$

where A is compact and connected, with boundary components $N^1, \ldots, N^r$, and each $B^j$ is an infinite repition of a compact (class 0) leaf $C^j$, and where $B^j$ meets A along $N^j$. Write the repitition for $B^j$ as

$$B^j = B_0^j \cup B_1^j \cup \cdots$$

with juncture $J^j \subset C^j$. We see from the definitions that for fixed j all the $B_i^j$ are diffeomorphic via the projection map p. Furthermore the compactness of the $B_i^j$ together with condition (c) in the definition of infinite repitition shows that, for a fixed j, all the $B_i^j$ are diffeomorphic to a fixed model $E^j$, obtained by splitting $C^j$ along $J^j$ and doubling $J^j$, and these diffeomorphisms $f_i^j : B_i^j \longrightarrow J^j$ all share a single pair (k,K) of dilation constants. Thus in the description

$$(*) \quad L = A \cup B^1 \cup \cdots \cup B^r$$

each noncompact component $B^j$ consists of a "quasi-isometric repetition" of compact pieces $J^j$. We formalize this notion:

<u>Definition</u> Given a noncompact manifold $B^p$ whose boundary is a compact connected manifold $N^{p-1}$, we say $B^p$ is a <u>quasi-isometric repetition</u> of the compact manifold-with-boundary $A^p$ (called the <u>model</u>) if

1.) $A^p$ has just two boundary components, diffeomorphic to $N^{p-1}$ and denoted $\partial-$, $\partial+$,

2.) There is a collection of diffeomorphisms $f_k : A \longrightarrow B$ where

a) the images $f_k(A)$, called <u>pieces</u>, fit together along their ends $f_k(\partial+)$ and $f_k(\partial-)$ to exhaust B,

b) the diffeomorphisms $f_k$, k = 1, 2, ..., all share a common pair (k,K) of dilation constants.

With this definition it is clear that the expression (*) for L is a quasi-isometry invariant.

We will give details later for the construction of the following foliation. It contains leaves of linear growth which fail to be quasi-isometry repetitions.

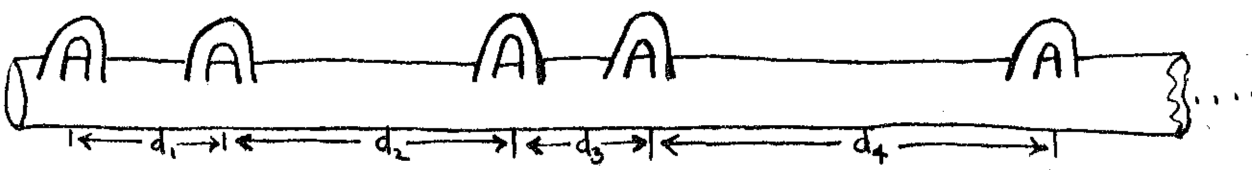
Example 1. There exists a codimension two foliation $\mathfrak{F}_1$ of a compact manifold $M^4$ having the following properties:

1.) $\mathfrak{F}_1$ has one compact leaf T, a torus.

2.) T is in the limit set (asymptote) of every leaf of $\mathfrak{F}_1$. (In particular T is the only minimal leaf)

3.) Each leaf of $\mathfrak{F}_1$ other than T has two ends.

4.) At least one end of each leaf has the quasi-isometry type of a cylinder $R^+ \times S^1$, with handles attached, spaced at intervals $d_k$, where the sequence $d_k$ is unbounded as k increases (see figure below).




By considering the product $M^4 \times S^1 = M^5$, foliated by surfaces (leaf of $\mathfrak{F}_1$) $\times$ (point of $S^1$), we obtain a foliation $\mathfrak{F}_2$ of the compact manifold $M^5$. The leaves of $\mathfrak{F}_2$ are nowhere dense, have linear growth, and fail to be quasi-isometry repetitions. This example shows that in general codimension, one does not have the counterpart to [C-C Thm 1]; that is, nowhere dense leaves having linear growth need not be of class 1. The non-compact leaves of $\mathfrak{F}_1$ or $\mathfrak{F}_2$ fail to have ends which are quasi-isometry repetitions of compact pieces, because the increasing lengths between handles make it impossible to obtain a single pair of dilation constants for the family of maps from the model to the **pieces** . (See the definition of quasi-isometry repetition.) "Ends" of a non-compact manifold, in particular of a leaf in a foliation. will be defined in the appendix.

It is one of the purposes of this thesis to give some answer to question 2 **for foliations** ., where leaf dimentions and codimensions are arbitrary, and the foliations are assumed to be of class $C^1$. We will take P in question 2 to be the property "L is a minimal leaf".

Definition. A minimal leaf in a foliation is a leaf whose closure in the foliated manifold is a minimal set of leaves. A minimal set of leaves in a foliated compact manifold M is a union of leaves which is closed in M, and has no proper subsets which are closed unions of leaves.

We will define a notion of quasi-homogeneous,(which will be a quasi-isometry property of Riemannian p-manifolds $L^p$) and show that any minimal leaf must be q-homogeneous . It will turn out that the leaves of Example 1 are not q-homogeneous. This implies that, viewing these leaves as Riemannian 2-manifolds, they cannot appear as minimal leaves in any compact foliation at all, though they do appear in the example as non-minimal leaves.

Definition. A Riemannian p-manifold L is said to be quasi-homogeneous(or q-homogeneous) if there exist dilation constants $(k,K)$ such that for each $r > 0$ there is $R > 0$, so that any "cover ball" $\widetilde{B}(x,r)$ can be "cover immersed" in any metric ball $B(y,R)$, and all the maps

$$f:\widetilde{B}(x,r) \longrightarrow B(y,R)$$

which arise this way share the common pair $(k,K)$ of dilation constants.

In this definition, a "cover ball" $\widetilde{B}(x,r)$ is the set of paths from x in L of length at most r, where two such paths are identified if they end at the same point and are homotopic rel boundary through paths of length at most r.

Also a cover immersion is an immersion of such a cover ball $\widetilde{B}(x,r)$ into L which has the property that $f(\gamma) = f(\gamma')$ implies that the paths $\gamma$ and $\gamma'$ end at the same point.

Theorem 2. If L is a minimal leaf in a compact foliated manifold, then L is quasi-homogeneous with its metric inherited from the foliated manifold.

The proof of this theorem relies on two things. One is the idea that leaves in a foliation pull apart in a continuous way. (This is formalized in Lemma 1) The other is the following theorem, generalizing some results of Birkhoff concerning flows on metric spaces.

Theorem 1. Let L be a leaf of the foliation $\mathfrak{F}$ of the compact manifold M. Then L is minimal if and only if L is recurrent.

The definition of recurrent follows:

Definition. Let L be a leaf of a compact foliated manifold $M^m$. Let $\hat{d}$ be the distance function on M, and d the induced distance function on L. (d comes from the Riemannian metric which L inherits from M.) Then L is said to be recurrent if given $\varepsilon > 0$ there is $T(\varepsilon) > 0$, such that the entire leaf L is within $\varepsilon$ (in the $\hat{d}$-metric) of the leaf-ball $B_d(x, T(\varepsilon))$. This T must depend only on $\varepsilon$ and not on the point $x \in L$.

Theorem 1 allows us to pass from the property "L is minimal" to the intermediate property "L is recurrent". Then to show Theorem 2, we use the recurrence condition along with the normal bundle to the foliation, in order to define maps between nearby leaves which serve as the cover immersions in the definition of quasi-homogeneous.

Theorem 2 gives a restriction on the possible quasi-isometry types of manifold L which can occur as minimal leaves in foliated compact manifolds. Later we will exhibit a class $E(p,q)$ $p,q > 1$, of examples of foliations, where in each case most of the leaves fail to be quasi-homogeneous.

THE NORMAL BUNDLE AND MAPS BETWEEN NEARBY LEAVES

Give $M^m$ a $C^\infty$ Riemannian metric $\gamma$ inducing a distance function $\hat{d}$ on M. The leaves $L_\cdot \in \mathfrak{F}$ inherit Riemannian metrics and corresponding distance functions $d_\iota$. We will always write d for metrics in leaves. The leaves are p-dimensional manifolds (possibly non-properly) imbedded in M. Tangent planes to leaves are each p dimensional subspaces of TM; the collection of complements to these is $\vec{n}$, the "normal bundle" to the foliation $\mathfrak{F}$. Thus for any $x \in M^m$, $\vec{n}(x)$ is a q-dimensional plane in $TM_x$ transverse to the leaves of $\mathfrak{F}$. If the leaves of $\mathfrak{F}$ are $C^2$ smooth, the bundle $\vec{n}$ is $C^1$ smooth. Even if the leaves are only $C^1$ smooth, there still exists a smooth complement to the bundle of tangent planes to leaves; pick one and call it $\vec{n}$ in what follows.

The exponential map $\exp: TM \longrightarrow M$ is defined in a neighborhood of M = zero-section(TM). Restriction of exp to the sub-bundle $\vec{n}$ gives a map defined in a neighborhood of M = zero-section($\vec{n}$):

$$\exp: \vec{n} \longrightarrow M,$$

and exp has rank m along the zero section M. The fiber dimension of n is q, the leaves have dimension p, and $p+q = m$. It follows that, if L is a leaf of $\mathfrak{F}$ and K a compact part of L,

$$\exp: \vec{n}|_K \longrightarrow M^m$$

is an embedding near K. More precisely, given such $K \subset L$, there is a disc bundle of size $\delta > 0$ around the zero section of $\vec{n}|_K$, denoted $\vec{n}_\delta(K)$, s.t. exp embeds $\vec{n}_\delta(K)$ into $M^m$.

If $\delta$ is sufficiently small, each disc $\vec{n}_\delta(y)$ with $y \in K$ is then mapped by exp to a q-disc in M transverse to the leaves of $\mathcal{F}$. We will sometimes write $\vec{n}_\delta(y)$ instead of $\exp(\vec{n}_\delta(y))$ when the context implies $\vec{n}_\delta(y) \subset M$. Now we make precise the idea that leaves in a compact foliated manifold pull apart in a continuous way; in fact, such pulling apart is uniform:

Lemma 1. Let $\varepsilon$ and A be given positive numbers. Then there are positive numbers $\varepsilon_1 < \varepsilon$ and $\delta$, such that:

1.) For any $x \in M$, $\vec{n}_{\varepsilon_1}[B(x,A)] \xrightarrow{\exp} M$ is an immersion.

2.) Given $\sigma$ a smooth path from x in the leaf $L_x$ with length $\|\sigma\|$ less than A, then for any $y \in \vec{n}_\delta(x')$, $\sigma$ lifts via $\vec{n}_{\varepsilon_1}[B(x,A)]$ to a path $\bar{\sigma}$ from y in the leaf $L_y$, and we have $\big|\|\sigma\| - \|\bar{\sigma}\|\big| < \varepsilon$.

3.) The map defined by path lifting

$$f: \widetilde{B}(x,A) \longrightarrow L_y \qquad \gamma \longmapsto \bar{\gamma}(1) \in L_y$$

is an immersion with dilation constants $(2^{-\varepsilon}, 2^{\varepsilon})$.

Proof. Statement 1 follows from the fact that exp has the right rank along the zero-section of $\vec{n}$. Statement 3 follows from 2 applied to sufficiently small $\varepsilon$, by compactness of M and smoothness of exp. The long proof of Statement 2 occupies the following pages.

For any positive numbers $(A,\varepsilon,\delta)$ let $H(A,\varepsilon,\delta)$ denote the statement:

$H(A,\varepsilon,\delta)$: If $y \in \vec{n}_\delta(x)$ and $\gamma_x$ is a path from x in $L_x$ with length $\|\gamma_x\| \leqslant A$, then $\gamma_x$ lifts via $\vec{n}$ to $\bar{\gamma}_x^y$ a path in $L_y$, and $\hat{d}(\gamma_x t, \bar{\gamma}_x^y t) \leqslant \varepsilon$, for all $t \in [0,1]$.

Then Lemma 1.2 says that given $A, \varepsilon$, there exists $\delta$ so that $H(A,\varepsilon,\delta)$ is true. In other words, if $S(A)$ is the statement "For each $\varepsilon$ there is $\delta$ so that $H(A,\varepsilon,\delta)$ is true," then Lemma 1.2 says $S(A)$ holds for all positive A.

To begin with, note that $S(A)$ implies $S(B)$ when $B < A$.

We also assert:

Claim 1: $S(A)$ implies $S(2A)$.

proof: Assume $S(A)$ holds. Let $\varepsilon_0$ be given. We must produce a $\delta$ so that $H(2A,\varepsilon_0,\delta)$ holds. We proceed as follows. First, apply $S(A)$ to $\varepsilon = \varepsilon_0$, producing $\delta_1 < \varepsilon_0$ so that $H(A,\varepsilon_0,\delta_1)$ holds. Then apply $S(A)$ to $\varepsilon = \delta_1$, producing a $\delta_2$ so that $H(A,\delta_1,\delta_2)$ holds. We claim that $\delta = \delta_2$ makes $H(2A,\varepsilon_0,\delta)$ true. To see this, let $\gamma_x$ be a path from x in $L_x$ with length $\|\gamma_x\| < 2A$. We can write $\gamma_x = \alpha_x \beta_{x'}$ where $\alpha_x$ and $\beta_{x'}$ are paths of length at most A. (The subscripts here indicate starting points of paths.) The path $\beta_{x'}$ begins at $x' = \alpha_x(1)$. Now to check $H(2A,\varepsilon_0,\delta)$ for the path $\gamma_x$, we pick any $y \in \vec{n}_\delta(x)$ and note first that $H(A,\delta_1,\delta_2)$ implies that $\alpha_x$ lifts to $\bar{\alpha}_x^y$ a path in $L_y$ from y, and we have

1.) $\hat{d}(\alpha_x t, \bar{\alpha}_x^y t) \leqslant \varepsilon_0$, for $t \in [0,1]$.

In particular at $x' = \alpha_x(1)$ and $y' = \overline{\alpha}_x^y(1)$ we have $y' \in \vec{n}_{\delta_1}(x')$ and so $H(A, \varepsilon_0, \delta_1)$ implies that $\beta_{x'}$ lifts via $\vec{n}$ to $\overline{\beta}_{x'}^{y'}$ and we have

2.) $\hat{d}(\beta_{x'}t, \overline{\beta}_{x'}^{y'}t) < \varepsilon_0$, for $t \in [0,1]$.

Then (1) and (2) together insure that the whole path $\gamma_x = \alpha_x \beta_{x'}$ satisfies

$$\hat{d}(\gamma_x t, \overline{\gamma}_x^y t) < \varepsilon_0, \text{ for } t \in [0,1], \text{ as long as } y \in \vec{n}_\delta(x).$$

This proves that $H(2A, \varepsilon_0, \delta)$ is true, hence proves Claim 1.

We have shown: (a) $S(A)$ implies $S(2A)$,

(b) $S(A)$ implies $S(B)$ when $B < A$.

We will show: (c) $S(A)$ holds for some positive $A$.

When (c) is shown, $S(A)$ will be shown for all positive $A$, by repeated application of (a) and (b).

In order to show (c) we must first show the following claim, which is weaker than (c), but allows us to define later a certain continuous function on $M$; this function will enable us to show (c). Initially we pick some fixed $\varepsilon_0$ small enough that $\vec{n}_{\varepsilon_0}|$ restricted to leaves of $\mathcal{F}$ is immersed by the exponential map.

Claim 2. There exists some positive $A$ and a $\delta > 0$ such that:

If $y \in \vec{n}_\delta(x)$ and $\gamma_x$ is of length $\|\gamma_x\| < A$, $\gamma_x$ a path in $L_x$, then $\gamma_x$ lifts via $\vec{n}_{\varepsilon_0}$ to $\overline{\gamma}_x^y$ a path in $L_y$ and we have

$$\hat{d}(\gamma_x t, \overline{\gamma}_x^y t) < \varepsilon_0.$$

(This claim is weaker than (c) since the quantifiers differ. In (c) they read $(\exists A)(\forall \varepsilon)(\exists \delta)$ whereas here they are $(\forall \varepsilon)(\exists A)(\exists \delta)$ )

Proof of claim 2.

Recall that $\vec{n}_{\epsilon_0}$ is the image under exp of a tubular neighborhood U of the zero-section of the normal bundle $\nu$ to the foliation. We can pull the foliation $\mathcal{F}$ on M back via exp to a foliation $\mathcal{F}'$ on U of dimension $p = \dim(\mathcal{F})$. The codimension of $\mathcal{F}'$ is $\dim(U) - p$ which is 2q. M itself is identified with the zero-section of $\nu$ and so is a compact subset of U; the foliation $\mathcal{F}'$ when restricted to M is just the original foliation $\mathcal{F}$. Since M is compact and contained in the foliated set U, there exist finitely many distinguished foliation neighborhoods $\{W_i\}$ for $\mathcal{F}'$, such that $M \subseteq \bigcup_i W_i$. Then the sets $V_i = M \cap W_i$ are finite in number and cover M. Hence we can pick some $A > 0$ so that $B_d(x, A) \subset B_{\hat{d}}(x, A) \subset$ some $V_i$, for each point $x \in M$. This is the A we will use to show claim 2; it remains to produce the $\delta$ appropriate $\delta$ to satisfy the claim, which we do now.

For each $x \in M \subset U$, choose open subsets $N_x$ and $N'_x$ of U such that $x \in N_x \subset \overline{N}_x \subset N'_x \subset$ some $W_i$. (The $N_x$ are m+q-dimensional sets.) Then the $N_x$ give an open cover of M. We assume the $N_x$ are small enough to give a cover subordinate to the cover $V_i$ above. In other words, each $N_x \cap M$ should be contained in one of the $V_i = W_i \cap M$ mentioned above. Now select $N_1, \ldots, N_k$ a finite subcover of M.

Under the exponential map, the metric $\hat{d}$ on M pulls back to a metric $\hat{d}^*$ on fibers of $\nu$. On any of the closed sets $\overline{N}_i \cap M$ each point $z \in \overline{N}_i \cap M$ is at a positive distance from the intersection $\nu(z) \cap (N'_i)^{comp}$ So the continuous function $z \longrightarrow d^*(z, \nu(z) \cap N_i'^{c})$ takes on a minimum

$\delta_i$ on the closed set $N_i \cap M$. If we set $\delta = \min \delta_i$, then this $\delta$, together with the A chosen above, satisfies the requirements of claim 2. Namely, if $y \in \vec{n}_\delta(x)$ where $x \in N_i$, and $B(x,A) \subset V_i$, then by our choice of $\delta$, y is forced to lie inside $N_i$, hence y lies in $W_i$, and so the plaque containing y also lies in $W_i$. (By a "plaque" is meant a leaf of $\mathcal{F}'|W_i$. Generally a plaque is a leaf of a foliation restricted to a distinguished chart.) Then any path from x of length less than A is contained in $B(x,A) \subset$ (plaque containing x) $\subset W_i$, and so lifts via $\vec{n}$ to a path in the plaque containing y. Since the latter plaque lies in $\vec{n}_{\varepsilon_0}$ we have shown claim 2.

Proof of (c) using Claim 2:

Recall we have chosen a fixed $\varepsilon_0$ so that $\vec{n}_{\varepsilon_0}|$(leaves of $\mathcal{F}$) immerses under exp. Claim 2 says there is $A_0$ and $\delta_0$ so that:

If $y \in \vec{n}_{\delta_0}(x)$ and $\gamma_x$ is a path in $L_x$ of length $\|\gamma_x\| < A_0$ then $\gamma_x$ lifts via $\vec{n}$ to $\bar{\gamma}_x^y$ and we have

$$\partial(\gamma_x t, \bar{\gamma}_x^y t) < \varepsilon_0.$$

We now show (c) from this; that is, we show the statement $S(A_0)$.

$S(A_0)$: For each $\varepsilon$ there is $\delta$ so that $H(A_0, \varepsilon, \delta)$ holds.

Let $\varepsilon > 0$ be given. Claim 2 allows us to define the following map:

$$f : \mathcal{V}_{\delta_0} \longrightarrow [0, \infty)$$

$$(x,\ y \in \vec{n}_{\delta_0}(x)) \longrightarrow \max_{t \in B(x,A)} \hat{d}(t, t')$$

For each $t \in B(x,A)$ we pick a path $\gamma_x$ from x to t of length $\leq A$, and then put $t' = \bar{\gamma}_x^y(1)$. $\mathcal{V}_{\delta_0}$ denotes a neighborhood of the zero section M of $\mathcal{V}$ on which the map f is defined. Such a neighborhood exists, by claim 2.

Now note that $f$ is continuous and $f = 0$ on M, where M is regarded as the zero-section of $\nu$. Thus $f^{-1}([0, \epsilon/2])$ is some neighborhood of M in $\nu$. It follows that there exists a $\delta$ small enough that $y \in \vec{n}_\delta(x)$ implies $f(x,y) < \epsilon/2$, which implies $\max_{t \in B(\epsilon, A)} \hat{d}(t,t') < \epsilon/2$. This proves (c) and ends the proof of Lemma 1.

PROOF OF THEOREM 1

forward implication: A minimal leaf must be recurrent.

Assume L is a minimal leaf with closure $\Sigma$. This means that $\Sigma$ is a minimal set of leaves, or equivalently that if $L'$ is any leaf in $\Sigma$ then closure$(L') = \Sigma$. Now assume that L is not recurrent. Then it is possible to find $a > 0$, $T_n \longrightarrow \infty$, and points $p_n, q_n \in L$, such that

(*) $q_n$ is further than a (in the g-metric) from $B_d(p_n, T_n)$.

Since $\Sigma$ is a compact subset of M we can pass to subsequences and assume $p_n \longrightarrow p$, $q_n \longrightarrow q$ where $p, q \in \Sigma$. Consider the leaf $L_p$.

> <u>Claim</u>. Every point of $L_p$ is at least at distance $a/3$ from q.

(This claim implies a contradiction to minimality of $\Sigma$, since $\overline{L_p}$ is a closed invariant set, which is a proper subset of $\Sigma$ because it fails to contain q.)

Proof of claim:

Pick a point x on $L_p$ and connect it to p via a path $\sigma$. $[\sigma(0) = p]$ If n is large enough, lemma 1 implies $\sigma$ will lift via $\vec{n}$ to a path $\bar{\sigma}$ in L = leaf through $p_n$. The endpoint $\bar{\sigma}(1)$ will lie in $B_d(p_n, T_n)$ as long as $T_n$ is larger than say $2\|\sigma\|$. Further increasing n we can ensure (putting $z = \bar{\sigma}(1)$) that

$$g(z,x) < a/3$$
$$g(q_n,q) < a/3$$

and (*) gives that $g(z,q_n) > a$.

To finish the proof of the claim, we apply the triangle inequality to the sequence of points $(z,x,q,q_n)$ of M, using the metric g (we supress the "g" in the distances):

$$(z,q_n) < (z,x) + (x,q) + (q,q_n).$$

When we reverse this inequality we obtain

$$(x,q) > (z,q_n) - (z,x) - (q,q_n) > a - a/3 - a/3 = a/3.$$

Thus every point x of $L_p$ is at least at distance a/3 from q as claimed. This proves the forward implication.

reverse implication: A recurrent leaf must be minimal.

Assume L is a recurrent leaf with closure $\Sigma$, and that L is not minimal. Then some leaf K of $\Sigma$ has closure $\overline{K} \neq \Sigma$. L is not contained in $\overline{K}$, otherwise we would have $\overline{K} = \Sigma$. So pick a point x on L at a positive distance $a > 0$ (g-metric) from the compact set $\overline{K}$. Pick also a point y on the leaf K. Since K is a leaf of $\Sigma$ = closure of L, we can pick a sequence $y_n$ of points of $\vec{n}(y) \cap L$, with $y_n \longrightarrow y$.

The recurrence assumption produces T(a/3) large enough that any ball of radius T in L approximates the whole leaf to within a/3. Now apply lemma 1 with $\epsilon = a/3$, $A = T$. It is then clear that for large enough n, paths $\sigma$ in $B_d(y_n,T)$ of length less than T will lift via $\vec{n}$ to paths $\bar{\sigma}$ in $B_d(y,T)$, satisfying

$$g[\sigma(t),\bar{\sigma}(t)] < a/3, \text{ for } t \in [0,1].$$

The ball $B(y_n, T)$ must approximate all of L to within a/3. In particular, x is within a/3 of some point z of $B(y_n, T)$. But if $\sigma$ is a path in L from $y_n$ to z of length $< T$, then the endpoint of its lift $z' = \overline{\sigma}(1)$ will be within a/3 of the point z. This gives

$$(x, z') < (x, z) + (z, z') < a/3 + a/3 = 2a/3,$$

contradicting the fact that x is at distance a from K and $z' \in K$. This contradiction completes the proof of the reverse implication, ending the proof of theorem 1.

PROOF OF THEOREM 2: A minimal leaf must be quasi homogeneous.

By theorem 1 we can assume L is recurrent. We will show it is quasi-homogeneous for dilation constants (1/2,2). So let $r > 0$ be a given positive number; we must produce $R > 0$, so as to satisfy the conditions of quasi-homogeneous.

First note that we can prove the existence of R under the assumption that $r > 1$, since for smaller r the same big R will work. So assume given $r > 1$.

Set $r_1 = r + 1$. We apply lemma 1.3 to $r_1$ and any $\varepsilon$ small enough that $[2^{-\varepsilon}, 2^{\varepsilon}] \subset [1/2, 2]$. This ensures the maps f coming from path lifting will have dilation numbers (1/2,2). Specifically, lemma 1.3 gives us $\delta_1 > 0$ (we also assume $\delta_1 < 1/2$), small enough so that whenever x,y are points of M with $y \in \vec{n}_{\delta_1}(x)$, we can always define a path lifting map

$$f: \tilde{B}_d(x, r_1) \longrightarrow L_y,$$

as stated in the lemma.

Now we choose $\delta_2$ small enough so that for $x \in M$ we have

$$B_{\hat{d}}(x, \delta_2) \subset \vec{n}_{\delta_1}[B_d(x, \delta_1)].$$

Such a $\delta_2$ exists because M is compact and is the zero section of $\vec{n}$. We now apply recurrence with its $\varepsilon = \delta_2$, producing $T(\delta_2)$ with the property that any d-ball of radius $T(\delta_2)$ in L is $\delta_2$-close ($\hat{d}$ metric) to the entire leaf L. Finally we set $R = T(\delta_2) + 2r_1$. We claim that this R satisfies the conditions of quasi-homogeneity.

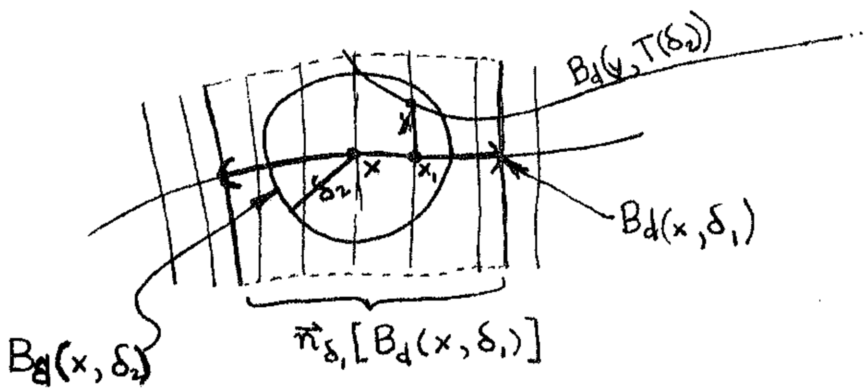
To check that this is so, assume we are given a cover ball $\widetilde{B}_d(x,r)$ and a metric ball $B_d(y,R)$, situated at random in the leaf L. By recurrence, some point $y_1$ of $B_d(y,T(\delta_2))$ is $\delta_2$-close to the point x. This implies (see sketch below) there is $x_1 \in L_x$ with $y_1 \in \vec{\pi}_{\delta_1}(x_1)$, and $d(x,x_1) < \delta_1$. Our choice of $\delta_1$ then gives the existence of a path lifting map

$$f: \ \widetilde{B}_d(x_1,r_1) \longrightarrow L_{y_1}.$$

Since f must have dilation bounds (1/2,2) it follows that the image $f(\widetilde{B}_d(x_1,r_1))$ is contained in $B_d(y_1,2r_1)$. Finally, our choice of $r_1$ implies that

$$\widetilde{B}_d(x,r) \subset \widetilde{B}_d(x_1,r_1), \text{ since } d(x,x_1) < \delta_1 \ ,$$

and our choice of R implies $B_d(y_1, 2r_1) \subset B_d(y,R)$. Thus the map for quasi-homogeneity is the restriction of the above f to the cover ball $\widetilde{B}_d(x,r)$. The map f is a "cover immersion", since it is defined by path lifting to fibers of $\vec{\pi}$, and $\vec{\pi}$ restricted to $B_d(x,r)$ is embedded by exp. $\blacksquare$



DESCRIPTION OF EXAMPLE 1

Start with the torus $T^2$ viewed as the unit square in the x-y plane, with the usual side identifications. Pick an angle $\theta$ which is an irrational multiple of $\pi$. Let $\vec{a}$ denote the unit vector field on $T^2$ all of whose vectors make the angle $\theta$ with the x-axis. Each orbit of $\vec{a}$ is dense in $T^2$.

Choose a point $x_0$ in the interior of the square and a small $\epsilon$-disc around $x_0$, which we call D. Let f be a $C^\infty$ function defined on D which is zero only at $x_0$ and is 1 off a subdisc $D'$ containing $x_0$. We now slow down the vector field $\vec{a}$ near $x_0$ by multiplying lengths by f:

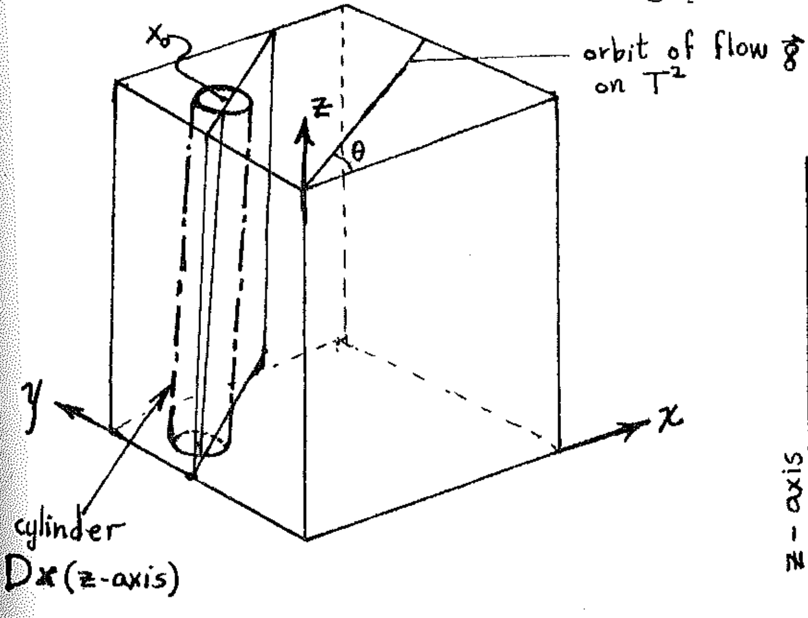
$$\vec{g}(x,y) = \begin{cases} f(x,y)\cdot\vec{a}, & \text{if } (x,y)\in D, \\ \vec{a} & \text{otherwise.} \end{cases}$$

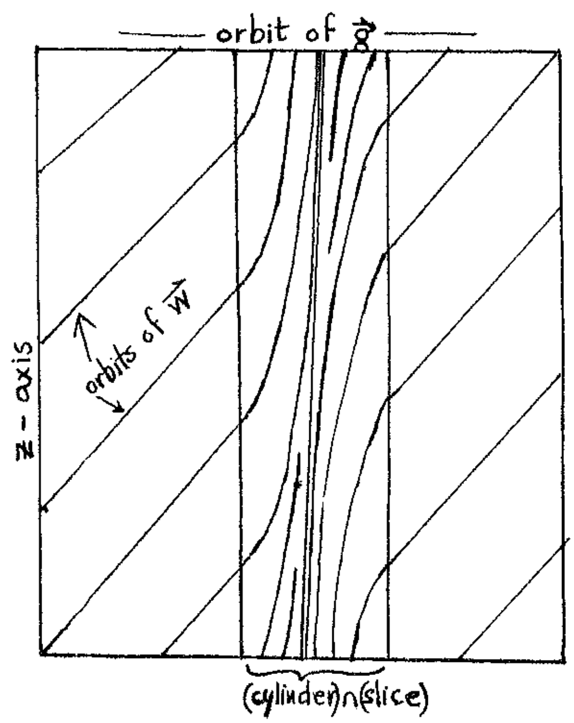
As it stands, $\vec{g}$ is a flow on the unit square in $R^2$. Use the same letter $\vec{g}$ to denote the flow on the unit cube in $R^3$ where now $\vec{g}(x,y,z) = \vec{g}(x,y)$. This is the old flow acting trivially in the z-direction.

We also use the unit flow in the z-direction, which we call dz. This is a vector field on the unit cube which points straight up, having no x- or y-components.

We now add these two flows to obtain $\vec{w} = \vec{g} + dz$. This is a flow on the unit cube in $R^3$, which is invariant under unit translation of the three axes, and so defines a flow on the torus $T^3$ viewed as the cube with identifications.

The flow $\vec{w}$ has now no stationary points. Adjust the lengths of the vectors so that $\vec{w}$ is a unit flow. There will be one closed orbit in $\vec{w}$, namely $x_0 \times$ (z-axis). There are two special types of orbit in $\vec{w}$ which $\alpha$-limit and $\omega$-limit on the closed orbit. All other orbits have the property that with time $\longrightarrow +\infty$ or $-\infty$, they spend arbitrarily long times near the closed orbit, only to escape eventually, cross the x-z plane some number of times, and then return again near the closed orbit. Note that the closer an orbit passes to the closed orbit, the more vertical it becomes, making more and more passes in the vertical direction before escaping finally from the cylinder $D \times$ (z-axis), only to return in time to this cylinder. The following pictures illustrate this behaviour.




<u>slice near $x_0$</u>

flow becomes nearly vertical.

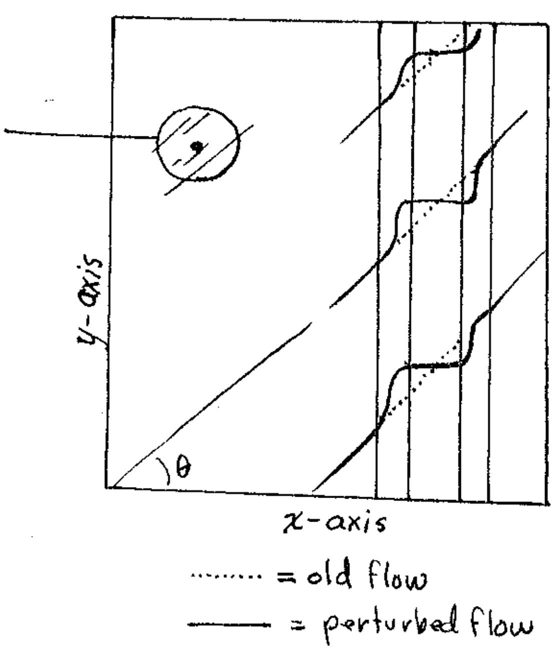
For definiteness, assume the cylinder D × (z-axis) lies in the left half $x < 0$ of the cube. Then on the right half of the cube the flow $\vec{w}$ goes in a constant direction. Pick now four numbers $a < b < c < d$ between 1/2 and 1, where $d-a$ is small. We want to adjust $\vec{w}$ to a flow $\vec{w}_2$ by straightening it out to go in the dx direction between b and c. That is,

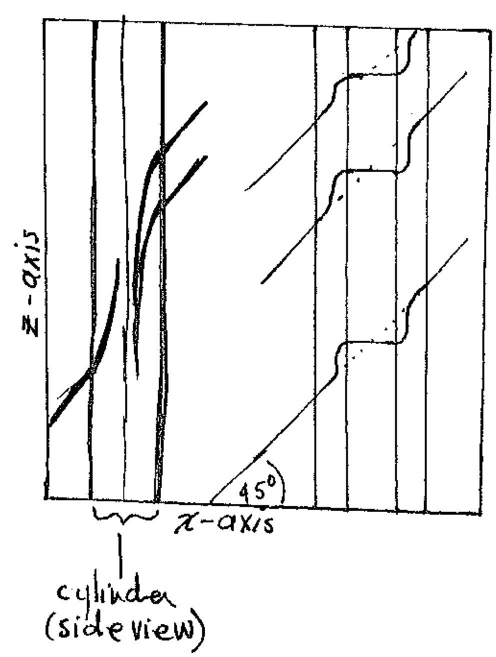
1.) Outside $(a,d)\times(y\text{-axis})\times(z\text{-axis}), \vec{w}_2 = \vec{w}$. (outside the "slab")

2.) Each orbit of $\vec{w}_2$ enters and leaves the slab at the same points as the corresponding orbit of $\vec{w}$,

3.) In the interval $[b,c]$ crossed with (y-z plane), the perturbed flow $\vec{w}_2$ is just dx.

It is clear we can change $\vec{w}$ to $\vec{w}_2$ in this way, so that $\vec{w}_2$ is $C^\infty$. The purpose of this change is to make the flow especially simple in the region $[b,c]\times$(y-z plane), because the next step is to perform a surgery operation involving the tori $b_1\times$(y-z plane) and $b_2\times$(y-z plane) (where $b < b_1 < b_2 < c$), so that these tori are now at right angles to the flow $\vec{w}_2$.



The tori just mentioned we will denote $T^2(b_1)$ and $T^2(b_2)$. They are situated in the unit cube of $R^3$ perpendicular to the x-axis, and so also to the flow $\vec{w}_2$, which is dx in the slab $[b,c]\times$(y-z plane); this slab contains both tori. The orbits of $\vec{w}_2$ have the property that they cross the pair of tori infinitely often in either direction (except for the special orbits limiting on the closed orbit, which do so in one direction and cross the tori infinitely often in the other). They spend arbitrarily long times between crossings for time going to $+\infty$ or $-\infty$, in the sense that given a long time T we can find a time interval $(t, t+T)$ in which the orbit remains in the cylinder $D\times$(z-axis), and so stays away from the pair of tori. After we perform the surgery operation mentioned above, these long periods in which an orbit stays away from the pair of tori, will produce long cylinders in the surface which comes from the orbit.

We are constructing a foliation of a manifold $M^4$. We will obtain $M^4$ by removing tubes around $T^2(b_1)$ and $T^2(b_2)$, which are two-dimensional compact submanifolds of $T^4 = T^3\times S^1$. Then we will identify the two holes in $T^4$ along the circles bounding the two tubes, after a flip of orientation.

So first consider $T^4 = T^3\times S^1$ foliated by two-dimensional surfaces (orbit of $w_2$)$\times S^1$. In this foliation there is one toral leaf, coming from the closed orbit of $\vec{w}_2$. All other leaves are infinite cylinders, which course around $T^4$ similarly to the way the orbits of $\vec{w}_2$ course around $T^3$.

The tori $T^2(b_1)$ and $T^2(b_2)$ are thought of as at the level $t = 1/2$, where t is the parameter of $S^1$ in the product $T^4 = T^3 \times S^1$. Thus each torus is orthogonal to the dx and dt directions. Each cylindrical leaf crosses the two tori, intersecting each in one point: these points have co-ordinates $(b_1, y, z, 1/2)$ and $(b_2, y, z, 1/2)$ where y and z depend on the leaf, or rather on the particular passage of the leaf through the two tori.

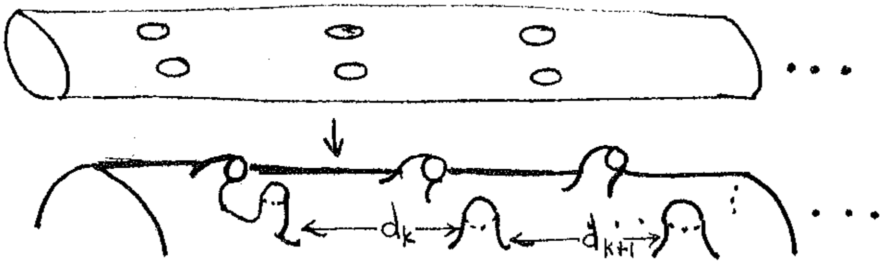
Now we choose a very small radius r, so that the intervals of radius r around $b_1$ and $b_2$ are disjoint and contained in the interval $[b,c]$ (where the flow $\vec{w}_2$ goes in the dx direction). This r will be the radius of the tubular neighborhoods removed around the tori. The tube around $T^2(b_1)$, for instance, consists of a union of two-dimensional discs, with centers at the points of $T^2(b_1)$ and radius r, and situated in the x-t plane. That is, for each point $(b_1, y, z, 1/2)$ of $T^2(b_1)$, the two-disc in question is the following subset of $T^4$:

$$D^1_{y,z} = \left\{(x,y,z,t) \in T^4 \colon (x-b_1)^2 + (t-1/2)^2 \leqslant r^2\right\}$$

The union of these discs $D^1_{y,z}$ for y and z ranging over the y-z plane constitutes the tubular neighborhood of $T^2(b_1)$. The tube around $T^2(b_2)$ is defined analogously, using discs $D^2_{y,z}$.

To complete the construction, we remove the interiors of the tubes around the two tori, then identify the two boundary components of $[T^4 - (\text{two tubes})]$. This identification should reverse orientation and sew together the corresponding circles, $\partial D^1_{y,z}$ and $\partial D^2_{y,z}$.

The reason for the flip in orientation is so we end up with a cylinder-with-handles as typical leaf.




We have already shown why the distances between these handles, namely $d_k$ above, are unbounded as $k \longrightarrow \pm\infty$. This completes the description of example 1.

FURTHER REMARKS ON EXAMPLE 1.

Remark 1. Let L be a noncompact leaf in the foliation of example 1. Then considered as a Riemannian 2-manifold, L is not quasi-homogeneous. [It follows that the quasi-isometry type of L cannot appear as a minimal leaf in any compact foliated manifold.]

This follows from four facts about L:

Fact 1: There is some $A > 0$ so that L does not contain any simply connected metric balls of radius $> A$.

(This fact is obvious from the construction of example 1.)

Fact 2: For every $c > 0$ there is some $C > 0$ such that, if $\gamma_1$ and $\gamma_2$ are homotopic paths (rel endpoints) in L, $\|\gamma_i\| < c$, there exists a homotopy between them through paths of length $< C$.

(This fact is proved in the appendix using fact 1 and a plaque decomposition.)

Fact 3: $\pi_1(L)$ is a countably generated free group.

(This is because L is an open oriented surface with an infinite number of handles.)

Fact 4: L contains arbitrarily large metric balls which lie in subsets of L homeomorphic to cylinders.

(These cylinders correspond to the increasing distances between handles in leaves of example 1.)

The following lemma will also be used:

PIGEON HOLE LOOP LEMMA: Given any collection of loops in $L^2$ based at a common point, if there is a bound on lengths of loops in the collection, then the collection can represent only finitely many classes in $\pi_1(L)$.

(This is proved in the appendix using plaques.)

We are using the term "cover ball", with the notation $\tilde{B}(x,r)$, to mean equivalence classes of (piecewise smooth) paths from x in L, which have length $<$ r. Two such paths are called equivalent if they are homotopic through paths of length less than r.

This notion is to be distinguished from that of a "ball in the universal cover". Notation for the latter will be $B(\tilde{L},x,r)$. It consists of equivalence classes of paths from x of length $<$ r, where in this case paths are called equivalent if they are homotopic (no restriction on lengths of paths used).

Now we assume that facts 1-4 hold for the Riemannian 2-manifold L, and in addition that L is quasi-homogeneous. We will show this arises in a contradiction.

That L is quasi-homogeneous means there are dilation constants (k,K) so that for every r there is R such that any cover ball $\tilde{B}(x,r)$ can be cover-immersed in any metric ball $B(y,R)$; all such immersions $f:\tilde{B}(x,r) \longrightarrow B(y,R)$ share dilation (k,K).

Applying fact 2 to the radius $r_0$, we produce for each $c = r_0$ a homotopy bound $C = C_0$. Then applying quasi-homogeneity to the radius $r \mp C_0$, we obtain a larger radius $R = R_1$.

(Quasi-homogeneity produces for each r an R such that certain conditions hold. Fact 2 is similar in that for each c it produces C so that certain conditions hold. When we apply either of these to particular constants, we will indicate the application as above; for example "from r = D we produce R = K" would mean that we applied quasi homogeneity to the radius D, and obtained a larger radius appropriate to the conditions of quasi-homogeneity, and we call this larger radius K.)

Thus for each $r_0$ and any $x, y$ L we have the mappings

$$(1)\ \widetilde{B}(x, r_0) \subset \widetilde{B}(x, C_0) \xrightarrow{f} B(y, R_1)$$

The image balls $B(y, R_1)$ can be chosen to lie in cylindrical subsets of L. In fact, using fact 4 we can choose a sequence $z_k$ of points of L and radii $D_k$ tending monotonically to $\infty$, such that each ball $B(z_k, D_k)$ is contained in $\mathcal{C}_k$, a cylindrical subset of L, with the $\mathcal{C}_k$ pairwise disjoint. Then for large k, $D_k > R_1$ and so the mappings in (1) apply to $y = z_k$.

Now given a map f from (1) above, its restriction to $\widetilde{B}(x, r_0)$ can be factored through $B(\widetilde{L}, x, r_0)$ producing maps

$$\widetilde{B}(x, r_0) \xrightarrow{inc} B(\widetilde{L}, x, r_0) \xrightarrow{\hat{f}} B(y, R_1).$$

The first map is the natural inclusion coming from the fact that the equivalence relation on paths of $B(\widetilde{L}, x, r_0)$ is weaker than that for paths of $\widetilde{B}(x, r_0)$). To show the second map is well defined, we note that if $\gamma$ and $\gamma'$ are paths from x which are homotopic, then there is a homotopy $H_t$ between them through paths of length less than $C_0$, so that $H_t$ goes through paths which remain inside $B(x,C_0)$. So each path $H_t$ is taken by $f:\widetilde{B}(x,C_0) \longrightarrow B(y,R_1)$ to a path in $B(y,R_1)$. Since all the image paths $f(H_t)$ end at the same point, we see that $\hat{f}(\gamma)=\hat{f}(\gamma')=f(H_t(1))$.

Now we apply fact 1, and claim there must be $r_0$ large enough that all the maps $\hat{f}:B(\widetilde{L},x,r_0)$ produced as above must fail to be one-to-one. If this were not so, the simply connected balls $B(\widetilde{L},x,r_0)$ would be carried by the immersions $\hat{f}$ (which are 1-1 and so diffeomorphisms) to simply connected subsets of L. These subsets would for large $r_0$ contain simply connected balls of radius greater than A of fact one, because all the f's share the same fixed dilation (k,K). We pick such an $r_0$ and fix it in what follows.

Now we choose a base point $x_0$ on L. Momentarily let $A_{2r_0}$ denote the set of (piecewise smooth) loops in L based at $x_0$, of length $< 2r_0$. Apply the "pigeon hole loop lemma" to this class of loops. We see that these loops can only represent a finite collection of classes in $\pi_1(L,x_0)$, and so we can choose a generator $[\delta]$ in the free group $\pi_1(L,x_0)$ which is not used in the expression of any loop of $A_{2r_0}$ by generators.

This loop $[\delta]$ is independent of any homotopy class coming from loops at $x_0$ of length less than $2r_0$, in the sense that there can be no non-trivial word in $[\delta]$ and generators used to express loops at $x_0$ of length less than $2r_0$, which reduces to 1.

With $r_0$ fixed as above and $\delta$ chosen, a loop at $x_0$, we define the following auxilliary constants:

$E_1 = \max(2r_0, \|\delta\|)$

$E_2 = 8E_1$

$E_3 = KE_2$ (where K comes from quasi-homogeneity constants $(k,K)$)

$E_4 = C(E_3)$ (we apply fact 2 to $c = E_3$ producing $C = E_4$)

$E_5 = KE_4$ (K from quasi-homogeneity)

These constants chosen, we apply quasi-homogeneity once more, to the constant $E_5$. We obtain say $R = R_6$. For large enough k, the radius $D_k$ of balls contained in cylinders $\mathcal{C}_k$ is so large that $D_k > R_6$. Hence we have the mappings:

$$\widetilde{B}(x_0, r_0) \xrightarrow{inc} \widetilde{B}(x_0, E_5) \xrightarrow{f''} B(z_k, D_k) \xrightarrow{inc} \mathcal{C}_k.$$

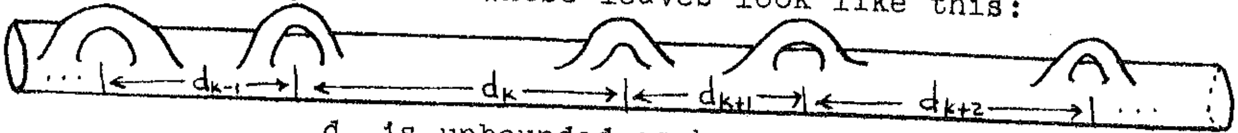
Let f denote the restriction of f'' to the set $\widetilde{B}(x_0, r_0)$.

By choice of $r_0$ the map $\hat{f}$ arising from factoring f through $B(\widetilde{L}, x_0, r_0)$, cannot be 1-1. This means for each k there are paths $\alpha_k$ and $\beta_k$ of length $< r_0$, which are not homotopic in L and yet $\hat{f}(\alpha_k) = \hat{f}(\beta_k)$. Therefore $\gamma_k = \alpha_k\beta_k^{-1}$ is a closed loop based at $x_0$ which is carried to a loop by f, and the length $\|\gamma_k\| < 2r_0$. It follows from construction of $[\delta]$ that it is independent of all the classes $[\gamma_k]$. Also, $\gamma_k$ is a non-trivial loop in $\pi_1(L, x_0)$ because the paths $\alpha_k$ and $\beta_k$ are not homotopic.

Now the loop $\delta$ gets carried by $f_k$ to a path $\bar{\delta}$ starting at $f_k(x_0)$. We cannot say that $\bar{\delta}$ is a closed loop. However, from the fact that $\gamma_k$ goes by $f_k$ to a loop, it follows that the loop $\delta\gamma_k\delta^{-1}$ is carried to a loop based at $f_k(x_0)$. Call the latter loop $\overline{\delta\gamma_k\delta^{-1}}$. For large enough k, the two loops $\overline{\gamma}_k$ and $\overline{\delta\gamma_k\delta^{-1}}$ are contained in the cylinder $\mathcal{C}_k$, and so they commute in that cylinder. This implies there is a homotopy from the commutator $[\overline{\gamma}_k, \overline{\delta\gamma_k\delta^{-1}}]$ to the trivial loop based at $f_k(x_0)$. Recalling the definitions of the constants $E_1, \ldots, E_5$, we see the following: First, since $\delta$ and $\gamma_k$ are each no longer than $E_1$, we have that the commutator $[\gamma_k, \delta\gamma_k\delta^{-1}]$ is no longer than $E_2$. Since $f_k$ has dilation bound K, the image loop $[\overline{\gamma}_k, \overline{\delta\gamma_k\delta^{-1}}]$ is no longer than $E_3$. We have observed that this loop is homotopic to the constant loop at $f_k(x_0)$; therefore there is a homotopy $\widetilde{H}_t$ to the constant loop, through loops no longer than $E_4$. Finally, since the map $f_k$ is locally a diffeomorphism, we see that if we begin to perform the homotopy near t = 0, the loop $\widetilde{H}_t$, which starts as the image by $f_k$ of the commutator $[\gamma_k, \delta\gamma_k\delta^{-1}]$ at $x_0$, remains above a loop $H_t$ which has length at most $KE_4 = E_5$, so that $H_t$ is a loop at $x_0$ which remains, for $t \in [0,1]$, within the cover ball $\widetilde{B}(x_0, E_5)$. We are thus assured that the homotopy $\widetilde{H}_t$ "pulls back" by $f_k$ to a homotopy $H_t$ from the commutator $[\gamma_k, \delta\gamma_k\delta^{-1}]$ to the constant loop at $x_0$. This is a contradiction, because $\delta$ was chosen in such a way that no such relation as $[\gamma_k, \delta\gamma_k\delta^{-1}] = 1$ is possible.

This establishes Remark 1.

Recall from the Description of Example 1 that we have constructed a compact 4-manifold $M^4$ with a $C^\infty$ foliation $\mathcal{F}$ of codimension 2, most of whose leaves look like this:




$d_k$ is unbounded as $k \to \pm\infty$.

We generate examples $E(p,q)$, $p > 1$, $q > 1$, from example 1. $E(2,2)$ is just example 1 itself. To get an example for the pair (dimension, codimension) = $(p,q)$ with $p,q > 1$, we take the manifold $M^4 \times S^{p-2} \times S^{q-2}$, foliated by leaves which are products (leaf of $\mathcal{F}$) $\times S^{p-2} \times$ (point of $S^{q-2}$). Call the resulting manifold $M(p,q)$ and the foliation so constructed call $\mathcal{F}(p,q)$.

Remark 2. If $p > 1$, $q > 1$, then the foliation $\mathcal{F}(p,q)$ of the compact manifold $M(p,q)$ contains leaves which fail to be quasi-homogeneous. [It follows that these leaves, considered as Riemannian p-manifolds, cannot appear as minimal leaves in compact foliated manifolds.]

This is shown by noting that facts 1-4, holding for most leaves of the foliation $\mathcal{F}$ of $M^4$, have counterparts holding for the corresponding leaves of $\mathcal{F}(p,q)$. The differences are that in case $p = 3$, fact 3 must be replaced by

(fact 3'): $\pi_1(L)$ is $Z \oplus$ (countable free group)

and fact 4 must be replaced in all cases by

(fact 4'): L contains arbitrarily large metric balls which lie in subsets of L homeomorphic to $I \times S^1 \times S^{p-2}$.

APPENDIX

Plaque Decompositions. Let $M^n$ be a compact manifold with a foliation $\mathcal{F}$ of leaf dimension p, codimension q. Then using the distinguished coordinate charts for $\mathcal{F}$, we can construct a finite cover $M = \bigcup_i V_i$ where each $V_i$ is the image of a diffeomorphism $f: D^p \times D^q \longrightarrow V_i \subset M$. Here $D^p$ and $D^q$ denote the unit balls in $R^p$ and $R^q$ centered at 0. The $V_i$ are called flow boxes. They can be chosen so that each "plaque" $\mathcal{P} = f_i(D^p \times \{*\})$ is a geodesically convex subset of the leaf of $\mathcal{F}$ on which it lies, and so there are no triple intersections of plaque boundaries. It seems appropriate to call the images $f_i(\{*\} \times D^q)$ by the term "coplaques". Then the definition of foliation implies that when we change flow boxes, plaques are preserved, whereas coplaques generally are not.

Proof of Pigeon Hole Loop Lemma. Say we are given a collection A of loops based at $x_0 \in L^2$, with a bound K on the lengths of loops in the collection. Then all loops of A remain inside the metric ball $B_d(x_0, K)$. Let $p_1, \ldots, p_k$ be the set of plaques in a plaque decomposition of $\mathcal{F}$ which intersect $B_d(x, K)$. The plaques $p_j$ divide each other into "sectors" $s_q$, where each $s_q$ is defined:

$$s_q = p_i \cap (p_{j_1} \cup \cdots \cup p_{j_r}) \cap (p_{j_{r+1}}^{\text{complement}} \cup \cdots \cup p_{j_{k-1}}^{\text{complement}})$$

Each such sector has vertices $v_1^q, \ldots, v_{n_q}^q$ which are points where two plaque boundaries meet. Now for each triple $(s_q, v_i^q, v_j^q)$ of a sector with two of its vertices, choose two deformation retracts $R_1^{q,i,j}$ and $R_2^{q,i,j}$ from $s_q$ onto each boundary component $\partial s_q - \{v_i^q, v_j^q\}$. We obtain a finite collection of such retractions for all sector-vertex triples occurring in the sectors $s_q$. Thus for all these retractions there is a global bound D on distortion.

Now each loop in A can be deformed to one which enters and leaves sectors at vertices. This will increase the length bound K to some K'. Call the set of deformed loops A'. It follows by applications of the retractions chosen above, that each loop in A' is homotopic to some loop of length $\leq$ DK' which goes along plaque boundaries, entering and leaving at vertices. But it is clear that there can be only finitely many homotopy classes of such loops along boundaries which have length bounded by DK'. This shows the pigeon hole loop lemma.

Proof of fact 2 for a 2-manifold which satisfies fact 1.

This proof is similar to the above, but slightly more complicated because the assertion is made independent of endpoints. So let $\gamma_1$, $\gamma_2$ be paths in L from p to q of length $\leq$ c, which are homotopic rel endpoints. We wish to produce C so that $\gamma_1$ and $\gamma_2$ are homotopic through paths of length at most C, and this C should be independent both of the paths $\gamma_1$ and $\gamma_2$ as well as of the endpoints p and q.

First, we can assume $\gamma_1$ and $\gamma_2$ together bound a region R which is simply connected, since otherwise we could argue separately on the regions produced.

From fact 1 we know R contains no metric ball of size $> A$. This allows us to obtain a universal bound N (in terms of c and A) on the number of plaques meeting R: Each point in R must be $< A$ from some point of $\gamma = \gamma_1 \gamma_2^{-1}$. (Otherwise there is a ball of size $> A$ contained in R) Therefore R is contained in the metric ball $B(p, A + 2\|\gamma\|)$. It is clear we can find a universal upper bound on the number of plaques meeting any $B(x,r)$ where r is fixed and x varies in M.

Since there are only finitely many flow boxes in the plaque decomposition of $\mathfrak{F}$, it follows there are only finitely many "sector boxes", contained in intersections of neighboring flow boxes. For each sector box we choose a distinguished sector $s_q$. We have a finite collection $s_q$ of distinguished sectors. Any other sector is diffeomorphic to one of these, and the diffeomorphisms arising this way all share a single distortion bound $K_1$.

We proceed as before, first adjusting $\gamma_1$ and $\gamma_2$ to enter and leave sectors at vertices, choosing deformation retracts onto boundary components, etc. Thus we deform $\gamma_1$ and $\gamma_2$ into some path along plaque boundaries joining p to q. The distortion bound $K_1$, along with distortion bounds for the retractions on the distinguished sectors, together ensure there will be a large C as required, since as remarked above there is a bound N on the number of plaques used.

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