991-6

On the subalgebras of continuous function lying between $A(D^n)$ and $C(T^n)$

by

Richard Taylor

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

Richard Taylor

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Committee Chairman

Thesis Advisor

The dissertation is accepted by the Graduate School

Original Signed By

JACOB BIGELEISEN, DEAN

Dr. Jacob Bigeleisen
Dean of the Graduate School

Abstract of the Dissertation

On the subalgebras of continuous

function lying between $A(D^n)$ and $C(T^n)$

by

Richard Taylor

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1978

In this thesis, I outline the basics of Gelfand theory, and just that part of function theory and harmonic analysis necessary to state and prove certain results concerning the closed subalgebras of continuous functions, lying in between $A(D^n)$ and $C(T^n)$, $n \geq 2$. Interest in these algebras is due, in part, to the failure of Wermers maximality theorem, which says that $A(D)$ is maximal in $C(T)$ (making the problem vacuous in case $n = 1$), in dimensions $n > 1$: at best this theorem yields the relative maximality of $A(D^n)$ in $C(T^n)$ (Theorem III.2.8).

The classical work on spectral synthesis, due to Peter-Weyl and others, shows that any closed translation invariant subspace of the continuous functions on a compact

abelian group is generated by a set of characters of the group. Since T^n is such a group, in pointwise multiplication, all closed translation invariant sub-algebras of $C(T^n)$ are easily characterized, along with their maximal ideal spaces.

It is the purpose of this thesis to see what can be said, in general, about any algebra B, $A(D^n) \subset B \subset C(T^n)$. To this end I classify those of the form A(K), where $T^n \subset K \subset D^n$, yielding a "Hartogs like" extension theorem for πm_B(notation: page 10) of this form.

I wish to thank Professor R. G. Douglas for his invaluable help during both the research and writing of this thesis, and Pat Belus, Glendora Milligan and Barbara Ginther for the tedious tasks of typing and correcting the manuscript.

Table of Contents

List of Symbols

Let $E \subset C^n$ be compact.

$\pi_j : \mathbb{C}^n \to \mathbb{C}$ is defined by $\pi_j(z_1, \cdots, z_n) = z_j$.

$P^n \equiv$ space of formal polynomials in variables $z_1, \cdots, z_n$.

$P^n_E \equiv$ subspace of $C(E)$ given by restricting P^n to E.

$R^n_E \equiv \{P/Q: P, Q \in P^n_E, Q(z) \neq 0 \text{ for all } z \in E\}$.

$\mathfrak{A}^n(E) \equiv \{f: E \to C$: There is a neighborhood, U, of E and an analytic function, $g: U \to C$ such that $f \equiv g$ on $E\}$.

$\mathfrak{A}^n_1(E) \equiv \{f: E \to C$: for all $z \in E$, there is a neighborhood, U_z of z, and an analytic function $g: U_z \to C$ such that $f = g$ on $U_z \cap E\}$.

$P(E) \equiv \overline{P^n_E}$

$R(E) \equiv \overline{R^n_E}$

$A(E) \equiv \overline{\mathfrak{A}^n(E)}$

$A_1(E) \equiv \overline{\mathfrak{A}^n_1(E)}$

$\hat{E} \equiv$ polynomial convex hull of E - i.e.,

$\{\zeta \in C^n: |P(\zeta)| \leq \sup_{z \in E} |P(z)| \text{ for all } P \in P^n\}$.

$\hat{E} \equiv$ rational convex hull of E - i.e.,

$\{\zeta \in C^n: |P(\zeta)| \leq \sup_{z \in E} \left|\frac{P(z)}{Q(z)}\right| \text{ for all } P, Q \in P^n$ such that $Q(z) \neq 0$ for all $z \in E\}$.

Let X be locally compact, Hausdorff.

$C_o(X) \equiv$ set of continuous functions on X vanishing at infinity.

$C_B(X)$ ≡ set of continuous bounded functions on X.

$M(X)$ ≡ space of complex Borel measures on X.

Let $Y \subset X$, $A \subset C(X)$.

$A|_Y$ ≡ subset of $C(Y)$ given by restricting A to Y.

$\overline{A|_Y}$ denotes the closure in $C(Y)$ of $A|_Y$.

For any two spaces A and B, $A \approx B$ [resp. $A \underset{\sim}{\subseteq} B$] will mean A identifies with B [resp. a subset of B].

If $A \subset B$ are Banach algebras and $f \in B$, then

$A(f)$ ≡ minimal closed subalgebra of B containing A and f.

CHAPTER I

Background

§0. Introduction.

We present, here, the function and Banach algebra theory necessary to state and prove the more specialized results of later chapters on subalgebras of $C(E)$, $E \subset C^n$ compact, and in particular, on $C(T^n)$, $n \geq 1$. The role played by the theory of one and several complex variables is not ignored, and thus some proofs of classical results, stemming from complex variable theory, are presented.

§1. Abstract Banach Algebras.

(i) Motivation. A Banach algebra is a Banach space, B, with a multiplication satisfying $\|fg\| \leq \|f\| \|g\|$ for any two elements, $f, g \in B$. Interest in Banach algebras arose largely from the subject of Harmonic analysis: $L^1(G)$, G a locally compact group is a Banach algebra of principal concern, where multiplication is convolution (Rudin-[1]). Every Banach algebra represents itself as a space of operators on itself: $Tf(g) \equiv fg$ is the operator induced by f; thus algebras of operators on Banach spaces form another important class of Banach algebra.

If B is a commutative Banach algebra, then B is often isomorphic (as will be shown) to a subalgebra of $C_o(X)$, X locally compact and Hausdorff, whose completion in the supremum norm is a function algebra.

(ii) The Spectrum and the Gelfand-Mazur Theorem.

Let A be any algebra with unit, I.

Definition. For $f \in A$, $\sigma_A(f)$ denotes the set of all $\lambda \in \mathbb{C}$ such that $f - \lambda I$ is not invertible in A. We define $R_A(f) \equiv \mathbb{C} - \sigma_A(f)$.

If $A \equiv C(X)$, X compact-Hausdorff, then clearly $\sigma_A(f) = f(X)$ for each $f \in A$, which motivated this definition.

For B a normed linear space and $S \subset B$, let $[S]_B$ denote the smallest closed subspace of B containing S.

I.1.1. Lemma. Let B be a normed linear space, $\Omega \subset \mathbb{C}$ open and connected, and $F : \Omega \to B$ analytic. If $K \subset \Omega$ has a limit point in Ω, then $[F(z) : z \in K]B = [F(z) : z \in \Omega]B$.

Proof: Fix $\phi \in B^*$(the dual of B). If $\phi(F(z)) = 0$ for each $z \in K$, then $\phi \circ F : \Omega \to \mathbb{C}$ is an analytic function vanishing on a set with a limit point from which we conclude that $\phi(F(z)) = 0$ for all $z \in \Omega$. The proof is complete by the Hahn-Banach theorem. QED.

I.1.2. Theorem. Let $f \in A \subset B$, B a Banach algebra with unit I, A a closed subalgebra containing I. Then

(1) $\sigma_A(f)$ is a nonempty closed, bounded, subset of $\mathbb{C}$; $\sigma_A(f) \subset$ ball of radius $\|f\|$ and

(2) Boundary $\sigma_A \subset \sigma_B(f) \subset \sigma_A(f)$.

<u>Proof</u>: (1) To show $\sigma_A(f)$ is bounded - in fact, lies in the ball of radius $\|f\|$, assume $|\lambda| > \|f\|$. Then $f - \lambda I = \lambda(\lambda^{-1}f - I)$. But $\|\lambda^{-1}f\| < 1$, so that $\lambda^{-1}f - I$ and thus $f - \lambda I$ is invertible in A (we've used the elementary fact that the unit ball about I, in A, consists of invertible elements). Thus $\lambda \in R_A(f)$ for $|\lambda| > \|f\|$.

Now, for each $\lambda \in R_A(f)$, $f - \lambda I$ has an inverse in A denoted by $(f - \lambda I)^{-1}$; standard calculations show that $\lambda \to (f - \lambda I)^{-1}$ is analytic from $R_A(f)$ into A and will vanish at infinity. Thus, by Liouville's theorem, $\sigma_A(f) = \phi$ implies that $(f - \lambda I)^{-1} = 0$ for all $\lambda \in C$, a contradiction. Thus $\sigma_A(f) \neq \phi$. It is closed because the set of invertible elements of A form an open set. This forces $C - \sigma_A(f)$ to be open.

(2) Clearly $\sigma_B(f) \subset \sigma_A(f)$. Now suppose there is a $\zeta \in$ Boundary $\sigma_A(f) - \sigma_B(f)$. Then there is an open ball, U, such that

(a) $\zeta \in U \subset R_B(f)$

(b) $U \cap R_A(f)$ is open and non-empty.

(c) $\sigma_A(f) \cap U \neq \phi$.

(a) implies that $\lambda \to (f - \lambda I)^{-1}$ is analytic from U into B. Thus, (b) and Lemma I.1.1 with $\Omega \equiv U$, $K \equiv U \cap R_A(f)$ imply that $(f - \lambda I)^{-1} \in A$ for each $\lambda \in U$. But this contradicts (c).

QED.

I.1.3. <u>Corollary</u> (Gelfand-Mazur). Let B be a Banach algebra for which each non-zero element has an inverse. Then $B \cong \mathbb{C}$.

<u>Proof</u>: We show that $\{\lambda I: \lambda \in \mathbb{C}\}$ equals B. If not, there is an $f \in B$ such that $f - \lambda I \neq 0$ for all $\lambda \in \mathbb{C}$, from which we conclude that $f - \lambda I$ is invertible for all $\lambda \in \mathbb{C}$, ie - $\sigma_B(f) = \phi$ contradicting Theorem I.1.2.(1).

I.1.4. <u>Corollary</u>. In the notation of Lemma I.1.1, $\sigma_A(f) = \sigma_B(f) \cup \bigcup_{j=1}^{k} \Omega_j$, where Ω_j is some bounded component of $R_B(f)$.

<u>Proof</u>: This follows from Theorem 1.1.2.(2) and elementary point set topology. QED.

(iii) <u>The Maximal Ideal Space</u>.

Let B be a commutative Banach algebra (not necessarily with unit) and M_B denote the set of multiplicative linear functionals on B: those linear $\phi : B \to \mathbb{C}$ for which $\phi(fg) = \phi(f)\phi(g)$ for each f, g $\in$ B. It is easily shown that $\|\phi\| \leq 1$ for such ϕ, that is $M_B \subset (B^*)_1$ the unit ball in B^* and that M_B is closed in the weak-* topology. Since $(B^*)_1$ is weakly compact and Hausdorff, so is M_B. We endow M_B with this topology. Note that the zero functional, denoted "0", lies in M_B, so that $M_B \neq \phi$.

Let $\hat{f}(h) \equiv h(f)$ for $h \in M_B$, $f \in B$; then $f \to \hat{f}$ is a homomorphism from B into $C(M_B)$, and M_B is clearly a maximal set onto which $\hat{B}$ "extends" itself to a point separating algebra of complex valued functions (If $M_B \subset M'$ is a larger such set and $x \in M'$, then $f \to \hat{f}(x)$ is multiplicative on B, so that there exists an $h \in M_B$ for which $h(f) \equiv \hat{f}(h) = \hat{f}(x)$ for each $f \in B$. But then $\hat{B}$ is not point separating on M'). Thus M_B is a "domain of holomorphy" for $\hat{B}$.

We define $m_B \equiv M_B - \{0\}$, the maximal ideal space of B, which may or may not be empty. Endowed with the relative topology, m_B is locally compact, and $\hat{B} \subset C_o(m_B)$ since $\hat{f}(0) = 0$ implies that $\hat{f}|m_B$ vanishes at infinity for each $f \in B$. From the last paragraph it is clear that m_B is a maximal set onto which $\hat{B}$ extends itself to a point separating, non-vanishing, algebra of complex valued functions.

In case $B \subset C_o(X)$ is nonvanishing, X locally compact-Hausdorff, each $x \in X$ defines an element $h_x \in m_B$ via $h_x(f) \equiv f(x)$, $f \in B$. Thus if B is point separating, then $X \subsetneqq m_B$. If $B \equiv C_o(X)$, then it is well known that $X \cong m_B$. Note that M_B identifies with the one point compactification of X. In particular, it follows that for B a function algebra, $m_B \neq \phi$. That this is always the case for a <u>commutative</u> Banach algebra <u>with</u> <u>unit</u> is shown below.

For the rest of the section, B is commutative with unit (although it is clear that many statements hold in more generality).

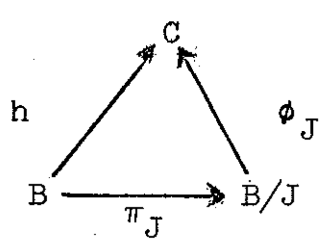
If $J \subset B$ is a closed ideal, B/J is a ring which is a Banach algebra in the quotient norm. Since the invertible elements of B form an open set disjoint from any proper ideal, the closure of a proper ideal is also a proper ideal. Thus any maximal (proper) ideal, $J \subset B$, is closed; since (as shown in ring theory) B/J is a field, B/J is thus a Banach field. It follows, by Corollary 1.1.3, that there exists an isomorphism

$$\phi_J : B/J \to \mathbb{C}$$

which is easily shown to be unique. If

$$\pi_J : B \to B/J$$

is the natural projection, then $\phi_J \circ \pi_J$ is a non-zero multiplicative linear functional on B, i.e.- $\phi_J \circ \pi_J \in m_B$. Conversely, if $h \in m_B$, (as shown in ring theory) $J \equiv \ker h$ is a maximal ideal and



commutes (using uniqueness of ϕ_J). Thus m_B is in one to one correspondence with the set of maximal ideals of B (hence its name). By Zorn's lemma, this is nonempty. Thus we've shown

(a) $m_B \neq \emptyset$

and

(b) For each $f \in B$, f is not invertible in B if and only if f is contained in a proper, maximal ideal, which in turn holds if and only if $\hat{f}(h) = 0$ for some $h \in m_B$. We summarize this below.

I.1.5. <u>Theorem</u>. Let B be commutative, with unit, I. Then

(1) $m_B \neq \emptyset$ and compact.

(2) For $f \in B$, $\sigma_B(f) = \{h(f) : h \in m_B\}$ and thus is also a compact, non-empty subset of C.

(3) $\hat{f}$ is nonvanishing on m_B if and only if f is invertible in B.

<u>Proof</u>: (1) Because M_B is itself compact, m_B is compact if and only if $\{0\}$ is open in M_B. This is, in fact, the case when B has a unit I: For

$$h(I) = \hat{I}(h) = \begin{cases} 1, & h \in m_B \\ 0, & h = 0 \end{cases}$$

forcing $\{0\}$ to be open in M_B. Thus m_B is compact and non-empty ((a) above).

(2) We note that, by (b), $\lambda \in \sigma_B(f)$ if and only if $h(f-\lambda I) = 0$, i.e.– $h(f) = \lambda$ for some $h \in m_B$.

(3) To say f is invertible in B means $0 \notin \sigma_B(f)$. By (2), this occurs if and only if f is nonvanishing on m_B.

QED.

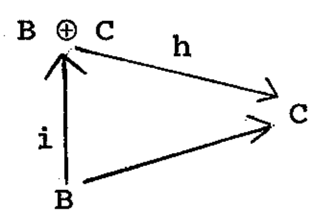
Note that Theorem I.1.5 contains Theorem I.1.2(1) in the case when B is commutative.

(iv) A Remark on Banach Algebras Without Unit.

For B commutative without unit, $B \oplus C$ is a Banach algebra in point wise addition and scalar multiplication, where the product operation is given by

$$(f,\lambda)(g,\beta) \equiv (fg + \lambda g + \beta f, \lambda\beta).$$

$B \oplus C$ has the unit $I \equiv (0,1)$. Define $i(f) \equiv (f,0), f \in B$, then



exhibits a homeomorphism between $m_B \oplus C$ and M_B: Clearly $h \circ i \in M_B$ for each $h \in m_{B\oplus C}$; conversely, if $h_1 \in M_B$, then $h(f,\lambda) \equiv h_1(f) + \lambda$ is an element of $m_{B\oplus C}$, for which $h \circ i = h_1$. Thus the desired homeomorphism is $h \to h \circ i = h_1$.

We define, for $f \in B$, $\sigma_B(f) \equiv \sigma_{B\oplus C}(f)$, which, by I.1.5(2), and the argument above, is compact and equal to $\{h(f) : f \in M_B\}$.

The theory of maximal ideals and spectrum for an arbitrary commutative algebra, B, easily reduces to that for $B \oplus C$.

(v) Semisimple Algebras.

For Commutative B, define $\phi : B \to C(m_B)$ by $\phi(f) \equiv \hat{f}$; then $J \equiv \ker \phi$ is a closed ideal and $\phi \circ \pi : B/J \to \hat{B} \approx \widehat{B/J}$ is an isomorphism. If $J = \{0\}$, B is called semisimple. This is equivalent to saying that the intersection of all kernals of multiplicative linear functionals is trivial. These kernals are precisely the regular maximal ideals of B, which are the same as the maximal ones if B has a unit.

We note that, in general, $\|f\|_B \geq \|\hat{f}\|_\infty \equiv \sup_{h \in m_B} |\hat{f}(h)|$. However, $\overline{\hat{B}}$ is a closed subalgebra of $C(m_B)$ and it is easily shown that $m_{\overline{\hat{B}}} = m_{B/J}$. Thus a semisimple algebra $(J = \{0\})$ is isomorphic to a dense subalgebra of a function algebra with the same maximal ideal space.

(vi) The Silov Boundary.

For B commutative, ∂B is defined to be the (unique) least closed subset, X, of m_B such that $\hat{f}$ achieves its supremum on X for each $f \in B$. Clearly, $f \to f_{|\partial B}$ is an isometry from $\overline{\hat{B}}$ into $C(\partial B)$, so that $\overline{\hat{B}}$ represents itself as a function algebra on ∂B. For B commutative, define, for $f \in B$,

$$\|f\|_s \equiv \sup_{h \in m_B} |h(f)| \quad (= \|\hat{f}\|_\infty).$$

I.1.6. Theorem. $\|f\|_s = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}}$. This is the spectral radius formula, proved in any standard text - e.g. (Loomis - [1]).

If $A \subset B$, A a subalgebra, then $\pi : m_B \to m_A$ denotes restriction : $\pi(h) \equiv h_{|A}$. π is clearly continuous.

I.1.7. Corollary. If B is commutative, $A \subset B$ a subalgebra, then $\partial A \subset \pi(\partial B)$.

Proof: Fix $f \in A$. By Theorem I.1.6,

$$\sup_{h\in m_A}|h(f)| = \lim_{n\to\infty}\|f^n\|^{1/n} = \sup_{h\in m_B}|h(f)| = \sup_{h\in\partial B}|\pi(h)(f)| = \sup_{h\in\pi\partial B}|h(f)|.$$

Clearly, since this holds for each $f \in A$, $\partial A \subset \pi\partial B$. QED.

(vii) Automorphisms of a Banach Algebra.

Let B be a commutative Banach algebra.

Definition. Let Hom B denote the semigroup of (linear) homomorphisms of B. Let Aut B denote the group of invertible elements of Hom B.

An element, $\varphi \in$ Hom B, induces a map, $\tilde{\varphi}$, of m_B, given by $\tilde{\varphi}(h) \equiv h\circ\varphi$. Clearly, if B is semisimple, then $\varphi \to \tilde{\varphi}$ is one to one.

The following may be directly verified

(a) $\widetilde{\varphi\circ\psi} = \tilde{\psi}\circ\tilde{\varphi}$

(b) $\widehat{\varphi(f)} = \hat{f}\circ\tilde{\varphi}$, for $\varphi \in \text{Hom } B$, $f \in B$.

By (a), $\widetilde{\text{Hom } B}$ is a semigroup, via composition of which $\widetilde{\text{Aut } B}$ is a subgroup, so that, a priori, $\tilde{\varphi}$ is a bijection of $\mathfrak{m}_B$ onto $\mathfrak{m}_B$ for each $\varphi \in \text{Aut } B$. In fact, (b) shows that $\tilde{\varphi}$ is a homeomorphism of $\mathfrak{m}_B$ onto $\mathfrak{m}_B$, $\varphi \in \text{Aut } B$.

I.1.8. Lemma. $\tilde{\varphi}\partial B = \partial B$ for $\varphi \in \text{Aut } B$.

Proof: Directly from the definitions.

1.1.9. Lemma. If $A \subset B$ and $\pi\colon \mathfrak{m}_B \to \mathfrak{m}_A$ is restriction, then $\tilde{\varphi}(\pi\mathfrak{m}_B) \subset \pi(\mathfrak{m}_B)$ for each $\varphi \in \text{Hom } B$ such that $\varphi(A) \subset A$.

Proof: Direct from the definitions.

§2. The Theory of Function Algebras.

(i) Some Properties of a Function Algebra on its Maximal Ideal Space.

Certain theorems on holomorphic functions generalize, in part, to shed light on the local behavior on $\mathfrak{m}_A$ of an algebra, $A \subset C(X)$, X compact Hausdorff. This includes a generalized maximum modulus principle and the related result that says if $f \in C(\mathfrak{m}_A)$ is locally A-approximatable, then we "might as well" assume $f \in A$.

Here, we state these theorems and prove some elementary related theorems and a corollary. These will play a crucial role later on.

I.2.1. Theorem. Let $A \subset C(X)$ be a closed subalgebra, X compact Hausdorff, viewed as an algebra on its maximal ideal space.

(1) Let $E \subset m_A$ be compact. Then for $f \in A$, $\hat{f}_{|E}$ achieves its maximum on (Boundary E) $\cup$ ($\partial B \cap E$).

(2) If $f \in C(m_A)$ is A-approximatable at each point $h \in m_A - z(f)$, then $m_{A(f)} = m_A$, $\partial A(f) = \partial A$, that is, $A(f)$ is a function algebra on m_A such that each $h \in m_A$ extends uniquely to a multiplicative linear functional on $A(f)$.

(3) If $f_1, \cdots, f_n \in A$, and g is analytic in a neighborhood of $\sigma(f_1, \cdots, f_n)$, then there is exactly one $f \in A$ such that

$$\hat{f}(h) = g(\hat{f}_1(h), \cdots, \hat{f}_n(h)), h \in m_A.$$

<u>Proof</u> (Gamelin-[1]). For A a commutative Banach Algebra, $I \subset A$ an ideal, we let $z(I)$ denote those $h \in m_A$ for which $h(g) = 0$ for all $g \in I$.

I.2.2. <u>Lemma</u>. Let $I \subset A \subset B$, B a commutative Banach algebra, A a closed subalgebra, I a closed ideal in A such that $IB \subset A$. Then each $h \in m_A - z(I)$ extends uniquely to a multiplicative linear functional on B.

<u>Proof:</u> Fix such an h, and choose $f_o \in I$ such that $h(f_o) \neq 0$. If h does extend to $\bar{h} \in m_B$, we must have, for each $g \in B$, $h(f_o)\bar{h}(g) = \bar{h}(f_o g) = h(f_o g)$ so that

$$(*) \quad \bar{h}(g) = h(f_o g)/h(f_o).$$

Thus the uniqueness.

It remains to show that $\bar{h}$, defined by (*), is linear and multiplicative on B. Linearity is clear. To show

multiplicativity: for $g_1, g_2 \in B$, $\bar{h}(g_1g_2) = h(f_\circ g_1g_2)/h(f_\circ)$
$= h(f_\circ g_1g_2)h(f_\circ)/h(f_\circ)^2 = h(f_\circ g_1 f_\circ g_2)/h(f_\circ)^2$
$= h(f_\circ g_1)h(f_\circ g_2)/h(f_\circ)^2 = \bar{h}(g_1)\bar{h}(g_2)$. QED.

I.2.3. Theorem. Let Y be a set, and $A \subset B$ be uniformly closed algebras of functions on Y. Assume $\partial A \subset Y$ and X is dense in ∂A (this means that each $f \in A$ approximates its supremum on X). If $I \subset A$ is an ideal such that $z(I) \cap Y \subset X$ and $IB \subset A$, then X is dense in ∂B.

Proof: We give two proofs.

Proof 1: Since $IB \subset A$, we have $I_{|X} \overline{B_{|X}} \subset A_{|X}$. Thus, by Lemma I.2.2, each $y \in Y - X$ extends to $h_y \in m_{\overline{B_{|X}}}$. But for $g \in B_{|X}$, we must have $h_y(g) = g(y)$. Thus, since h_y is bounded on $B_{|X}$, this gives $|g(y)| \leq \|g_{|X}\|_\infty$.

Proof 2: Let f_X denote $f_{|X}$, for $f \in B$. Fix $y \in Y - X, g \in B$ and assume (*) $|g(y)| > \|g_X\|_\infty = 1$. Choose $f \in I$ such that $f(y) \neq 0$. Then $|f(y)||g(y)|^n =$
$|(fg^n)(y)| \leq \|f_X g_X^n\|_\infty \leq \|f_X\|_\infty \|g_X\|^n_\infty = \|f_X\|_\infty$ for all $n \geq 1$. By (*), the left side approaches infinity as n gets large, a contradiction. QED.

I.2.4. Corollary. Let A be as in Theorem I.2.1, and assume $X = \partial A$. Define $Y \equiv X \cup (m_A - z(I))$, $I \subset A$ an ideal. If $B \subset C_B(Y)$ extends $A_{|Y}$ and is locally A - approximatable on Y, then $\partial B = X$.

<u>Proof</u>: Define A_1 to be the (closed) subalgebra of $C(m_A)$ consisting of all continuous functions locally A-approximatable on Y. Define B_1 to be the (closed) subalgebra of $C_B(Y)$ generated by B and $A_1|_Y$. By Theorem I.2.1(2) we have

(a) $$\partial A_1 = \partial A = X$$

Thus, $A_1 \approx A_1|_Y$ and

(b) $$A_1 \subset B_1$$

(c) $$I_1 B_1 \subset A_1$$

where I_1 is the ideal in A_1 generated by I. Thus the hypothesis of Theorem I.2.3 are satisfied so that $\partial B_1 = X$. Since $\partial A = X \subset Y$ and $A|_Y \subset B \subset B_1$, we have $\partial B = X$. QED.

(ii) Representing Measures.

Let X be locally compact and Hausdorff, $B \subset C_o(X)$ a closed subalgebra. Note that, as expected, $\partial B \subsetneqq X$: for $|f(h)| \le \|f\|_\infty = \sup_{x \in X} |f(x)|$ for each $f \in B$, $h \in \mathfrak{m}_B$. Thus, we might as well assume $\partial B = X$, although this assumption will not be used.

For $h \in \mathfrak{m}_B$ there exists a $\mu \in M(X)$ (complex Borel measures on X) such that

$$h(f) = \int_X f d\mu \text{ for each } f \in B$$

and $\|\mu\| = \|h\|$ by the Hahn-Banach theorem. If X is compact, $1 \in B$, then $\|\mu\| = \|h\| = 1$ and $\|\mu\| = 1 = h(1) = \int_X 1 d\mu = \mu(X)$, showing that μ is positive. Conversely, if μ is chosen to be positive, then $\|\mu\| = \mu(X) = \int_X 1 d\mu = h(1) = 1$. Such a positive measure, μ, "representing" h is naturally called a representing measure for h.

(iii) Polynomial and Rational Convexity.

Let Y be a set, B an algebra of complex valued functions on Y, and $K \subset Y$ such that $f_{|K}$ is bounded for each $f \in B$.

Definition. $\hat{K}(B)$ denotes the set of all $y \in Y$ for which $|f(y)| \le \|f\|_K^\infty$ for each $f \in B$.

$\check{K}(B)$ denotes the set of all $y \in Y$ for which $|f(y)| \le \|\frac{f}{g}\|_K^\infty |g(y)|$ for each $g \in B$ bounded below on K.

$\hat{K}(B)$, the B-polynomially convex hull of K, coincides with the set of $y \in Y$ for which $f_{|K} \to f(y)$ extends to be the multiplicative on $B_{|K} = P_B(K)$. $\check{K}(B)$, the B-rational convex hull of K, coincides with the set of $y \in Y$ for which $(f/g)|K \to f(y)/g(y)$ extends to be multiplicative on $R_B(K)$, the space of rational functions on K; note that this space is automatically an algebra, since B is.

Remark. (1) For B an algebra of functions on a locally compact-Hausdorff space X, and Y denoting m_B, $K \subset Y$, then $\hat{K}(B)$ is clearly equal to $m_{P_B}(\check{K})$, and $\check{K}(B)$ equal to $m_{R_B}(K)$.

Definition. K is B-polynomially convex if $K = \hat{K}(B)$; K is B-rationally convex if $K = \check{K}(B)$.

In case $Y \equiv C^n$ (n possibly infinite) $K \subset C^n$ compact, $B \equiv \mathcal{P}^n$ (the algebra of polynomials in $z_1, \cdots, z_n$, acting on C^n) then, since $f_{|K}$ is bounded for each $f \in \mathcal{P}^n$, $\hat{K} \equiv \hat{K}(\mathcal{P}^n)$ is defined and equal to $m_{P(K)}$: for $h \in m_{P(K)}$, $\zeta \equiv (h(\pi_1), \cdots, h(\pi_n))$, then $Q(\zeta) = h(Q)$ for Q a polynomial in $\pi_1, \cdots, \pi_n$, showing that ζ represents h on P(K) and so lies in $\hat{K}$ by definition. Similar statements hold for $\check{K} \equiv \check{K}(\mathcal{P}^n)$. It therefore follows that $E \subset C^n$, $\zeta \in C^n$, $\zeta \in \hat{E}$ (respectfully $\check{E}$) if, and only if, $\pi_1 - \zeta_1, \cdots, \pi_n - \zeta_n$ generate a proper ideal in P(K) (respectfully R(K)).

I.2.5. Lemma. Let $K \subset C^n$ be compact. Then $\zeta \in \check{K}$ if, and only if, for each polynomial, P such that $P(\zeta) = 0$, $P(z) = 0$ for some $z \in K$.

Proof (Gamelin - [2]): Fix $\zeta \in C^n$. If P is a polynomial such that $P(\zeta) = 0$ and $P(z) \neq 0$ for all $z \in K$, then clearly $\zeta \notin \check{K}$. Conversely, if $\zeta \notin \check{K}$, then, by the remark proceeding Lemma I.2.5, there are quotients, $P_j/Q_j, Q_j(z) \neq 0$ for all $z \in K$, such that the function $R(z)$ defined by

$$\sum_{j=1}^{n} \frac{P_j(z)}{Q_j(z)} (z_j - \zeta_j) \text{ is nonvanishing for } z \in K.$$

Then $Q_1(z), \cdots, Q_n(z)R(z)$ does the job. QED.

Let $E \subset C$ be compact.

I.2.6. Theorem. Assume $P(E) \subset B \subset C(E)$, B a closed subalgebra. Let $K \equiv \pi m_B \subset \hat{E}$ where $\pi\colon m_B \to m_{P(E)} \approx \hat{E}$ is restriction. Then $R(K) \subset A(K) \subset B$.

Proof: Clearly $R(K) \subset A(K)$. Note that Theorem I.1.5(3) shows that $R(K) \subset B$. However, we can do better and show that $A(K) \subset B$.

Fix $g \in \mathfrak{A}(K)$. We have $\sigma_B(\pi_1, \cdots, \pi_n) = K$, so that, upon application of Theorem I.2.1 (3), we see that $g \in B$. Thus $\mathfrak{A}(K) \subset B$ and, therefore, its closure, $A(K) \subset B$. QED

I.2.7. Corollary. If $K \subset C^n$ is rationally or polynomially convex, then $A(K) = R(K)$ or $P(K)$, respectfully.

Proof: Let $B \equiv R(K)$ or $P(K)$, respectfully, and define $E \equiv \partial B$. Then the hypothesis of Theorem I.2.6 hold. Thus $A(K) \subset B$. But clearly $B \subset A(K)$. QED.

Remark .(2) For $E \subset K \subset \hat{E}$, note the inclusions and relations:

$$P(E) \approx P(\hat{E}) = A(\hat{E}),\ \hat{E} \approx \mathfrak{m}_{P(E)}$$

$$P(E) \subset A(E)\ .$$

In many cases, $A(E) = C(E)$ - e.g. $E = T^n, n \geq 1$.

We will often write $A(\hat{E})$ in place of $P(E)$ to conform with convention; the identification of $A(\hat{E})$ with a closed subalgebra of $C(E)$, as opposed to one of $C(\hat{E})$ will be clear from the context.

(iv) Some Deeper Properties of Function Algebras: Maximality, Relative Maximality, and Extensions of "Half Plane" Algebras.

Definitions: Let X_1, X_2 be compact, Hausdorff, $x_1 \in X_1, x_2 \in X_2$.

(a) For $f \in C(X_1 \times X_2)$, define $f^{x_2} \in C(X_1)$ by $f^{x_2}(x) \equiv f(x, x_2)$, $f_{x_1} \in C(X_2)$ by $f_{x_1}(x) = f(x_1, x)$.

(b) For $A_j \subset C(X_j)$ closed subspaces $(j=1,2)$ $A_1 \otimes A_2 \subset C(X_1 \times X_2)$ is the closed subspace generated by $\{fg : f \in A_1, g \in A_2\}$.

$$A_1 \circledast A_2 \equiv \{f \in C(X_1 \times X_2) : f^{x_2} \in A_1 \text{ for all } x_2 \in X_2$$
$$\text{and } f_{x_1} \in A_2 \text{ for all } x_1 \in X_1\}.$$

If A_1, A_2 (above) are algebras, then so is $A_1 \otimes A_2$, $A_1 \circledast A_2$. In this case, for $h_1 \in \mathfrak{m}_{A_1}$, $h_2 \in \mathfrak{m}_{A_2}$, $h_1 \otimes h_2 \in \mathfrak{m}_{A_1 \otimes A_2}$ is defined by

$$h_1 \otimes h_2\left(\sum_{j=1}^{n} f_j g_j\right) \equiv \sum_{j=1}^{n} h_1(f_j)h_2(g_j),$$

easily shown to be well defined and bounded.

Clearly $(h_1, h_2) \to h_1 \otimes h_2$ maps $\mathfrak{m}_{A_1} \times \mathfrak{m}_{A_2}$ one to one and onto $\mathfrak{m}_{A_1 \otimes A_2}$.

(c) Let $E \subset X_1$ be closed. Then $(A_1)_E \equiv A_1 + I_E$, I_E denoting the ideal of continuous functions vanishing on E.

If A_1 is an algebra, so is $I_E + A_1$. Then $A_1 + I_E$ is closed if, and only if, $A_1|_E$ is closed in $C(E)$.

(d) $A_1 \subset C(X)$ is maximal if $A_1 \subset B \subset C(X)$, B a closed subalgebra, implies $A_1 = B$.

$A_1 \subset C(X)$ is relatively maximal if $A_1 \subset B \subset C(X)$ and $\pi\mathfrak{m}_B = \pi\mathfrak{m}_{A_1}$ (where $\pi\colon \mathfrak{m}_B \to \mathfrak{m}_{A_1}$ is restriction) implies $A_1 = B$.

We now prove the key lemma in our study of relative maximality.

<u>Definition</u>. If $f \in C(X)$, $E \subset X$, then f "peaks" on E if $f(x)=1$ for each $x \in E$, $|f(x)| < 1$ for $x \in X-E$.

Let $B \subset C(X)$ be a closed subalgebra such that $X = \mathfrak{m}_B$.

I.2.8. <u>Lemma</u> (Gamelin - [3]). Suppose f peaks on $E \subset \mathfrak{m}_B$. Then

(1) $B|_E = \overline{B|_E}$

(2) $h \in \mathfrak{m}_B$, $h(f) = 1$ implies that supp $\mu \subset E$ for

each representing measure, μ, for h.

Proof: Suppose $f(x) = 1$ for all $x \in E$ and $|f(x)| < 1$ for all $x \in X-E$. Consider the map $r : B \to C(E)$ given by $r(g) = g_{|E}$. This induces the map r': $B/\ker r \to C(E)$. To show (1), we must show r' has closed range; since r' is clearly bounded it suffices to show it is bounded below. For $g \in B$, $g + \ker r \in B/\ker r$, and $\|g + \ker r\| \equiv \inf_{u \in \ker r} \|g + u\|_{\infty}^{X}$. Since for each $n \geq 1$, $g(f^n-1) = 0$ on E, this is less then or equal to $\|g + g(f^n-1)\|_{\infty}^{X} = \|gf^n\|_{\infty}^{X}$ for each $n \geq 1$. But $\|gf^n\|_{\infty}^{X} \to \|g\|_{\infty}^{E}$ as $n \to \infty$. Thus, $\|g + \ker r\| \leq \|g\|_{\infty}^{E}$. Since $g \in B$ was arbitrarily chosen, r is bounded below, and (1) follows.

To show (2), let $\mu \in M(X)$ represent $h \in m_B$, chosen such that $h(f) = 1$. For each $n \geq 1$, $1 = h(f^n) = \int_X f^n d\mu$ $= \int_{E-X} f^n d\mu + \int_E f^n d\mu$, which approaches $\int_E f^n d\mu = \mu(E)$. Thus, $\mu(E) = 1$. Since $\mu(X) = 1$, clearly (2) follows.

Finally, (3) follows from (2) by choosing μ above, such that $\text{supp}\, \mu \subset \partial B$. QED.

Henceforth, we let μ_x denote point mass at $x \in X$ and we will assume that if $A \subset C(X)$ is a closed subalgebra, then each $x \in X$ is a peak point for A.

I.2.9. Lemma. Let $A_1 \otimes A_2 \subset B \subset C(X_1 \times X_2)$, B a closed subalgebra. Then for all $x_1 \in X_1$, $A_2 \subset B_{x_1}$ is closed in $C(X_2)$, and if $x_1 \otimes h \in m_{A_1 \otimes A_2}$ extends to B and $\mu \in M(X_1 \times X_2)$ represents $x_1 \otimes h$ on B, then supp $\mu \subset \{x_1\} \times X_2$. Thus, $\mu = \mu_{x_1} \otimes \sigma$, $\sigma \in M(X_2)$; a priori, σ represents h on A_2 and is multiplicative on B_{x_1}. Thus h extends to B_{x_1}.

Proof: Choose $\varphi \in A_1$ which peaks at x_1 : $\varphi(x_1) = 1$, $|\varphi(x)| < 1$ for $x \neq x_1$. Then $\varphi \otimes 1$ peaks on $\{x_1\} \times X_2$. Thus, by Lemma I.2.8 (2), supp $\mu \subset \{x_1\} \times X_2$. Elementary measure theory then shows that $\mu = \mu_{x_1} \otimes \sigma$, $\sigma \in M(X_2)$. The rest is just computation. QED.

I.2.10. Lemma. Each $h_1 \otimes h_2 \in m_{A_1 \otimes A_2}$ extends to $A_1 \hat{\otimes} A_2$

Proof: If μ_j represemts h_j (j=1,2) then $\mu_1 \otimes \mu_2$ represents $h_1 \otimes h_2$ on $A_1 \otimes A_2$ and is easily seen to be multiplicative on $A_1 \hat{\otimes} A_2$. QED.

We consider this extension, clearly independent of the representing measures μ_1 and μ_2, the canonical one, and denote it by $h_1 \hat{\otimes} h_2 \in m_{A_1 \hat{\otimes} A_2}$.

The following theorem is essentially contained in Rudin - [2].

I.2.11. Theorem. If $A_j \subset C(X_j)$ (j=1,2) are relatively maximal subalgebras, then so is $A_1 \hat{\otimes} A_2$.

Proof: Let $A_1 \hat{\otimes} A_2 \subset B \subset C(X_1 \times X_2)$, and assume $\pi m_B = m_{A_1 \hat{\otimes} A_2}$. Then $A_2 \subset B_{x_1} \subset C(X_2)$ for all $x_1 \in X_1$. Fix $x_1 \in X$, $h \in m_{A_2}$.

By hypothesis, $x_1 \circledast h$ extends to B. Thus, by Lemma I.2.9 h extends to B_{x_1}. Since this holds, thus, for all $h \in m_{A_2}$, relative maximality of A_2 shows that $B_{x_1} = A_2$. Similarly, for $x_2 \in X_2$, $B^{x_2} = A_1$. Thus $B = A_1 \circledast A_2$. QED.

Although the problem of characterizing all extensions of relatively maximal subalgebras, $B \subset C(X)$, is far from solved a complete solution exists in the case $B \equiv C(X_1) \circledast A$ where $A \subset C(X_2)$ is maximal.

Definition. Let $A \subset C(X)$ be a function algebra. Then $E \subset X$ is antisymmetric (for A) if $f \in A$ and $f_{|E}$ is real valued implies $f_{|E}$ is constant.

I.2.12. Theorem. Fix $\mu \in (A^{\perp})_1$, an extreme point. Then supp μ is antisymmetric. For the proof see Gamelin-[4].

In what follows $\mathscr{A}$ denotes the collection of all antisymmetric sets for A.

I.2.13. Corollary (Bishop): $A = \{f \in C(X): f_{|E} \in \overline{A_{|E}}$ for each $E \in \mathscr{A}\}$.

Proof: If $f_{|E} \in \overline{A_{|E}}$, $E \in \mathscr{A}$, then $\int_X f d\mu = 0$ for each extreme point measure $\mu \in (B^{\perp})_1$ by Theorem I.2.12. Thus $\int_X f d\mu = 0$ for each $\mu \in (B^{\perp})_1$, by the Krein-Milman Theorem. Thus $f \in A$ by the Hahn Banach Theorem. QED.

I.2.14. <u>Theorem</u>. Let $A \subset C(X_2)$ be a maximal closed subalgebra, and assume $C(X_1) \otimes A \subset B \subset C(X_1 \times X_2)$, B a closed subalgebra. Then there is a closed subset $E \subset X$ such that

(1) $B = \{f \in C(X_1 \times X_2): f_x \in A \text{ for each } x \in E\}$

$= (C(X_1) \otimes A)_{E \times X_2}$ and

(2) $m_B = (X_1 \times X_2) \cup (E \times m_A)$ where $E \times m_A \underset{\sim}{\subseteq} m_B$ via, $(x,h)(f) \equiv h(f_x)$ which extends $x \otimes h$ on $C(X_1) \otimes A$.

<u>Proof</u>: Clearly each antisymmetric set, $V \subset X_1 \times X_2$, for B, satisfies $V \subset \{x_1\} \times X_2$ for some $x_1 \in X_1$. Thus, by Corrollary I.2.13, $B = \{f \in C(X_1 \times X_2): f_x \in \overline{B_x}$ for each $x \in X_1\}$.

But for $x \in X_1$, $A \subset B_x \subset C(X_2)$, so that by maximality of A, $\overline{B_x}$ is either A or $C(X_2)$. Let $E \equiv \{x \in X_1: \overline{B_x} = A\}$. Then E is closed, and $B = \{f \in C(X_1 \times X_2): f_x \in A$ for each $x \in E\}$. Thus clearly $(C(X_1) \otimes A)_{E \times X_2} \subset B$. Equality follows from an argument similar to that in the first paragraph. Finally (2) follows directly from (1). QED.

(v) <u>The Role of Compactification in the Study of m_A</u>.

Quite often a function algebra, B, on a compact set, X, is known to extend itself to a locally compact set, Y, so that

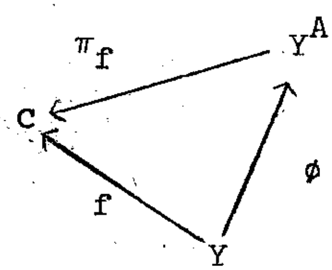
$$X \subset Y \subset \overline{Y} \subsetneqq \mathfrak{m}_B .$$

Then $\overline{Y}$ is, up to homeomorphism, a certain compactification of Y, defined below.

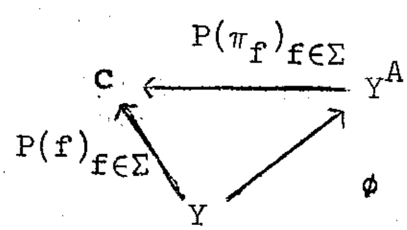
Definition. Let Y be locally compact and Hausdorff, $A \subset C_B(Y)$ a point separating algebra. Let $\Sigma \subset A$ be a generating set, and define $\phi(y) \equiv (f(y))_{f\in\Sigma}$, $y \in Y$. The A - compactification of Y, denoted Y^A, is (represented by) the closure in C^Σ (in product topology) of $\phi(Y)$.

Note that $\phi: Y \to C^\Sigma$ is an embedding, since $y_\alpha \to y$ in Y, if, and only if, $f(y_\alpha) \to f(y)$ for each $f \in \Sigma$ (using the Stone-Weierstrass Theorem). By the Tychonoff product theorem, Y^A is, indeed, compact. Clearly Y^A is well defined up to homeomorphism.

Define $\pi_f: C^\Sigma \to C$ by $\pi_f((z_g)_{g\in\Sigma}) \equiv z_f$. The diagram



commutes for each $f \in \Sigma$. Thus, so does



for P a polynomial in $\{\pi_f\}_{f\in\Sigma}$. Since $\emptyset(X)$ is dense in $Y^A \equiv \overline{\emptyset(X)}$, $\emptyset(P(f)_{f\in\Sigma}) \equiv P(\pi_f)_{f\in\Sigma}$ is a well defined isometry and thus extends uniquely to an isometry from $\bar{A}$ onto $P(Y^A)$.

We clearly have $Y \subseteq Y^A \subseteq m_{\bar{A}}$. In case Y is compact to begin with, then $\emptyset$ maps Y homeomorphically onto $Y^A \subset C^\Sigma$. Thus, $B \approx P(Y^A)$ and $m_B \approx m_{P(Y^A)} = \widehat{P(Y^A)}$. Thus, the determination of m_A amounts, conceptually, to the determination of the polynomially convex hull of a certain compact subset, namely Y^A, of C^Σ.

We now apply this to the problem of uniqueness of extension of maximal ideals. Let $B \supseteq A$ be commutative Banach algebras. As already noted,

$$\pi\colon m_B \to m_A,\ \pi(h) \equiv h_{|A}$$

is continuous.

For $A \subset B \subset C(X)$, $\partial A = X$, π is rarely one to one. Thus, the question, when is $\pi^{-1}(h)$ a unique point?, and if not, why not?, are relevant to the study of extending algebras for A.

<u>Remarks</u>. (1) For X compact, $A \subset B \subset C(X)$ closed algebras, B generated by A and a collection of quotients, $f/g, f, g \in A, g(x) \neq 0$ for each $x \in X$, it is clear that $h \in m_A$, card $\pi^{-1}(h) > 1$ implies $h(g) = 0$: for if $\bar{h} \in \pi^{-1}(h)$, $k \equiv f/g$ then $kg = f$, so that

$$h(f) = \bar{h}(f) = \bar{h}(k)\bar{h}(g)$$

$$= \bar{h}(k)h(g), \text{ and thus,}$$

$$\bar{h}(k) = h(f)/h(g) \text{ if } h(g) \neq 0.$$

(2) If E is defined by $z(g) \cap \pi m_B \subset m_A - X$ then E is closed, and therefore $m_A - E$ is locally compact, on which $\widehat{f/g} = \hat{f}/\hat{g}$ is bounded for any one of the denominators g in (1). Thus $B \subsetneqq C_B(m_A - E)$, and we may apply the notion of compactification to help describe $\pi^{-1}(E)$. The following theorem is helpful in predicting the size and shape of $\pi^{-1}(z(g) \cap \pi m_B)$ for $g \in B$ when $B = A(f/g)$.

I.2.15. Theorem. Let X be compact Hausdorff, metrizable, $A \subset B \subset C(X)$ function algebras. Choose $E \subset \pi(m_B) \subset m_A$ such that card $\pi^{-1}(h) = 1$ for each $h \in Y \equiv \pi(m_B) - E$. Then

$$\hat{\partial B}\big|_{\pi^{-1}(E)} \subsetneqq Y^B - Y \subsetneqq \pi^{-1}(E).$$

Proof: Note that $Y^B - Y$ identifies with the set of sequences $(h_j)_1^\infty \subset Y$ such that h_j is approaching E as $j \to \infty$ and $\lim_j \hat{f}(h_j)$ exists for each $f \in B$, and sequences $(h_j^{(1)})_1^\infty$ $(h_j^{(2)})_1^\infty$ identify if $\lim_j \hat{f}(h_j^{(1)}) = \lim_j \hat{f}(h_j^{(2)})$ for each $f \in B$. It is clear that $Y^B - Y \subsetneqq \pi^{-1}(E)$, since each $(h_j)_1^\infty \in Y^B - Y$ extends $h = \lim_j h_j \in E$.

Assume the first inclusion of the theorem were false. Then there exists an $f \in B$, $h_\circ \in \pi^{-1}(E)-(Y^B-Y)$, $\varepsilon > 0$ such that

(*) $\quad |\hat{f}(h_\circ)| \geq |\hat{f}(h)| + \varepsilon$ for each $h \in Y^B - Y$.

Choose $(h_j)_1^\infty \in Y$, h_j approaching E. Since m_B is compact, there exists a convergent subsequence $(h_{j_i})_1^\infty$; this means $\hat{f}(h_{j_i})$ converges for each $f \in B$. Thus, by the identification made above, $(h_{j_i})^\infty$, identifies itself with an element, h, of $Y^B - Y$ and

$$\left|\lim_{i\to\infty} \hat{f}(h_{j_i})\right| + \varepsilon = |\hat{f}(h)| + \varepsilon \leq |\hat{f}(h_\circ)| .$$

Let $(V_j)_{j=1}^\infty$ be open sets in πm_B such that $E = \bigcap_{j=1}^\infty V_j$, $V_1 \supset V_2 \supset \cdots$. We claim that for some $j \equiv j_0$, (*) holds for each $h \in V_j - E$. For assume not: then for each j, there exists $h_j \in V_j - E$ such that (*) is false for $h \equiv h_j$. Clearly h_j approaches E; thus, as shown above, there exists a subsequence $(h_{j_i})_1^\infty$ such that (*) holds, a contradiction.

Choose V open such that $E \subset V \subset \overline{V} \subset V_{j_\circ}$. Then (*) holds for each $h \in \overline{V} - E$, and, in particular, for $h \in$ Boundary V. But $\pi^{-1}(E) \subset \pi^{-1}(V)$ open in m_B, and Theorem I.2.1(1) implies that $\sup_{h \in \pi^{-1}(V)} |\hat{f}(h)|$ is achieved on Boundary $\overline{\pi^{-1}(V)} \subset$ Boundary $\pi^{-1}(V) \subset \pi^{-1}$(Boundary V) $\approx$ Boundary V, a contradiction, since we showed that (*) must hold for points in Boundary V. QED.

Remarks. (3) It follows that $m_{\hat{B}|\pi^{-1}(E)} \approx \widehat{Y_B - Y}$, where $Y_B - Y \subsetneqq C^\Sigma$ identifies with all limits $(\hat{f}(h))_{f\in\Sigma}$ as $h \to E$.

(4) If E is $\hat{A}$-polynomially convex, then clearly $m_{\hat{B}|\pi^{-1}(E)} = \pi^{-1}(E)$. Thus $\pi^{-1}(E) \approx \widehat{Y_B - Y}$ by (3).

I.2.16. Corollary. In the notation of Theorem I.2.15, if $E = \{h_\circ\}$, then card $\pi^{-1}(h_\circ) > 1$, if, and only if, for some $f \in B$, $\hat{f}$ has no continuous extension to a neighborhood of $h_\circ$.

<u>Proof</u>: Clearly $\pi^{-1}(h\ ) = m_{P_B}(\{h_\circ\})$ contains more then one point if, and only if, $\partial P_B(\pi^{-1}(h_\circ))$ does. By the theorem, this holds if, and only if, $Y^B - Y$ contains more then one point, i.e. - there exist sequences $(h_j^{(1)})_1^\infty$, $(h_j^{(2)})_1^\infty$, $h_j^{(i)}$ approaching $h_\circ (i=1,2)$ such that $\lim_j \hat{f}(h_j^{(1)}) \neq \lim_j \hat{f}(h_j^{(2)})$ for some $f \in B$. But this means, for such f, that $\hat{f}$ has no continuous extension to a neighborhood of $h_\circ$. QED.

Note that for $x \in X$, $\pi^{-1}(h_x)$ is trivial. For if $h \in \pi^{-1}(h_x)$, let μ be a representing measure for h_x. Then $\int_X f d\mu = h_x(f) = f(x)$ for each $f \in A$. But there exists an $f \in A$ peaking at x, forcing supp $\mu = \{x\}$. Thus, for each $f \in B$, $h(f) = \int_X f d\mu = f(x) = h_x(f)$, showing that h_x extends uniquely to B.

CHAPTER II

Algebras with Conformal Structure

§0. Introduction.

Banach spaces acted on by groups are the easiest ones to study structure wise. In particular, if B is a Banach algebra acted on by a group G, that is $G \subseteq \text{Aut } B$, then m_B is also acted on by G (see I.§1.(vii)) and in some cases is thus easily characterized, along with B. These "translation invariant" algebras form the building blocks of a larger, more interesting class of algebra some of which will be discussed.

Before going on, we must present some basic facts from harmonic analysis. Let G be a locally compact abelian group and m Haar measure on G. Then $\hat{G}$, the group of continuous homomorphisms, $\lambda: G \to T$, identifies with $m_{L_1(G)}$: for $\lambda \in \hat{G}$, $f \to \int_G f(x)\lambda(-x)dm(x)$ is multiplicative in convolution, and all multiplicative linear functionals on $L^1(G)$ arise in this manner for a unique $\lambda \in \hat{G}$. An element of $\hat{G}$ is called a character.

A theorem of Pontryagin states that

(1) $\hat{G}$ separates points on G, and

(2) Each $\lambda \in \hat{\hat{G}}$ is defined by evaluation at a point of G. Put another way: $\hat{\hat{G}} = G$ (Pontryagin duality).

The uniqueness theorem for Fourier transforms states that if, for $\mu \in M(G)$

$$\hat{\mu}(\lambda) \equiv \int_G \lambda(-x)d\mu(x) = 0$$

for each $\lambda \in \hat{G}$ then $\mu \equiv 0$. Note that, for G compact, this implies that the characters on G form a complete orthonormal bases on $L^2(G)$: it only remains to verify that

$$\int_G \lambda dm = \begin{cases} 1, \lambda = 1 \\ 0, \lambda \neq 1 \end{cases} .$$

This is a mere exercise. For a complete reference, see Rudin - [3].

For A a Banach space, $F : G \to A$ a continuous function such that $x \to \|F(x)\| \in L^1(G)$, we define the "generalized" Fourier transform of F : $\hat{F}(\lambda) \equiv \int_G F(x)\lambda(-x)dm(x)$. In general, $\int_G F(x)dm(x)$ exists <u>whenever</u> F is continuous and

$$\int_G \|F(x)\|dm(x) < \infty \text{ (i.e. } x \to \|F(x)\| \in L^1(G)) .$$

§1. <u>Spectral Synthesis in Banach Spaces Acted on by a Group</u>.

Let A be a Banach space, and G a locally compact, abelian group and consider a map

$$A \times G \to A$$

$$(f, x) \to f_x ,$$

continuous in each variable separately such that

(i) $(f_x)_y = f_{xy}$

(ii) $(\alpha f+\beta g)_x = \alpha f_x + \beta f_y$

for each $f, g \in A$, $x, y \in G$. If $B \subset A$ satisfies $f_x \in B$ for each $f \in B$, then B is called translation invariant. For $f \in A$, let f^T denote the Banach space valued function, $x \to f_x$, on G.

II.1.1. <u>Theorem</u>. Let $\phi \in C(G)$ satisfy $\phi(1) = 1$, and assume $x \to \|\phi(x)f_x\|_A$ lies in $L^1(G)$ for each $f \in A$. Then

(1) A is generated by elements, $\widehat{\phi f}^T(\lambda) \equiv \int_G \phi(x)f_x\lambda^{-1}(x)dm(x)$, for $f \in A$, $\lambda \in \hat{G}$.

(2) If G is compact then A is generated by $\{\hat{f}^T(\lambda): \lambda \in \hat{G}, f \in A\}$ and $\hat{f}^T(\lambda)_x = \lambda(x)\hat{f}^T(\lambda)$, $x \in G$, $f \in A$.

<u>Proof</u>: (1) Suppose $L \in A^*$ and $L(\widehat{\phi f}^T(\lambda)) = 0$ for each $f \in A$, $\lambda \in \hat{G}$. Then for such f and λ,

$$\int_G L(\phi(x)f_x)\lambda^{-1}(x)dm(x)$$

$$= L(\int_G \phi(x)f_x\lambda^{-1}(x)dm(x)) = L(\widehat{\phi f}^T(\lambda)) = 0,$$

from which it follows that $L(\phi(x)f_x) = 0$ for each $x \in G$ by the uniqueness theorem. Thus $L(f) = \phi(1)L(f_1) = 0$, since $\phi(1) = 1$. Since this is so for each $f \in A$, and $L \in A^*$ was arbitrary, the Hahn Banach Theorem now applies to complete the proof.

(2) We may choose $\phi(x) = 1$. For $\lambda \in \hat{G}$, $y \in G$,

$$\hat{f}^T(\lambda)_y = \left(\int_G f_x \lambda^{-1}(x)\,dm(x)\right)y$$

$$= \int_G (f_x)_y \lambda^{-1}(x)\,dm(x) = \int_G f_{xy} \lambda^{-1}(x)\,dm(x)$$

$$= \int_G f_x \lambda^{-1}(xy^{-1})\,dm(x) = \lambda(y) \int_G f_x \lambda^{-1}(x)\,dm(x)$$

$$= \lambda(y)\hat{f}^T(\lambda).$$ QED.

Remark. By observing the proof of (1) and choosing ϕ, if possible, such that $\phi(x) \neq 0$ for each $x \in G$, we may conclude that for $\Sigma \subset A$ any set whose translates generate A, $\{\widehat{\phi f}^T(\lambda) : f \in \Sigma,\ \lambda \in \hat{G}\}$ in fact, generates A.

§2. A Study of Translation Invariant Subalgebras of Continuous Functions on Compact Abelian Groups: The Classical Group Theoretic Results.

Let G be compact, abelian. For $f \in C(G)$, $f_x(y) \equiv f(xy)$ defines a group action of G on $C(G)$ satisfying (i) and (ii) of II.§1, with $A \equiv C(G)$.

II.2.1. Theorem. Let $A \subset C(G)$ be a closed subspace. Then A is translation invariant if, and only if, A is generated by characters. Furthermore, in this case a character λ lies in A if, and only if, $\hat{f}(\lambda) \neq 0$ for some $f \in A$.

Proof: Let $A \subset C(G)$ be translation invariant. By Theorem

II.1.1 (2), A is generated by functions of the form $\hat{f}^T(\lambda)(x) = \hat{f}^T(\lambda)(x\cdot 1) = \hat{f}^T(\lambda)_x(1) = \lambda(x)\hat{f}^T(\lambda)(1)$ $= \lambda(x)\hat{f}(\lambda)$, $\lambda \in \hat{G}$, $f \in A$. Thus A is generated by just those characters $\lambda \in \hat{G}$ for which $\hat{f}(\lambda) \neq 0$ for some $f \in A$. Conversely, if A is generated by characters, then, since $\lambda_x(y) = \lambda(x)\lambda(y)$ is scaler multiplication by $\lambda(x)$, A is translation invariant. This shows the first statement. As for the second, only note that if $\lambda \in A$, $\hat{\lambda}(\lambda) = 1 \neq 0$. QED.

Now consider a semigroup $\Sigma \subset \hat{G}$ such that $\Sigma \cup \Sigma^{-1}$ generates $\hat{G}$: each $\lambda \in \hat{G}$ can be written as $\beta\gamma^{-1}$, $\beta, \gamma \in \Sigma$. $A(\Sigma)$ denotes the closed subalgebra of $C(G)$ generated by Σ (acting as characters on G). Hom Σ denotes the semigroup of non-zero homomorphisms $h : \Sigma \to D$ topologized weakly with respect to Σ : $h_\alpha \to h$ if, and only if, $h_\alpha(\lambda) \to h(\lambda)$ for each $\lambda \in \Sigma$. Then

$$G \subset \text{Hom}\,\Sigma \text{ via } x \to h_x,\ h_x(\lambda) \equiv \lambda(x).$$

To show this, note that if $h_x = h_y$, then $\lambda(x) = \lambda(y)$ for each $\lambda \in \Sigma$, and thus for each $\lambda \in \hat{G}$ (since Σ generates $\hat{G}$). Thus $x = y$ since the characters separate points on G (Pontryagin duality).

If $\Sigma = \hat{G}$, then the above embedding is onto, since each homomorphism of $\hat{G}$ into D arises from an element of G, again, by Pontryagin duality. The following theorem is due to Arens and Singer - [1] .

II.2.2. <u>Theorem</u>. $\mathrm{Hom}\,\Sigma = \mathfrak{m}_{A(\Sigma)}$

$$G = \partial A(\Sigma)\,.$$

<u>Proof</u>: We identify $\partial L^1(\Sigma)$ with G, and $\mathfrak{m}_{L^1(\Sigma)}$ with $\mathrm{Hom}\,\Sigma$: If $h \in \mathrm{Hom}\,\Sigma$, one can compute that

$$h(f) \equiv \hat{f}(h) \equiv \sum_{\lambda\in\Sigma} f(\lambda)h(\lambda), \; f \in L^1(\Sigma)$$

is multiplicative in convolution. It is a classical result that all multiplicative linear functionals on $L^1(\Sigma)$ arise in this manner. Thus $\mathfrak{m}_{L^1(\Sigma)} = \mathrm{Hom}\,\Sigma$.

Now, if $\Sigma = \hat{G}$, then (by previous remarks) $\mathfrak{m}_{L^1(\Sigma)} = \mathrm{Hom}\,\hat{G} = G$ and thus $\partial L^1(\hat{G}) \subset G$. By Corollary I.1.7, since $L^1(\Sigma) \subsetneqq L^1(\hat{G})$, we thus have $\partial L^1(\Sigma) \subset G$. Equality follows by noting that $\widehat{L_1(\Sigma)}$ is invariant with respect to translations by elements of G.

Finally, note that the uniform closure of $L^1(\Sigma)$ on its maximal ideal space, $\mathrm{Hom}\,\Sigma$, coincides with $A(\Sigma)$. Thus, by the remark in the last paragraph of I.§1.(v),

$$\mathfrak{m}_{A(\Sigma)} = \mathrm{Hom}\,\Sigma$$

$$\partial A(\Sigma) = G\,. \qquad \text{QED.}$$

Now let $\Sigma \subset \hat{G}$ be an arbitrary collection of characters.

II.2.3. <u>Lemma</u>. For $f \in C(G)$, $f \in A(\Sigma)$ if and only if $\hat{f}(\lambda) = 0$ for each $\lambda \in \hat{G} - \Sigma$.

Proof (Rudin - [4]): For $f \in A(\Sigma)$, $\lambda \notin \Sigma$, $\hat{f}(\lambda) = \int_G f(x)\lambda^{-1}(x)\,dm(x)$. Approximating f by polynomials, $\sum_{\beta\in\Sigma} a_\beta \beta$, we can approximate this integral by $\sum_{\beta\in\Sigma} a_\beta \int_G \beta(x)\lambda^{-1}(x)\,dm(x)$, which equals zero since $\beta\lambda^{-1} \neq 1$ implies $\int \beta\lambda^{-1}(x)\,dm(x) = 0$. Conversely, assume $\hat{g}(\lambda) = 0$ for each $\lambda \notin \Sigma$. To show $g \in A(\Sigma)$ we assume $\mu \in M(G)$ is such that $\mu \perp A(\Sigma)$ and show that $\int_G g\,d\mu = 0$ (from which the Hahn-Banach Theorem applies to complete the proof).

Define $\tilde{g}(y) \equiv g(y^{-1})$.

$$\int_G g\,d\mu = \int_G \tilde{g}(y^{-1})\,d\mu(y) = \tilde{g} * \mu(1).$$

But $\hat{\tilde{g}}(\lambda^{-1}) = \int_G g(x^{-1})\lambda(x)\,dm(x) = \hat{g}(\lambda) = 0$, $\lambda \notin \Sigma$, and $\hat{\mu}(\lambda^{-1}) = \int_G x(\lambda)\,d\mu(x) = 0$, $\lambda \in \Sigma$. Thus $\widehat{\tilde{g} * \mu}(\lambda^{-1}) = \hat{\tilde{g}}(\lambda^{-1})\hat{\mu}(\lambda^{-1}) = 0$ for each $\lambda \in \hat{G}$, so that $\tilde{g} * \mu = 0$ almost everywhere with respect to Haär measure. Noting continuity of $\tilde{g} * \mu$, we thus have $\int_G g\,d\mu = \tilde{g} * \mu(1) = 0$. QED.

Now assume Σ is a semigroup. Then Σ induces an ordering on G via $\lambda \leq \beta$ if, and only if, $\lambda\beta^{-1} \in \Sigma$. This ordering is <u>total</u> if $\lambda \in \Sigma$ or $\lambda^{-1} \in \Sigma$ for each $\lambda \in \hat{G}$; <u>Archimedean</u> if for each $\lambda \in \hat{G}$, there exists an n such that $\lambda^n \geq \beta$. If Σ induces an Archimedean ordering, then it also induces a total ordering such that $\lambda, \lambda^{-1} \in \Sigma$ implies $\lambda \equiv 1$; furthermore, if $\Sigma' \supset \Sigma$ is another semigroup, then $\Sigma' = \hat{G}$.

II.2.4. Lemma. Assume $1 \in \Sigma$, Σ Archimedean, $\mu \in M(G)$ is positive, and

$$\int_G \beta \, d\mu = \begin{cases} 0, & \beta \in \Sigma, \ \beta \neq 1 \\ 1, & \beta = 1. \end{cases}$$

Then $\mu \equiv m$, Haar measure on G.

Proof: Since $\Sigma \cup \Sigma^{-1} = \hat{G}$ and μ is positive, we have for $\beta \notin \Sigma$, $\beta^{-1} \in \Sigma$ and thus

$$\int_G \beta \, d\mu = \overline{\int_G \beta^{-1} d\mu} = 0.$$

Thus

$$\hat{\mu}(\beta) = \begin{cases} 0, & \beta \neq 1 \\ 1, & \beta = 1, \end{cases}$$

identifying μ as Haar measure, m. QED.

II.2.5. Theorem (Wermer's Maximality Theorem (Rudin - [5]). If Σ is Archimedean, then $A(\Sigma)$ is a maximal, proper, subalgebra of $C(G)$.

Proof: Suppose $A(\Sigma) \subsetneq B \subsetneq C(G)$, B a closed subalgebra. Fix $\lambda \in \Sigma$, $\lambda \neq 1$. If $\lambda^{-1} \in B$, then since we observed that $\lambda^{-1} \notin \Sigma$, Σ, λ^{-1} generate a semigroup $\Sigma^1 \not\supseteq \Sigma$ which we observed must therefore be equal to $\hat{G}$. Thus $\hat{G} \subseteq B$ and $B = C(G)$ by the Stone-Weierstrass Theorem.

Now suppose $\lambda^{-1} \notin B$. Then λ is not invertible in B, so that by Theorem I.1.5 (2), there is an $h \in m_B$ such that $h(\lambda) = 0$. For $\beta (\neq 1) \in \Sigma$, there exists an $n \geq 0$ such that $\beta^n \lambda^{-1} = \gamma \in \Sigma$; thus $\beta^n = \lambda\gamma$, and $(h(\beta))^n = h(\lambda)h(\gamma) = 0$; thus $h(\beta) = 0$ for each $\beta \in \Sigma$, $\beta \neq 1$. Since $h(1) = 1$, Lemma II.2.4 shows that m represents h. Thus, for each $f \in B$, $\lambda \notin \Sigma$, we have

$$\int_G f\lambda^{-1} dm = h(f)h(\lambda^{-1}) = 0.$$

Finally, by Lemma II.2.3, we conclude that $f \in A(\Sigma)$. QED.

§3. Application to the Study of Certain Subalgebras of $C(T^2)$.

(i) The Translation Invariant Ones.

Since our main interest centers around subalgebras of $C(T^n)$, we discuss the simplest of these before going on – the translation invariant ones. By Theorem II.2.1, these are precisely those generated by a <u>semigroup</u> of characters. Such semigroups identify with subsemigroups $\Sigma \subset \mathbb{Z}^n$: for $K \in \mathbb{Z}^n$ corresponds to the character $\pi_1^{K_1} \cdots \pi_n^{K_n}$ on T^n. We are interested in algebras B such that $A(D^n) \subset B \subset C(T^n)$. Thus we assume $P^+ \subset \Sigma \subset \mathbb{Z}^n$, where $P^+ = \{K \in \mathbb{Z}^n : K_j \geq 0, j=1,\cdots,n\}$.

$A(\Sigma)$ is generated by $\{\pi_1^{K_1} \cdots \pi_n^{K_n} : K \in \Sigma\}$. By Theorem II.2.2.,

$m_{A(\Sigma)} \approx \text{Hom}\ \Sigma$, so we have the map

$$\pi : \text{Hom}\ \Sigma \to D^n \approx m_{A(D^n)} \approx \text{Hom}(P^+).$$

Clearly π (Hom Σ) is the subsemigroup of D^n consisting of precisely those $\zeta \in D^n$ which extend to homomorphisms $h_\zeta : \Sigma \to D$.

We consider examples only for n = 2, since they are the simplest. Fix $\zeta \in \pi(\text{Hom } \Sigma)$. If $\zeta_1 \neq 0$, $\zeta_2 = 0$, then each $(n,m) \in \Sigma$ satisfies $m \geq 0$, so that $A(\Sigma)$ is generated by a semigroup of quotients, π_2^m/π_1^n $(m, n \geq 0)$ along with P^+. Since π_1^n does not vanish at $\zeta = (\zeta_1, 0)$, remark (1) Chapter I. §2. (v) shows that card $\pi^{-1}(\zeta_1, 0) = 1$. Similar statements hold if $\zeta = (0, \zeta_2)$, $\zeta_2 \neq 0$.

We've shown, thus, that (0,0) is the only point that may blow up in Hom $\Sigma = \mathfrak{m}_{A(\Sigma)}$ - equivalently, have <u>more than one</u> extension to a homomorphism $h_\zeta : \Sigma \to D$. By Corollary I.2.16, this occurs if, and only if, for some $(n,m) \in \Sigma$, $\pi_1^n \pi_2^m |_{\pi(\text{Hom } \Sigma) - \{(0,0)\}}$ has no continuous extension to π(Hom Σ).

The simplest concrete, yet conceptually general, example of this phenomenon occurs when $\Sigma \equiv \{(n,m): n + m \geq 0\}$. In this case, $A(\Sigma) = A(P^+)(\pi_2/\pi_1) = A(\pi_2/\pi_1)$, and $\pi(\text{Hom } \Sigma) - \{(0,0)\} = \{\zeta \in D^2: |\zeta_2|\ |\zeta_1|^{-1} \leq 1\}$. Since (0,0) is a limit point of this set, we have

$$\pi(\mathfrak{m}A(\frac{\pi_2}{\pi_1})) = \pi(\text{Hom } \Sigma) = \{\zeta \in D^2: |\zeta_2| \leq |\zeta_1|\} .$$

Now, by Theorem I.2.15, $\partial A(\pi_2/\pi_1)|_{\pi^{-1}\{(0,0)\}}$ is contained in the set of limits of ζ_2/ζ_1 as ζ approaches 0 with

$|\zeta_2| \le |\zeta_1|$, $\zeta_1 \neq 0$. Clearly such limits lie in D. Thus, if α is such a limit, then $\{\zeta \neq 0, \zeta \in D^2: \zeta_2/\zeta_1 = \alpha\} \subset \pi(\mathrm{Hom}\ \Sigma)$. Thus,

$$C_\alpha \equiv \{\zeta \in D^2: \zeta_2 - \alpha\zeta_1 = 0\} \subset \pi(\mathrm{Hom}\ \Sigma).$$

Conversely, if such a set, C_α, lies in $\pi(\mathrm{Hom}\ \Sigma)$, then as ζ approaches 0, $\zeta \in C_\alpha$, $\zeta \neq 0$, $\zeta_2/\zeta_1 \to \alpha$, trivially. Since $\pi m_A(\pi_2/\pi_1) = \bigcup_{\alpha \in D} C_\alpha$, we see that these limits exhaust all of D; since D is polynomially convex, we see, using Remark 4, Chapter I. §2. (v) that $A(\pi_2/\pi_1)\big|_{\pi^{-1}\{(0,0)\}} \approx A(D)$ and its maximal ideal space identifies with D. Thus, we see that

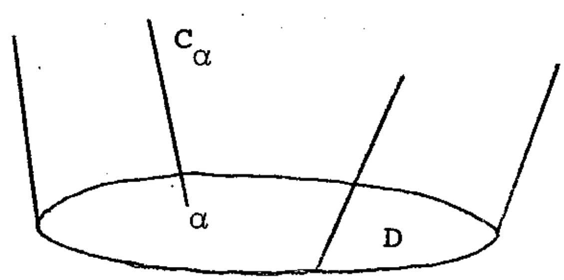
$$m_{A(\pi_2/\pi_1)} = \mathrm{Hom}\ \Sigma = \pi^{-1}\Big(\bigcup_{\alpha\in D} C_\alpha\Big) = \bigcup_{\alpha\in D}\pi^{-1}(C_\alpha)$$

$$= \pi^{-1}(0) \cup \Big(\bigcup_{\alpha\in D}\pi^{-1}(C_\alpha - \{(0,0)\})\Big)$$

$$= D \cup \bigcup_{\alpha\in D} C_\alpha$$

where C_α meets D at the point α. Note this is homeomorphic to D^2.

Diagram:



A line protruding from the point $\alpha \in D$ represents C_α.

(ii) Extensions of Half Plane Subalgebras of $C(T^2)$.

Now we classify all extensions of the "half plane" algebras

$$A(\Sigma_{a,b}) \subset C(T^2) \quad \text{where}$$

(a) $\Sigma_{a,b} \equiv \{(n,m) \in \mathbb{Z}^2 : an + bm \geq 0\}$, a,b positive real numbers. Clearly $A(D^2) \subset A(\Sigma)$. These algebras are translation invariant, since they are generated by characters. Their maximal ideal spaces were discussed in general at the beginning of II.§3.

The classification of the closed algebras B such that

(b) $$A(\Sigma_{a,b}) \subset B \subset C(T^2)$$

will be shown to follow from Theorem 1.2.14 (Theorems II.3.1 and II.3.1′).

First we consider the special case $a = 0$, $b \neq 0$. Clearly $\Sigma_{0,b} = \mathbb{Z} \times \mathbb{Z}^+$. Since

$$A(\mathbb{Z}) = C(T), \; A(\mathbb{Z}^+) = A(D)$$

we have

$$A(\Sigma_{0,b}) = C(T) \otimes A(D) \subset C(T^2).$$

Noting that $A(D)$ is maximal in $C(T)$ (by Theorem II.2.5), Theorem I.2.14 applies verbatim to characterize all extending algebras defined by (b), with $a = 0$.

<u>Theorem</u> II.3.1. If $C(T) \otimes A(D) \subset B \subset C(T^2)$ then there is a closed set $E \subset T$ such that

$$B = \{f \in C(T^2): f_\zeta(z)$$

extends to be analytic on D for each $\zeta \in E\}$, and

$$m_B \approx T^2 \cup E \times D.$$

For $\Sigma_{a,b}$ defined by (a), two cases arise:

1-(ii) a/b is rational

2-(ii) a/b is irrational.

In case a,b satisfy (1), Theorem II.3.1′ will reduce the characterization of the algebras defined by (b) to Theorem II.3.1. In case a,b satisfy (2), the line $an + bm = 0$ passes through no lattice point so that $\Sigma_{a,b}$ induces an Archimedean ordering on $\mathbb{Z}^2$. Thus, $A(\Sigma)$ is maximal in $C(T^2)$ by Theorem II.2.5 and thus the only extensions given by (b) are the trivial ones: $B = A(\Sigma_{a,b})$ or $B = C(T^2)$. Thus we consider case 1-(ii).

<u>Theorem</u> II.3.1′. Assume a and b are relatively prime integers and let B be a closed algebra satisfying (b). Choose integers α and β such that $\beta a - \alpha b = 1$ (possible because a and b are relatively prime). Then there exists a closed set $E \subset T$ such that

$$B = \{f \in C(T^2): f(\zeta^\alpha z^a, \zeta^\beta z^b)$$

extends to an analytic function, $\tilde{f}_\zeta$, on D for each $\zeta \in E\}$, and

$$m_B \approx T^2 \cup E \times D$$

where for $(\zeta,z) \in E \times D$, $h_{(\zeta,z)} \in m_B$ is defined by

$$h_{(\zeta,z)}(f) = \tilde{f}_\zeta(z).$$

<u>Proof:</u> Consider the map given by

$$T^2 \xrightarrow{\emptyset} T^2$$

$$(z,\zeta) \longrightarrow (\zeta^\alpha z^a, \zeta^\beta z^b).$$

We show that $\emptyset$ is an isomorphic homeomorphism onto T^2. Since $\emptyset$ is clearly a continuous homomorphism, and T^2 contains no proper subset homeomorphic to itself, it suffices to show that ker $\emptyset$ is trivial. Suppose $(\zeta^\alpha z^a, \zeta^\beta z^b) = (1,1)$. We must only show $z = \zeta = 1$. For each $(n,m) \in \mathbb{Z}^2$, we have

$$z^{an} = \zeta^{-\alpha n}$$

$$z^{bm} = \zeta^{-\beta m}$$

so that

$$z^{an+bm} = \zeta^{-\alpha n-\beta m}.$$

The definition of α and β simply says that

$$(*) \ \det \binom{ab}{\alpha\beta} = 1$$

so that the simultaneous equations

$$an + bm = k$$
$$\alpha n + \beta b = \ell$$

have integral solutions for each pair of integers k, ℓ. Thus $z^k = \zeta^{-\ell}$ for each integer pair k, ℓ, implying that $z = \zeta = 1$, and we are done.

Now consider the adjoint, $\emptyset^*$, of $\emptyset$ given by

$$C(T^2) \xrightarrow{\emptyset^*} C(T^2)$$
$$f \longrightarrow f \circ \emptyset .$$

Then $\emptyset^*$ is clearly an isometric isomorphism of the Banach algebra $C(T^2)$ onto itself. Let $\Sigma \equiv \Sigma_{a,b}$. We claim that

$$(**) \qquad \emptyset^*(A(\Sigma)) = A(D) \otimes C(T).$$

For $(n,m) \in \Sigma$, we have

$$(\overset{**}{*}) \qquad \emptyset^*(\pi_1^n \pi_2^m) = \pi_1^{an+bm} \pi_2^{\alpha n+\beta m} \in A(D) \otimes C(T)$$

since $an + bm \geq 0$ defines Σ. Thus $\emptyset^*(A(\Sigma)) \subset A(D) \otimes C(T)$. Since the set $\{\pi_1^k \pi_2^\ell : k \in \mathbb{Z}, \ \ell \in \mathbb{Z}\}$ spans a dense subalgebra of $A(D) \otimes C(T)$ (by the Stone-Weierstrass Theorem), to show containment in the other direction, namely $A(D) \otimes C(T) \subset \emptyset^*(B)$, we must show that $\pi_1^k \pi_2^\ell \in \emptyset^*(A(\Sigma))$ for each $k \in \mathbb{Z}^+$, $\ell \in \mathbb{Z}$. But this follows from (*) and $(\overset{**}{*})$. Thus (**) is verified.

From (**) we conclude that $A(D) \otimes C(T) \subset \phi^*(B) \subset C(T^2)$. Thus by Theorem II.3.1, there is a closed set $E \subset T$ such that $\phi^*(B) = \{f \in C(T^2): f_\zeta(z)$ extends to be analytic on D for each $\zeta \in E\}$. Thus $B = \{g \in C(T^2): \phi^*(g)_\zeta(z)$ extends to be analytic on D for all $\zeta \in E\}$. But this is just the desired characterization of B, since

$$\phi^*(g)_\zeta(z) = \phi^*(g)(\zeta, z) = g(\phi(\zeta, z)) = g(z^a\zeta^\alpha, z^b\zeta^\beta).$$

QED.

(iii) <u>The "Big Disk" Approach to Half Plane Algebras and their Extensions.</u>

Now we show that all the algebras discussed in (ii) are intersections of "Disk" algebras. In particular, we show that if $\Sigma_{a,b} \subset \mathbb{Z}^2$ induces an Archimedean ordering (i.e. $-\frac{a}{b}$ is irrational), then $A(\Sigma_{a,b})$ is the "Big Disk" Algebra. Define

$$C^+ \equiv \{z \in C: \operatorname{Im} z > 0\}$$

and

$$A(C^+) \equiv \{f \in C_B(\overline{C^+}): f \text{ is analytic on } C^+\}.$$

We first prove the following crucial fact:

<u>Lemma</u> II.3.2. $A(C^+)$ is a uniformly closed subalgebra of $C_B(\overline{C^+})$ such that $\mathbb{R}$ is dense in its Silov boundary (i.e.- each $f \in A(C^+)$ approximates its supremum on $\mathbb{R}$).

<u>Proof:</u> Since uniform limits of bounded (resp. analytic) functions are bounded (resp. analytic) we see that $A(C^+)$ is uniformly closed in $C_B(\overline{C^+})$. Clearly, $A(C^+)$ is an algebra.

To verify the second statement, we use the fact that if $Q: \overline{C^+} \longrightarrow C$ is defined by $Q(z) = \frac{1}{z+i}$, then $Q \in A(C^+)$ and vanishes at infinity.

For $f \in A(C^+)$, the maximum modulus principle implies that f approximates its supremum on $\mathbb{R} \cup \{\infty\}$. Since Q vanishes at ∞, Qf thus achieves its supremum on $\mathbb{R}$. We've shown that $\partial(QA(C^+)) \subset \mathbb{R}$. Note also that $Z(QA(C^+)) \equiv \{z \in C^+ : Q(z) = 0\} = \emptyset$.

Thus the hypothesis of Theorem I.2.3 are satisfied with $Y = \overline{C^+}$, $X = \mathbb{R}$, $B = A(C^+)$, $A = I = QB$. By the conclusion of that theorem, $X = \mathbb{R}$ is dense in $\partial B = \partial A(C^+)$. But this is just the desired result. QED.

Now we define, for $\tau \in T^2$, a,b positive real numbers, the function

$$\overline{C^+} \xrightarrow{\emptyset_\tau^{a,b}} D^2$$
$$z \longrightarrow (\tau_1 e^{iaz}, \tau_2 e^{ibz}).$$

For fixed a,b we will simply write $\emptyset_\tau$ for $\emptyset_\tau^{a,b}$.

A simple computation shows that $\emptyset_\tau(z) \in D^2$ for $z \in C^+$ (so that the definition makes sense) and that $\emptyset_\tau(\mathbb{R}) \subset T^2$. Note also that $\emptyset_1(\mathbb{R})$ is a subgroup of T^2 so that T^2 is the disjoint union of the cosets $\{\emptyset_\tau(\mathbb{R}) : \tau \in T^2\}$.

Now, for $\tau \in T^2$, we define the subalgebra $A_\tau^{a,b}$, of $C(T^2)$, by $A_\tau^{a,b} = \{f \in C(T^2) : f(\emptyset_\tau(t))$ (t real) extends to a bounded analytic function on $C^+\}$.

By Lemma II.3.2., A_τ is a uniformly closed subalgebra of $C(T^2)$.

Now Theorem II.3.1 is contained in the following:

<u>Theorem</u> II.3.3. Fix $a,b > 0$. Then the closed subalgebras of $C(T^2)$ extending $A(\Sigma_{a,b})$ are precisely the intersections of maximal algebras of the form $A_\tau^{a,b}$, $\tau \in T^2$. If $\Sigma_{a,b}$ induces an Archimedean ordering on $\mathbb{Z}^2$, then $A(\Sigma_{a,b}) = A_\tau^{a,b}$ for each $\tau \in T^2$.

<u>Proof</u>: If a,b satisfy case 1-(ii) and the hypotheses of Theorem II.3.1$'$ hold, then

$B = \bigcap_{\tau \in E_1} A_\tau^{a,b}$ where $E_1 = \{(\zeta^\alpha, \zeta^\beta):\ \zeta \in T\}$. To see this, simply let $z = e^{it}$ (t real) in the conclusion of Theorem II.3.1$'$.

Assume now that a,b satisfy case 2-(ii) and let $\Sigma \equiv \Sigma_{a,b}$. Then Σ induces an Archimedean ordering on $\mathbb{Z}$, and we proceed to show that $A(\Sigma) = A_\tau^{a,b}$ for each $\tau \in T^2$. Again, we may assume $a = 1$, b is irrational. Fix $\tau \in T^2$, and define $f_{n,m} \equiv \pi_1^n \pi_2^m$ for $(n,m) \in \Sigma$. Then for real t, we have $f_{n,m} \circ \emptyset_\tau(t) = (\tau_1 e^{it})^n (\tau_2 e^{ibt})^m = \tau_1^n \tau_2^m e^{i(n+bm)t}$. Since b is irrational, $n + bm > 0$ and thus $e^{i(n+bm)z}$ is bounded for $\operatorname{Im} z > 0$. Thus $f_{n,m} \circ \emptyset_\tau$ extends to be bounded and analytic in C^+, i.e. $f_{n,m} \in A_\tau^{1,b}$. Since $A(\Sigma)$ is generated by polynomials in the functions $f_{n,m}$, we have $A(\Sigma) \subset A_\tau^{1,b}$. Equality follows from the maximality of $A(\Sigma)$ (Theorem II.2.5).

It remains to show that $A_\tau^{a,b}$ is maximal in $C(T^2)$ for each $\tau \in T^2$, a,b positive real numbers. But $A_\tau^{a,b}$ is essentially the disk algebra in case a,b satisfies 1-(ii) (II.3.1′) and the "Big Disk" algebra in case a,b satisfies 2-(ii) (as shown above). In either case, Theorem II.2.5 applies to conclude that $A_\tau^{a,b}$ is maximal. QED.

It is interesting to note that in the Archimedean case ($a = 1$, b irrational) $\phi_1^{1,b}(\mathbb{R})$ is a <u>dense</u> subgroup of T^2 (Arens and Singer [2]) so that each $\phi_\tau^{1,b}(\mathbb{R})$ is a dense coset of the subgroup $\phi_1^{1,b}(\mathbb{R})$. We will proceed to show that the maximal ideal space of $A(\Sigma_{1,b})$ is the disjoint union of the "Big Disks" $\phi_\tau^{1,b}(\overline{C^+})$ $(\tau \in T^2)$ and the "point at infinity."

In fact, we have a characterization of $m_A(\Sigma_{a,b})$ for $a,b > 0$ independent of Theorem II.2.2.

<u>Theorem</u> II.3.4. For $a,b > 0$ and B a closed algebra extending $A(\Sigma_{a,b})$, let $E_1 \equiv \{\tau \in T^2 \colon A_\tau^{a,b} \supset B\}$. Then

(1) $$m_B \approx \bigcup_{\tau\in E_1} \phi_\tau^{a,b}(\overline{C^+}) \cup \begin{cases} E, & a/b \text{ rational } (1\text{-(ii)}) \\ \{\infty\}, & a/b \text{ irrational } (2\text{-(ii)}) \end{cases}$$

where $E = \{\zeta \in T \colon (\zeta^\alpha, \zeta^\beta) \in E_1\}$, α,β as in Theorem II.3.1′.

(2) $$\pi m_B = \overline{\bigcup_{\tau\in E_1} \phi_\tau^{a,b}(\overline{C^+})}\,.$$

Note that $E_1 = T^2$ in case 2-(ii). We first prove the following lemma.

Lemma II.3.5. Fix $(z_1, z_2) \in D^2$, $|z_1| \neq 0$, $|z_2| \neq 0$. Then the following two statements are equivalent.

(i) $|z_1|^n |z_2|^m \leq 1$ for each $(n,m) \in \Sigma_{a,b}$

(ii) $\dfrac{\ln|z_1|}{\ln|z_2|} = \dfrac{a}{b}$.

To see this, let $c = -\ln|z_1|$, $d = -\ln|z_2|$. Then $|z_1|^n |z_2|^m \leq 1$ is equivalent to $n \ln|z_1| + m \ln|z_2| \leq 0$, that is $cn + dm \geq 0$, or $(n,m) \in \Sigma_{c,d}$.

Now (i) can be rephrased as $\Sigma_{a,b} \subset \Sigma_{c,d}$. Since $\Sigma_{a,b}$ is maximal proper subsemigroup of $\mathbb{Z}^2$, this is equivalent to $\Sigma_{a,b} = \Sigma_{c,d}$, which clearly occurs if and only if $\frac{c}{d} = \frac{a}{b}$, i.e. (ii) holds. QED.

Now to the proof of the theorem. Let $\Sigma \equiv \Sigma_{a,b}$. Define $S \equiv \{(z_1, z_2) \in D^2 : z_1 \neq 0,\ z_2 \neq 0,\ \frac{\ln|z_1|}{\ln|z_2|} = \frac{a}{b}\}$, and let $\pi : m_{A(\Sigma)} \longrightarrow D^2$ be projection. First we consider the case $B = A(\Sigma)$.

We proceed to show

(a) $\bigcup_{\tau \in T^2} \phi_\tau^{a,b}(\overline{C^+}) \subset \pi(m_{A(\Sigma)}) \subset \bar{S}$. The first set inclusion follows from Theorem II.3.3 noting that $\phi_\tau^{a,b}(\overline{C^+}) \subset \pi m_{A_\tau^{a,b}}$ for each $\tau \in T^2$. The last set inclusion follows from the lemma and two facts: for each $h \in m_{A(\Sigma)}$ we have

(1′) $|h(\pi_1^n \pi_2^m)| \leq 1$ for $(n,m) \in \Sigma$ and

(2′) $h(\pi_1) = 0$ if and only if $h(\pi_2) = 0$.

Statement (2′) follows from the observation that there are integers $n,m,k,\ell > 0$ such that $\frac{\pi_2^n}{\pi_1^m} \in A(\Sigma)$ (i.e. $(-m,n) \in \Sigma$) and $\frac{\pi_1^k}{\pi_2^\ell} \in A(\Sigma)$ (i.e. $(k,-\ell) \in \Sigma$). By (2′), each $z \in \pi\, m_{A(\Sigma)} - \{(0,0)\}$ satisfies $z_1 \neq 0$, $z_2 \neq 0$ (i.e. the hypothesis of the lemma). Thus by (1′) and the lemma, $\pi m_{A(\Sigma)} - \{(0,0)\} \subset S$. Since $(0,0) \in \bar{S}$, we have $\pi m_{A(\Sigma)} \subset \bar{S}$, which is the last set inclusion of (a).

Now one can compute the following:

(b) $S \subset \bigcup_{\tau \in T^2} \phi_\tau^{a,b}(\overline{C^+})$.

Putting (a) and (b) together, we get

$$\bigcup_{\tau \in T^2} \phi_\tau^{a,b}(\overline{C^+}) \subset \pi(m_{A(\Sigma)}) \subset \overline{\bigcup_{\tau \in T^2} \phi_\tau^{2,b}(\overline{C^+})}.$$

Since the middle term is itself closed, (2) is verified with $B = A(\Sigma)$.

To verify (1), first we note that

(c) $\pi^{-1}(\zeta)$ is a singleton for each $\zeta \in \pi(m_{A(\Sigma)})$ such that $\zeta \neq (0,0)$ (this follows from (2′) and remark (1)-(v) of I-2-(v)) and

(d) $\overline{\bigcup_{\tau \in T^2} \phi_\tau^{a,b}(C^+)} = \bigcup_{\tau \in T^2} \phi_\tau^{a,b}(\overline{C^+}) \cup \{(0,0)\}$, the union being disjoint (this follows from simple, yet tedious computations). By (c), (d) and (2) we have

$$m_{A(\Sigma)} \approx \bigcup_{\tau \in T^2} \phi_\tau^{a,b}(\overline{C^+}) \cup \pi^{-1}\{(0,0)\}.$$

Now it remains to examine $\pi^{-1}\{(0,0)\}$.

First we consider case 1-(ii). $A(\Sigma)$ is generated by $\pi_1, \pi_2, \pi_2^m\pi_1^{-n}, \pi_1^n\pi_2^{-m}$ for two positive integers n,m such that $an - bm = 0$. It thus suffices to check the spectrum of $\pi_2^n\pi_1^{-n}$ in $\widehat{A(\Sigma)}_{|\pi^{-1}\{(0,0)\}}$. But clearly if we choose, for each $\zeta \in T$, $\tau = \tau(\zeta) \in T^2$ such that $\tau_2^m\tau_1^{-n} = \zeta$, then

$$\pi_2^m\pi_1^{-n} \circ \phi_\tau^{a,b}(z) = (\tau_1 e^{iaz})^{-n}(\tau_2 e^{ibz})^m = \zeta e^{i(-an+bm)z} = \zeta$$

since $an - bm = 0$. Thus $\pi_2^m\pi_1^{-n}$ takes the constant value ζ on $\phi_{\tau(\zeta)}^{a,b}(\overline{C^+})$, for each $\zeta \in T$. But

$$(*) \qquad (0,0) \in \bigcap_{\zeta \in T} \phi_{\tau(\zeta)}^{a,b}(\overline{C^+}).$$

Thus T lies in the desired spectrum (see Theorem I.2.15 with $A = A(D^2)$, $B = A(\Sigma)$, $E = \{(0,0)\}$, second inclusion: we just showed that $T \subsetneq Y^B - Y$ in the notation of that theorem). Since $(\pi_2^m\pi_1^{-n})^{-1} = \pi_1^n\pi_2^{-m} \in A(\Sigma)$, clearly T equals the spectrum of $\pi_2^m\pi_1^{-n}$ in $\widehat{A(\Sigma)}_{|\pi^{-1}\{(0,0)\}}$ and case 1-(ii) is taken care of.

Next we consider 2-(ii). Noting that (*) still holds and using Theorem I.2.15, first inclusion, we must only check that for each $(n,m) \in \Sigma$,

(e) $\lim_{\text{Im} z \to \infty} \pi_1^n \pi_2^m \circ \phi_\tau^{1,b}(z) = 0.$

Since b is irrational, $(n,m) \in \Sigma$ means $n + bm > 0$. Thus for $(n,m) \in \Sigma$, the left side of (e) equals

$$\lim_{\text{Im} z \to \infty} \tau_1^n \tau_2^m e^{i(n+bm)(\text{Re} z + i \text{Im} z)} = \tau_1^n \tau_2^m \lim_{\text{Im} z \to \infty} e^{i(n+bm)\text{Re} z} e^{-(n+bm)\text{Im} z} = 0$$

since $|e^{i(n+bm)\text{Re}\, z)}| = 1$ and $e^{-(n+bm)\text{Im} z}$ approaches 0 as $\text{Im}\, z \to \infty$. Thus (e) is verified and case 2-(ii) is taken care of.

Now for an arbitrary algebra B extending $A(\Sigma_{a,b})$ case 1-(ii) only remains (since only the trivial extensions exist for case 2). But (1) and (2) follow readily from Theorem II.3.1$'$, and a discussion analogous to the one above shows that $\pi^{-1}\{(0,0)\}$ identifies with the subset E of T.

QED.

We note that although the map π: $m_B \to D^2$ was shown to be non-injective for extensions B of $A(\Sigma_{a,b})$ with $a \neq 0$, $b \neq 0$, a/b rational, π is injective for the algebras $\phi^*(B)$, ϕ^* defined in Theorem II.3.1$'$. Since ϕ^* is an isometric isometry, we see that the nature of the map π is not an algebraic property of B.

(iv) <u>A comment on a More General Class of Subalgebra of</u> $C(T^2)$.

Finally, we must mention that all translation invariant algebras extending $A(D^n)$ including the half plane algebras considered in (ii) and (iii) are special cases of algebras generated by semigroups of "generalized characters", $(z_1-\zeta_1)^n(z_2-\zeta_2)^m$, where $\zeta \equiv (\zeta_1,\zeta_2) \in \text{Int } D^n$. These are just those algebras invariant with respect to all translations about ζ (these automorphisms of D^n are discussed in Chapter III. §1. (ii)). Thus all observations of this section apply to this more general class of algebra.

CHAPTER III

Rational Subalgebras of $C(T^n)$ Extending $A(D^n)$

§0. Introduction.

A rational algebra containing A, or a rational extension of A, is one generated by A and a set of inverses of elements of A.

For $E \subset C^n$, compact, $P(E) \subset B \subset C(E)$, B a rational extension of P(E), then m_B identifies with a rationally convex set, $K \subset C^n$, such that $E \subset K \subset \hat{E} \approx m_{P(E)}$ and we have $B \approx R(K)$: for let π: $m_B \to \hat{E}$ be restriction; since (by Theorem I.1.5(3)) the generating functions of B do not vanish on m_B, π is one to one (using Remark (1), I.§2.(v)) and $B \subseteq R(K)$ where $K = \pi(m_B)$. But, by Theorem I.2.6, $R(K) \subset B$. Thus $B \approx R(K)$ and $K = m_{R(K)}$ is therefore rationally convex.

We characterize all rational extensions, B, of $A(D^n)$: $A(D^n) \subsetneq B \subsetneq C(T^n)$, along with their maximal ideal spaces, for $n \geq 2$ (case $n = 1$ is trivial by Wermer's Maximality Theorem) are characterized. Thus we deduce certain restrictions on the shape of πm_B if B is not a rational extension - improved upon in Chapter IV.

§1. Invariance of $A(\frac{1}{f})$ Under Continuous Deformation of f, where $f \in A$, A a Closed Subalgebra of a Banach Algebra B.

Let $A \subset B$ be Banach algebras, $1 \in A$ the unit. Let Σ_A

denote the semigroup of elements of A invertible in B. For $f, g \in \Sigma_A$, define $f \sim g$ if, and only if, f and g lie in the same connected component of Σ_A (here, path connected and connected are synonomous). If $\Sigma_A/\sim$ denotes the set of $\sim$ equivalence classes and $\tilde{f}$ denotes the equivalence class of f, then $\tilde{f}\,\tilde{g} \equiv \widetilde{fg}$ is a well defined operation making $\Sigma_A/\sim$ into a semigroup with unit $\tilde{1}$.

III.1.1. <u>Theorem</u>. For $f, g \in \Sigma_A$, $f \sim g$ implies $A(\frac{1}{f}) = A(\frac{1}{g})$.

<u>Proof</u>: Let $f \sim g$. Then there exists a continuous $\phi:[0,1] \to \Sigma_A$ such that $f = \phi(0)$, $g = \phi(1)$. Let $E \equiv \{t \in [0,1]: \frac{1}{g(t)} \in A\,\frac{1}{f})\}$. Clearly E is closed. If $t_0 \in E$, then, a priori — and this is the crux of the proof — $\phi(t_0)$ is invertible in $A(\frac{1}{f})$ and therefore so is $\phi(t)$ for t near t_0. Thus E is open. We conclude that $E = [0,1]$. Thus, $\frac{1}{g} = \frac{1}{\phi(1)} \in A(\frac{1}{f})$. Similarly, $\frac{1}{f} \in A(\frac{1}{g})$. QED.

III.1.2. <u>Corollary</u>. Suppose A is commutative. Then, for f, $g \in \Sigma_A$, $\frac{1}{f} \in A(\frac{1}{g})$ if, and only if, $\tilde{f}$ divides $\tilde{g}^N$ in Σ_A for some $N \geq 0$ (this just means that $f\ell \sim g^N$ for some $\ell \in \Sigma_A$).

<u>Proof</u>: First suppose $\frac{1}{f} \in A(\frac{1}{g})$. Choose $\varepsilon > 0$ such that

(a) $\|\frac{1}{f} - v\| < \varepsilon$ implies v is invertible for all $v \in B$.

Since A is commutative, so is $A(\frac{1}{g})$, which is thus the closure of $\{g^{-N}\ell : \ell \in A, N \geq 0\}$. Thus, there exists $\ell \in A$, $N \geq 0$

such that

(b) $\|\frac{1}{f} - g^{-N}\ell\| < \varepsilon$. Then $\phi(t) \equiv \frac{t}{f} + \frac{(1-t)\ell}{g^N}$ is invertible for $0 \le t \le 1$, by (a). Thus

$$\psi(t) \equiv \phi(t)fg^N \in \Sigma_A \text{ for } 0 \le t \le 1.$$

Since $\psi(0) = \ell f$ $\psi(1) = g^N$ we have $f\ell \sim g^N$. Conversely, if $f\ell \sim g^N$, $f, g, \ell \in \Sigma_A$ then by Theorem III.1.1, $A(\frac{1}{f\ell}) = A(\frac{1}{g^N})$. But therefore $\frac{1}{f} = \ell(f\ell)^{-1} \in A(\frac{1}{f\ell}) = A(\frac{1}{g^N})$ $= A(\frac{1}{g})$. QED.

Note that the converse did not depend on A being commutative.

Let G be a connected semigroup with unit, e, and consider a map

$$B \times G \to B$$

$$(f,x) \to f_x \text{ such that}$$

(i) $(fg)_x = f_x g_x$

(ii) $1_x = 1$

(iii) $(f,x) \to f_x$ is continuous in each variable separately.

III.1.3. <u>Corollary</u>. Let $A \subset B$ be a commutative subalgebra containing 1. Then A translation invariant implies $A(\frac{1}{f})$ is translation invariant.

Proof: Assume A is so invariant. By (i) and (ii), therefore Σ_A is invariant. Thus, by the connectedness of G, and (iii) (with f held fixed), $\Lambda f \equiv \{f_x : x \in G\}$ is connected in Σ_A; since it contains $f = f_e$, clearly $f_x \sim f$ for all $x \in G$. Thus, by Corollary III.1.2 or Theorem III.1.1,

(a) $\frac{1}{f_x} \in A(\frac{1}{f})$, $x \in G$.

Fix $x \in G$, $k \in A(\frac{1}{f})$. Then $(f^{-Nj})(\ell j) \to k \in A(\frac{1}{f})$, $\ell_j \in A$, and (iii) (with x held fixed) implies that

(b) $(f_x)^{-Nj}(\ell j)_x \to k_x$.

But $(\ell j)_x \in A$ by assumption, and thus (a) implies that $(f_x)^{-Nj}(\ell j)_x \in A(\frac{1}{f})$ for all j, and finally (b) shows that $k_x \in A(\frac{1}{f})$, since $A(\frac{1}{f})$ is closed. Since $k \in A(\frac{1}{f})$ and $x \in G$ were chosen arbitrarily, we are done. QED.

§2. The Role of Aut P(E) in Studying Extending Algebras for P(E).

(i) The General Case.

Definition. Let G_e denote the connected component of e in the topological semigroup, G.

For B a commutative Banach algebra, topologize Hom B (defined-I. §1. (vii)) weakly with respect to B: $\varphi_\alpha \to \varphi$ if, and only if, $f \circ \varphi_\alpha \to f \circ \varphi$ for all $f \in B$.

Definition. For $A \subset B$, let G_A^B denote the subsemigroup of

Hom B consisting of the <u>continuous</u> elements of Hom B, leaving A invariant.

Then $(G_A^B)_e$ satisfies (i), (ii), and (iii) preceding Corollary III.1.3. If B is a function algebra, then we let G^B denote $G^B_{\{0\}}$.

III.2.1. <u>Corollary</u>. If $A \subset B$, B a commutative Banach algebra, then $(G_A^B)_e$ leaves $A(\frac{1}{f})$ invariant for each $f \in \Sigma_A$.

<u>Proof</u>: Direct application of Corollary III.1.3. QED.

Now we fix $E \subset \mathbf{C}^n$, E compact.

<u>Definition</u>. Hom E denotes the set of maps $\varphi : E \to E$ for which $f \circ \varphi \in A(E)$ for each $f \in A(E)$. Aut E denotes the group of invertible elements of Hom E. Clearly Hom E is a semigroup of continuous functions from E into E (of which Aut E is a subgroup) defining a semigroup action on A(E) via the map

$$A(E) \times \text{Hom } E \to A(E)$$

$$(f,\varphi) \to f \circ \varphi \approx f_\varphi .$$

We topologize Hom E weakly with respect to A(E). Now we have the topological inclusions

$$\text{Hom } E \subsetneqq \text{Hom } A(E) \subsetneqq \text{Hom } C(E)$$

$$\text{Aut } E \subsetneqq \text{Aut } A(E) \subsetneqq \text{Aut } C(E).$$

Remarks. (1) For $E = T^n$ ($n \geq 1$), $A(E) = C(E)$ and therefore Hom $E = C(E,E) =$ Hom $C(E) =$ Hom $A(E)$. Similarly for Aut(E) with $C(E,E)$ replaced by Homeo (E,E).

(2) For E polynomially convex, $A(E) = P(E)$ and since $E \approx m_{P(E)}$ we have Hom $E \approx$ Hom $A(E)$, Aut $E \approx$ Aut $A(E)$.

Definition. $\mathrm{Hom}_p(E)$ denotes the set $\varphi \in$ Hom E such that $\varphi \circ P(E) \subset P(E)$.

III.2.2. Lemma. Let $A \equiv P(E)$ and fix $f \in A$, $f(x) \neq 0$ for all $x \in E$. Then the connected component of e in $\mathrm{Hom}_p(E)$ leaves $A(\frac{1}{f})$ invariant. Thus $\mathrm{Hom}_p(E)$ induces an action on $\hat{E}$ leaving $m_{A(\frac{1}{f})} \subseteq \hat{E}$ invariant.

Proof: Direct application of Corollary III.2.1 with $B = A(E)$. QED.

Now assume $E = \partial P(E)$. Then Aut $\hat{E} \approx$ Aut $A(\hat{E})$ by Remark (2)-III.§2.(i) which, by Theorem I.1.8, leaves E invariant. It follows that

$$\mathrm{Aut}\ \hat{E} \subset \mathrm{Aut}\ E, \text{ via } \varphi \to \varphi_{|E}.$$

For $K \subset \hat{E}$, let $\tilde{K}$ denote the least compact set, containing K, fixed by $\mathrm{Aut}_e(\hat{E})$(the connected component of e in Aut $(\hat{E})$).

For $P(E) \subset B \subset C(E)$, let $\tilde{B}$ denote the largest rational extension of P(E) contained in B. By Theorem I.1.5(3), each $f \in P(E)$ is invertible in B if and only if $f(z) \neq 0$ for each $z \in \pi m_B$. Thus, by definition, $\tilde{B} = R(\pi m_B)$.

III.2.3. Theorem. Let $P(E) \subset B \subset C(E)$, B any closed subalgebra. Then if $\pi : \mathfrak{m}_B \to \hat{E}$ is restriction, we have

(1) $\tilde{B}$ is fixed by $\mathrm{Aut}_e(\hat{E})$

and

(2) $\widetilde{\pi\mathfrak{m}_B} \subset \widecheck{\pi(\mathfrak{m}_B)}$.

Proof: Clearly $\mathfrak{m}_{\tilde{B}} = \widecheck{\pi(\mathfrak{m}_B)}$, since $\tilde{B} = R(\pi\mathfrak{m}_B)$. By Lemma III.2.2, therefore, $\tilde{B}$ and $\widecheck{\pi(\mathfrak{m}_B)}$ are fixed by $\mathrm{Aute}(\hat{E})$. This shows (1) and (2). QED.

Before we apply all this to characterize all rational extensions of $A(D^n)$ $(n \geq 2)$, we describe the one parameter subgroups of Aut D^n, and prove the relative maximality of $A(D^n)$.

(ii) The One Parameter Subgroups of Aut D^n.

For $E \subset C^n$, compact, if $\phi : R \to \mathrm{Aut}\, E$ is a continuous homomorphism, we obtain the group action

$$A(E) \times R \to A(E)$$

$$(f,x) \to (f,\phi(x)) \equiv f \circ \phi(x).$$

Since, as observed in III.§2.(i), $\mathrm{Aut}\, E \subsetneq \mathrm{Aut}\, \partial P(E)$ when E is polynomially convex, we obtain the continuous homomorphism

$$\phi_{|\partial P(E)} : R \to \mathrm{Aut}\, \partial P(E),$$

inducing the one parameter action

$$A(\partial P(E)) \times R \to A(\partial P(E)).$$

Note that a one parameter subgroup of Aut(E) has connected image, and thus lies in $\mathrm{Aut}_e(E)$. This fact enables the proceeding results to apply.

The automorphisms of D are precisely those functions $\varphi : D \to \mathbb{C}$ of the form $\varphi(z) \equiv \alpha \dfrac{z - \zeta}{z\bar{\zeta} - 1}$ for $|\alpha| = 1$, $|\zeta| < 1$ (Cartan - [1]). These act homogeneously on D, and therefore their tensor products,

$$(*) \qquad \varphi(z_1, \cdots, z_n) \equiv \varphi_1(z_1)\cdots\varphi_n(z_n), \varphi_j(z_j) \equiv \alpha_j \frac{z_j - \zeta_j}{z_j\bar{\zeta}_j - 1}$$

act homogeneously on D^n, $n \geq 1$. However, we must do better.

The simplest examples of (*) are the translations about 0:

$$T_\alpha(z_1, \cdots, z_n) \equiv (\alpha_1 z_1, \cdots, \alpha_n z_n), \quad |\alpha_j| = 1 \quad (j = 1, \cdots, n).$$

Now we show that for every u, v $\in$ Int D^n, there is a continuous one parameter subgroup of Aut D^n one of whose elements takes u onto v.

III.2.4. Theorem. Fix v,u $\in$ Int D^n. Then there is a $\varphi \in$ Aut D^n of the form (*) such that $\varphi^{-1} \circ T_{-1} \circ \varphi(v) = u$. Thus $x \to \varphi^{-1} \circ T_x \circ \varphi$ is the desired one parameter subgroup of $\mathrm{Aut}(D^n)$.

Proof: By taking tensor products, the theorem reduces to the case $n = 1$. Thus, $u, v \in D$. Define $\varphi_1(z) \equiv \frac{z-u}{z\bar{u}-1}$, $\zeta \equiv \varphi_1(v)$, so that

$$\varphi_1(v) = \zeta,\ \varphi_1^{-1}(0) = u.$$

It suffices to find a φ_2 of form (*) such that $\varphi_2^{-1} \circ T_{-1} \circ \varphi_2(\zeta) = 0$, for then $\varphi \equiv \varphi_2 \circ \varphi_1$ does the job. If

$$\psi(z) \equiv \frac{z-\zeta}{z\bar{\zeta}-1},$$

then $\psi(\zeta) = 0$ and we must only find a φ such that

$$\psi = \varphi^{-1} \circ T_{-1} \circ \varphi .$$

First we obtain a particular fixed point for ψ. $t\zeta = \psi(t\zeta)$, or $t\zeta = \frac{(t\zeta)-\zeta}{(t\zeta)\bar{\zeta}-1}$ is equivalent to $t^2|\zeta|^2 - 2t + 1 = 0$, which has a solution in $(0,1)$. We conclude that ψ has a fixed point, $\tau = t\zeta$, $0 < t < 1$, so that $\frac{\tau}{|\tau|} = \frac{t\zeta}{|t\zeta|} = \frac{\zeta}{|\zeta|}$ or

$$\text{(a)} \quad \tau|\zeta| = |\tau|\zeta,\ \psi(\tau) = \tau.$$

Now we define

$$\varphi(z) \equiv \frac{z-\tau}{z\bar{\tau}-1}$$

and claim that this φ will do the job. We must only show that ψ and $\varphi^{-1}\circ T_{-1}\circ\varphi$ send 0 to the same point (for one easily verifies that the later is also of the form $\varphi(z) \equiv \frac{z-w}{z\bar{w}-1}$, and clearly such a gadget is determined by $w = \varphi(0)$), i.e.-that (noting $\varphi = \varphi^{-1}$) $\varphi(-\varphi(0)) = \zeta$. But $\varphi(0) = \tau$, so that this says $\varphi(-\tau) = \zeta$ or

$$\text{(b)} \quad \frac{2\tau}{|\tau|^2+1} = \zeta .$$

But $\frac{\tau-\zeta}{\tau\bar{\zeta}-1} = \psi(\tau) = \tau$: equivalently $2\tau = \frac{\tau^2|\zeta|^2+\zeta^2}{\zeta}$. But by (a) this is $2\tau = \frac{|\tau|^2\zeta^2+\zeta^2}{\zeta} = (|\tau|^2+1)\zeta$, ie $\zeta = \frac{2\tau_1}{|\tau|^2+1}$, which is (b), and we are done. QED.

We have shown that, for each $v, u \in D^n$, there exists a $\varphi \in \text{Aut } D^n$ of form (*) and a $\zeta_o \in T^n$, namely

$$\zeta_o = (-1,\cdots,-1)$$

such that

$\varphi^{-1}\circ T_{\zeta_o}\circ\varphi(v) = u$. One easily verifies that

$$t \to \varphi^{-1}\circ T_{(e^{\pi it},\cdots,e^{\pi it})}\circ\varphi$$

is a continuous homomorphism into Aut D^n, which takes v onto u for t = 1.

(iii) Relative Maximality of a Certain Class of Subalgebra of $C(T^n)$, $n \geq 1$.

We define, now, for each integer n, 2^n subalgebras of

$C(T^n)$ which will be shown to be relativly maximal in $C(T^n)$.
We need some notation: $K_0 \equiv D$, $K_1 \equiv T$. Define, for $j = 1,2$, $A_j \subset C(T)$ by $A_j \equiv A(K_j)$. If $\delta = (\delta_1,\cdots,\delta_n)$, $\delta_i = 0$ or 1 $(i=1,\cdots,n)$, then define $A_\delta^{(n)} \subset C(T^n)$ by $A_\delta^{(n)} = \bigotimes_{j=1}^{n} A_{\delta_j}$.
Then

$$A(D^n) = A^{(n)}_{(0,\cdots,0)} \subset A_\delta^{(n)} \subset C(T^n) = A(T^n) = A^{(n)}_{(1,\cdots,1)}.$$

If $\Sigma_0 \equiv \mathbb{Z}^+$, $\Sigma_1 \equiv \mathbb{Z}$, then $K_j = \text{Hom}\,\Sigma_j$, $A(K_j) = A(\Sigma_j)$ $(j = 1,2)$.

Thus, for $\delta = (\delta_1,\cdots,\delta_n)$,

$$A_\delta^{(n)} = \bigotimes_{i=1}^{n} A(K_{\delta_i}) = \bigotimes_{i=1}^{n} A(\Sigma_{\delta_i}) = A(\bigtimes_{i=1}^{n} \Sigma_{\delta_i}).$$

Thus, with $K_\delta^{(n)} \equiv \bigtimes_{j=1}^{n} K_{\delta_j}$, we have

$$\mathfrak{m}_{A_\delta^{(n)}} = \text{Hom}\,(\bigtimes_{j=1}^{n} \Sigma_{\delta_j}) = \bigtimes_{j=1}^{n} \text{Hom}\,\Sigma_{\delta_j} = \bigtimes_{j=1}^{n} K_{\delta_j} = K_\delta^{(n)}.$$

We record these and other pertinent facts below.

III.2.5. Lemma. Fix $\delta = (\delta_1,\cdots,\delta_n)$.

(1) $\mathfrak{m}_{A_\delta^{(n)}} = K_\delta^{(n)}$

(2) $A(K_\delta^{(n)}) = R(K_\delta^{(n)}) = A_\delta^{(n)} = A(\bigtimes_{i=1}^{n} \Sigma_{\delta_i})$.

Proof: By Lemma I.2.6, since (1) is already verified, we have $R(K_\delta^{(n)}) \subset A(K_\delta^{(n)}) \subset A_\delta^{(n)}$. But clearly, $A_\delta^{(n)} \subset R(K_\delta^{(n)})$. Thus (2) follows. QED.

We now attack the more function theoretic question of relative maximality of $A_\delta^{(n)}$ and related ideas. Clearly this will generalize Wermer's maximality theorem which shows $A_{(0)} \approx A(D)$ is, in fact, maximal. Although $A(D^n)$ for $n > 1$ is far from being maximal, enough properties are retained to assure the weaker property of relative maximality.

Note that each $\zeta \in T$ is a peak point for $A(D)$: $\frac{z\bar{\zeta}+1}{2}$ peaks at ζ. Thus, for each $\zeta \equiv (\zeta_1, \cdots, \zeta_n) \in T^n$, $\prod_{i=1}^{n} (\frac{z_i\bar{\zeta}_i+1}{2})$ peaks at ζ.

By Lemma II.2.4, $0 \in D$ has the unique representing measure, $m \in M(T)$ (Haär measure), on $A(D)$. Since Aut D acts homogeneously on Int D, each $\zeta \in D$ has a unique representing measure on $A(D)$, denoted M_ζ.

III.2.6. Lemma. Let $\zeta^1 \equiv (\zeta_1, \cdots, \zeta_{m-1})$, $\zeta^2 \equiv (\zeta_{m+1}, \cdots, \zeta_n)$ and $\zeta \equiv (\zeta^1, \tau, \zeta^2) \in D^n$. Assume $|\zeta_i| = 1 (i=1, \cdots, m-1, m+1, \cdots, n)$, $|\tau| < 1$, and let μ be a representing measure for ζ on $A(D^n)$. Then $\mu = \mu_{\zeta^1} \otimes m_\tau \otimes \mu_{\zeta^2}$.

Proof: Apply Lemma I.2.9 with $X_1 = T^{n-1}, X_2 = T, A_1 = A(D^{n-1})$, $A_2 = A(D), x_1 = (\zeta^1, \zeta^2) \in X_1$, and apply above remarks. QED.

III.2.7. Lemma. If $\delta = (\delta_1, \cdots, \delta_n)$ then

$$A_\delta^{(n)} = A_{(\delta_1, \cdots, \delta_{n-1})}^{(n-1)} \otimes A_{\delta_n}.$$

Proof: By Lemma III.2.5(2) $A_\delta^{(n)} = A(\Sigma)$ where $\Sigma \equiv \mathop{X}_{i=1}^{n} \Sigma_{\delta_i}$. By Lemma II.2.3., $A(\Sigma) = \{f \in C(T^n) : \hat{f}(k_1, \cdots, k_n) = 0$ whenever $k_i \notin \Sigma_{\delta_i}$ for some $i\}$. Now fix $f \in A_{(\delta_1, \cdots, \delta_{n-1})}^{(n-1)} \otimes A_{\delta_n}$ and $k_1, \cdots, k_n \in \mathbb{Z}$. If $k_i \notin \Sigma_{\delta_i}$ for $i \leq n-1$, then for each

$z_n \in T$, $f(\cdot, z_n) \in A^{(n-1)}_{(\delta_1, \cdots, \delta_{n-1})}$ so that $\int_{T^{n-1}} f(z, z_n)$ $z_1^{-k_1} \cdots z_{n-1}^{-k_{n-1}} dz = \hat{f}_{z_n}(k_1, \cdots, k_{n-1}) = 0$. Thus,

$$\hat{f}(k_1, \cdots, k_n) = \int_T (\int_{T^{n-1}} f(z, z_n) z_1^{-k_1} \cdots z_{n-1}^{-k_{n-1}} dz) dz_n = 0.$$

Similarly, if $k_n \notin \Sigma_{\delta_n}$, $\hat{f}(k_1, \cdots, k_n) = 0$. Thus $f \in A(\Sigma) = A_\delta^{(n)}$. We've shown $A^{(n-1)}_{(\delta_1, \cdots, \delta_{n-1})} \otimes A_{\delta_n} \subset A_\delta^{(n)}$. The opposite inequality is clear. QED.

III.2.8. Theorem (Rudin - [2]). If $\delta = (\delta_1, \cdots, \delta_n)$ then $A_\delta^{(n)}$ is relatively maximal in $C(T^n)$ (for definition see p.19d).

Proof: Trivial for $n = 1$: by Wermer's maximality theorem applied to $A(D)$, this is maximal and, a priori, relatively maximal; $C(T)$ is trivially relatively maximal in itself. The theorem now follows by induction on n, using Theorem I.2.11 and Lemma III.2.7. QED.

(iv) Characterization of all Rational Extensions of $A(D^n)$ in $C(T^n)$.

III.2.9. Lemma. Let $T^n \subset K \subset D^n$. Then K is invariant with respect to each automorphism (*), the one parameter groups, if, and only if K is a union of sets of the form $K_\delta^{(n)}$.

Proof: Observe that each set $K_\delta^{(n)}$ is so invariant, and therefore so is any union of such sets. Conversely, suppose K is so invariant. For each $\zeta \in K$, it suffices to find a δ^ζ such that $\zeta \in K^{(n)}_{\delta^\zeta} \subset K$, for then $K = \bigcup_{\zeta \in K} \{\zeta\} \subset \bigcup_{\zeta \in K} K^{(n)}_{\delta^\zeta} \subset \bigcup_{\zeta \in K} K = K$, so that $K = \bigcup_{\zeta \in K} K_{\delta^\zeta}$. To this end, fix $\zeta \in K$. Define

$$\delta_j \equiv \begin{cases} 0, & |\zeta_j| < 1 \\ 1, & |\zeta_j| = 1 \end{cases}.$$

Then $\zeta \in K_\delta^{(n)}$, where $\delta = (\delta_1, \cdots, \delta_n)$. Fix

$$\text{(a) } \tau_j \in \begin{cases} \text{Int } K_{\delta_j}, & \delta_j = 0 \\ K_{\delta_j}, & \delta_j = 1 \end{cases} = \begin{cases} \text{Int } D, & |\zeta_j| < 1 \\ T, & |\zeta_j| = 1 \end{cases}$$

and choose $\varphi_j \in \text{Aut } D$ such that $\varphi_j(\zeta_j) = \tau_j$ (which exists by the choice of τ_j). Then $\varphi \equiv \varphi_1 \otimes \cdots \otimes \varphi_n$ takes ζ to $\tau \equiv (\tau_1, \cdots, \tau_n)$. Since τ was chosen to be any point satisfying (a), and K is closed, we've shown that $\zeta \in K_\delta^{(n)} \subset K$.

QED.

III.2.10. Lemma. Let K, as in Lemma III.2.9, be invariant with respect to each one parameter group of automorphisms. Then K is a subsemigroup of D^n, if, and only if, $K = K_\delta^{(n)}$ for some δ.

Proof: Sets of the form $K_\delta^{(n)}$ are clearly semigroups.

Conversely, suppose K is a subsemigroup of D^n. To show K is of the desired form, it suffices to show that any two $K_\delta^{(n)}$'s contained in K both lie in a third. For then the set of all $K_\delta^{(n)}$'s contained in K (of which there are at most 2^n) contains a maximal element (one contained in all the rest) which must equal K by Lemma III.2.9. To this end, say $K_{\delta^1}^{(n)}, K_{\delta^2}^{(n)} \subset K$. By assumption $K_{\delta^1}^{(n)} K_{\delta^2}^{(n)} = K_{\delta^1\delta^2}^{(n)} \subset K^{(n)}$. But

$K^{(n)}_{\delta^j} \subset K^{(n)}_{\delta^1\delta^2}$ $(j=1,2)$. QED.

III.2.11. Theorem. Let $A(D^n) \subset B \subset C(T^n)$, B a closed subalgebra. Then the following statements are equivalent.

(1) B is invariant with respect to all one parameter sub-groups of Aut D^n.

(2) πm_B is rationally convex.

(3) $B = A^{(n)}_\delta$, $m_B = K^{(n)}_\delta$ for some δ.

In particular, such a B is determined by its maximal ideal space, $K^{(n)}_\delta$.

Proof: (1) ⇒ (2): If (1) holds, then πm_B is invariant by Lemma I.1.9. Since B is, a priori, translation invariant, m_B is a semigroup using II.2.1 and II.2.2, and thus so is πm_B, which is therefore, by III.2.10, of the form $K^{(n)}_\delta$ which is rationally convex.

Theorem (2) ⇒ (3): If (2) holds, then with $K \equiv \pi m_B$ and Theorem I.2.6, we get $R(K) \subset B$ and $K = m_{R(K)}$. Also, $R(K)$ satisfies (1) by Theorem III.2.3. Thus, by the argument in the last paragraph, $K = K^{(n)}_\delta$ for some δ. But then $R(K) = A^{(n)}_\delta$ by Lemma III.2.5(2). Finally $B = A^{(n)}_\delta$ by Theorem III.2.8 (Relative maximality of $A^{(n)}_\delta$).

(3) ⇒ (1): Clear by observation. QED.

III.2.12. Theorem. Let $A(D^n) \subset B \subset C(T^n)$. Then

$$\widehat{\pi \mathfrak{m}_B} = \mathop{\times}_{j=1}^{n} \pi_j(\pi \mathfrak{m}_B).$$

Proof: Let $K \equiv \pi \mathfrak{m}_B$. First note that $\pi_j(K) = \sigma_B(\pi_j) = T$ or D by Corollary I.1.4. Thus $\mathop{\times}_{j=1}^{n} \pi_j(K)$ is rationally convex. Thus it remains to show that

(a) $K \subset \mathop{\times}_{j=1}^{n} \pi_j(K) \subset \check{K}$, for this will force $\check{K} = \mathop{\times}_{j=1}^{n} \pi_j(K)$, the desired result.

By Remark(1)-I.2.(iii) $\check{K} = \mathfrak{m}_{R(K)}$ and thus Theorem III. 2.11. implies that $K \subset \check{K} = K_\delta^{(n)}$ for some δ. Therefore, $\pi_j(K) \subset K_{\delta_j}$ $(j=1,\cdots,n)$, i.e. $\mathop{\times}_{j=1}^{n} \pi_j(K) \subset \mathop{\times}_{j=1}^{n} (K_{\delta_j}) = K_\delta^{(n)} = \check{K}$ which is (a). QED.

We end this section with a characterization (up to maximal ideal space) of all algebras $B, A(T^2) \subset B \subset C(T^2)$ such that $\pi \mathfrak{m}_B \subset$ Boundary D^2.

III.2.13. Theorem. Let $A(D^2) \subset B \subset C(T^2)$. Then either

(a) B extends an algebra of the form $A_\delta^{(2)}$, $\delta \neq (0\ 0)$ ($A_\delta^{(2)} = A(D \times T)$, $A(T \times D)$, or $A(T^2)(= C(T^2))$) and thus is completely described by Theorem II.3.1 as $(A_\delta^{(2)})_V$, $V = E \times T$ or $T \times E$ respectively (notation defined in I. §2.(iv)-(c)) or

(b) $\widehat{\pi \mathfrak{m}_B} = D^2$.

If $\pi m_B \subset$ Boundary D^2, then π is an injection onto a set of the form $E_1 \times D \cup D \times E_2 \cup T^2$, and B has the same maximal ideal space as $(C(T)\otimes A(D))_{E_1 \times T} \cap (A(D)\otimes C(T))_{T\times E_2}$. In case (a), $E_1 = \emptyset$ or $E_2 = \emptyset$. Otherwise case (b) prevails.

<u>Proof</u>: Suppose $f^{-1} \in B$ for some $f \in A(D^2)$ not invertible in $A(D^2)$. Then $A \subset A(f^{-1}) \subset B$. The algebra $A(f^{-1})$ is a rational extension and thus of the form $A_\delta^{(2)}$ by Theorem III.2.11. Clearly $A_\delta^{(2)} \supsetneq A(D^n)$. Thus (a) holds.

Now suppose otherwise: $f^{-1} \in B$ implies $f^{-1} \in A(D^2)$. This is equivalent to (b), for $f^{-1} \in B$ (resp. $A(D^2)$) says $f(z) \neq 0$ for all $z \in \pi m_B$ (resp. D^2). Now use Lemma I.2.5.

In Case (a), say $B \supset A(T\times D) = C(T) \otimes A(D)$. With the notation of Chapter I, §2.(iv).(c), Theorem III.3.1$'$ shows that

$$B = \{f \in C(T^2): f_{z_1} \in A(D)$$

$$\text{for all } z_1 \in E\} = A(T\times D)_{E\times T},$$

where $E \equiv \{z_1 \in T : f_{z_1} \in A(D) \text{ for all } f \in B\}$.

If $\pi m_B \subset$ Boundary D^2, then the desired conclusion follows directly from Lemma III.2.6, which characterizes all representing measures for maximal ideals on B.

If, say $E_2 = \emptyset$, then $\pi(m_B) = T^2 \cup E_1 \times D \subset T \times D$. Thus, $C(T) \otimes A(D) \subset A(T\times D) \subset A(\pi m_B) \subset B$ by Theorem I.2.6 which is case (a).

If $E_1 \neq \emptyset$, $E_2 \neq \emptyset$, then B cannot satisfy (a) since it has the wrong maximal ideal space. Thus case (b) prevails. QED.

Recall that the dual of R is R: $\lambda(x) = \ell^{i\lambda x}$, $x, \lambda \in R$.

§1. Translation Partials

(i) The Case for a General Banach Space A.

Let $\mathfrak{A} \equiv \{f \in A: x \to f_x \text{ is analytic near } 0\}$.

It can be shown from Theorem II.1.1(1) that if $|f_x|$ is bounded in x for each $f \in A$, then $\mathfrak{A}$ is dense in A (hint: let $\phi(x) \equiv 1/1+x^2$ and compute the translate of $\widehat{\phi f}(\lambda)$, $\lambda \in R$). However, this fact will not be used.

For $f \in \mathfrak{A}$, let $\dfrac{d_y^n f_x}{dx^n}$ denote the nth order derivative of $x \to f_x$ at $y \in R$; that this exists for each $y \in R$ is seen below.

IV.1.1. Lemma. (1) $\mathfrak{A}$ is translation invariant. Thus $f \in \mathfrak{A}$ implies f has a power series expansion about each $y \in R$, so that $\dfrac{d_y^n f_x}{dx^n}$ exists as a function from R into A.

(2) For $f \in \mathfrak{A}$, $y \in R$,

$$\left(\frac{d_0^n f_x}{dx^n}\right)_y = \frac{d_y^n f_x}{dx^n} = \frac{d^n (f_y)_x}{dx^n}.$$

Since the derivatives of analytic functions are analytic, we see that $f \in \mathfrak{A}$ implies $\dfrac{d_0^n f_x}{dx^n} \in \mathfrak{A}$ for all $n \geq 0$.

Proof: Fix $f \in \mathfrak{A}$, $y \in R$. Then there is a neighborhood, N, of 0, and $f^{(n)} \in A$ $(n = 0,1,2,\cdots)$ such that

CHAPTER IV

Some Deeper Properties of πm_B

§0. Introduction.

We return to the group action defined in Chapter II.§1, this time letting $G = R$: the action becomes

$$A \times R \to A$$

$$(f,x) \to f_x.$$

In particular, we examine the action

$$A(E) \times R \to A(E) \quad \text{defined by}$$

$$(f,x) \to (f,\phi(x)) \equiv f(\phi(x))$$

of III,§2.(ii) where ϕ: $R \to \text{Aut } E$ is not only continuous, but also analytic (as will be made clear).

When $T^n \subset K \subset D^n$, $\partial A(K) = T^n$, we show, via use of this group action, that $A(K) = R(K)$. This yields immediately the corollary: if $K \cap \text{Int } D^n \neq \phi$, then $f \in A(K)$ implies that f extends "analytically" to all of D^n. It also sheds light on the more general class of algebras of form $A_1(K)$: the locally power series approximatable functions on K.

In discussing any one parameter action on a space A, we adapt the notation of Chapter II.§1. In particular, $\Sigma \subset A$ is translation invariant if $f_x \in \Sigma$ for all $f \in \Sigma, x \in R$; $f_x = (f,\phi(x))$ (if ϕ is fixed).

$$f_x = \sum_{n=0}^{\infty} f^{(n)} \frac{x^n}{n!}, \quad x \in N.$$

Thus, by (i) and (ii) of II.§1, and continuity of translation,

$$(f_y)_x = f_{y+x} = f_{x+y} = (f_x)_y$$

$$= (\sum_{n=0}^{\infty} f^{(n)} \frac{x^n}{n!})_y = \sum_{n=0}^{\infty} f_y{}^{(n)} \frac{x^n}{n!}, \quad x \in N.$$

This clearly shows (1). To show (2), simply note that

$$f^{(n)} = \frac{d_0^n f_x}{dx^n} \quad \text{and}$$

$$f_y{}^{(n)} = \frac{d_0^n (f_y)x}{dx^n} = \frac{d_0^n f_{y+x}}{dx^n} = \frac{d_y^n f_x}{dx^n}. \qquad \text{QED.}$$

IV.1.2. <u>Theorem</u>. Let $B \subset \mathfrak{A}$ be a subspace. Then

(1) $\overline{B}$ is translation invariant implies $\frac{d_0 f_x}{dx} \in \overline{B}$ for all $f \in B$.

(2) $\frac{d_0 f_x}{dx} \in B$ for all $f \in B$ implies $\overline{B}$ is translation invariant.

<u>Proof</u>: If $f_y \in \overline{B}$ for all $y \in R$, then $\frac{d_0 f_x}{dx} = \lim_{y \to 0} \frac{f_y - f}{y} \in \overline{B}$. This shows (1).

To show (2) we use induction. Assume $\frac{d_0 f_x}{dx} \in B$ for all $f \in B$. We first show that therefore

(*) $\quad \frac{d_0^n f_x}{dx^n} \in B, \; n \geq 0.$

For if (*) holds for $n = m$ and $f \in B$, we have

$$\frac{d_0^{m+1} f_x}{dx^m} = \frac{d_0}{dy}\left(\frac{d_y^m f_x}{dx^m}\right) = \frac{d_0}{dy}\left(\frac{d_0^m f_x}{dx^m}\right)_y \text{ (by Lemma IV.1.1(2))} \in B,$$

applying the inductive hypothesis to the case $n = 1$. Now, if $f \in B$, we have, $f_y = \sum_{n=0}^{\infty} \frac{d_0^n f_x}{dx} \frac{y^n}{n!} \in \overline{B}$ by (*) for small y. Lemma I.1.1 now applies to show $f_y \in \overline{A}$ for all $y \in R$. QED.

(ii) Application to the Case $A = A(E)$.

Fix $E \subset C^n$, compact.

Let $\phi : R \to \text{Aut } E$ be a homomorphism, and $q : R \times E \to E$ defined by $q(x,z) \equiv \phi(x)(z)$. Let q_j denote the jth coordinate of $q(j = 1,\cdots,n)$.

Definition. ϕ is analytic if there exists a compact neighborhood, $0 \in N \subset R$, such that $q_j \in \mathfrak{A}^{n+1}(N\times E)$.

The homomorphism $R \to \text{Aut } D^n$ in Theorem III.2.4 is easily seen to be analytic since each $\varphi \in \text{Aut } D^n$ has analytic extension to a neighborhood of D^n.

IV.1.3. Lemma. Let ϕ be analytic. Then

(1) $\mathfrak{A}(E) \subset \mathfrak{A}$.

(2) $f \in \mathfrak{A}(E) \Rightarrow \frac{d_0 f_x}{dx} \in \mathfrak{A}(E)$.

Proof: Fix $f \in \mathfrak{A}(E)$. Then f extends analytically to

$\tilde{f} : V \to \mathbf{C}$, $E \subset V$, open.

Since ϕ is analytic,

$$q : N \times E \to C$$

extends analytically to

$$\tilde{q} : \Omega \to C, \; N \times E \subset \Omega,$$

open in C^{n+1} (q N as above). Since $\tilde{q}(N\times E) \subset E \subset V$ there exist neighborhoods $U \supset N$, $W \supset E$ such that $U \times W \subset \Omega$ and $\tilde{q}(U\times W) \subset V$. Then

$$\tilde{f}\circ\tilde{q} : U \times W \to C$$

is analytic, so that, by the Riemann extension theorem (Gunning and Rossi - [1]), so is

$$G : U \times U \times W \to C, \text{ where}$$

$$G(x,y,z) \equiv \frac{\tilde{f}\circ\tilde{q}(x,z) - \tilde{f}\circ\tilde{q}(y,z)}{x - y} .$$

Fix a compact neighborhood, W_1, of E, $E \subset W_1 \subset W$. On $N \times N \times W_1$, $G(x,y,z)$ is uniformly continuous. Thus, for fixed $x \in N$,

$$F_x(z) \equiv \lim_{y \to x} G(x,y,z) \text{ is}$$

analytic on Int (W_1), being a uniform limit of functions analytic on Int (W_1). But

$$F_{x|E} = \lim_{y \to x} \frac{f_y - f_x}{y - x} = \frac{d_y f_x}{dx} ,$$

the limit taken in the supremum norm in $A(E)$, for each $x \in N$. This clearly shows (1) and (2). QED.

§2. An Extension Theorem.

(i) Discussion.

This section is devoted to a study of the properties of $A(K)$ for a certain class of sets K, $E \subset K \subset \hat{E}$, where E is acted on by lots of analytic homomorphisms. In particular, we show that if $\partial A(K) = E$, E sufficiently nice, e.g. $E \equiv T^n$, then $A(K) = P(K)$. This is equivalent to saying that for each $f \in \mathfrak{A}(K)$, there exists a $g \in A(\hat{E})$ such that $f = g_{|K}$ — that is, each f analytic near K "extends analytically" to $\hat{E}$. Classical examples of such sets are $K \equiv$ Boundary $\hat{E}$ whenever this is connected. Hartogs' Theorem states that for each $f \in \mathfrak{A}(K)$, f extends analytically to a neighborhood of $\hat{E}$. Here we show that for certain sets E, a piece of Boundary $\hat{E}$ can be removed and replaced by a chunk of Interior $\hat{E}$ to obtain a set K which satisfies a similar extension theorem.

(ii) Equivalence of Algebras of the Form $A(K)$ and $R(K)$, $T^n \subset K \subset D^n$, $\partial A(K) = T^n$.

As in III.2.(ii), we fix a compact set $E \subset C^n$ such that $E = \partial P(E)$. Fix an analytic homomorphism

$$\phi : R \to \text{Aut } \hat{E}.$$

Recall the inclusion $\operatorname{Aut} \hat{E} \subset \operatorname{Aut} E$ (III. §2.(ii)) and the corresponding homomorphism $\phi_{|E} : R \to \operatorname{Aut} E$. We note that $\phi_{|E}$ is, a priori, analytic whenever ϕ is.

We let f_x denote either $f\circ\phi(x)$ or $f\circ(\phi_{|E})(x) = f\circ(\phi(x)_{|E})$, depending on whether $f \in A(\hat{E})$ or $f \in A(E)$ respectively. No confusion should result.

IV. 2. 1. Lemma. If $E \subset K \subset \hat{E}$, $\partial A(K) = E$, then $A(K)$ is translation invariant.

Proof: Fix g analytic in a neighborhood, Ω, of K. Since $x \to \phi_j(x)$ is continuous from R into $A(\hat{E})(j=1,\cdots,n)$, $\phi(x)(K) \subset \Omega$ for small x. Thus $g_x \equiv g\circ\phi(x) \in \mathfrak{A}(K) \subset A(K)$ for small x, so that $(g_{|E})x \in A(K)_{|E} \approx A(K)$ for small x. Since, by Lemma IV.1.3(1), $x \to (g_{|E})_x$ is analytic from R into A(E), Lemma I.1.1 shows that $(g_{|E})_x \in A(K)$ for all $x \in R$. QED.

IV.2.2. Lemma. If $A_1 \subset \mathfrak{A}(E)$ is an $\mathfrak{A}(\hat{E})$ module satisfying $\frac{\partial f}{\partial z_j} \in A_1$ for each $f \in A_1$, then $\overline{A_1}$ is translation invariant.

Proof: By Theorem IV.1.2(2) it suffices to show that $\frac{d_0 f_x}{dx} \in A_1$ for each $f \in A_1$. But, by the chain rule, for $f \in A_1$,

$$\frac{d_0 f_x}{dx}(z) = \frac{d_0 f(\phi(x)(z))}{dx}$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) \frac{d_0\phi_j(x)}{\partial x}(z) .$$

By assumption, $\frac{\partial f}{\partial z_j} \in A_1$ $(j=1,\cdots,n)$ and Lemma IV.1.3.(2) shows that $\frac{d_0\phi_j(x)}{\partial x} \in \mathfrak{A}(\hat{E})$. Thus the sum lies in A_1 (since we assumed A_1 is an $\mathfrak{A}(\hat{E})$ module). QED.

Note that Lemma IV.2.2 contains IV.2.1 which was presented for its simplicity (e.g. it avoids use of translation partials) and elegance.

<u>Definition</u>. For $K, V \subset C^n$, K compact, let $\mathfrak{A}_V(K)$ denote the set of all $f \in \mathfrak{A}(K-V)$ such that all mixed partials of f are bounded on $K-V$. Let $A_V(K)$ denote $\overline{\mathfrak{A}_V(K)}$.

Clearly $\mathfrak{A}_\phi(K) = \mathfrak{A}(K)$. Note that $\mathfrak{A}_V(K)$ is an algebra closed under $\frac{\partial}{\partial z_j}$ $(j=1,\cdots,n)$.

IV.2.3. <u>Lemma</u>. Let $P(E) \subset B \subset C(E)$, B a closed subalgebra. Let π: $m_B \to \hat{E}$ be restriction. Fix $I \subset B$ an ideal. Then $A_{z(I)-E}(\pi m_B)$ is translation invariant.

<u>Proof</u>: Let $V \equiv z(I)-E$, $K \equiv \pi m_B$. By Corollary I.2.4, $\partial A_V(K)$. Since, thus, $\mathfrak{A}(\hat{E}) \subsetneq \mathfrak{A}_V(K) \subsetneq \mathfrak{A}(E)$, Lemma IV.2.2 applies to show that $A_V(K) \approx A_V(K)|_E$ is translation invariant. QED.

We apply these results in the case $E = T^n$, and use the fact that the one parameter homomorphisms, $\phi : R \to \text{Aut } D^n$, are analytic.

IV.2.4. <u>Theorem</u>. Let $D^n \subset K \subset T^n$.

(1) If $\partial A(K) = T^n$, then $A(K) = A_\delta^{(n)}$ for some $\delta = (\delta_1,\cdots,\delta_n)$.

In particular, by Theorem III.2.11, we see that algebras of the form $A(K)$ are precisely those of the form $R(K)$.

(2) If $K = \pi m_B$, $A(D^n) \subset B \subset C(T^n)$, $I \subset B$ an ideal, then $A_{(z_B(I)-T^n)}(K) = A_\delta^{(n)}$ for some δ.

Proof: Using Theorem III.2.11, (1) follows directly from Lemma IV.2.1, and (2) from IV.2.3. QED.

(iii) A Generalization of Hartogs' Theorem.

When $K \cap \text{Int } D^n \neq \emptyset$, we get the following generalization of Hartogs' Theorem (stated in IV.§2.(i)).

IV.2.5. Theorem. Let E satisfy $\partial P(E) = E$ and assume

(i) there is a dense subset, Ω, of $\hat{E}$ such that for each ζ^1, $\zeta^2 \in \Omega$ there exists an analytic homomorphism, $\phi : R \to \text{Aut } \hat{E}$ such that $\phi(x)(\zeta^1) = \zeta^2$ for some $x \in R$, and

(ii) $A(\hat{E})$ is relatively maximal in $C(E)$.

Then,

(1) if $E \subset K \subset \hat{E}$, $K \cap \Omega \neq \emptyset$ and $\partial A(K) = E$, then $A(K) = P(K) \approx A(\hat{E})$

and

(2) if $P(E) \subset B \subset C(E)$, B a closed subalgebra such that $\pi m_B \cap \Omega \neq \emptyset$, $I \subset B$ an ideal, then

$A^1 \equiv A_{z(I)-E}(\pi m_B) = P(\pi m_B) \approx A(\hat{E})$.

Proof: Let B_1 be either $A(K)$ (in(1)) or A^1 (in (2)). By (ii), it suffices to show that $\pi m_{B_1} = \hat{E}$. By Lemma IV.2.1 (applied to (1)) and IV.2.3 (applied to (2)), B_1 is invariant with respect to every analytic, one-parameter action induced by an analytic homomorphism, $\phi : R \rightarrow \text{Aut } \hat{E}$. Thus so is πm_{B_1}. Since this intersects Ω, (i) implies $\pi m_{B_1} = \hat{E}$. QED.

Discussion. As already mentioned, (1) expresses an extension result, whereas (2) may be rephrased as follows. Let $K = \pi m_B - (z(I)-E)$ and $f \in \mathfrak{A}(K)$. Then f extends to lie in $A(\hat{E})$ if, and only if, all mixed partials of f are bounded on K.

If we assume, in IV.2.5.(1), that Boundary K is connected, then Hartogs' Theorem yields a stronger result: $A(\text{Boundary } K) \approx A(\hat{E})$. For each $f \in A(\text{Boundary } K)$ can be approximated by functions in $\mathfrak{A}(\text{Boundary } K)$ which extend analytically to a neighborhood of K (by Hartogs' Theorem); thus $A(\text{Boundary } K) = A(K) \approx A(\hat{E})$ by Theorem IV.2.5.

If E_1, E_2 both satisfy the hypothesis of Theorem IV.2.5, then so does $E_1 \times E_2$, noting that $\widehat{E_1 \times E_2} = \hat{E}_1 \times \hat{E}_2$. To verify (i) for $\hat{E}_1 \times \hat{E}_2$, simply note that if $x \rightarrow \phi_i(x)$ is an analytic homomorphism $(i=1,2)$, then so is $x \rightarrow \phi_1(x) \otimes \phi_2(x)$. The verification of (ii) for $\hat{E}_1 \times \hat{E}_2$ is merely Theorem I.2.11, noting that $\mathfrak{A}(\hat{E}_1 \times \hat{E}_2) = \mathfrak{A}(\hat{E}_1) \otimes \mathfrak{A}(\hat{E}_2)$ by a basic theorem in complex variables. Thus if we knew that (i) and (ii) of

Theorem IV.2.5 held for $E \equiv S^n$ or $E \equiv I \equiv [0,1]$, then the conclusion of the theorem would hold for the class of objects, $T^n \times S^{m_1} \times \cdots \times S^{m_k} \times I^p$, $n, m_j, p \geq 0$. (Note that $S^{m_1} \times \cdots \times S^{m_k} \neq S^{m_1 + \cdots + m_k}$).

The relevance of IV.2.5.(2) needs some explanation. If we assume that $z(I) \cap E = \phi$, and $z(I) \cap \pi(m_B)$ is a singleton, things begin to make sense.

Let $A(D^2) \subsetneq B \subset C(T^2)$. If $B = A(\pi_2/\pi_1)$, its maximal ideal space projects onto $K \equiv$ closure of
$\{\zeta \in D^2: \zeta \neq 0, |\zeta_2/\zeta_1| \leq 1\} = \{\zeta \in D^2: |\zeta_2| \leq |\zeta_1|\}$.
Clearly $\pi_1(z) = 0$ implies $\pi_2(z) = 0$, $z \in K$. Thus, $z(\pi_1) = z(\pi_1, \pi_2) = (0,0)$, and we may apply the extension theorem to any function analytic <u>near</u> $K - \{(0,0)\}$: if such an f extends analytically to a neighborhood of K, then, a priori, all its mixed partials are bounded on $K - \{(0,0)\}$. The point is that this is <u>enough</u>, and, in fact, implies that f must have been an element of $A(D^2)$ to begin with!

Using Corollary I.2.16, we have: For $f \in B$, $\hat{f}$ extends <u>continuously</u> to (0,0) if, and only if, all extensions of (0,0) to B agree on f; we've shown that $\hat{f}$ extends analytically to (0,0) if, and only if, $f \in A(D^2)$. An example of a function analytic and bounded near $K - \{(0,0)\}$, having no analytic extension to a neighborhood of K is $\hat{f}_n \equiv \pi_2^n/\pi_1$, for $\dfrac{\partial^n f_n}{\partial z_2^n}$ blows up at (0,0). Note, however, that f_1 has no continuous extension to all of K, so that (0,0) "blows up" in $m_{A(f_n)}$, while f_n, $n \geq 2$ does have a continuous extension because $f_n(z) = z_2^n z_1 = z_2^{n-1}(z_2/z_1)$ approaches 0 as z approaches (0,0) in K.

(iv) The Algebra of Locally Power Series Approximatable Functions on $K, T^n \subset K \subset D^n$.

Fix $K \subset C^n$, compact. Let $\mathfrak{A}_1(K)$ denote all $f \in C(K)$ having the property that for all $\zeta \in K$ there is a neighborhood N of ζ, a function u analytic in N, such that $u|_{N \cap K} = f|_{N \cap K}$. Let $A_1(K)$ denote $\overline{\mathfrak{A}_1(K)}$.

Let U_K denote the set of $\zeta \in K$ for which there is a function analytic near ζ, vanishing on K.

Note that U_K is open in K. Thus $K - U_K$ is compact. Also

$$(*) \qquad \mathfrak{A}_1(K)|_{K - U_K} \subset \mathfrak{A}(K-U_K).$$

For each $\zeta \in K - U_K$, $f \in \mathfrak{A}_1(K)$, f must have a unique analytic extension to a neighborhood of ζ, lest $\zeta \in U_K$. It is not hard to show that f therefore extends analytically to a neighborhood of K.

IV.2.6. Lemma. Let $E \subset K \subset \hat{E}$, $\partial A_1(K) = E$, and assume $U_K \cap E = \phi$, and that U_K has the global property that there exists an $f \in \mathfrak{A}(\hat{E})$, $f \equiv 0$, vanishing on it. Then,

(1) $\partial A(K-U_K) = E$. Thus $A_1(K) \subset A(K-U_K) \subset C(E)$ and

(2) each $h \in m_{A_1(K)}$ such that $h(f) \neq 0$ has a unique extension to $A(K-U_K)$.

Proof: For each $g \in \mathfrak{A}(K-U_K)$, fg is locally analytic on $K-U_K$

(in fact $fg\,|_{K-U_K} \in \mathfrak{A}(K-U_K)$) and extends uniquely to a continuous function, still denoted fg, on K, vanishing on U_k. Since U_k is a relatively open subset of K, fg is thus locally zero at each point of U_K. Thus we've shown that $fg \in \mathfrak{A}_1(K)$ and since g is arbitrary,

(a) $fA(K-U_K) \subset A_1(K)$.

The definition of U_K shows that $z(f) \cap (K-U_K)$ is nowhere dense in $K-U_K$. Thus,

(b) $(K-z(f)) \cup E$ is dense in $K-U_K$.

Now (a), along with Theorem I.2.3, shows that for each $g \in A(K-U_K)$, $g|_{(K-z(f)) \cup E}$ achieves its supremum on E. Thus, by (b), so does g, and (1) follows. Finally (a) and Lemma I.2.2 imply (2). QED.

IV.2.7. <u>Theorem</u>. Assume $T^n \subset K \subset D^n$, $\partial\mathfrak{A}_1(K) = T^n$. Then

(1) if $U_K = \emptyset$, then $A_1(K) = A_\delta^{(n)}$ for some δ.

(2) If some $f \in \mathfrak{A}(D^n)$ vanishes on U_K, then $A_1(K) \subset A_\delta^{(n)}$ (some δ) and $\mathfrak{m}_{A_1}(K)$ projects onto $K_\delta^{(n)} \cup E$ where $E \subset z(f)$, and the only points in this projection that blow up in $\mathfrak{m}_{A_1}(K)$ lie in z(f).

<u>Proof</u>: By (*), $\mathfrak{A}_1(K) \subset \mathfrak{A}(K)$ if $U_K = \emptyset$. Clearly $\mathfrak{A}(K) \subset \mathfrak{A}_1(K)$. Thus $U_K = \emptyset$ implies $A_1(K) = A(K)$. This shows (1).

By Lemma IV.2.6(1) (with $E = T^n$) and Theorem IV.2.4(1), $A(K-U_K) = A_\delta^{(n)}$ for some δ, and thus $A_1(K) \subset A_\delta^{(n)}$. Therefore, $K_\delta^{(n)} \subset m_{A_1}(K)$. By IV.2.6(2) we have: $h \in m_{A_1}(K) - K_\delta^{(n)}$ implies $h(f) = 0$. This shows (2). QED.

BIBLIOGRAPHY

1. ARENS, R., and SINGER, I.

   [1] Generalized Analytic Functions, Trans. Amer. Math. Soc. 81 (p.379-393), Theorems 4.1 (p.382) and 4.6 (p.383).
   [2] Generalized Analytic Functions (pp.379-393), Lemma 5.6 (p.387).

2. CARTAN, H.

   [1] Elementary Theory of Analytic Functions of One or Several Complex Variables, Addison-Wesley Pub. Co., Inc., Reading, MA., VI.2.6-Prop.6.2 (p.184).

3. GAMELIN, T.W.

   [1] Uniform Algebras, Prentice-Hall, Inc., Englewood Cliffs, N.J., Ch. III.4 - Theorem 4.1 (p.77).
   [2] Uniform Algebras, Ch. III, Sec. 2, Lemma 2.4 (p.69).
   [3] Uniform Algebras, Ch. II, Sec. 12, Lemma 12.3 (p.57).
   [4] Uniform Algebras, Ch. II, Sec. 13, Theorem 13.1 (p.60).

4. GUNNING, R. and ROSSI, H.

   [1] Analytic Functions of Several Complex Variables, Prentice-Hall, Inc., Englewood Cliffs, N.J., Ch. I. Sec. c - Theorem 3 (p.19).

5. LOOMIS, L.H.

   [1] An Introduction to Abstract Harmonic Analysis, D. Van Nostrand Co., Inc., New York, Ch. IV, Sec. 24, Theorem 24A (p.75).

6. RUDIN, W.

   [1] Fourier Analysis on Groups, Interscience Pub., a division of John Wiley and Sons, New York, 1.1.6 (p.3).

[2] Function Theory in Poly Disks, W.A. Benjamin, Inc., New York, Theorem 2.2.2 (p.22).

[3] Fourier Analysis on Groups, 1.2 (p.6) and 1.7 (p.27).

[4] Fourier Analysis on Groups, 8.7.3 (p.217).

[5] Fourier Analysis on Groups, Theorem 9.2.2 (p.233).