Joint Quasitriangularity of 2-tuples of Essentially Normal Essentially Commuting Operators on Infinite Dimensional Hilbert Spaces

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Gail Annette Kaplan

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We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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This paper extends the definition of quasitriangularity to n-tuples of essentially normal, essentially commuting operators and studies their general properties. The paper then concentrates on a special class of 2-tuples of operators. It culminates with necessary and sufficient conditions for quasitriangularity in this setting.

This dissertation is dedicated to Dr. Dorothy Bernstein and Dr. Geraldine Coon who taught me to love mathematics, and to my parents who taught me to love life.

Table of Contents

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I. INTRODUCTION.

This paper will deal with the notion of quasitriangularity. In 1967 Halmos introduced the concept; the germ of the idea but not the name had been used previously by Aronszajn and Smith in their work on the existence of nontrivial invariant subspaces for compact operators. Operator theorists continued to ponder the question of the existence of a nontrivial invariant subspace for a larger class of operators. In 1966 Bernstein and Robinson extended the class to include all operators that are polynomially compact; T is polynomially compact if there exists a polynomial p such that p(T) is compact. These authors used nonstandard analysis in their proofs. After studying their paper Halmos proved their results using standard techniques. He found the notion of compactness employed only twice and in one instance the weaker hypothesis of quasitriangularity was sufficient. Hence, the birth of his 1967 paper. In their 1970 paper Douglas and Pearcy contributed to the structure theory of quasitriangular operators. Several Roumanian mathematicians developed theorems describing further properties of this class of operators. Their major result is elegantly stated: the operator H is quasitriangular if and only if $\mathrm{ind}(H - \lambda) \geq 0$ whenever $H - \lambda$ is semi-Fredholm. Brown, Douglas and Fillmore in their 1973 paper obtained the same result for essentially normal operators by applying their work on extensions of C*-algebras. In the early seventies Voiculescu extended the notion of quasitriangularity to n-tuples of operators which pairwise commute.

In this paper I will generalize the notion of quasitriangularity to n-tuples of essentially normal operators which pairwise essentially commute. Many of the basic properties of quasitriangular n-tuples are simple generalizations of the results for a single operator. However, a necessary and sufficient condition for quasitriangularity in the case of an n-tuple, even in very limited cases is quite difficult. I will examine closely the properties of 2-tuples of operators (U,H) where U is essentially unitary, H is essentially self-adjoint, U and H essentially commute, and the joint essential spectrum is a subset of the cylinder $\mathbb{T} \times [0,1]$ where $\mathbb{T}$ denotes the unit circle. The reason for the restriction to this case is threefold. First, it is possible to use the Brown, Douglas, and Fillmore theory; second, this setting is the first nontrivial situation; and third, the index in three space completely characterizes the problem. In four space the problem is much more complex.

To study these 2-tuples it is necessary to carefully examine the structure of their joint essential spectrum X. I will construct a set $\tilde{X} \subseteq \mathbb{R}^2$ which is homeomorphic to X. The region $\tilde{X}$ is an annulus. The complement of $\tilde{X}$, the set $\mathbb{C}|\tilde{X}$, has one unbounded component, O_∞, plus any number of bounded components, $\{O_i\}_{i\geq 0}$. We can think of these components as holes in the spectrum. The major result is necessary and sufficient conditions for the joint quasitriangularity of the 2-tuple (U,H) previously described. These conditions are expressed in terms of the indices associated with the holes in the spectrum.

II. BACKGROUND MATERIAL: EXTENSIONS

The work of Brown, Douglas, and Fillmore on extensions of C^*-algebras is an essential ingredient for the work in this paper. For the sake of completeness I present a summary of the needed results [3].

First it is necessary to define the concept of an extension. Let $\mathcal{L}(\mathcal{H})$ denote the linear operators on the Hilbert space $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the subset of $\mathcal{L}(\mathcal{H})$ consisting of the compact operators, and $\mathfrak{D}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ the Calkin algebra.

Definition II-A. Let X be compact and metrizable. An extension of $\mathcal{K}$ by $C(X)$ is a pair (E,ϕ) where E is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $\mathcal{K}(\mathcal{H})$ and I, and ϕ is a $*$-homomorphism of E onto $C(X)$ with kernel $\mathcal{K}$. Extensions (E_1,ϕ_1) on $\mathcal{H}_1$ and (E_2,ϕ_2) on $\mathcal{H}_2$ are equivalent if there exists a $*$-isomorphism $\psi : E_1 \to E_2$ such that $\phi_1 = \phi_2\psi$. The set of equivalence classes of extensions of $\mathcal{K}$ by $C(X)$ is denoted by $\mathrm{Ext}(X)$.

Note that the definition implies that $\mathcal{E}/\mathcal{K}$ is isomorphic to $C(X)$. The following equivalent definition is a more workable one.

Definition II-B. An extension of $\mathcal{K}$ by $C(X)$ is a $*$-monomorphism τ of $C(X)$ into the Calkin algebra $\mathfrak{D}$ such that $\tau(1) = 1$. Extensions τ_1 and τ_2 are equivalent if there is a $*$-isomorphism $\mu : \mathfrak{D}(\mathcal{H}_1) \to \mathfrak{D}(\mathcal{H}_2)$ induced by a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$\mu\tau_1 = \tau_2$. This means that for any operator $T \in \mathcal{H}_1$, $\mu(\pi(T)) = \pi(UTU^*)$, where π denotes the standard projection from $\mathcal{L}(\mathcal{H})$ onto $\mathfrak{D}(\mathcal{H})$.

For X a compact metric space Ext X turns out to be an abelian group under the operation of addition given by the following definition.

Definition II-C. The sum of extensions (E_1, ϕ_1) on $\mathcal{H}_1$ and (E_2, ϕ_2) on $\mathcal{H}_2$ of $\mathcal{K}$ by $C(X)$ is the extension of $\mathcal{K}$ by $C(X)$ defined by $(E, \phi) = (E_1, \phi_1) + (E_2, \phi_2)$ where $E = \{(T_1 \oplus T_2) + K : T_i \in E_i, \phi_1(T_1) = \phi_2(T_2), K \in \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2)\}$ and $\phi((T_1 \oplus T_2) + K) = \phi_1(T_1) = \phi_2(T_2)$.

For subsets of the plane the identity in the group Ext(X) is given by the extension generated by operators of the form $N + K$ where $\sigma_e(N) = X$, N is normal, and K is compact. This is called the trivial extension and can be shown to satisfy the following definition.

Definition II-D. The extension (E, ϕ) of $\mathcal{K}$ by $C(X)$ is trivial if there exists a $*$-monomorphism σ of $C(X)$ into E such that $\sigma(1) = I$ and $\phi\sigma$ is the identity on $C(X)$. Equivalently, $\tau : C(X) \to \mathfrak{D}(\mathcal{H})$ is trivial if there exists a $*$-monomorphism $\sigma : C(X) \to \mathcal{L}(\mathcal{H})$ such that $\sigma(1) = I$ and $\pi\sigma = \tau$.

The following theorem allows us to conclude that the trivial extension is unique.

Theorem II-E. If X is a compact metric space then there exists a trivial extension of $\mathcal{K}$ by C(X) and any two trivial extensions are equivalent.

Proof. [3, p.81]

Corollary II-F. If Z is a sep rable commutative C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, then there exists an orthonormal basis $\{\phi_n\}$ of $\mathcal{H}$ such that each operator in Z is a compact pertubation of an operator that is diagonal relative to $\{\phi_n\}$. In particular, the same conclusion holds for any countable family of mutually commuting self-adjoint operators on $\mathcal{H}$.

The classification of essentially normal operators provided much of the motivation for the BDF work on extensions. Given an essentially normal operator T with $\sigma_e(T) = X$, the problem reduces to calculating Ext(X) and determining the element of Ext(X) which T represents. The solution is elegantly stated in their paper by the following theorem.

Theorem II-G. If T_1 and T_2 are essentially normal operators on $\mathcal{H}$, then a necessary and sufficient condition that T_1 be unitarily equivalent to some compact pertubation of T_2 is that T_1 and T_2 have the same essential spectrum X and $\operatorname{ind}(T_1 - \lambda I) = \operatorname{ind}(T_2 - \lambda I)$ for all $\lambda \notin X$.

Proof. [3, p.118]

In dealing with the classification of n-tuples $T = (T_1, T_2, \ldots, T_n)$ of essentially normal, essentially commuting operators whose joint essential spectrum is homeomorphic to a subset of $\mathbb{R}^2$ the process remains virtually identical. We must compute the joint essential spectrum of the n-tuple and then the problem is transformed into calculating Ext(X) where $X = \sigma_e(T)$ and determining the element of Ext(X) which T represents.

Let us examine the situation in more detail. By definition every element $\tau \in \operatorname{Ext}(X)$ is a $*$-isomorphism, $\tau : C(X) \to \mathfrak{D}$. The map τ induces a map $\gamma_1(\tau) : C(X)^{-1}/C(X)_0^{-1} = \pi^1(X) \to \mathfrak{D}^{-1}/\mathfrak{D}_0^{-1} = \mathbb{Z}$ where $C(X)^{-1}$ denotes the invertible functions in C(X), $C(X)_0^{-1}$ denotes the connected component of the identity in $C(X)^{-1}$, $\mathfrak{D}^{-1}$ denotes the invertible elements in $\mathfrak{D}$, $\mathfrak{D}_0^{-1}$ denotes the connected component of the identity in $\mathfrak{D}$, and $\pi^1(X)$ is the first cohomotopy group of X, the group of homotopy classes of continuous maps from X into $\mathbb{C}\backslash\{0\}$. For any function $f : X \to \mathbb{C}\backslash\{0\}$, the index of $\tau(f)$ depends only on the equivalence class $[\tau]$ of τ and the homotopy class $[f]$ of f. We readily see that $\operatorname{ind} \tau(fg) = \operatorname{ind} \tau(f)\tau(g) = \operatorname{ind} \tau(f) + \operatorname{ind} \tau(g)$ so $\gamma_1(\tau) : \pi^1(X) \to \mathbb{Z}$ is a homomorphism. Moreover, $\operatorname{ind}(\tau_1 + \tau_2)(f) = \operatorname{ind}(\tau_1(f) \oplus \tau_2(f)) = \operatorname{ind} \tau_1(f) + \operatorname{ind} \tau_2(f)$ so γ_1 is a homomorphism. When is γ_1 a bijective map? This question is answered by the major result in the BDF paper [3].

Theorem II-H. For X a compact subset of the plane γ_1 is bijective.

Proof. [3, p. 116]

III. GENERAL PROPERTIES OF QUASITRIANGULAR n-tuples

First, a short review of the well known finite dimensional theory will be presented. In a finite dimensional Hilbert space every linear operator can be expressed as a square matrix. By the appropriate choice of a basis this operator can be written as an upper triangular matrix; hence, every operator on a finite dimensional space is upper triangular. In the remainder of this paper the term triangular will mean upper triangular. Another way to express this property is that for every operator A on a finite dimensional space there exists an increasing sequence of projections $\{P_k\}$ such that dim $P_k = k$ and each P_k is invariant under A, equivalently $P_k A P_k - A P_k = 0$ for all k. This well known result is simple to prove as illustrated by the following proposition.

Proposition III-A. For every operator A on a finite dimensional Hilbert space $\mathcal{H}$ there exists an increasing sequence $\{P_k\}_{k=1}^{\dim \mathcal{H}}$ of projections with dim $P_k = k$ and $(I - P_k)AP_k = 0$ for $k = 1,2,\ldots,\dim \mathcal{H}$.

Proof. For dim $\mathcal{H} = 1$ the statement is trivally true. Assume the proposition holds for dim $\mathcal{H} = n - 1$. It is necessary to show we can construct the sequence of projections when dim $\mathcal{H} = n$. Define P_1 as projection onto the subspace spanned by any fixed eigenvector v of A. Express $\mathcal{H}$ as the direct sum of $P_1\mathcal{H}$ and its orthogonal complement $(P_1\mathcal{H})^\perp$. Since $\dim(P_1\mathcal{H})^\perp = n - 1$, we know by the induction hypothesis that there exists an increasing sequence of projections $\{F_k\}_{k=1}^{n-1}$ with dim $F_k = k$ and $F_k A F_k - A F_k = 0$ for all k. Define $\mathcal{H}_k$ as the subspace spanned by $F_{k-1}\mathcal{H}$ and the eigenvector v for $k = 2, \ldots, n$ and $\mathcal{H}_1 = P_1\mathcal{H}$. Then the projections P_k onto the subspaces $\mathcal{H}_k$ satisfy the desired properties.

When we allow the dimension of $\mathcal{H}$ to become countably infinite, the situation drastically changes. First, we must define triangularity in this new setting.

Definition III-B. An operator A on the seperable Hilbert space $\mathcal{H}$ is said to be triangular if there exists an increasing sequence of finite rank projections $\{P_k\}$ converging strongly to the identity such that dim $P_k = k$ and $P_k\mathcal{H}$ is invariant under A for each k, $\|P_k A P_k - A P_k\| = 0$ for all k.

Note that when $\mathcal{H}$ is finite dimensional this agrees with the previous definition. In the setting of an infinite dimensional $\mathcal{H}$ it is no longer possible to triangularize every operator, nor is it even possible to "almost triangularize" every operator. In this paper the words almost and essentially will mean the property holds modulo the compact operators. For instance, the operators in a set S will be said to almost compute if AB - BA is compact for all elements A and B in S. The property of an operator being almost triangular is called quasitriangularity. Halmos [9] shows that an operator is unitarily equivalent to a triangular plus a compact if and only if it satisfies the following definition.

<u>Definition III-C</u>. An operator A on a seperable Hilbert space $\mathcal{H}$ is said to be quasitriangular if there exists an increasing sequence of projections $\{P_k\}$ converging strongly to the identity such that $\lim_{k\to\infty}\|P_k A P_k - AP_k\| = 0$.

The question I wish to consider is under what conditions is it possible to simultaneously triangularize n operators modulo the compacts. In this paper $T = (T_1, T_2, \ldots, T_n)$ will denote an n-tuple of operators on the separable Hilbert space $\mathcal{H}$, $\mathcal{P}$ the set of finite rank projections, and $\mathcal{K}$ the compact operators on $\mathcal{H}$.

<u>Definition III-1</u>. The n-tuple T is said to be quasitriangular if $T_1, T_2, \ldots, T_n$ essentially commute and there exists an increasing sequence of finite rank projections $\{P_k\}$ converging strongly to the identity such that $\lim_{k\to\infty}\|T_i P_k T_i - T_i P_k\| = 0$ for $i = 1, 2, \ldots, n$.

It is a trivial consequence of this definition that each operator T_i is quasitriangular. The motivation for the condition that the operators essentially commute is twofold; it is suggested by the finite dimensional case and enables us to utilize the BDF theory. Voiculescu [12] requires that the operators actually commute in his definition of quasitriangularity.

When Halmos initiated the study of quasitriangular operators in his 1967 paper, he showed that it is useful to have two alternative definitions for the concept. I will also formulate two additional definitions for quasitriangular n-tuples. They are patterned after Halmos' definitions. Definition III-2 differs from definition III-1 only by the absence of the word increasing.

Definition III-2. The n-tuple T is said to be quasitriangular if $T_1, T_2, \ldots, T_n$ essentially commute and there exists a sequence of finite rank projections $\{P_k\}$ converging to the identity such that $\lim_{k\to\infty} \|P_k T_i P_k - T_i P_k\| = 0$ for $i = 1,2,\ldots,n$.

Since the set of finite rank projections is partially ordered, we can speak of $\lim_{P\to I, P\in\mathcal{P}} \inf \|PTP - TP\| = 0$. By this we will mean that given any positive number ϵ and projection $P \in \mathcal{P}$, there exists a projection $P_\epsilon \in \mathcal{P}$ such that $P \leq P_\epsilon$ and $\|P_\epsilon T_i P_\epsilon - T_i P_\epsilon\| < \epsilon$ for $i = 1,2,\ldots,n$. The third definition employs this concept.

Definition III-3. The n-tuple T of essentially commuting operators is said to be quasitriangular if $\lim_{P\to I, P\in\mathcal{P}} \inf \|PTP - TP\| = 0$.

The following proof of the equivalence of the preceeding definitions is adapted from Halmos' proof for one operator. The only modification is the choice of γ in theorem III-F. Definition III-2 follows trivially from definition III-1, the only difference being the omission of the word increasing. The next lemma shows that definition III-1 is a simple consequence of definition III-3.

Lemma III-D. Let T denote an n-tuple of essentially commuting operators on $\mathcal{H}$. If $\liminf_{P\to I, P\in\mathcal{P}} \|PTP - TP\| = 0$, then there exists an increasing sequence of finite rank projections $\{P_k\}$ converging strongly to the identity such that $\lim_{k\to\infty} \|P_k T_i P_k - T_i P_k\| = 0$ for $i = 1,\ldots,n$.

Proof. Let $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and P_0 denote the projection onto the subspace generated by $\{e_0\}$. Using the definition of lim inf we know we can find a projection P_1 of finite rank satisfying $e_1 \in \mathcal{R}(P_1)$, $P_0 \leq P_1$ and $\|P_1 T_i P_1 - T_i P_1\| < 1$ for $i = 1,\ldots,n$. Next we obtain a projection P_2 of finite rank satisfying $e_2 \in \mathcal{R}(P_2)$, $P_1 \leq P_2$, and $\|P_2 T_i P_2 - T_i P_2\| < \frac{1}{2}$ for $i = 1,\ldots,n$. In general we find a projection P_m of finite rank satisfying $e_m \in \mathcal{R}(P_m)$, $P_{m-1} \leq P_m$, and $\|P_m T_i P_m - T_i P_m\| < \frac{1}{2^m}$ for $i = 1,\ldots,n$. We have now produced an increasing sequence of finite rank projections $\{P_k\}$ converging to the identity and satisfying $\lim_{k\to\infty} \|P_k T_i P_k - T_i P_k\| = 0$ for $i = 1,\ldots,n$.

To complete the proof of the equivalency of the three definitions, it is necessary to show that definition III-3 follows from definition III-2. This is the nontrivial implication; its proof utilizes the following lemma which shows that when two projections of the same rank are close, in the sense of the norm, it is possible to find a unitary operator close to the identity which transforms one projection onto the other. This lemma appears in Halmos' paper [9]. I present his proof for the sake of completeness.

Lemma III-E. Let E and F be projections of the same finite rank satisfying $\|E - F\| < \epsilon < 1$. Then there exists a unitary operator W satisfying $W^*EW = F$ and $\|I - W\| \le 2\,\epsilon^{\frac{1}{2}}$.

Proof. We can find a unitary operator W_0 satisfying $W_0^*EW_0 = F$ since the rank of E equals the rank of F. Decompose $\mathcal{H}$ as the direct sum of the range of E and the range of I - E. Now consider each operator as a 2 x 2 matrix with respect to this decomposition.

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad W_0^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

Since $F = W_0^*EW_0$, the matrix of F is

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix}.$$

We know that $W_0^*W_0 = I$ since W_0 is unitary. Performing this matrix computation we obtain

$$W_0^*W_0 = \begin{pmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{pmatrix} = I.$$

Thus we have the following relations:

$$(\star)\quad \begin{aligned} A^*A + C^*C &= I \\ A^*B + C^*D &= 0 \\ B^*A + D^*C &= 0 \\ B^*B + D^*D &= I \end{aligned}$$

We know that $\|E - F\| < \epsilon$ and that the norm of a matrix dominates the norm of each entry.

Thus $E - F = \begin{pmatrix} 1 - A^*A & -A^*B \\ - B^*A & -B^*B \end{pmatrix}$ implies $\|1 - A^*A\| < \epsilon$,

and $\|B^*B\| < \epsilon$. Combining these with ($\star$) we obtain $\|C^*C\| < \epsilon$ and $\|1 - D^*D\| < \epsilon$. These estimates will be necessary to show that $\|1 - W\|$ is small. Next, let U and V be unitary operators on the range of E and the range of $1 - E$, respectively. We then obtain a unitary operator $W_1 = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ on $\mathcal{H}$

which commutes with E. The following simple computation shows that the operator W_1W_0 transforms E onto F.

$$(W_1W_0)^*E(W_1W_0) = W_0^*W_1^*EW_1W_0 = W_0^*W_1^*W_1EW_0 = W_0^*EW_0 = F$$

Since the norm of a matrix is dominated by the square root of the sum of the squares of the norms of its entries, we can make $\|1 - W_1W_0\| = \left\|\begin{pmatrix} 1 - UA & - UB \\ - VC & 1 - VD \end{pmatrix}\right\|$ small by choosing U and V so that $\|1 - UA\|$ and $\|1 - VD\|$ are small. We readily see that $\|-VC\|^2 = \|C\|^2 = \|C^*C\| < \epsilon$ and $\|UB\|^2 = \|B\|^2 = \|B^*B\| < \epsilon$ for any unitaries U and V. Applying the usual polar decomposition we can express A as the product QP where Q is a unitary operator and P is the positive operator $(A^*A)^{\frac{1}{2}}$. Let U be the inverse of Q so that $1 - UA = 1 - Q^{-1}QP = 1 - P$. Since $\|A\| < 1$ we know $0 \le P^2 \le P \le 1$ which implies $0 \le 1 - P \le 1 - P^2$. Using these relations we obtain $\|1 - UA\| = \|1 - P\| \le \|1 - P^2\| = \|1 - A^*A\| < \epsilon$. By using an identical argument for D, we obtain $\|1 - VP\| \le \|1 - D^*D\| < \epsilon$ where V is the inverse of the unitary element in the polar decomposition of D. Since $\|1 - W_1W_0\|$ $\le \sqrt{\|1 - UA\|^2 + \|UB\|^2 + \|VC\|^2 + \|1 - VD\|^2} = \sqrt{4\epsilon} = 2\epsilon^{\frac{1}{2}}$, the proof is complete by setting $W = W_1W_0$.

Our goal of showing the equivalency of the three definitions of quasitriangularity will be achieved when we prove the following theorem which shows definition III-2 implies definition III-3.

Theorem III-F. Let T be an n-tuple of operators on $\mathcal{H}$ and $\{P_k\}$ a sequence of finite rank projections converging to the identity and satisfying $\lim\limits_{k\to\infty}\|P_kT_iP_k - T_iP_k\| = 0$ for $i = 1,\ldots,n$. Then $\liminf\limits_{P\to I, P\in\mathcal{P}}\|PTP - TP\| = 0$.

Proof. It is necessary to show that given $\epsilon > 0$ and $P_0 \in \mathcal{P}$ there exists $P \in \mathcal{P}$ such that $P_0 \le P$ and $\|PT_iP - T_iP\| < \epsilon$ for $i = 1,\ldots,n$. Let δ be a fixed positive number to be determined later by certain properties of the operators T_i. Let $\{e_1, e_2, \ldots, e_{n_0}\}$ be an orthonormal basis for the range of P_0. Since $P_n \to I$ and $\|P_kT_iP_k - T_iP_k\| \to 0$ for $i = 1,\ldots,n$ we can find an integer m such that $\|e_j - P_m e_j\| < \delta/\sqrt{n_0}$ for $j = 1,\ldots,n_0$ and $\|P_m T_i P_m - T_i P_m\| < \epsilon$ for $i = 1,\ldots,n$. The first inequality implies that the set $\{P_m e_j\}_{j=1}^{n_0}$ is linearly independent. Let F_0 be the projection onto the subspace spanned by the vectors $\{P_m e_j\}$ so that $F_0 \le P_m$. I claim that $\|P_0 - F_0 P_0\| \le \delta$. To see this let f be any vector in the range of P_0, so $f = \sum_{j=1}^{n_0} \alpha_j e_j$. The following computation shows $\|P_0 - F_0 P_0\| \le \delta$.

$$\|f - F_0 f\|^2 = \|\sum_j \alpha_j (e_j - P_n e_j)\|^2 \le \sum_j |\alpha_j|^2 \, \|e_j - P_n e_j\|^2$$

$$\le \sum_j |\alpha_j|^2 \cdot \sum_j \|e_j - P_n e_j\|^2 \le \|f\|^2 \cdot n_0 (\delta/\sqrt{n_0})^2$$

$$= \|f\|^2 \delta^2.$$

Since $e_j = P_m^{\perp} e_j + P_m e_j$ we obtain $F_0 e_j = P_m e_j$ for $j = 1,\ldots,n_0$. Thus, the range of F_0 equals the range of $F_0 P_0$ and F_0 maps the range of P_0 onto the range of F_0. Let S represent the restriction of F_0 to the range of P_0. S is a linear operator mapping a space of finite dimension n_0 onto a space of the same finite dimension, so S is invertible. The next step is to find an upper bound for the norm of S^{-1}. Let $f \in \mathcal{R}(P_0)$. Then $\|F_0 f\| \ge \|f\| - \|f - F_0 f\| \ge \|f\| - \delta\|f\| = (1 - \delta)\|f\|$. If $f \in \mathcal{R}(P_0)$, then $F_0 f = Sf$ so $\|Sf\| \ge (1 - \delta)\|f\|$. Thus, $\frac{1}{1-\delta} \|Sf\| \ge \|f\| = \|S^{-1}(Sf)\|$. Hence $\|S^{-1}\| \le \frac{1}{1-\delta}$. The inequality $\|P_0 - F_0 P_0\| < \delta$ tells us that F_0 is close to P_0 on the range of P_0. We now want to show that F_0 is close to P_0 on the range of $P_0^{\perp}$. $\mathcal{R}(P_0)$ will denote the range of P_0.

To achieve this and let $f \perp \mathcal{R}(P_0)$ so that $P_0 f = 0$. Let $g = S^{-1}F_0 f$; $g \in \mathcal{R}(P_0)$ since S^{-1} maps the range of F_0 onto the range of P_0. Thus $F_0 g = S_q = F_0 f$. We are now able to make the following computations.

$$\|g\| = \|S^{-1}F_0 f\| \le \frac{1}{1-\delta}\|f\|.$$

$$\|F_0 f - P_0 g\| \le \|F_0 f - F_0 P_0 g\| + \|F_0 P_0 g - P_0 g\|$$

$$\le 0 + \delta\|g\|$$

$$\le \frac{\delta}{1-\delta}\|f\|.$$

Hence, $\|F_0 f\|^2 = (F_0 f, f) \le |(F_0 f - P_0 g, f)| + |(P_0 g, f)|$

$$\le \frac{\delta}{1-\delta}\|f\|^2 + 0.$$

Therefore, $\|F_0(1 - P_0)\| \le (\frac{\delta}{1-\delta})^{\frac{1}{2}}$. Combining this inequality with the previous inequality $\|P_0 - F_0 P_0\| \le \delta$, we obtain $\|P_0 - F_0\| \le \delta + (\delta/1-\delta)^{\frac{1}{2}}$. Set $\delta = \delta + (\delta/1-\delta)^{\frac{1}{2}}$. At this point δ is chosen so that $\delta < 1$ to enable us to apply lemma III-E which states there exists a unitary operator W satisfying $W^* P_0 W = F_0$ and $\|1 - W\| \le 2\sqrt{\delta}$. Set $P = WP_m W^*$. We know $F_0 \le P_m$ so $W^* P_0 W \le P_m$ and $P_0 \le WP_m W^* = E$. It now remains to show that $\|PT_i P - T_i P\| < \epsilon$ for $i = 1, \ldots, n$. We know $\|PT_i P - T_i P\| = \|(WP_m W^*)T_i(WP_m W^*) - T_i(WP_m W^*)\|$. Since the right hand side of this equation is a continuous function of W, we can make the above expression close to $\|P_m T_i P_m - T_i P_m\|$ by choosing W close to 1. However, $\|P_m T_i P_m - T_i P_m\|$ is dominated by δ so if we make sure $\delta < \epsilon/2$ (redefining δ if necessary) we will have the desired inequality.

We have now seen that all three definitions of quasitriangularity are equivalent. Halmos shows that the set of quasitriangular operators is closed. The next theorem shows that this property is also enjoyed by the set of quasitriangular n-tuples.

Theorem III-4. The set of quasitriangular n-tuples is closed.

Proof. Let $\{(T_1^k, T_2^k, \ldots, T_n^k)\}$ be a sequence of quasitriangular n-tuples converging to $(T_1, T_2, \ldots, T_n)$. First, I will show that $T_1, T_2, \ldots, T_n$ essentially commute. Since $T_i^k T_j^k - T_j^k T_i^k$ is compact and the compacts are a closed set it is sufficient to show that given $\epsilon > 0$ there exists a positive integer K such that $k \geq K$ implies $\|(T_i^k T_j^k - T_j^k T_i^k) - (T_i T_j - T_j T_i)\| < \epsilon$. We have the following string of inequalities.

$$\|(T_i^k T_j^k - T_j^k T_i^k) - (T_i T_j - T_j T_i)\|$$

$$= \|T_i^k T_j^k - T_i T_j^k + T_i T_j^k - T_j^k T_i^k + T_j T_i^k - T_j T_i^k - T_i T_j + T_j T_i\|$$

$$\leq \|(T_i^k - T_i)T_j^k\| + \|T_i(T_j^k - T_j)\| + \|(T_j - T_j^k)T_i^k\| + \|T_j(T_i - T_i^k)\|.$$

Let $M = \sup_k \|T_i^k\|$. Choose K sufficiently large so that for $k \geq K$, $\|T_i^k - T_i\| < \epsilon/4M$, $\|T_j^k - T_j\| < \epsilon/4\|T_i\|$, $\|T_j - T_j^k\| < \epsilon/3M$, and $\|T_i - T_i^k\| < \epsilon/4\|T_j\|$. For such a K, $k \geq K$ implies $\|(T_i^k T_j^k - T_j^k T_i^k) - (T_i T_j - T_j T_i)\| < \epsilon$. Next it must be shown that

$T_1, \ldots, T_n$ are mutually quasitriangular. This is accomplished by a slight modification of Halmos' argument. Let k_0 be sufficiently large so that $\|T_i^{k_0} - T_i\| < \epsilon/3$ for $i = 1, \ldots, n$. Definition III-2 tells us that given $\epsilon > 0$ and $P_0 \in \rho$ we can find $P \in \rho$ such that $P_0 \leq P$ and $\|T_i^{k_0}P - PT_i^{k_0}P\| < \epsilon/3$ for $i = 1, \ldots, n$. For each i we have $\|PT_iP - T_iP\| \leq \|PT_i^{k_0}P - PTP\| + \|PT_i^{k_0}P - T_i^{k_0}P\| + \|T_i^{k_0}P - T_iP\| < \epsilon$. Thus, $(T_1, \ldots, T_n)$ is a quasitriangular n-tuple.

Since the quasitriangular n-tuples are closed, the complement of the set is open. My next task will be to find the radius of nonquasitriangularity for the n-tuple $(T_1, T_2, \ldots, T_n)$. The following is adapted from the version for a single operator presented by Apostol, Foias, and Voiculescu [2].

Definition III-G. For any n-tuple T define $q(T)$ $= \liminf_{P \in \rho, P \to I} \max_{i=1,\ldots,n} \{\|PT_iP - T_iP\|\}$; q is called the modulus of quasitriangularity. We say that the projections $\{P_k\}$ implement the modulus of quasitriangularity for the n-tuple T if $P_k \to I$ and $\lim_{k \to \infty} \max_{i=1,\ldots,n} \{\|P_k^{\perp}T_iP_k\|\} = q(T)$.

Definition III-5. For any n-tuple T of essentially commuting operators let $r(t) = \inf\{\max_{1 \leq i \leq n} \|T_i - S_i\| : S$ is a quasitriangular n-tuple$\}$. $r(T)$ is the distance from T to the quasitriangular n-tuples.

Notice that an n-tuple T of essentially commuting operators is quasitriangular if and only if $r(T) = 0$. The following development will show that $r(T) = q(T)$.

Construction III-6. Our first task is to construct a sequence of finite dimensional subspaces X_k satisfying

1. $P_{X_k} \to I$,

2. $X_k + T_i X_k \subseteq X_{k+1}$ for $i = 1, \ldots, n$, and

3. $\lim_{k\to\infty} \max_{i=1,\ldots,n} \{\|P^{\perp}_{X_k} T_i P_{X_k}\|\} = q(T)$.

Let $\{P_k\}$ be an increasing sequence of projections implementing the modulus of quasitriangularity of T and let Y_k be the subspace $P_k\mathcal{H}$. Let $X_1 = Y_{k_1}$ where k_1 is chosen sufficiently large so that $\max_{i=1,\ldots,n} \{\|P^{\perp}_{Y_{k_1}} T_i P_{y_k}\|\} < \frac{1}{2} + q(T)$. Let X_2 be the subspace generated by S_2 and Y_{k_2} where S_2 is the subspace generated by X_1 and $T_i X_1$ for $i = 1, \ldots, n$ and where $k_2 > k_1$ is chosen sufficiently large so that $\|P^{\perp}_{Y_{k_2}} T_i P_{Y_{k_2}}\| < \frac{1}{2^2} + q(T)$ and

$d(Y_{k_2}, S_2) < \frac{1}{2^2} \|T_i\|$ for $i = 1, \ldots, n$ where $d(Y_{k_2}, S_2)$ $= \|(I - P_{Y_{k_2}}) P_{S_2}\|$. This choice is possible because $P_{Y_{k_2}} \to I$ and $\{P_{Y_k}\}$ implements the modulus of quasitriangularity of T. Each $x \in X_2$ can be expressed as $x_y + x_y^{\perp}$ where $x_y \in Y_{k_2}$, $x_y^{\perp} \in Y_{k_2}^{\perp}$, and $\|x_y^{\perp}\| < \frac{1}{2^2} \cdot \|T_i\|$ for $i = 1, \ldots, n$. Hence, for each i we have the following string of inequalities.

$$\|P^{\perp}_{X_2} T_i P_{X_2} x\| \le \|P^{\perp}_{X_2} T_i P_{X_2} x_y\| + \|P^{\perp}_{X_2} T_i P_{X_2} x_y^{\perp}\|$$

$$\le \|P^{\perp}_{Y_{k_2}} T_i P_{Y_{k_2}} x_y\| + \|T_i\| \cdot \|x_y^{\perp}\|$$

$$\le \frac{1}{2^2} + q(T) + \|T_i\| \cdot \frac{1}{2^2} \cdot \|T_i\|$$

$$\le \frac{1}{2} + q(T).$$

In general let X_k be the subspace generated by S_ℓ and Y_{k_ℓ} where S_ℓ is the subspace generated by $X_{\ell-1}$ and $T_i X_{\ell-1}$ for $i = 1,\ldots,n$ and where $k_\ell > k_{\ell-1}$ is chosen sufficiently large so that $d(Y_{k_\ell}, S_\ell) < 1/2^\ell \|T_i\|$ and $\|P^{\perp}_{Y_{k_\ell}} T_i P_{Y_{k_\ell}}\| < \frac{1}{2^\ell} + q(T)$ for $i = 1,\ldots,n$. Each $x \in X_\ell$ satisfies $x = x_y + x_y^{\perp}$ where $x_y \in Y_{k_\ell}$, $x_y^{\perp} \in Y^{\perp}_{k_\ell}$ and $\|x_y^{\perp}\| < \frac{1}{2^\ell}\|T_i\|$ for $i = 1,\ldots,n$. Thus we have

$$\|P^{\perp}_{Y_\ell} T_i P_{X_\ell}\| \le \|P^{\perp}_{X_\ell} T_i P_{X_\ell} x_y\| + \|P^{\perp}_{X_\ell} T_i P_{X_\ell} x_y^{\perp}\|$$

$$\le \|P^{\perp}_{Y_{k_\ell}} T_i P_{Y_{k_\ell}} x_y\| + \|T_i\| \cdot \|x_y^{\perp}\|$$

$$\le \frac{1}{2^\ell} + q(T) + \|T_i\| \cdot 1/2^\ell \|T_i\|$$

$$\le 1/2^{\ell-1} + q(T).$$

This completes the construction of the desired subspaces $\{X_k\}$.

Lemma III-7. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of operators on $\mathcal{H}$ which essentially commute and X_k an increasing sequence of subspaces satisfying

1. $P_{X_k} \to I$ and 2. $X_k + T_i X_k \subseteq X_{k+1}$ for $i = 1, \ldots, n$.

Set $T_i^{\ell} = \sum_{k=\ell}^{\infty} P_{X_k}^{\perp} T_i P_{X_k}$ for $i = 1, \ldots, n$. Then $(T_1 - T_1^{\ell}, T_2 - T_2^{\ell}, \ldots, T_n - T_n^{\ell})$ is a quasitriangular n-tuple.

Proof. Since $P_{X_k} T_i P_{X_k} = (P_{X_{k+1}} - P_{X_k}) T_i (P_{X_k} - P_{X_{k-1}})$ the summands have orthogonal initial spaces and orthogonal ranges; therefore, each T_i^{ℓ} is well defined. It is sufficient to show that $(T_i - T_i^{\ell}) P_{X_k} = P_{X_k}(T_i - T_i^{\ell}) P_{X_k}$ for $i = 1, \ldots, n$. For each i and $\ell < k$ we obtain the following string of equalities. Note that the condition $\ell < k$ does not matter as we could always define new projections $Q_m = P_{X_{k+m}}$ and $(T_i - T_i^{\ell}) Q_m = Q_m (T_i - T_i^{\ell}) Q_m$ for all m, $i = 1, \ldots, n$. First we see that

$$\begin{aligned} P_{X_k}^{\perp} T_i P_{X_k} &= (P_{X_{k+1}} - P_{X_k}) T_i P_{X_k} \\ &= P_{X_{k+1}} T_i P_{X_k} - P_{X_k} T_i P_{X_k} \\ &= P_{X_{k+1}} T_i P_{X_k} - P_{X_{k+1}} T_i P_{X_{k-1}} + P_{X_k} T_i P_{X_{k-1}} - P_{X_k} T_i P_{X_k}. \end{aligned}$$

We also need the following relation.

$$P_{X_k} T_1^{\ell} P_{X_k} = P_{X_k} \left(\sum_{n=\ell}^{\infty} P_{X_n}^{\perp} T_1 P_{X_n} \right) P_{X_k}$$

$$= \sum_{n=\ell}^{\infty} P_{X_k} P_{X_n}^{\perp} T_1 P_{X_n} P_{X_k}$$

$$= \sum_{n=\ell}^{k-1} P_{X_n}^{\perp} T_1 P_{X_n} .$$

We are now able to obtain the desired equality.

$$(T_1 - T_1^{\ell}) P_{X_k} = T_1 P_{X_k} - T_1^{\ell} P_{X_k}$$

$$= T_1 P_{X_k} - \left[\sum_{n=\ell}^{\infty} P_{X_n}^{\perp} T_1 P_{X_n} \right] P_{X_k}$$

$$= T_1 P_{X_k} - \sum_{n=\ell}^{\infty} (P_{X_{n+1}} - P_{X_n}) T_1 (P_{X_n} - P_{X_{n-1}}) P_{X_k}$$

$$= T_1 P_{X_k} - \sum_{n=\ell}^{k} (P_{X_{n+1}} - P_{X_n}) T_1 (P_{X_n} - P_{X_{n-1}}) P_{X_k}$$

$$= T_1 P_{X_k} - (I - P_{X_k}) T_1 P_{X_k} - \sum_{n=\ell}^{k-1} (P_{X_{n+1}} - P_{X_n}) T_1 (P_{X_n} - P_{X_{n-1}})$$

$$= P_{X_k} T_1 P_{X_k} - P_{X_k} T_1^{\ell} P_{X_k}$$

$$= P_{X_k} (T_1 - T_1^{\ell}) P_{X_k} .$$

The following lemma shows that $q(T)$ is a lower bound for $r(T)$.

Lemma III-8. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of essentially commuting operators in $\mathfrak{L}(\mathcal{H})$ and $S = (S_1, \ldots, S_n)$ a quasitriangular n-tuple of operators in $\mathfrak{L}(\mathcal{H})$. Then $q(T) \leq r(t)$.

Proof. $q(T) = q(T - S + S)$

$$= \liminf_{P \to I} \|(I - P)(T - S + S)P\|$$

$$\le \liminf_{P \to I} \|(I - P)(T - S)P\| + \liminf_{P \to I} \|(I - P)SP\|$$

$$\le \liminf_{P \to I} \|T - S\| + q(S)$$

$$= \|T - S\|.$$

Corollary III-9. $q(T) \le \inf \{\|T - S\| : S \text{ is quasitriangular}\}$.

Theorem III-10. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of essentially commuting operators in $\mathcal{L}(\mathcal{H})$. Then $q(T) = \inf \{\|T - S\| : S \text{ is quasitriangular}\}$.

Proof. After applying the previous corollary we need only show that $\inf \{\|T - S\| : S \text{ is quasitriangular}\}$ is bounded above by $q(T)$. Construct finite dimensional subspaces by the method described in construction III-6 so that:

1. $X_k + T_i X_k \subseteq X_{k+1}$ for $i = 1, \ldots, n$,

2. $P_{X_k} \to I$, and

3. $\max_{i=1,\ldots,n} \{\|P_{X_k}^{\perp} T_i P_{X_k}\|\} \to q(T)$ as $k \to \infty$.

Set $T_i^{\ell} = \sum_{j=\ell}^{\infty} P_{X_k}^{\perp} T_i P_{X_k}$. By the previous lemma $(T_1 - T_1^{\ell}, T_2 - T_2^{\ell}, \ldots, T_n - T_n^{\ell})$ is a quasitriangular n-tuple so $\inf \{\|T - S\| : S \text{ is quasitriangular}\} \le \|T - (T - T^{\ell})\|$ $= \|T^{\ell}\| = \sup_{k>\ell} \|P_{X_k}^{\perp} T P_{X_k}\|$. Therefore, $\inf \{\|T - S\| : S$ is quasitriangular$\} \le \lim_{k\to\infty} \sup_{k>\ell} \|P_{X_k}^{\perp} T P_{X_k}\| = q(T)$.

Let us now turn our attention to the detailed examination of two interesting examples of quasitriangular n-tuples. Let S^{2n-1} denote the unit sphere in $\mathbb{R}^{2n}$, $L^2(S^{2n-1})$ the space of square integrable complex valued functions on S^{2n-1}, and $H^2(S^{2n-1})$ the Hardy space consisting of the closure of the analytic polynomials, cf. [4]. We can construct an orthonormal basis by setting $e_k = \alpha_k Z_k$ where $k = (k_1,\ldots,k_n)$ is an n-tuple of nonnegative integers, $|k| = k_1 + k_2 + \ldots + k_n$, $k! = k_1!k_2!\ldots k_n!$, $\alpha_k = \frac{1}{\sqrt{2\pi^n}} \sqrt{\frac{(n+|k|-1)!}{k!}}$, and $Z^k = Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n}$ where Z is any point in $\mathbb{C}^n$. Let T_{Z_i} represent the Toeplitz operator that multiplies each element of $H^2(S^{2n-1})$ by Z_i and $T_{Z_i}^*$ its adjoint. Let us now examine the structure of these operators to determine the joint quasitriangularity of $(T_{Z_1}^*,\ldots,T_{Z_n}^*)$. First, let us fix i and then decompose $\mathcal{H}$ as an infinite direct sum $\bigoplus_k \mathcal{H}_k^i$ so that each summand is invariant under T_{Z_i}. This is accomplished by letting $\mathcal{H}_k^i$ be the subspace spanned by the basis elements e_k where k_j is fixed for $j \ne i$ and k_i in any nonnegative integer. Observe that

$$
\begin{aligned}
T_{Z_i} e_k &= T_{Z_i} \alpha_k (Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n}) \\
&= \alpha_k (Z_1^{k_1} Z_2^{k_2} \ldots Z_n^{k_n}), \text{ so } T_{Z_i} \mathcal{H}_k^i \subseteq \mathcal{H}_k^i.
\end{aligned}
$$

We can think of the basis elements e_k forming a square array with k_i increasing along a vertical or horizontal line determined by the integers k_j, $j \neq i$. Each $\mathcal{H}_k^i$ is spanned by the elements on that line. More formally, $\mathcal{H} = \oplus_k \mathcal{H}_k^i$ where $k = (k_1,\ldots,k_n)$ is an n-tuple of nonnegative integers and $\mathcal{H}_k^i$ is the subspace spanned by e_k satisfying the condition k_j is fixed for $j \neq i$ and $k_i = 0,1,2,\ldots$. Since each $\mathcal{H}_k^i$ is invariant under T_{Z_i}, the matrix of T_{Z_i} with respect to the decomposition is block diagonal; $T_{Z_i} = \oplus_k T_{Z_i}|_{\mathcal{H}_k^i}$.

<u>Proposition III-11.</u> Each block is a weighted shift.

<u>Proof.</u> Let $e_k \in \mathcal{H}_k^i$, $k = (k_1,\ldots,k_n)$ and $k^i \pm 1 = (k_1,k_2,\ldots,k_{i-1},k_i \pm 1,k_{i+1},\ldots,k_n)$. Then

$$T_{Z_i}e_k = T_{Z_i}\left\{\left[1/\sqrt{2\pi^n} \cdot \sqrt{\frac{(n+|k|-1)!}{k!}}\right] Z^k\right\}$$

$$= \frac{1}{\sqrt{2\pi^n}} \cdot \sqrt{\frac{(n+|k|-1)!}{k!} \cdot \frac{(n+|k|)(k_i+1)}{(n+|k|)(k_i+1)}}\ Z^{k^i+1}$$

$$= \sqrt{\frac{k_i+1}{n+|k|}}\ \frac{1}{\sqrt{2\pi^n}}\ \sqrt{\frac{(n+|k|)!}{(k^i+1)!}}$$

$$= \sqrt{\frac{k_i+1}{n+|k|}} \cdot e_{k^i+1}$$

Thus, $T_{Z_i}e_k = \alpha_k e_{k^i+1}$ where $\alpha_k = \sqrt{\frac{k_i+1}{n+|k|}}$ and $T_{Z_i}|_{\mathcal{H}_k^i}$ is a weighted shift.

Using the preceeding computation it is easy to determine the value of $T^*_{Z_i}$ on each basis element e_k.

$$(\star)\quad \left(T_{Z_i}\big|_{\mathcal{H}^i_k}\right)^* e_k = \begin{cases} 0 & k_i = 0 \\ \alpha^i_{k-1}\, e^i_{k-1} & k_i \neq 0 \end{cases}$$

Proposition III-12. The n-tuple $(T^*_{Z_1}, T^*_{Z_2}, \ldots, T^*_{Z_n})$ is quasitriangular.

Proof. First, note that whenever a Hilbert space is spanned by common eigenvectors of an n-tuple of essentially commuting operators, the n-tuple must be quasitriangular. Construct a sequence of finite rank projections converging strongly to the identity to implement the joint quasitriangularity of $T^*_{Z_1}, \ldots T^*_{Z_n}$ by defining P_m as projection onto the subspace spanned by $\langle e_k \rangle_{|k| \le m}$. It follows from $(\star)$ that the range of P_m is invariant under each $T^*_{Z_i}$. Furthermore, direct computation shows that the $T^*_{Z_i}$ commute. Hence, $(T^*_{Z_1}, \ldots, T^*_{Z_n})$ is a quasitriangular n-tuple.

It is interesting to examine the individual operators T_{Z_i} in the n-tuple $(T_{Z_1}, \ldots, T_{Z_n})$. Various mathematicians showed that it is possible for the direct sum of two operators to be quasitriangular when neither of the summands is quasitriangular. The following proposition is due to Apostol, Foias, and Voiculescu.

Proposition III-4. Let $V \in \mathcal{L}(\mathcal{H})$ be a non-unitary isometry and $\alpha \neq \beta$ real numbers. If we set $T_{\alpha,\beta} = (V + \alpha) \oplus (V^* + \beta)$ then $T_{\alpha,\beta}$ and $T^*_{\alpha,\beta}$ are nonquasitriangular.

Proof. See [1, p.168]

The next proposition provides an example of a quasitriangular operator that can be written as an infinite direct sum of non-quasitriangular operators.

Proposition III-13. The operator T_{Z_i} is quasitriangular but the restriction of T_{Z_i} to $\mathcal{H}_k^i$ is a nonquasitriangular operator.

Proof. Let $T_{Z_i} = \begin{pmatrix} \ddots & & \\ & T_i^k & \\ & & \ddots \end{pmatrix}$ where $T_{Z_i}^k = T_{Z_i}|_{\mathcal{H}_k^i}$.

The adjoint of the operator $T_{Z_i}^k$ has nonzero kernel since it sends the basis element $e_{k_1,k_2,\ldots,k_{i-1};\ 0,k_{i+1},\ldots,k_n}$ to zero. By showing that $T_{Z_i}^k$ is bounded below we will be able to apply Douglas and Pearcy's theorem [7, p.177] which states that if an operator is bounded below and its adjoint has nontrivial kernel, then the operator is not quasitriangular. We know from a previous computation that

$$T_{Z_i}^k e_k = \alpha_k e_{k^i+1} \quad \text{where } \alpha_k = \frac{k_i+1}{n+|k|} = \frac{(k_i+1)(n+|k|-k_i)}{(n+|k|)(n+|k|-k_i)}$$

$$= \frac{(n+|k|-k_i)k_i + (n+|k|-k_i)}{(n+|k|)(n+|k|-k_i)} \geq \frac{k_i + n + |k| - k_i}{(n+|k|)(n+|k|-k_i)}$$

$$= \frac{n+|k|}{(n+|k|)(n+|k|-k_i)} = \frac{1}{n+|k|-k_i} = \frac{1}{(n+k_1+\cdots+k_{i-1}+\cdots+k_n)}.$$

The last expression is constant in $\mathcal{H}_k^i$ since in this subspace k_j

is fixed for $j \neq i$. The next computation shows that $T^k_{Z_i}$ is bounded below on the basis elements e_k. $\|T^k_{Z_i} e_k\| = \|\alpha_k e_{k^i+1}\| = |\alpha_k| \cdot \|e_{k^i+1}\| = |\alpha_k| \cdot \|e_k\| \geq \frac{1}{n+|k|-k_i} \cdot \|e_k\|$. To show $T^k_{Z_i}$ is bounded below on all of $\mathcal{H}^i_k$, let $x \in \mathcal{H}^i_k$. Then $x = \sum_\ell X_\ell e_\ell$, so that $\|T^k_{Z_i} X\| = \|T^k_Z (\sum_\ell X_\ell e_\ell)\|$

$$= \left\|\sum_\ell X_\ell \sqrt{\frac{\ell_i+1}{n+|\ell|}}\, e_\ell\right\| = \left\{\sum_\ell \left(X_e \sqrt{\frac{\ell_i+1}{n+|\ell|}}\right)^2\right\}^{\frac{1}{2}}$$

$$\geq \left\{\sum_\ell \left(X_\ell \frac{1}{n+|\ell|-\ell_i}\right)^2\right\}^{\frac{1}{2}} = \frac{1}{n+|\ell|-\ell_i} \cdot \left(\sum_\ell X_\ell^2\right)^{\frac{1}{2}} = \frac{1}{n+|\ell|-\ell_i} \cdot \|X\|.$$

We are now able to conclude that $T^k_{Z_i}$ is not quasitriangular. Since the essential spectrum of T_{Z_i} is a disk we know that T_{Z_i} must be quasitriangular.

Is the n-tuple $(T_{Z_1}, \ldots, T_{Z_n})$ quasitriangular? For even n an affirmative answer follows from the theory of extension. For arbitrary n, Voiculescu provided the answer in a private communication. He indicated the following proof. We proceed by defining the strong left essential spectrum.

<u>Definition III-I.</u> The strong left essential spectrum of an n-tuple T, denoted by $\sigma_{s\ell e}(T)$ is a subset of $\mathbb{C}^n$; λ is in $\sigma_{s\ell e}(T)$ if there exists an orthonormal sequence $\{\phi_j\}_{j \in N}$ such that $\lim_{m\to\infty}(\max_{1\leq j\leq n} \|(T_j - \lambda_j)\phi_m\|) = 0$.

Voiculescu, claims that for any n-tuple $T = (T_1, \ldots, T_n)$ of essentially commuting operators and an n-tuple $Y = (Y_1, \ldots, Y_n)$

of commuting normal operators satisfying $\sigma(Y) = E(\sigma_{s\ell e}(T))$ where $E(\sigma_{s\ell e}(T))$ denotes the least Stein compact set containing $\sigma_{s\ell e}(T)$, $q(T) = q(T \oplus Y)$. For the sake of completeness I will outline the construction of the proof Voiculescu presents.

The following two propositions are needed.

Proposition III-J. Let $\lambda \in \sigma_{s\ell e}(T)$ and let $\mathcal{H}^1$ be another Hilbert space, $X \subset \mathcal{H}$, $Y \subset \mathcal{H} \oplus \mathcal{H}^1$ be finite dimensional subspaces such that $Y \supset X \oplus 0$. Then there is a finite dimensional subspace $K \subset \mathcal{H} \oplus \mathcal{H}^1$, $K \supset Y + (T_j \oplus \lambda_j I)Y$ for $j = 1,\ldots,n$ and an isometry $U : K \to \mathcal{H}$ such that for $Z = U(Y)$ and $W = U|_Y$ we have $\|U(T_j \oplus \lambda_j I)|_Y W^*|_Z - T_j|_Z\| \le \epsilon$ for $j = 1,\ldots,n$ where $\epsilon > 0$ is given. In particular $|\|(I - P_Z)T_j P_Z\| - \|(I - P_Y)(T_j \oplus \lambda_j I)P_Y\|| \le \epsilon$, $\|W(P_Y(T_j \oplus \lambda_j I)|Y) W^*|_Z - P_2 T|_2\| \le \epsilon$, and $q(T \oplus \lambda) = q(T)$.

Proof [12]. The proof begins by replacing T by $T - \lambda$ if necessary so that it can be assumed that $\lambda = 0$. Then the finite dimensional subspaces X_1, X_2, and X_3 are defined so that $X_1 \subset X_3 \subset \mathcal{H}$, $X_2 \subset \mathcal{H}^1$, $X_1 \oplus X_2 \supset Y$, and $T_j(X_1) \subset X_3$ for $j = 1,\ldots,n$. The desired subspace K is defined to be $X_3 \oplus X_2$. The existence of an isometry $V : X_2 \to \mathcal{H}$ satisfying $\|T_j P_{V(X_2)}\| \le \epsilon$ and $V(X_2) \perp X_3$ is known since 0 is in $\sigma_{s\ell e}(T)$. The isometry $U : K \to \mathcal{H}$ is defined to be the inclusion map on

X_3 and V on X_2. All of the desired results are obtained by a long string of inequalities based on the properties of the above subspaces and maps.

The next proposition relates the distance between the Taylor spectrum [12] and the strong left essential spectrum to the modulus of quasitriangularity.

<u>Proposition III-K</u>. Let $N \in L^n(\mathcal{H})$ be an n-tuple of normal operators and let K denote the Taylor spectrum of N. Let d denote the distance $d(Z,Z^1) = \max_{1 \le j \le n} |Z_j - Z_j^1|$ in $\mathbb{C}^n$ and suppose that $\sup\{d(Z,\sigma_{s\ell e}(T)) \, | Z \in K\} < \epsilon$. Then $|q(T \oplus) - q(t)| < \epsilon$.

<u>Proof [12]</u>. It is possible to find $Z^{(k)} \in \sigma_{s\ell e}(T)$ and pairwise disjoint Borel subsets Σ_k of K satisfying $\bigcup_{k=1}^{\infty} \Sigma_k = K$ and $d(Z^{(k)}, \Sigma_k) < \epsilon$. Define the n-tuple $N^1 = (N_1^1, N_2^1, \ldots, N_n^1)$ by $N_j^1 = Z_j^{(1)} P_1 + Z_j^{(2)} P_2 + \cdots + Z_j^{(n)} P_n$ where P_n denotes the spectral projection of N corresponding to Σ_k. We can now conclude that $\|N - N^1\| < \epsilon$ and $q(T \oplus N^1) = q(T \oplus Z^{(1)} \oplus \cdots \oplus Z^{(n)})$ $= q(T)$ by applying the previous proposition.

It is now possible to prove the desired result.

<u>Theorem III-L</u>. Let $X \in L^n(\mathcal{H})$ and let Y be a normal n-tuple of commuting operators such that $\sigma(Y) = E(\sigma_{s\ell e}(X))$. Then $q(X) = q(X \oplus Y)$.

Proof [12]. The inequality $q(X) \geq q(X \oplus Y)$ is trivial so it is necessary to show the reverse inequality. The proof is based on the construction of the following objects. The open neighborhood Λ of $\sigma_{s\ell e}(X)$ is a subset of $\mathbb{C}^n$ satisfying $\sup_{x \in \Lambda} d(Z, \sigma_{s\ell e}(X)) < \epsilon$. For an ordinal p we construct the open set Λ_p using transfinite induction. Let $\Lambda_0 = \Lambda$ and $\Lambda_p = \bigcup_{p'<p} \Lambda p'$ if p is a limit ordinal. In the case $p = p' + 1$, let $\Lambda_p = \pi(\tilde{\Lambda}_{p'})$ where $\tilde{\Lambda}_{p'}$ is the envelope of holomorphy of Λ and π is the projection obtained by extending the coordinate functions from $\Lambda_{p'}$ to $\tilde{\Lambda}_{p'}$. Define H_p as the direct sum of a countable infinity of copies of $L^2(\Lambda_p)$ and $Z_p \in L^n(H)$ as the normal n-tuple given by multiplication by $Z_1, \ldots, Z_n$. $H_{p'}$ denotes the subspace of antianalytic functions to $\mathcal{H}_p$ and $\mathcal{H}_{p''} = \mathcal{H}_p \ominus \mathcal{H}_{p'}$. Note that $\mathcal{H}_{p''}$ is an invariant subspace of Z_p. $Z_{p'}$ denotes the compression of Z_p to $H_{p'}$ and $Z_{p''}$ the restriction of Z_p to $\mathcal{H}_{p''}$. By using the preceeding proposition III-K it is sufficient to show $q(X) \leq q(X \oplus Z_p) + \epsilon$ for all p. This is done by repeated use of that proposition and approximation arguments.

Now let us return to the n-tuple of Toeplitz operators $T = (T_{Z_1}, \ldots, T_{Z_n})$ and apply these results. T is an n-tuple of essentially commuting operators whose essential spectrum is the sphere. It follows by applying work of BDF [3, p.68] that under these conditions $\sigma_{s\ell e}(T) = S^{2n-1} = \{\lambda \in \mathbb{C}^n : |\lambda| = 1\}$. The smallest compact Stein set containing the sphere, denoted

$E(S)$, can be thought of as the smallest domain of holomorphy containing S. Thus $E(S) = B = \{\lambda \in \mathbb{C}^n : |\lambda| \leq 1\}$. Using the work of BDF on extensions, we know that Ext(B) is trivial since B is contractable. Therefore, any n-tuple of essentially normal, essentially commuting operators with joint essential spectrum equal to B must be quasitriangular. Consider once again the n-tuple $T = (T_{Z_1}, \ldots T_{Z_n})$. Let Y be a normal n-tuple of operators satisfying $\sigma(Y) = \sigma_{\ell e}(Y) = E(\sigma_{\ell e}(T)) = B$. Then Voiculescu's result enables us to conclude that $q(T) = q(T \oplus Y) = 0$ since $T \oplus Y$ generates the trivial extension.

Examining direct sums more closely we find that the quasitriangularity of a direct sum implies the quasitriangularity of each summand under certain circumstances. A brief introduction to the notion of polynomial convexity is necessary.

Definition III-M [8]. A domain $D \subseteq \mathbb{C}^n$ is polynomially convex if for every compact subset K of D, the set $\hat{K} = \{Z \in \mathbb{C}^n : |f(Z)| \leq \|f\|_K$ for all polynomials $f\}$ is contained in D. $\hat{K}$ is called the polynomially convex hull of K. K is polynomially convex if $K = \hat{K}$.

Theorem III-N. (Oka-Weil approximation [8].) Let K be a compact polynomially convex subset of $\mathbb{C}$. Then every function analytic in a neighborhood of K can be approximated uniformly on K by polynomials.

Lemma III-14. The cross product of n polynomially convex sets in polynomially convex.

Proof. Let $K_1, \ldots, K_n$ be polynomially convex subsets of $\mathbb{C}^n$. To show $\overset{n}{\underset{i=1}{X}} K_i$ is polynomially convex, I must show that $\overset{n}{\underset{i=1}{X}} K_i = S$ where $S = \{(Z_1, \ldots, Z_n) : |p(Z)| \leq \sup_{Z \in \overset{n}{\underset{i=1}{X}} K_i} |p(Z)|$ for all polynomials $p\}$. Clearly, $\overset{n}{\underset{i=1}{X}} K_i \subseteq S$. Let $Z = (Z_1, \ldots, Z_n) \in S$ and ${}^i p$ be any polynomial defined on K_i. Then ${}^i p$ can be considered as a polynomial ${}^i p^*$ defined on the cross product and independent of the Z_j variable for $j \neq i$. We know $|{}^i p(Z_i)| = |{}^i p^*(Z_1, \ldots, Z_n)|$ $\leq \sup_{Z \in \overset{n}{\underset{i=1}{X}} K_i} |{}^i p^*(Z)| = \sup_{Z_i \in K_i} |{}^i p(Z_i)|$. Hence, $Z_i \in K_i$. Similar reasoning shows $Z_j \in K_j$ for $j \neq i$.

Proposition III-15. If $T_1 \oplus T_2$ is quasitriangular and $\sigma_\ell(T_2)$ can be separated by open contractable sets, then T_1 and T_2 are quasitriangular.

Proof. It is sufficient to show that whenever $T_i - \lambda$ is Fredholm, $\operatorname{ind}(T_i - \lambda) \geq 0$ for $i = 1,2$. Let $i = 1$. Suppose $T_1 - \lambda$ is Fredholm. We know that we can separate $\sigma_e(T_1)$ and $\sigma_e(T_2)$ by open contractable sets O_1 and O_2. Suppose $\lambda \in O_1$. Then $\operatorname{ind}(T_2 - \lambda) = 0$, but $\operatorname{ind}(T_1 - \lambda) + \operatorname{ind}(T_2 - \lambda) \geq 0$ so $\operatorname{ind}(T_1 - \lambda)$ $\geq -\operatorname{ind}(T_2 - \lambda) = 0$. Now suppose $\lambda \notin O_1$. Then $\operatorname{ind}(T_1 - \lambda) = 0$. An identical argument works for $i = 2$.

We need to use the notion of polynomial convexity for a condition on the spectrum to see when the quasitriangularity of a 2-tuple of direct sums implies the quasitriangularity of the 2-tuples of summands. Note that for subsets of the plane polynomial convexity is equivalent to simple connected.

Proposition III-16. If $(S_1 \oplus S_2, T_1 \oplus T_2)$ is a quasitriangular 2-tuple such that $\sigma_e(S_1,T_1)$ and $\sigma_e(S_2,T_2)$ can be separated by polynomially convex sets P_1 and P_2 then (S_1,T_1) and (S_2,T_2) are quasitriangular 2-tuples.

Proof. Define analytic functions f and g by setting $f(Z) = Z$ on $\sigma_e(S_1,T_1)$, $f(Z) = 0$ on $\sigma_e(S_2,T_2)$, $g(Z) = Z$ on $\sigma_e(S_2,T_2)$, and $g(Z) = 0$ on $\sigma_e(S_1,T_1)$. The polynomial convexity of $P_1 \cup P_2$ implies that f and g can be uniformly approximated by polynomials $\{f_n\}$ and $\{g_n\}$ respectively. Using the functional calculus, $(f_n(S_1 \oplus S_2), f_n(T_1 \oplus T_2))$ and $(g_n(S_1 \oplus S_2), g_n(T_1 \oplus T_2))$ are quasitriangular 2-tuples. Since the quasitriangular 2-tuples form a closed set, $\lim_{n\to\infty}(f_n(S_1 \oplus S_2), f_n(T_1 \oplus T_2)) = (S_1,T_1)$ and $\lim_{n\to\infty}(g_n(S_1 \oplus S_2), g_n(T_1 \oplus T_2)) = (S_2,T_2)$ are quasitriangular.

In the finite dimensional case every operator is triangular; however, every n-tuple is not jointly triangular. The following proposition shows that in this setting commutivity is a sufficient condition for joint triangularity. First, a lemma.

<u>Lemma III-17</u>. Let $T_1, \ldots, T_n$ be commuting linear operators on a finite dimensional Hilbert space. Then $T_1, \ldots, T_n$ have a common invariant subspace of dimension one.

<u>Proof</u>. For $n = 1$, the invariant subspace is that spanned by any eigenvector. Assume the hypothesis is true for $T_1, \ldots, T_{n-1}$ ane let the desired subspace be spanned by the vector X, so that $T_i X = \lambda_i X$ for $i = 1, \ldots, n-1$. Consider the set $S = \{X, T_n X, T_n^2 X, \ldots, T_n^{\ell} X\}$ where ℓ is the largest integer so that S is linearly independent. T_n restricted to the subspace generated by S has an eigenvector $y = \sum_{k=0}^{\ell} \alpha_k T_n^k X$. For $i = 1, \ldots, n-1$,

$$T_i y = T_i\left(\sum_{k=0}^{\ell} \alpha_k T_n^k X\right) = \sum_{k=0}^{\ell} \alpha_k T_n^k T_i X = \lambda_i\left(\sum_{k=0}^{\ell} \alpha_k T_n^k X\right) = \lambda_i y.$$

Thus, the subspace spanned by the vector y is invariant under $T_1, \ldots, T_n$.

<u>Proposition III-18</u>. Let $T_1, \ldots, T_n$ be mutually commuting operators on $\mathcal{H}$, $\dim \mathcal{H} = n < \infty$. Then there exists an increasing sequence of subspaces $\{M_i\}_{i=1}^{n}$, $\dim M_i = i$ such that each M_i is invariant under each T_j.

<u>Proof</u>. For $\dim \mathcal{H} = 1$ the statement is trivially true. Assume the hypothesis for $\dim \mathcal{H} = k - 1$. Let $\dim \mathcal{H} = k$. By the previous lemma there exists a subspace M_1 of dimension one which is invariant under each T_i, $i = 1, \ldots, n$. Consider the quotient space $\mathcal{H}/M_1$ which has dimension $k - 1$. The inductive hypothesis holds so there exist subspaces $\{N_j\}_{j=1}^{k-1}$ of $\mathcal{H}/M_1$ with $\dim N_j = j$ for $j = 1, \ldots, k - 1$ that are invariant under $T_i|_{\mathcal{H}/M_1}$

for $i = 1,\ldots,n$. Define $M_{\ell+1}$ as the subspace spanned by N_ℓ and M_1 for $\ell = 1,\ldots,k-1$. The dimension of $M_{\ell+1}$ is $\ell + 1$ and $M_{\ell+1}$ is invariant under each T_i so the conditions are satisfied.

When we allow the dimension of $\mathcal{H}$ to become countably infinite, commutivity is not a sufficient condition for joint quasitriangularity. However, if we add an additional hypothesis we obtain the desired conclusion.

<u>Proposition III-M</u>. If $T_1, T_2, \ldots, T_n$ are commuting normal operators on $\mathcal{H}$, then $(T_1,\ldots,T_n)$ is a quasitriangular n-tuple.

<u>Proof</u>. This result follows from the BDF proof that the trivial extension is unique. See Corollary II-F.

Let us now examine the joint quasitriangularity of an n-tuple $H = (H_1,\ldots,H_n)$ of self-adjoint operators which almost commute. Let $X = \sigma_e(H)$. Consider any real polynomial p defined on the joint essential spectrum X of H, a subset of $\mathbb{R}^n$. The assumptions on $H_1,\ldots,H_n$ imply that $p(H_1,\ldots,H_n)$ (which is well defined up to a compact pertubation) is an essentially self-adjoint operator, and therefore, the index of $p(H_1,\ldots,H_n) - \lambda$ is zero whenever $\lambda \notin \sigma_e(p(H_1,\ldots,H_n))$. Hence, by applying the Stone-Weierstrass Theorem we can conclude that the map $\gamma_1 : \text{Ext}\, X \to \text{Hom}(\pi^1(X),\mathbb{Z})$ is identically zero. Since we know that for subsets of $\mathbb{R}^3$ γ_1 is an isomorphism, it is possible to conclude that any 3-tuple of self-adjoint almost commuting operators which is jointly quasitriangular generate the trivial extension.

In a personal communication from Larry Brown, he shows that the situation differs drastically for 4-tuples of operators. Using the example described in [6,p.516], Brown defines operators S and T which are essentially normal and essentially commute. He constructs the operators so that the real and imaginary parts, S_r, S_i, T_r, and T_i respectively, do not form a quasitriangular 4-tuple, but $(S_i \oplus S_i, S_r \oplus S_r, T_i \oplus T_i, T_r \oplus T_r)$ is a quasitriangular 4-tuple and generates the trivial extension. The example is based on the fact that Ext(X) exhibits torsion. Thus, in higher dimensions there are other phenomena which occur and must be considered. Kaminker and Shoeket [10] discuss higher order topological invariants in their work on the behavior of Ext(X) for higher dimensional spaces.

IV. EXAMINATION OF 2-TUPLES OF OPERATORS

We will now examine a restricted class of operators, 2-tuples of essentially commuting operators; one operator will be essentially unitary and one operator will be essentially self-adjoint. Since the essential spectrum of the unitary is a subset of the unit circle and the essential spectrum of the self-adjoint is real, the joint essential spectrum is a subset of a cylinder. The joint essential spectrum can be considered as a subset of $\mathbb{T} \times [0,1]$ with only slight modification; normalize the self-adjoint operator.

The following example shows the relevance of an index condition for the joint quasitriangularity of two operators.

Let U_+ denote the forward shift and U_+^* the backward shift. Consider the 2-tuple $(U,H) = ((U_+^*)^2, \frac{1}{3}I) \oplus (U_+, \frac{2}{3}I)$. The joint essential spectrum of (U,H) is the union of two circles, $\sigma_e(U,H) = \mathbb{T} \times \{\frac{1}{3}, \frac{2}{3}\}$. Since $\operatorname{ind}(U_+) = -1$, we know U_+ is not quasitriangular. Hence, $(U_+, \frac{2}{3}I)$ is not quasitriangular. Since $\sigma_e(U_+^*)^2, \frac{1}{3}I)$ and $\sigma_e(U_+, \frac{2}{3}I)$ can be separated by polynomially convex sets, proposition III-16 implies that (U,H) is not quasitriangular.

Let us now move to a more general setting by examining a pair of operators (U,H) satisfying the following conditions: $\pi(U)$ is unitary, $\pi(H)$ is self-adjoint, $[U,H] \in \mathcal{K}$, and

$\sigma_e(U,H) = X \subseteq \mathbb{T} \times [0,1]$. The question to consider is when is (U,H) a quasitriangular 2-tuple?

First, we will study the structure of the joint essential spectrum, the set X. We can examine X as a subset of the cylinder which sits in $\mathbb{R}^3$ or we can consider the subset $\tilde{X}$ of $\mathbb{R}^2$ homeomorphic to X obtained in the following manner. Let $p : \mathbb{T} \times [0,1] \to \mathbb{R}^2$ be defined by $p(e^{i\theta},\lambda) = e^{i\theta}(\lambda + 1)$. We obtain an annulus as seen in diagram IV-1. Let $p(X) = \tilde{X}$, a subset of the Z plane.

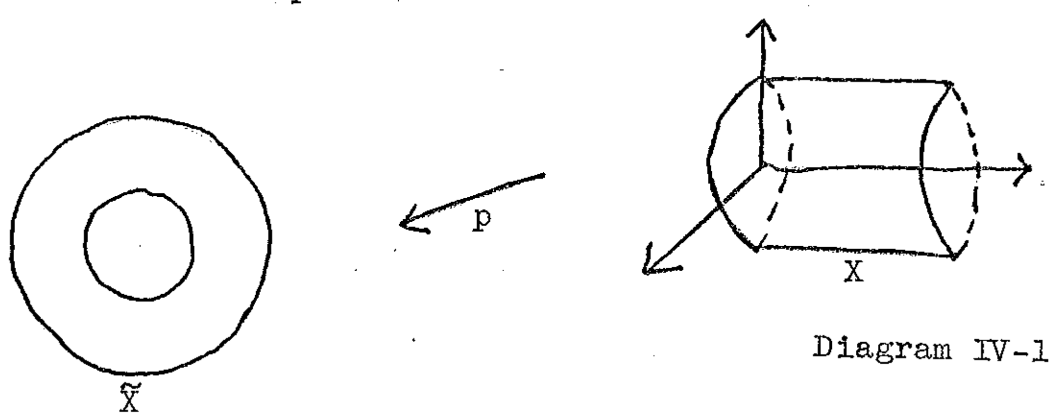

Diagram IV-1

Note that X is homeomorphic to $\tilde{X}$ so $\text{Ext}(X) = \text{Ext}(\tilde{X})$. Let τ represent the extension generated by (U,H) in Ext X and $\tilde{\tau}$ the corresponding extension in $\text{Ext}(\tilde{X})$. Since $\tilde{X}$ is a subset of $\mathbb{R}^2$ the extension $\tilde{\tau}$, and correspondingly the extension τ is determined solely by the index of operators in the algebra generated by U, H, I, and $\mathcal{K}$. Recall that $\gamma_1 : \text{Ext}(\tilde{X}) \to \text{Hom}(\pi^1(\tilde{X}),\mathbb{Z})$ and since $\tilde{X} \subseteq \mathbb{R}^2$, γ_1 is an isomorphism. The set $\mathbb{C}\backslash\tilde{X}$ can be written as $O_\infty \cup (\bigcup_{i\geq 0} O_i)$ where O_∞ represents the unbounded component and each O_i represents a bounded component of the complement of the spectrum.

<u>Definition IV-2</u>. Let (Z_1,λ_1) and (Z_2,λ_2) be points in $\mathbb{T} \times [0,1]\backslash X$. Let $p((Z_1,\lambda_1)) = p_1$ and $p((Z_2,\lambda_2)) = p_2$ be located in components O_1 and O_2 respectively where p denotes the homeomorphism from $\mathbb{T} \times [0,1]$ to $\mathbb{R}^2$. If $\lambda(p_1) > \lambda(p_2)$ where $\lambda(p_i) = \lambda_i$ we say $O_2 \leq O_1$.

<u>Construction IV-3</u>. Let p_1 and p_2 be defined as in the preceeding definition. Let K_{p_1} denote the line passing thru (0,1) and (Z_1,λ_1), let $L_{p_1p_2}$ denote the line joining (Z_1,λ_1) and (Z_2,λ_2) and let $\pi_{K_{p_1}}, \pi_{L_{p_1p_2}} : \mathbb{R}^3 \to \mathbb{C}$ denote the projections which send the lines K_{p_1} and $L_{p_1p_2}$ respectively to the origin. Geometrically we are projecting along these lines to the plane Z = 0. See diagram IV-4. Notice that $L_{p_1p_2} \cap X = \emptyset$ and $K_{p_1} \cap X = \emptyset$ due to the concexity of $\mathbb{T} \times [0,1]$; and therefore, $0 \notin \pi_{K_{p_1}}(X)$ and $0 \notin \pi_{L_{p_1p_2}}(X)$.

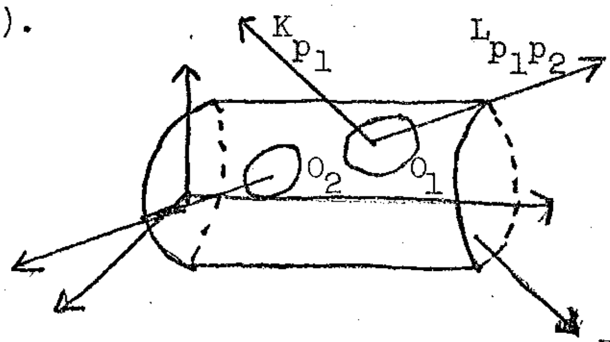

Diagram IV-4

To prove the major result, we need the following definition.

<u>Definition IV-5</u>. Two components O_1 and O_2 are similar if there exist points $p_1 \in O_1$ and $p_2 \in O_2$ such that $\lambda(p_1) = \lambda(p_2)$.

Definition IV-6. Two components O_1 and O_2 are said to be equivalent if there exists a sequence $\{O^K\}_{k=1}^{n}$ such that O^i is similar to O^{i+1}, $O^1 = O_1$ and $O^n = O_2$. Let $[O_i]$ represent the equivalence class which contains O_i.

Definition IV-7. Let p be any point in the component O of $\mathbb{C}\backslash\tilde{X}$. We define the function $f_O : \tilde{X} \to \mathbb{C}\backslash\{0\}$ by $f_O(Z) = Z - p$.

We are now able to prove the general result.

Theorem IV-8. Let (U,H) be a 2-tuple satisfying $\pi(U)$ is unitary, $\pi(H)$ is self-adjoint, $[U,H] \in \mathcal{K}$, and $X = \sigma_e(U,H) \subseteq \mathbb{T} \times [0,1]$. Then (U,H) is quasitriangular if and only if

1. For every component $O_i \in \mathbb{C}\backslash\tilde{X}$, $0 \le \gamma_1(\tau)[f_{O_i}]$, and
2. For every pair of components O_1 and O_2 with $O_2 \le O_1$, $\gamma_1(\tau)[f_{O_1}] \le \gamma_1(\tau)[f_{O_2}]$.

Proof. First let us assume (U,H) is a quasitriangular 2-tuple. Using the notation of construction IV-3 we know that $0 \notin \pi_{K_{p_1}}(X)$. These facts imply that $\pi_{K_{p_1}}(U,H)$ is Fredholm and $\pi_{K_{p_1}}(U,H)$ is quasitriangular where $\pi_K(U,H)$ is defined by the functional calculus up to a compact perturbation. Using similar reasoning $\pi_{L_{p_1p_2}}(U,H)$ is Fredholm and quasitriangular. But every Fredholm quasitriangular operator must have nonnegative index. Therefore $0 \le \text{ind } \pi_{K_{p_1}}(U,H)$ and $0 \le \pi_{L_{p_1p_2}}(U,H)$. But considering the orientation of the components O_1 and O_2 relative to the lines $L_{p_1p_2}$ and K_{p_1} yields that $[\pi_{L_{p_1p_2}}] = [f_{O_2}] - [f_{O_1}]$ and $[\pi_{K_{p_1}}] = [f_{O_1}]$. Therefore, $0 \le \text{ind}\ \pi_{L_{p_1p_2}}(U,H) = \gamma_1(\tau)\ [\pi_{L_{p_1p_2}}] = \gamma_1(\tau)[f_{O_2}] - \gamma_1(\tau)[f_{O_1}]$ and $0 \le \text{ind}\ \pi_{K_{p_1}}(U,H) = \gamma_1(\tau)[\pi_{K_{p_1}}] = \gamma_1(\tau)[f_{O_1}]$.

For the reverse implication we must show that if conditions 1 and 2 are satisfied then (U,H) is a quasitriangular 2-tuple. Using the fact that each component O_i is an open set condition 2 allows us to conclude that for any components O_1 and O_2 in the same equivalence class, $\gamma_1(\tau)[f_{O_1}] = \gamma_1(\tau)[f_{O_2}]$. The index is a nonnegative integer valued function that is decreasing as we move to the right along the cylinder. Let $[O_1],\ldots,[O_n]$ represent the equivalence classes of the components of $\mathbb{C}\backslash\tilde{X}$ where $n_i = \gamma_1(\tau)[f_{O_i}]$ and $n_i \le n_{i-1}$. For each equivalence class $[O_i]$ let $\lambda^i \inf = \inf\limits_{\substack{p\in O \\ O\in[O_i]}} \lambda(p)$ and $\lambda^i \sup = \sup\limits_{\substack{p\in O \\ O\in[O_i]}} \lambda(p)$. For (U,H) we can draw a "spectral picture" which shows the components and the indices. For example see diagram IV-9.

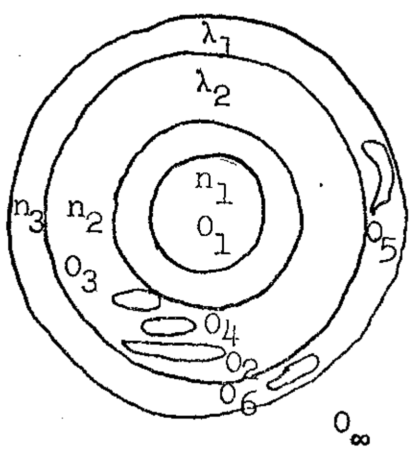

Diagram IV-9

I will now construct a quasitriangular 2-tuple of operators $(\overline{U},\overline{H})$ which generates the same extensions as (U,H). This will be accomplished if $(\overline{U},\overline{H})$ and (U,H) have the same indices and the same joint essential spectrum. It will then be possible to conclude that (U,H) is also quasitriangular. Let (U_0,H_0) generate a trivial extension with $\sigma_e(U_0,H_0) = X$. Consider the sequence of 2-tuples $\{((U^*)^{n_i-n_{i+1}}, \lambda_i I)\}_{i=1}^{\ell-1}$. The 2-tuple $((U^*)^{n_i-n_{i+1}}, \lambda_i I)$ has the circle of radius 1 with center $(\lambda_i,0)$ for its joint essential spectrum. Let $(\overline{U},\overline{H}) = (U_0,H_0) \oplus [\bigoplus_{i=1}^{\ell-1}((U^*)^{n_i-n_{i+1}}, \lambda_i I)]$. Then $(\overline{U},\overline{H})$ and (U,H) have the same "spectral picture" and thus generate the same extension. More formally, let $\overline{\tau}$ and τ represent the extensions generated by $(\overline{U},\overline{H})$ and (U,H) respectively. Then $[\overline{\tau}] = [\tau]$ since all corresponding indices are identical.

V. PROJECTIONS FOR THE FUTURE

In conclusion let us examine some of the potential settings for future study. It should be possible to extend the theory developed in this paper to subsets of more general manifolds. A simple generalization would be to certain types of surfaces of revolution. Given a pair of essentially normal, pairwise essentially commuting operators (T_1, T_2) whose joint essential spectrum X is a subset of $\mathbb{C} \times \mathbb{R}$ define $P = \{p(z,x) : p$ is a polynomial, $p(z,x) \neq 0$ on $X\}$. Consider P as a subset of $C(X)^{-1}$, the continuous invertible functions on X. The set P generates a subsemigroup π^+ of $\pi^1(X)$. We know that if (T_1, T_2) is quasitriangular, $f(T_1, T_2)$ is a quasitriangular operator for any $f \in \pi^+$. More formally, $\gamma_1(\pi^+) \subseteq \mathbb{Z}^+ \cup \{0\}$ is a necessary condition for the quasitriangularity of the pair (T_1, T_2). We conjecture that this is also a sufficient condition. The situation with no restrictions on the spectrum is much more complex due to torsion. Ultimately, we would hope to extend the results to n-tuples of essentially normal, pairwise essentially commuting operators with few, if any, restrictions on the structure of their joint essential spectrum.

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