SPECTRUM OF THE RANDOM WALK ON THE FUNDAMENTAL GROUP

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Frank David Eisenberg

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Abstract of the Dissertation

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In [5] Eells introduced the notion of a probability measure on $C_{ab}(M)$ - the space of continuous maps $[0,1] \to M$ with endpoints $a,b \in M$. Based on this he defined a symmetric probability distribution on the fundamental group.

We will consider the random walk based on the above mentioned probability distribution which, in general, is determined by its matrix of transition probabilities, $\|p_{ij}\|$. Here p_{ij} denotes the probability of going from one element in $\pi_1(M)$ to another in one step under the given distribution.

Letting $\lambda = \sup_{\overline{\lambda} \in \text{Spectrum of } \|p_{ij}\|} |\overline{\lambda}|$ our main result will be that if M is compact with strictly negative curvature then $\lambda < 1$.

A group G is called amenable if there exists a linear

functional, B, on the space of bounded real valued functions on G with

1) $\inf_{x \in G} f(x) \leq Bf \leq \sup_{x \in G} f(x)$ and

2) defining $g : G \to R$ by $g(y) = f(xyz)$, $x,y,z \in G$
$B(f) = B(g)$, f a bounded real valued function on G.

It turns out that $\lambda < 1$ is equivalent to $\pi_1(M)$ being not amenable. This in turn is equivalent to proving a certain Mean Value Inequality on the space of bounded real valued functions on $\pi_1(M)$. The functions we deal with in the proof will be the sum of angles in a triangle, one side of which remains on the invariant line, L_a, of $a \in \pi_1(M)$.

Once we know that the fundamental group is not amenable we obtain results regarding the structure of $\pi_1(M)$ based on the theory of amenable groups.

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Introduction

In this article we will be concerned with a compact, connected, oriented Riemannian manifold M, with curvature $K \leq c < 0$.

We consider a symmetric random walk on the fundamental group $\pi_1(M)$. Every such random walk is completely determined by its matrix of transition probabilities, $\|p_{ij}\|$, where p_{ij} denote the probability of going from one element to another in one step.

$$\text{Let } \lambda = \sup_{\bar{\lambda} \in \text{Spectrum of } \|p_{ij}\|} |\bar{\lambda}|$$

We would like to know, can λ take on the value one as raised in [9]. Our main result will be that it cannot. Our method of proof will be based on Stochastic Riemannian Geometry which is used to define the probability measure on $\pi_1(M)$.

From the above result we will obtain certain immediate consequences regarding the structure of the fundamental group and its subgroups. In particular we have that there exists a subgroup, H, of $\pi_1(M)$ such that H is not cyclic, abelian, or finite.

It should be remarked here that these results follow also from a theorem of Eberlein [3]. If $\lambda = 1$ then $\pi_1(M)$ cannot have a free noncyclic proper subgroup, but this contradicts Eberlein's main theorem.

§1.

Let M be an oriented compact Riemannian manifold

Let $C_{ab}(M)$ denote the space of continuous functions from $(0,1) \to (a,b)$ with endpoints a,b. We begin by introducing a measure, called Weiner Measure, on $C_{ab}(M)$.

Let Δ be the Laplace operator on M.

The heat operator, $L = \Delta - \frac{\partial}{\partial t}$, has a unique fundamental solution h. Some properties of h are as follows: [5]

1) $h : M \times M \times R(>0) \to R(>0)$ is a smooth function with $h_t : M \times M \to R(>0)$ and

$$h_t(x,y) = h_t(y,x) \qquad x,y \in M, \quad t > 0 .$$

2) $L_x(h) = L_y(h)$. Here L_x denotes L operating on the first variable; L_y is L operating on the second variable.

3) $\int_M h_t(x,y)dy = 1$ for all $x \in M$, $t > 0$.

4) $h_{t+s}(x,y) = \int h_t(x,z)h_s(z,y)dz$

h_t is called the heat density.

$C_{ab}(M)$ topologized with the topology of uniform convergence is a complete metric space. The metric is given by

$$\langle x,y \rangle = \sup d(x(s),y(s)) \quad s \in (0,1)$$

where $x,y \in C_{ab}(M)$ and d is the distance function with respect to the Riemannian metric on M.

Let $\underline{t} = (0 \le t_1 < t_2 < \cdots < t_n < 1)$

$M^{\underline{t}} = M \times M \times \cdots \times M$ n times.

Let B be a Borel subset of $M^{\underline{t}}$. Define $\rho_{\underline{t}} : C_{ab}(M) \to M^{\underline{t}}$ by

$$\rho_{\underline{t}}(x) = (x(t_1), x(t_2), \cdots, x(t_n))$$

The fibres, $\rho_{\underline{t}}^{-1}(B)$ generate the σ-algebra of Borel sets in $C_{ab}(M)$. [5]

Define a measure on the Borel sets of $C_{ab}(M)$ as follows: For any such fibre $\rho_{\underline{t}}^{-1}(B)$ let

$$\omega_{ab}(\rho_{\underline{t}}^{-1}(B)) = \int_B h_{t_1}(a,m_1)h_{t_2-t_1}(m_1,m_2)\cdots$$
$$\cdots h_{t_n-t_{n-1}}(m_{n-1},m_n)h_{1-t_n}(m_n,b)dm_1\cdots dm_n$$

We then have the following:

1) $\omega_{ab}(C_{ab}(M)) = h_1(a,b)$ by 4) above.

2) Every nonvoid open subset of C_{ab} has strictly positive ω_{ab} - measure.

3) ω_{ab} determines a countably additive measure, called Weiner measure, on $C_{ab}(M)$, since:

a) $w_{ab}(\rho_{\underline{t}}^{-1}(B)) \ge 0$ B a Borel set in M .

b) $w_{ab}(\bigcup_i(\rho_{\underline{t}}^{-1}(B)) = \int_{\bigcup_i B_i} h_{t_1}(a,m_1)\cdots h_{1-t_n}(m_n,b)dm_1\cdots dm_n$

$$= \sum_i \int_{B_i} h_1(a,m_1)\cdots h_{1-t_n}(m_n,b)dm_1\cdots dm_n = \sum_i w_{ab}(\rho_{\underline{t}}^{-1}(B_i))$$

c) $w_{ab}(C_{ab}(M)) = h_1(a,b)$; we normalize so that w_{ab} has total measure 1.

§2.

In this section we will give some preliminary facts regarding probability which we will need in order to define and solve our problem. In the latter part we will say what it means for a group to be amenable and also cite some results pertaining to amenable groups.

Definition: A Markov Process is a sequence of random variables (measurable functions), $\{X_n\}$ $n=1,\cdots$ on the Probability Space $(\Sigma,\mathcal{B},Pr)$ with the following property:

$$Pr[X_{n+1}=c_{n+1}|X_1=c_1,\cdots,X_n=c_n] = Pr[X_{n+1}=c_{n+1}|X_n=c_n]$$

where $c_i \in S$ the State Space and $Pr[X_i=c_i] > 0$.

In other words the conditional probability of an event knowing the states at previous moments is the same as that knowing just the last state.

Definition: A Markov Chain is a Markov Process with a denumerable number of states.

To be given a random walk means to know for each point of the State Space a set of probabilities for transition to another point of the space.

Definition: For any Markov Chain $\{X_n\}$ $n=1\cdots$ we define a row vector and a square matrix $\|p_{ij}\|$ by:

$$\pi_1 = Pr[X_1=c_i],\ p_{ij} = Pr[X_{n+1}=c_j | X_n=c_i]$$

where $Pr[X_n=c_i] > 0$.

$\|p_{ij}\|$ is called the matrix of transition probabilities for the Markov Chain.

By basic properties of the probability function we have then, $0 \le p_{ij} \le 1$ and $\sum_j p_{ij} = 1$. p_{ij} is the probability of going from c_i to c_j in one step under the given probability distribution Pr.

<u>Definition</u>: Let G be a countable group.

$P : G \to [0,1]$ is a symmetric probability distribution on G if :

$$P(x) = P(x^{-1}),\ x \in G$$

$$\sum_{x \in G} P(x) = 1 .$$

To this probability distribution $P(x)$ we associate a matrix of transition probabilities, $M(G,P) = \|p_{ij}\|$ where we set $p_{ij} = P(x_i^{-1}x_j)$, $x_i, x_j \in G$.

$M(G,P)$ corresponds to a particular random walk on G as follows: If $y \in G$ was reached after the n^{th} step then yx will be reached at the next step with probability $P(x)$;

$$p_{y,yx} = P[X_{n+1}=yx | X_n=y] = P(y^{-1}yx) = P(x).$$

One can consider $\tilde{M} = M(G,p)$ as a bounded linear operator on the Hilbert Space $\ell^2(G)$ of functions $h(x)$ ($x \in G$ $h(x)$ real) with $\Sigma|h(x)|^2 < \infty$ by setting

$$\tilde{M}(h(x)) = \sum_j p_{x_i x_j} h(x_j) \qquad x_i, x_j \in G$$

<u>Lemma</u>: $\sum_j p_{x_i x_j} h(x_j)$ is bounded.

Proof. We must show that $\|\tilde{M}h\|^2 \leq K\|h\|$ K constant, $\|\cdot\|$ the norm in ℓ^2.

By Cauchy's inequality, $\langle A,B\rangle^2 \leq \|A\|^2\|B\|^2$ where A,B are vectors with components in ℓ^2 and $\langle\ ,\ \rangle$ is the inner product in ℓ^2. Letting

$$A = (\sqrt{p_{x_i x_1}}, \sqrt{p_{x_i x_2}}, \sqrt{p_{x_i x_3}} \cdots), \qquad B = (\sqrt{p_{x_i x_1}}\, h(x_1) \sqrt{p_{x_i x_2}}\, h(x_2);$$

we have

$$(\sum_j p_{x_i x_j} h(x_j))^2 \leq (\sum_j p_{x_i x_j})(\sum_j p_{x_i x_j} h^2(x_j))$$

$$\|\tilde{M}h\|^2 = \sum_i (\sum_j p_{x_i x_j} h(x_j))^2 \leq \sum_i (\sum_j p_{x_i x_j})(\sum_j p_{x_i x_j} h^2(x_j))$$

$$= \sum_i \sum_j p_{x_i x_j} h^2(x_j) \quad (\text{since } \sum_j p_{x_i x_j} = 1)$$

$$= \sum_j \sum_i p_{x_i x_j} h^2(x_j) = \sum_j h^2(x_j) \quad (\text{since } p_{x_i x_j} = p_{x_j x_i})$$

$$= \|h\| .$$

So $K = 1$ and $\tilde{M}$ is a well defined operator on $\ell^2(G)$

<u>Definition</u>: The Spectrum of $\tilde{M}$ is the set of numbers, $\overline{\lambda}$,

where $\tilde{M} - \bar{\lambda}I$ does not have an inverse. (Here I denotes the unit operator on $\ell^2(G)$.)

<u>Definition</u>: The Spectral Radius, λ, is defined to be

$$\lambda = \sup_{\bar{\lambda} \in \text{Spectrum of } \tilde{M}} |\bar{\lambda}|$$

We now digress to define amenability of groups and state two of their properties.

<u>Definition</u>: A linear functional, B, acting on the space of all bounded real valued functions on a group G is called amenable if:

1) $\inf_{x \in G} f(x) \leq Bf \leq \sup_{x \in G} f(x)$

2) Define $g : G \to R$ by $g(y) = f(xyz)$; $x,y,z \in G$. Then $B(f) = B(g)$ for all f, f a bounded real valued function on G.

<u>Definition</u>: A group G is called amenable if there exists an amenable functional on the space of bounded real valued functions on G.

A finite, or Abelian group is amenable [2]. We will see that a free group on two generators is not.

<u>Theorem.(Kesten)</u>: [8]. Let G be a countable group and p an arbitrary fixed symmetric probability measure on G such

that $F = \{x \in G | p(x) > 0\}$ generates G. A necessary and sufficient condition for G to be amenable is that $\lambda = 1$.

Theorem (Folner) : [6]. Let F be any translation invariant space of bounded real valued functions on a group G. A necessary and sufficient condition for G to be amenable is that

$$\sup_{x \in G} H(x) \geq 0 \text{ for all H of the form}$$

(1) $$H(x) = h_1(x)-h_1(a_1xb_1)+h_2(x)-h_2(a_2xb_2)+\cdots+h_n(x)-h_n(a_nxb$$

$$h_i \in F,\ a_i,b_i \in G \text{ arbitrary}.$$

We can use Folners theorem to show that a free group on two generators is not amenable.

Let G be the free group on two generators a,b. Let E be the set of elements of G which end on a power of a. We will show that there does not exist an amenable functional on the space spanned by the characteristic function, $f(x)$, of E and it translates, $f(xa_1)$, $a_1 \in G$ ($f(x) = 1$ if $x \in E$, 0 otherwise).

It suffices to prove that $\sup_x H(x) < 0$ for the function

$$H(x) = -f(x) + f(xa^2b) + f(x) - f(xa) - f(x) + f(xa^2b^2)$$

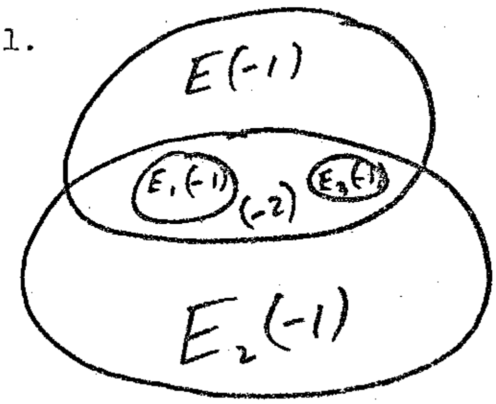
$$= f(xa^2b) - f(xa) + f(xa^2b^2) - f(x)$$

$f(xa^2b)$ = characteristic function of $E_1 = Eb^{-1}a^{-2}$

$f(xa)$ = characteristic function of $E_2 = Ea^{-1}$

$f(xa^2b^2)$= characteristic function of $E_3 = Eb^{-2}a^{-2}$

Now E_2 consists partly of elements ending on $a^r (r\neq 0, r\neq -1)$, partly of elements ending with b^s $(s\neq 0)$ and of the unit element. Also we have $E \cup E_2 = G$, $E_1 \cap E_3 = \phi$, $E_1 \subset E \cap E_2$, $E_3 \subset E \cap E_2$. We evaluate $H(x)$ on each of the disjoint areas in the picture and find that $H(x) \leq -1$.



Definition: Let $c_0, c_1 : [\alpha,\beta] \to M$ be a differentiable curve with $c_0(\alpha) = c_1(\alpha) = p, c_0(\beta) = c_1(\beta) = q$. A continuous map $H : [\alpha,\beta] \times [0,1] \to M$ is called a (p,q)-homotopy between c_0 and c_1 if $H_s : [\alpha,\beta] \to M$, where $H_s(t) = H(t,s)$, is a differentiable curve from p to q and $H_0(t) = c_0(t)$, $H_1(t) = c_1(t)$.

Definition: c_0 and c_1 are called homotopic-(p,q) if there is a (p,q)-homotopy between them.

Homotopic-(p,q) is an equivalence relation in the set Ω_{pq}, of all differentiable curves from p to q.

The equivalence classes under the relation "homotopic-(p,p)" form a group, the Fundamental group of M based at $p, \pi_1(M,p)$. If $\langle\alpha\rangle$, $\langle\beta\rangle$ are elements of $\pi_1(M,p)$, $\alpha,\beta : [0,1] \to M$, then the group operation is defined by :

$$\langle\alpha\rangle\langle\beta\rangle = \langle\alpha\beta\rangle$$

where

$$\alpha\beta = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Lemma: M compact $\Rightarrow \pi_1(M,p)$ countable.

Proof. Any open cover of M has a finite subcover. In particular we can choose a finite open cover $\{U_1, \cdots, U_n\}$ of M such that each U_i is a convex ball (M locally homeomorphic to R^n).

If $U_i \cap U_j \neq \phi$, pick a point, $p_{ij} \in U_i \cap U_j$ arbitrary (a p_{ij}-point). There are at most finitely many p_{ij}-points.

For each pair of distinct p_{ij}-points choose a path (which we call a p_{ij}-path) joining them. There are at most a finite number of p_{ij}-paths.

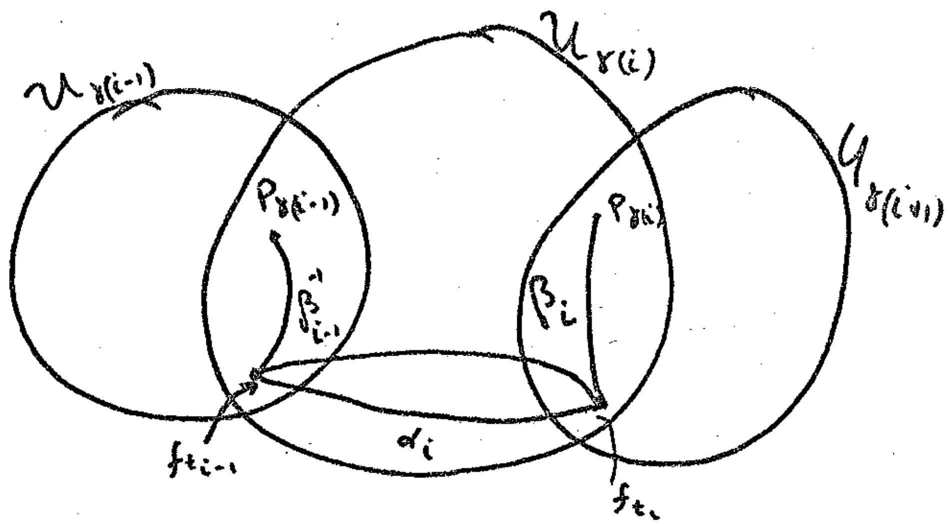
Choose a p_{ij}-point, p_k, as base point of the fundamental group. We will show that any element of $\pi_1(M,p_k)$ is equivalent to a finite product of p_{ij}-paths. This will complete the proof by the above remark.

Let $f : [0,1] \to M$ represent an element, α, of $\pi_1(M,p_k)$. Partition $[0,1]$, $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$, such that $f([t_{i-1},t_i]) \subset U_{\gamma(i)}$. Let α_i be the equivalence class of the path $f([t_{i-1},t_i])$.

Then $\alpha = \alpha_1 \cdots \alpha_n$

Each point $f(t_i)$ lies in $U_{\gamma(i)} \cap U_{\gamma(i+1)}$. Let β_i be the minimal geodesic joining $f(t_i)$ and the p_i-point in $U_{\gamma(i)} \cap U_{\gamma(i+1)}$. $\beta_i \subset U_{\gamma(i)} \cap U_{\gamma(i+1)}$ since the $\{U_i\}$ are convex.

Now $\alpha = \alpha_1\beta_1\beta_1^{-1}\alpha_2\beta_2\beta_2^{-1}\alpha_3\cdots\alpha_{n-1}\beta_{n-1}\beta_{n-1}^{-1}\alpha_n$ and each $\beta_{i-1}^{-1}\alpha_i\beta_i$ is equivalent to a p_{ij}-path in $U_{\gamma(i)}$ since the U_i are simply connected. So α is equivalent to a finite product of p_{ij}-paths and we are done



§3.

Consider the random walk on $G = \pi_1(M,b)$ defined in the previous section. We will prove that λ is necessarily less than 1.

To do this we begin by constructing a symmetric probability measure on $\pi_1(M,b)(=\pi_1)$. By Kesten's theorem it suffices to show that π_1 is not amenable, which, by Folner's theorem is equivalent to exhibiting a function, $H(x)$, of the form

(1) such that

$$\sup_x H(x) < \varepsilon < 0$$

Recall that $H(x) = h_1(x) - h_1(a, xb_1) + h_2(x) - h_2(a_2 xb_2) + \cdots + h_n(x) - h_n(a_n xb_n)$ where $h_i \in F$, F bounded real valued function on G, $a_i b_i \in G$ arbitrary.

It will turn out that the h's will be chosen as the sums of the angles of triangles, one side of which will be on the invariant line of $a \in \pi_1$.

We construct our probability measure as follows:

Let $\gamma \in \pi_1(M,b)$ and let $\Omega^{\gamma}_{bb}(M)$ be the corresponding equivalence class in $\Omega_{bb}(M)$. In Section 1 we defined Weiner Measure on the space $C_{ab}(M)$.

Setting $a = b$, we have

$$\omega_{bb}(\Omega_{bb}(M) = h_1(b,b)$$

where h is the fundamental solution to the heat equation. Define $p : \pi_1 \to R(>0)$ by

$$p(\gamma) = \frac{\omega_{bb}(\Omega^{\gamma}_{bb}(M)}{h_1(b,b)}$$

Since $\sum_{\gamma \in \pi_1} p(\gamma) = 1$ and $p(\gamma) = p(\gamma^{-1})$ (γ and γ^{-1} are in the same equivalence class), p is a symmetric probability measure on π_1.

Let $\overline{M}$ denote the universal covering space of M.

<u>Definition:</u> A decktransformation of $\overline{M}$ is a homeomorphism h, $h : \overline{M} \to \overline{M}$ with $\pi \circ h = \pi$, where $\pi : \overline{M} \to M$ is the projection map.

<u>Definition:</u> Let $c_0, c_1 : [\alpha,\beta] \to M$ be differentiable curves with $c_0(\alpha) = c_0(\beta) = p_0$, $c_1(\alpha) = c_1(\beta) = p_1$. $H : [\alpha,\beta] \times [0,1] \to M$ is called a free homotopy between c_0 and c_1 if the map $H_s : [\alpha,\beta] \to M$ with $H_s(t) = H(t,s)$ is a differentiable curve with $H_s(\alpha) = H_s(\beta)$, $H_0(t) = c_0(t)$, and $H_1(t) = c_1(t)$.

<u>Definition:</u> Differentiable curves c_0 and c_1 are called free homotopic if there exists a free homotopy between them.

"Free homotopic" is an equivalence relation in the set of differentiable closed curves from $[\alpha,\beta] \to M$.

The set of decktransformations forms a group under composition of maps and π_1 is isomorphic to this group of decktransformations of $\overline{M}$.

We view π_1 as the group of decktransformations of $\overline{M}$.

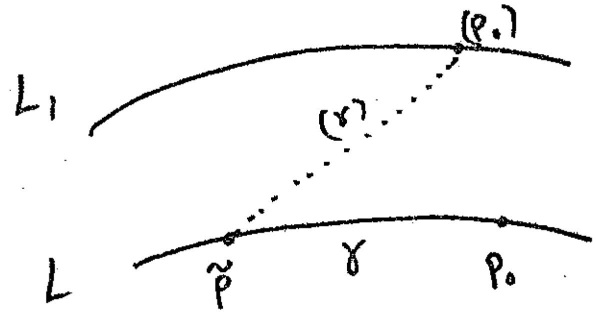
<u>Definition:</u> $c : [\alpha,\beta] \to M$ is an invariant line of the homeomorphism $h : M \to M$ if c is a geodesic ray with

$$h(c(t_0)) = c(t_1) \qquad t_0, t_1 \in [\alpha,\beta] .$$

We will construct the invariant line of $a \in \pi_1(M,p)$.

Let c_0 represent a (p,p)-homotopyclass $a \in \pi_1(M,p)$. c_0 is free homotopic to a closed geodesic c. $L = \pi^{-1}(c)$ is a geodesic ray in $\overline{M}$.

Let f_{c_0} be the decktransformation induced by c_0 (as a representative of $a \in \pi_1(M,p)$), $\tilde{p} \in L$ with $\pi(\tilde{p}) = p = c(t)$, $f_{c_0}(\tilde{p}) = p_0$. Now if $L_1 = \pi^{-1}(c)$ also covers c, then $f_{c_0}(\tilde{p}) = p_0$ cannot be in L_1 since by construction f_{c_0} comes from a homotopyclass of c, and the geodesic joining $\tilde{p}$ and p_0, γ, must therefore project to c. This can only happen if $p_0 \in L$. This means that $f_{c_0}(\tilde{p})$ remains in L and L is invariant under f_{c_0}. L is called an invariant line of $a \in \pi_1(M,p)$ in $\overline{M}$.



We proceed to construct our function H(x) and show that it satisfies the desired inequality.

Let L_a be the invariant line of $a \in \pi_1$ and pick $p \in L_a$ arbitrary. An exact choice of p will be made towards the end of the paper.

We will show that

(2) $h_1(x) < \frac{1}{n+1} \sum_0^n h_1(a^i x) - \varepsilon, \ \varepsilon > 0$

for h_1 a suitable bounded real valued function on π_1

Define $h : \overline{M} \to R$ by $h(q) = h_1 \circ x(q)$ where $q \in \overline{M}$ and x a decktransformation on $\overline{M}$.

It suffices to prove that

(3) $h(q) < \frac{1}{n+1} \sum_0^n h(a^i q) - \varepsilon, \ \varepsilon > 0$ for all $q \in \overline{M}$.

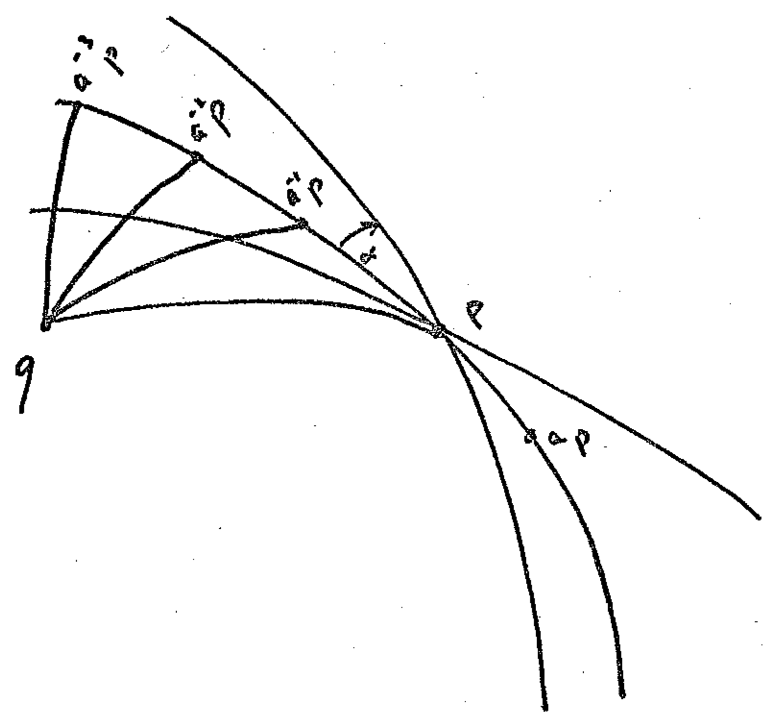
Let $B_\delta(p)$ be a δ-neighborhood of p. Denote by $C_{a,p}$ the double cone centered at p making an angle α with L_a where $1 \leq \cos\alpha \leq 1 + \delta$, $\delta > 0$.

Let $q \in \overline{M}$ be any point outside the union of $C_{a,p}$ and $B_\delta(p)$.

Let $\overline{h} : \overline{M} \to R$ be defined as follows: $\overline{h}(q)$ = sum of the angles in the triangle $\Delta(p, a^{-1}p, q)$. We have, since L_a is invariant, a an isometry,

$$\begin{aligned} \overline{h}(aq) &= \text{sum of angles in the triangle } \Delta(p, a^{-1}p, aq) \\ &= \text{sum of angles in } \Delta(a^{-1}p, a^{-2}p, q) \\ \overline{h}(a^2q) &= \text{sum of angles in } \Delta(a^{-2}p, a^{-3}p, q) \end{aligned}$$

In general, $\overline{h}(a^i q)$ = sum of angles in $\Delta(a^{-i}p, a^{-(i+1)}p, q)$



We show first that $\overline{h}(q) < \frac{1}{n+1}\sum_{0}^{n} \overline{h}(a^i q) - \varepsilon$

Lemma I: $\overline{h}(q) \le \pi - \eta \quad \eta > 0$

Proof. First we show that for our compact manifold we can always assume $-1 \le K \le -\varkappa$, K the Riemannian curvature of M.

Let X,Y be differentiable vectorfields with $X_p = v$, $Y_p = w$.

$K(v,w) = \frac{k(v,w)}{k_1(v,w)}$ where v,w span a 2-plane in M_p

$$k(v,w) = \langle R(v,w)w,v\rangle, \quad k_1(v,w) = \|w\|^2\|v\|^2 - \langle v,w\rangle^2$$

Let π : TM → M be the projection map. Since M is compact,

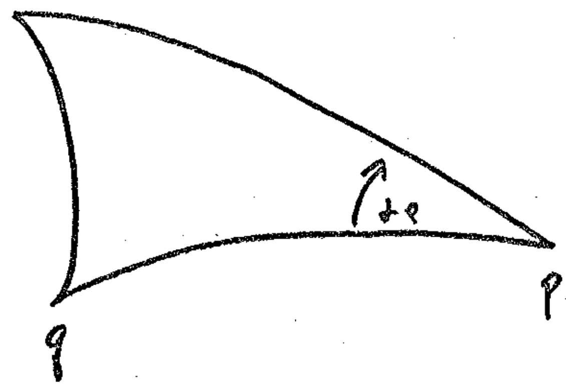
$$\{(v,w) \mid v,w \in \pi^{-1}(M),\ \|v\| = 1\ \|w\| = 1\ \langle v,w\rangle = 0\}$$

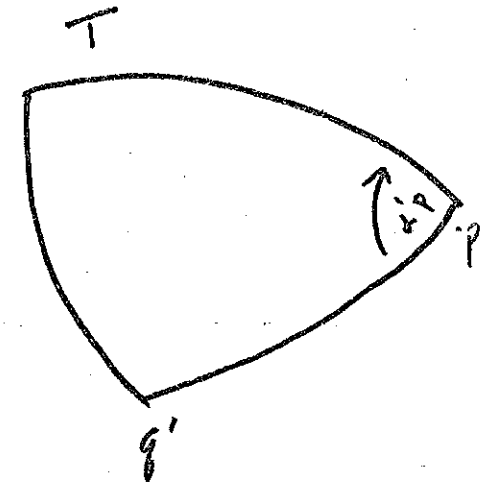
is also compact. Since the map k : TM x TM → R with (v,w) → k(v,w is continuous ($\langle , \rangle$ a tensor and X,Y differentiable vector fields) K is bounded. There exist, then, real numbers λ,μ such that $\mu \le K \le \lambda$ $(\mu \le \lambda < 0)$. Without loss of generality we could have used $\mu\langle\ ,\ \rangle$ in place of $\langle , \rangle$ for the fundamental tensor and the above inequality becomes $-1 \le K \le \frac{\lambda}{\eta}(= -\varkappa)$.

So we assume $-1 \le K \le -\varkappa$. Consider the same situation in $H_{-\varkappa}$ -the hyperbolic 2-plane of constant curvature $-\varkappa$- and compare our triangle $\Delta(p,a^{-1}p,q)$ to a triangle T in $H_{-\varkappa}$ with sides the same length as $\Delta(p,a^{-1}p,q)$.

Let α_p be the angle at p in $\Delta(p,a^{-1}p,q)$; let α_p' in T correspond to α_p.

Since $K < 0$ $\alpha_p' > \alpha_p$. [7].





The area A of T attains a positive minimum since $q' \in H_{-\varkappa}$ (corresponding to $q \notin \overline{M}$) is outside $B'_\delta(p') \in H_{-\varkappa}$ (corresponding to $B_\delta(p)$) and $\alpha'_p > \alpha_p > \alpha$.

By Gauss-Bonnet, for T we have

$$\Sigma \text{ angles in } T - \pi = -\int_A \varkappa = -\varkappa A < 0$$

Since $K < 0$, Σ angles of $\Delta(p,a^{-1}p,q)$ is strictly less than the sum of the angles in T, i.e.

$$\overline{h}(q) < \Sigma \text{ angles in } T = \pi - \eta,\ \eta > 0.$$

Lemma II: $\overline{h}(q) < \frac{1}{n+1} \sum_{i=0}^{n} \overline{h}(a^i q) - \eta/2,\ \eta > 0.$

Proof. By construction,

$$\frac{1}{n+1} \sum_{0}^{n} \overline{h}(a^i q) = \frac{n}{n+1}\pi + \frac{\text{corner } \sphericalangle_n\text{'s}}{n+1} + \frac{1}{n+1} \sum_{j=0}^{n} \text{center } \sphericalangle\text{'s}.$$

By corner $\sphericalangle$'s we mean the angle at $a^{-(n+1)}p$ in $\Delta(a^{-n}p, a^{-(n+1)}p, q)$ and the angle at p in $\Delta(p,a^{-1}p,q)$. The center $\sphericalangle$'s are the angles at q in $\Delta(a^{-j}p, a^{-(j+1)}p, q)$. (See diagram).

By the triangle inequality,

$$4)\quad \frac{1}{n+1}\sum_{i=0}^{n} \overline{h}(a^i q) \geq \frac{n}{n+1}\pi + \frac{\text{corner } \sphericalangle\text{'s}}{n+1} + \frac{\text{center } \sphericalangle}{n+1}$$

where center $\sphericalangle$ denotes the angle at q of $\Delta(p, a^{-(n+1)}p, q)$.

By Lemma I,

$$\overline{h}(q) - \frac{1}{n+1}\sum_{0}^{n} \overline{h}(a^i q) < \pi - \frac{n}{n+1}\pi - \frac{\text{corner } \sphericalangle\text{'s}}{n+1} - \frac{\text{center } \sphericalangle}{n+1} - \eta$$

$$\leq \frac{1}{n+1}(\pi - \text{corner } \sphericalangle\text{'s} - \text{center } \sphericalangle) - \eta$$

Choose N, such that for $n > N$, $\frac{1}{n+1}(\pi - \text{corner } \sphericalangle\text{'s} - \text{center } \sphericalangle) <$ $\eta/2$, and

$$\overline{h}(q) - \frac{1}{n+1}\sum_{0}^{n} \overline{h}(a^i q) < -\eta/2.$$

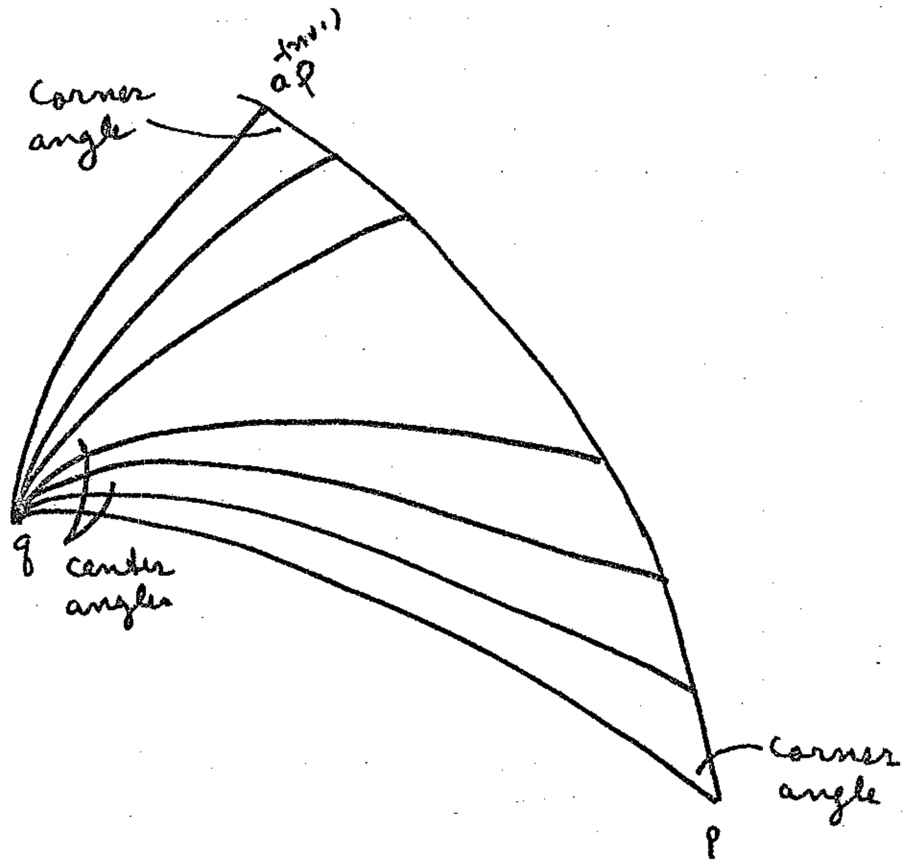
Theorem: Let M be a connected compact manifold with $K \leq c < 0$. Then $\lambda < 1$.

Proof: It suffices to prove (3) globally. In other words we will show that

$$H(q) = h(q) - \frac{1}{n+1}\sum_{0}^{n} h(a^i q) < -\varepsilon \qquad \varepsilon > 0$$

for all $q \in \overline{M}$ and suitable h.

Let L_e be another invariant line in $\overline{M}$ not meeting L_a and let $\hat{h}$, e, $\hat{p}$ correspond to $\overline{h}$, a, p respectively. Let $C_{a,p}$ and $C_{e,\hat{p}}$ be as previous notation. (We will show in the next lemma that we can always find two invariant lines such that the double cones making a small angle, α, with each do not intersect.)



Let $$I = (n+1)\overline{h}(q) - \sum_{0}^{n} \overline{h}(a^i q)$$

$$II = (n+1)\hat{h}(q) - \sum_{0}^{n} \hat{h}(e^i q) \ .$$

Let $$H(q) = I + II.$$

Now, by Lemma 2 we have that if $q \in C_{a,p}$ $II < -\eta/2$, but for $q \in C_{a,p}$ we have

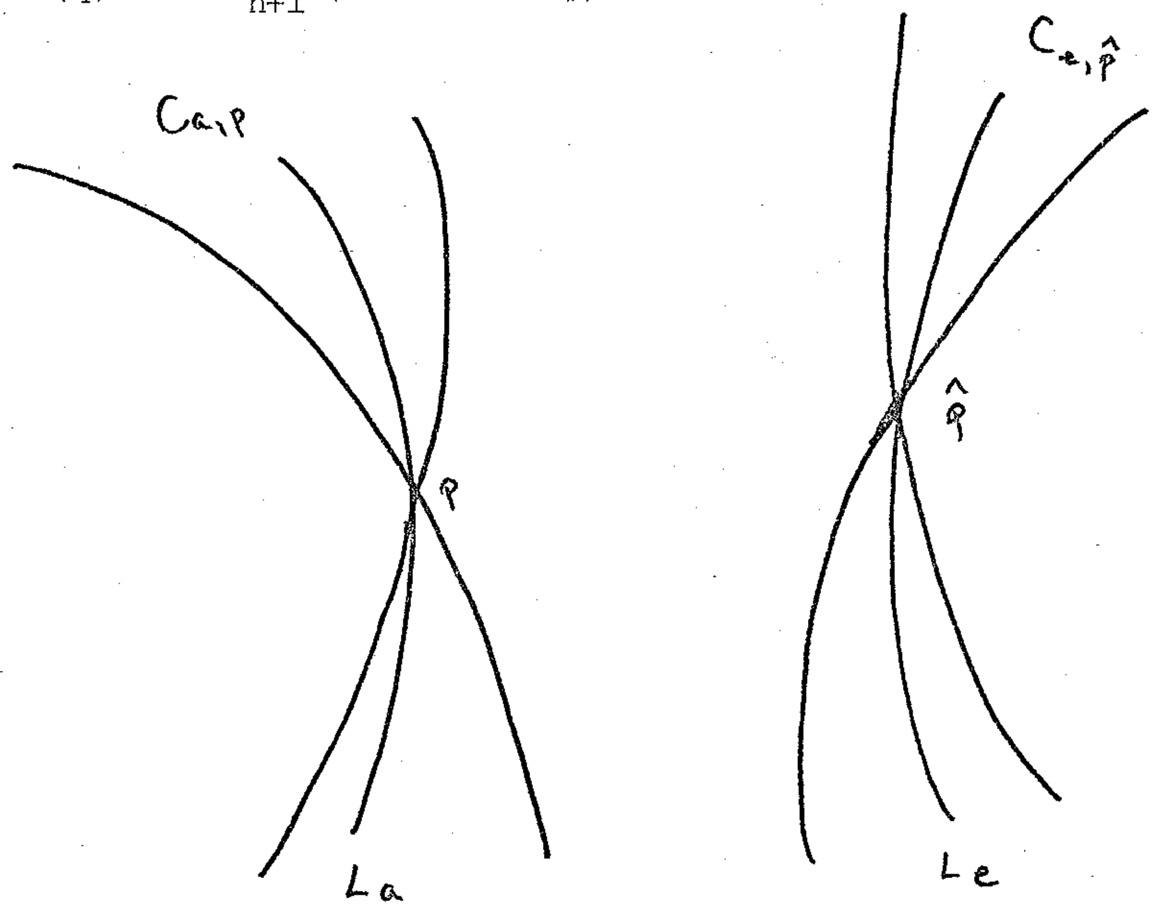
$$\overline{h}(q) - \frac{1}{n+1} \sum_{i=0}^{n} \overline{h}(a^i q) = \overline{h}(q) - \frac{n}{n+1}\pi - \frac{\text{corner } \sphericalangle\text{'s}}{n+1}$$

$$- \frac{1}{n+1} \sum_{i=0}^{n} \text{center } \sphericalangle\text{'s}$$

$$\leq \overline{h}(q) - \frac{n}{n+1}\pi - \frac{\text{corner } \sphericalangle\text{'s}}{n+1} - \frac{\text{center } \sphericalangle}{n+1}$$

(by the triangle inequality)

$$\leq \overline{h}(q) - \pi + \frac{1}{n+1}(\pi - \text{center } \sphericalangle)$$



Now, $\overline{h}(q) - \pi < 0$; and $\exists N_1 \ni n > N_1 \Rightarrow \frac{1}{n+1}(\pi\text{-center } \dot{\sphericalangle}) < \eta/4$. So if $n > \max(N,N_1)$, $H(q) < -\eta/2 + \eta/4 = -\eta/4 = -\varepsilon$.

We shave shown that $H(q) < -\varepsilon < 0$ for all $q \in \overline{M}$ implying that π_1 is not amenable, which by Kesten's theorem is equivalent to $\lambda < 1$.

We will now show that we can always find two invariant lines, L_m and L_n, such that the double cones making a small angle, α, with them at $\tilde{p} \in L_m$ and $\tilde{q} \in L_n$, don't meet.

We first fix some notation and cite facts that we will need to prove the next Lemma.

If $c : [\alpha,\beta] \to M$ is a geodesic then $\dot{c}|_q$ denotes the tangent vector at $q = c(\alpha_1)$ $\alpha \le \alpha_1 \le \beta$. If c_1 and c_2 are two geodesics which meet at a point, q, with tangent vectors $w_1 = \dot{c}_1|_q$, $w_2 = \dot{c}_2|_q$ then by $\sphericalangle(w_1,w_2)$ we mean the angle between w_1 and w_2. $d(\cdot,\cdot)$ will denote the distance function with respect to the Riemannian metric.

Definition: Let X be a parallel vectorfield along a geodesic c, with $X|_{c(\alpha)} = x$ ($x \in M_{c(\alpha)}$ the tangent space at $c(\alpha)$). We say that x is parallel translated along c to $x_1 \in M_{c(\beta)}$ if $x_1 = X_{c(\beta)}$.

Definition: Two geodesics $c_1, c_2 : [0,1] \to M$ are called nearby if $d(c_1(u_i), c_2(u_i)) < \varepsilon$, $\varepsilon > 0$ where $0 \le u_i \le 1$.

Definition: Let $c : [\alpha,\beta] \to M$ be a differentiable curve. A variation of c is a differentiable map $V : [\alpha,\beta] \times [-\gamma,\gamma] \to M$ with $V_0(t) = V(t,0) = c(t) \quad \alpha \le t \le \beta$.

Let $L(V_{\gamma'}) = L(\gamma') \quad -\gamma \le \gamma' \le \gamma$ where $L(V_{\gamma'})$ is the length of the curve $V_{\gamma'}$. Let $X = V_* D_1$, $Y = V_* D_2$, $\tilde{Y} = Y - \langle Y,X\rangle X$ where

$$D_1 = \text{derivative with respect to first variable}$$

$$D_2 = \text{derivative with respect to second variable} .$$

We recall the Variation Formulae.

$$L'(0) = \langle Y,X\rangle_{|(t,0)}\Big|_{\alpha}^{\beta}$$

$$L''(0) = \int_{\alpha}^{\beta} \langle \nabla_{D_1}\tilde{Y},\nabla_{D_1}\tilde{Y}\rangle - \langle R(Y,X)X,Y\rangle_{|t,0}\, dt + \langle \nabla_{D_2}Y,X\rangle_{|(t,0)}\Big|_{\alpha}^{\beta}$$

<u>Lemma</u>: Let M be a compact connected Riemannian manifold with curvature $K \le c < 0$. Then for every L_m there exists an L_n, $\tilde{p} \in L_m$ $\tilde{q} \in L_n$, such that

$$C_{m,\tilde{p}} \cap C_{n,\tilde{q}} = \emptyset .$$

<u>Proof</u>: Our method of proof is as follows:

Given L_m we pick $p \in L_m$ arbitrary. We then construct L_n and choose $q \in L_n$ in such a way that the minimal geodesic joining p and q is approximately of unit length. We then show, using the Variation Formulae that the distance between L_m and L_n attains a minimum at points $\tilde{p}$ and $\tilde{q}$ in neighborhoods of p and q respectively.

Again using the Variation Formulae we show that the distance between any geodesics $\ell(t) \subset C_{m,\tilde{p}}$ and $\ell_2(t) \subset C_{n,\tilde{q}}$ attains a positive minimum.

Let $L_m = c(t)$ where $c : [-\alpha,\alpha] \to \overline{M}$ is parameterized by arclength. Pick $p \in L_m$ arbitrary. Let $\mu : [0,1] \to \overline{M}$ be a geodesic with $\mu(0) = p$ and $\langle\dot{\mu},\dot{c}\rangle_{|p} = 0$. Let $q_0 = \mu(\alpha_1)$ $0 < \alpha_1 \leq 1$ where $\int_0^{\alpha_1} \|\dot{\mu}(\tau)\| d\tau = 1$.

Let $c_1 : [0,1] \to \overline{M}$ be a geodesic with $c_1(s) = q_0$, $0 < s < 1$, and $\langle\dot{c},\dot{\mu}\rangle_{|q_0} = 0$.

<u>Definition</u>: Let $c : [0,1] \to M$ be a geodesic in M with $c(z) = p$. By "invariant lines are dense in the geodesics" we mean as follows:

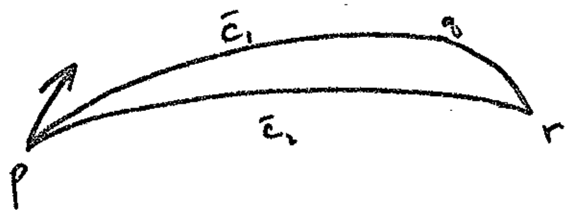
For all $\varepsilon,\delta > 0$ there exists an invariant line $c' : [-\alpha,\alpha] \to M$, such that $c'(t) \in B_\delta(p)$ and $\langle\dot{c}', v'\rangle_{|c'(t)} < \varepsilon$ where v' is $\dot{c}_{|p}$ parallel translated to $c'(t)$ along the minimal geodesic joining p and $c'(t)$.

$K < 0$ implies the invariant lines are dense [10].

So, for all $\varepsilon,\delta > 0$ there exists an invariant line, $L_n = c_2(t)$ $(c_2:[-\alpha,\alpha] \to \overline{M})$ such that for $q_1 = c_2(0) \in B_\delta(q_0)$ $\sphericalangle\langle\dot{c}_1,v\rangle_{|q_0} < \varepsilon$ where v is $\dot{c}_2(0)$ parallel translated to q_0 along μ_1, the minimal geodesic joining q_0 and q_1.

Let $\gamma : [0,1] \to \overline{M}$ be the minimal geodesic joining p and $c_2(0)$.

The following sublemma will state, essentially, that if we have two nearby geodesics, $\bar{c}_1$ and $\bar{c}_2$, starting at p, and $x \in M_p$, then parallel translating x along $\bar{c}_1$ to q then along the minimal geodesic joining q and $\bar{c}_2$ (at r), makes a small angle with x parallel translated along $\bar{c}_2$ to r.



<u>Sublemma</u>: Let $\bar{c}_1, \bar{c}_2 : [0,1] \to \bar{M}$ be nearby geodesics with $\bar{c}_1(0) = \bar{c}_2(0)$. Let $\bar{c} : [0, s_0] \to \bar{M}$ be the minimal geodesic joining $\bar{c}_1(t_0)$ and $\bar{c}_2(t_0)$. Denote by $\bar{V} : [0, t_0] \times [0,1] \to \bar{M}$ a family of geodesics with $\bar{V}(t,0) = \bar{c}_1(t)$, $\bar{V}(t,1) = \bar{c}_2(t)$ and $\bar{V}(t,s)$ = the minimal geodesic joining $\bar{c}_1(0)$ and $\bar{c}(s)$. $0 \le s \le s_0$.

Let $x \in M_p$, $X(t_0, 0)$ = x parallel translated to $\bar{c}_1(t_0)$ along $\bar{c}_1(t)$; $X(t_0, s_0) = X(t_0, 0)$ parallel translated to $\bar{c}(s_0)$ along $\bar{c}(s)$. $X(0, s_0)$ = x parallel translated along $\bar{c}_2(t)$ to $\bar{c}(s_0)$. $V = V_* D_1$ $W = V_* D_2$.

$$\text{Then } |X(t_0, s_0) - X(0, s_0)| < \bar{\varepsilon}, \ \bar{\varepsilon} > 0.$$

<u>Proof</u>: By the fundamental theorem of calculus,

$$X(t,s) = \int_0^s \nabla_W X + X(t,0)$$

Since X is parallel,

$$X(t_0,s_0) - X(0,s_0) = \int_0^{t_0} D_1\left(\int_0^{s_0} \nabla_W X - X(t,0)\right) = \int_0^{t_0}\int_0^{s_0} \nabla_V \nabla_W X$$

$$= \int_0^{t_0}\int_0^{s_0} \nabla_V\nabla_W X - \nabla_W\nabla_V X \quad (\text{since } \nabla_V X = 0,\ X \text{ parallel})$$

$$= \int_T R(V,W)X \text{ where } T \text{ is the triangle with vertices } p, \bar{c}_1(t_0),\ \bar{c}(s_0).$$

$|X(t_0,s_0) - X(0,s_0)| < \bar{\varepsilon}$, $\bar{\varepsilon} > 0$ since by construction the area of T is small.

We continue with the proof of the lemma.

Define $F(t) = d(c(0), c_2(t))$; then $F(0) = L(\gamma)$ and $F'(0) = \cos \measuredangle(\dot{c}_2(0),\dot{\gamma})|_{c_2(0)}$ by the first variation. We show first that F attains a minimum for some $\tilde{t}$ near 0. Since $K \leq c < 0$ the second variation is strictly positive. It suffices, therefore, to show that $F'(\tilde{t}) = 0$ since $F'' \geq \delta > 0$.

Parallel translating $\dot{c}_2$ and $\dot{\gamma}$ from q_1 to q_0 (along μ_1) preserves angles, therefore,

$$\langle \dot{c}_2,\dot{\gamma}\rangle|_{q_1} = \langle c_2,\dot{\gamma}\rangle|_{q_0}.$$

Also, since $\mu(0) = \gamma(0)$ and μ and γ are nearby geodesics (their endpoints are close and $\bar{M}$ simply connected), by the sublemma, $\dot{\gamma}$ and $\dot{\mu}$ make almost the same angle with $\dot{c}_2$ at q_0, i.e.

$$\langle \dot{c}_2,\dot{\gamma}\rangle|_{q_0} \approx \langle \dot{c}_2,\dot{\mu}\rangle|_{q_0} \text{ (here } \approx \text{ signifies that } |\langle \dot{c}_2,\dot{\gamma}\rangle|_{q_0} - \langle \dot{c}_2,\dot{\mu}\rangle|_{q_0}| < \varepsilon,\ \varepsilon > 0)$$

But $\langle \dot{c}_2, \dot{\mu} \rangle_{|q_0} \approx \langle \dot{c}_1, \dot{\mu} \rangle_{|q_0}$ by density and $\langle \dot{c}_1, \dot{\mu} \rangle_{|q_0} = 0$.

We have, so far, that $|F'(0)| < \varepsilon_1, \varepsilon_1 > 0$, $F'' \geq \delta > 0$ so that

$$F'(t) = F'(0) + \int_0^t F''(s)ds \geq -\varepsilon_1 + \delta t, \quad t > 0$$

and

$$F'(t) = F'(0) + \int_t^0 F''(s)ds \leq \varepsilon_1 + \delta t, \quad t < 0.$$

So there exists $\tilde{t}$, $|\tilde{t}| < \frac{\varepsilon_1}{\delta}$ with $F'(\tilde{t}) = 0$. There exists, therefore, a minimal geodesic, ξ, which makes a right angle with L_n at $c_2(\tilde{t})$ and joints p and $c_2(\tilde{t})$.

We show next that the distance between the invariant line attains a positive minimum by proving that the minimum is attained in neighborhood of p and q. Define

$$f(t) = d(L_n, c(t)). \quad \text{Then } f(0) = L(\xi) \text{ and}$$

$$f'(0) = \cos \sphericalangle (\dot{c}(0), \dot{\xi})_{|p} \,.$$

Since ξ and μ are nearby geodesics, $\langle \dot{c}, \dot{\xi} \rangle_{|p} \approx \langle \dot{c}, \dot{\mu} \rangle_{|p} = 0$. So $|f'(0)| < \varepsilon_2$, $\varepsilon_2 > 0$, and there exists s, $|s| < \varepsilon_3$, $\varepsilon_3 > 0$, with $f'(s) = 0$ by the previous argument.

Since $f'' > 0$, $f(t)$, the distance between the invariant lines, attains a positive minimum and we have constructed two distinct invariant lines.

It remains to show that the double cones don't meet.

By the previous construction we have a minimal geodesic,

$\tilde{\xi}$, from $\tilde{p} = c(s)$ to $\tilde{q} \in L_n$. Let $\tilde{p}$ and $\tilde{q}$ be the vertices of the double cones to L_m and L_n respectively. To prove they don't meet it suffices to show that any two geodesics $\ell(t) \in C_{m,\tilde{p}}$ and $\ell_2(t) \in C_{n,\tilde{p}}$ making an angle less than ε with L_m and L_n at $\tilde{p}$ and $\tilde{q}$, respectively, don't meet.

Let $\ell, \ell_2 : [-\beta, \beta] \to M$ be such geodesic with $\ell(0) = \tilde{p}$, $\ell_2(0) = \tilde{q}$, $\sphericalangle(\dot{c}, \ell)_{|\tilde{p}} < \varepsilon$, $\sphericalangle(c_2, \ell_2)_{|\tilde{q}} < \varepsilon$.

Define $g(t) = d(\tilde{p}, \ell_2(t))$. Then $g'(0) = \cos \sphericalangle(\dot{\tilde{\xi}}, \dot{\ell}_2)_{|\tilde{q}}$.

By previous arguments, $\langle \dot{\tilde{\xi}}, \dot{\ell}_2 \rangle_{|\tilde{q}} \approx \langle \dot{\tilde{\xi}}, \dot{c}_2 \rangle_{|\tilde{q}} = 0$ and $g'' > 0$, so that $-\varepsilon_4 \leq g'(0) \leq \varepsilon_4$ and there exists a t_0, $|t_0| < \frac{\varepsilon_4}{\delta}$, and $g'(t_0) = 0$. g attains a minimum at t_0. Let ξ_1 be the minimal geodesic joining $\tilde{p}$ and $\ell_2(t_0)$.

Finally, denote by $G(t) = d(\ell_2, \ell(t))$ the distance between the two geodesics

$$G'(0) = \cos \sphericalangle(\dot{\ell}, \dot{\xi}_1)_{|\tilde{p}} .$$

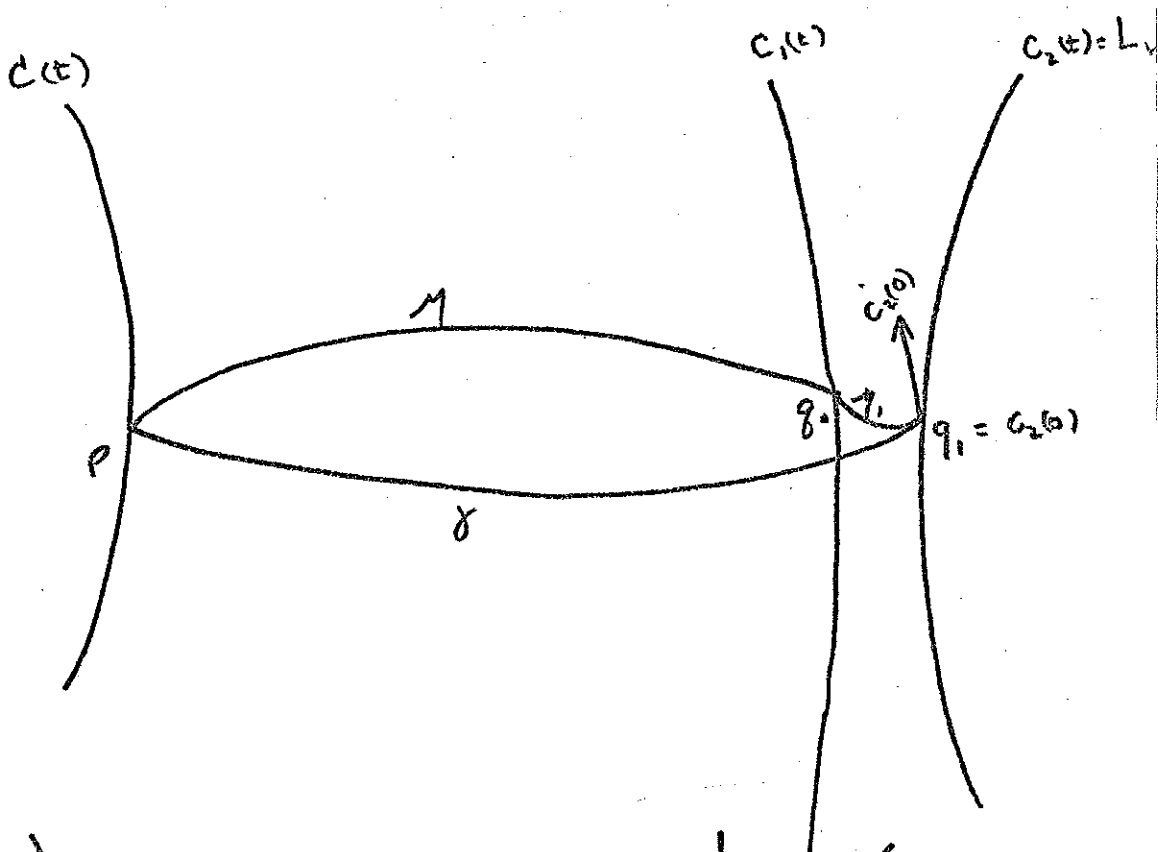
Now, $\sphericalangle(\dot{\tilde{\xi}}, \dot{c})_{|\tilde{p}} = 90^o$, $\sphericalangle(\dot{c}, \dot{\ell})_{|\tilde{p}} < \varepsilon$ by construction. Since ξ_1 and $\tilde{\xi}$ are nearby geodesics

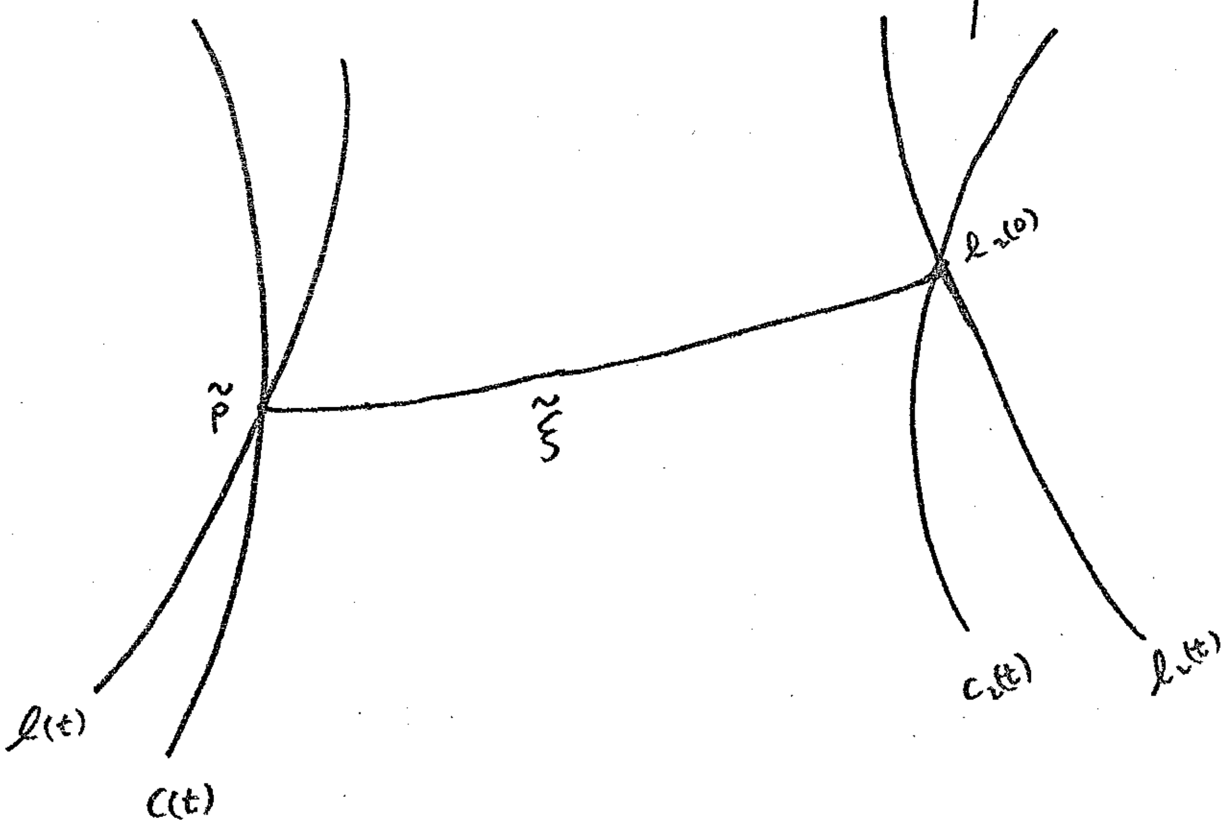
$$\sphericalangle(\dot{\xi}_1, \dot{\tilde{\xi}})_{|\tilde{p}} < \varepsilon_5 \text{ and } \cos \sphericalangle(\dot{\xi}, \dot{\ell})_{|\tilde{p}} \approx 90^o .$$

So $G'(0)$ is near 0 and there exists s', $|s'| < \frac{\varepsilon_6}{\delta}$ with $G'(s') = 0$.

We have then, that the distance between the double cones attains a positive minimum proving the Lemma.

In the proof of the theorem, we let $p = \tilde{p}$, $e = \tilde{q}$, $m = a$ and $n = e$.


c(t)
c₁(t)
$c_2(t) = L_v$
$\dot{c}_2(0)$
μ
p
q_0
η_1
$q_1 = c_2(0)$
γ


$\tilde{p}$
$\tilde{\xi}$
$\ell_2(0)$
$\ell(t)$
c(t)
$c_2(t)$
$\ell_2(t)$

Since $\pi_1(M)$ is not amenable, it cannot be Abelian, a well known result. A necessary and sufficient condition for a group G to be amenable is that every finitely generated subgroup of G is amenable [2]. It follows then that $\pi_1(M)$ has either no proper subgroups or at least one which is not amenable, e.g., not cyclic, Abelian or finite.

References

1. Chung, Markov Chain with Stationary Transition Probability. Springer Verlag, (1967).

2. Day, Mahlon M., Amenable Semigroups, Illinois J. of Math., (1957), 509-544.

3. Eberlein, Patrick, Some Properties of the Fundamental Group of a Fuchsian Manifold, Inventions Math 19, (1973), 5-13

4. Eells, J., Random Walk on the Fundamental Group. Proc. Symp. Math., Vol. 27 part 2, (1973). 211-17.

5. Eels, J. and Elworthy, K., Weiner Integration on Certain Manifolds. C.I.M.E. (IV) (1970) 67-94.

6. Falner, E., On Groups with Full Banach Mean Value. Math. Scan (3) (1955), 243-54.

7. Gromoll, D., Klingenberg, W., Meyer, W., Riemannsche Geometric in Grosen. Springer Verlag Lecture Notes (55).

8. Kesten, H., Full Banach Mean Value on Countable Groups. Math. Scan. (7) (1959), 146-56.

9. Milnor, J., A Note on Curvature and Fundamental Group, J. Diff. Geom 2 (1968), 1-7.

10. Smale, S., Differentiable Dynamical Systems. AMS Bull. (73) (1967). 747-817.