UNIFORMIZATION OF HYPERELLIPTIC SURFACES

A Dissertation presented

by

Daniel Martin Gallo

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

December, 1979

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

Daniel Martin Gallo

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Henry Laufer

Henry Laufer — Committee Chairman

Irwin Kra

Irwin Kra — Thesis Advisor

Bernard Maskit

Bernard Maskit

Daniel Z. Freedman

Daniel Z. Freedman

The dissertation is accepted by the Graduate School

Charles W Kim

Dean of the Graduate School

Abstract of the Dissertation

Uniformization of Hyperelliptic Surfaces

by

Daniel Martin Gallo

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1979

Let S be a closed Riemann surface of genus $g \geq 2$. Suppose S is hyperelliptic and $J : S \to S$ is the hyperelliptic involution. Let S be represented as D/Γ where D is the unit disc and Γ is a fixed point free Fuchsian group of the first kind. Moreover, let $j(z) = -z$ be a lift of the involution to D.

We show that Γ has a fundamental polygon (called a hyperelliptic polygon) $P \subset D$ with the following properties: P is a $4g$ sided, simple, convex hyperbolic polygon which is j-invariant. The weierstrass points of S correspond to the midpoints of the sides of P, the vertices of P and the origin.

Let $T(\Gamma)$ be the Teichmüller space of Γ. It is known that

$T(\Gamma)$ is a complex manifold of dimension 3g-3. For each $[\mu] \in T(\Gamma)$ there exists a well defined Jordan domain $D_\mu \subset \mathbb{C} \cup \{\infty\}$ and a quasi-Fuchsian group Γ^μ acting on D_μ such that $D_\mu/_{\Gamma^\mu} \cong S_\mu$ is a Riemann surface of genus g. The map j acts in a natural manner on $T(\Gamma)$ and the fixed point set $T(\Gamma)^j$ corresponds to a component of the hyperelliptic locus in $T(\Gamma)$.

The Bers fiber space $F(\Gamma) = \{([\mu],z) \in T(\Gamma) \times \mathbb{C} \cup \{\infty\};$ $[\mu] \in T(\Gamma)$ and $z \in D_\mu\}$ is also a complex manifold of dimension 3g-2. For each $[\mu] \in T(\Gamma)^j$ we show the existence of a canonical fundamental polygon $P_{\mu'} \subset D_\mu$ for Γ^μ. The vertices of $P_{\mu'}$ lie on holomorphic sections of $F(\Gamma)$ over $T(\Gamma)^j$. Using Teichmüller theory we compute the polynomial of the surface corresponding to a specific hyperelliptic polygon.

We then study topological properties of dissections of hyperelliptic surfaces of the type given by hyperelliptic polygons. We consider the possibility of prescribing for a surface of genus 2 (necessarily, hyperelliptic) the periods of holomorphic differentials on half the loops of such a dissection.

To my parents.

Table of Contents

Acknowledgements

I would like to express my gratitude to Professors Irwin Kra and Bernard Maskit for their guidance and above all for their patience.

I. INTRODUCTION

Canonical polygons for finitely generated Fuchsian groups were first studied by Fricke (1920's) and later by Keen (1966). In this thesis we study a different type of polygon which is canonical for Fuchsian groups whose orbit space is a hyperelliptic Riemann surface. The involution of the surface is represented by a symmetry of the polygon and the Weierstrass points are displayed symmetrically. We call such a polygon a hyperelliptic polygon.

It turns out that hyperelliptic polygons were first considered by Whittaker [18] in 1899. Whittaker's proofs however, are incomplete and his treatment relies mainly on specific examples.

By the use of quasiconformal maps, in particular the Ahlfors-Bers solution to the mapping problem for a given dilatation (and the continuity of that solution when the dilatation depends linearly on a real parameter t), we are able to prove the existence of hyperelliptic polygons for all hyperelliptic surfaces.

In Chapter II, after establishing the preliminaries, we count real parameters for hyperelliptic polygons. Their number coincides with the dimension of the Teichmüller space of hyperelliptic surfaces.

In II.4 we prove the existence of a simple, convex hyperelliptic polygon for any given hyperelliptic surface. This polygon gives a specific set of generators for the Fuchsian group which uniformizes the hyperelliptic surface.

In Chapter III we look at uniformizations of hyperelliptic surfaces by quasi-Fuchsian groups and find similar fundamental polygons. In fact we show that if the quasi-Fuchsian group belongs to the identity component of the hyperelliptic locus in the Teichmuller space, then the vertices of the polygon lie on holomorphic sections of the Bers fiber space over the Teichmuller space. Finally, using a theorem of Kravetz, we compute the polynomial of the surface corresponding to a specific hyperelliptic polygon.

In Chapter IV we consider topological dissections of hyperelliptic surfaces of the type given by hyperelliptic polygons. We compute the intersection matrix of such a dissection and establish a relation among periods of two C^1 closed differentials.

In the last two sections we consider the possibility of prescribing for a surface of genus 2 (necessarily, hyperelliptic), the periods of holomorphic differentials on half the loops of such a dissection.

II. UNIFORMIZATION AND HYPERELLIPTIC SURFACES

II.1 Preliminaries. We will work with groups Γ whose elements are Mobius transformations $T(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$. Hence the elements of Γ are conformal self maps of the extended complex plane $\mathbb{C} \cup \{\infty\}$.

If $z \in \mathbb{C} \cup \{\infty\}$, let $\Gamma_z = \{\gamma \in \Gamma; \gamma(z) = z\}$. We say Γ is discontinuous at z if

i) Γ_z is finite, and

ii) there is a neighborhood U of z such that

$$\gamma(U) = U \text{ for all } \gamma \in \Gamma_z, \text{ and}$$

$$\gamma(U) \cap U = \emptyset \text{ for all } \gamma \in \Gamma - \Gamma_z.$$

Let $\Omega = \Omega(\Gamma) = \{z \in \mathbb{C} \cup \{\infty\}; \Gamma \text{ is discontinuous at } z\}$. We call Ω the region of discontinuity of Γ and say that Γ is discontinuous if $\Omega \neq \emptyset$. We call $\Lambda = \Lambda(\Gamma) = \mathbb{C} \cup \{\infty\} - \Omega$ the limit set.

It can be shown (Ford [9]) that card $\Lambda = 0,1,2$ or ∞.

If card $\Lambda \leq 2$ then Γ is called an elementary group. If Ω is not empty and card $\Lambda > 2$ then Γ is called a (non-elementary) Kleinian group.

If there exists a component Σ of the region of discontinuity of a Kleinian group Γ which is invariant (i.e. $\gamma(\Sigma) = \Sigma$ for all $\gamma \in \Gamma$), then Γ is called a function group.

If Γ is a Kleinian group the projection $p : \Omega \to \frac{\Omega}{\Gamma}$ onto the set of orbits of Γ in Ω induces a unique conformal structure on $\frac{\Omega}{\Gamma}$.

Let S be a Riemann surface and Γ a function group with invariant component Σ. If Γ acts freely on Σ and $\frac{\Sigma}{\Gamma} = S$ we say that Γ <u>uniformizes</u> S.

Let Δ be an open subset of $\mathbb{C} \cup \{\infty\}$ that is invariant under a discontinuous group Γ and $\Delta \subset \Omega$. By a <u>fundamental domain</u> R for Γ in Δ we will mean an open subset of Δ with the following properties:

i) whenever $\gamma z_1 = z_2$ for some $\gamma \in \Gamma$, $z_1, z_2 \in R$ then $\gamma = \mathrm{id}$.

ii) every point $z \in \Delta$ is equivalent to a point of $\overline{R}$ ($\overline{R}$ is the closure of R).

iii) the two dimensional Lebesgue measure of $\overline{R} - R$ is zero.

If Γ is a Kleinian group and there is a circle $C \subset \mathbb{C} \cup \{\infty\}$ such that the interior of C is fixed by Γ, then Γ is called <u>Fuchsian</u>. In this case $\Lambda \subset C$ and Γ is called of the <u>first kind</u> if $\Lambda = C$, otherwise it is of the <u>second kind</u> (see [12]).

We will consider only Fuchsian groups of the first kind which leave the open unit disc D fixed. We note that if T(z) is a Mobius transformation belonging to such a group then $T(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ and $|a|^2 - |b|^2 = 1$ (see [9]). We denote by Aut D the group of all Mobius transformations which fix D.

By Koebe's uniformization theorem [17] every compact surface S of genus $g \geq 2$ may be obtained via a fixed point free Fuchsian group Γ as $\frac{D}{\Gamma}$.

We will assume throughout that $g \geq 2$. (For uniformizations of surfaces of genus 0 or 1 see [11].) A Riemann surface S of genus g is hyperelliptic if there exists a conformal self map $J : S \to S$ such that $J^2 = id$, and J has $2g + 2$ fixed points. It is known that every hyperelliptic surface corresponds to the surface of a polynomial of two variables z and w: $w^2 = \prod_{i=1}^{2g+2} (z - a_i)$ where the $a_i \in \mathbb{C}$ are distinct (see [17]).

There are various other characterizations of hyperelliptic surfaces (see [17]). The only ones we will use are the existence of an involution with $2g + 2$ fixed points or equivalently the existence of a representation of the surface as a twofold cover of the sphere branched at $2g + 2$ points.

The Poincaré metric on the unit disc is defined by $ds = \frac{2|dz|}{1-|z|^2}$. In this metric geodesics are arcs of circles orthogonal to the unit circle (including straight lines through the origin).

Let $a_1, \ldots, a_{n+1}$, $n > 1$, be a set of points such that $a_j \neq a_k$, $1 \leq j, k \leq n$ and $a_{n+1} = a_1$, which are joined together (in order) by Jordan arcs $A_1, \ldots, A_n$. By a polygon P we will mean the closed curve formed by the arcs $A_1, \ldots, A_n$. The points $a_1, \ldots, a_n$ are called the vertices of P and the arcs $A_1, \ldots, A_n$ are called the sides of P.

If a polygon P is contained in $\overline{D}$ and its sides are hyperbolic line segments then we say P is a hyperbolic polygon.

A hyperbolic polygon is simple if no two non-adjacent sides intersect and adjacent sides intersect only at a vertex. For a simple hyperbolic polygon P, it is easy to construct a homeomorphism from the unit circle onto P. Thus P is a Jordan curve and we denote by P^o that component of the complement of P which is contained in D. We say a simple hyperbolic polygon is a fundamental polygon for a Fuchsian group Γ if P^o is a fundamental domain for Γ in D.

A simple hyperbolic polygon is called convex if any two points $z_1, z_2 \in P^o \cup P$ may be joined by a geodesic line segment which is contained in $P^o \cup P$.

We will work with simple hyperbolic polygons $P \subset D$ which have $4g$ sides and satisfy the following properties:

i) P is invariant under the transformation $j(z) = -z$.

ii) (We denote the sum of the interior angels of P by $\hat{P}$.) $\hat{P} = 2\pi$.

iii) If we give P the counterclockwise orientation and label the sides $A_1, \ldots, A_{4g}$ (in counterclockwise order) then there exist Mobius transformations $T_i \in \text{Aut } D$, $i = 1, \ldots, 2g$, such that $T_i(A_i) = A_{2g+i}$ with the reverse orientation. (Note that by the invariance property $A_{2g+i} = j(A_i)$,[1] $i = 1, \ldots, 2g$.)

We call such a polygon a hyperelliptic polygon (Fig. 1).

In what follows we will prove that every hyperelliptic surface may be uniformized by a Fuchsian group Γ which has

[1] $j(A_i)$ refers to the point set with no implied orientation.

a hyperelliptic polygon as a fundamental polygon.

II.2 Poincaré's Theorem. We will need the following theorem of Poincaré to prove that every hyperelliptic polygon is a fundamental polygon for a Fuchsian group which uniformizes a hyperelliptic surface. Since such polygons lie in D, we will omit consideration of polygons with vertices on the unit circle.

Poincaré's Theorem. Let $P \subset D$ be a simple, hyperbolic polygon which satisfies the following:[2]

i) the sides of P are identified in pairs by elements of Aut D which generate a group G.

ii) the vertices of P distribute themselves into sets of G-equivalent vertices which we call cycles, and the sum of the angles of a cycle is equal to $2\pi/\nu$, where ν is a positive integer.

Then G is a Fuchsian group and P is a fundamental polygon for G. (For a proof see [15].)

In order to apply Poincaré's theorem to hyperelliptic polygons we need to verify condition ii).

Lemma II.2.1. Let P be a hyperelliptic polygon. Then there is one equivalence class of vertices.

Proof. Let A_1 be the first side of the polygon. We label its vertices a_1 and a_2 and denote the oriented hyperbolic line joining a_1 to a_2 by $\langle a_1, a_2 \rangle$. We label the rest of the vertices in counterclockwise order. Thus $A_i = \langle a_i, a_{i+1} \rangle$,

[2] If a polygon contains vertices on the unit circle there is also a completeness condition which must be satisfied (see [15]).

$i = 1,\ldots,4g-1$, and $A_{4g} = \langle a_{4g}, a_1 \rangle$ (see Fig. 1). (Note that $-a_i = a_{2g+i}$, $i = 1,\ldots,2g$.)

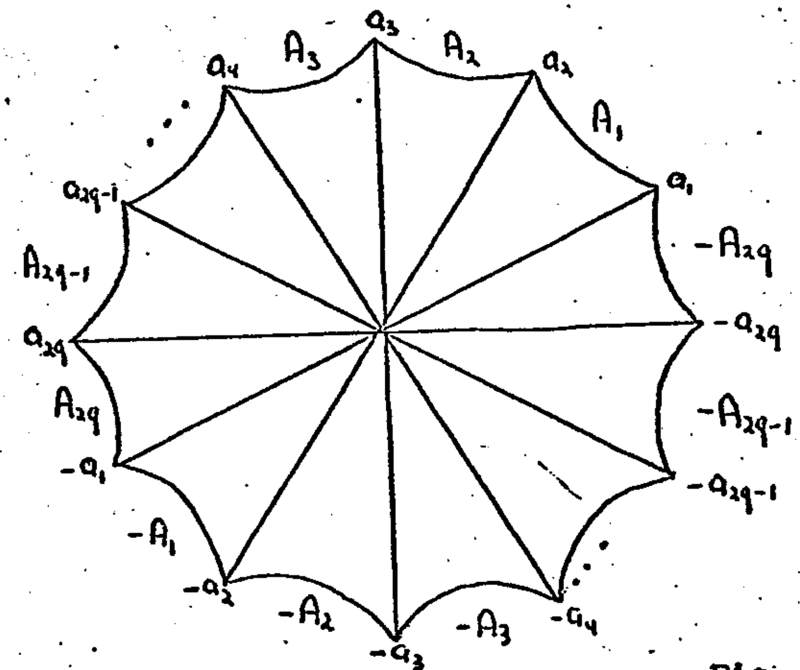


Fig. 1

Let $T = \prod_{n=0}^{2g-2} T_{2g-n}^{\epsilon_n}$ where $\epsilon_n = \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$[3]. By tracing the identifications in Fig. 1, we obtain that $T(a_1) = a_2$. Thus a_1 is equivalent to a_2. By relabeling the transformations suitably we obtain that a_i is equivalent to a_{i+1}, $i = 1,\ldots,4g-1$. Thus all vertices are equivalent.

Corollary II.2.2. Let P be a hyperelliptic polygon. Then the group Γ generated by the T_i, $i = 1,\ldots,2g$, is a Fuchsian group and P is a fundamental polygon for Γ.

Proof. Lemma II.2.1 and Poincaré's theorem yield the

[3] When multiplying two Mobius transformations R and S we will use the notation $S \circ R$ to mean R followed by S. Thus $\prod_{n=0}^{m} S_n = S_m \circ \cdots \circ S_1 \circ S_0$.

desired result.

Let P be a hyperelliptic polygon. By Corollary II.2.2, the group generated by the T_i, $i = 1,\dots,2g$, is Fuchsian. We call Γ the group associated to P.

Lemma II.2.3. Let P be a hyperelliptic polygon with side pairing transformations T_i, $i = 1,\dots,2g$. Then $T_i(z) = \dfrac{\alpha_i z+\beta_i}{\overline{\beta}_i z+\alpha_i}$ where $|\alpha_i|^2 - |\beta_i|^2 = 1$ and $\alpha_i \in \mathbb{R}$.

Proof. Each T_i leaves D fixed and is thus of the form $T_i(z) = \dfrac{\alpha_i z+\beta_i}{\overline{\beta}_i z+\overline{\alpha}_i}$. Since $T_i(A_i) = j(A_i)$ with the reverse

orientation, $T_i(a_i) = -a_{i+1}$ and $T_i(a_{i+1}) = -a_i$, where a_i and a_{i+1} are the vertices of $A_i = \langle a_i, a_{i+1}\rangle$ and $a_i \neq a_{i+1}$. Therefore $T_i(a_i) = \dfrac{\alpha_i a_i+\beta_i}{\overline{\beta}_i a_i+\overline{\alpha}_i} = -a_{i+1}$ and $T_i(a_{i+1}) = \dfrac{\alpha_i a_{i+1}+\beta_i}{\overline{\beta}_i a_{i+1}+\overline{\alpha}_i} = -a_i$

and we obtain the following equations

$$-a_i(\overline{\beta}_i a_{i+1}+\overline{\alpha}_i) = \alpha_i a_{i+1}+\beta_i \tag{II.2.1}$$

$$-a_{i+1}(\overline{\beta}_i a_i+\overline{\alpha}_i) = \alpha_i a_i+\beta_i \tag{II.2.2}$$

Subtracting (II.2.1) from (II.2.2) we get $(a_i - a_{i+1})\overline{\alpha}_i = \alpha_i(a_i - a_{i+1})$. Since $a_i - a_{i+1} \neq 0$, $\overline{\alpha}_i = \alpha_i$ and $\alpha_i \in \mathbb{R}$.

Lemma II.2.4. Under the same hypothesis of Lemma II.2.3, j conjugates each generator T_i to its inverse. Moreover,

$$T_{i_n}^{\epsilon_{i_n}} \circ \ldots \circ T_{i_1}^{\epsilon_{i_1}} \circ j = j \circ T_{i_n}^{-\epsilon_{i_n}} \circ \ldots \circ T_{i_1}^{-\epsilon_{i_1}},$$

where $1 \leq i_k \leq 2g$, $k \in \mathbb{N}^+$ and $\epsilon_{i_k} = \mp 1$.

Proof. We use induction on n. If $\epsilon_{i_1} = 1$ then by Lemma II.2.3

$$T_{i_1}(z) = \frac{\alpha_{i_1} z+\beta_{i_1}}{\overline{\beta}_{i_1} z+\alpha_{i_1}} \text{ where } \alpha_{i_1} \in \mathbb{R}. \text{ Thus}$$

$$T_{i_1} \circ j(z) = T_{i_1}(-z) = \frac{-\alpha_{i_1} z+\beta_{i_1}}{-\overline{\beta}_{i_1} z+\alpha_{i_1}} = \frac{-(\alpha_{i_1} z+\beta_{i})}{\overline{\beta}_{i_1} z-\alpha_{i_1}}$$

$$= -T_{i_1}^{-1}(z) = j \circ T_{i_1}^{-1}(z)$$

and j conjugates T_{i_1} to its inverse.

Assume $T_{i_{n-1}}^{\epsilon_{i_{n-1}}} \circ \ldots \circ T_{i_1}^{\epsilon_{i_1}} \circ j = j \circ T_{i_{n-1}}^{-\epsilon_{i_{n-1}}} \circ \ldots \circ T_{i_1}^{-\epsilon_{i_1}}$. Let $T = T_{i_{n-1}}^{\epsilon_{i_{n-1}}} \circ \ldots \circ T_{i_1}^{\epsilon_{i_1}}$ and $T^* = T_{i_{n-1}}^{-\epsilon_{i_{n-1}}} \circ \ldots \circ T_{i_1}^{-\epsilon_{i_1}}$. If $\epsilon_{i_n} = 1$ then again using Lemma II.2.3 we obtain

$$\text{(II.2.3)} \quad T_{i_n}(j \circ T^*(z)) = T_{i_n}(-T^*(z)) = -T_{i_n}^{-1}(T^*(z)) = j \circ T_{i_n}^{-1} \circ T^*(z).$$

Since $T \circ j = j \circ T^*$, $T_{i_n}(j \circ T^*(z)) = T_{i_n} \circ T_{i_{n-1}}^{\epsilon_{i_{n-1}}} \circ \ldots \circ T_{i_1}^{\epsilon_{i_1}} \circ j(z)$.

Now from (II.2.3) we obtain

$$T_{i_n} \circ T_{i_{n-1}}^{\epsilon_{i_{n-1}}} \circ \dots \circ T_{i_1}^{\epsilon_{i_1}} \circ j(z) = j \circ T_{i_n}^{-1} \circ T^*(z) = j \circ T_{i_n}^{-1} \circ T_{i_{n-1}}^{-\epsilon_{i_{n-1}}} \circ \dots \circ T_{i_1}^{-\epsilon_{i_1}}(z)$$

which is the desired result. If $\epsilon_{i_n} = -1$ a similar computation yields $T_{i_n}^{-1} \circ T_{i_{n-1}}^{\epsilon_{i_{n-1}}} \circ \dots \circ T_{i_1}^{\epsilon_{i_1}} \circ j(z) = j \circ T_{i_n} \circ T_{i_{n-1}}^{-\epsilon_{i_{n-1}}} \circ \dots T_{i_1}^{-\epsilon_{i_1}}(z)$.

Corollary II.2.5. Let P be a hyperelliptic polygon with associated Fuchsian group Γ. Then $\frac{D}{\Gamma}$ has an involution J.

Proof. Define $J : \frac{D}{\Gamma} \to \frac{D}{\Gamma}$ by $J(\{z\}) = \{j(z)\}$. To show that J is well defined suppose $\{z_0\} = \{z_1\}$. Then there exists a $T \in \Gamma$ such that $T(z_0) = z_1$ and by Lemma II.2.4 $j \circ T(z_0) = T^* \circ j(z_0)$ where $T^* \in \Gamma$. Thus $J(\{z_0\}) = J(\{z_1\})$.

In order to prove that D/Γ is hyperelliptic we will show that the involution J of Corollary II.2.5 has 2g + 2 fixed points. Trivially $\{o\}$ is a fixed point. Since the vertices a_i, $i = 1,\dots,4g$, are equivalent and their orbit $\{a_i\}$ contains the negative of each vertex, $\{a_i\}$ is a fixed point of J. We will show that the hyperbolic midpoints c_i of the sides A_i, $i = 1,\dots,2g$, are also fixed points.

Lemma II.2.6. Let A be a Mobius transformation of the form $A(z) = \frac{\alpha z+\beta}{\overline{\beta}z+\alpha}$ where $|\alpha|^2 - |\beta|^2 = 1$, $\beta \neq 0$ and $\alpha \in \mathbb{R}$. Then $A(z) = -z$ has a unique solution $z_0 \in D$.

Proof. The solutions to $\frac{\alpha z+\beta}{\overline{\beta}z+\alpha} = -z$ are $\frac{-\alpha \mp 1}{\overline{\beta}}$.

$|\frac{-\alpha+1}{\bar{\beta}}| = \sqrt{\frac{\alpha^2-2\alpha+1}{\alpha^2-1}}$ (resp. $|\frac{-\alpha-1}{\bar{\beta}}| = \sqrt{\frac{\alpha^2+2\alpha+1}{\alpha^2-1}}$) and since $\alpha > 1$, $\alpha^2 - 1 > \alpha^2 - 2\alpha + 1$ (resp. $\alpha^2 + 2\alpha + 1 > \alpha^2 - 1$). Thus $1 > \frac{\alpha^2-2\alpha+1}{\alpha^2-1}$ (resp. $\frac{\alpha^2+2\alpha+1}{\alpha^2-1} > 1$) and $1 > |\frac{-\alpha+1}{\bar{\beta}}|$ (resp. $1 < |\frac{-\alpha-1}{\bar{\beta}}|$).

The generating transformations T_i, $i = 1,\ldots,2g$, of a group Γ associated to a hyperelliptic polygon are of the form $T_i(z) = \frac{\alpha_i z+\beta_i}{\bar{\beta}_i z+\alpha_i}$, where $\alpha_i \in \mathbb{R}$. By Lemma II.2.6, $T_i(z) = -z$ has a unique solution $c_i \in D$. $\{c_i\}$ is thus a fixed point of J. To show that the $\{c_i\}$, $i = 1,\ldots,2g$, are distinct we need to know that they are not congruent modulo Γ.

<u>Lemma</u> II.2.7. Let T_i map $A_i = \langle a_i, a_{i+1}\rangle$ onto $\langle -a_i, -a_{i+1}\rangle$ and let c_i be the hyperbolic midpoint of $\langle a_i, a_{i+1}\rangle$. Then $T_i(c_i) = -c_i$.

<u>Proof</u>. Since c_i is the hyperbolic midpoint of A_i (see Fig. 1) $\int_{a_i}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{a_{i+1}}^{c_i} \frac{|dz|}{1-|z|^2}$ where the line integral is along the unique geodesic joining the limits of integration. But the Poincaré metric is invariant under the transformations j and T_i. Hence

$$\int_{-a_i}^{-c_i} \frac{|dz|}{1-|z|^2} = \int_{a_i}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{a_{i+1}}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{T_i(a_{i+1})}^{T_i(c_i)} \frac{|dz|}{1-|z|^2} .$$

$$\text{Thus } \int_{-a_i}^{-c_i} \frac{|dz|}{1-|z|^2} = \int_{T_i(a_{i+1})}^{T_i(c_i)} \frac{|dz|}{1-|z|^2} = \int_{-a_i}^{T_i(c_i)} \frac{|dz|}{1-|z|^2}$$

and $T_i(c_i) = -c_i$.

We have shown that the c_i, $i = 1,\ldots,2g$, lie on a fundamental polygon for Γ. They are distinct since the sides A_i intersect only at vertices and none of the c_i is a vertex.

Since P is a fundamental polygon and identifications occur in pairs it is obvious that the c_i are not congruent. Similarly 0 is not congruent to any of the vertices or the hyperbolic midpoints and none of the hyperbolic midpoints is congruent to a vertex.

Thus $\{0\}$, $\{a_1\}$, $\{c_1\},\ldots,\{c_{2g}\}$ are distinct fixed points of J. Since it is well known that a conformal automorphism of a compact surface of genus g has at most 2g+2 fixed points, these are all the fixed points of J.

We have proven the following:

<u>Lemma</u> II.2.8. Let P be a hyperelliptic polygon with associated Fuchsian group Γ. Then D/Γ is hyperelliptic and the fixed points of the hyperelliptic involution are $\{0\}$, $\{a_1\}$ where a_1 is the first vertex, and $\{c_i\}$, $i = 1,\ldots,2g$, where c_i is the hyperbolic midpoint of the side A_i.

II.3. Parameters. Before proving that any hyperelliptic surface of genus g may be uniformized by a fixed point free Fuchsian group which has a hyperelliptic fundamental polygon, one would like to know that the number of parameters which determine hyperelliptic polygons agrees with the dimension of the Teichmüller space of hyperelliptic surfaces (see Chapter III).

In this section we informally construct a space of hyperelliptic polygons which is disconnected and not well defined. Our construction will show that this space depends on 4g-2 real parameters. On the other hand, the Teichmuller space of hyperelliptic surfaces is a complex manifold of dimension 2g-1.

Let $R_1,\ldots,R_{2g}$ be rays through the origin such that $\sum_{i=1}^{2g-1} \alpha_i < \pi$, where $\alpha_i > 0$, $i = 1,\ldots,2g-1$, is the angle between R_i and R_{i+1}. Clearly there are 2g-1 real parameters which determine this construction. We let $R_1 = R$, the real axis (Fig. 2).

We now fix a choice of rays $R_1,\ldots,R_{2g}$ and construct a 2g-1 real parameter family of polygons with vertices on the given rays satisfying properties i) and ii) of II.1, page 6.

Let t_i and $-t_i$, $i = 1,\ldots,2g$, be the points of intersection of R_i with $|z| = 1$. Let L_1 be the arc of a circle orthogonal to $|z| = 1$ which joins the points $-t_{2g}$ and t_2 and lies inside $\overline{D}$. (We call L_1 the hyperbolic line joining $-t_{2g}$ and t_2 (Fig. 3).)

L_1 intersects R_1 at a point r_1, $0 < |r_1| < 1$.

Choose $a_1 \in \mathbf{R}_1$ such that $r_1 < a_1 < 1$ and construct the polygon $P_1 = \langle a_1, t_2, t_3, \ldots, t_{2g}, -a_1, -t_2, \ldots, -t_{2g}, a_1 \rangle$ (Fig. 4). ($\langle z_1, z_2, \ldots, z_n, z_1 \rangle$ denotes the hyperbolic polygon formed by joining the points z_i, $i = 1, \ldots, n$, in their given order by hyperbolic lines.) Each angle at the vertices of P_1 is 0 except for the angle β_1 at the vertices a_1 and $-a_1$. Since $r_1 < a_1 < 1$, $\beta_1 < \pi$ and $\hat{P}_1$ is less than 2π.

Let L_2 be the hyperbolic line joining a_1 to t_3 and let t_2' be the point of intersection of L_2 with R_2. Let $P_1' = \langle a_1, t_2', t_3, \ldots, t_{2g}, -a_1, -t_2', -t_3, \ldots, -t_{2g}, a_1 \rangle$ (Fig. 5).

$\hat{P}_1' > 2\pi$ since the angle β_2 at the vertices t_2' and $-t_2'$ is equal to π.

For each choice of $r \in R_2$ such that $|t_2'| \leq |r| \leq |t_2|$ we obtain a polygon

$P_1^r = \langle a_1, r, t_3, \ldots, t_{2g}, -a_1, -r, -t_3, \ldots, -t_{2g}, a_1 \rangle$ and $P_1^{t_2} = P_1$, $P_1^{t_2'} = P_1'$ (Fig. 6).

As r varies from $|t_2'|$ to $|t_2|$ along R_2, $\widehat{P_1^r}$ is strictly decreasing from $k_1 > 2\pi$ to $2\beta_1 < 2\pi$. (This last fact may be shown by isolating the triangles $\langle a_1, r, t_3, a_1 \rangle$. As r varies the sum of the interior angles of $\langle a_1, r, t_3, a_1 \rangle$ is strictly decreasing since the hyperbolic area of $\langle a_1, r, t_3, a_1 \rangle$ is strictly increasing (Fig. 7).[4])

[4] The continuity of $\widehat{P_1^r}$ will be shown in Lemma II.4.15.

Thus for some $r_2 \in R_2$, $|t_2'| < |r_2| < |t_2|$, $\hat{P}_1^{r_2} = 2\pi$ and $\hat{P}_1^{r} < 2\pi$ for all $r \in R_2$ such that $|r_2| < |r| < |t_2|$.

Choose $a_2 \in R_2$ such that $|r_2| < |a_2| < |t_2|$.

We now repeat the same process to find an interval from which to choose $a_3 \in R_3$.

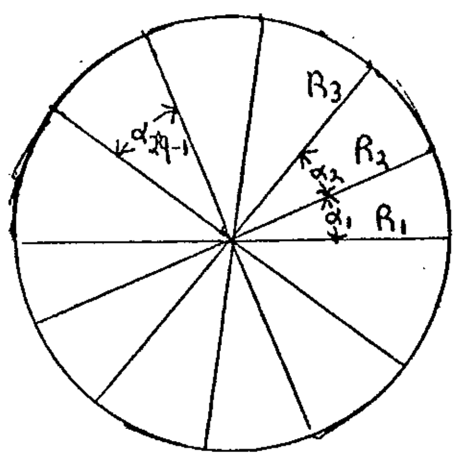


Fig. 2

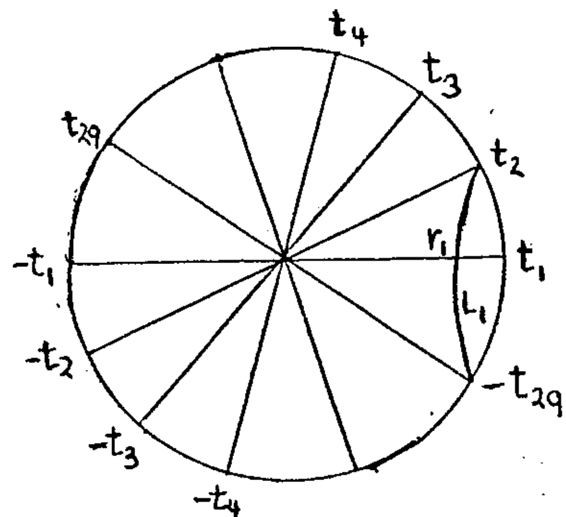


Fig. 3

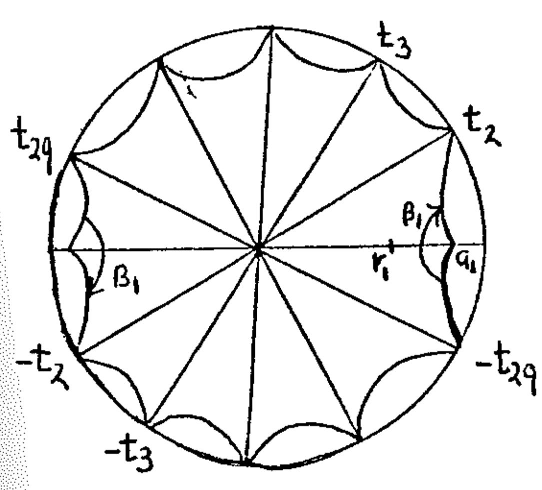


Fig. 4

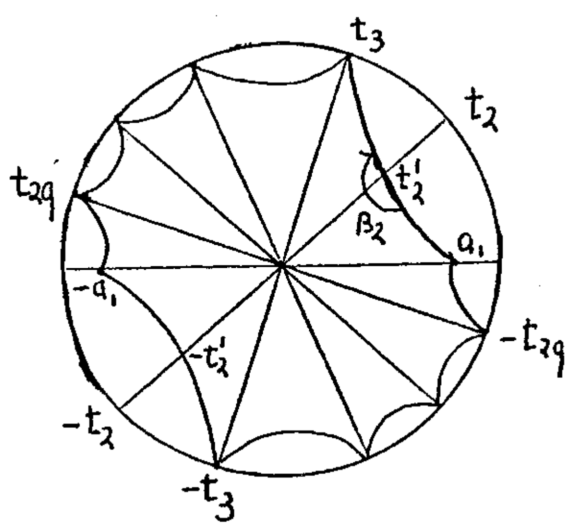


Fig. 5

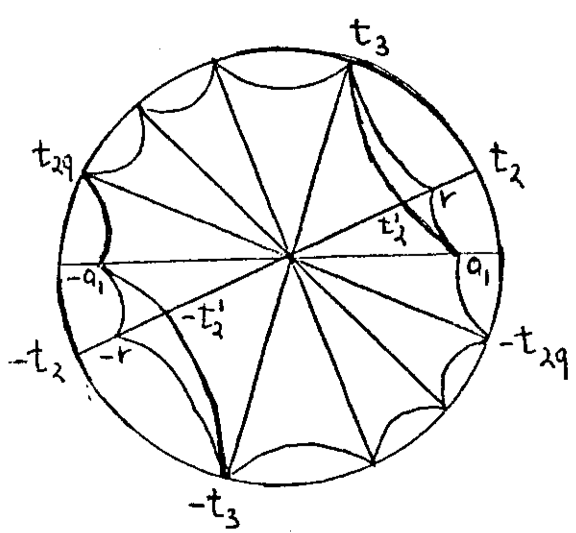


Fig. 6

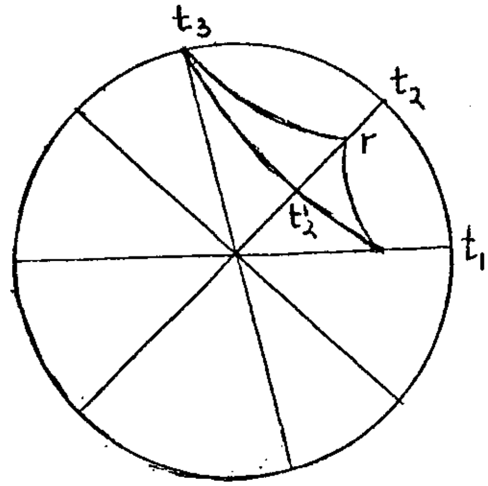


Fig. 7

Let L_3 be the hyperbolic line joining a_2 to t_4 and let t_3' be the point of intersection of L_3 with R_3. Construct the polygons $P_2' = \langle a_1, a_2, t_3', t_4, \ldots t_{2g}, -a_1, -a_2, -t_3', \ldots, -t_{2g}, a_1\rangle$ and $P_2 = \langle a_1, a_2, t_3, t_4, \ldots, t_{2g}, -a_1, -a_2, -t_3, -t_4, \ldots, -t_{2g}, a_1\rangle$. For each $r \in R_3$ such that $|t_3'| < |r| < |t_3|$ let $P_2^r = \langle a_1, a_2, r, t_4, \ldots, t_{2g}, -a_1, -a_2, -r, -t_4, \ldots, -t_{2g}, a_1\rangle$.

As in the previous case there is an $r_3 \in R_3$, $|t_3'| < |r_3| < |t_3|$, such that $\widehat{P_2^{r_3}} = 2\pi$; and for all $r \in R_3$ such that $|r_3| < |r| < |t_3|$, $\widehat{P_2^{r}} < 2\pi$ (Fig. 8).

We continue this process and choose points $a_i \in R_i$, $i = 1, \ldots, 2g-1$, such that $\widehat{P_{2g-2}^{a_{2g-1}}} < 2\pi$ (Fig. 9). Let L_{2g} be the

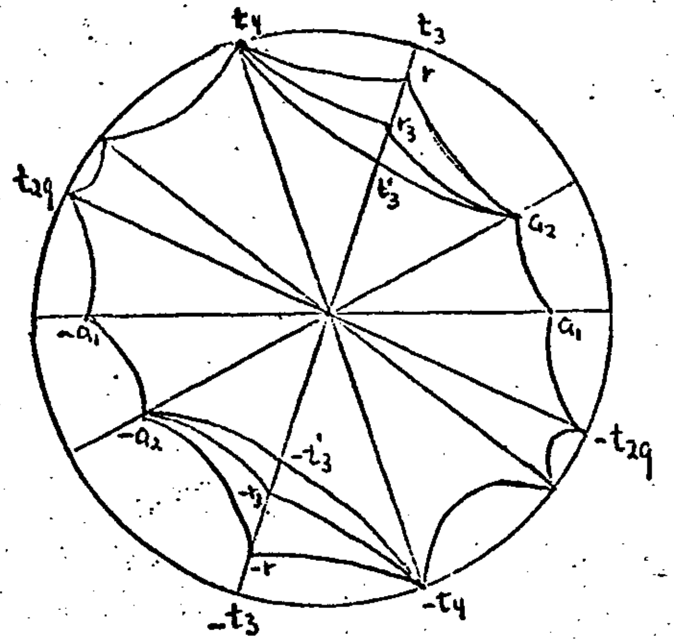

t4
t3
r
r3
t'3
a2
t2q
a1
-a1
-a2
-t2q
-t'3
-r3
-r
-t3
-t4

Fig. 8

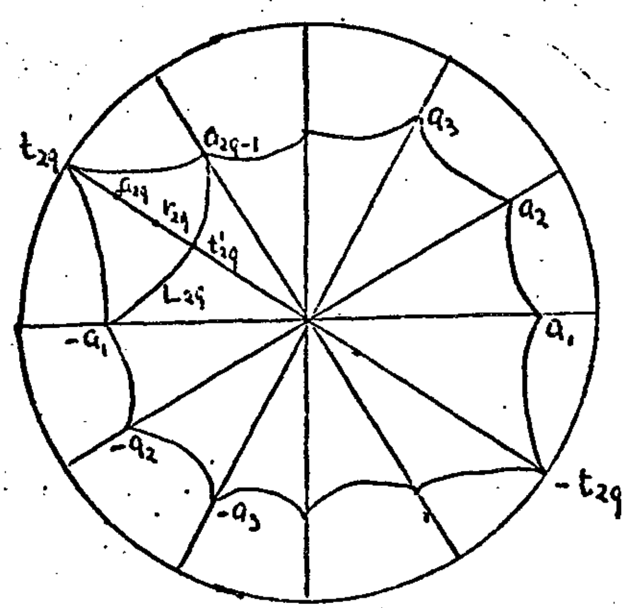

t2q
a2q-1
a3
a2q
r2q
t'2q
a2
L2q
-a1
a1
-a2
-a3
-t2q

Fig. 9

hyperbolic line joining a_{2g-1} to $-a_1$ and let t'_{2g} be the intersection of L_{2g} with R_g. As before there exists a uniquely determined point $a_{2g} \in R_{2g}$ such that $P_{2g-1}^{\hat{a}_{2g}} = 2\pi$.

Thus we have constructed a 4g-2 real parameter family of polygons satisfying properties i) and ii).

As it stands we need to know the existence of Mobius transformations pairing opposite sides of each polygon (property iii)) before we can claim they are hyperelliptic polygons.

The following Lemma shows that they all satisfy property iii).

<u>Lemma</u> II.3.1. Let $a, b \in D - \{0\}$, $a \neq b$. Then there exists a unique Mobius transformation A which fixes D such that $A(a) = -b$ and $A(b) = -a$.

Proof. Since both determinants

$$D_1 = \left| \frac{1-|b|^2}{1-|a|^2} \right| > 0 \qquad D_2 = \begin{vmatrix} \frac{1-|b|^2}{1-|a|^2} & \frac{1-b\bar{a}}{1-a\bar{b}} \\ \frac{1-a\bar{b}}{1-b\bar{a}} & \frac{1-|a|^2}{1-|b|^2} \end{vmatrix} = 0$$

are ≥ 0, we have sufficient conditions for the existence of an analytic map A with the desired properties (Ahlfors [1]).

By Pick's lemma and the equality $\frac{|A(a)-A(b)|}{|1-\overline{A(a)}\,A(b)|} = \frac{|a-b|}{|1-\bar{a}b|}$, A is fractional linear (Ahlfors [1]).

To prove uniqueness let B be another transformation with the same properties. By Lemma II.2.7 both A and B map the hyperbolic midpoint of the line joining a to b onto the hyperbolic midpoint of the line joining -a to -b. Thus $B^{-1}A$ fixes the line joining a to b and $B^{-1}A = \mathrm{id}$.

We include a geometric proof of the existence part of Lemma II.3.1.

Proof. We will first construct a transformation $T \in \mathrm{Aut}\, D$ such that $T(a) = r > 0$ and $T(b) = -r$. Let c be the hyperbolic midpoint of the line segment $\langle a,b \rangle$ and let $A_1 \in \mathrm{Aut}\, D$ be such that $A_1(c) = 0$ (see Ford [9]). A_1 maps the line segment $\langle a,b \rangle$ onto a line segment through the origin with zero as midpoint. Let A_2 be a rotation about the origin such that

$A_2 \circ A_1(a) = r > 0$ and $A_2 \circ A_1(b) = -r$. Let

$T = A_2 \circ A_1$ and $A = j \circ T^{-1} \circ j \circ T$.

Now $A(a) = j \circ T^{-1} \circ j \circ T(a) = j \circ T^{-1}(-r) = -b$.

Similarly $A(b) = -a$ and trivially $A \in \mathrm{Aut}\, D$.

II.4. Continuity argument. Let S be a surface of genus g with a hyperelliptic involution J and let $w_1, \ldots, w_{2g+2}$ be the Weierstrass points of S. We denote this data by the symbol $(S, J, w_1, \ldots, w_{2g+2})$.

In this section we will construct a hyperelliptic polygon and, by means of quasiconformal maps, deform the polygon to obtain a new hyperelliptic polygon whose associated Fuchsian group uniformizes S.

We begin by defining quasiconformal maps. A homeomorphism w of the open unit disc D onto itself is quasiconformal (or μ-conformal) if it has generalized locally square integrable derivatives which satisfy the equation $w_{\bar{z}} = \mu(z)w_z$ almost everywhere on D, where μ is a measurable complex valued function on D with ess. sup. $|\mu(z)| = k < 1$.

We say μ is a Beltrami coefficient for a Fuchsian group Γ if $\mu \circ A = \frac{A'}{\overline{A'}}\mu$ for all $A \in \Gamma$.

The following theorem will be needed.

Theorem II.4.1 (Ahlfors-Bers [3]). Let μ be a measurable complex valued function on D with ess. sup. $|\mu(z)| = k < 1$. Then there exists a unique quasiconformal self map w^μ of D satisfying the Beltrami equation $w^\mu_{\bar{z}} = w^\mu_z \mu(z)$ and normalized by $w^\mu(0) = 0$, $w^\mu(1) = 1$.

We now construct a specific hyperelliptic polygon which is somewhat easier to work with. Subdivide the unit disc D by lines $R_1, \ldots, R_{2g}$ through the origin such that the angle between R_i and R_{i+1} is $\frac{2\pi}{4g}$. Let R_1 be on the real axis (Fig. 10).

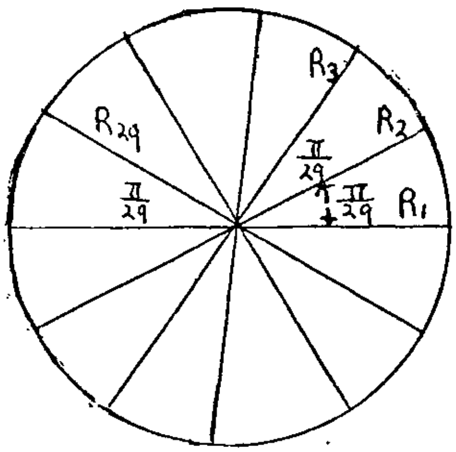


Fig. 10

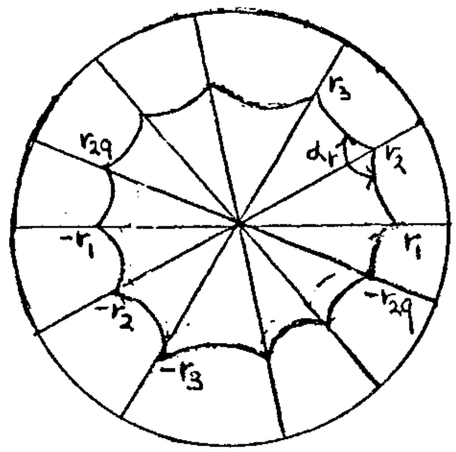


Fig. 11

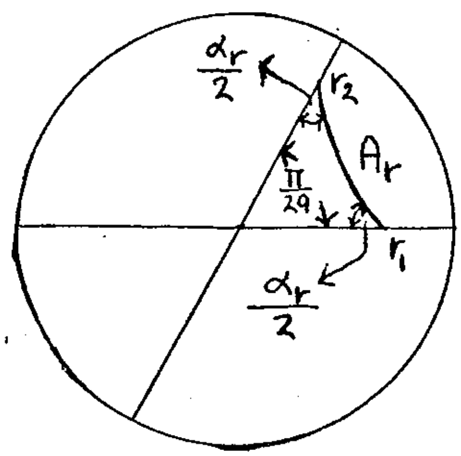


Fig. 12

For each $0 < r < 1$ choose points r_i, $-r_i \in R_i$ such that $|r_i| = r$ and construct the non-eculidean polygon $P_r = \langle r_1, \ldots, r_{2g}, -r_1, \ldots, -r_{2g}, r_1 \rangle$ (Fig. 11).

The interior angle α_r at each vertex of P_r is strictly increasing from 0 to $\Pi - \frac{2\pi}{4g}$ as r decreases from 1 to 0. We show this fact by isolating the hyperbolic triangle $A_r = \langle 0, r_1, r_2, 0 \rangle$ (Fig. 12). The hyperbolic area $|A_r|$ of A_r is strictly decreasing to 0 as $r \to 0$. Thus $|A_r| = \Pi - \frac{2\pi}{4g} - \alpha_r \to 0$ as $r \to 0$ and $\alpha_r \to \Pi - \frac{2\pi}{4g}$.[5]

The sum of the interior angles of P_r is thus strictly increasing from 0 to $4g(\Pi - \frac{2\pi}{4g}) = \Pi(4g-2) > 2\pi$. Hence there exists an $r_0 \in (0,1)$ for which $\hat{P}_{r_0} = 2\pi$. We let $P_{r_0} = P^*$ and call it the <u>perfectly symmetric polygon</u>.

If we label the sides of P^* by A_i^*, $i = 1, \ldots, 4g$, (in counterclockwise order) then Lemma II.3.2 guarantees the existence of Mobius transformations T_i^*, $i = 1, \ldots, 2g$, identifying opposite sides so that P^* is a hyperelliptic polygon.

Let Γ^* be the group associated to P^* (as defined in II.2) and denote the Weierstrass points of $\frac{D}{\Gamma^*}$ as follows: $w_i^* = \{c_i^*\}$, $i = 1, \ldots, 2g$, where c_i^* is the hyperbolic midpoint of A_i^*, $w_{2g+1}^* = \{0\}$ and $w_{2g+2}^* = \{a_j^*\}$ where a_j^* is any vertex.

We extend the group Γ^* by the map $z \mapsto -z$ to obtain the group $\{j, \Gamma^*\}$. As is well known, $\frac{D}{\{j,\Gamma^*\}}$ is conformally a sphere with 2g+2 distinguished points $\{w_1^*\}, \ldots, \{w_{2g+2}^*\}$, each of which

5 The continuity of α_r will be shown in Lemma II.4.15.

is the fixed point of an elliptic element of order two in $\{j,\Gamma^*\}$.

Choose a quasiconformal map $f: \frac{D}{\{j,\Gamma^*\}} \to \frac{S}{J}$ with the property that $f(\{w_i^*\}) = w_i$, $i = 1,\ldots,2g+2$. Such a map exists by Bers' theorem [5]. The map f defines on P* a function $\mu(z) = \frac{f_{\bar z}}{f_z}$ and we extend μ to all of D by requiring that it be compatible with $\{j,\Gamma^*\}$. μ is thus a Beltrami coefficient for $\{j,\Gamma^*\}$.

Throughout the remainder of this section P^*, f, and $(S,J,w_1,\ldots,w_{2g+2})$ will remain fixed.

<u>Lemma</u> II.4.2. Let μ be a Beltrami coefficient for the group $\{j,\Gamma^*\}$ and let w^μ be the unique normalized quasiconformal self map of D such that $w^\mu_{\bar z} = w^\mu_z\mu(z)$. Then $w^\mu \circ j = j \circ w^\mu$.

<u>Proof</u>. By Bers' theorem [5] $w^\mu \circ j \circ (w^\mu)^{-1}$ is an elliptic transformation of order two which fixes D with fixed point $w^\mu(0) = 0$. Thus $w^\mu \circ j \circ (w^\mu)^{-1} = j$.

Let $a_1^*,\ldots,a_{4g}^*$ be the vertices of P* and let $t \in [0,1]$. Trivially $t\mu$ is also a Beltrami coefficient for the group $\{j,\Gamma^*\}$. We define $P_{t\mu} = \langle w^{t\mu}(a_1^*),\ldots,w^{t\mu}(a_{4g}^*),w^{t\mu}(a_1^*)\rangle$.

For convenience we denote each vertex $w^{t\mu}(a_i^*)$, $i = 1,\ldots,4g$, by a_i^t. Each side $A_i^t = \langle a_i^t, a_{i+1}^t\rangle$, $i = 1,\ldots,4g-1$, is oriented from the lower to the higher index. When $i = 4g$ we orient $A_i^t = \langle a_{4g}^t, a_1^t\rangle$ from a_{4g}^t to a_1^t.

The angle $0 \leq \alpha_i^t < 2\pi$ at the vertex a_i^t to the left of the line A_{i-1}^t, $i = 2,\ldots,4g$, is called the <u>interior angle</u> of $P_{t\mu}$ at a_i^t. When $i = 1$, $0 \leq \alpha_1^t < \Pi$ is the angle to the left of A_{4g}^t. $\hat{P}_{t_\mu}$ is the sum of the interior angles of $P_{t\mu}$.

<u>Lemma</u> II.4.3. Let $t \in [0,1]$. Then the following properties hold.

i) $P_{t\mu}$ is j-invariant.

ii) The transformation $T_i^t = w^{t\mu} \circ T_i^* \circ (w^{t\mu})^{-1}$, $i = 1,\ldots,2g$, identify opposite sides of $P_{t\mu}$.

<u>Proof</u>.

i) This is a trivial consequence of Lemma II.4.2 and the fact that j is an isometry.

ii) Consider the sides $A_i^t = \langle a_i^t, a_{i+1}^t \rangle$, $i = 1,\ldots,2g$. Then $T_i^t(a_i^t) = w^{t\mu} \circ T_i^* \circ w^{t\mu -1}(w^{t\mu}(a_i^*)) = w^{t\mu} \circ T_i^*(a_i^*)$ $= w^{t\mu}(-a_{i+1}^*) = -a_{i+1}^t$. In the same manner $T_i^t(a_{i+1}^t) = -a_i^t$. By Bers' theorem [5] the elements T_i^t are Mobius transformations which preserve the unit disc. Since A_i^t is the geodesic joining a_i^t with a_{i+1}^t $T_i^t(A_i^t) = -A_i^t$.

Before proceeding we make two observations.

First, the group $\Gamma^t = \{T_i^t\}$, $i = 1,\ldots,2g$, is a Fuchsian group (see Bers [5]). Second, since the transformations T_i^t

identify opposite sides of $P_{t\mu}$, the same argument used in II.2.1 shows that all vertices are equivalent under Γ^t.

In order to prove that $P_{t\mu}$ is a simple polygon we use a continuity argument. We will show in Lemmas II.4.13 and II.4.14 that $A = \{t \in [0,1] : P_{t\mu}$ is a simple polygon$\}$ is both open and closed and thus equal to $[0,1]$.

We will need the preliminary Lemmas II.4.4-11.

Lemma II.4.4. Let $G = \{\gamma_1, \gamma_2, \gamma_3\}$ be a Fuchsian group generated by three elliptic transformations γ_1, γ_2 and γ_3 of order two whose fixed points are not equivalent in G. Then the fixed points are not colinear.

Proof. Suppose the fixed points z_1, z_2 and z_3 lie on a hyperbolic line L. Since γ_1, γ_2 and γ_3 are elliptic of order two, L is invariant under G. Since G is Fuchsian, the unit circle is also invariant under G. Thus the two points of intersection of L with the unit circle is a closed invariant set. Since any closed invariant set consisting of at least two points must contain all the limit points (see Ford [9]), G has at most two limit points. G is thus a finitely generated elementary Fuchsian group. This is a contradiction since the only finitely generated elementary Fuchsian groups are either cyclic or have exactly two elliptic conjugacy classes (see Greenberg [10]). On the other hand, G has three non-conjugate elliptic elements.

Lemma II.4.5. No two distinct sides of $P_{t\mu}$ are colinear.

Proof. Suppose the two sides $A_i^t = \langle a_i^t, a_{i+1}^t \rangle$ and $A_l^t = \langle a_l^t, a_{l+1}^t \rangle$, $0 < i, l \leq 4g$, $i \neq l$ are colinear.

According to Lemma II.4.3 the transformations T_i^t and T_l^t identify each side with its corresponding opposite side. Thus the transformations $-T_i^t$ and $-T_l^t$ are elliptic of order two with fixed points at the midpoints of the sides A_i^t and A_l^t respectively.

Since all vertices of $P_{t\mu}$ are equivalent there exists a transformation $T_{a_l} \in \Gamma^t$ such that $T_{a_l}(a_l^t) = -a_l^t$. Thus $-T_{a_l}$ is also elliptic of order two and its fixed point is a_l^t.

Let $G = \{-T_i^t, -T_l^t, -T_{a_l}\} \subset \{j, \Gamma^t\}$ be the group generated by $-T_i^t$, $-T_l^t$ and $-T_{a_l}^t$. Note that no two of these elements can be conjugate since the corresponding elements in $\{j, \Gamma^*\}$ are not conjugate (Lemma II.2.8). Moreover, by our original assumption the fixed points of these elements are colinear. This contradicts Lemma II.4.4.

Corollary II.4.6. Let α_i^t be the interior angles of $P_{t\mu}$, $i = 1, \ldots, 4g$. Then $\alpha_i^t \neq 0$ and $\alpha_i^t \neq \pi$

Proof. It suffices to note that if $\alpha_i^t = 0$ or π then the two sides of $P_{t\mu}$ which determine α_i^t are colinear. This contradicts Lemma II.4.5.

Throughout the remainder of the section we will adopt the following conventions.

Given distinct points $z_1, z_2 \in D$, $L_{z_1}^{z_2}$ denotes the hyperbolic line determined by z_1 and z_2 which is oriented from z_1 to z_2.

Given a hyperbolic line segment $\langle z_1, z_2 \rangle$, we say that $\langle z_1, z_2 \rangle$ separates the set A from the set B if A and B lie on distinct sides of $L_{z_1}^{z_2}$.

The notation $U_{z_1}^{\epsilon}$ will mean an ϵ-neighborhood centered at z_1.

Lemma II.4.7 and Corollary II.4.8 are obvious and we omit the proofs.

Lemma II.4.7. Let $a, b, c \in D$ be distinct points and let c lie to the left of L_a^b. Then there exists $\epsilon > 0$ such that the ϵ-neighborhoods U_a^{ϵ}, U_b^{ϵ} and U_c^{ϵ} have the following properties.

i) U_a^{ϵ}, U_b^{ϵ} and U_c^{ϵ} are pairwise disjoint.

ii) Given $a_1 \in U_a^{\epsilon}$, $b_1 \in U_b^{\epsilon}$ and $c_1 \in U_c^{\epsilon}$ then c_1 lies to the left of $L_{a_1}^{b_1}$.

Corollary II.4.8. Let $a, b, c, d \in D$ be distinct points and let L_a^b separate c from d. Then there exists $\epsilon > 0$ such that the ϵ-neighborhoods U_a^{ϵ}, U_b^{ϵ}, U_c^{ϵ} and U_d^{ϵ} have the following properties.

i) U_a^{ϵ}, U_b^{ϵ}, U_c^{ϵ} and U_d^{ϵ} are pairwise disjoint.

ii) Given $a_1 \in U_a^{\epsilon}, b_1 \in U_b^{\epsilon}, c_1 \in U_c^{\epsilon}$ and $d_1 \in U_d^{\epsilon}$ the line $L_{a_1}^{b_1}$ separates c_1 from d_1.

Lemma II.4.9. Let α_i^t, $i = 1,\ldots,4g$, be the interior angles of $P_{t\mu}$. Then $0 < \alpha_i^t < \pi$.

Proof. We will use a continuity argument. We assume i is fixed and for simplicity $i \neq 1$. By relabeling suitably the same proof may be used when $i = 1$.

Let $B_i = \{t \in [0,1]$ such that $0 < \alpha_i^t < \pi\}$. We note first that B_i is not empty since each interior angle of P^* lies in the interval $(0,\pi - \frac{2\pi}{4g})$ (see p.23).

To show that B_i is open let $t_o \in B_i$. Thus $a_{i+1}^{t_o}$ lies to the left of $A_{i-1}^{t_o} = \langle a_{i-1}^{t_o}, a_i^{t_o}\rangle$. Let $\epsilon > 0$ be such that the ϵ-neighborhoods $U^\epsilon_{a_{i-1}^{t_o}}$, $U^\epsilon_{a_i^{t_o}}$ and $U^\epsilon_{a_{i+1}^{t_o}}$ satisfy the properties of Lemma II.4.7. (Fig. 13).

For each m, $m = i-1, i, i+1$, the map $t \xrightarrow{fm} w^{t\mu}(a_m^*)$ is continuous (see [3]). Thus there exist $0 < \delta_m$ such that $w^{t\mu}(a_m^*) \in U^\epsilon_{a_m^{t_o}}$ for all $|t_o - t| < \delta_m$, $t \in [0,1]$. Let $\delta = \min\{\delta_m\}$. Then for all $|t_o - t| < \delta$, $t \in [0,1]$, $w^{t\mu}(a_m^*) \in U^\epsilon_{a_m^{t_o}}$. Thus the vertex a_{i+1}^t lies to the left of the geodesic line determined by A_{i-1}^t and $0 < \alpha_i^t < \pi$. Consequently, B_i is open.

In the above we have assumed $0 < t_o < 1$. A similar argument works for $t_o = 0$ and $t_o = 1$.

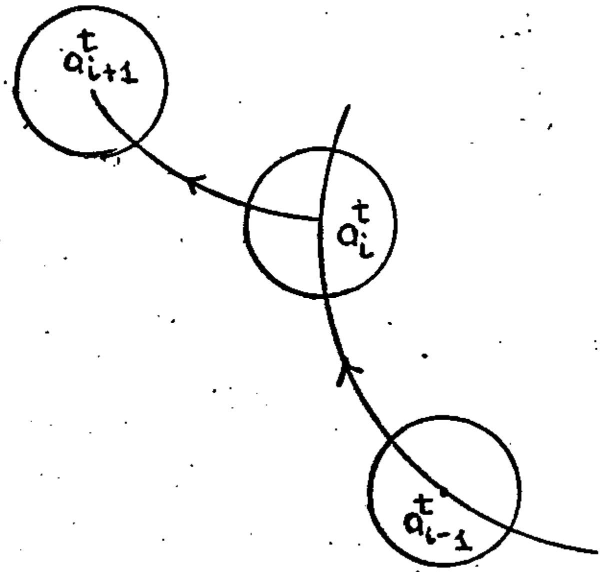


Fig. 13

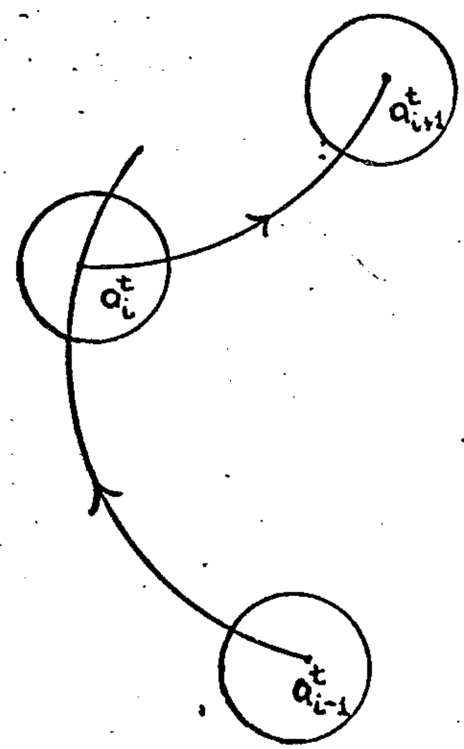


Fig. 14

To show that B_i is closed let $t_k \to t$ where $t_k \in B_i$. By Corollary II.4.6 we know that $\alpha_i^t \neq 0$ or π. Thus we assume $\alpha_i^t > \pi$ and a_{i+1}^t lies to the right of A_{i-1}^t (Fig. 14).

Let $\epsilon > 0$ be such that the ϵ-neighborhoods $U^\epsilon_{a^t_{i-1}}$, $U^\epsilon_{a^t_i}$ and $U^\epsilon_{a^t_{i+1}}$ satisfy the properties of Lemma II.4.7 (with the orientation reversed). Let $t_{k_n} \in B_i$ be such that $w^{t_{k_n}\mu}(a_m^*) \in U^\epsilon_{a^t_m}$, $m = i-1, i, i+1$. Thus the point $a_{i+1}^{t_{k_n}}$ lies to the right of $A_{i-1}^{t_{k_n}}$ and $\alpha_i^{t_{k_n}} > \pi$. This contradicts $t_{k_n} \in B_i$.

We have shown that $0 < \alpha_i^t < \pi$ and B_i is closed. Since B_i is both open and closed $B_i = [0,1]$.

<u>Corollary</u> II.4.10. Let $P_{t\mu}$ $t \in [0,1]$ be simple. Then $P_{t\mu}$ is convex.

<u>Proof</u>. By Lemma II.4.9 $0 < \alpha_i^t < \pi$, $i = 1,\ldots,4g$. A simple polygon with this property is convex (see Magnus [14]).

<u>Lemma</u> II.4.11. Let $P \subset D$ be a simple, convex, hyperbolic polygon with sides A_i, $i = 1,\ldots,4g$. Then there exist convex neighborhoods H_i of A_i with the following property:

$$H_k \cap H_\ell = \emptyset \text{ whenever } A_k \cap A_\ell = \emptyset,\ 1 \le k,\ell \le 4g.$$

<u>Proof</u>. For each fixed side A_i, $i = 1,\ldots,4g$, of P, we

denote by $A_{i_1}, \ldots, A_{i_{4g-3}}$ the sides of P which are disjoint from A_i.

Clearly, since P is convex, there exists for each side A_i a line L_i which separates A_i from $\bigcup_{j=1}^{4g-3} A_{i_j}$ (Fig. 15). Let H_i' (resp. H_i'') be the open half plane determined by L_i which contains A_i (resp. $\bigcup_{j=1}^{4g-3} A_{i_j}$). Note that H_i' and H_i'' are disjoint, open, convex sets.

Let $H_i = H_i' \cap (\bigcap_{j=1}^{4g-3} H_{i_j}'')$. We note that H_i is open and convex. Moreover, $A_i \subset H_i'$ and since A_{i_j}, $j = 1, \ldots, 4g-3$, is not adjacent to A_i, $A_i \subset H_{i_j}''$. Thus $A_i \subset H_i$.

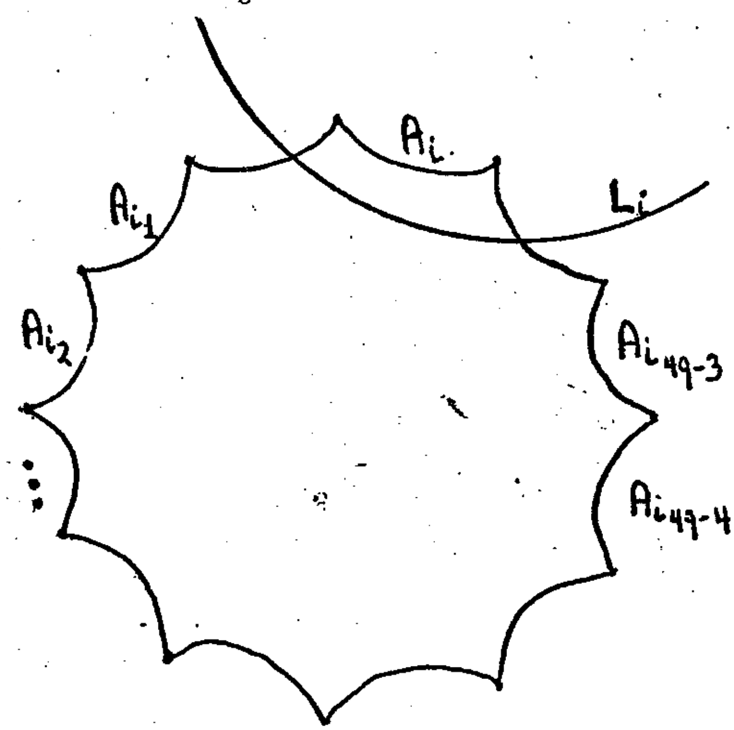


Fig. 15

Suppose A_k and A_ℓ, $1 \leq k, \ell \leq 4g$, are disjoint sides. If $z \in H_\ell$, then $z \in H_\ell'$ and $z \notin H_\ell''$. But $A_\ell = A_{k_r}''$ for some $1 \leq r \leq 4g-3$. Thus $z \notin H_{k_r}''$ and $z \notin H_k = H_k' \cap (\bigcap_{j=1}^{4g-3} H_{k_j}'')$. Hence $H_\ell \cap H_k = \emptyset$.

Corollary II.4.12. Let $P \subset D$ be a simple, convex, hyperbolic polygon with vertices a_i, $i = 1,\ldots,4g$. Then there exists $\epsilon > 0$ such that the ϵ-neighborhoods $U_{a_i}^\epsilon$ have the following properties.

i) $U_{a_i}^\epsilon \cap U_{a_j}^\epsilon = \emptyset$ whenever $a_i \neq a_j$, $1 \leq i,j \leq 4g$

ii) Given $z_i \in U_{a_i}^\epsilon$, $i = 1,\ldots,4g$, the non-adjacent sides of polygon $\langle z_1,\ldots,z_{4g},z_1\rangle$ are disjoint.

Proof. Let H_i, $i = 1,\ldots,4g$, be convex neighborhoods as described in Lemma II.4.11. Thus $a_i \in H_i \cap H_{i-1}$, $i = 2,\ldots,4g$, and $a_1 \in H_1 \cap H_{4g}$. Choose $\epsilon > 0$ small enough so that $a_i \in U_{a_i}^\epsilon \subset H_i \cap H_{i-1}$, $i = 2,\ldots,4g$, and $a_1 \in U_{a_1}^\epsilon \subset H_1 \cap H_{4g}$.

Clearly, the $U_{a_i}^\epsilon$, $i = 1,\ldots,4g$, are pairwise disjoint.

We need to verify property ii).

Let $\langle z_i,z_{i+1}\rangle$ and $\langle z_j,z_{j+1}\rangle$, $1 \leq i, j < 4g$, be non-adjacent sides of $\langle z_1,\ldots,z_{4g},z_1\rangle$.

Clearly, $A_i = \langle a_i,a_{i+1}\rangle$ and $A_j = \langle a_j,a_{j+1}\rangle$ are not adjacent. Thus $H_i \cap H_j = \emptyset$. But $U_{a_i}^\epsilon$, $U_{a_j}^\epsilon \subset H_i$ and $U_{a_j}^\epsilon$, $U_{a_{j+1}}^\epsilon \subset H_j$, thus $\langle z_i,z_{i+1}\rangle \subset H_i$ and $\langle z_j,z_{j+1}\rangle \subset H_j$. Therefore $\langle z_i,z_{i+1}\rangle \cap\langle z_j,z_{j+1}\rangle = \emptyset$. The same argument works if one of the sides is of the form $\langle z_{4g},z_1\rangle$.

Therefore $\langle z_i, z_{i+1}\rangle \cap \langle z_j, z_{j+1}\rangle = \emptyset$. The same argument works if one of the sides is of the form $\langle z_{4g}, z_1\rangle$.

We are now in a position to prove that the set A previously defined is both open and closed. We note first that A is not empty since $0 \in A$.

Lemma II.4.13. A is open.

Proof. Let $t_o \in A$. Thus $P_{t_o\mu}$ is simple and by Corollary II.4.10 it is convex. Let $\epsilon > 0$ be such that the ϵ-neighborhoods $U^{\epsilon}_{a_i^{t_o}}$, $i = 1,\ldots,4g$, satisfy the properties of Corollary II.4.12.

As in the proof of Lemma II.4.9 we choose $0 < \delta$ such that $w^{t\mu}(a_i^*) \in U^{\epsilon}_{a_i^{t_o}}$ for all $|t_o - t| < \delta$, $t \in [0,1]$ and $i = 1,\ldots,4g$. Thus the non-adjacent sides of $P_{t\mu}$ are disjoint. By Lemma II.4.5 the adjacent sides of $P_{t\mu}$ are not colinear. Thus $P_{t\mu}$ is simple.

In the above we have assumed $0 < t_o < 1$. A similar argument works for $t_o = 0$ and $t_o = 1$.

Lemma II.4.14. A is closed.

Proof. Let $t_k \to t$ where $t_k \in A$. We need to show that $P_{t\mu}$ is simple. By Lemma II.4.5 no two sides of $P_{t\mu}$ are colinear thus the proof will follow from the definition of a simple

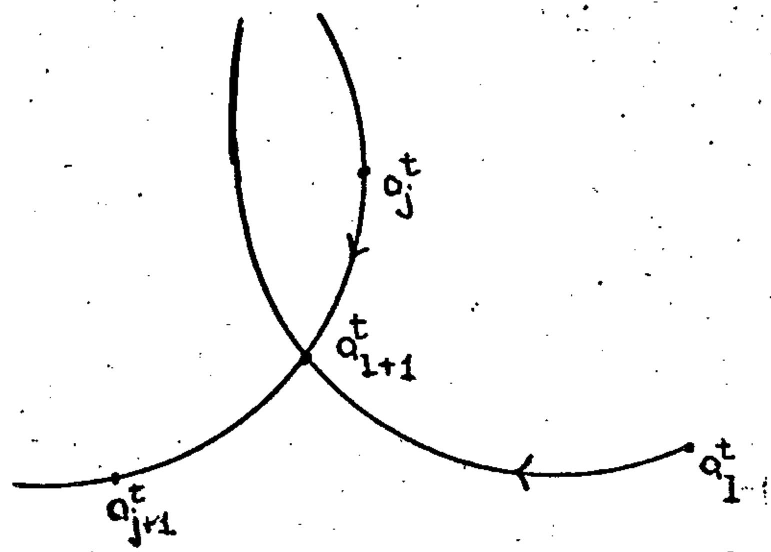


Fig. 16

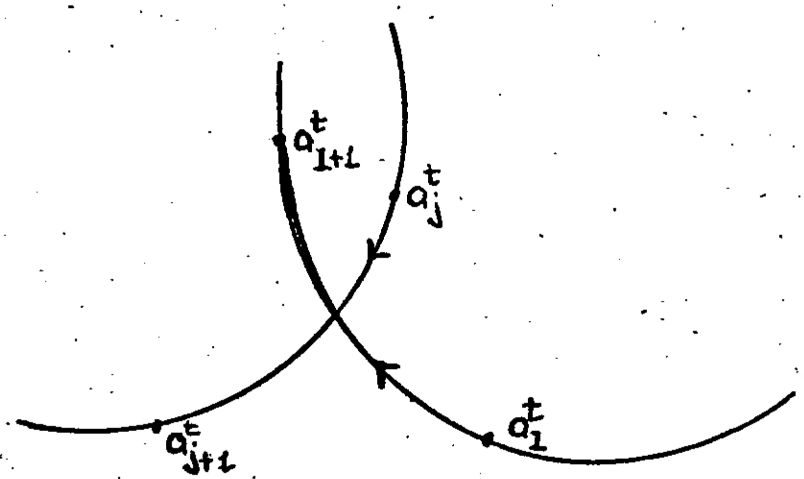


Fig. 17

polygon when we prove that no two non-adjacent sides of $P_{t\mu}$ intersect at exactly one point.

We assume that there exist sides A_j^t and A_ℓ^t $0 < j, \ell \leq 4g$, $j \neq \ell - 1, \ell, \ell + 1$ which intersect in a single point (Figs. 16 and 17). The point of intersection may be a vertex as in Fig. 16 or an interior point of both sides as in Fig. 17. In either case we may assume that the side A_ℓ^t separates a_j^t from a_{j+1}^t.

Let $\epsilon > 0$ be such that the ϵ-neighborhoods satisfy the properties of Corollary II.4.8.

Let $t_{k_n} \in A$ be such that $w^{t_{k_n}\mu}(a_m^*) \in U^\epsilon_{a_m^t}$, $m = j-1, j, \ell-1, \ell$. Thus the line $A_1^{t_{k_n}}$ separates the point $a_j^{t_{k_n}}$ from $a_{j+1}^{t_{k_n}}$. This is a contradiction since by Corollary II.4.10 $P_{t_{k_n}\mu}$ is convex and no side can separate two vertices.

<u>Lemma</u> II.4.15. Let α_j^t, $j = 1,\ldots,4g$ be the interior angles of $P_{t\mu}$. Then α_j^t is continuous function of t.

<u>Proof</u>. Suppose $z_1,z_2,z_3 \in D$ are distinct points. We first construct a Mobius transformation T which fixes D such that $T(z_2) = 0$ and $T(z_1) \in \mathbb{R}^+$. Let $T(z) = \dfrac{az-az_2}{-\overline{a}\overline{z}_2z+\overline{a}}$ where

$$a = \sqrt{\frac{1}{1-|z_2|^2}}\, e^{-i\theta/2} \text{ and } \theta = \arg\left(\frac{z_1-z_2}{-\overline{z}_2z_1+1}\right).$$

We need to verify that T has the required properties.

1. $|a|^2 - |a|^2|z_2|^2 = |a|^2(1-|z_2|^2) = \dfrac{1-|z_2|^2}{1-|z_2|^2} = 1.$

Thus T fixes D.

2. $T(z_2) = az_2 - az_2 = 0$

3. $T(z_1) = \dfrac{az_1-az_2}{-\overline{a}\overline{z}_2 z_1+\overline{a}} = \dfrac{a}{\overline{a}}\, re^{i\theta} = r.$

(Note that $r = \left|\dfrac{z_1-z_2}{-\overline{z}_2 z_1+1}\right| > 0.$)

Join z_1 to z_2 (resp. z_2 to z_3) by a geodesic l_1 (resp. l_2) oriented from z_1 to z_2 (resp. z_2 to z_3). Since T is a conformal map, the angle formed at z_2 to the left of the line l_1 is equal to

(II.4.1) $\qquad 2\pi - \arg T(z_3)$

(In the above formula we use the principal branch of the argument lying between 0 and 2π.)

Now consider the points $w^{t\mu}(a^*_{j-1})$, $w^{t\mu}(a^*_j)$ and $w^{t\mu}(a^*_{j+1})$. From (II.4.1) we obtain

(II.4.2) $\qquad \alpha^t_j = 2\pi - \arg\left(\dfrac{a_t w^{t\mu}(a^*_{j+1}) - a_t w^{t\mu}(a^*_j)}{-a_t \overline{w^{t\mu}(a^*_j)}\, w^{t\mu}(a^*_{j+1}) + \overline{a}_t}\right),$

where $a^t_j = \sqrt{\dfrac{1}{1-|w^{t\mu}(a^*_j)|^2}}\; e^{-i\theta_t/2}$ and

$$(II.4.3) \qquad \theta_j^t = \arg\left(\frac{w^{t\mu}(a_{j-1}^*)-w^{t\mu}(a_j^*)}{-\overline{w^{t\mu}(a_j^*)}w^{t\mu}(a_{j-1}^*)+1}\right).$$

By Lemma II.4.9 $0 < \alpha_j^t < \pi$. Thus the values of the argument function in formula (II.4.2) lie in the open interval $(\pi, 2\pi)$ and the argument function is therefore continuous. In formula (II.4.3) the argument function is allowed to vary continuously with respect to t. All other functions considered are continuous, thus α_j^t is a continuous function of t. (Note that we have again assumed that $j \neq 1$. The same proof may be used when $j=1$ by a suitable relabeling of indices.)

We also note that in general if $z_1^t, z_2^t, z_3^t \in D$ are three distinct points varying continuously with respect to a parameter t, then α^t (defined analogously) is also a continuous function of t provided α^t lies in an interval properly contained in $(0, 2\pi)$. In particular, this applies to the construction of the perfectly symmetric polygon in this section and the polygons constructed in section II.3.

<u>Lemma</u> II.4.16. Let $t \in [0,1]$. Then $\hat{P}_{t\mu} = 2\pi$.

<u>Proof</u>. Since the elements T_i^*, $i = 1,\ldots,2g$, generate the group Γ^* the elements T_i^t also generate a Fuchsian group and satisfy the same relation as the T_i^* (Bers [5]). Thus

$$(T_2^t)^{-1}\circ\ldots\circ T_{2g-1}^t\circ(T_{2g}^t)^{-1}\circ(T_1^t)^{-1}\circ\ldots\circ T_{2g-2}^t\circ(T_{2g-1}^t)^{-1}\circ T_{2g}^t\circ T_1^t = id.$$

Let α_i^t, $i = 1,\ldots,4g$, be the interior angles of $P_{t\mu}$. Since

$(T_2^t)^{-1}\circ\ldots\circ(T_{2g}^t)^{-1}\circ(T_1^t)^{-1}\circ\ldots\circ T_{2g}^t\circ T_1^t(A_1^t) = A_1^t$ the angle $\hat{P}_{t\mu} = \Sigma\ \alpha_i^t$ fills out a circle and is a multiple of 2π. But $\hat{P}_{t\mu}$ is a continuous function of t (Lemma II.4.15) and $\hat{P}_{t\mu} = 2\pi$.

<u>Corollary</u> II.4.17. P_μ is a hyperelliptic polygon and is a fundamental polygon for Γ^μ. $\frac{D}{\Gamma^\mu}$ is conformally equivalent to S.

<u>Proof</u>. Since A is both open and closed $A = [0,1]$ and P_μ is a simple polygon. From Lemma II.4.4 and II.4.16 we obtain that P_μ is j-invariant, the transformations T_i^μ, $i = 1,\ldots,2g$, identify opposite sides of P_μ and $\hat{P}_\mu = 2\pi$. Using Poincaré's theorem P_μ is a fundamental polygon for Γ^μ

To prove the last assertion we construct the diagram (with commutative squares)

$$\begin{tikzcd}
 & D \arrow[r, mapsto, "w^\mu"] \arrow[d] & D \arrow[d] \\
 & \frac{D}{\Gamma^*} \arrow[r, mapsto, "h_1^\mu"] \arrow[d] & \frac{D}{\Gamma^\mu}, \arrow[d] \\
\frac{S}{J} \arrow[r, mapsto, "f^{-1}"] & \frac{D}{\{j,\Gamma^*\}} \arrow[r, mapsto, "h_2^\mu"] & \frac{D}{\{j,\Gamma^\mu\}}
\end{tikzcd}$$

where h_1^μ and h_2^μ are the induced maps.

The maps f and h_2^μ have the same dilation. Thus $h_2^\mu \circ f^{-1} : \frac{S}{J} \to \frac{S}{\{j,\Gamma^\mu\}}$ is conformal (see Bers [5]). Consequently S and $\frac{D}{\Gamma^\mu}$ are twofold covers of $\frac{S}{J}$ branched over the same 2g+2 points and are conformally equivalent.

For clarity we restate Corollary II.4.17 as a theorem.

<u>Theorem</u> II.4.18. Let $(S,J,w_1,\ldots,w_{2g+2})$ be a hyperelliptic surface. Then there exists a hyperelliptic polygon P with associated Fuchsian group Γ such that S is conformally equivalent to $\frac{D}{\Gamma}$.

The conformal equivalence is given by a map $\hat{h}$ with the property that $\hat{h}(w_i) = \{c_i\}$, $i = 1,\ldots,2g$, $\hat{h}(w_{2g+1}) = \{0\}$ and $\hat{h}(w_{2g+2}) = \{a_j\}$ where a_j is any vertex of P.

<u>Corollary</u> II.4.19. Let $(S,J,w_1,\ldots w_{2g+2})$ be a hyperelliptic surface. Then S may be uniformized by a Fuchsian group generated by elements T_i, $i = 1,\ldots,2g$, where $T_i(z) = \frac{a_i z+b_i}{\bar{b}_i z+a_i}$, $a_i \in \mathbb{R}$, and

$$(\text{II.4.4})\quad T_2^{-1}\circ\ldots\circ T_{2g-1}\circ T_{2g}^{-1}\circ T_1^{-1}\circ\ldots\circ T_{2g-2}\circ T_{2g-1}^{-1}\circ T_{2g}\circ T_1 = \text{id}.$$

Moreover any uniformization of S by a Fuchsian group (with no elliptic elements) has a set of generators satisfying (II.4.4).

<u>Proof</u>. Let P be a hyperelliptic polygon whose associated group Γ uniformizes S. The pairing transformations T_i, $i = 1,\ldots,2g$, generate Γ and $T_i(z) = \frac{a_i z+b_i}{\bar{b}_i z+a_i}$, where $a_i \in \mathbb{R}$

by Lemma II.2.3. If Γ' is another uniformization of S (without elliptic elements) then it is well known that $\Gamma' = T \circ \Gamma \circ T^{-1}$ for some motion T. The elements $T_i' = T \circ T_i \circ T^{-1}$ generate Γ' and satisfy (II.4.4).

III. MODULI OF HYPERELLIPTIC SURFACES

III.1 Teichmüller spaces. In order to study the moduli of hyperelliptic surfaces we need to establish some facts about Teichmüller spaces. All the material in this section is expository.

We first define quasi-Fuchsian groups and then give brief descriptions of the Teichmüller space and the Bers fiber space of a Fuchsian group. (For a more detailed description see [4].)

Let Γ be a discontinuous group of Mobius transformations and let C be an oriented Jordan curve such that the domains D' and D'', interior and exterior to C respectively, are invariant under Γ. Γ is called a quasi-Fuchsian group and the domains D' and D'' are called the invariant components of Γ. In this case $\Lambda(\Gamma)$ is contained in C and Γ is said to be of the first kind if $\Lambda(\Gamma) = C$, otherwise it is of the second kind (see [12]).

As in Chapter II, D will denote the open unit disc. We denote by Aut D the group of all Mobius transformations which fix D.

Let Γ be a quasi-Fuchsian group with an invariant component D' and let $\pi : D \to D'$ be a Riemann map. Define the group

$$G = \{\gamma \in \text{Aut } D; \gamma = \pi^{-1} \circ T \circ \pi \text{ for some } T \in \Gamma\}.$$

G is a Fuchsian group (see Kra [12]) and is called the Fuchsian equivalent of Γ. Clearly $\frac{D}{G} \cong \frac{D'}{\Gamma}$.

A polygon will be called a smooth Jordan polygon if it is a simple polygon whose sides are analytic arcs.

Let Γ be a Fuchsian group with no parabolic elements. $L^\infty(\Gamma)$ is the space of Beltrami differentials for Γ (II.4) with norm $\|\mu\|_\infty$ = ess. sup.$\{|\mu(z)| \; ; \; z \in D\}$.

The open unit ball $M(\Gamma)$ of $L(\Gamma)$ is the space of Beltrami coefficients for Γ.

Let $\mu \in M(\Gamma)$ and extend μ to all of $\mathbb{C} \cup \{\infty\}$ by requiring that $\mu|D^c = 0$. We denote by w_μ the unique quasiconformal self map of $C \cup \{\infty\}$ which fixes 1, -1 and i, and satisfies the Beltrami equation $(w_\mu)_{\bar{z}} = \mu(z)(w_\mu)_z$ (see [3]).

We say that $\mu, \nu \in M(\Gamma)$ are equivalent (and write $\mu \sim \nu$) if and only if $w_\mu = w_\nu$ on the unit circle. $T(\Gamma)$ is the space of equivalence classes in $M(\Gamma)$. Φ is the projection of $M(\Gamma)$ onto $T(\Gamma)$. The domain $w_\mu(D)$ depends only on the equivalence class $\Phi(\mu)$ of μ.

The Bers fiber space is defined by $F(\Gamma) = \{(\Phi(\mu), z) \in T(\Gamma) \times \mathbb{C} \cup \{\infty\}; \; \mu \in M(\Gamma) \text{ and } z \in w_\mu(D)\}$.

Both $T(\Gamma)$ and $F(\Gamma)$ are complex manifolds (see [4]).

The group Γ acts discontinuously on $F(\Gamma)$ as a group of biholomorphic self maps by the rule

$$(\Phi(\mu), z)\gamma = (\Phi(\mu), \gamma^\mu(z)),$$

where $\mu \in M(\Gamma)$, $z \in w_\mu(D)$ and $\gamma^\mu = w_\mu \circ \gamma \circ (w_\mu)^{-1}$ (see Bers [4]).

$V(\Gamma) = \frac{F(\Gamma)}{\Gamma}$ is also a complex manifold and the projection $(\Phi(\mu), z) \mapsto \Phi(\mu)$ is a holomorphic map from $V(\Gamma)$ onto $T(\Gamma)$. The inverse image of $\Phi(\mu)$ is the closed surface $\dfrac{w_\mu(D)}{w_\mu \circ \Gamma \circ (w_\mu)^{-1}}$. Note that $\Gamma^\mu = w_\mu \circ \Gamma \circ (w_\mu)^{-1}$ is a quasi-Fuchsian group with invariant component $w_\mu(D)$.

Let Γ be a fixed point free Fuchsian group such that D/Γ is a closed surface of genus g, and let H be a non-trivial group of automorphisms of D/Γ. $p : D \to D/\Gamma$ is the natural projection map.

Let $\Gamma' = \{\gamma \in \text{Aut } D;\ p\circ\gamma = h\circ p \text{ for some } h \in H\}$. It is well known that Γ' is a Fuchsian group, Γ is a normal subgroup of Γ' and $\Gamma'/\Gamma \simeq H$.

We will now define actions of Γ'/Γ on $T(\Gamma)$ and of Γ' on $F(\Gamma)$.

Γ' acts on the space $M(\Gamma)$ of Beltrami coefficients for Γ by

$$\text{(III.1.1)} \qquad \mu\cdot g = (\mu\circ g)\overline{g'}/g' \text{ for all } \mu \in M(\Gamma), \quad g \in \Gamma'.$$

The subgroup Γ acts trivially on $M(\Gamma)$ and the action of $g \in \Gamma'$ depends only on its equivalence class modulo Γ. Therefore (III.1.1) induces an action of H on $T(\Gamma)$ by the rule

$$\Phi(\mu)\cdot\alpha(g) = \Phi(\mu\cdot g) \text{ for all } \mu \in M(\Gamma),\ g \in \Gamma',$$

where $\alpha : \Gamma' \mapsto H$ is the quotient map.

We now define the action of Γ' on $F(\Gamma)$ by the rule

$$(\Phi(\mu),z)\cdot g = (\Phi(\mu\cdot g),\ g^{\mu}\ (z)),$$

where $\mu \in M(\Gamma)$, $z \in w_{\mu}(D)$, $g \in \Gamma'$, and $g^{\mu} \circ w_{\mu\cdot g} = w_{\mu} \circ g$.

III.2 The hyperelliptic locus. In this section we consider the subspace of Teichmüller space which consists of hyperelliptic surfaces. In order to do so we will first define the Teichmüller modular group.

Using the material from Chapter II, we then obtain new results on uniformizations of hyperelliptic surfaces by quasi-Fuchsian groups.

Let θ be an automorphism of the Fuchsian group Γ with the property that $\theta(\gamma) = w \circ \gamma \circ w^{-1}$ for all $\gamma \in \Gamma$, where w is a quasiconformal self map of D. θ is called a geometric automorphism.

The map w induces a biholomorphic self map of $M(\Gamma)$ by sending $\mu \in M(\Gamma)$ into the Beltrami coefficient of $w_{\mu} \circ w^{-1}$. It is easy to check that this mapping preserves equivalence classes and hence induces a biholomorphic self map $\theta^* : T(\Gamma) \to T(\Gamma)$ which depends only on the conjugacy class of θ modulo the group of inner automorphisms of Γ.

Thus the (Teichmüller) modular group Mod Γ, which is defined to be the quotient of the group of geometric automorphisms by the normal subgroup of inner automorphisms, acts on $T(\Gamma)$ as a group of biholomorphic self maps.

Let $H = \{\Phi(\mu) \in T(\Gamma);\ \dfrac{w_{\mu}(D)}{\Gamma^{\mu}}$ is hyperelliptic$\}$. Thus if

$\Phi(\mu) \in H$ then $\frac{w_\mu(D)}{\Gamma^\mu}$ has an involution J_μ. The map $w_\mu : D \to w_\mu(D)$ induces a quasiconformal map $f_\mu : \frac{D}{\Gamma} \to \frac{w_\mu(D)}{\Gamma^\mu}$. Hence $f_\mu^{-1} \circ J_\mu \circ f_\mu$ is a quasiconformal self map of $\frac{D}{\Gamma}$ and we may lift it to a geometric automorphism of Γ. In this manner J_μ induces a self map J_μ^* of $T(\Gamma)$.

Let $T(\Gamma)^{J_\mu^*}$ be the fixed point set of the map J_μ^*. J_μ^* and J_ν^* are called <u>equivalent</u> if $T(\Gamma)^{J_\mu^*} = T(\Gamma)^{J_\nu^*}$. Kravetz [13] has shown that $T(\Gamma)^{J_\mu^*}$ is a component of H and that when $g > 2$ there are infinitely many inequivalent J_μ^*. It is not hard to show that each component of H contains a copy from each conformal equivalence class of hyperelliptic surfaces. When $g = 2$ it is well known that $T(\Gamma)^{J_\mu^*} = T(\Gamma)$.

<u>Proposition</u> III.2.1. Let Γ be a fixed point free quasi-Fuchsian group and D' an invariant component of Γ. Let $\frac{D'}{\Gamma}$ be hyperelliptic with involution J and let $\{z_o\}$ be a Weierstrass point of $\frac{D'}{\Gamma}$. Let $E : D' \to D'$ be a lift of the involution such that $E^2 = id.|D'$ and $E(z_o) = z_o$. Then there exists a fundamental region $R \subset D'$ which is bounded by a smooth Jordan polygon P' invariant under E. Moreover, if we label the sides of P' by $A_1', \ldots, A_{4g}'$ (in order), there exist generating transformations $T_1', \ldots, T_{2g}'$ for Γ such that $T_i'(A_i') = E(A_i')$, $i = 1, \ldots, 2g$.

Proof. Let $\pi : D \mapsto D'$ be a Riemann map such that $\pi(0) = z_0$. Then $\pi^{-1} o E o \pi$ is a conformal self map of D with a fixed point at zero, thus $\pi^{-1} o E o \pi = j|D$ (see [9]). Let G be the Fuchsian equivalent of Γ. Thus $\frac{D}{G}$ is hyperelliptic and by Theorem II.4.16, G has a fundamental polygon $P \subset D$ with sides $A_1, \ldots, A_{4g}$ and generating transformations $T_i \in G$, $i = 1, \ldots, 2g$, such that $T_i(A_i) = -A_i$. Let $R = \pi(P^o)$. It is easy to verify that R is a fundamental domain for Γ bounded by $\pi(P) = P'$ a smooth Jordan polygon with sides $\pi(A_i) = A_i'$. The transformations $T_i' = \pi o T_i o \pi^{-1}$ generate Γ and $T_i'(A_i') = E(A_i')$.

Lemma III.2.2. Let $\Phi(\mu) \in H \subset T(\Gamma)$ and let J_μ be the involution of $\frac{w_\mu(D)}{\Gamma^\mu}$. Let $\{z_0\}$ be a Weierstrass point of $\frac{w_\mu(D)}{\Gamma^\mu}$. Then there exists a lift $E : w_\mu(D) \to w_\mu(D)$ of the involution J_μ such that $E^2 = id.|w_\mu(D)$ and $E(z_0) = z_0$.

Proof. Let E' be a lift of J_μ to $w_\mu(D)$. Since $\{z_0\}$ is a Weierstrass point $E'(z_0) = T(z_0)$. But $T^{-1}E'$ is also a lift of J_μ and $T^{-1}E'(z_0) = z_0$. Let $E = T^{-1}E'$, then $E^2 = S|w_\mu(D)$ where $S \in \Gamma^\mu$. Since E^2 has a fixed point $S = id$.

Corollary III.2.3. Let $\Phi(\mu) \in H \subset T(\Gamma)$. Then there exists a fundamental region $R \subset w_\mu(D)$ as described in Proposition III.2.1.

Proof. Lemma III.2.2 and Proposition III.2.1 yield the desired result.

Let G be a Fuchsian group such that $\frac{D}{G}$ is hyperelliptic. It is well known that G is conjugate to a Fuchsian group Γ which is j-invariant. We call Γ a normalized hyperelliptic group.

If Γ is a normalized hyperelliptic Fuchsian group then $j(z) = -z$ is a lift of the involution J of $\frac{D}{\Gamma}$. Let $\langle J \rangle$ be the group of conformal self maps of $\frac{D}{\Gamma}$ generated by J. Let Γ' be the group of all lifts of elements of $\langle J \rangle$ to D. It is easy to verify that $\{j,\Gamma\} = \Gamma'$ and $\frac{\Gamma'}{\Gamma} \cong \langle J \rangle$

We will assume throughout the rest of this section that Γ is a normalized hyperelliptic Fuchsian group and $\Gamma' = \{j,\Gamma\}$.

We now turn our attention to the fundamental regions in the fibers over $T(\Gamma)^j \subset H \subset T(\Gamma)$. We call $T(\Gamma)^j$ the identity component of H.

Proposition III.2.4. Let $\mu \in M(\Gamma)$ be an even coefficient (i.e. $\mu = \mu o j$). Then

i) $w_\mu(D)/\Gamma^\mu$ is hyperelliptic and $j^\mu = w_\mu o j o (w_\mu)^{-1}$ is an elliptic Mobius transformation of order 2 that fixes $w_\mu(D)$. Moreover, $j^\mu | w_\mu(D)$ is a lift of the hyperelliptic involution of $w_\mu(D)/\Gamma^\mu$.

ii) the quasi-Fuchsian group Γ^μ has a fundamental region $R \subset w_\mu(D)$ whose boundary is a smooth j^μ-invariant Jordan polygon with sides $A_1^!,\ldots,A_{4g}^!$. Moreover, Γ^μ is generated by transformations $T_i^!$, $i = 1,\ldots,2g$, such that $T_i^!(A_i^!) = j^\mu(A_i^!)$.

Proof.

i) Since μ is even, the mapping $\gamma \to \gamma^{\mu}$ (where $\gamma \in \{j,\Gamma\}$) maps elliptic elements onto elliptic elements of the same order (see [4]). Since the group $\{j^{\mu},\Gamma^{\mu}\}$ is quasi-Fuchsian, it follows trivially that $j^{\mu}|w_{\mu}(D)$ induces an involution of $w_{\mu}(D)/_{\Gamma^{\mu}}$ with $2g + 2$ fixed points.

ii) The proof follows from part i of this lemma and Proposition III.2.1.

Corollary III.2.5. Let $\mu \in M(\Gamma)$ such that $\Phi(\mu) \in T(\Gamma)^{j}$. Then there exists a $\mu' \in \Phi(\mu)$ such that $j^{\mu'}$ is an elliptic Mobius transformation of order 2, $w_{\mu}(D)$ is $j^{\mu'}$-invariant, and the group Γ^{μ} has a $j^{\mu'}$- invariant fundamental region as defined in Proposition III.2.4.

Proof. Let $\Phi(\mu) \in T(\Gamma)^{j}$, thus $\mu \sim \mu o j$. In order to apply Proposition III.2.4 we need to show that there exists a $\mu' \in \Phi(\mu)$ such that $\mu' = \mu' o j$.

We first show that in general if $\nu_1 \sim \nu_2$ then $\nu_1 o j \sim \nu_2 o j$. Let w_{ν_1} and w_{ν_2} be the normalized solutions to the mapping problem for ν_1 and ν_2 respectively. Then $w_{\nu_1} o j$ and $w_{\nu_2} o j$ are solutions for $\nu_1 o j$ and $\nu_2 o j$. Since $w_{\nu_1}(z) = w_{\nu_2}(z)$ on the unit circle we have that $w_{\nu_1} o j = w_{\nu_2} o j$ on the unit circle. Thus for some $0 < \theta < 2\pi$, $e^{i\theta} w_{\nu_1} o j$ and $e^{i\theta} w_{\nu_2} o j$ are normalized solutions for $\nu_1 o j$ and $\nu_2 o j$. Hence $\nu_1 o j \sim \nu_2 o j$.

Now let $\mu' \in \Phi(\mu)$ be the unique Teichmüller differential with minimal maximal dilation (see [13]). By the above argument $\mu' \circ j \sim \mu'$. But the maximal dilation of μ' is the same as that of $\mu' \circ j$, hence $\mu' \circ j = \mu'$. The proof now proceeds as that of Proposition III.2.4 using the coefficient μ'.

The fundamental regions for the groups Γ^{μ}, where $\Phi(\mu) \in T(\Gamma)^j$ may be described more precisely.

Let $z_0 \in D$ such that $\{z_0\}$ is a Weierstrass point of D/Γ. As in the proof of Lemma III.2.2, there exists an elliptic element of order two, $T_{z_0} \in \Gamma'$ such that $T_{z_0}(z_0) = z_0$.

Since j and T_{z_0} are equivalent modulo Γ, $T(\Gamma)^j = T(\Gamma)^{T_{z_0}}$.

As in the proof of Corollary III.2.5, if $\Phi(\mu) \in T(\Gamma)^j$, there exists a $\mu' \in \Phi(\mu)$ such that $\mu' = \mu' \circ j$. Thus, we define the section $s_{z_0} : T(\Gamma)^j \to F(\Gamma)$ by $s_{z_0}(\Phi(\mu))$ $= (\Phi(\mu), w_{\mu'}(z_0))$.

<u>Lemma</u> III.2.6. The map $s_{z_0} : T(\Gamma)^j \to F(\Gamma)$ is a holomorphic section.

<u>Proof</u>. It follows from a theorem of Rauch (see [6]) that the natural map $f : T(\Gamma') \to T(\Gamma)^j \subset T(\Gamma)$ is a complex analytic embedding. Thus the map $f^* : F(\Gamma') \to F(\Gamma)$ given by $f^*(\Phi(\mu),z) = (f(\Phi(\mu)),z)$ is also a complex analytic embedding.

It is well known that $s^*_{z_0} : T(\Gamma') \to F(\Gamma')$ defined by

$s^*_{z_0}(\Phi(\mu)) = (\Phi(\mu), w_\mu(z_0))$ is a holomorphic section (see [7]). Now $s_{z_0} = f^* \circ s^*_{z_0} \circ f^{-1}$.

(Note that when $g = 2$, s_{z_0} is a global section, i.e. $s_{z_0} : T(\Gamma) \to F(\Gamma)$).

When $z_0 = 0$ we call s_{z_0} the zero section. Thus Corollary III.2.5 states that every group Γ^μ such that $\Phi(\mu) \in T(\Gamma)^J$ has a fundamental region symmetric about a point in the zero section. We will show that the vertices of these regions may also be chosen to lie on holomorphic sections.

Before continuing we need to define the Poincaré metric $\lambda_{D(\mu)}$ for $w_\mu(D)$. $\lambda_{D(\mu)}$ is defined as follows:

Let $\lambda_D = \dfrac{1}{1-|z|^2}$ be the Poincaré metric on the unit disc. Choose $\pi : D \to w_\mu(D)$, a Riemann map and define $\lambda_{D(\mu)}$ by

$$\lambda_{D(\mu)}(\pi(z)) \; |\pi'(z)| = \lambda_D(z) \text{ for } z \in D.$$

$\lambda_{D(\mu)}$ is well defined (see Kra [12]). In fact geodesics in $w_\mu(D)$ are the images under π of geodesics in D.

By Theorem II.4.18, Γ has a hyperelliptic fundamental polygon with vertices $a_1, \ldots, a_{4g}$. Let T_{a_i} be a lift of the involution J on $\frac{D}{\Gamma}$ such that $T_{a_i}(a_i) = a_i$, $i = 1, \ldots, 4g$. Thus $s_{a_i}(T(\Gamma)^J) = \{(\Phi(\mu), w_{\mu'}(a_i)) : \Phi(\mu) \in T(\Gamma)^J\}$ is a holomorphic section.

Let $P_{\mu'} \subset w_\mu(D)$ be the polygon formed by joining the points $w_{\mu'}(a_i)$ (in order), $i = 1,\ldots,4g$, by geodesics in $\lambda_{D(\mu)}$.

Theorem III.2.7. Let $\Phi(\mu) \in T(\Gamma)^j$. Then $P_{\mu'} \subset w_\mu(D)$ is a simple $j^{\mu'}$-invariant polygon and $P^o_{\mu'}$ (the interior of $P_{\mu'}$) is a fundamental region for Γ^μ.

Proof. Let $\pi : D \to w_\mu(D)$ be a Riemann map such that $\pi(0) = w_{\mu'}(0)$ and $\pi(1) = 1$ (see [2]). As in the proof of Proposition III.2.1, $\pi^{-1} \circ j^{\mu'} \circ \pi = j$. $w = \pi^{-1} \circ w_{\mu'}$ is a quasiconformal self map of D (see [2]) which is j-invariant and satisfies $w(0) = 0$. w extends to a self map $\hat{w}$ of $\mathbb{C} \cup \{\infty\}$ with the following properties: $\hat{w}(0) = 0$, $\hat{w}(1) = 1$ and $\hat{w}(\frac{1}{\bar{z}}) = \frac{1}{\overline{\hat{w}(z)}}$ (see [2]).

Thus $\hat{w} \circ \Gamma \circ \hat{w}^{-1}$ is a Fuchsian group and by Theorem III.4.8 $P_{\hat{w}} = \langle w(a_1),\ldots,w(a_{4g}), w(a_1)\rangle$ is a hyperelliptic polygon. Now $\pi(P_{\hat{w}}) = P_{\mu'}$ and it is easy to verify that $P_{\mu'}$ has the required properties.

In order to simplify our results for the identity component $T(\Gamma)^j$, we construct a fiber space over $T(\Gamma)^j$ in which the fibers are j-invariant.

Let $\mu \in M(\Gamma')$ and ϕ_μ be the unique quasiconformal automorphism of $\mathbb{C} \cup \{\infty\}$ which fixes 0, 1 and ∞, and satisfies the

Beltrami equation $(\phi_\mu)_{\bar z} = \mu(z)(\phi_\mu)_z$ where $\mu|D^c = 0$ (see [3]).

<u>Lemma</u> III.2.8. Let $\mu, \nu \in M(\Gamma')$. Then $\mu \sim \nu$ if and only if $\phi_\mu = \phi_\nu$ on the unit circle.

<u>Proof</u>. Since w_μ and ϕ_μ (resp. w_ν and ϕ_ν) are solutions of the same Beltrami equation, there exists a Mobius transformation α_μ (resp. α_ν) such that $\alpha_\mu \circ w_\mu = \phi_\mu$ (resp. $\alpha_\nu \circ w_\nu = \phi_\nu$) (see [3]).

If $\phi_\mu = \phi_\nu$ on the unit circle, then $(\alpha_\nu)^{-1} \circ \alpha_\mu \circ w_\mu = w_\nu$ on the unit circle. Since $w_\nu \circ (w_\mu)^{-1}$ fixes 1, -1 and i, we obtain that $(\alpha_\nu)^{-1} \circ \alpha_\mu = \mathrm{id.}$ and $w_\mu = w_\nu$ on the unit circle.

Suppose conversely that $w_\mu = w_\nu$ on the unit circle. Then

(II.2.1) $$\alpha_\nu \circ (\alpha_\mu)^{-1} \circ \phi_\mu = \phi_\nu$$

on the unit circle. Since the transformations $\phi_\mu \circ j \circ (\phi_\mu)^{-1}$ and $\phi_\nu \circ j \circ (\phi_\nu)^{-1}$ both are of order 2 and fix 0 and ∞, they are equal. We obtain hence from (II.2.1)

(II.2.2) $$\phi_\mu \circ j \circ (\phi_\mu)^{-1} = \phi_\nu \circ j \circ (\phi_\nu)^{-1}$$
$$= \alpha_\nu \circ (\alpha_\mu)^{-1} \circ \phi_\mu \circ j \circ (\phi_\mu)^{-1} \circ \alpha_\mu \circ (\alpha_\nu)^{-1}$$

on the image of the unit circle under ϕ_ν.

Since the transformations of (II.2.2) are Mobius, (III.2.2) holds for all $z \in \mathbb{C} \cup \{\infty\}$. Thus $\phi_\mu \circ j \circ (\phi_\mu)^{-1}$ and $\alpha_\nu \circ (\alpha_\mu)^{-1}$ commute and must have the same fixed points (see [9]), or since they are of order 2, one interchanges the fixed points of the other. We assume the latter possibility does not occur. By definition, α_ν and α_μ fix 1. Thus $\alpha_\nu \circ (\alpha_\mu)^{-1} = \mathrm{id}$. The desired result follows from (II.2.1).

We must verify that $\alpha_\nu \circ (\alpha_\mu)^{-1}$ cannot interchange the two points $0, \infty$. Since w_μ and w_ν agree on the unit circle,

$$w_\mu \circ \gamma \circ (w_\mu)^{-1} = w_\nu \circ \gamma \circ (w_\nu)^{-1} \quad \text{for all } \gamma \in \Gamma'$$

In particular $W = (w_\nu)^{-1} \circ w_\mu$ commutes with each $\gamma \in \Gamma'$. Let $\gamma = j$. Thus W either fixes 0 and ∞ or permutes them. It is well known that a quasiconformal map that commutes with Γ' must map each component of Γ' onto itself (see [16]). In particular $w_\mu(0) = w_\nu(0)$ and $w_\nu(\infty) = w_\nu(\infty)$.

The relations $\alpha_\mu \circ w_\mu = \phi_\mu$ (and the similar one for ν) show that $\alpha_\mu(w_\mu(0)) = \alpha_\nu(w_\nu(0))$ and $\alpha_\mu(w_\mu(\infty)) = \alpha_\nu(w_\nu(\infty))$. Thus α_μ and α_ν agree on the three points $w_\mu(0)$, $w_\mu(1) = 1$ and $w_\mu(\infty)$. Hence they are the same Mobius transformation.

We now define $\widetilde{F}(\Gamma') = \{(\Phi(\mu), z) \in T(\Gamma') \times \mathbb{C};\ \mu \in M(\Gamma')$ and $z \in \phi_\mu(D)\}$. Note that by Lemma III.2.8 $\widetilde{F}(\Gamma')$ is well defined.

<u>Lemma</u> III.2.9. $\widetilde{F}(\Gamma')$ is a manifold isomorphic to $F(\Gamma')$ (i.e. there exists a biholomorphic map $f : F(\Gamma') \to \widetilde{F}(\Gamma')$ such that the following diagram commutes

$$\begin{tikzcd} F(\Gamma') \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{F}(\Gamma') \arrow[dl, "\tilde{p}"] \\ & T(\Gamma') & \end{tikzcd},$$

where p and $\tilde{p}$ are the natural projection maps).

Proof. We first construct a Mobius transformation α_μ, such that $\alpha_\mu(1) = 1$, $\alpha_\mu(w_\mu(0)) = 0$ and $\alpha_\mu(w_\mu(\infty)) = \infty$. We consider three cases.

Case 1. $w_\mu(0) \neq \infty$ and $w_\mu(\infty) \neq \infty$. Then

$$\alpha_\mu(z) = \frac{a_\mu z + b_\mu}{c_\mu z + d_\mu}$$

where

$$a_\mu = w_\mu(\infty) - 1 \quad , \quad b_\mu = (1 - w_\mu(\infty))w_\mu(0),$$

$$c_\mu = (w_\mu(0) - 1) \quad \text{and} \quad d_\mu = w_\mu(\infty)(1 - w_\mu(0))$$

We need to verify that α_μ has the required properties.

1) $$\alpha_\mu(1) = \frac{(w_\mu(\infty)-1) + (1-w_\mu(\infty))w_\mu(0)}{(w_\mu(0)-1) + (w_\mu(\infty)(1-w_\mu(0))} = 1$$

(Note that neither the numerator nor the denominator can be 0 since $w_\mu(0) \neq 1$ and $w_\mu(\infty) \neq 1$.)

2) $$\alpha_\mu(w_\mu(0)) = \frac{(w_\mu(\infty)-1)w_\mu(0) + (1-w_\mu(\infty))w_\mu(0)}{(w_\mu(0)-1)w_\mu(0) + w_\mu(\infty)(1-w_\mu(0))} = 0$$

(Note that since $w_\mu(\infty) \neq 1$ and $w_\mu(0) \neq w_\mu(\infty)$ the numerator can never be 0.)

<u>Case 2.</u> $w_\mu(0) = \infty$. Then $\alpha_\mu(z) = \dfrac{1 - w_\mu(\infty)}{1 - w_\mu(\infty)}$

and

1) $$\alpha_\mu(1) = \frac{1 - w_\mu(\infty)}{1 - w_\mu(\infty)} = 1$$

2) $$\alpha_\mu(w_\mu(0)) = \frac{1 - w_\mu(\infty)}{w_\mu(0) - w_\mu(\infty)} = 0$$

3) $$\alpha_\mu(w_\mu(\infty)) = \frac{1 - w_\mu(\infty)}{w_\mu(\infty) - w_\mu(\infty)} = \infty$$

<u>Case 3.</u> $w_\mu(\infty) = \infty$. Then $\alpha_\mu(z) = \dfrac{z - w_\mu(0)}{1 - w_\mu(0)}$

and

1) $$\alpha_\mu(1) = \frac{1 - w_\mu(0)}{1 - w_\mu(0)} = 1$$

2) $$\alpha_\mu(w_\mu(0)) = \frac{w_\mu(0) - w_\mu(0)}{1 - w_\mu(0)} = 0$$

$$3) \quad \alpha_\mu(w_\mu(\infty)) = \frac{w_\mu(\infty) - w_\mu(0)}{1 - w_\mu(0)} = \infty$$

In all cases α_μ has the required properties. Since ϕ_μ is unique, we thus have $\alpha_\mu \circ w_\mu = \phi_\mu$.

Let $f : F(\Gamma') \to \widetilde{F}(\Gamma')$ be defined by $f(\Phi(\mu), z) = (\Phi(\mu), \alpha_\mu(z))$. Thus $f^{-1} : \widetilde{F}(\Gamma') \to F(\Gamma')$ is given by $f^{-1}(\Phi(\mu), z) = (\Phi(\mu), (\alpha_\mu)^{-1}(z))$.

Since $w_\mu(0)$ and $w_\mu(\infty)$ are the fixed points of the elliptic transformation j^μ, they depend (homomorphically) on $\Phi(\mu) \in T(\Gamma')$ (see [3]). Thus, from our previous computations in Cases 1, 2 and 3 of this lemma, we obtain that the coefficients of α_μ depend (holomorphically) on $\Phi(\mu) \in T(\Gamma')$.

It follows that $\widetilde{F}(\Gamma')$ is a manifold (see [4]) and f is a biholomorphic map. The commutativity of the above diagram is trivial. The proof is complete.

We define the space

$$\widetilde{F}(T(\Gamma)^j) = \{(\Phi(\mu), z) \in T(\Gamma)^j \times \mathbb{C};\ \mu \in M(\Gamma) \text{ and } z \in \phi_{\mu'}(D)\}.$$

($\widetilde{F}(T(\Gamma)^j)$ is the pullback of $\widetilde{F}(\Gamma')$ via the natural map $g : T(\Gamma)^j \to T(\Gamma')$ where $g(\Phi(\mu)) = \Phi(\mu')$.)

<u>Proposition</u> III.2.10. Let $\Phi(\mu) \in T(\Gamma)^j$. Then the

following hold:

i) $\phi_\mu(D)$ is a j-invariant domain.

ii) Let $\Gamma^{\phi_\mu} = \phi_\mu \circ \Gamma \circ (\phi_\mu)^{-1}$. Γ^{ϕ_μ} has a fundamental region $R \subset \phi_\mu(D)$ whose boundary is a smooth j-invariant Jordan polygon with sides $A'_1, \ldots, A'_{4g}$. Moreover, Γ^{ϕ_μ} is generated by transformations T'_i, $i = 1, \ldots, 2g$, such that $T'_i(A'_i) = j(A'_i)$.

iii) Let $P \subset D$ be a hyperelliptic fundamental polygon for Γ and let $a_1, \ldots, a_{4g}$ be the vertices of P. Let $\tilde{s}_{a_i} : T(\Gamma)^j \to \mathcal{F}(T(\Gamma)^j)$ be defined by $\tilde{s}_{a_i}(\Phi(\mu)) = (\Phi(\mu), \phi_{\mu'}(a_i))$, $i = 1, \ldots, 4g$. The maps $\tilde{s}_{a_i}$ are holomorphic sections.

iv) Let $P_{\phi_{\mu'}} \subset \phi_\mu(D)$ be the polygon formed by joining the points $\phi_{\mu'}(a_1), \ldots, \phi_{\mu'}(a_{4g})$ (in order) with geodesics in $\phi_{\mu'}(D)$.[6] $P_{\phi_{\mu'}}$ is a j-invariant fundamental polygon for Γ^{ϕ_μ}.

<u>Proof</u>. We note first that $\phi_\mu \circ j \circ (\phi_\mu)^{-1} = j$, $\phi_\mu(D) = \alpha_\mu(w_\mu(D))$ and $\Gamma^{\phi_\mu} = \alpha_\mu \circ \Gamma^\mu \circ (\alpha_\mu)^{-1}$. The assertions now follow from Lemma III.2.9, Corollary III.2.5 and Theorem III.2.7.

---

[6] The Poincaré metric for $\phi_\mu(D)$ is defined in the same manner as that for $w_\mu(D)$.

III.3 The polynomial corresponding to the perfectly symmetric polygon. The perfectly symmetric polygon P^* with sides $A_1^*,\dots,A_{4g}^*$ was defined in II.4. We note that

$$A_2^* = e^{i\frac{2\pi}{4g}} A_1^*,\ A_3^* = e^{i\frac{4\pi}{4g}} A_1^*,\dots,A_j^* = e^{i\frac{(j-1)2\pi}{4g}} A_1^*,$$

$j = 1,\dots,4g.$[7]

Let Γ^* be the group generated by the transformations $T_i^* \in \text{Aut } D$ such that $T_i^*(A_i^*) = A_{i+2g}^*$ with opposite orientation, $i = 1,\dots,2g$. Let $\gamma^* : D \to D$ be the map given by $\gamma^*(z) = (e^{i\frac{2\pi}{4g}})z$, thus

$$\text{(III.3.1)} \qquad T_j^* = (\gamma^*)^{j-1} \circ T_1^* \circ (\gamma^*)^{-(j-1)},\ j = 2,\dots,2g.$$

From (III.3.1) it is clear that $\gamma^{*-1} \circ \Gamma^* \circ \gamma^* = \Gamma^*$ and γ^* induces an automorphism γ of $\frac{D}{\Gamma^*}$(where $\gamma(\{z\}) = \{\gamma^*(z)\}$) such that the following diagram commutes (here p^* is the projection $p^* : D \to D/\Gamma^*$).

$$\begin{tikzcd} D \arrow[r, mapsto, "\gamma^*"] \arrow[d, "p^*"] & D \arrow[d, "p^*"] \\ \frac{D}{\Gamma^*} \arrow[r, mapsto, "\gamma"] & \frac{D}{\Gamma^*} \end{tikzcd}$$

From the definition of γ^* it is obvious that $\gamma^{4g} = id$.

Each map γ^k, $1 \le k \le 4g$, $k \ne 2g$, fixes the points $\{0\}$ and $\{a_j^*\}$, where a_j^* is any vertex, and no other points. When $k = 2g$, γ^k is the hyperelliptic involution J and its fixed points are $\{0\}$, $\{a_j^*\}$ and $\{c_i^*\}$, $i = 1,\dots,2g$, where c_i^* is the hyperbolic midpoint of A_i^*.

---

7 Here $e^{i\theta}A_j^* = \{e^{i\theta}z ; z \in A_j^*\}$

Consider the twofold cover $^{D}/_{\Gamma^*} \xrightarrow{\tilde{\mathfrak{J}}} {}^{D}/_{\Gamma^*}/\langle J\rangle$ where $\langle J\rangle$ is the group generated by J and $\mathfrak{J}$ is the natural projection.

From (III.3.1) and Lemma 11.2.4 we obtain that $\gamma^{*-1}\circ\{J,\Gamma^*\}\circ\gamma^* = \{J,\Gamma^*\}$. Thus γ^* induces an automorphism $\hat{\gamma}$ of $^{D}/_{\Gamma^*}/\langle H_J\rangle \cong \mathbb{C} \cup \{\infty\}$ (where $\hat{\gamma}(\{\{z\}\}) = \{\gamma(\{z\})\}\})$ such that the following diagram commutes:

$$\begin{tikzcd}
^{D}/_{\Gamma^*} \arrow[r, "\gamma"] \arrow[d, "\mathfrak{J}"] & ^{D}/_{\Gamma^*} \arrow[d, "\mathfrak{J}"] \\
\mathbb{C}\cup\{\infty\} \arrow[r] & \mathbb{C}\cup\{\infty\}
\end{tikzcd}$$

Since $\gamma^{2g} = J$, it follows that $\hat{\gamma}^{2g} = \mathrm{id}$. Hence $\hat{\gamma}$ is elliptic of order 2g and its fixed points are $\mathfrak{J}(\{0\})$ and $J(\{a_1\})$. Moreover,

$$\text{(III.3.2)} \qquad \mathfrak{J}(\{c_{n+1}\}) = \mathfrak{J}(\gamma(\{c_n\})) = \hat{\gamma}(\mathfrak{J}(\{c_n\})), \; n = 1,\ldots,2g-1$$

and

$$\text{(III.3.3)} \qquad \mathfrak{J}(\{c_1\}) = \mathfrak{J}(\gamma(\{c_{2g}\})) = \hat{\gamma}(\mathfrak{J}(\{c_{2g}\})).$$

By normalizing, we may assume that $\mathfrak{J}(\{0\}) = 0$, $\mathfrak{J}(\{a_1\}) = \infty$, $\mathfrak{J}(\{c_1\}) = 1$ and $\hat{\gamma}$ is a rotation about the origin by $\frac{\pi}{g}$. From (III.3.2) and (III.3.3) we obtain that $\mathfrak{J}(\{c_{n+1}\}) = e^{i\frac{n\pi}{g}}$, $n = 0,\ldots,2g-1$.

The surface corresponding to the polynomial $w^2 = z\prod_{n=0}^{2g-1}(z-e^{i\frac{n\pi}{g}})$ is branched over 0, ∞ and the 2g points $e^{i\frac{n\pi}{g}}$, $n = 0,\ldots,2g-1$. Since the conformal equivalence class of a hyperelliptic surface is uniquely determined by the two-fold cover, we have that the polynomial corresponding to ${}^{D}/\Gamma^*$ is $w^2 = z\prod_{n=0}^{2g-1}(z-e^{i\frac{n\pi}{g}})$, $n = 0,\ldots,2g-1$.

The same result may be obtained by means of Teichmüller theory in the following manner:

Let $A = \langle\gamma\rangle$ be the group generated by γ and consider the covering ${}^{D}/\Gamma^* \xrightarrow{\tilde{\gamma}} ({}^{D}/\Gamma^*)/A$. $\tilde{\gamma}$ is branched at the points $\{0\}$, $\{a_j^*\}$ and $\{c_i^*\}$ (these are the only fixed points of the elements of A). We now compute the branching orders.

Since $\{0\}$ and $\{a_j^*\}$ are fixed by every element of A, the branching order of $\tilde{\gamma}$ at each of these points is $4g-1$. The only element of A which fixes the $\{c_i^*\}$ is the involution γ^{2g}. Thus the branching order at each $\{c_i^*\}$ is one.

We compute the genus g' of $({}^{D}/\Gamma^*)/A$ by the Riemann-Hurwitz relation (see [17]). Thus $2g - 2 = 2n(g'-1) + B$ where $n = 4g$ and B is the sum of the branching orders. Since $B = 10g - 2$ we obtain $g' = 0$.

Let $T(\Gamma^*)$ be the Teichmüller space of Γ^* and let Γ' be the group of lifts to D of the elements of A. Thus $\Gamma' = \{g \in \text{Aut } D;\ p^* \circ g = g \circ p^* \text{ for some } h \in A\}$ where $p^* : D \to {}^{D}/\Gamma^*$ is the natural projection. The group ${}^{\Gamma'}/\Gamma^* \cong A$ acts on $T(\Gamma^*)$ in the manner defined in III.1. $T(\Gamma^*)^A$ (the fixed point set of A in $T(\Gamma^*)$) corresponds to surfaces admitting a group isomorphic to A as a group of conformal automorphisms.

Theorem III.3.7 (Kravetz [13]). The set of points in $T(\Gamma)$ left fixed by every member of Γ'/Γ is precisely $T(\Gamma')$. (Here Γ' is the group of lifts of a group of automorphism of D/Γ.)

Using Theorem III.3.7 and our previous computation $T(\Gamma^*)^A = T(0,3)$ (here $T(0,3)$ is the Teichmüller space of a three times punctured sphere) and there is only one conformal equivalence class of surfaces in $T(\Gamma^*)^A$.

Let S^* be the surface corresponding to the polynomial $w^2 = z\prod_{n=0}^{2g-1}(z - e^{\frac{in\pi}{g}})$. S^* admits the automorphism

$$(w,z) \xrightarrow{\alpha} (e^{i\frac{\pi}{2g}}w, e^{i\frac{\pi}{g}}z)$$

which is of order $4g$.

Trivially $\alpha^k, 1 \leq k \leq 4g, k \neq 2g$, fixes $(0,0)$, (∞,∞) and no other points. Since $\alpha^{2g}(w,z) = (-w,z)$, α^{2g} is the hyperelliptic involution.

Clearly there exists a $\mu \in M(\Gamma^*)$ such that $\Phi(\mu) \in T(\Gamma^*)^A$ and $w_\mu(D)/_{\Gamma^\mu} \cong S^*$. Since there is only one conformal equivalence class of surfaces represented in $T(\Gamma^*)^A$, $S^* \approx D/\Gamma^*$.

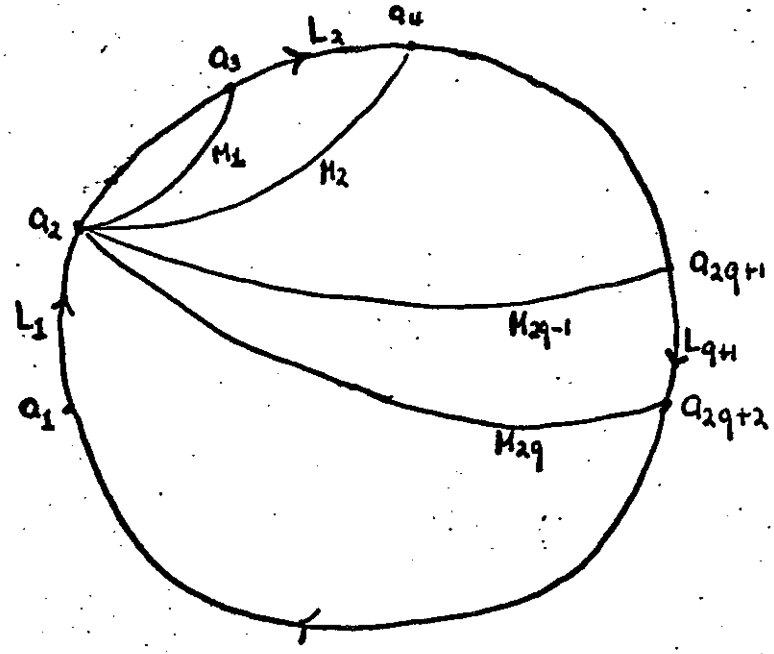
IV. TOPOLOGICAL PROPERTIES

IV.1 A topological dissection. In Chapter II, by means of uniformization theory, we were able to find hyperelliptic polygons for hyperelliptic surfaces. In this chapter we will find dissections of hyperelliptic surfaces into polygons of the same type without using uniformizing groups. To be more precise, given a hyperelliptic surface S with involution J, we will exhibit a set of J-invariant loops $\gamma_1,\ldots,\gamma_{2g}$ on S such that by cutting along these loops one obtains a 4g sided J invariant polygon.

Suppose S is the hyperelliptic surface corresponding to the polynomial $w^2 = z\prod_{i=1}^{2g+2}(z-a_i)$. Let $\gamma \subset \mathbb{C} \cup \{\infty\}$ be an oriented Jordan curve such that the points $a_1,\ldots,a_{2g+2}$ lie on γ and are ordered according to the orientation of γ (Fig. 18).

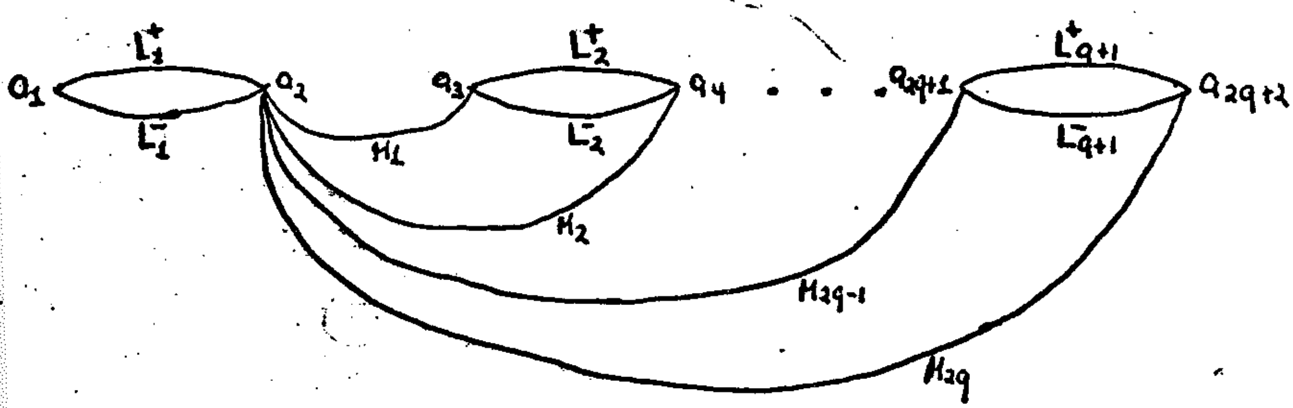
Let L_i, $i = 1,\ldots,g+1$, be the arc of γ which joins a_{2i-1} to a_{2i}. Let M_i, $i = 1,\ldots,2g$, be an arc (whose interior is contained in the interior component of γ) joining a_2 to a_{2+i}. We assume the arcs M_i intersect only at a_2 (Fig. 18).

We make cuts along the arcs L_i labeling one side of each cut + and the other - as in Fig. 19. It is well known that if we take another sphere with the same cuts (we label the points on the second sphere $a_1',\ldots,a_{2g+2}'$, the cuts $L_1',\ldots,L_{g+1}'$ and the arcs $M_1',\ldots,M_{2g}'$) and attach the two



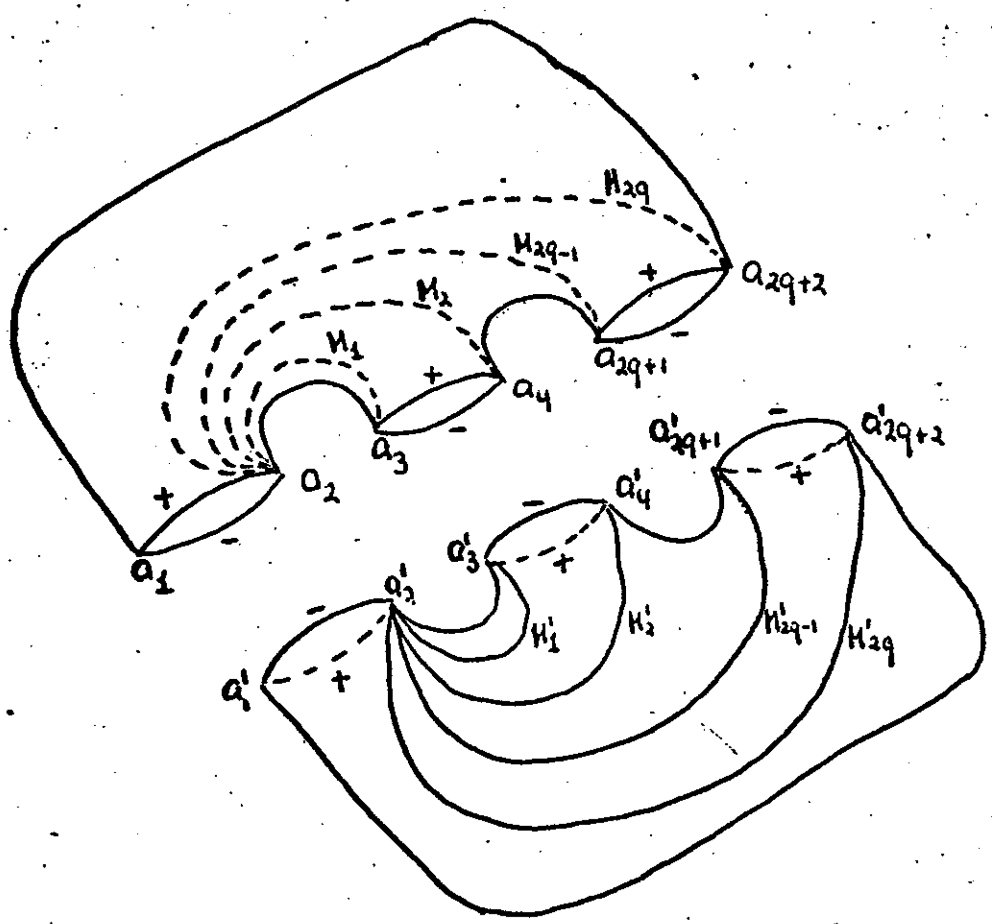
a_4
L_2
a_3
M_1
M_2
a_2
a_{2q+1}
L_1
M_{2q-1}
L_{q+1}
a_1
a_{2q+2}
M_{2q}

Fig. 18



L_1^+
a_1
a_2
L_1^-
a_3
L_2^+
a_4
L_2^-
a_{2q+1}
L_{q+1}^+
a_{2q+2}
L_{q+1}^-
M_1
M_2
M_{2q-1}
M_{2q}

Fig. 19



H_{2q}
H_{2q-1}
H_2
H_1
a_{2q+2}
a_{2q+1}
a_4
a_3
a_2
a_1
a'_{2q+1}
a'_{2q+2}
a'_4
a'_3
a'_2
a'_1
H'_1
H'_2
H'_{2q-1}
H'_{2q}

Fig. 20

spheres along corresponding cuts (identifying + with -), we obtain a conformal representation of S. Note that the involution J is the interchange of spheres (Fig. 20).

On the surface S the loops $\gamma_i = M_i'M_i$ (M_i followed by M_i') are simple and J-invariant (Fig. 20). We will dissect S along these loops.

We cut each sphere along the arcs M_i and M_i' respectively and then make the proper identifications to obtain the dissection of S.

On the first sphere we start at the point a_2 and cut in the direction of a_{i+2}, labeling the right hand side of the cut M_i^+ and the left hand side M_i^- (Fig. 21). We make similar conventions on the second sphere

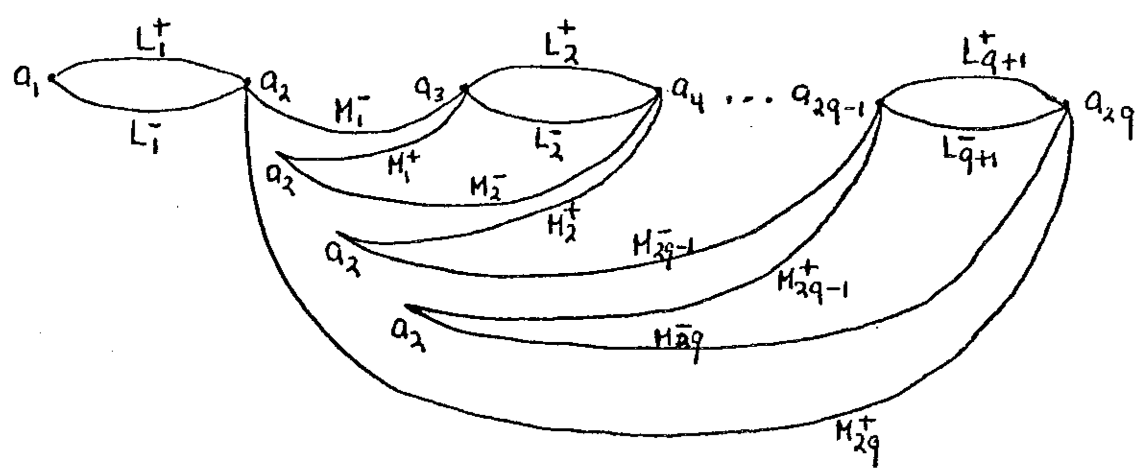


Fig. 21

We now attach the two spheres along the cuts L_i and L_i' to obtain the dissection of S given by Fig. 22. (For clarity, in Figs. 22 and 23 we have drawn all arcs as straight lines.)

The sides of Fig. 22 are relabeled as follows:

$$\gamma_1 = M_1'^{+}M_1^{-}, \quad \gamma_2 = M_2^{+}M_2'^{-}, \ldots, \gamma_i = \begin{cases} M_i'^{+}M_i^{-} & i \text{ odd} \\ M_i^{+}M_i'^{-} & i \text{ even} \end{cases} \quad i=1,\ldots,2g$$

$$\gamma_{2g+1} = M_1^{+}M_1'^{-}, \; \gamma_{2g+2} = M_2'^{+}M_2^{-}, \ldots, \gamma_{2g+i} = \begin{cases} M_i^{+}M_i'^{-} & i \text{ odd} \\ M_i'^{+}M_i^{-} & i \text{ even} \end{cases} .$$

The desired $4g$ sided polygon is thus obtained in Fig. 23.

We note that on this polygon the involution J corresponds to a conformal self map which identifies the side γ_i with γ_{i+2g} and has a fixed point at a_1. The cuts L_i, $i = 1,\ldots,g+1$, which appear in Fig. 22 correspond to loops around the handles of S.

IV.2 A relation among periods of closed C^1 differentials.

Proposition IV.2.1. Let ω and η be closed C^1 differentials on a compact Riemann surface S of genus g and let

$$\int_{\gamma_i}\omega = A_i, \quad \int_{\gamma_i}\eta = A_i', \quad \text{where } \gamma_i,\; i = 1,\ldots,2g,$$

are the loops corresponding to the dissection in IV.1.

Then $\iint_S \omega \wedge \eta = \sum_{i,j} \epsilon_{ij} A_i' A_j$, where $1 \le i,\; j \le 2g$,

and $$\epsilon_{ij} = \begin{cases} 1 & j < i \\ 0 & j = i \\ -1 & j > i \end{cases} .$$

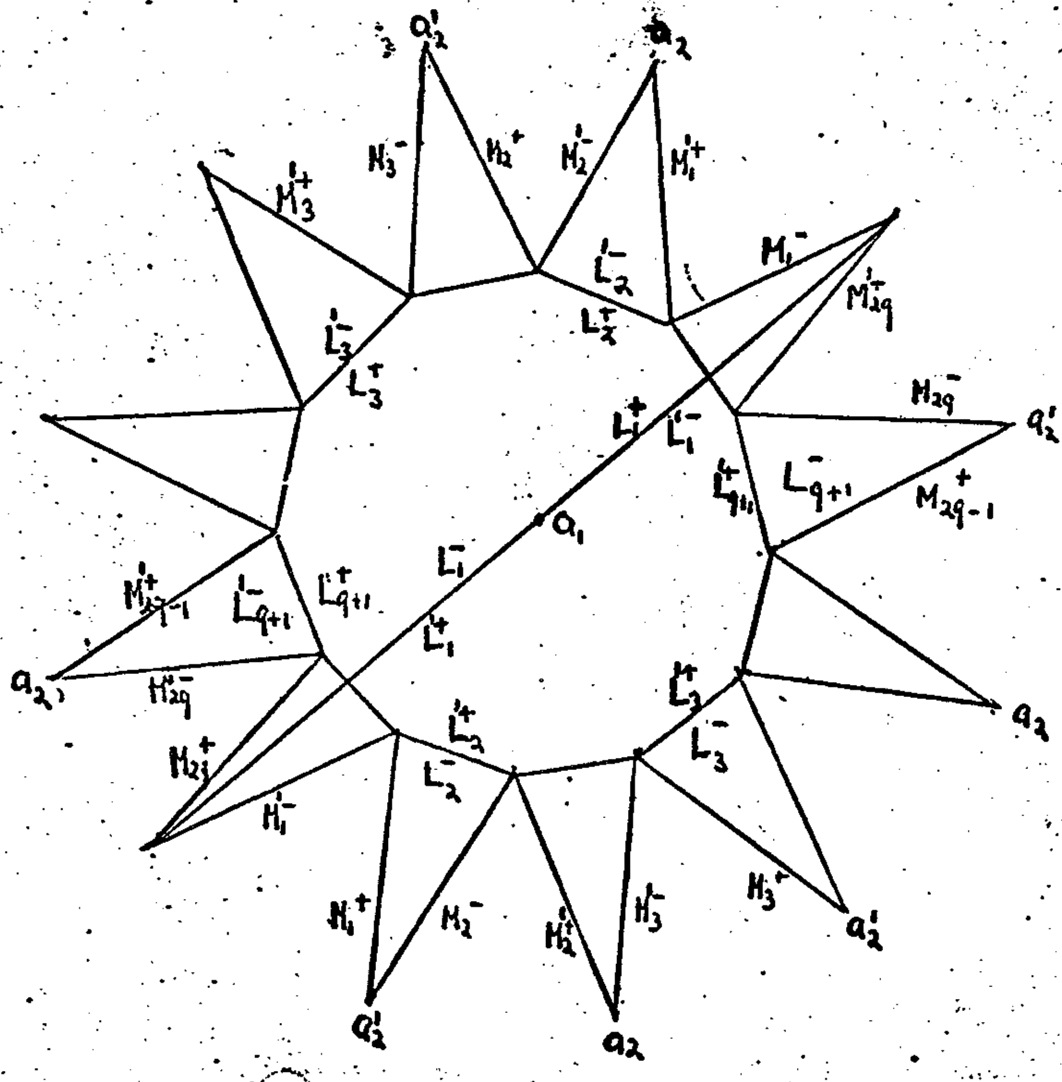
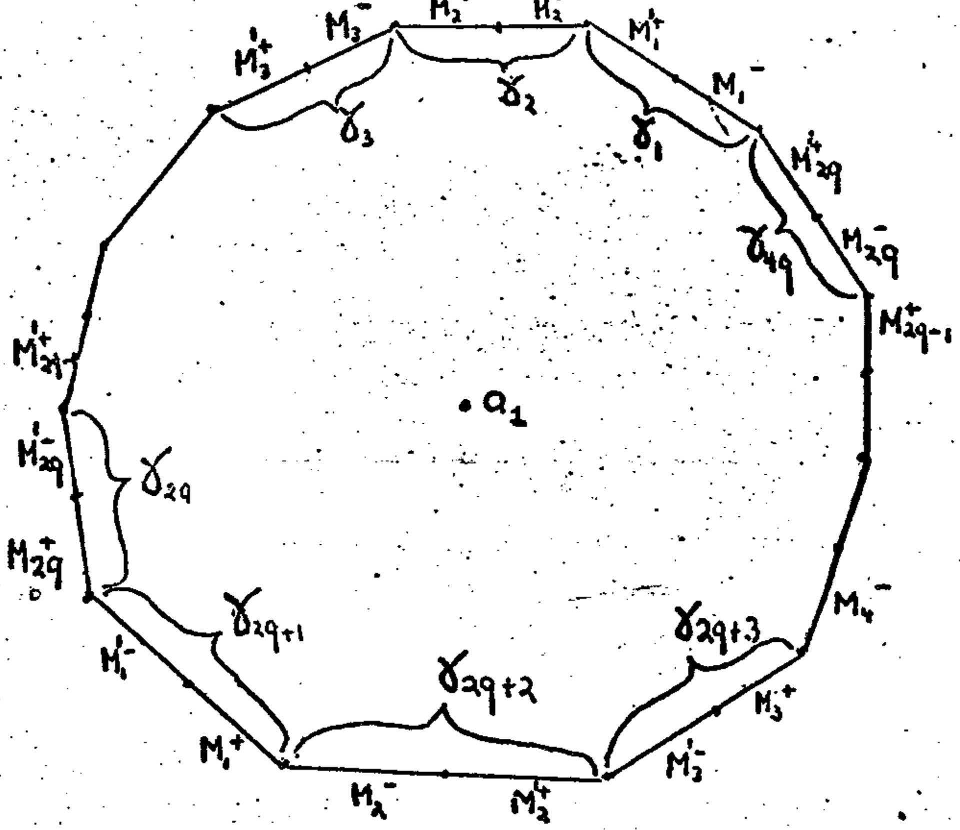


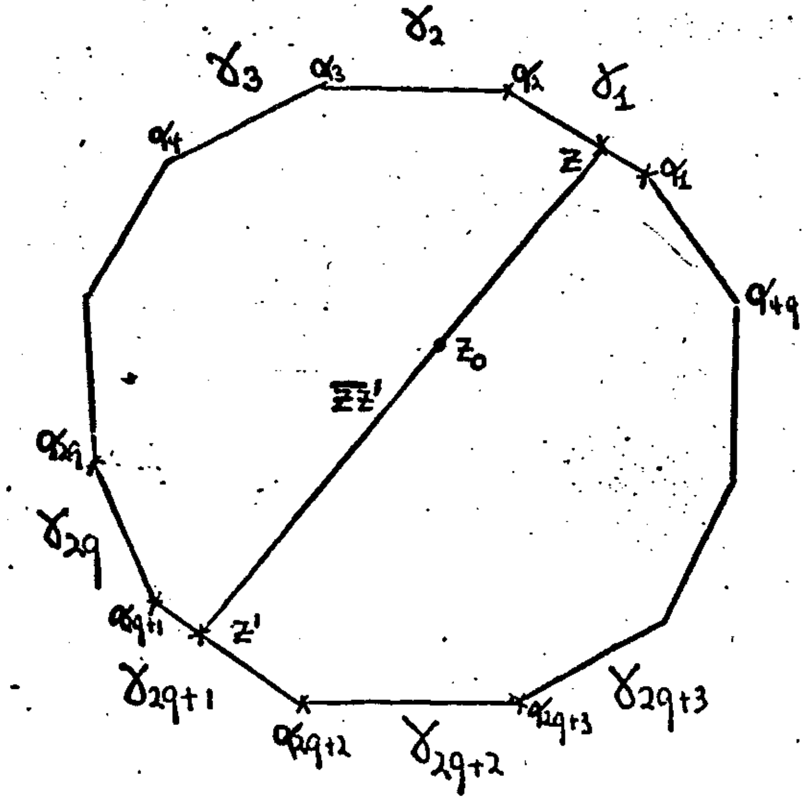
Fig. 22



M_2^+
$M_2'^-$
M_3^-
$M_3'^+$
$M_1'^+$
M_1^-
$M_{2q}'^+$
M_{2q}^-
M_{2q-1}^+
γ_1
γ_2
γ_3
γ_{4q}
a_1
$M_{2q-1}'^+$
$M_{2q}'^-$
M_{2q}^+
γ_{2q}
γ_{2q+1}
γ_{2q+2}
γ_{2q+3}
$M_1'^-$
M_1^+
M_2^-
$M_2'^+$
$M_3'^-$
M_3^+
M_4^-

Fig. 23

Proof. We dissect S along the loops γ_i to obtain the polygon P of IV.1 (Fig. 24). Here we have labeled the vertices $\alpha_1,\ldots,\alpha_{4g}$.



(Fig. 24)

Since the interior of P is simply connected, $w = df$ in the interior of P. Thus, by Stokes' theorem

$$(IV.2.1)\qquad \iint_S w \wedge \eta = \int_P f\eta = \sum_{i=1}^{2g}\left(\int_{\gamma_i} f\eta + \int_{\gamma_i^{-1}} f\eta\right).$$

Let z_o lie in the interior of P and let z and z' be equivalent points on γ_i and γ_i^{-1} respectively. Join z to z' by a curve $\overline{zz'}$ passing through z_o and lying in the interior of P (Fig. 24). Then $f(z) = \int_{z_0}^{z} \omega$ where the integral is taken along the segment of $\overline{zz'}$ from z_0 to z. Thus

$$(\text{IV.2.2}) \quad \int_{\gamma_i} f\eta + \int_{\gamma_i^{-1}} f\eta = \int_{\gamma_i} ((\int_{z_0}^{z} \omega) - (\int_{z_0}^{z'} \omega))\eta = -\int_{\gamma_i} (\int_{\overline{zz'}} \omega)\eta .$$

Suppose $z \in \gamma_1$. Then $\overline{zz'}$ is homologous to $\overline{z\alpha_2} + \gamma_2 + \ldots + \gamma_{2g} + \overline{\alpha_2 z'}$. Thus, since $\int_{\overline{z\alpha_2}} \omega = -\int_{\overline{\alpha_2 z'}} \omega$, we obtain

$$\int_{\overline{zz'}} \omega = \int_{\overline{zz'}} df = f(z') - f(z) = \int_{\gamma_2} \omega + \ldots + \int_{\gamma_{2g}} \omega .$$

When $z \in \gamma_i$ a similar computation yields

$$(\text{IV.2.3}) \qquad f(z') - f(z) = \sum_{i<j} \int_{\gamma_i} \omega - \sum_{j<i} \int_{\gamma_j} \omega$$

Now (IV.2.1), (IV.2.2) and (IV.2.3) yield the desired result.

IV.3. <u>The intersection matrix</u> P_0. We orient the polygon P constructed in IV.1 in a counterclockwise direction. Thus each side γ_i, $i = 1,\ldots,2g$, corresponds to an oriented loop in S. The positive direction of γ_1 is from a_2 to a_3 on the first sphere and from a_3' to a_2' on the second sphere (Fig. 25).

In general the orientation of γ_i will be positive from a_2 to a_{2+i} on the first sphere, and positive from a_{2+i}' to a_2' on the second sphere when i is odd. When i is even, the orientation will be positive from a_2' to a_{2+i}' on the second sphere, and positive from a_{2+i} to a_2 on the first sphere (Fig. 25).

In order to compute the intersection matrix of the loops γ_i we need to give a precise definition of the intersection number of two loops. We state the following lemmas without proof (a proof may be found in [8]).

<u>Lemma</u> IV.2.1. Let c be a simple closed curve on the Riemann surface S. Then there exists a real closed differential η_c with the property $\int_c \alpha = (\alpha, {}^*\eta_c)$.[8]

<u>Lemma</u> IV.2.2. Let a and b be two homology cycles on S. The intersection number of a and b is $a \cdot b = \iint_S \eta_a \wedge \eta_b$, and the following properties hold

$$a \cdot b = -b \cdot a$$

$$(a+b) \cdot c = a \cdot c + b \cdot c \quad \text{and } a \cdot b \in \mathbb{Z} \, .$$

If we isolate the intersection point a_2 of the loops γ_1 and γ_2 (Fig. 26) we note that $\gamma_1 \cdot \gamma_2 = 1$ (a proof may be found in [8]).

Similarly, we obtain $\gamma_1 \cdot \gamma_3 = -1$ (Fig. 26).

8 Here ${}^*\eta_c$ is the harmonic conjugate of η_c and $(\alpha, {}^*\eta_c) = \iint_D \alpha \wedge - \bar{\eta}_c$

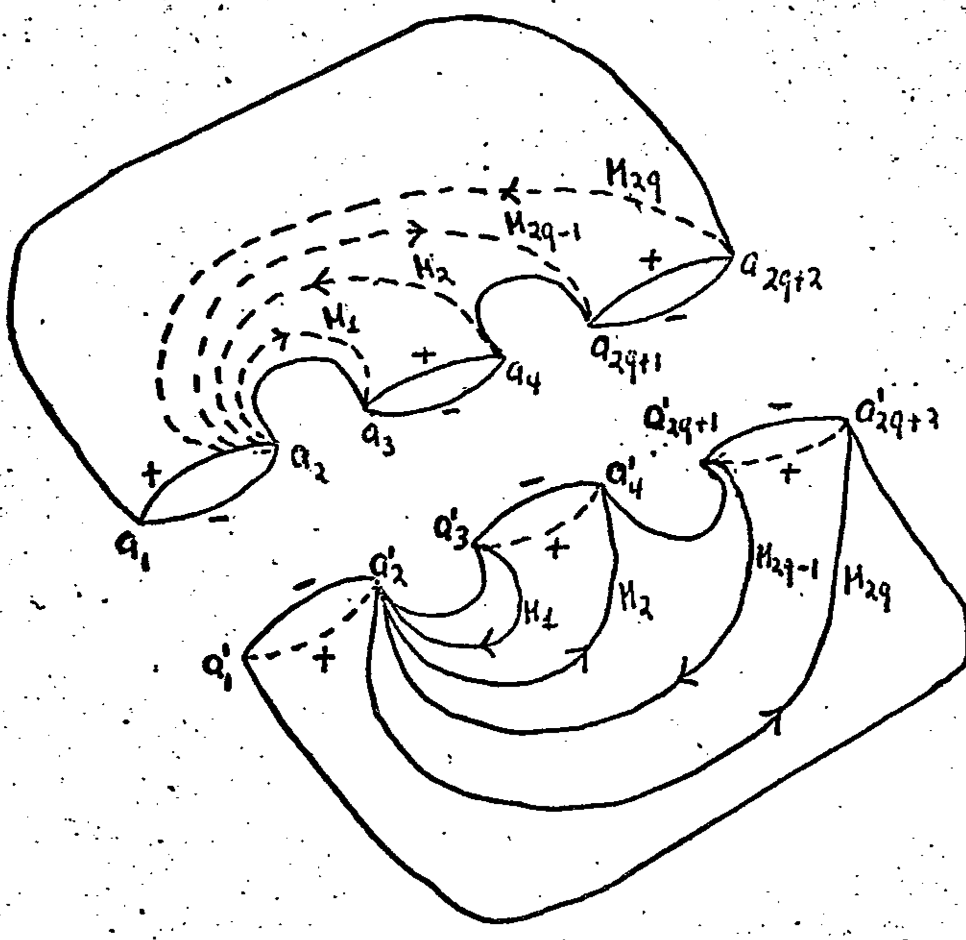
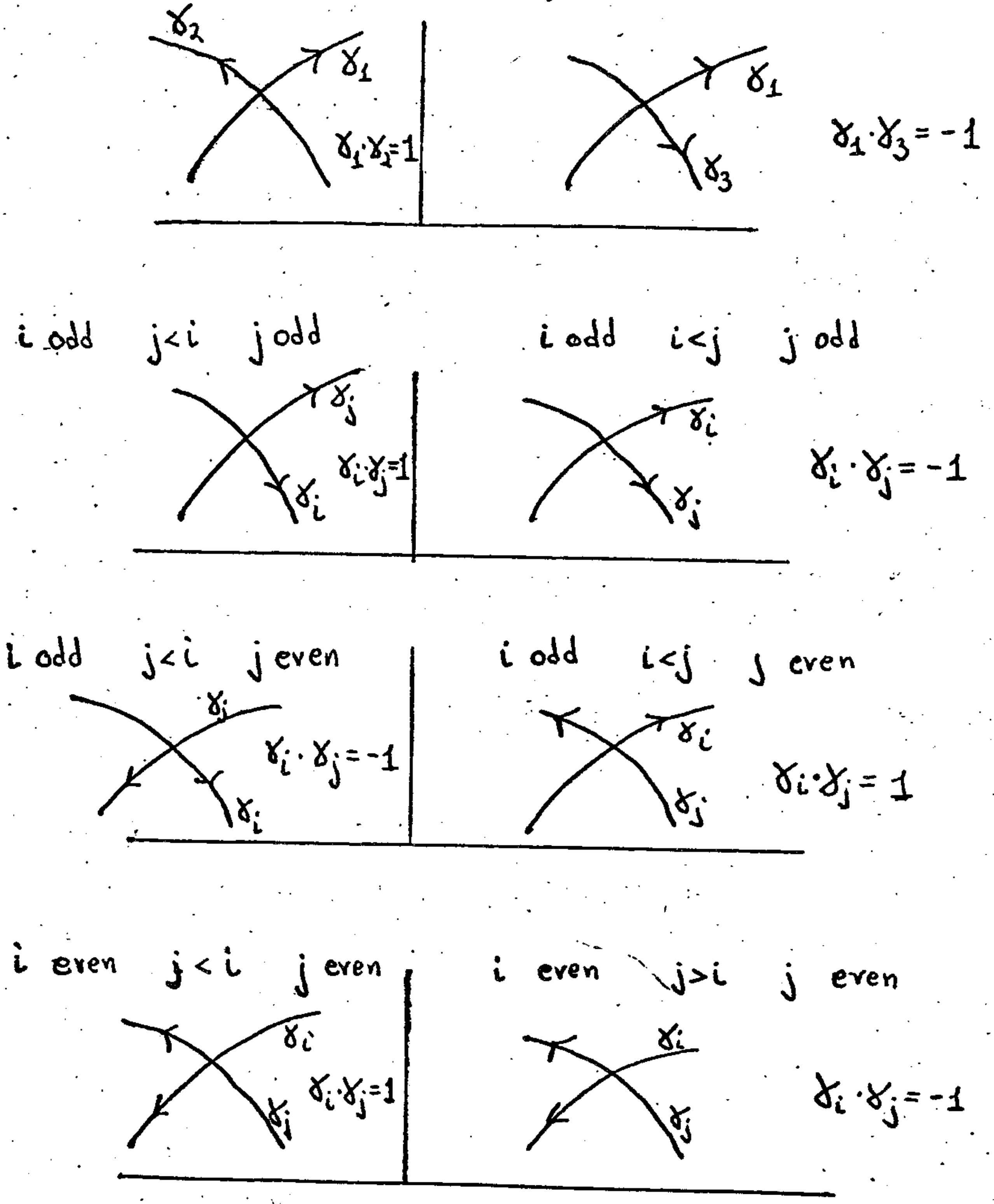


Fig. 25



γ_2
γ_1
$\gamma_1 \cdot \gamma_2 = 1$
γ_1
γ_3
$\gamma_1 \cdot \gamma_3 = -1$
i odd $j<i$ j odd
γ_j
γ_i
$\gamma_i \cdot \gamma_j = 1$
i odd $i<j$ j odd
γ_i
γ_j
$\gamma_i \cdot \gamma_j = -1$
i odd $j<i$ j even
γ_j
γ_i
$\gamma_i \cdot \gamma_j = -1$
i odd $i<j$ j even
γ_i
γ_j
$\gamma_i \cdot \gamma_j = 1$
i even $j<i$ j even
γ_i
γ_j
$\gamma_i \cdot \gamma_j = 1$
i even $j>i$ j even
γ_i
γ_j
$\gamma_i \cdot \gamma_j = -1$

Fig. 26

In general the intersection numbers of the loops γ_i, $i = 1,\ldots,2g$, are the following (see Fig. 26):

$$\gamma_1 \cdot \gamma_j = \begin{cases} 0 & j = 1 \\ -1^j & j > 1 \end{cases}$$

$$\gamma_2 \cdot \gamma_j = \begin{cases} -1^j & j < 2 \\ 0 & j = 2 \\ -1^{j+1} & j > 2 \end{cases}$$

$$\gamma_3 \cdot \gamma_j = \begin{cases} -1^{j+1} & j < 3 \\ 0 & j = 3 \\ -1^j & j > 3 \end{cases}$$

$$\vdots$$

$$\gamma_i \cdot \gamma_j = \begin{cases} -1^{j+n_1} & j < i \\ 0 & j = i \\ -1^{j+n_2} & j > i \end{cases},$$

where $n_1 = \begin{cases} 1 & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$ and $n_2 = \begin{cases} 0 & i \text{ odd} \\ 1 & i \text{ even} \end{cases}$.

This information may be collected in a matrix which we call the intersection matrix $P_o = (\gamma_i \cdot \gamma_j)$. Note that by Lemma IV.3, P_o is skew-symmetrix i.e. $\gamma_i \cdot \gamma_j = -\gamma_j \cdot \gamma_i$. We write the matrix P_o below, its entries are $\gamma_i \cdot \gamma_j$, $1 \le i, j \le 2g$.

$i \downarrow \ \backslash \ \rightarrow j$	γ_1	γ_2	γ_3	γ_4	$\cdots$	γ_j	$\cdots$	γ_{2g}
γ_1	0	1	-1	1				1
γ_2	-1	0	1	-1				-1
γ_3	1	-1	0	1				1
γ_4	-1	1	-1	0				-1
$\vdots$								
γ_i					$\gamma_i \cdot \gamma_j$			
$\vdots$								
γ_{2g}	-1	1	-1	1				0

IV.4. <u>A canonical homology basis.</u> To every compact Riemann surface of genus $g \ge 2$, one may associate a set of homology generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ with the intersection matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where O is the $g \times g$ zero matrix and I is the $g \times g$ identity matrix. Such a set is called a canonical homology basis.

A canononical homology basis appears in Fig. 27.

In IV.1 we associated to every hyperelliptic surface of genus $g \geq 2$ a set of loops $\gamma_1,\ldots,\gamma_{2g}$ (Fig. 25) based at a Weierstrass point a_2. (In fact, our construction yields a similar set of loops for any compact surface of genus $g \geq 2$, since topologically such a surface may be obtained in the same manner.)

We will now express the homology classes determined by the loops $\gamma_1,\ldots,\gamma_{2g}$ in terms of the canonical homology loops of Fig. 27 in the following manner:

We first replace each canonical homology loop α_i (resp. β_i), $i = 1,\ldots,g$, by a homologous loop α_i' (resp. β_i') based at a_2 (Fig. 28). (A set of loops such as $\alpha_1',\ldots,\alpha_g'$, $\beta_1',\ldots,\beta_g'$ is traditionally called a canonical dissection of S (see [11]).) We then write each loop γ_i, $i = 1,\ldots,2g$, in terms of the loops $\alpha_1',\ldots,\alpha_g'$, $\beta_1',\ldots,\beta_g'$.

We will take products of loops as elements of $\pi_1(S,a_2)$ Since we are only interested in homology, we will write products additively and take equivalence classes of elements in $\pi_1(S,a_2)$ modulo the commutator subgroup of $\pi_1(S,a_2)$.

<u>Lemma</u> IV.4.1. $\gamma_1 = -\alpha_1'$

<u>Proof</u>. The assertion is obvious by inspection (Figs. 25 and 28).

<u>Lemma</u> IV.4.2. $\gamma_i = (-\gamma_{i-1} - \alpha_{i-1}')$, whenever $i < 2g$ is odd.

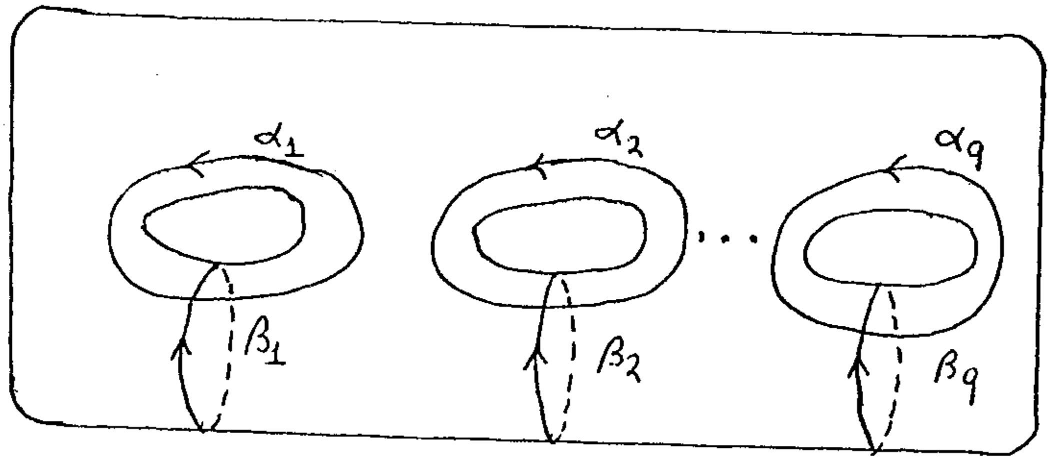
<u>Proof</u>. The loops $-\gamma_{i-1}$ and $-\alpha_{i-1}'$ appear in Fig. 29. Their sum appears in Fig. 30.

<u>Lemma</u> IV.4.3. $\gamma_i = (\sum_{j=1}^{i/2} \alpha_j') - \beta_i' + \beta'_{\frac{i+2}{2}}$, whenever $i < 2g$ is even.

<u>Proof</u>. The loops α_j' and α_j'' (Fig. 31) are homologous since they are both homologous to α_j. Thus we obtain that $\sum_{j=1}^{i/2} \alpha''_j = \sum_{j=1}^{i/2} \alpha_j'$ (Figs. 32 and 33). $(\sum_{j=1}^{i/2} \alpha_j'') - \beta_i'$ is computed in Fig. 39. The final sum appears in Fig. 35.

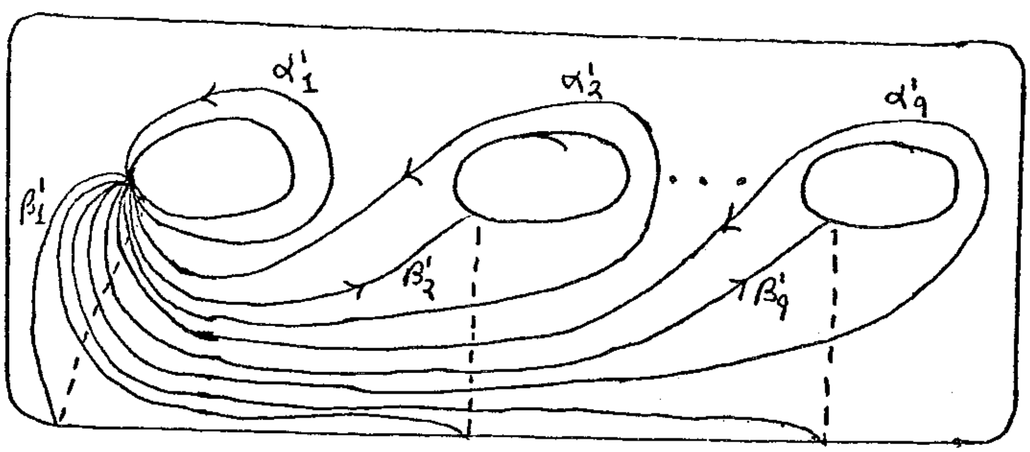
<u>Lemma</u> IV.4.4. $\gamma_{2g} = -\gamma_{2g-1} - \beta_g'$

<u>Proof</u>. The loops γ_{2g-1} and β_g' appear in Fig. 36. Their sum appears in Fig. 37.



α_1
α_2
α_q
β_1
β_2
β_q

Fig. 27



α_1'
α_2'
α_q'
β_1'
β_2'
β_q'

Fig. 28

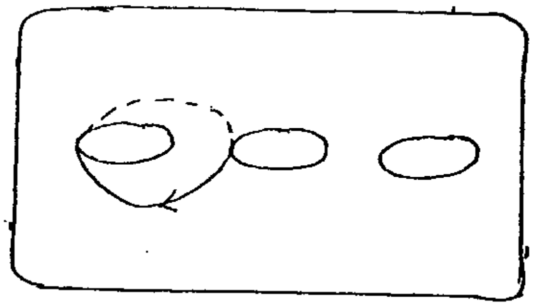


Fig.29(a) $-\gamma_{i-1}$ (i odd)

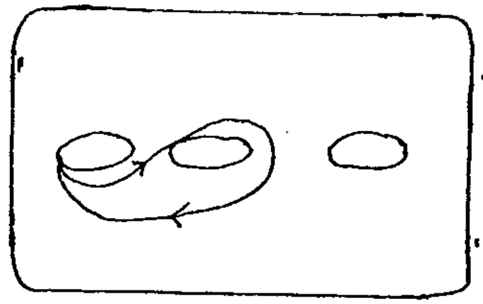


Fig.29(b) $-\alpha'_{i-1}$

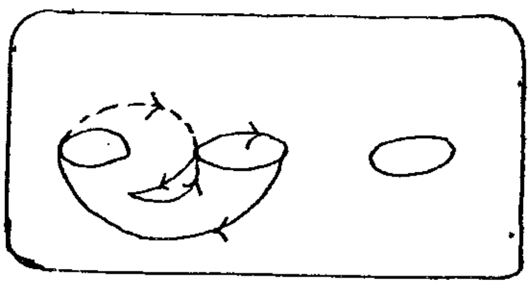


Fig. 30(a) $-\gamma_{i-1}-\alpha'_{i-1}$

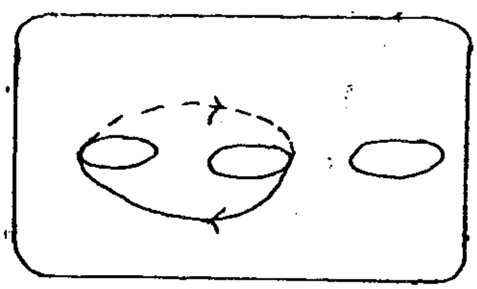


Fig. 30(b) $-\gamma_{i-1}-\alpha'_{i-1}$

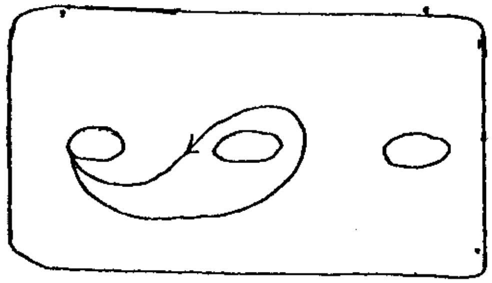


Fig.31(a) α_j'

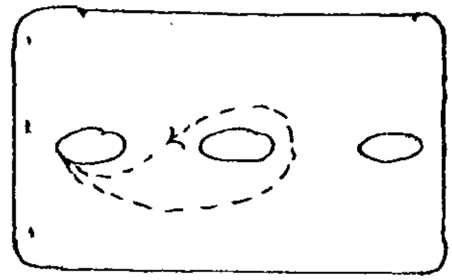


Fig.31(b) α_j''

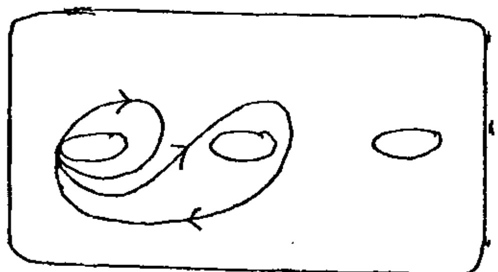


Fig.32(a) $-\sum_{j=1}^{i/2} \alpha_j'$ (i even)

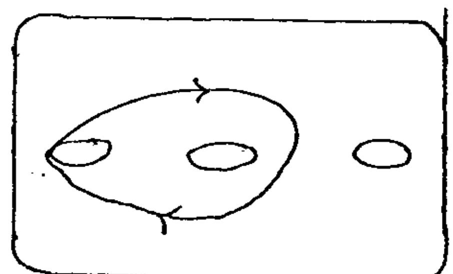


Fig.32(b) $-\sum_{j=1}^{i/2} \alpha_j'$ (i even)

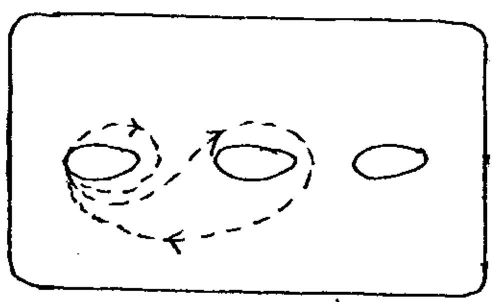


Fig.33(a) $-\sum_{j=1}^{i/2} \alpha_j''$

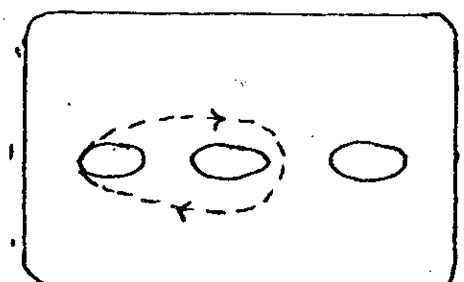


Fig.33(b) $-\sum_{j=1}^{i/2} \alpha_j''$

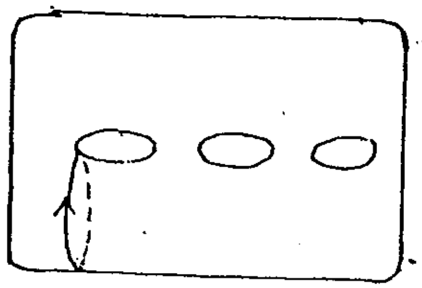


Fig.34(a) β_1'

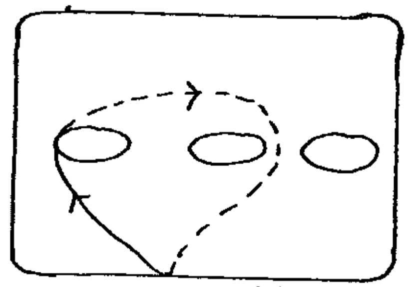


Fig.34(b) $-\sum_{j=1}^{i/2}\alpha_j'' + \beta_1'$

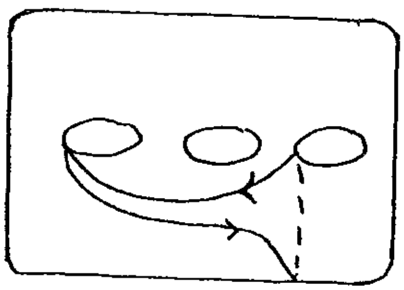


Fig.35(a) $-\beta'_{\frac{i+2}{2}}$

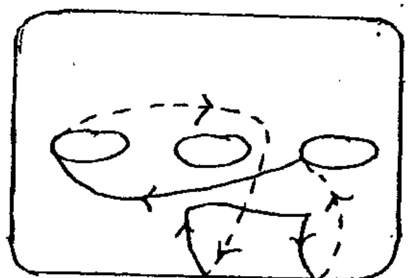


Fig.35(b) $-\sum_{j=1}^{i/2}\alpha_j'' + \beta_1' - \beta'_{\frac{i+2}{2}}$

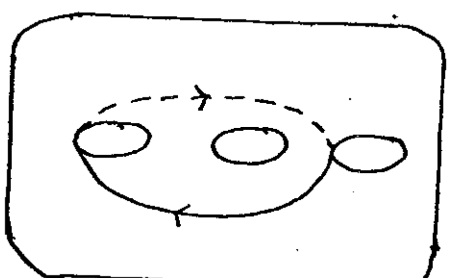


Fig.35(c) $-\sum_{j=1}^{i/2}\alpha_j'' + \beta_1' - \beta'_{\frac{i+2}{2}}$

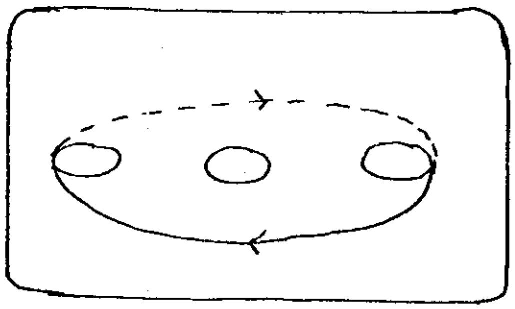


Fig.36(a) γ_{2q-1}

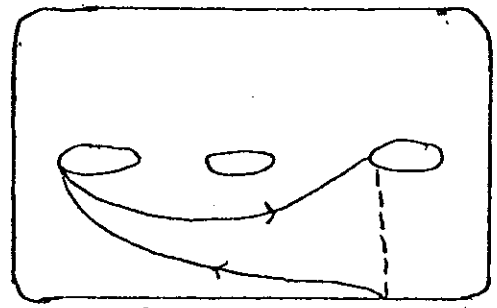


Fig.36(b) β'_q

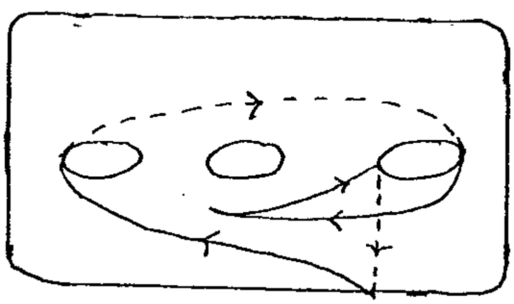


Fig.37(a) $\gamma_{2q-1}+\beta'_q$

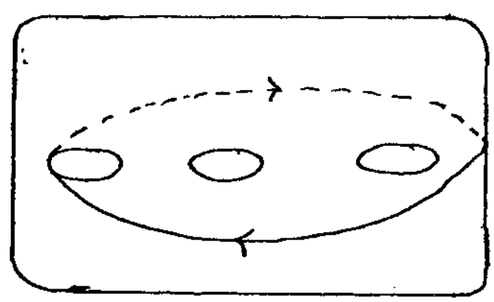


Fig.37(b) $\gamma_{2q-1}+\beta'_q$

IV.5 Prescribing periods for g=2. It is well known that to every canonical homology basis one may assign a unique basis of holomorphic differentials $\eta_1, \ldots, \eta_g$ with the property $\int_{\alpha_j} \eta_k = \delta_{jk}$. Moreover for this basis, the matrix $\pi = (B_{jk})$ with $B_{jk} = \int_{\beta_j} \eta_k$ is symmetric with positive definite imaginary part. We say such a set of differentials is normalized.

Using the relationship we have just established between the set of loops $\gamma_1, \ldots, \gamma_{2g}$ and a canonical homology basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, one may compute the periods of a normalized set of holomorphic differentials for this basis over the loops $\gamma_1, \ldots, \gamma_{2g}$.

We will compute these periods for g=2.

In this case

$$\int_{\alpha_1} \eta_1 = 1, \quad \int_{\alpha_2} \eta_1 = 0, \quad \int_{\alpha_2} \eta_2 = 0 \text{ and } \int_{\alpha_2} \eta_2 = 1$$

We also have from IV.4

$$\gamma_1 = -\alpha_1$$

$$\gamma_2 = \alpha_1 - \beta_1 + \beta_2$$

$$\gamma_3 = -\alpha_1 - \alpha_2 + \beta_1 - \beta_2$$

$$\gamma_4 = \alpha_1 + \alpha_2 - \beta_1$$

Thus

$$\int_{\gamma_1}\eta_1 = \int_{-\alpha_1}\eta_1 = -1$$

$$\int_{\gamma_2}\eta_1 = \int_{\alpha_1}\eta_1 + \int_{-\beta_1}\eta_1 + \int_{\beta_2}\eta_1$$

$$= 1 - B_{1,1} + B_{2,1}$$

$$\int_{\gamma_3}\eta_1 = \int_{-\alpha_1}\eta_1 + \int_{\beta_1}\eta_1 + \int_{-\beta_2}\eta_1 + \int_{-\alpha_2}\eta_1$$

$$= -1 + B_{1,1} - B_{2,1}$$

$$\int_{\gamma_4}\eta_1 = \int_{\alpha_1}\eta_1 + \int_{\alpha_2}\eta_1 + \int_{-\beta_1}\eta_1$$

$$= 1 - B_{1,1}$$

$$\int_{\gamma_1}\eta_2 = 0, \int_{\gamma_2}\eta_2 = -B_{1,2} + B_{2,2}$$

$$\int_{\gamma_3}\eta_2 = B_{1,2} - B_{2,2} - 1 \text{ and } \int_{\gamma_4}\eta_4 = 1 - B_{1,2}.$$

We collect this information in a matrix π_0.

	γ_1	γ_2	γ_3	γ_4
η_1	-1	$1-B_{1,1}+B_{2,1}$	$-1+B_{1,1}-B_{2,1}$	$1-B_{1,1}$
η_2	0	$-B_{1,2}+B_{2,2}$	$B_{1,2}-B_{2,2}-1$	$1-B_{1,2}$

Given a pair (γ_i,γ_j), $i, j = 1,\ldots,4$, $i \neq j$ we say that periods are prescribable over (γ_i,γ_j) if there exists a basis ω_1,ω_2 of holomorphic differentials such that the matrix of periods over (γ_i,γ_j) has the following form

	γ_i	γ_i
ω_i	1	0
ω_2	0	1

It is obvious that such a basis will exist if the matrix

$$I_{i,j} = \begin{array}{c|c|c|} & \gamma_i & \gamma_j \\ \hline \eta_i & \int_{\gamma_i}\eta_1 & \int_{\gamma_i}\eta_1 \\ \hline \eta_2 & \int_{\gamma_i}\eta_2 & \int_{\gamma_j}\eta_2 \\ \hline \end{array}$$

is invertible. In that case one may solve for ω_1,ω_2 in terms of η_1,η_2.

We will show that given any compact surface of genus $g=2$ and a topological dissection given by curves $\gamma_1,\ldots,\gamma_4$ as already defined, there always exists a pair (γ_i,γ_j) on which periods are prescribable. In fact, we can make a stronger statement.

<u>Lemma</u> IV.5.1. Let S be a compact surface of genus $g=2$. Then if periods are not prescribable over (γ_1,γ_2), they are

prescribable over the remaining pairs (γ_1,γ_3), (γ_1,γ_4), (γ_2,γ_3) and (γ_2,γ_4).

Proof. We examine the possible cases.

Suppose $\operatorname{Det} I_{1,2} = B_{1,2} - B_{2,2} = 0$.

Then i) $\operatorname{Det} I_{1,3} = 1+B_{2,2} - B_{1,2} = 1$,

ii) $\operatorname{Det} I_{1,4} = -1+B_{1,2} \neq 0$.

Since $\operatorname{Im} \pi \geq 0$, $\operatorname{Im} B_{2,2} \neq 0$. By our original assumption $\operatorname{Im} B_{1,2} = \operatorname{Im} B_{2,2}$, thus $\operatorname{Im} B_{1,2} \neq 0$ and $\operatorname{Det} I_{1,4} \neq 0$.

iii) $\operatorname{Det} I_{2,3} = -1+B_{1,1} - B_{2,1} \neq 0$.

If $-1+B_{1,1} - B_{2,1} = 0$ then $\operatorname{Im} B_{1,1} = \operatorname{Im} B_{2,1}$. Since $\operatorname{Im} B_{1,2} = \operatorname{Im} B_{2,2}$, we obtain $\operatorname{Im} B_{1,1} = \operatorname{Im} B_{2,1} = \operatorname{Im} B_{1,2} = \operatorname{Im} B_{2,2}$. This is a contradiction since $\operatorname{Im} \pi$ is invertible.

iv) $\operatorname{Det} I_{2,4} = (1-B_{1,2})\,(1-B_{1,1}+B_{2,1})$

$$= (\operatorname{Det} I_{1,4})\,(\operatorname{Det} I_{2,3}) \neq 0.$$

In all cases the determinant is not zero, thus the corresponding periods are prescribable.

References

1. L.V. Ahlfors, Conformal Invariants, McGraw Hill, New York, 1973.

2. L.V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, Princeton, 1966.

3. L.V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math., 72 (1960), 385-404.

4. L. Bers, Fiber spaces over Tiechmuller spaces, Acta Math., 130 (1973), 89-126.

5. L. Bers, Uniformization by Beltrami equations, Comm. Pure Appl. Math., 14 (1961), 215-228.

6. C. Earle, On moduli of closed surfaces with symmetries, Advances in the theory of Riemann surfaces, Ann. of Math. Studies, 66 (1971), 119-130.

7. C. Earle and I. Kra, On sections of some holomorphic families of closed Riemann surfaces, Acta Math., 137 (1976), 49-79.

8. H. Farkas and I. Kra, Riemann Surfaces, to appear.

9. L. Ford, Automorphic Functions, (second ed.) Chelsea, New York, 1951.

10. L. Greenberg, Maximal Fuchsian groups, Bull. Amer. Math. Soc., 69 (1963), 569-573.

11. L. Keen, Canonical polygons for finitely generated Fuchsian groups, Acta. Math. 115 (1966), 1-16.

12. I. Kra, Automorphic Forms and Kleinian Groups, W.A. Benjamin, 1972.

13. S. Kravetz, On the geometry of Teichmuller spaces and the structure of their modular groups, Ann. Acad. Sci. Fenn., 278 (1959), 1-35.

14. W. Magnus, Noneuclidean Tesselations and Their Groups, Academic Press., New York, 1974.

15. B. Maskit, On Poincare's theorem for fundamental polygons, Advances in Math., 7 (1971), 219-230.

16. B. Maskit, Self-maps of Kleinian groups, Amer. J. of Math., 93 (1971), 840-856.

17. G. Springer, Introduction to Riemann Surfaces, Addison-Wesley, 1957.

18. E. Whittaker, On the connexion of algebraic functions with automorphic functions, Phil. Trans., vol. 192 (1899), 1-32.