Singularities, Branched Coverings, and Characteristic Classes

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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A certain class of singularities of vector bundle homomorphisms is defined and the relationship between the characteristic classes of the vector bundles involved determined. These vector bundle homomorphisms have the property that near its' singularity subsets, (which are assumed to be submanifolds), the behavior of the homomorphism is controlled by the topology of the embedding of the singularities, and is an isomorphism elsewhere.

Examples are given that show that many naturally occuring maps between manifolds give rise to these types of homomorphisms, notably: branched coverings, dilatations, and some types of generic maps.

The basic results are then applied to these examples to obtain relations between the characteristic numbers of

manifolds related in these fashions. Some known results are re-obtained without appealling to deep results of Atiyah-Singer. Prominent among these is the signature formula for branched coverings due to Hirzebruch.

The method is to compute the difference of the two vector bundles in the appropriate Grothendieck ring in terms of local datum that arise from the homomorphism.

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Introduction

Consider the following problem: Given a mapping $f : M \to N$ of manifolds, (here M and N can belong to any of the categories; TOP, PL, C^∞, Real or Complex Analytic, or Algebraic and f is a morphism), determine relations between the invariants of the manifolds M and N and the invariants arising from the map f.

A first illustration of a solution to this problem is the classical Riemann-Hurwitz theorem on the holomorphic mapping $f : M \to N$ of compact Riemann surfaces. This formula can be written

$$2 - 2g(M) + w = d \cdot (2 - 2g(N))$$

where d is an integer, the degree of the mapping, and w is the index of ramification, i.e., the sum of the orders of the various pts. of ramification. The genera $g(M)$, $g(N)$ are analytic invariants of M and N themselves while d, w depend upon the mapping.

For non-singular, complex projective algebraic varieties a much more profound relationship exists between invariants of these manifolds and quantities depending upon a holomorphic mapping. This is the so-called Grothendieck-Riemann-Roch Theorem. This theorem contains, as a special case, the Riemann-Hurwitz formula.

In the Riemann-Hurwitz formula notice that the terms $2 - 2g(M)$, $2 - 2g(N)$ are, respectively, the Euler characteristics of M and N. Now, forgetting the holomorphic structure of the map, we obtain a formula relating the Euler characteristics of M and N to invariants which arise from f when f is considered to be smooth or PL.

Since the Euler characteristic is a "characteristic class" we see, at least in this special case, that if the singularities of the map are sufficiently nice and, the behavior of the map near to these singularities is of a special nature, it should be possible to make some kind of quantitative statement about the relationship of the characteristic classes of M and N, the <u>local</u> behavior of the map f, and the topology of the singularity subset in M (or in N).

Assume for a moment that M and N are smooth manifolds and f is a smooth mapping. Then there is an induced map of tangents:

$$\begin{array}{ccc} TM & \xrightarrow{df} & f^{!}TN \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

($f^{!}TN$ denotes the "Pulled back" bundle via the map f of the tangent bundle of N).

Suppose that df has the following properties:

$df : TM_m \to (f^! TN)_m$ is an isomorphism for all $m \in M$, except for m belonging to a finite collection $\bigcup_{i=1}^{q} Y_i$ of <u>disjoint</u>, closed sub-manifolds of M, along each of which df has constant rank, (the rank may differ on each Y_i), and the behavior of df near Y_i is "controlled" by the topology of the embedding $j_i : Y_i \to M$. Then one would expect that the difference of the characteristic classes of M and the pulled-back classes of N should be a sum of co-homology classes "concentrated" on the sub-manifolds Y_i. Specifically, assume $Y_i \subset M$ is, for each i, of codimension 1, (resp. of codimension 2, with an oriented normal bundle). Then for any point $y \in Y_i$ it should be possible to choose local co-ordinates $(u_1^i,\ldots,u_{n-1}^i,x_i),(u_1^i,\ldots,u_{n-1}^i) \in R^{n-1}, x_i \in \mathbb{R}$, (resp. $(u_1^i,\ldots,u_{n-2}^i,z_i),(u_1^i,\ldots,u_{n-2}^i) \in R^{n-2}, z_i \in \mathbb{C}$) about y, so that Y_i is given locally by $x_i = 0$ (resp. $z_i = 0$), and local co-ordinates $(v_1^i,\ldots,v_{n-1}^i,x_i')$, (resp. $(v_1^i,\ldots,v_{n-2}^i,z_i')$ about $f(y)$, such that $f(Y_i)$ is a submanifold of codimension 1, (resp. codimension 2, oriented normal bundle) given locally by $x_i' = 0$(resp. $z_i' = 0$), and df having the local description

$$df\left(\frac{\partial}{\partial u_j^i}\right) = \frac{\partial}{\partial v_j^i} \qquad j = 1,\ldots, n-1$$

$$df\left(\frac{\partial}{\partial x_i}\right) = x_i^{p_i} \cdot \frac{\partial}{\partial x_i'} \qquad i = 1,\ldots, q$$

or, (resp. df $(\frac{\partial}{\partial z_i}) = \frac{\partial}{\partial v^i_j}$

$$df(\frac{\partial}{\partial z_i}) = z_i^{p_i} \cdot \frac{\partial}{\partial z'_i}$$

where p_i is an integer ≥ 1.

More generally, let E,F be two real, (resp. complex) vector bundles of the same fibre dimension, say n, over a manifold M and α a vector bundle homomorphism between them. Suppose that except for a finite collection $\bigcup_{i=1}^{q} Y_i$ of codimension 1, (resp. codimension 2, oriented normal bundle), closed sub-manifolds of M α is an isomorphism, and α has constant rank, say $n - j_i$, along Y_i. Let $Ker_i(\alpha)$ be the kernal vector bundle of α restricted to Y_i of fibre dimension j_i, and let $s^i_1,\ldots,s^i_n$ and $t^i_1,\ldots,t^i_n$ be local spanning sections of E and F respectively in a neighborhood of any point $y \in Y_i$ chosen in such a way as to have $s^i_{n-j_i+1},\ldots,s^i_n$ span $Ker_i(\alpha)$. Then α, near Y_i should have the form

$$\alpha(s^i_k) = t^i_k \qquad k = 1,\ldots,n - j_i$$

$$\alpha(s^i_\ell) = x_i^{p_i} \cdot t^i_\ell \qquad \ell = n - j_i + 1,\ldots,n$$

in the real case (Y_i of codimension 1), or

$$\alpha(s^i_k) = t^i_k \qquad k = 1,\ldots, n - j_i$$

$$\alpha(s^i_\ell) = z_i^{p_i} \cdot t^i_\ell \qquad \ell = n - j_i + 1,\ldots,n$$

in the complex case (Y_i of codimension 2, oriented normal bundle). Under these conditions we say that α is a local wrapping homomorphism of degree p_i about Y_i.

It will be shown that many naturally occuring maps have this kind of property.

Since the characteristic classes that will be of interest here are stable invariants, to find formulae it is sufficient to have a representation of the difference bundle $F - E \in KO(M)$, (resp. $K(M)$) where $KO(M)$ (resp. $K(M)$) denotes the Grothendieck ring of real (resp. complex) vector bundles on M. Since $F - E$ is, in some sense, zero on $M - \bigcup_i Y_i$ it is reasonable to expect, just as in cohomology, that $F - E$ is in the image of a Gysin type homomorphism

$$\bigoplus_i KO(Y_i) \to KO(M), \text{ (resp. } \bigoplus_i K(Y_i) \to K(M)).$$

In fact, in these special cases the Gysin homomorphism in co-homology and the "push forward" homomorphism $j_!$ are essentially the same. It is this fact that allows us to write the main result of this paper concerning local wrapping type homomorphisms. This result, which will be referred to as "the main result" in the future, may be stated in the following fashion:

In $KO(M)$, (resp. $K(M)$) the following identity holds:

$$F - E = j_!\left(\sum_i(\lambda_i^{p_i}+\lambda_i^{p_i-1}+\cdots+\lambda_i)\mathrm{Ker}_i(\alpha)\right)$$

where $j_! = j_!^1 + \cdots + j_!^q$ is the direct sum of the individual "push forward" homomorphisms, λ_i is the normal (real) line bundle of Y_i in M, (resp. λ_i is the naturally associated complex line bundle to the codimension 2 sub-manifold Y_i), and $\lambda_i^{p_i}$ denotes $\lambda_i \overbrace{\otimes \cdots \otimes}^{p_i} \lambda_i$, $\otimes$-product over $\mathbb{R}$, (resp. over $\mathbb{C}$).

There is an interesting combination of the above two types of homomorphisms. Consider real vector bundles and homomorphisms having the property that α is an isomorphism except on a finite collection $\bigcup_i Y_i$ of codimension 2 closed sub-manifolds with oriented normal bundles λ_i, λ_i is considered as a complex line bundle as usual. Near Y_i demand that α be the realization of a homomorphism of the second kind mentioned above. Specifically, it is required that the bundle $Ker_i(\alpha)$ be the realization of a complex vector bundle, call it K_c^i, then we obtain the second result:

$$F - E = \phi[j_!(\sum_i(\lambda_i^{p_i} + \lambda_i^{p_i-1} + \cdots + \lambda_i) \cdot K_c^i)]$$

where ϕ: $K(M) \to KO(M)$ is the forgetful homomorphism.

I give now some indication of the type of results one can obtain by applying the above formulae.

Suppose $b : M^n \to N^n$ is a local branched covering of compact complex analytic manifolds, i.e. there is a sequence $Y_1, \ldots, Y_q$ of q disjoint codimension 1 (complex) closed, and analytic sub-manifolds of M and open neighborhoods $U_i \supset Y_i$ such that $b(U_i)$ is open and the restriction of b exhibits

$U_i - Y_i$ as a p_i - sheeted covering space of $b(U_i) - b(Y_i)$. The integer $(p_i - 1)$ is called the branching order of b along Y_i.

It can be shown that for any point $y \in Y_i$ there one local co-ordinates (ξ, Z_i), $\xi \in \mathbb{C}^{n-1}$, $Z_i \in \mathbb{C}$ about y and local co-ordinates (ξ', Z_i') about f(y) such that b has the local description

$$\xi' = \xi \qquad Z_i' = Z_i^{p_i}$$

where Y_i is given locally by $Z_i = 0$. Taking the differential clearly exhibits the bundle map thus obtained as a local wrapping homomorphism of degree $(p_i - 1)$ about Y_i.

Let td(M), (resp. td(N)), denote the (total) Todd class of M, (resp. N). Recall its definition. Let C(M) $C(M) = \prod_j (1+d_j)$, $d_j \in H^2(M, \mathbb{Z})$, be a formal factorization of the (total) Chern class of M. Then set

$$td(M) = \prod_j \frac{d_j}{1 - \exp(-d_j)}$$

$$= 1 + td_1(M) + \cdots$$

Note that for a compact Riemann surface an easy calculation gives

$$td(M) = 1 + \tfrac{1}{2}c_1(M) = 1 + \tfrac{1}{2}(2-2g(M))[M] \quad [M] \in H_2(M, \mathbb{Z})$$

is the canonical generator.

Now, to each of the sub-manifolds Y_i there corresponds a (holomorphic) line bundle L_i such that $c_1(L_i) = y_i$, y_i is the Poincaré dual of the homology class determined by the sub-manifold Y_i. Applying the main result, and noting that $Ker_i(db) = \lambda_i$ = normal line bundle of Y_i in M, we see that in K(M) the following relation holds: $b^! TN - TM = \sum_i (L_i^{p_i} - L_i)$. Now upon taking the (total) Todd class of both sides of this relation we get the following:

$$b^* td(N) \prod_i (1-\exp(-y_i)) = td(M) \prod_i (y_i).$$

In particular, let's assume that the manifolds in question are algebraic, (i.e. non-singular, irreducible, projective algebraic varieties), in which case it is well known that the Todd and arithmetic genera agree, (Hirzebruch-Riemann-Roch Theorem), recall that the Todd genus is obtained by evaluation of the top dimensional term of the (total) Todd class on the preferred generator of $H_{2n}(M, \mathbb{Z})$. Using the above relation one can now determine the relationship between the arithmetic genera of two such manifolds related in this manner.

As a specific example, consider the case of compact Riemann surfaces. Computing we easily obtain:

$$b^* c_1(N) = \sum_i (p_i - 1) y_i + c_1(M)$$

which gives, upon evaluation on the fundamental 2-cycle of M,

$$d(2-2g(N)) = w + (2-2g(M))$$

$w = \sum_i (p_i - 1)$, and d is the degree of the mapping.

Another important class of examples of maps which give rise to wrapping homomorphisms are the "dilatations" or "Blow-ups". A "dilatation" of a manifold X^n along a sub-manifold Y gives rise to a commutative diagram of manifolds and maps:

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ \uparrow j' & & \uparrow j \\ Y' & \longrightarrow & Y \end{array} .$$

X' is an n-dimensional manifold and σ is a proper map. Blow ups exist in the smooth, analytic, and PL categories. In any case, the restriction of σ to the compliment of Y' in X' is a bi-morphism onto the compliment of Y in X, and σ on Y' is the projection of the "projectified" normal bundle of Y in X onto Y.

It will be shown that the induced map of tangents satisfies the conditions for a wrapping homomorphism of degree 1 along Y' .

In this case we obtain the following formula:

Let cc( ) denote the (total) Chern class in the complex case or the (total) Stiefel-Whitney class in the C^∞, real analytic, or PL case. Then

$$cc(X') = \sigma^* cc(X) \left(\frac{\beta^* cc(y)}{v} [(1+v) \sum_i (1-v)^i \beta^* cc_{n-m-i}(N) - \beta^* cc(N)] \right)$$

m = codimension of Y in X, and $v = cc_1(\lambda)$, where λ is the normal line bundle of Y' in X', N is the normal bundle of Y in X.

A special case of this result was obtained by Ian Porteous in his thesis. If X is algebraic then this is the result that he obtained verifying earlier conjectures of Todd-Segre. He obtained his result by algebraic methods, i.e., by application of the Grothendieck-Riemann-Roch Theorem. In their paper [1] Atiyah-Hirzebruch extend Porteous' result to arbitrary compact complex manifolds, they also settle the real analytic case as well. Since they construct their "real Grothendieck element" by means of resolutions of sheaves of analytic functions this method does not carry over to the C^∞ case. They then prove a differentiable analogue of the Grothendieck-Riemann-Roch Theorem which allows one to extend the results of Porteous to this case. Again they use sheaf theoretic methods, this time with germs of C^∞ functions. The approach taken in this paper avoids this and is more geometric in nature. In any case there is nothing new to be added in this context.

Possibly the most important application is to branched coverings of smooth or PL manifolds. Suppose that $b : M^n \to N^n$ is a branched covering of C^∞ closed and oriented manifolds, branched along q codimension 2 closed (and oriented) sub-manifolds $Y_1, \ldots, Y_q$, (associated to the sub-manifolds Y_i

are the q-complex line bundles λ_i), with order of branching $p_i - 1$ respectively.

Let $L_1, \ldots, L_q$ be complex line bundles on M corresponding to the co-homology classes $y_1, \cdots, y_q$ dual to the homology classes determined by $Y_1, \ldots, Y_q$. Then according to the main result we have:

$$(*) \qquad b^! TN - TM = \phi(\sum_i L_i^{p_i} - L_i).$$

This formula leads to the following relationship between the characteristic classes of M and N. Let P( ) denote the (total) rational Pontrjagin class, then

$$b^* P(N) \cdot \prod_i (1+y_i^2) = P(M) \cdot \prod_i (1+p_i^2 y_i^2),$$

(cup product is understood), or for W( ), the (total) Stiefel-Whitney class

$$b^* W(N) \prod_i (1+y_i \bmod 2) = W(M) \prod_i (1+p_i y_i \bmod 2).$$

Sylvan Cappell has informed me that he has obtained the same formula relating the Stiefel-Whitney classes by combinatorial methods.

The virtue of this formulation lies in the fact that it may be used to find the relations existing between any of the various characteristic numbers of M and N.

In particular, let $\mathcal{L}$( ) denote the (total) Hirzebruch L class. Recall its definition. If $\prod_j (1+d_j^2) = P(M)$ is a formal factorization of the (total) Pontrjagin class of M, where $d_j \in H^2(M,\mathbb{Z})$, then set

$$\mathcal{L}(M) = \prod_j \frac{d_j}{\tan h(d_j)}$$
$$= 1 + \mathcal{L}_1(M) + \cdots$$

Now suppose that the dimension of M is divisible by 4, say 4k. Applying $\mathcal{L}(\quad)$ to both sides of the relation (*) yields

$$\mathcal{L}(M) = b^*\mathcal{L}(N)\cdot\prod_i \left(\frac{y_i}{\tanh(y_i)} \cdot \frac{\tanh(p_i y_i)}{p_i y_i} \right).$$

It is well known that by evaluation of $\mathcal{L}_k(M)$ on the fundamental cycle, [M], of M one obtains the signature of the manifold M.

Now, $b^* y_i' = p_i y_i$ where $y_i' \in H^2(N,\mathbb{Z})$ is dual to the homology class determined by $b(Y_i) \subset N$. In particular $y_i' = p_i x_i$. Rewriting the above formula, we obtain

$$\mathcal{L}(M) = b^*(\mathcal{L}(N)\prod_i \frac{x_i}{\tanh(x_i)} \cdot \frac{\tanh(p_i x_i)}{p_i x_i})$$

and evaluating on [M] gives

$$(**) \qquad \text{Signature}(M) = d[\mathcal{L}(N)\prod_i \frac{x_i}{\tanh(x_i)} \cdot \frac{\tanh(p_i x_i)}{p_i x_i}][N]$$

where d is the degree of the mapping. Note that for $q = 1$, $M \xrightarrow{b} N$ is a p-fold branched cover, branched along $Y' \subset N$. In this special, but extremely important case we obtain

$$(**)' \qquad \text{Signature}(M) = (\mathcal{L}(N) \ \tanh(px)\text{cotanh}(x))[N].$$

This is precisely the formula obtained by Thomas-Wood [2]. They obtain their formula by using elementary group representation theory and the G-signature theorem of Atiyah-Singer.

Thomas-Wood obtained this result during their investigation of the problem of representing homology classes by embedded sub-manifolds in codimension 2. They draw upon an earlier work of Hirzebruch [3] who was concerned with a different question. His investigation of the signature of branched coverings arose in connection work of Atiyah [4] and Kodaira [5] who studied the signature of ramified coverings in some special cases which are of interest because they show that the signature of the total space of a differentiable fibre bundle need not be equal to the product of the signatures of the base and fibre (this multiplicative property does hold however if the fundamental group of the base operates trivially on the co-homology of the fibre [6]). Briefly, Hirzebruchs result can be stated as follows:

$$\text{Signature}(M) = \text{Signature}\left(\frac{(1+U)^p - (1-U)^p}{(1+U)^p + (1-U)^p} \cdot \frac{1}{U}\right)$$

$$= p\cdot\text{Signature}(N) - \frac{p(p^2-1)}{3}\text{Signature}(U\circ U)+\cdots$$

where $U \subset N$ is a closed, and oriented sub-manifold of codimension 2 which realizes the co-homology class x, and $U \circ \cdots \circ U$ denotes the oriented self-intersection co-bordism class of U in N.

The equivalence of this formula to (**)' is contained in Thomas-Wood.

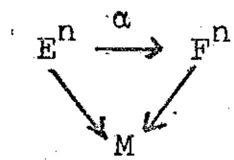
If one considers appropriate actions of the finite abelian group $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_q}$ on smooth manifolds one can, by using the G-signature theorem, obtain the result (**) and thereby generalize Hirzebruchs result to this case.

In the TOP category branched coverings exist, (one assumes that the branching sets $Y_1, \ldots, Y_q$ are locally flat in M). The methods of Hirzebruch-Thomas-Wood do not apply here. Wall [7] has extended the G-signature theorem to certain types of group actions on topological manifolds. Applying this theorem to this particular case yields (**).

On the other hand, if one assumes that the local branching sets $Y_1, \ldots, Y_q$ are locally flat and have normal vector bundles then it is possible to construct a "topological Riemann-Roch Theorem", (i.e. push forward homomorphisms in K-theory), and the proof of (*) goes through in this case avoiding the G-signature theorem.

The last application is to generic vector bundle maps. Since this topic will not be included in this paper I mention it only briefly and for the sake of completeness. This topic should be the subject of future work.

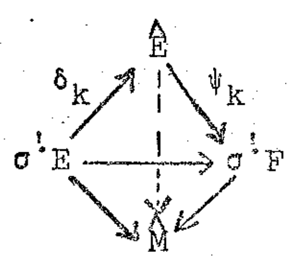
In this paragraph assume that all manifolds and maps are at least of class C^∞. The vector bundles may be either real or complex. Let



(n = fibre dimension)

denote a vector bundle homomorphism. Now any such vector bundle homomorphism can be viewed as a section s_α of the vector bundle HOM(E,F) over M. Let Σ^i denote the sub (fibre) bundle of HOM(E,F) whose fibre Σ^i_m at each point $m \in M$ consists of all those homomorphisms in $HOM(E,F)_m$ with kernal rank precisely equal to i. Define $\Sigma^i(\alpha) = s_\alpha^{-1}(\Sigma^i)$. A vector bundle homomorphism is generic just in case the map s_α is tranverse to all of the sub-manifolds Σ^i.

Assume α is generic and let $k = \max_i | \Sigma^i(\alpha) \neq \emptyset$. Then $\Sigma^k(\alpha)$ is a closed sub-manifold of M. Performing a dilatation of M along $\Sigma^k(\alpha)$ we obtain a new manifold $\hat{M}$ and a vector bundle $\hat{E}^n$ over $\hat{M}$ together with homomorphisms



such that this diagram commutes, δ_k satisfies the conditions for a wrapping homomorphism, (in the appropriate sense), and ψ_k is a generic vector bundle homomorphism with $\Sigma^k(\psi_k) = \emptyset$.

By repeated applications of this construction one can obtain a formula for the difference F - E lifted to some manifold obtained from M by a sequence of dilations. This should yield information on the Thom polynomials for some special types of higher order singularities.

§1. Preliminaries

A. Complex line bundles and homology.

In this section the relation between complex line bundles and homology is discussed. These results will be of fundamental importance in all that follows.

Let X be a "nice" space, by this I mean X is paracompact and having the homotopy type of a finite polyhedron. Denote by $\text{Line}_C(X)$ the set of all isomorphism classes of complex line bundles on X. Tensor product of elements of $\text{Line}_C(X)$ turns this set into an abelian group. The unit, 1, is the trivial line bundle. The inverse of an element λ is given by $\lambda^{-1} = \lambda^* = \text{HOM}(\lambda,1)$ as is easily checked by considering transition data.

The classification theorem [8] implies that the U(1)-bundles over X are in one-to-one correspondence with homotopy classes of continuous maps from X to BU(1), the classifying space for the group U(1). Therefore we have a bijection of sets

$$\text{Line}_C(X) = [X,BU(1)].$$

Since BU(1) is a $K(\mathbb{Z},2)$, $[X,BU(1)]$ has a group structure and is isomorphic to $H^2(X,\mathbb{Z})$. The correspondence $\lambda \mapsto c_1(\lambda)$, where $c_1(\lambda)$ is the Chern class of λ, induces an isomorphism

$$\text{Line}_C(X) \approx H^2(X,\mathbb{Z}).$$

Now, $c_1(\lambda\otimes\eta) = c_1(\lambda) + c_1(\eta)$ so we see that the correspondence is additive. It is clear from the above discussion that the map is surjective and it follows from elementary obstruction theory that it is injective as well.

Assume for a moment that X is a compact and oriented smooth manifold without boundary. Then we have the following result of Thom [9]: Every 2 dimensional integral co-homology class of such an X can be represented by a codimension 2 oriented, and closed sub-manifold, i.e. the sub-manifold determines a homology class which corresponds under Poincaré duality in X to the 2 dimensional cohomology class being represented.

This result can immediately be extended to the case where X is a TOP manifold, the 2 dimensional cohomology class being represented by a locally flat codimension two closed, and oriented sub-manifold.

Assume now that X is a C^∞ oriented and closed manifold. The connection between complex line bundles and homology is now clear. Suppose $Y \subset X$ is an orientable codimension 2 sub-manifold of X, then Y has a normal 2-plane bundle λ. Orient Y and its' normal bundle so as to span the given orientation of X. Now, the bundle λ has structure group the special orthogonal group SO(2) and is therefore the realization of a U(1) bundle, also denoted λ, on Y. There is then the following theorem [10]:

Let $j : Y \to X$ be an embedding of an oriented compact manifold Y as a codimension 2 sub-manifold of a compact and oriented manifold X. Let $y \in H^2(X,\mathbb{Z})$ be the cohomology class which corresponds under Poincaré duality to the oriented cycle Y and let λ be the normal SO(2) bundle of Y in X. Then considering λ as a complex line bundle we have that

$$c_1(\nu) = j^* y \in H^2(Y,\mathbb{Z}).$$

Hence λ is the restriction of a unique, (up to homotopy) complex line bundle L on X, i.e. we have $j^! L = \lambda$ and $c_1(L) = y$.

B. <u>The diagonal section.</u>

Let $\lambda \in \text{Line}_{\mathbb{C}}(Y)$ with projection $\pi : \lambda \to Y$. Consider the pulled back bundle,

$$\begin{array}{ccc} \pi^! \lambda & \longrightarrow & \lambda \\ \big\downarrow & & \big\downarrow \pi \\ \lambda & \xrightarrow[\pi]{} & Y \end{array}$$

Now, $\pi^! \lambda \subset \lambda \times \lambda$ consisting of all pairs $(\omega_x, \omega'_{x'})$ such that $x = x'$ and $\omega_x \in \pi^{-1}(x)$ $\omega'_{x'} \in \pi^{-1}(x')$. This bundle has a canonically defined section $s : \lambda \to \pi^! \lambda$ given by $s(\omega_x) = (\omega_x, \omega_x)$. This section vanishes precisely along the zero section of λ and is in fact transverse to it. This section is called the diagonal section.

If a line bundle (real or complex) admits a non-vanishing section then it is isomorphic to a trivial bundle. Hence, the bundle $\pi^! \lambda$ restricted to $\lambda_0 = \lambda$ - (zero section) is trivialized in a canonical way.

In a similar fashion the bundle $\pi^!(\lambda^p) = \pi^!(\overbrace{\lambda\otimes\cdots\otimes\lambda}^{p})$ admits a natural section s^p defined by $s^p(\omega_x) = (\omega_x, \omega_x\otimes\cdots\otimes\omega_x)$. This section also vanishes only along the zero section of λ but is not transverse to it if $p > 1$.

In a similar fashion any codimension one closed submanifold Y of a closed manifold X gives rise, via Poincaré duality, to an element $y \in H^1(X, \mathbb{Z}_2)$. In a manner analogous to the above we see that there is a real line bundle L on X whose Stiefel-Whitney class is the element y, and the restriction of the bundle L to Y is the normal bundle of Y in X. Also, diagonal sections and their powers can be defined in this case.

C. <u>The Case of a Non-compact Manifold.</u>

If one wishes to extend these results to the case of a non-compact X there are problems. For any pair (Z,W) of CW complexes, define $\mathcal{H}^q(Z,W) = \varprojlim_{\alpha} H^q(Z_\alpha, W_\alpha)$, ($\mathbb{Z}$, or $\mathbb{Z}_2$ coeffecients are understood), the Z_α runs through all finite subcomplexes of Z and $W_\alpha = W \cap Z_\alpha$.

In particular if (Z,W) is a pair of finite CW complexes then $\mathcal{H}^*(Z,W) = H^*(Z,W)$. All the elementary properties of

$H^*(Z,W)$ extend to $\mathcal{H}^*(Z,W)$. Thus $\mathcal{H}^*(Z,W)$ is a contravariant functor of homotopy type and $\mathcal{H}^*(Z,W)$ is a $\mathcal{H}^*(Z)$-module. It is easy to see that if Z and W are manifolds with a countable topology then the natural map $H^q(Z,W) \to \mathcal{H}^q(Z,W)$ is surjective.

Now for any CW complex X define $\mathcal{L}ine_C(X)$, (resp. $\mathcal{L}ine_{\mathbb{R}}(X)$) as $\varprojlim_\alpha \mathrm{Line}_{\mathbb{C}}(X_\alpha)$, (resp. $\varprojlim_\alpha \mathrm{Line}_{\mathbb{R}}(X_\alpha)$). An element $\mathcal{L} \in \mathcal{L}ine_{\mathbb{C}}(X)$, (resp. $\mathcal{L}ine_{\mathbb{R}}(X)$), assigns to any inclusion $i_\alpha : X_\alpha \to X$ an element $i_\alpha^! \mathcal{L} \in \mathrm{Line}_{\mathbb{C}}(X_\alpha)$, (resp. $\mathrm{Line}_{\mathbb{R}}(X_\alpha)$).

In particular if X is a manifold and Y is a closed codimension 2 sub-manifold with an oriented normal bundle, (resp. Y a codimension 1 closed sub-manifold) then there is a unique element $\mathcal{L}$ in $\mathcal{L}ine_{\mathbb{C}}(X)$, (resp. $\mathcal{L}ine_{\mathbb{R}}(X)$), such that $\mathcal{L}$ corresponds to $j_*(1) \in \mathcal{H}^2(X,\mathbb{Z})$, (resp. $\mathcal{H}^1(X,\mathbb{Z}_2)$), (see next paragraph), and for the inclusion $j : Y \to X$, $j^! \mathcal{L}$ gives the normal complex, (resp. real) line bundle of Y in X, with this in mind define $cc_1(\mathcal{L}) = j_*(1)$ in either case.

A few properties of the relative groups $\mathcal{H}^*(X,X-Y)$, where Y is a <u>closed</u> sub-manifold of X will be needed. If Y has an orientable normal bundle then $\mathbb{Z}$ coefficients are understood, otherwise use $\mathbb{Z}_2$. In this case we have "Thom homomorphisms" $\phi_* : \mathcal{H}^*(Y) \to \mathcal{H}^*(X,X-Y)$. There is also the natural map i^* induced by the map $i : (X,\emptyset) \to (X,X-Y)$, $i^* : \mathcal{H}^*(X,X-Y) \to \mathcal{H}^*(X)$. The composition $i^* \circ \phi_* : \mathcal{H}^*(Y) \to \mathcal{H}^*(X)$

is a "Gysin homomorphism" and is denoted by j_*.

Let X_0 be an open neighborhood of the sub-manifold Y in X, then there is the excision isomorphism $\sigma : H^*(X_0, X_0 - Y) \to H^*(X, X-Y)$.

D. The "Push Forward" Homomorphism in K-Theory.

In order to construct the "push forward" homomorphism it is first necessary to give the "difference bundle" construction of Atiyah-Hirzebruch [11]. The construction given here differs from the one given there, it is due to Bott [12].

Let X denote a finite CW complex, Y a subcomplex and E and F two complex vector bundles on X. Suppose also that we have an isomorphism, α, between the bundle E restricted to Y and F restricted to Y. The difference bundle will then be an element of the relative Grothendieck ring K(X,Y), denoted by $d(E,F;\alpha)$. This element is analogous to the difference co-cycle in cohomology in the sense that $d(E,F;\alpha) = 0$ if α extends to an isomorphism over all of X.

Set $Z = X_1 \cup_Y X_2$, i.e., Z is the space obtained from the disjoint union of two copies of X, say X_1 and X_2, joined along $Y \subset X_i$.

Take the bundle $E \cup_\alpha F$ over Z, i.e. take E over X_1, F over X_2, and glue them together using α on Y. Let $\pi : Z \to X$ be the natural projection given by the identity on each factor and let $s_i : X_i \to Z$ be the two inclusions of $X_i \subset Z$.

Now the sequence of maps

$$X \underset{\pi}{\overset{s_i}{\rightleftarrows}} Z \longrightarrow Z/X_i \approx X/Y$$

gives rise to an exact sequence

$$0 \to K(Z,X_2) \to K(Z) \underset{\pi^!}{\overset{s_2^!}{\rightleftarrows}} K(X) \to 0$$

which splits. Hence we may identify $K(X,Y)$ with the kernal of $s_2^!$ in $K(Z)$. Make this identification in what follows. Consider the element $E \cup_\alpha F - \pi^! F \in K(Z)$, clearly this belongs to the kernal of $s_2^!$ and therefore defines an element $d(E,F;\alpha) \in K(X,Y)$.

The main properties of the difference bundle that will be of use to us are summarized in the following proposition:

Proposition. (Atiyah-Hirzebruch [13])

(i) $d(E,F;\alpha)$ depends only upon the homotopy class of the isomorphism α.

(ii) if $i^! : K(X,Y) \to K(X)$ denotes the natural map, then $i^! d(E,F;\alpha) = E - F$.

(iii) If α extends to an isomorphism over all of X, then $d(E,F;\alpha) = 0$.

(iv) $d(E\oplus E', F\oplus F'; \alpha\oplus\alpha') = d(E,F;\alpha) + d(E',F';\alpha')$.

(v) $d(F,E;\alpha^{-1}) = - d(E,F;\alpha)$

(vi) If D is a vector bundle on X, then $d(E\otimes D, F\otimes D; \alpha\otimes 1) = d(E,F;\alpha)\cdot D$, (where the multiplication on the right makes use of the fact that $K(X,Y)$ is a $K(X)$-module).

(vii) $d(E,F;\alpha)$ is functorial.

There is a Grothendieck ring of real vector bundles over a nice space X. In exactly the same way one can construct difference elements for this case.

We are now ready to construct the push forward homomorphism, for more details see [1], let (Z,W) be any pair of CW complexes define $\mathcal{K}(Z,W) = \varprojlim_{\alpha} K(Z_\alpha, W_\alpha)$, where Z_α runs through all finite subcomplexes of Z and $W_\alpha = W \cap Z_\alpha$. All the elementary properties of the relative Grothendieck ring extend to $\mathcal{K}(Z,W)$. In particular there is a homomorphism of rings $Ch : \mathcal{K}^*(Z) \to \mathcal{H}^{**}(Z;Q)$, where $\mathcal{H}^{**}(Z,Q)$ is the direct product of the $\mathcal{H}^q(Z,Q)$. An element $\xi \in \mathcal{K}(Z)$ assigns an element $i_\alpha^! \xi \in K(Z_\alpha)$ for any inclusion $i_\alpha : Z_\alpha \to Z$. Thus one can define Chern classes of elements of $\mathcal{K}(Z)$ as elements of $\mathcal{H}^*(Z,\mathbb{Z})$.

Now let X be a smooth manifold, Y a closed differentiable sub-manifold of codimension 2, (this construction can be generalized to any codimension). Suppose that the normal bundle of Y in X is given an almost complex structure, and denote by λ the complex line bundle over Y thus obtained. Let E be any complex vector bundle on Y. The purpose here is to construct elements $\gamma_y^t(E) \in \mathcal{K}(X,X-Y)$ and $j_!(E) \in \mathcal{K}(X)$.

Using a Riemannian metric we can find an open neighborhood λ^o of Y in λ, (Y is identified with the zero section of λ), an open neighborhood X_o of Y in X and a differentiable homeomorphism of λ^o with X_o which is the identity on Y. Assume this identification has been made and denote by $\pi : X_o \to Y$ the projection map which corresponds to the bundle projection $\lambda \to Y$.

Consider the sequence of vector bundles on X_o :

$$0 \to \pi^!(1) \overset{c}{\to} \pi^!(\lambda^*) \to 0$$

where c is defined by

$$c(z,1) = (z,\tilde{z}) \qquad z \in X_o - Y \ ,$$

$\tilde{z}$ is the dual of the element z considered as an element of λ, and $c \equiv 0$ on Y. This sequence is exact on $X_o - Y$. Thus we may form the element $d(\pi^! 1, \pi^! \lambda^*; c) \in \mathcal{K}(X_o, X_o - Y)$. Since $\mathcal{K}(X_o,X_o-Y)$ is a $\mathcal{K}(X_o)$-module we may, for any element $E \in \mathcal{K}(Y)$, form the product $d(\pi^! 1, \pi^! \lambda^*, c)\cdot\pi^! E = d(\pi^! E, \pi^! \lambda^* \otimes E, c\otimes 1)$ belonging to $\mathcal{K}(X_o,X_o-Y)$. Finally, using the excision isomorphism $\sigma : \mathcal{K}(X_o,X_o-Y) \to \mathcal{K}(X,X-Y)$ define

$\gamma_Y^t(E) = \sigma(d(\pi^! E, \pi^! \lambda^* \otimes E, c\otimes 1))$. The assignment $E \mapsto \gamma_Y^t(E)$ gives a homomorphism $\phi_! : \mathcal{K}(Y) \to \mathcal{K}(X, X-Y)$. $\phi_!$ is analogous to the Thom homomorphism in cohomology. The image of $\gamma_Y^t(E)$ in the natural homomorphism $i^! : \mathcal{K}(X,X-Y) \to \mathcal{K}(X)$ is denoted by $j_!(E)$. Thus we have an <u>additive</u> homomorphism $j_! : \mathcal{K}(Y) \to \mathcal{K}(X)$, $j_!$ is the push forward homomorphism. It is similar to the Gysin homomorphism in cohomology. In particular we have the identity $j_!(E \cdot j^!(E')) = j_!(E) \cdot E'$, $E \in \mathcal{K}(Y)$, $E' \in \mathcal{K}(X)$ and $j : Y \to X$ is the embedding, also $j_!(1) = 1 - \mathcal{L}^*$, where $\mathcal{L}$ is the element of $\mathcal{L}ine_C(X)$ corresponding to the element $j_*(1) \in \mathcal{H}^2(X)$.

It can be shown that these definitions are independent of the choice of X_o and the identification of X_o with λ^o.

In just the same way we can define $\mathcal{KO}(X,X-Y)$. Suppose that Y is a <u>closed</u> codimension one sub-manifold of X. Denote by λ the real normal line bundle of Y in X. It is then possible to construct homomorphisms:

$$\gamma_Y^t : \mathcal{KO}(Y) \to \mathcal{KO}(X,X-Y)$$

$$j_! : \mathcal{KO}(Y) \to \mathcal{KO}(X)$$

just as for the complex case.

It is essential for applications that one have a formula for the characteristic classes of $j_!(E)$ in terms of the classes of E. This is easy to obtain in this case, the formula is the so-called "differentiable Riemann-Roch Theorem". The formula is [1], $cc(j_! E) = j_*(\frac{cc(E-\lambda^* \otimes E)}{cc_1(\lambda)})$, here $cc(\ )$ denotes the Chern class in the complex case, or the Stiefel-Whitney class in the real case. $cc_1(\lambda)$ is the element of $H^2(Y,\mathbb{Z})$ in the complex case or of $H^1(Y,\mathbb{Z}_2)$ in the real case, given by $j^*(j_*(1))$. Note: the characteristic classes of the difference E - F of two bundles is defined to be $cc(E-F) = \frac{cc(E)}{cc(F)}$.

To derive this formula note that it is sufficient to prove it for the element $j_!(1)$. This follows from the fact that $\phi_* : H^*(Y) \to H^*(X,X-Y)$ is an $H^*(Y)$-homomorphism, and the definition of $j_!(E)$.

Now, $j_!(E) = 1 - \mathcal{L}^*$ so

$$cc(j_!(1)) = cc(1-\mathcal{L}^*)$$

$$= \frac{1}{1 - j_*(1)}$$

which, by properties of j_*

$$= j_*(\frac{1}{cc_1(\lambda)} \cdot (\frac{1}{1 - cc_1(\lambda)})).$$

and

$$= j_*(\frac{cc(1-\lambda^*)}{cc_1(\lambda)}).$$

This formula will not actually be used in this form. The following one, which is a formal consequence of the above, will:

$$cc(j_!(E)) = 1 + j_*(\frac{cc(E-\lambda^* \otimes E) - 1}{cc_1(\lambda)}).$$

§2. Definitions.

A. Wrapping Homomorphisms (complex case)

Let X be a nice space, and let K be an i-dimensional complex vector bundle on X. Now for any element $\lambda \in \mathrm{Line}_{\mathbb{C}}(X)$ and integer $p \geq 1$ we have the following canonically defined homomorphism over the total space of the bundle λ :

$$\begin{array}{ccc} \pi^{!}K & \xrightarrow{\ w\ } & \pi^{!}(\lambda^{p}\otimes K) \\ & \searrow \quad \swarrow & \\ & \lambda & \end{array}$$

p

$w(\omega_x, k_x) = (\omega_x, \omega_x\otimes\cdots\otimes\omega_x\otimes k_x)$ where k_x belongs to the fibre of K at x, and $\pi : \lambda \to X$ is the bundle projection.

Note that $\Sigma^{j}(w) = \emptyset$ for $j > 0$ and $j \neq i$, $\Sigma^{i}(w) =$ the zero section of λ.

Let I of (fibre) dimension n-i be another complex vector bundle on X, then we can extend the above homomorphism to

$$\begin{array}{ccc} \pi^{!}I \oplus \pi^{!}K & \xrightarrow{\ W\ } & \pi^{!}I \oplus \pi^{!}(\lambda^{p}\otimes K) \\ & \searrow \quad \swarrow & \\ & \lambda & \end{array}$$

where $W = 1 \oplus w$.

Definition. The qunituple $\{\lambda, K, I, W, p\}$ will be called the local model for a wrapping homomorphism of degree p.

Notice that the bundle Ker(W) along the zero section of λ is precisely the bundle K, (X is identified with the zero section of λ), the bundle coker(W) is $\lambda^p \otimes K$, and the bundle I can be thought of as an image bundle.

Assume now that X is a TOP manifold and

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

a homomorphism of (complex) vector bundles of the same fibre dimension over X. If α satisfies the following conditions:

a) Except for a finite collection $Y_1, \ldots, Y_q$ of locally flat codimension 2 sub-manifolds with oriented normal vector bundles λ_ℓ, α is an isomorphism, and on each Y_ℓ α has constant rank.

Define the bundle $Im_\ell(\alpha)$ over Y_ℓ by the exactness of the sequences

$$0 \to Ker_\ell(\alpha) \to j_\ell^! E \to Im_\ell(\alpha) \to 0$$

$$0 \to Im_\ell(\alpha) \to j_\ell^! F \to Coker_\ell(\alpha) \to 0 \quad .$$

$\ell = 1, \ldots, q$, and $j_\ell : Y_\ell \to X$, and identify an open neighborhood X_0^ℓ of Y_ℓ in X with an open neighborhood λ_ℓ^0 of the zero section of λ_ℓ in the usual fashion, (the assumption is that this can be done), let $\pi : X_0^\ell \to Y_\ell$ be the projection coming from the bundle projection $\lambda_\ell \to Y_\ell$ regarded, as usual, as a complex line bundle.

b) There exists isomorphisms A_ℓ, B_ℓ such that the diagram

$$\begin{array}{ccc} E/_{X_o^\ell} & \xrightarrow{A_\ell} & \pi^! \mathrm{Im}_\ell(\alpha) \oplus \pi^! \mathrm{Ker}_\ell(\alpha) \\ \downarrow{\scriptstyle \alpha/_{X_o^\ell}} & & \downarrow{\scriptstyle W} \\ F/_{X_o^\ell} & \xrightarrow{B_\ell} & \pi^! \mathrm{Im}_\ell(\alpha) \oplus \pi^! (\lambda_\ell^p \otimes \mathrm{Ker}_\ell(\alpha)) \end{array}$$

commutes, then α will be called a local wrapping homomorphism of degree p_ℓ about Y_ℓ.

B. Real Case.

The definitions given in the preceeding sub-section can be extended to real vector bundles and homomorphisms in two distinct ways. The first way is to replace the word "complex" with "real" everywhere and demand that the submanifolds $Y_1, \ldots, Y_q$ have codimension 1, be locally flat, and make no orientability assumptions on the normal bundles. The second way is to retain the assumptions on the submanifolds $Y_1, \ldots, Y_q$, and rank assumptions on $X - \bigcup_\ell Y_\ell$, but demand that near each Y_ℓ, α be conjugate to the realization of the local model $\{\lambda_\ell, K_c, \mathrm{Im}_\ell(\alpha)\otimes\mathbb{C}, W, p_\ell\}$, where $\mathrm{Ker}_\ell(\alpha)$ is the realization of the complex vector bundle K_c and $\mathrm{Im}_\ell(\alpha)\otimes\mathbb{C}$

denotes the complexification of $\mathrm{Im}_\ell(\alpha)$.

C. Complex Analytic Case.

Suppose X is a complex analytic manifold of dimension n and D_ℓ, $\ell = 1,\ldots,q$, a collection of disjoint non-singular divisors on M. Then, in an appropriate sense, each D_ℓ determines q holomorphic line bundles $\{D_1\},\ldots,\{D_q\}$ on X, and also q-codimension one (complex) analytic sub-manifolds also denoted by D_ℓ. If $j_\ell : D_\ell \to M$ is the inclusion then $j_\ell^!\{D_\ell\}$ corresponds to the holomorphic normal bundle of D_ℓ in X. If X is compact and $h_\ell \in H^2(X,\mathbb{Z})$ is the cohomology class represented by the oriented (2n-2)-cycle D_ℓ, then $c_1(\{D_\ell\}) = h_\ell$. If X is not compact we must use the inverse limit singular theory defined in Section 1.

Now a holomorphic vector bundle homomorphism will be called a local wrapping homomorphism of degree p_ℓ about D_ℓ if for each ℓ it is bi-holomorphically conjugate to the local model

$$\{j_\ell^!\{D_\ell\},\ \mathrm{Ker}_\ell(\alpha),\ \mathrm{Im}_\ell(\alpha),\ W,\ p_\ell\}.$$

In any case we have the following:

Proposition. A vector bundle homomorphism $\alpha : E \to F$ over X satisfies the local equations given in the introduction if and only if it is a (local) wrapping homomorphism.

§3. Examples.

A. Generic Σ^1-type Singularities.

It is easy to see by considering local data that a wrapping homomorphism of real vector bundles with codimension one singularity subsets or of complex vector bundles with (complex) codimension 1 singularity subsets is almost never generic. They are generic however if the (fibre) dimensions of the kernal bundles are all one, and the local degree about Y_i is one also.

B. Banal Vector Bundle Homomorphisms.

Definition. [14] A vector bundle homomorphism

$$\begin{array}{ccc} E^n & \xrightarrow{\alpha} & F^n \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

of vector bundles of the same fibre dimension over a manifold X is called banal if

i) rank $\alpha_x \geq n - 1$ for all $x \in X$ and

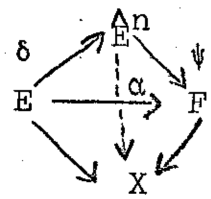
ii) $\Sigma^1(\alpha)$ is a closed submanifold of codimension 1, and if $x \in \Sigma^1(\alpha)$ then

$$\dim(ds_\alpha(TX_x) + T\Sigma^1_{s_\alpha}(x)) \geq \dim \Sigma^1 + 1,$$

where TX_x is the tangent space of X at x, and $T\Sigma^1_{s_\alpha}(x)$ is the tangent space to the sub-manifold $\Sigma^1 \subset HOM(E,F)$ at $s_\alpha(x)$

Levine proved the following:

<u>Lemma</u> (Levine [14]). Let $\alpha : E \to F$ be a banal vector bundle homomorphism over a manifold X. Then there exists another vector bundle $\hat{E}^n$ and homomorphisms



having the properties:

i) a. $\psi \circ \delta = \alpha$ and $\Sigma^1(\alpha) = \Sigma^1(\delta)$.

b. ψ is an isomorphism.

ii) let λ be the normal line bundle of $\Sigma^1(\alpha)$ in X, then $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\delta)$, and the sequence

$$0 \to \mathrm{Im}(\alpha) \to j^!\hat{E} \to \lambda \otimes \mathrm{Ker}(\alpha) \to 0$$

is exact.

It is easy to see that the homomorphism $\delta : E \to \hat{E}$ is a wrapping homomorphism of degree one about $\Sigma^1(\alpha)$.

C. <u>Ramified Coverings</u>.

Suppose $b : X \to X'$ is a local ramified (branched) covering of real manifolds, branched along $\bigcup_{i=1}^{q} Y_i$ of branching degree $(p_i - 1)$ on Y_i. Consider the induced map of tangents

$$\begin{array}{ccc} TX & \xrightarrow{db} & b^{!}TX' \\ & \searrow \quad \swarrow & \\ & X & \end{array},$$

from the local description of b given in the introduction it is easy to see that db is a local wrapping homomorphism of degree p_i-1 on each Y_i, in the second sense, i.e. db is the realization of the local wrapping homomorphism of complex bundles $\{\lambda_i, \lambda_i, TY_i \otimes \mathbb{C}, W, p_i-1\}$, where λ_i denotes the associated complex line bundle of Y_i in X.

D. Dilatations.

It is not so easy to see that a dilatation gives rise to a wrapping homomorphism.

The manifolds of this section can be smooth, real or complex analytic, and algebraic.

Let X^n be a manifold and Y a sub-manifold of dimension m. By dilatation of X along Y we obtain new manifolds X' of dimension n, and Y' of dimension n-1. Y' is a closed sub-manifold of X'. We also have the diagram of manifolds and maps

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ {\scriptstyle j'}\uparrow & & \uparrow{\scriptstyle j} \\ Y' & \xrightarrow{\beta} & Y \end{array}$$

which commutes, σ is a proper map and β is the restriction of σ to Y! σ is a bi-morphism except on Y', and $\beta : Y' \to Y$ has the structure of a projective fibre bundle over Y with fibres the projective space of dimension n-m-1.

Associated to this diagram there are a number of vector bundles. They are: the tangent bundles of X, X', Y, and Y' denoted by TX, TX', TY and TY' respectively, N the normal bundle of Y in X, λ the normal line bundle of Y' in X', and F, the kernal of $d\beta$.

Clearly we have the isomorphisms

$$j^! TX = TY \oplus N$$

$$j'^! TX' = TY' \oplus \lambda$$

$$= \beta^! TY \oplus F \oplus \lambda.$$

Suppose now that p' and $p = \sigma(p')$ are a pair of corresponding points under σ, and $p \in Y$. Then it is possible to choose local co-ordinates $x_1, \ldots, x_n$ on X about p and local co-ordinates $x_1', \ldots, x_n'$ on X' about p' so that locally σ has the form

$$x_i \circ \sigma = x_i' \qquad i = 1, \ldots, m$$

$$x_k \circ \sigma = x_k' \cdot x_n' \qquad k = m+1, \ldots, n-1$$

$$x_n \circ \sigma = x_n' .$$

The local equations of Y at p are $x_k = 0$, and the local equation of Y' at p' is $x'_n = 0$.

The map σ induces a map of tangents

$$\begin{array}{ccc} TX' & \xrightarrow{d\sigma} & \sigma^! TX \\ & \searrow \quad \swarrow & \\ & X' & \end{array}$$

From the local description of σ we see that $d\sigma$ is an isomorphism except along Y', at p' $d\sigma$ is given locally by

$$d\sigma\left(\frac{\partial}{\partial x'_i}\right) = \frac{\partial}{\partial x_i} \qquad i = 1, \ldots, m$$

$$d\sigma\left(\frac{\partial}{\partial x'_k}\right) = x'_n \cdot \frac{\partial}{\partial x_k} \qquad k = m+1, \ldots, n-1$$

$$(*) \qquad d\sigma\left(\frac{\partial}{\partial x'_n}\right) = \frac{\partial}{\partial x_n} + \sum_{k=m+1}^{n-1} x'_k \cdot \frac{\partial}{\partial x_k} .$$

First of all, the equation (*) is the equation of the natural embedding of the line bundle λ into $\beta^! N$. Secondly, $\operatorname{coker}(d\beta) = \lambda \otimes \ker(d\beta) = \lambda \otimes F$. It then follows that $\operatorname{Im}(d\beta) = \lambda \oplus \beta^! TY$.

On the other hand we may construct a degree one wrapping homomorphism of the bundles

$$\begin{array}{ccc} \pi^! \lambda \oplus \pi^! F & \xrightarrow{W} & \pi^! \lambda \oplus \pi^! (\lambda \otimes F) \\ & \searrow \quad \swarrow & \\ & X'_o & \end{array}$$

where X'_o is an open neighborhood of Y' in X' and $\pi : X'_o \to Y'$

is the projection coming from the bundle projection $\lambda \to Y'$. Now $d\sigma/_{X'_0}$ gives rise to a homomorphism

$$\begin{array}{ccc} \pi^!(\lambda \oplus F) & \xrightarrow{G} & \pi^!(\beta^! N) \\ & \searrow \quad \swarrow & \\ & X'_0 & \end{array},$$

(i.e. on X'_0 $d\sigma$ may be written as $1 \oplus G$, where $1 : \pi^!(\beta^! TY) \to \pi^!(\beta^! TY)$ is the identity, I have ignored this term).

<u>Proposition</u>. G is conjugate to W and is therefore a wrapping homomorphism of degree one about Y'.

<u>Proof</u>. Now the sequence

$$0 \to \lambda \to \beta^! N \to \lambda \otimes F \to 0$$

is exact [15], and the isomorphism $B : \lambda \oplus \lambda \otimes F \to \beta^! N$ is given locally by

$$B\left(\frac{\partial}{\partial x'_n} + 0\right) = \frac{\partial}{\partial x_n} + \sum_{k=m+1}^{n-1} x'_k \cdot \frac{\partial}{\partial x_k}$$

$$B\left(0 + \frac{\partial}{\partial x'_n} \otimes \frac{\partial}{\partial x'_k}\right) = \frac{\partial}{\partial x_k} \qquad k = m+1, \ldots, n-1.$$

Since W is given locally by

$$W\left(\frac{\partial}{\partial x'_n} + 0\right) = \frac{\partial}{\partial x'_n} + 0$$

and

$$W\left(0 + \frac{\partial}{\partial x'_k}\right) = 0 + x'_n \cdot \frac{\partial}{\partial x'_n} \otimes \frac{\partial}{\partial x'_k}\right) \qquad k = m+1, \ldots, n-1$$

it then follows that the diagram

$$\begin{array}{ccc} \pi^{!}\lambda \oplus \pi^{!}F & \xrightarrow{W} & \pi^{!}\lambda \oplus \pi^{!}(\lambda\otimes F) \\ \downarrow 1 & & \downarrow B \\ \pi^{!}\lambda \oplus \pi^{!}F & \xrightarrow{G} & \pi^{!}(\beta^{!}N) \end{array}$$

commutes. Q.E.D.

Thus any dilatation gives rise to a wrapping homomorphism in the appropriate sense.

§4. Statement and proof of the main theorem.

Theorem 1. Suppose

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \;\; \swarrow & \\ & X & \end{array}$$

is a local wrapping homomorphism of complex (resp. real, analogous sense) vector bundles with data p_i, Y_i, and λ_i, $i = 1,\ldots,q$. Then the identity

$$F - E = \mathcal{J}_!\Big(\sum_{i=1}^{q}(\lambda_i^{p_i} + \lambda_i^{p_i-1} + \cdots + \lambda_i)\cdot \mathrm{Ker}_i(\alpha)\Big)$$

holds in $\mathcal{K}(X)$, (resp. $\mathcal{K}\mathcal{O}(X)$).

The proof of theorem 1 will be divided into two lemmas. Note that it is sufficient to consider the case of one wrapping locus, also the proof in the real case is analogous to the complex one, just replace the word "complex" with the word "real" everywhere.

With the notation of the preceeding sections we can now state

Lemma A. As elements of $\mathcal{K}(X_o, X_o - Y)$ $\quad d(E/_{X_o}, F/_{X_o}; \alpha/_{X_o}) =$ $= d(\pi^!1, \pi^!\lambda^p; \beta_p)\cdot\pi^!K$, where $K = \mathrm{Ker}(\alpha/Y)$ and $\beta_p : \pi^!1 \to \pi^!\lambda^p$ is given by $\beta_p(z,1) = (z, \otimes\cdots\otimes z)$ for $z \in X_o - Y$ and $\beta_p \equiv 0$ on Y.

Proof. By assumption the diagram

$$\begin{array}{ccc} E/_{X_o} & \xrightarrow{A} & \pi^! I \oplus \pi^! K \\ {\scriptstyle \alpha/_{X_o}} \downarrow & & \downarrow {\scriptstyle W} \\ F/_{X_o} & \xrightarrow{B} & \pi^! I \oplus \pi^!(\lambda^p \otimes K) \end{array}$$

commutes, A and B are isomorphisms over all of X_o. Now $\alpha/_{X_o - Y}$ is an isomorphism so we may form the element

$$d(E/_{X_o}, F/_{X_o}; \alpha/_{X_o - Y}) \in K(X_o, X_o - Y).$$

Since $\alpha/_{X_o} = B^{-1}WA$, and by properties of the difference element we obtain

$$d(E/_{X_o}, F/_{X_o}; \alpha/_{X_o}) = d(\pi^! I \oplus \pi^! K, \pi^! I \oplus \pi^!(\lambda^p \otimes K); W)$$

and the right hand side gives

$$d(\pi^! K, \pi^!(\lambda^p \otimes K); \beta_p \otimes 1) = d(\pi^! 1, \pi^! \lambda^p; \beta_p) \cdot \pi^! K.$$

<u>Lemma B</u>. - $d(\pi^! 1, \pi^! \lambda^p; \beta_p) = d(\pi^!(1 \otimes (\lambda^p \oplus \lambda^{p-1} \oplus \cdots \oplus \lambda)), \pi^!(\lambda^* \otimes (\lambda^p \oplus \lambda^{p-1} \oplus \cdots \oplus \lambda)); c \otimes 1)$, (recall $c : \pi^! 1 \to \pi^! \lambda^*$ is given by $c(z,1) = (z, \tilde{z})$ $z \in X_o - Y$, $\tilde{z}$ is the dual of the element z, and $c \equiv 0$ on Y).

<u>Proof</u>. Consider the following diagram:

$$\begin{array}{ccc}
\pi^!(1\otimes(\lambda^p\oplus\cdots\oplus\lambda)) & \xrightarrow{c\otimes 1} & \pi^!(\lambda^*\otimes(\lambda^p\oplus\cdots\oplus\lambda)) \\
\Big\uparrow{\scriptstyle \beta_p\oplus\cdots\oplus\beta_1} & & \Big\downarrow{\scriptstyle e_p\oplus\cdots\oplus e_1 = e} \\
\pi^!(\underbrace{1\oplus\cdots\oplus 1}_{p}) & \xrightarrow[\beta_{p-1}\oplus\cdots\oplus 1]{} & \pi^!(\lambda^{p-1}\oplus\cdots\oplus 1)
\end{array}$$

where $e_i : \pi^!(\lambda^*\otimes\lambda^i) = \pi^!\lambda^{i-1}$ is the canonical isomorphism. This diagram commutes on $X_o - Y$, this is easy to see. Thus we have

$d(\pi^!(1\oplus\cdots\oplus 1),\ \pi^!(\lambda^{p-1}\oplus\cdots\oplus 1);\ \beta_{p-1}\oplus\cdots\oplus 1) =$

$= d(\pi^!(1\oplus\cdots\oplus 1),\ \pi^!(\lambda^{p-1}\oplus\cdots\oplus 1;\ e\circ c\otimes 1\circ\ (\beta_p\oplus\cdots\oplus\beta_1))$ which, by properties of the difference element gives

$d(\pi^!(1\oplus\cdots\oplus 1),\ \pi^!(\lambda^{p-1}\oplus\cdots\oplus 1);\ \beta_{p-1}\oplus\cdots\oplus 1) =$

$= d(\pi^!(1\oplus\cdots\oplus 1),\ \pi^!(\lambda^p\oplus\cdots\oplus\lambda);\ \beta_p\oplus\cdots\oplus\beta_1)\ +$

$d(\pi^!(\lambda^p\oplus\cdots\oplus\lambda),\ \pi^!(\lambda^*\otimes(\lambda^p\oplus\cdots\oplus\lambda);\ c\otimes 1)$. Expand both sides and cancel to obtain

$d(\pi^! 1,\ \pi^!\lambda^p;\ \beta_p) + d(\pi^!(\lambda^p\oplus\cdots\oplus\lambda),\ \pi^!(\lambda^*\otimes(\lambda^p\oplus\cdots\oplus\lambda);\ c\otimes 1) = 0,$

and the result follows.

We have now shown that in $K(X_o, X_o - Y)$ there is the identity $d(E/_{X_o},\ F/_{X_o};\ \alpha/_{X_o}) = -d(\pi^! 1, \pi^!\lambda^*; c)\cdot\pi^!((\lambda^p\oplus\cdots\oplus\lambda)K)$. Applying the excision isomorphism to each side of this equation and we find that in $K(X, X-Y)$ $d(E,F;\alpha) = -\gamma_Y^t((\lambda^p\oplus\cdots\oplus\lambda)K)$.

Apply the natural map

$$i^! : K(X,X-Y) \to K(X)$$

and we obtain the main result

$$F - E = j_!((\lambda^p \oplus \cdots \oplus \lambda)K). \qquad \text{Q.E.D.}$$

<u>Corollary</u>: Suppose

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a real vector bundle homomorphism that is the realization of a complex wrapping homomorphism of degree p about $Y \subset X$. Then in $KO(X)$ we have the identity

$$F - E = \phi(j_!((\lambda^p \oplus \lambda^{p-1} \oplus \cdots \oplus \lambda) \otimes K_c))$$

<u>Proof</u>. Consider the element $d(E,F;\alpha) \in K(X,X-Y)$ constructed in the usual way. By the assumption on the homomorphism α it is clear that

$$d(E,F;\alpha) = \phi(\sigma(d(\pi^! K_c, \pi^!(\lambda^p \otimes K); w).$$

Now, from the proof of the main theorem we have

$$-\sigma(d(\pi^! K_c, \pi^!(\lambda^p \otimes K_c); w) = \gamma_Y^t((\lambda^p \oplus \cdots \oplus \lambda)) \otimes K_c).$$

The result then follows by applying the natural map
$i^! : KO(X,X-Y) \to KO(X)$.

§5. Characteristic classes and wrapping homomorphisms. Statement and proof of results.

Theorem. Suppose

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a local wrapping homomorphism of complex, (resp. real, analogous sense), vector bundles with data p_i, Y_i, and λ_i, $i = 1,\ldots,q$. Then the characteristic classes $cc(E)$ and $cc(F)$ are related by the formula

$$cc(F-E) = \prod_{i=1}^{q} \left(1 + j_*^i\left[\frac{1}{cc_1(\lambda_i)} \cdot \frac{cc(\lambda_i^{p_i} \otimes K_i) - cc(K_i)}{cc(K_i)}\right]\right)$$

where $K_i = Ker_i(\alpha)$.

Proof. The proof consists of a straightforward application of the formula for the characteristic classes of elements of the form $j_!(\ \ )$.

Corollary. If $q = 1$, then there is the formula

$$cc(F) = cc(E) + j_*\left[\frac{cc(I)}{v} \cdot \left(\sum_{k=0}^{\ell} (1+pv)^k cc_{\ell-k}(K) - cc(K)\right)\right]$$

where $I = Im(\alpha)$, $K = Ker(\alpha)$, ℓ = dimension of the fibres of K, and $v = cc_1(\lambda)$.

Proof. Taking cc of both sides of the main identity gives

$$cc(F-E) = \frac{cc(F)}{cc(E)} = cc(j_!((\lambda^p \oplus \cdots \oplus \lambda)\cdot K))$$

$$= 1 + j_*[\frac{1}{v} \cdot \left(\frac{cc((\lambda^p \oplus \cdots \oplus \lambda)\cdot K)}{cc((\lambda^{p-1} \oplus \cdots \oplus 1)\cdot K)} - 1 \right)]$$

$$= 1 + j_*[\frac{1}{v} \cdot \frac{cc(\lambda^p \otimes K) - cc(K)}{cc(K)}]$$

multiply both sides by cc(E) to obtain

$$cc(F) = cc(E) + cc(E)\cdot j_*[\frac{1}{v} \cdot \frac{cc(\lambda^p \otimes K) - cc(K)}{cc(K)}]$$

$$= cc(E) + j_*[\frac{j^* cc(E)}{v} \cdot \frac{cc(\lambda^p \otimes K) - cc(K)}{cc(K)}]$$

since $j_*(j^*(a)\cdot b) = a\cdot j_*(b)$,

$$= cc(E) + j_*[\frac{cc(I)}{v} \cdot (cc(\lambda^p \otimes K) - cc(K))]$$

because $j^! E = I \oplus K$. Now substitution of the formula

$$cc(\lambda^p \otimes K) = \sum_{k=0}^{\ell} (1+pv)^k cc_{\ell-k}(K)$$

gives the desired result.

For example, if $\ell = 1$, so that $Ker(\alpha)$ is a line bundle on Y, the formula reduces to

$$cc(F) = cc(E) + p\cdot j_*(cc(I)).$$

§6. Applications

A. Banal vector bundle homomorphisms.

Suppose

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a banal vector bundle homomorphism then

$$cc(F) = cc(E) + j_*\left[\frac{cc(I)}{v} \cdot \left(\sum_{k=0}^{\ell} (1+v)^k cc_{\ell-k}(K) - cc(K)\right)\right].$$

The special case where the dimension of the kernal is 1 was obtained by Levine [14] and is

$$cc(F) = cc(E) + j_*(cc(I)) .$$

For generic vector bundle maps the formula is the same as Levine's.

B. Dilatations

Suppose X' is obtained from X by a dilatation along Y, then if $\sigma : X' \to X$ denotes the map we have

$$\sigma^! TX - TX' = j_!(\lambda \otimes F),$$

which holds in $K(X')$ in the complex case or in $KO(X')$ in the real case.

Corollary. [1], [15]. The behavior of the characteristic classes under a dilatation is expressed by the formula

$$cc(X') = \sigma^{*}cc(X) + j_{*}^{!}[\frac{\beta^{*}cc(Y)}{v}((1+v)\sum_{i=0}^{n-m}(1-v)^{i}\beta^{*}cc_{n-m-i}(N)-\beta^{*}cc(N))]$$

Proof. Since $\beta^{!}N = \lambda \oplus \lambda \otimes F$ we may write $\lambda \otimes F = \beta^{!}N - \lambda$. Then the formula becomes

$$\sigma^{!}TX - TX' = j_{!}(\beta^{!}N-\lambda)$$

or, what is equivalent

$$TX' - \sigma^{!}TX = j_{!}(\lambda-\beta^{!}N).$$

Taking cc( ) of both sides we obtain

$$\frac{cc(X')}{\sigma^{*}cc(X)} = 1 + j_{*}^{!}[\frac{1}{v}((1+v)\cdot \frac{\sum_{i}(1-v)^{i}\beta^{*}cc_{n-m-i}(N)}{\beta^{*}cc(N)} - 1)]$$

now multiply both sides of this equation by $\sigma^{*}cc(X)$ and use the fact that

$$\frac{1}{\beta^{*}cc(N)} = \beta^{*}cc(N)\cdot j'^{*}[\frac{1}{\sigma^{*}cc(X)}] ,$$

and the result follows.

C. Ramified Coverings (Real case).

In this section denote by $X \xrightarrow{b} X'$ a (local) branched covering of smooth and orientable closed manifolds with local data Y_i, p_i, and $\lambda_i (i=1,\ldots,q)$.

Now, according to the main result concerning these types of homomorphisms we have the

Theorem. The idenity

$$b^! TX' - TX = \phi\left(\sum_{i=1}^{q} L_i^{p_i} - L_i\right)$$

holds in $KO(X)$, where $L_i \in \mathrm{Line}_{\mathbb{C}}(X')$ corresponds to the sub-manifold Y_i.

Proof. It is sufficient to prove the result for a single branching locus. In this case, as we have already seen, the kernal of db along the branching locus $Y \subset X$ can be identified with the normal bundle λ of Y in X. Applying the main result to this situation yields

$$b^! TX' - TX = \phi(j_!((\lambda^{p-1} \oplus \cdots \oplus \lambda) \otimes \lambda)).$$

Since $\lambda = j^! L$ and $j_!(1) = 1 - L^*$, we have, using the fundamental function property of $j_!$,

$$b^! TX' - TX = \phi(L^p \oplus \cdots \oplus L^2 \otimes (1 - L^*))$$

$$= \phi(L^p - L).$$

Corollary. Let $P(\ \ )$, (resp. $W(\ \ )$), denote the total rational Pontrjagin class, (resp. the total Stiefel-Whitney class). Then

$$P(X) = b^{*}P(X')\prod_{i=1}^{q}(1+\sum_{k=1}^{\infty}(-1)^{k-1}(p_i^{2k-2}-p_i^{2k})y_i^{2k})$$

and

$$W(X) = b^{*}W(X')\prod_{i=1}^{q}(1+\sum_{k=1}^{\infty}(-1)^{k-1}(p_i^{k-1}-p_i^{k})(y_i \bmod 2)^{k}) .$$

Proof. Write

$$TX - b^{!}TX' = \sum_{i=1}^{q}\phi(L_i-L_i^{p_i}).$$

Applying P( ) to both sides we obtain

$$P(X) = b^{*}P(X')\prod_{i=1}^{q}\left(\frac{1+y_i^2}{1+p_i^2y_i^2}\right) .$$

Since

$$\frac{1+y_i^2}{1+p_i^2y_i^2} = 1+\sum_{k=1}^{\infty}(-1)^{k-1}(p_i^{2k-2}-p_i^{2k})y^{2k}$$

the result follows. The proof for the Stiefel-Whitney classes is similar.

Suppose now that the dimension of X' is devisible by four, say 4ℓ.

Corollary. The $\mathcal{L}$-classes are related by the formula

$$\left(\prod_{i=1}^{q}p_i\right)\mathcal{L}(X) = b^{*}\mathcal{L}(X')\prod_{i=1}^{q}\operatorname{cotanh}(y_i)\tanh(p_iy_i)$$

Proof. Since $TX - b^{!}TX' = \phi(\sum_{i=1}^{q}L_i-L_i^{p_i})$, taking $\mathcal{L}(\ )$ of both sides yields

$$\frac{\mathcal{L}(X)}{b^*\mathcal{L}(X')} = \prod_{i=1}^{q} \frac{y_i}{\tanh(y_i)} \cdot \frac{\tanh(p_i y_i)}{p_i y_i}$$

or

$$(\prod_{i=1}^{q} p_i)\mathcal{L}(X) = b^*\mathcal{L}(X') \prod_{i=1}^{q} \operatorname{cotanh}(y_i)\tanh(p_i y_i).$$

Now, $b_i^* y_i' = p_i y_i$ where $y_i' \in H^2(X',\mathbb{Z})$ is dual to the sub-manifold $b(Y_i) \subset X'$. In particular $y_i' = p_i \cdot x_i$ for some $x_i \in H^2(X',\mathbb{Z})$. We can then rewrite the above formula as

$$(\prod_{i=1}^{q} p_i)\mathcal{L}(X) = b^*[\mathcal{L}(X') \prod_{i=1}^{q} \operatorname{cotanh}(x_i)\tanh(p_i x_i)].$$

Evaluation of both sides on $[X]$ gives the

<u>Theorem.</u>

$$(\prod_{i=1}^{q} p_i)\text{Signature}(X) = d[\mathcal{L}(X') \prod_{i=1}^{q} \operatorname{cotanh}(x_i)\tanh(p_i x_i)][X'].$$

D. <u>Ramified Coverings (Complex case).</u>

<u>Theorem.</u> Suppose

$$b : X \to X'$$

is a local branched covering of complex analytic manifolds with data Y_i, λ_i, and p_i. Then the identity

$$b^! TX' - TX = \sum_{i=1}^{q} L_i^{p_i} - L_i$$

holds in $\mathcal{K}(X)$.

The proof of this is similar to the real case and will be omitted.

Corollary. The (total) Todd classes of X and X' are related by the formula

$$\left(\prod_{i=1}^{q} p_i\right) td(X) = b^* td(X') \prod_{i=1}^{q} \frac{1 - \exp(-p_i y_i)}{1 - \exp(-y_i)}$$

Proof. Write

$$TX - b^! TX' = \sum_i L_i - L_i^{p_i} .$$

Apply td( ) to both sides of the equation and the result follows.

Now, suppose that X and X' are algebraic manifolds, and denote by $\chi(\ )$ the arithmetic genus.

Theorem. If $X \xrightarrow{b} X'$ is a local branched covering then the arithmetic genus of X is given by

$$\left(\prod_{i=1}^{q} p_i\right) \chi(X) = d\left[td(X') \prod_{i=1}^{q} \frac{1 - \exp(-p_i x_i)}{1 - \exp(-x_i)}\right][X'].$$

Proof. We have

$$\left(\prod_{i=1}^{q} p_i\right) td(X) = b^* td(X') \prod_{i=1}^{q} \frac{1 - \exp(-p_i y_i)}{1 - \exp(-y_i)} .$$

Since there is an element $x_i \in H^2(X', \mathbb{Z})$ such that $b^* x_i = y_i$, the result follows by the Hirzebruch-Riemann-Roch theorem.

E.g. if $X \to X'$ is a branched covering of surfaces branched along a curve $C' \subset X'$ dual to $y' \in H^2(X',\mathbb{Z})$ with order of ramification $p - 1$, a calculation gives

$$\chi(X) = p[\chi(X') + \tfrac{1}{12}((1-3p+2p^2)x^2+3(1-p)xc_1(X'))[X']].$$

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