Convex Riemannian Manifolds of Non-negative Curvature

A Dissertation presented

by

Stephen David Kronwith

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Department of Mathematics

State University of New York

at

Stony Brook

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STEPHEN DAVID KRONWITH

We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

David Ebin, Associate Professor
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The dissertation is accepted by the Graduate School.

Herbert Weisinger, Dean, Graduate School

August, 1977

Abstract of the Dissertation

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During the past decade, exciting breakthroughs have occured in the study of complete, non-compact manifolds of non-negative curvature. Jeff Cheeger and Detlef Gromoll have shown that such a manifold contains a compact, totally geodesic submanifold S. In actuality, S has the stronger property of being convex, a set C being convex if for any point p in the closure of C, there is a number $\epsilon(p)$ with $0 < \epsilon(p) < r(p)$ such that the intersection of C with an open ball of radius $\epsilon(p)$ has the property that between any two points there is a unique minimal geodesic completely contained in the intersection which joins these points.

In order to study convex sets we introduce the notion of a convex manifold, a compact manifold with boundary whose interior is a smooth Riemannian manifold and which can be imbedded into a manifold of the same dimension as a convex set.

We first show that such a manifold of non-negative (positive) curvature has a complete metric of non-negative (positive) curvature on its interior. We then use this result to discover if a convex manifold of non-negative (positive) curvature can be isometrically imbedded into a complete, non-compact manifold of non-negative (positive) curvature, the converse of the already known theorems of Cheeger and Gromoll.

We answer the question partially in the general dimension case where the second fundamental form of the boundary is positive definite and the curvature is positive, and almost completely in the case of convex surfaces, the answer in all these cases being in the affirmative. The question remains open in the case when the boundary of our surface is a geodesic and the curvature along the boundary is not identically zero.

The techniques used are both analytical and geometrical; including a geometric construction of new convex manifolds from old.

to my loving parents

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Acknowledgments

I should first like to thank my advisor, Detlef Gromoll, for the guidance and patience he has shown me all throughout my graduate career. From the time I first met him as an undergraduate, until this day, he has always inspired me with the fact that one can be a good, even great, mathematician and still remain a decent human being in the process. For that I shall be grateful.

I would also like to thank John Thorpe, Paul Kumpel, David Ebin and my good friend José Andrade— they never said no to a plea for help.

And finally I'd like to thank Gary Aviv, my friend from the beginning of all this to the end, Harvey Lowy, who along with Gary kept me sane during the bad times and still managed to remain great friends, and lastly my dearest Helaine, who has had more to do with the completion of this work than either of us fully appreciate.

I-Notation and Basic Definitions

For the basic definitions of differential geometry and topology we will need, along with the well known theorems in the fields quoted without proof, we refer the reader to the bibliography. For example, (3) and (8) contain all the basics we will need. In addition, we make the following conventions in this paper:

By $(M^n, \langle , \rangle)$ we mean an n-dimensional, connected, C^∞ Riemannian manifold with C^∞ metric $\langle , \rangle$. Often, when the dimension of the manifold is unimportant, we will suppress the superscript; and, when it is clear what the metric is, we will merely just write M. At times, we will also use the classical line-element form of the metric, ds^2; that is, given $U \subset M$ an open set, x a coordinate system on U with x_i the i^{th} coordinate function, we have

$$ds^2|U = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j ,$$

where $g_{ij} = \langle X_i, X_j \rangle$ and $X_i = \frac{\partial}{\partial x_i}$, the canonical i^{th} coordinate vector field of the chart.

We define V(M) to be the space of C^∞ vector fields on M. Then we have the Levi-Civita Connection $\nabla : V(M) \times V(M) \longrightarrow V(M)$ and write $\nabla(X,Y) = \nabla_X Y$. ∇ satisfies

1) $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
2) $\nabla_X fY = X(f)Y + f\nabla_X Y$
3) $\nabla_{(X_1 + X_2)} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
4) $\nabla_{fX} Y = f\nabla_X Y$

where f is a C^∞ real valued function on M, the space of which is denoted by F(M).

If we define the <u>Torsion Tensor</u> T: V(M) x V(M) $\longrightarrow$ V(M) by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ where $[X,Y] \in V(M)$ denotes the <u>Lie Bracket</u> of X and Y ($[X,Y]f = XY(f) - YX(f)$, $f \in F(M)$), then ∇ is the unique connection satisfying T=0 and the Ricci Identity

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \text{ for all } X,Y,Z \in V(M).$$

We define the <u>Curvature Tensor</u> R: V(M) x V(M) x V(M) $\longrightarrow$ V(M), and write $R(X,Y,Z) = R(X,Y)Z$, by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z$.

Let $p \in M$, v_p and $w_p \in M_p$, (M_p being the <u>tangent space</u> to M at p). Then if v and w span a two dimensional linear subspace of M_p, call it σ, we define the <u>sectional curvature</u>, K_σ, of M with respect to the subspace by

$$K_\sigma = K(v,w) = \langle R(v,w)w, v\rangle / (\|v\|^2 \|w\|^2 - \langle v,w\rangle^2)$$

where $\|v\| = \langle v,v\rangle^{\frac{1}{2}}$.

As we see even here, when there is no cause for confusion, the base point of the tangent vector will be dropped from the notation.

Given a C^∞ function $f: M \longrightarrow N$, we denote by $f_{*p}: M_p \longrightarrow N_{f(p)}$ the induced map on the tangent spaces of M and N defined by $(f_* v_p)\varphi = v(\varphi \circ f)$ where $v \in M_p$ and $\varphi \in F(N)$.

Given an immersion $f: M \longrightarrow (N, \langle , \rangle)$, the metric induced by f, denoted by $f^*\langle , \rangle$ and $\langle\langle , \rangle\rangle$, is defined by $\langle\langle v,w \rangle\rangle = \langle f_* v, f_* w\rangle$. It is, of course, the unique metric on M which makes f an isometric mapping.

If $\gamma: I \longrightarrow M$, I an interval in $\mathbb{R}$, is a differentiable curve, we define $\dot{\gamma}$, the velocity or tangent vector of γ at t by $\dot{\gamma}(t) = \gamma_* \frac{d}{dt}$ where $\frac{d}{dt}$ is the canonical vector field on $\mathbb{R}$.

If $(M, \langle , \rangle)$ is given, $f \in F(M)$, we denote by ∇f, the gradient of f, that element of V(M) which uniquely satisfies

$$\langle \nabla f, X \rangle = Xf \quad \text{for all } X \in V(M).$$

The Hessian Tensor, H_f, is the tensorfield of type (1,1) on M defined by $H_f X = \nabla_X \nabla f$, $X \in V(M)$. The Hessian form, h_f, is the 2-form defined by

$$h_f(X,Y) = \langle \nabla_X \nabla f, Y \rangle.$$

Let $i: (M, \langle\,,\rangle) \longrightarrow (\tilde{M}, \langle\langle\,,\rangle\rangle)$ be an isometric immersion, ∇ and $\tilde{\nabla}$ the respective Levi-Cevita Connections. For a normal vectorfield N along i, we have the second fundamental form of i with respect to N, $\ell_N: V(M) \times V(M) \longrightarrow \mathbb{R}$, defined by

$$\ell_N(X,Y) = \langle \nabla_X N, Y \rangle .$$

Given $(M, \langle\,,\rangle)$ we define the distance function $\rho: M \times M \longrightarrow \mathbb{R}$ by

$$\rho(p,q) = \inf_c \left\{ L(c) \,\middle|\, c \in \Omega_{pq} \right\},$$

where $L(c) = \int_0^1 \| \dot{c}(t) \| \, dt$ is the length of the curve c, and Ω_{pq} is the set of all piecewise differentiable curves $c: [0,1] \longrightarrow M$ with $c(0) = p$, $c(1) = q$. It is well known that ρ defines a metric on M and the topology induced by this metric agrees with the given one on M.

Finally, given $p \in M$, we set $B_\epsilon(p) = \left\{ x \in M \,\middle|\, \rho(x,p) < \epsilon \right\}$ to be the open ball of radius ϵ about p.

All other definitions and notations will be explained as encountered later in the paper.

II-Convex Manifolds

1. Introduction

During the past decade, exciting breakthroughs have occured in the study of complete, non-compact manifolds of non-negative curvature, primarily due to the works of Gromoll and Meyer (9) and Cheeger and Gromoll (4). It was shown in the latter paper that such a manifold, M, contains a compact, totally geodesic submanifold S. In actuality, S has the stronger property of being convex in the sense of Definition II.1. S is called the 'soul' of M. It turns out that the inclusion $i: S \longrightarrow M$ is a homotopy equivalence. So the non-compact manifold has the homotopy type of a compact manifold. When $K(M) \geqslant 0$, S is actually diffeomorphic to the normal bundle of S. So when $K > 0$, M is diffeomorphic to $\mathbb{R}^n$, the soul here being a point.

Definition II.1: Given a manifold M, and a set $C \subset M$, C is called convex if for any point $p \in \overline{C}$, there is a number $\epsilon(p)$ with $0 < \epsilon(p) < r(p)$ such that $C \cap B_{\epsilon(p)}(p)$ has the property that between any two points there is a unique minimal geodesic completely contained in $C \cap B_{\epsilon(p)}(p)$ which joins these points. Here $r(p)$ is the convexity radius of M at p, and $\overline{C}$

denotes the closure of C in M.

Using this definition, it can be shown that a closed convex set $C \subset M$ is an imbedded topological manifold with smooth interior int(C) and possibly non-smooth boundary (which might be empty). Here and in the future, smooth will mean C^∞.

Due to the importance of convex sets in the work on non-compact manifolds of non-negative curvature, it is the goal of this paper to better understand their structure. In doing this, we define the abstract notion of a convex manifold.

<u>Definition II.2</u>: A compact topological manifold, M^n, with connected boundary ∂M, is called <u>C^r convex</u> (or is said to have <u>C^r convex boundary</u>), $r=0,1,\ldots,\infty$ if

a) int(M) is a smooth Riemannian manifold.

b) There is a Riemannian manifold N^n and an isometric imbedding $i: M \longrightarrow N$ which is C^∞ on int(M) and C^r on ∂M such that i(M) is a convex subset of N.

Basically, this is not a new notion, and some information has been known for some time. Two important facts which we will not prove but which can be found in the literature (for example in (2)) are:

a) Let $C \subset M$ be convex, $\dim(M) = 2$. If $p \in \partial C$, $v \in (\partial C)_p$, then the geodesic $\gamma : [-\delta, \delta] \longrightarrow M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$, stays to the outside of the interior of C. That is, for sufficiently small δ, $\gamma(t) \notin \text{int}(C)$ for all $t \in [-\delta, \delta]$.

b) If ∂C is C^2, the second fundamental form of an outward pointing normal field along the boundary is positive semi-definite.

An intuitive way of seeing (b) is the following: (see figures 1 and 2).

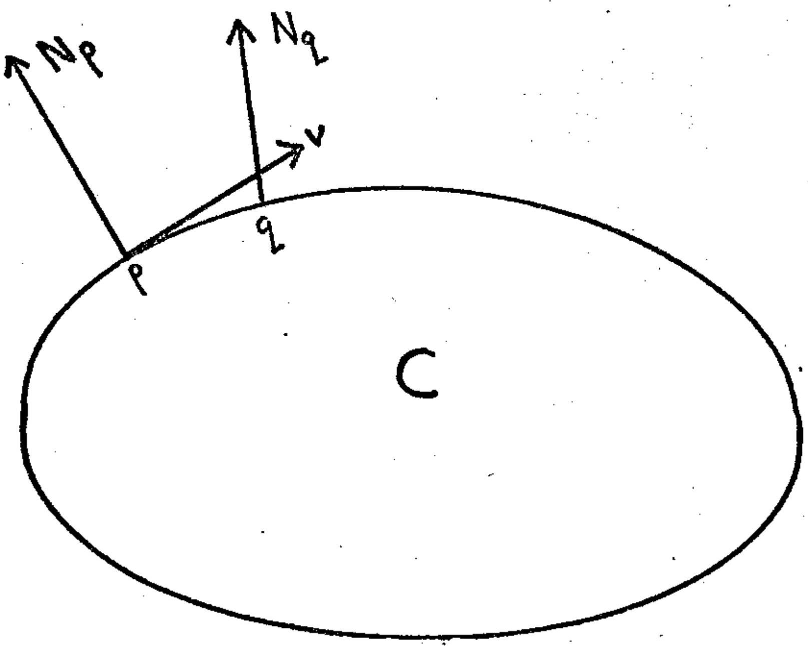

fig. 1

Since the second fundamental form measures the direction in which the normal field falls as it moves in the direction of a tangent vector $v \in (\partial C)_p$, it is in the convex case that the normal field N falls in the v direction as it moves along the boundary in that direction.

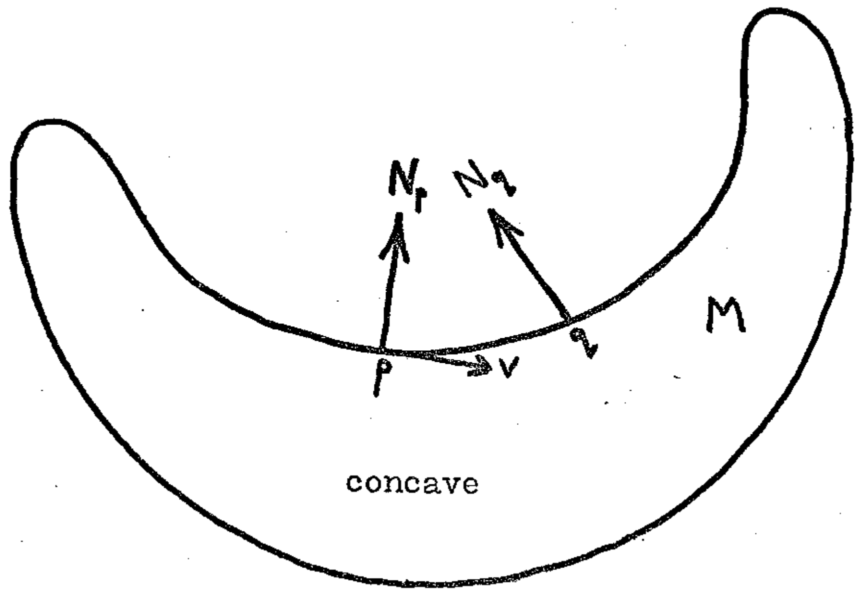

fig. 2

We have only discussed these cases for dim(M) = 2 because it is more easily seen and it is this dimension in which we will need these results later on in the paper.

In the next section we investigate the structure of convex manifolds of non-negative (positive) curvature. We will show that the interior of such a manifold can be given a complete metric of non-negative (positive) curvature. Moreover, and vital to our later discussions of convex surfaces, this new metric will agree with the old one off an arbitrarily small one-sided tubular neighborhood of the boundary. We will also show that under certain conditions, M can be imbedded into a complete, non-compact manifold of corresponding curvature conditions. Finally, it will be shown that in almost all dimensions, a positively curved convex manifold with smooth boundary is diffeomorphic to the standard disc D^n. If the boundary is not smooth, we get a homeomorphism.

In chapter III we use these results to answer the following question: since the study of convex manifolds arose from the work on complete, non-compact manifolds of non-negative curvature, is the reverse direction true? That is, given a convex manifold of non-negative (positive) curvature, can it be isometrically imbedded into a complete, non-compact manifold of non-negative (positive) curvature. In chapter II we answer the question partially for arbitrary dimension, and, for the most part, in chapters III and IV we completely answer it in the case of surfaces. In chapter III, our techniques will be mostly analytical; in chapter IV they will be more geometrical and the results there will yield an alternate proof of those in the previous chapter.

Finally, let's note that our definition of convexity is a local one. Often one speaks of a globally convex manifold, M. That is, M has the property that any two points can be joined by a unique minimal geodesic. It is clear that in the compact cases in which we are working, global convexity and convexity are equivalent.

2. The interior structure and results

Having defined the concept of a convex manifold, we will now tackle the first problem noted in the introduction; that is, the determination of the interior structure of these manifolds. The first theorem proved will deal with the case $K > 0$. And though this result will be the easier to come by of the two structure theorems in this section, it will be the second result which we use in the course of the next chapter. Never the less, the technique is important for both results and vital to the imbedding theorems to follow.

If one calls a convex manifold in which the second fundamental form of the outward normal is positive definite by strictly convex, then the major result of this chapter is the following

Theorem II.1: Let M be strictly convex with C^r boundary, $r \geq 2$. Also, let M be positively curved. Then M can be C^r isometrically imbedded into a complete, non-compact manifold of positive curvature.

In attempting to prove this theorem, we first prove the first of the interior structure theorems.

Theorem II.2: Let $(M, \langle , \rangle)$ be a convex manifold of strictly positive curvature, ∂M not necessarily smooth. Then there exists a complete metric of positive curvature on int(M).

proof- We first define the function $f: \text{int}(M) \longrightarrow \mathbb{R}$ by $f(x) = \rho(x, \partial M)$ where ρ is the Riemannian distance function on M. Then it is known (compare (6)) that f can be approximated by a C^∞ convex function $\tilde{f}$, where convex means that the hessian form, $h_{\tilde{f}}$, is negative semi-definite.

Let $\psi = 1/\tilde{f}$. Then $\lim_{x \to \partial M} \psi(x) = \infty$. We also have that

$$
\begin{aligned}
&h_\psi(v_p, v_p) = h_{1/\tilde{f}}(v, v) = \langle \nabla_v \nabla(1/\tilde{f}), v \rangle = \\
&\langle \nabla_v(-\nabla\tilde{f}/\tilde{f}^2), v \rangle = -\langle \nabla_v \nabla\tilde{f}/\tilde{f}^2, v \rangle = \\
&-\{\langle v(1/\tilde{f}^2)\nabla\tilde{f} \;+\; 1/\tilde{f}^2(\nabla_v\nabla\tilde{f}), v \rangle = \frac{2\,v(\tilde{f})^2}{\tilde{f}^3} \\
&-1/\tilde{f}^2(h_{\tilde{f}}(v, v)) > 0.
\end{aligned}
$$

Now, we define $H: \mathrm{int}(M) \times \mathbb{R} \longrightarrow \mathbb{R}$ by $H(x,t) = \varphi(x) - t$. Since $H_* = (\varphi_*, -\mathrm{id})$ we have that H is a regular map and by the implicit function theorem, $H^{-1}(0) = \tilde{M}$ = graph of φ, is, with the metric induced from the product metric on $\mathrm{int}(M) \times \mathbb{R}$, a Riemannian submanifold of $\mathrm{int}(M) \times \mathbb{R}$, and, as we will show in the Lemma following this theorem, the metric is a complete one on $\tilde{M}$.

Let $\tilde{K}$ denote the curvature of $\tilde{M}$. Let $\tilde{v}_{(p,t_0)} = v_p + \alpha\, d/dt|_{t_0}$, $\tilde{w}_{(p,t_0)} = w_p + \beta\, d/dt|_{t_0}$, $\alpha, \beta \in \mathbb{R}$ be linearly independent tangent vectors in $\tilde{M}_{(p,t_0)}$, σ their span. Then by the Gauss Equations (8), we have

$$\tilde{K}_\sigma = K_\sigma^{\mathrm{int}(M) \times \mathbb{R}} + 1/\|\nabla H\|^2 \det \begin{vmatrix} h_H(\tilde{v},\tilde{v}), h_H(\tilde{v},\tilde{w}) \\ h_H(\tilde{v},\tilde{w}), h_H(\tilde{w},\tilde{w}) \end{vmatrix}.$$

To evaluate the determinant we note that

$$h_H(\tilde{v},\tilde{v}) = \langle \tilde{\nabla}_{\tilde{v}} \nabla H, \tilde{v} \rangle^{\sim} = \langle \tilde{\nabla}_{v + \alpha\, d/dt}(\nabla\varphi, -d/dt), v + \alpha\, d/dt \rangle^{\sim} = \langle \nabla_v \nabla\varphi, v \rangle + \langle \nabla_{\alpha\, d/dt}(-d/dt), \alpha\, d/dt \rangle = \langle \nabla_v \nabla\varphi, v \rangle + 0 = h_\varphi(v,v).$$

So then the curvature is positive by the Cauchy-Schwartz inequality for positive semi-definite forms on a vector space.

So we have $\tilde{K}_\sigma > 0$. Let $G: \mathrm{int}(M) \longrightarrow \tilde{M}$ be defined by $G(x) = (x, \psi(x))$. If we give int(M) the induced metric from $\tilde{M}$, it is readily seen that G is an isometry. So int(M) is given a complete metric of positive curvature.

As stated, we still must prove the following

<u>Lemma II.1</u>: The construction of the graph, as given above, yields a complete metric.

<u>proof</u>- As we saw above, the graph of ψ, $\tilde{M}$, sits in int(M) x $\mathbb{R}$. The product distance function, $\rho_{(\text{int}(M) \times \mathbb{R})}$ satisfies

$$(1)\quad \rho_{(\text{int}(M) \times \mathbb{R})}\left[((p,\psi(p)),(q,\psi(q))\right]^2 = \left[\rho_{\text{int}(M)}(p,q)\right]^2 + |\psi(p) - \psi(q)|^2 .$$

Let $\{(x_n, \psi(x_n)\}$ be a Cauchy sequence in $\tilde{M}$. Then, for all $\epsilon > 0$, there exists an N such that for n,m $> N$ we have $\rho_{(\text{int}(M) \times \mathbb{R})}\left[(x_n, \psi(x_n)),(x_m, \psi(x_m))\right] < \epsilon$.

By (1) we can easily see that both $\{x_n\}$ and $\{\psi(x_n)\}$ are also Cauchy-sequences in int(M) and $\mathbb{R}$ respectively. Now we ask, what is the range of ψ? Well, $\tilde{f}$, as defined in the theorem, takes on a maximum on M by compactness and since $\tilde{f} \geq 0$ and is zero on ∂M, this maximum must accur at an interior point of M, say α where $f(\alpha) = \beta$. So the range of $\psi = [\beta^{-1}, \infty)$ and since $[1/\beta, \infty)$ is complete, $\psi(x_n)$ converges to, say, a point $\ell \in \mathbb{R}$.

What about $\{x_n\}$? It is clear that it can do one of the two following things:

a) $\{x_n\}$ can approach some $p \in \text{int}(M)$. Well, if this happens, it is trivial to see that $\{(x_n, \psi(x_n))\}$ converges to (p, ℓ) by continuity.

b) $\{x_n\}$ approaches some $q \in \partial M$. If this happens, $\psi(x_n) \longrightarrow \infty$ by continuity. But, $\psi(x_n)$ is a Cauchy-sequence and therefore bounded in $\mathbb{R}$, so, in reality, this possibility cannot occur.

So the only alternative is (a) and therefore $\widetilde{M}$ is complete and we're done.

<u>Corollary II.1</u>: With the same hypothesis as in Theorem II.2, int(M) is diffeomorphic to $\mathbb{R}^n$.

<u>proof</u>- By Cheeger and Gromoll (4), the soul of int(M) is a point.

The problem with the foregoing graph technique is that the new metric is, of course, not the old one. Worse still is that this new metric does not agree with the given one anywhere on int(M).

What we do next is improve our technique so that the new metric will agree with the old one off an arbitrarily small, one-sided tubular neighborhood of the boundary. This is the result that will prove useful in what's to follow in the next chapters. The only change in hypothesis is the demand for smooth boundary. But since this is usually the only type of boundary ever talked about, the trade off is not a bad one.

Theorem II.3: Let $(M, \langle , \rangle)$ be convex, ∂M smooth and $K \geq 0$ $(K > 0)$. Then given $\epsilon > 0$ sufficiently small, there exists a complete metric of non-negative (positive) curvature on int(M) such that this new metric agrees with the old one off the one-sided ϵ-tubular neighborhood of the boundary.

proof- Again we let $f\colon \mathrm{int}(M) \longrightarrow \mathbb{R}$ be $f(x) = \rho(x, \partial M)$. Now we have that f is continuously convex, but in the $K \geq 0$ case, there is no approximation theorem as we had in the strict case. In fact, such a theorem is still being sought. The best result known to date is again due to Green and Wu (7). But since ∂M is smooth, there is a one-sided tubular neighborhood of the boundary which will exclude any points of the cut locus of M and therefore on which f will be C^{∞} (cf. 3). Call this neighborhood $B_\epsilon(\partial M)$, that is, $B_\epsilon(\partial M) =$

$\{ x \in \mathrm{int}(M) \mid \rho(x, \partial M) < \epsilon \}$.

Again, let $\psi = 1/f$. ψ is C^{∞} on $B_{\epsilon}(\partial M)$ and $\lim_{x \to \partial M} \psi(x) = \infty$. As before, for $p \in B_{\epsilon}(\partial M)$, we have $h_{\psi}(v,v) \geq 0$.

Now define $g: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} \int_{1/\epsilon}^{t} e^{-1/(s^2 - 1/\epsilon^2)} \, ds, & t > 1/\epsilon, \\ 0 & \text{else.} \end{cases}$$

g is C^{∞} with $d^n g/dt^n \big|_{1/\epsilon} = 0$ for all n.

Let $\tilde{\psi}: \mathrm{int}(M) \longrightarrow \mathbb{R}$ be defined by

$$\tilde{\psi}(x) = \begin{cases} \psi(x), & x \in B_{\epsilon}(\partial M) \\ \gamma(x) & \text{else,} \end{cases}$$

where γ is any C^{∞} extension of ψ outside of $B_{\epsilon}(\partial M)$.

Finally, let $\Gamma = g \circ \tilde{\psi}: \mathrm{int}(M) \longrightarrow \mathbb{R}$.

We have that $\lim_{x \to \partial M} \Gamma(x) = \infty$, Γ is C^{∞} and in $B_{\epsilon}(\partial M)$ we have

$$\begin{aligned} h_{\Gamma}(v,v) = h_{g \circ \tilde{\psi}}(v,v) &= \langle \nabla_v (g' \circ \tilde{\psi}) \nabla \tilde{\psi}, v \rangle \\ &= (g' \circ \tilde{\psi}) \langle \nabla_v \nabla \tilde{\psi}, v \rangle + v(g' \circ \tilde{\psi}) \langle \nabla \tilde{\psi}, v \rangle \\ &= (g' \circ \tilde{\psi}) h_{\tilde{\psi}}(v,v) + (g'' \circ \tilde{\psi}) v(\tilde{\psi}) v(\tilde{\psi}) \\ &= (g' \circ \tilde{\psi}) h_{\tilde{\psi}}(v,v) + (g'' \circ \tilde{\psi}) v(\tilde{\psi})^2. \end{aligned}$$

In $B_{\epsilon}(\partial M)$, $h_{\tilde{\psi}}$ is non-negative and since g'

and g'' are also non-negative, $h_\Gamma(v,v) \geq 0$. If $p \in M - B_\epsilon(\partial M)$, g' and $g'' = 0$ so $h_\Gamma = 0$. By continuity then, $h_\Gamma(v,v) \geq 0$ for all $p \in \overline{M - B_\epsilon(\partial M)}$. So Γ is (non-strictly) concave.

We now proceed as before by taking the graph of Γ and our theorem is proved.

proof of Theorem II.1: By definition, $M^n \subset N^n$. Since $K(M) > 0$ and the second fundamental form of the boundary is positive definite, by continuity N can be chosen to be a compact convex manifold of positive curvature. Then, by applying Theorem II.3 to N, we get our result.

If the strict curvature and boundary conditions are relaxed, we cannot as yet use Theorem II.3, for an extension is not guaranteed. It is precisely this problem for surfaces that is dealt with in chapter III. All we can say here is the following

Corollary II.2: If M is convex, $K(M) > 0$ and M can be imbedded into a convex $\tilde{M}$ with $K(\tilde{M}) \geq 0$ ($K(\tilde{M}) > 0$), then M can be imbedded into a complete, non-compact manifold of non-negative (positive curvature).

proof- Just apply Theorem II.3 as above to the ambient manifold.

We now prove directly a theorem shown to be true in slightly greater generality by Cheeger and Gromoll (4).

Corollary II.3: Let M^n be convex with possibly non-smooth, simply connected, boundary, $K > 0$. Then M is homeomorphic to the disc D^n for $n > 5$.

proof- As we have just shown, int(M) is diffeomorphic to $\mathbb{R}^n$, and therefore contractible. So, given $\epsilon > 0$ sufficiently small, $B'_\epsilon(\partial M) = \{ p \in M \mid \rho(p, \partial M) > \epsilon \}$, is contractible. But if ϵ is small enough, we can connect all points on $\overline{B'_\epsilon(\partial M)}$ to ∂M by unique minimal geodesics and then it is clear that by retracting the boundary along these trajectories, we get that M is itself contractible (see figure 3).

Then by Smale (15), we get the result.

Corollary II.4: M^n convex, ∂M smooth. Then if $K > 0$ and $n \neq 4, 5$, $n \geq 2$, M is diffeomorphic to D^n.

proof- There exists only one differentiable structure up to diffeomorphism on D^n.

Now that we have determined the interior

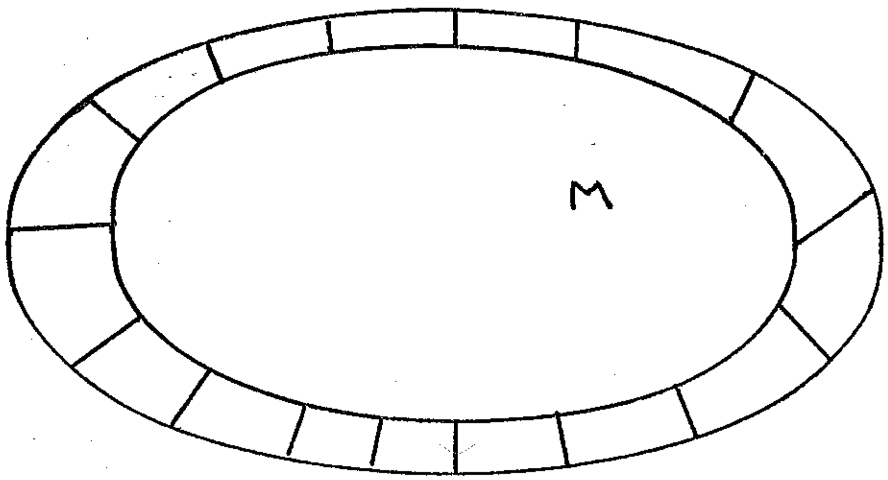

fig. 3

structure of convex manifolds, we can begin the classification of surfaces as noted in the introduction.

We will need some preliminary notions first and these will be dealt with in the beginning of the next chapter. But before we begin chapter III, it may be interesting to note some further questions related to this chapter which still remain open:

1) As noted, can a continuously convex function on a manifold of non-negative curvature be approximated by a smooth one?

2) Can Theorem II.3 be proved without the smoothness condition on the boundary? (Of course, an affirmative answer to (1) answers affirmatively for (2).

III- Convex Surfaces And Their Imbeddings

1. Introduction

In this chapter we study one of the most neglected dimensions in modern differential geometry—2. We concern ourselves with convex surfaces and the possibility of imbedding them into complete non-compact surfaces of the same curvature conditions. Also, imbeddings into compact surfaces are studied.

To initiate the study, we first review two known topics in differential geometry—Fermi coordinates and geodesic curvature. These notions can be found in many books but we will exhibit and prove facts about them in our context.

2. Fermi Coordinates

Let N be a subset of T(M), the tangent bundle of M, such that if $(m, \gamma) \in N$, then $\exp_m \gamma$ is defined. As we know, N is an open set and exp is C^∞ on N. In particular, Let $\hat{M} = \{ (m,0) \in T(M) \mid m \in M \}$. Then there is an open set $\hat{N}$ in T(M) such that $\hat{M} \subset \hat{N} \subset N$. It is then known that if we define $G: \hat{N} \longrightarrow M \times M$ by $G(p, \gamma) = (p, \exp_p \gamma)$, then G is C^∞ and G_* is non-singular and onto at all points (p,0) in T(M).

Now let σ be a C^{∞} curve in M that is univalent on the open interval $I \subset \mathbb{R}$. Let $e_1 \ldots e_n$ be the C^{∞} fields on σ that are independent at each $\sigma(t)$ where $e_n(t) = T_{\sigma}(t)$, the tangent vector to σ at $\sigma(t)$. Let $z_1, \ldots, z_n$ be the dual base to $e_1, \ldots, e_n$ for each t. By the above, there is a neighborhood V of $\hat{M} \subset T(M)$ such that G is a diffeomorphism of V onto a neighborhood N_M of the diagonal in M x M. Let $U = \{(m,Y) \text{ in } V \mid m = \sigma(t) \text{ and } z_n(Y) = 0 \text{ for some } t \in I\}$. Then $F = G_{|U}$ is a one-to-one C^{∞} map of the submanifold U into M x M. Moreover, F_* is non-singular at each point of U, so F is an imbedding of U into M x M. The map $H = \pi_2 \circ F$ then gives a one-to-one C^{∞} map of U onto an open neighborhood W of the image set $\sigma(I)$. (Here, π_2 is projection onto the second factor).

Define Fermi-coordinates $y_1, \ldots, y_n$ on p in W by letting $H^{-1}(p) = (\sigma(t), Y)$ in W and $y_i(p) = z_i(Y)$, $i = 1, \ldots, n-1$ and $y_n(p) = t$.

Now let M^n be a Riemannian manifold, φ a coordinate map on M with domain U and $x_i = u_i \circ \varphi$ the i^{th} coordinate function, u_i the i^{th} coordinate function on $\mathbb{R}^n$. Also let $X_i = \frac{\partial}{\partial x_i}$. The coordinate system $x_1, \ldots, x_n$ is <u>orthogonal</u> if $\langle X_i, X_j \rangle = 0$ for

$i \neq j$. We have the following

Lemma III.1: (Gauss). Let γ be an arbitrary univalent curve in a surface M parametrized by arc-length on (a,b) and let X be the unit tangent to γ and Y be a unit C^{∞} field along γ such that $\langle X,Y \rangle = 0$. Then the Fermi-coordinate system induced by Y on a neighborhood of γ is an orthogonal coordinate system about γ .

proof- Let φ be the Fermi-coordinate map from the neighborhood U of γ onto a set $V \subset \mathbb{R}^2$ (see figure 1).

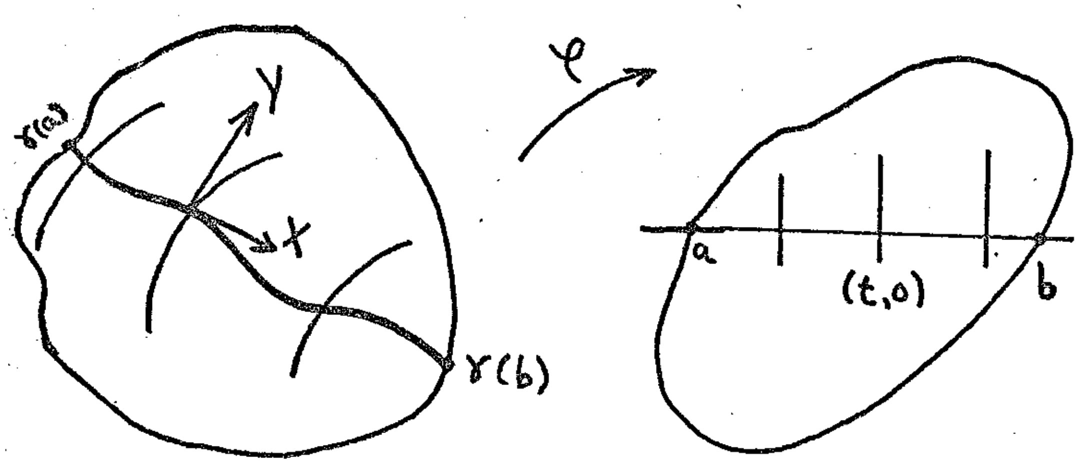

fig. 1

Then for (t,s) in V, $\varphi^{-1}(t,s) = \exp_{\gamma(t)} sY$. We let X and Y be the coordinate fields on U which extend X and Y along γ . Since the y-curves are geodesics parametrized by arc-length, $\nabla_Y Y = 0$ and $\langle Y,Y \rangle = 1$. Then $Y\langle X,Y \rangle = \langle \nabla_Y X, Y \rangle + \langle X, \nabla_Y Y \rangle = \langle \nabla_Y X, X \rangle = \frac{1}{2} X \langle Y,Y \rangle = 0$, since $T = 0$ implies that $\nabla_Y X = \nabla_X Y$. Then $\langle X,Y \rangle$ is constant along the y-curves and since $\langle X,Y \rangle = 0$ on γ, we have $\langle X,Y \rangle = 0$ on U.

So, given the curve, we have a coordinate system (x,y) such that the line element $ds^2 = dy^2 + g dx^2$, where $g = \langle X,X \rangle$. What we will be dealing with in the coming sections, is the case where γ is a simple closed curve homeomorphic to a circle and parametrized by arc-length, where γ bounds a convex manifold which is imbedded into an ambient surface. We will attempt to extend g to a $\tilde{g}$ past the boundary γ which is the $y = 0$ curve such that g agrees with $\tilde{g}$ on γ up to certain order and such that certain other conditions are satisfied. It will be clear from the context that solutions of the problem locally in an open neighborhood of a point on the curve is tantamount, by the compactness of γ and agreement of g and $\tilde{g}$, to proving the assertions globally on the whole curve, joining up at the endpoints. So we will always confine ourselves (as will be seen later) to local extensions along a univalent part of the boundary γ.

A straightforward calculation shows that if we have an orthogonal coordinate system about a curve γ, $ds^2 = dy^2 + gdx^2$, then the curvature is given by the formula

$$K = (-1/\sqrt{g})(\partial^2\sqrt{g}/\partial y^2).$$

3. <u>Geodesic Curvature</u>

Let M be an oriented surface. If σ is an oriented C^∞ curve in M with unit tangent T, let T,N be the orthonormal oriented base along σ and define the <u>geodesic curvature</u> of σ to be the C^∞ function k with $\nabla_T T = kN$. k is defined since $T\langle T,T\rangle = 0 = \langle\nabla_T T,T\rangle$, so $\nabla_T T$ is a multiple of N.

<u>Lemma III.2</u>- If (x,y) is an oriented orthogonal coordinate system on $U \subset M$, where $ds^2 = dy^2 + gdx^2$, then the geodesic curvature k_y of the y-coordinate curve is given by $(-1/2g)(\partial g/\partial y)$.

<u>proof</u>- Let X,Y be the coordinate vector fields of the orthogonal system, $X \circ \gamma(t) = T(t)$. Then $k_y = \langle\nabla_{X/\sqrt{g}}X/\sqrt{g}, T\rangle$. Now $\partial g/\partial y = Y\langle X,X\rangle = 2\langle\nabla_Y X,X\rangle$. Since the torsion is zero we have that $\partial g/\partial y = 2\langle\nabla_X Y,X\rangle$. Since $\langle X,Y\rangle = 0$ we get

$X\langle X,Y\rangle = 0 = \langle\nabla_X X,Y\rangle + \langle X,\nabla_X Y\rangle$. So $\partial g/\partial y =$ $-2\langle\nabla_X X,Y\rangle$. So $-1/2g\,(\partial g/\partial y) = \langle\nabla_X X,Y\rangle/g$. But $k_y = 1/\sqrt{g}\,\langle\nabla_X(1/\sqrt{g})X,Y\rangle = \langle X(1/\sqrt{g})X +$ $1/\sqrt{g}\,\nabla_X X,Y\rangle = 1/g\langle\nabla_X X,Y\rangle$ so our lemma is proved.

4. Convex Surfaces

We now take up the discussion of convex manifolds of dimension two. The major question taken up will be if convex surfaces of non-negative (positive) curvature can be isometrically imbedded into complete surfaces of non-negative (positive) curvature. The procedure will be to first imbed our surface into another convex surface of the same curvature conditions as a proper subset. Then by invoking Theorem II.3, we get the desired result. This section will deal mainly with the analytic tools discussed in the foregoing sections. This will suffice for a while but will fail in certain cases. In chapter IV a more geometrical approach will be developed to continue on.

As an immediate consequence of Theorem II.1 is the

Theorem III.1: Let M be a strictly convex surface of positive curvature. Then M can be imbedded into a complete, non-compact surface of positive curvature.

Without loss of generality (by the existence of orientable covers), we will assume that all surfaces discussed from here on are orientable.

Let $\gamma = \partial M$ be parametrized by arc-length. We pick once and for all an orientation on M such that $\dot{\gamma}$ and the global unit normal vector field N form an oriented base along γ.

As before, we define the geodesic curvature of γ, k_γ, as $\langle \nabla_{\dot{\gamma}} \dot{\gamma}, N \rangle$. Since $\dot{\gamma} \langle \dot{\gamma}, N \rangle = 0$, we get then that $\langle \nabla_{\dot{\gamma}} \dot{\gamma}, N \rangle = -\langle \dot{\gamma}, \nabla_{\dot{\gamma}} N \rangle = -\langle \ell_N \dot{\gamma}, \dot{\gamma} \rangle$, where ℓ_N is the second fundamental form of $\dot{\gamma}$ with respect to N. By the definition of M being convex, ℓ_N is positive semi-definite, so therefore $k \leq 0$.

Choose an orthogonal coordinate system on γ, such that the y-coordinate curves are the arc-length geodesics perpendic ular to γ. We have the line element $ds^2 = dy^2 + g dx^2$ where $g = \langle X, X \rangle$, $X = \partial / \partial x_i$.

What we will be doing in this section is extending g past the boundary, keeping the metric in the same form, keeping the curvature conditions, and for some $y = y_0$ curve, having $k_{y_0} = 0$.

<u>Theorem III.2</u>: Let M be a strictly convex surface of positive curvature. Then M can be imbedded into a compact surface (without boundary) of positive curvature.

<u>proof</u>- We want to extend g to a $\bar{\bar{g}}$ with C^∞ agreement on y = 0 with $k_{y_0} = 0$ for some y_0.

First, we canextend g to $\bar{g}$ in $(-\alpha, \alpha) \times [0, \epsilon]$ for sufficiently small ϵ, α small, by continuity such that $K > 0$ and $k_{\gamma_y} < 0$ for all $y \in [0, \epsilon]$. Recalling the last sections we have

$$k_{\gamma_y} = -1/2\bar{g}(\partial \bar{g}/\partial y) \text{ , } K = -1/\sqrt{\bar{g}}\,(\partial^2/\partial y^2(\sqrt{\bar{g}}))$$

If we set $u = \sqrt{\bar{g}}$ we have that $k \leq 0$ is equivalent to $-1/2u^2(\frac{\partial}{\partial y}(u^2)) = -1/u\,(\partial u/\partial y) \leq 0$. So having $\partial \bar{g}/\partial y \geqslant 0$ is equivalent to $\partial u/\partial y \geqslant 0$. Also, $K \geqslant 0$ is, in the same way, equivalent to $\partial^2 u/\partial y^2 \leq 0$.

Therefore, we are faced with the following problem: we wish to extend $\bar{g}$ to $\bar{\bar{g}}$ on $(-\alpha, \alpha) \times [0, \delta]$ where $\delta \leq \epsilon$ such that

1) $\bar{\bar{g}}|_{(x,0)} = \bar{g}(x,0)$ up to all orders,
2) $\frac{\partial \bar{\bar{g}}}{\partial y}(x, y_0) \equiv 0$, $y_0 \leq \delta$,
3) $\partial^2 \bar{\bar{g}}/\partial y^2 \leq 0$

Now define $\bar{\bar{g}}$ on $(-\alpha, \alpha) \times [0, \delta]$, $\delta \leq \epsilon$, by

$\bar{\bar{g}}(x,y) = \bar{g}(x,P(y))$ where P is a C^∞ extension of $f(y) = y$ whose maximum value, λ, occurs before ϵ and has the properties

1) $P(0) = 0$,
2) $P^k(0) = 0$ for all $k > 1$, $P'(0) = 1$,
3) $P^k(y_0) = 0$ for all k (see figure 1).

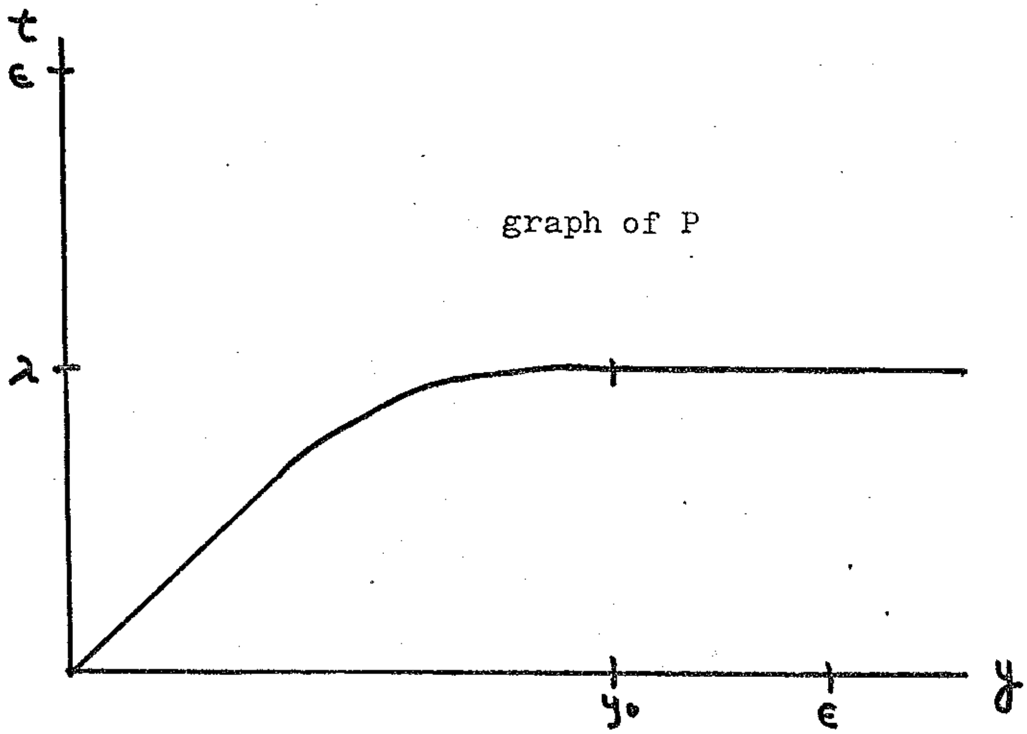

fig. 1

Then we have

a) $\bar{\bar{g}}(x,0) = \bar{g}(x,P(0)) = \bar{g}(x,0)$,

b) $\partial \bar{\bar{g}}/\partial y(x,y) = \partial \bar{g}/\partial y(x,P(y)P'(y)$ so $\partial \bar{\bar{g}}/\partial y(x,0) = \partial \bar{g}/\partial y(x,0)$,

c) $\partial^2 \bar{\bar{g}}/\partial y^2(x,y) = \partial \bar{g}/\partial y(x,P(y)P''(y) +$

$(P'(y))^2\, \partial^2 g/\partial y^2(x,P(y))$ so $\partial^2\bar{\bar{g}}/\partial y^2(x,0) =$
$\partial^2\bar{g}/\partial y^2(x,0)$,

d) $\partial^k(\bar{\bar{g}})/\partial y^k(x,0) = \partial^k\bar{g}/\partial y^k(x,0)$ for all k,

e) $\partial\bar{\bar{g}}/\partial x = 0 = \partial\bar{g}/\partial x(x,0)$ and $\partial^k\bar{\bar{g}}/\partial x^k =$
$0 = \partial^k\bar{g}/\partial x^k(x,0)$ for all k, since $\bar{g}$ is constant along the x, y = o curve,

f) Since $\partial\bar{\bar{g}}/\partial x$ and $\partial\bar{\bar{g}}/\partial y$ and higher partial derivatives are continuous, mixed partials are equal. This can be used, for example to show that

$$\partial^2\bar{\bar{g}}/\partial y\partial x\,(x,0) = \partial^2\bar{\bar{g}}/\partial x\partial y(x,0) =$$
$$\partial^2\bar{\bar{g}}/\partial x\partial y(x,y)\big|_{y=0} = \partial/\partial x(\partial\bar{g}/\partial y(P'y)) =$$
$$P'(y)(\partial/\partial x)\partial\bar{g}/\partial y\big|_{y=0} = P'(0)\,\partial^2\bar{g}/\partial y\,\partial x(x,0) =$$
$$\partial^2\bar{g}/\partial y\partial x(x,0).$$

We can equally show that $\bar{\bar{g}}$ and $\bar{g}$ agree up to all orders at y = 0.

g) $\partial^2\bar{\bar{g}}/\partial y^2 = \partial\bar{g}/\partial y P''(y) + (P'(y))^2\,\partial^2\bar{g}/\partial y^2$. Since $\partial\bar{g}/\partial y > 0$, $P''(y) < 0$ on $[0,\delta]$, and $\partial^2\bar{g}/\partial y^2 < 0$, we have $\partial^2\bar{\bar{g}}/\partial y^2(x,y) < 0$. So K remains positive.

Now take the double of $\overline{M}$, where $\overline{M}$ is M union this added collar up to $y = y_0$. In the first copy of $\overline{M}$, $ds^2 = dy^2 + \grave{g}dx^2$ as it is in the second copy. What we need is the following fact:

If $g: (-\alpha, \alpha) \times [0,a] \longrightarrow \mathbb{R}$, then when is the function $h: [-\alpha,\alpha] \times [-\delta,\delta] \longrightarrow \mathbb{R}$ defined by

$$h(x,y) = \begin{cases} g(x,y), & y \geq 0, \\ g(x,-y), & y \leq 0 \end{cases}$$

C^∞ at $y = 0$?

It is clear that the answer is precisely when $\partial^k g/\partial y^k(x,0) = 0$ for all k which our g satisfies at y_0, so the copies of $\bar{M}$ fit along their common boundary differentiably and $\bar{M}$ is therefore diffeomorphic to S^2.

<u>Theorem III.3</u>- Let M be strictly convex surface where $K \geq 0$. Then M can be imbedded into a convex $\tilde{M}$ where $K(\tilde{M}) \geq 0$ and $k_{\partial \tilde{M}} < 0$ and this imbedding is C^2.

First, let's notice that a C^2 extension is the best we can hope for. For if p is a point on the boundary with $K_p = 0$ and $\partial K/\partial y < 0$, there is no way to extend so that there is C^3 agreement along the boundary and K still non-negative since $\partial K/\partial y$ is just the third derivative of g and having the third derivative continous forces $K < 0$ in any extension.

<u>proof</u>- Again, look at our orthogonal coordinates. We are given $\partial g/\partial y(x,0) > 0$ and $\partial^2 g/\partial y^2(x,0) \leq 0$.

We need to extend g to $\bar{g}$ which agrees with g C^2 on $y = 0$ and satisfies the same inequalities.

Define $g_x : [0,\epsilon] \longrightarrow \mathbb{R}$ by $g_x(y) = g(x,0) + \partial g/\partial y(x,0)y + \partial^2 g/\partial y^2(x,0)y^2/2$. Then let $\bar{g}(x,y) = g_x(y)$.

1) $\bar{g}(x,0) = g_x(0) = g(x,0)$,

2) $\partial\bar{g}/\partial y(x,y) = \lim_{h \to 0} \frac{\bar{g}(x,y+h) - \bar{g}(x,y)}{h}$

$= \lim_{h \to 0} \frac{g_x(y+h) - g_x(y)}{h} = g_x'(y)$, so $\partial\bar{g}/\partial y(x,0) = g_x'(0) = \partial g/\partial y(x,0)$,

3) $\partial\bar{g}/\partial x(x,y) = \lim_{h \to 0} \frac{\bar{g}(x+h,y) - \bar{g}(x,y)}{h} = \lim_{h \to 0} \frac{g_{x+h}(y) - g_x(y)}{h} = \lim_{h \to 0} \Big[g(x+h,0) +$

$\partial g/\partial y\,(x+h,0)y + \partial^2 g/\partial y^2(x+h,0)y^2/2 - \Big(g(x,0) + \partial g/\partial y(x,0)y + \partial^2 g/\partial y^2(x,0)y^2/2 \Big)\Big] / h$

$= \lim_{h \to 0} \frac{g(x+h,0) - g(x,0)}{h} + \lim_{h \to 0} \Big(\big[\partial g/\partial y(x+h,0) - \partial g/\partial y(x,0)\big]y\Big)/h + \lim_{h \to 0} \Big[\Big(\partial^2 g/\partial y^2(x+h,0) - \partial^2 g/\partial y^2(x,0)\Big)\frac{y^2}{2}\Big]/h$

$= \partial g/\partial x(x,0) + \partial^2 g/\partial x\,\partial y(x,0)y + \partial^3 g/\partial x\,\partial y^2(x,0)y^2/2$,

so $\partial\bar{g}/\partial x$ exists and is continuous. Also $\partial\bar{g}/\partial x(x,0) = \partial g/\partial x(x,0)$. In a similar fashion we get existence of all partials and mixed partials and agreement in a C^2 fashion on $y=0$.

Now note that $\partial^2\bar{g}/\partial y^2 = g_x''(y) = \partial^2 g/\partial y^2(x,0)$ ≤ 0. So the curvature is extended constantly on the perpendicular lines. Let $\alpha = \inf\{y \mid \partial\bar{g}/\partial y > 0\}$. Then if $\beta = \min(\epsilon, \alpha)$, $\bar{g}$ extended to $(-\alpha, \alpha) \times [0,\beta]$ satisfies what we want.

<u>Corollary III.1</u>: With the same hypothesis of Theorem III.3, we can imbed M into a compact surface diffeomorphic to S^2, in the C^2 sense.

<u>proof</u>- Same as in Theorem III.2

<u>Corollary III.2</u>: With the same hypothesis, M can be C^2 imbedded into a complete manifold of non-negative curvature.

<u>proof</u>- Since M is C^2 imbedded into a convex surface, we proceed as before and apply Theorem II.3 to the ambient surface.

<u>Corollary III.3</u>: Let M be convex, $K \geq 0$, $k \leq 0$. Then if we have $K_{|\partial M} = 0$, M can be imbedded into a complete surface of non-negative curvature in a C^2 sense.

<u>proof-</u> From the proof of Theorem III.3, we can extend g to $\bar{g}$ to keep $K \geq 0$. But $\partial\bar{g}/\partial y = \partial g/\partial y(x,0) + (\partial^2 g/\partial y^2)y_{|(x,0)} = |k| + 0$. So if $k \leq 0$ and $K_{|\partial M} = 0$,

we can keep k negative.

The problem still remains to tackle $K > 0$, $k \leq 0$. But, using orthogonal coordinates, we see that if $k_p = 0$, the positivity of K forces k to increase and therefore we cannot use the technique. The same is true for the general case $K \geq 0$, $k \leq 0$ when the hypothesis of Corollary III.3 is not satisfied.

We first tried to vary the curves chosen to evaluate the geodesic curvature from the y = constant curves to the curve $\gamma(t)$ where $x \circ \gamma(t) = t$, $y \circ \gamma(t) = \alpha(t)$, α variable. But we get

Lemma III.3: Let (x,y) be a Fermi-chart as before. Let $\gamma(t)$ be the curve satisfying $x \circ \gamma(t) = t$, $y \circ \gamma(t) = \alpha(t)$ where t is the arc-length parameter of the y = 0 curve, and $\alpha(t)$ is a non-negative, real valued function. Then if k_{y_0} represents the geodesic curvature of the $y = y_0$ curve, the geodesic curvature of γ, k_γ, is

$$1/\Gamma\Delta\left[\left(\alpha'' - \alpha' X(g)/2g + k_{(y \circ \gamma(t))}\left(1 + 2\alpha'^2/g\right)\right]$$

where $\Delta = \|\dot{\gamma}\|$, $\Gamma = 1 + (\alpha'^2)/g$.

proof- By our definition of γ, we can see that $\dot{\gamma} = X + fY$ where $X = \partial/\partial x$, $Y = \partial/\partial y$ and $f \circ \gamma(t) = \alpha'(t)$. Let $\Delta = \|\dot{\gamma}\|$. Then $T = \dot{\gamma}/\Delta$ is the unit tangent field along γ. Note that $\dot{\gamma}(f) = \gamma_* d/dt(f) = d/dt(f \circ \gamma) = \alpha''$. Now

$$\nabla_T T = \nabla_{\dot{\gamma}/\Delta} \dot{\gamma}/\Delta = 1/\Delta \nabla_{\dot{\gamma}} \dot{\gamma}/\Delta =$$
$$1/\Delta(\dot{\gamma}(1/\Delta)\dot{\gamma} + 1/\Delta \nabla_{\dot{\gamma}} \dot{\gamma}) = 1/\Delta(-\dot{\gamma}(\Delta)\dot{\gamma}/\Delta^2 +$$
$$1/\Delta \nabla_{\dot{\gamma}} \dot{\gamma}) = -1/\Delta^3 \dot{\gamma}(\Delta)\dot{\gamma} + 1/\Delta \nabla_{\dot{\gamma}}(X + fY) =$$
$$-1/\Delta^3 \dot{\gamma}(\Delta)\dot{\gamma} + 1/\Delta(\nabla_{\dot{\gamma}} X + \alpha'' Y + f \nabla_{\dot{\gamma}} Y) =$$
$$-1/\Delta^3 \dot{\gamma}(\Delta) + 1/\Delta(\nabla_X X + f\nabla_X Y + \alpha'' Y + f\nabla_X Y).$$

Let $N = -f/g\,(X) + Y$. We see that $\langle \dot{\gamma}, N \rangle = 0$. Let $\Gamma = \| N \| = 1 + \alpha'^2/g$. Then

$$k_\gamma(t) = \langle \nabla_T T, N \rangle = 1/\Gamma\Delta(\langle \nabla_X X + f\nabla_X Y + \alpha'' Y$$
$$+ f\nabla_X Y, -f/g(X) + Y\rangle) = 1/\Gamma\Delta\,(-f/g\langle \nabla_X X, X\rangle +$$
$$\langle \nabla_X X, Y\rangle - f^2/g\langle \nabla_X Y, X\rangle + f\langle \nabla_X Y, Y\rangle + \alpha'' -$$
$$f^2/g\langle \nabla_X Y, X\rangle + f\langle \nabla_X Y, Y\rangle) = 1/\Gamma\Delta\,(-\alpha' X(g)/2g +$$
$$k_{y=\gamma(t)}\,(1 + 2\alpha'^2/g) + \alpha'')$$ since $\langle X, \nabla_X Y\rangle \gamma(t) = k_{y=\gamma(t)}$, as proved before.

As one can see, this is a very unwieldy formula and our technique seems unable to be effective.

In the next chapter we develop geometric techniques to continue on.

IV-H-convex Extensions

1. Introduction

As we saw in chapter III, the analysis breaks down when we reach the $K > 0$, $k \leq 0$ case. The use of orthogonal coordinates leads us nowhere. The most general case, (except for the special instance of Corollary III.3), that is, $K \geq 0$, $k \leq 0$, looks even more hopeless.

What we do in this chapter is to tackle these problems with a more geometric approach. We provide a construction of new convex surfaces from old, called H-convex extensions. It will be through this technique that the most general results of this paper will come.

Recall, before we begin, that with M compact and convex, there exists an $r > 0$ such that each ball $B_r(p)$ is strongly convex for each $p \in M$, (cf. 8).

2. The Construction and Results

Definition IV.1: A convex surface is called H-convex if its boundary consists of a piecewise smooth geodesic.

Theorem IV.1: Let M be a convex surface

with $\psi : [0,1] \longrightarrow \partial M$ an arc-length parametrization of the boundary. If $K > 0$ ($K \geq 0$) and if there exists a point $p \in \partial M$ with $k_p < 0$, then M can be non-properly imbedded (C^2 imbedded) into an H-convex surface of positive (non-negative) curvature.

<u>proof</u>- First, of course, we have M sitting in an unbounded surface N with the same curvature conditions. If $K > 0$, we have that this is a C^{∞} imbedding; if $K \geq 0$, our imbedding is C^2 as described in Theorem III.3. Since M is compact, there is a compact $V \subset N$ with $M \subset V$. Let d be the elementary length of V (see (8) for the definition) and let u be the smallest parameter value such that any geodesic tangent to M at its boundary at p lies to the outside of $B_r(p)$ for all parameter values less than u, where this is the p with $k_p < 0$. Now let $\delta = \min(u,r,d)$ where r is as in the remark at the end of the introduction.

About our point p choose the ball $B_{\delta/2}(p)$. We might as well assume that $\psi(o) = p$. Let $v = \dot{\psi}(0)$. By convexity, the minimal geodesic α, with $\alpha(0) = p$ and $\dot{\alpha}(0) = v$ lies to the outside of M for all $t \in [0, \delta/2]$. Let $\alpha(\frac{\delta}{2}) = q \in \partial B_{\delta/2}(p)$ (see figure 1).

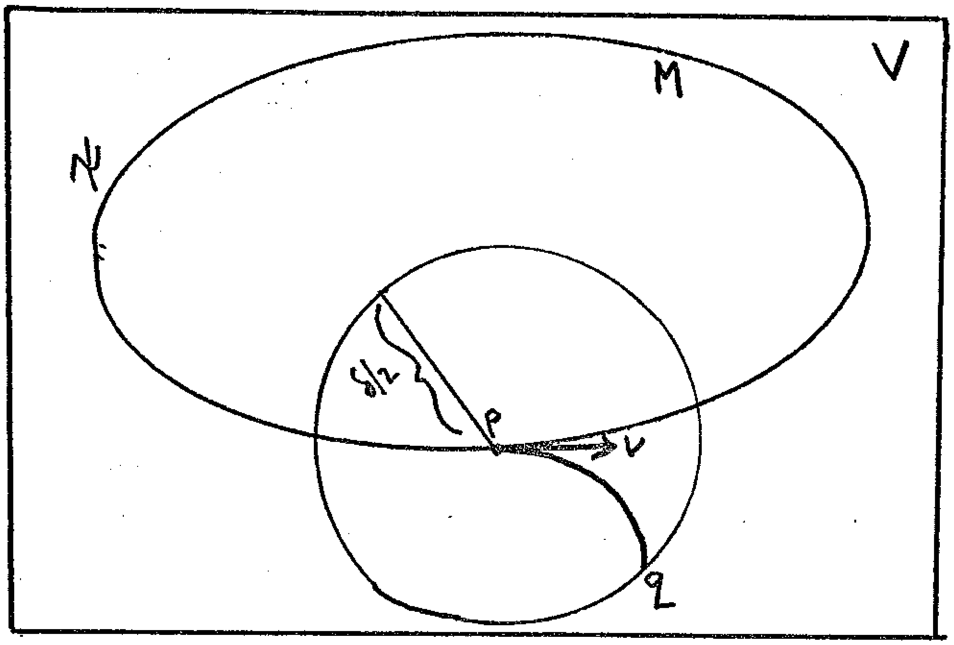

fig. 1

Let t_1 be the point such that $t_1 > 0$ and $\gamma(t_1) \cap \overline{B_{\delta/2}(p)} \neq \emptyset$. Let $\gamma(t_1) = s$. We choose points $t_0 \in [0, t_1]$ and $r_0 \in M$ as follows:

a) <u>point of type 1</u>- If the interval $[t_0, t_1]$ is such that $k \mid [t_0, t_1] = 0$ and $k \mid [t_0 - \epsilon, t_1] \neq 0$ for all $\epsilon > 0$, then let $r_0 = \gamma(t_0)$ (see figure 2).

b) <u>point of type 2</u>- If $k_{\gamma(t_1)} < 0$, let $r_0 = s = \gamma(t_1)$ (see figure 3).

Let β be the unique minimal geodesic from q to r_0. By the triangle inequality, $\rho(r_0, q) < \delta$.

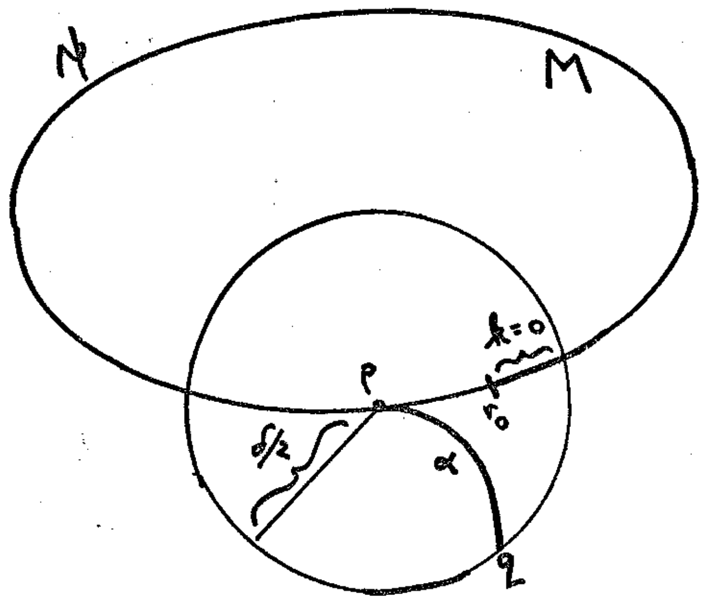

fig. 2

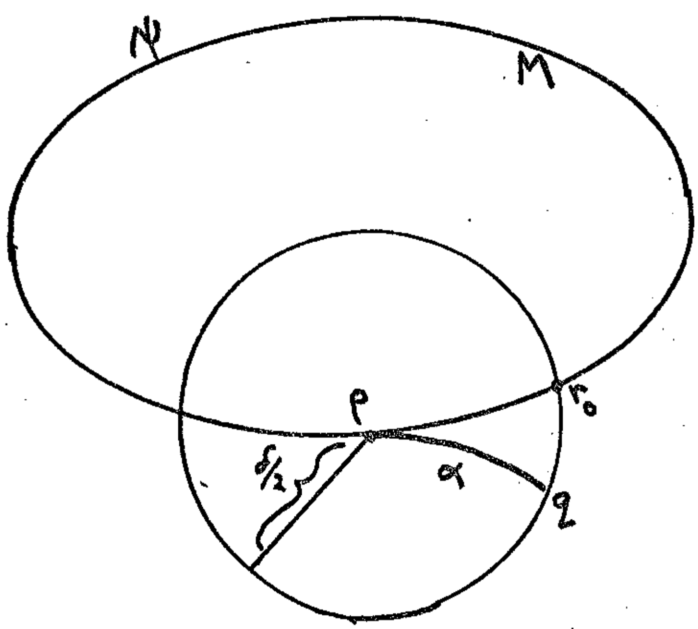

fig. 3

so β exists and lies in $B_\delta(p)$ (see figures 2' and 3').

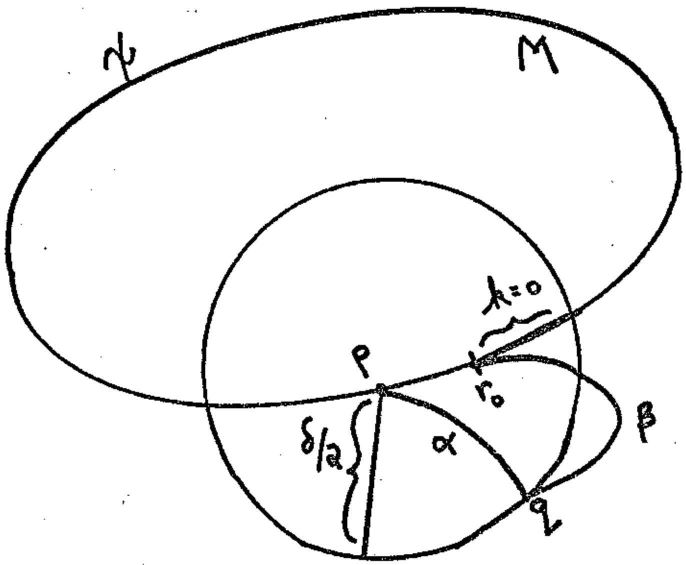

fig. 2'

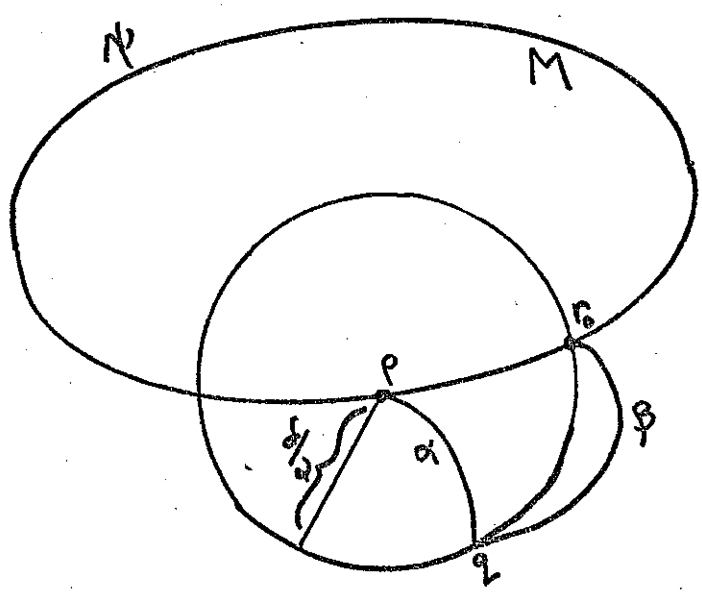

fig. 3'

Now we have the triangle (p,q,r_0). If β is already tangent to ψ, call the triangle and its interior Δ_p. If not, depending upon our orientation, either $\dot{\psi}(t_1)$ or $-\dot{\psi}(t_1)$ is interior to the triangle (p,q,r_0). Let that interior vector be called v (see figure 4).

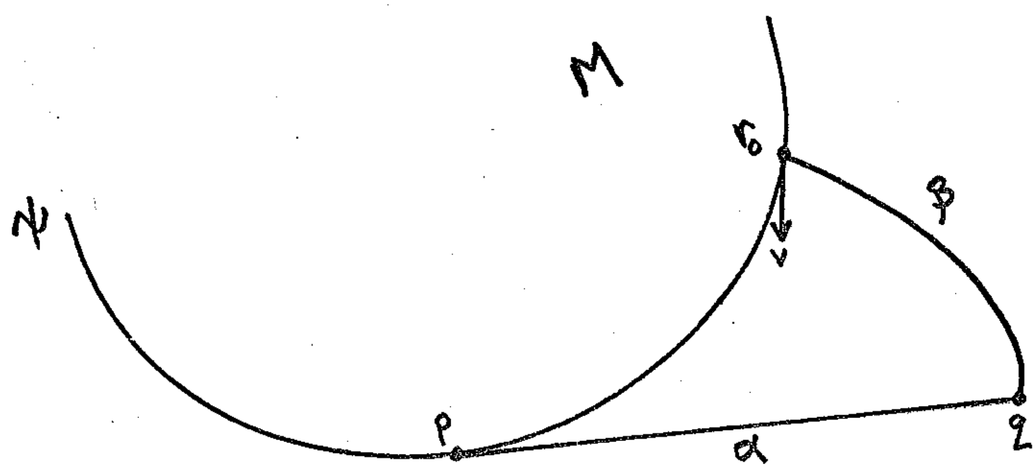

fig. 4

By the parametrization of ψ, v is of unit length. Let $\gamma(t) = \exp_{r_0} tv,\ t \in [0,w]$. This geodesic will have to leave the triangle sooner or later. The question is, how can it do this?

a) It can't leave by crossing ψ, since ψ is not a geodesic by choice of p, and in our neighborhood, convexity forces γ away from ψ.

b) If γ leaves by crossing β (see figure 5), we would have a conjugate point on the minimal geodesic β with respect to r_0 at some point s_0 of intersection, contradicting our choice of δ.

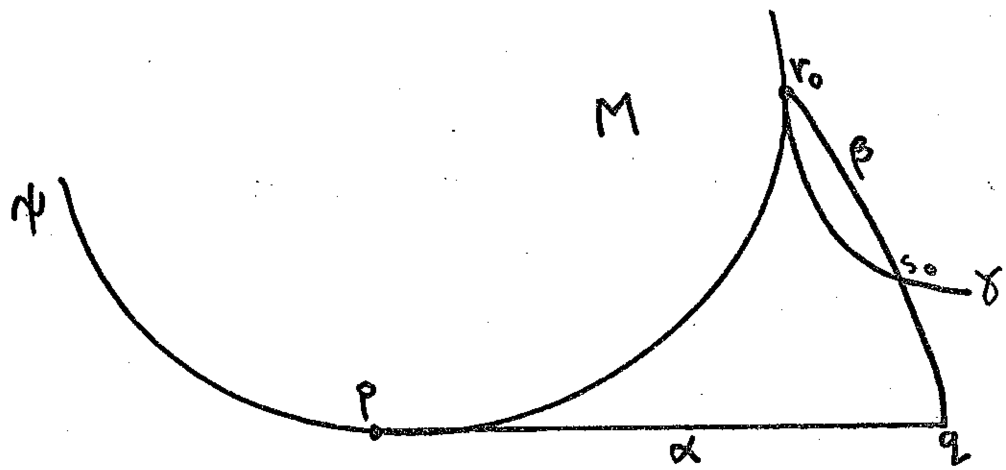

fig. 5

So γ must leave the triangle by crossing α and making a convex angle θ at the point of intersection, x. We call this triangle now which is contained in $B_\delta(p)$ and its interior by Δ_p (see figure 6).

Now, if r_0 is a point of type 1, we continue

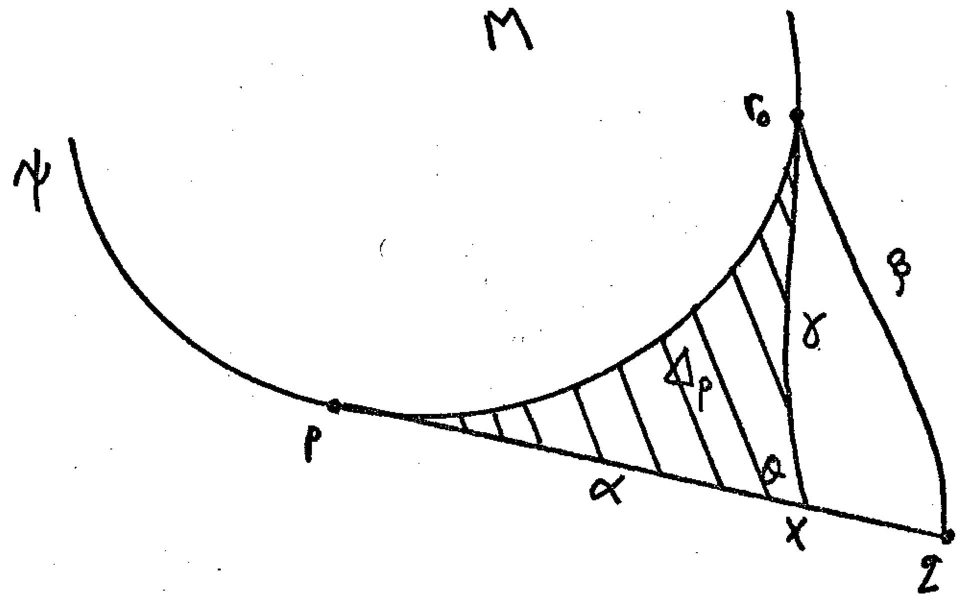

fig. 6

along ψ until we find a point p_2 (call now $p = p_1$) where $k_{p_2} < 0$ and start again. If there is no such point, we are done with this part of the construction.

If r_0 is of type 2, we start the process over again at r_0, choosing another ball and continuing as before.

By the compactness of ψ, we can continue around ψ coming up with $\tilde{M} = M \cup \bigcup_{i \in I} \Delta p_i$ where I is a finite set. $\tilde{M}$ is an H-convex extension of M and we're done. Figure 7 shows a typical end product of the construction. Q.E.D.

Now that we have our extension, we wish to

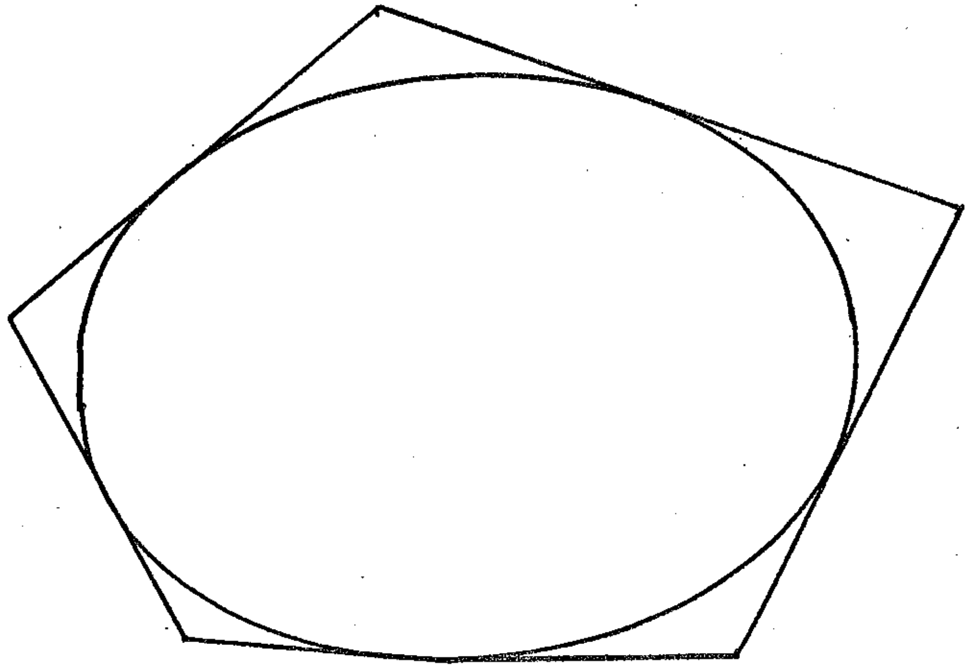

fig. 7

make it a proper one; that is, we want an H-extension $\hat{M}$ of M such the $\partial M \cap \partial \hat{M} = \emptyset$.

Let's look at three successive geodesics of $\partial \tilde{M}$. Call them α, β, and γ. (see figure 8).

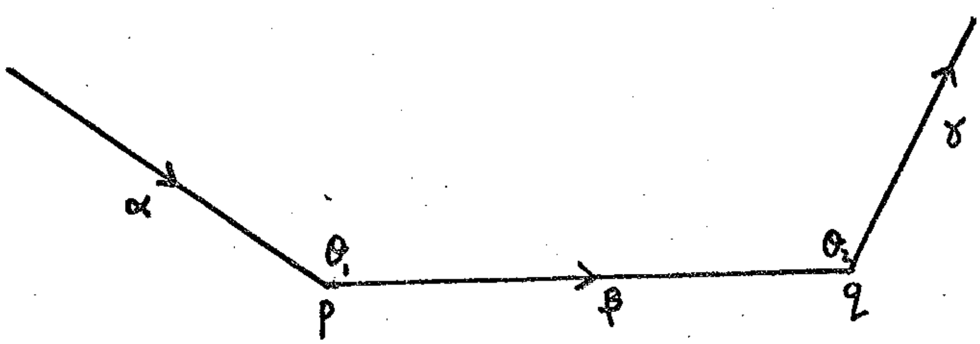

fig. 8

We call the convex angle where α meets β at the point p by θ_1; the convex angle where β meets γ at the point q we call θ_2.

Now pick an outward unit vector X at p such that $\measuredangle(X, -\dot{\alpha}(p)) = \measuredangle(X, \dot{\beta}(p)) = (2\pi - \theta_1)/2$. Next pick an outward unit vector Y at q such that $\measuredangle(Y, -\dot{\beta}(q)) = \measuredangle(Y, \dot{\gamma}(q)) = (2\pi - \theta_2)/2$ (see figure 9).

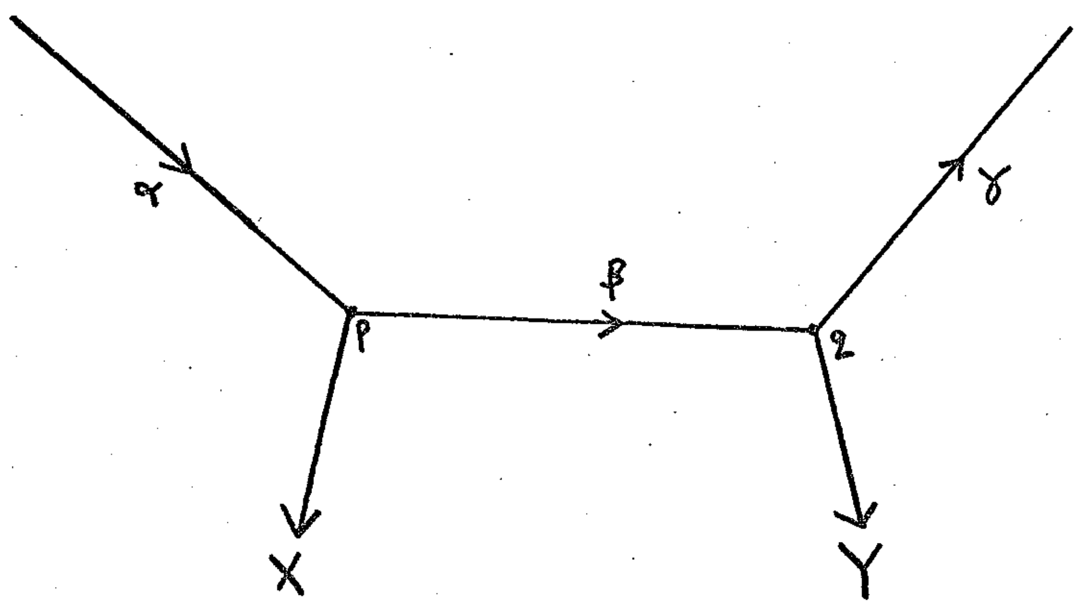

fig. 9

Now if p and q are not conjugate to each other, there exists a Jacobi field J along β with J(p) = X and J(q) = Y. The Jacobi field does not vanish anywhere along β as long as there are no conjugate points at all along β. If all the geodesic segments of our H-convex M are minimal, then there is no problem with conjugate points. Certainly, all those geodesics added on to M are minimal by construction. However, as in the case of points of type 1, they may hook up with a geodesic which is part of the original boundary of M. If the sum of the lengths of these geodesics is too long, conjugate points may occur. So we must make sure that any geodesic segment of ∂M has, first, no conjugate points to begin with, and second, when hooked up with added geodesics of the construction, no conjugate points will occur. Well, it is clear that all we have to ask is that there are no conjugate points on our original ∂M since, by going out a small enough parameter value when needed in the construction, we can assure that when the two geodesics are hooked up, there will be no conjugate points.

So we have our non-vanishing Jacobi field J. This field, in turn, generates a variation through geodesics, so we get a geodesic β' as shown in figure 10 by going outward, say, a parameter distance of 1.

We now continue with this process at all corners to get our proper H-extension (see figure 11).

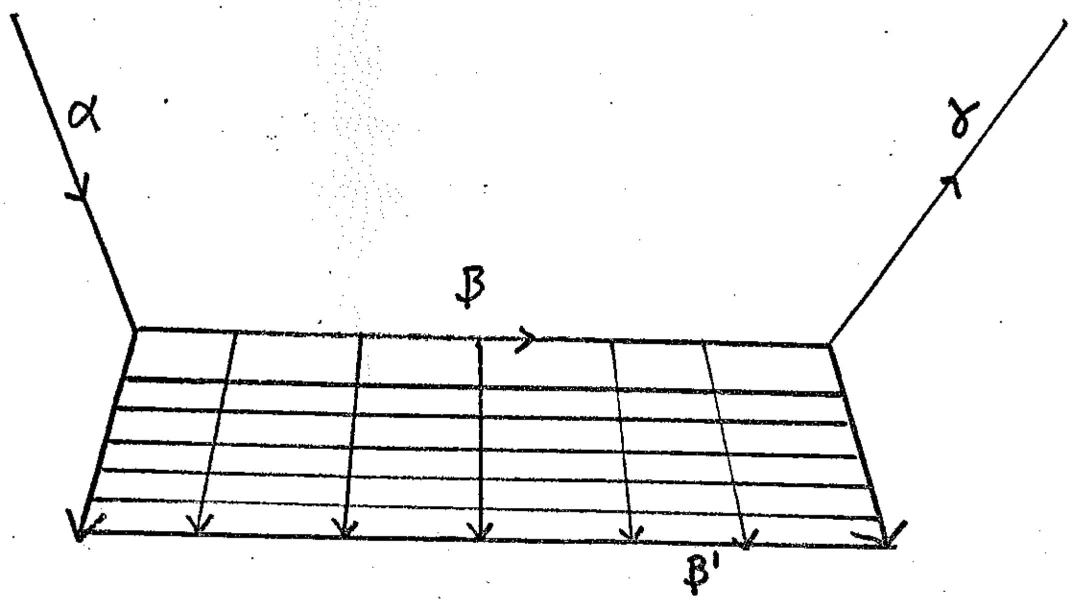

fig. 10

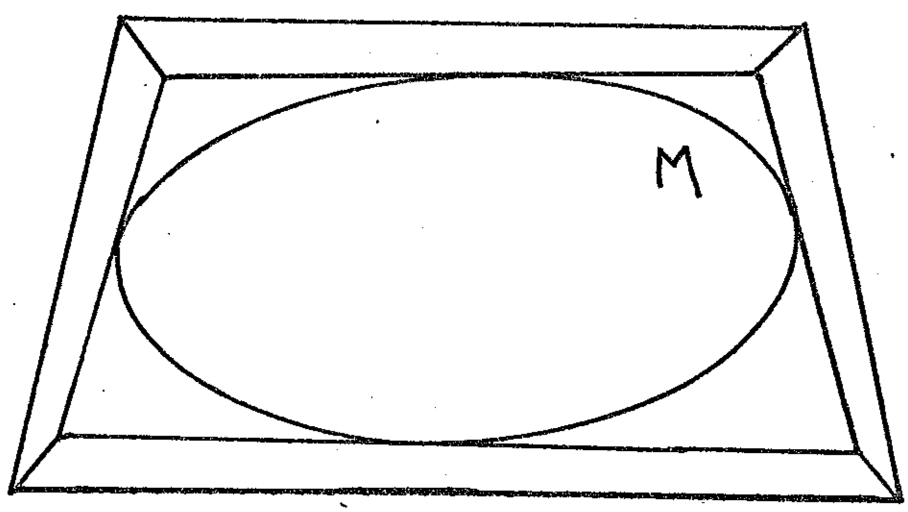

fig. 11

So we have M sitting properly in an H-convex $\hat{M}$. What is left to do is smooth out the convex corners keeping convexity. In $\mathbb{R}^2$ it is straightforward to do this and similarly in our case (see figure 12).

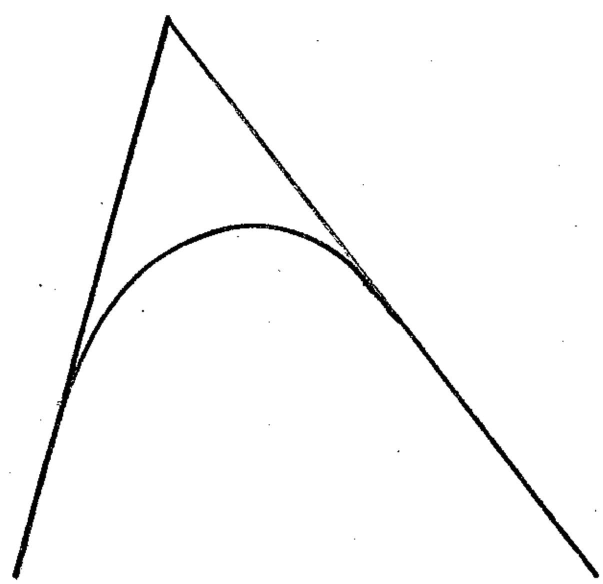

fig. 12

So, we get as final product a smooth convex surface M' properly containing M where the extension is C^{∞} if $K > 0$, C^2 if $K \geq 0$.

We have thus proved the following

<u>Theorem IV.2</u>: Let M be convex, $\dim(M) = 2$, with $K > 0$ ($K \geq 0$) and a point p such that $k_p < 0$.

Then, if there are no conjugate points along the boundary ∂M, M can be properly C^{∞} (C^2) imbedded into a convex surface of positive (non-negative) curvature.

Using Theorem II.3, we then get the

<u>Corollary IV.1</u>: Let M be a convex surface, $K > 0$ ($K \geq 0$). If there is a point $p \in \partial M$ such that $k_p < 0$ and no conjugate points on ∂M, then M can be C^{∞} (C^2) imbedded into a complete, non-compact surface and also a compact surface without boundary diffeomorphic to S^2 with $K > 0$ ($K \geq 0$).

<u>Corollary IV.2</u>: Let M be a convex surface with $K > 0$ ($K \geq 0$). Let $A = \{ p \in \partial M \mid k_p = 0 \}$. If A is discrete, then the results of Theorem IV.2 and Corollary IV.1 hold.

<u>proof</u>- There are no points of type 1, therefore no geodesic segments and therefore no conjugate points.

The existence of conjugate points implies that locally, geodesics intersect, so we get a converse to the existence of proper convex extensions in the

<u>Corollary IV.3</u>: M convex, $K \geq 0$. If there exists conjugate points on the boundary, then there does not exist any proper convex extension of M arbitrarily close to M.

Of course, Corollary IV.3 is not true for sufficiently large proper, or sufficiently small non-proper extensions since on the paraboloid the convex set shown in figure 13 sits in the convex half-paraboloid and in a sufficiently large ball about r even though the boundary contains the conjugate points p and q if the geodesic segment is large enough.

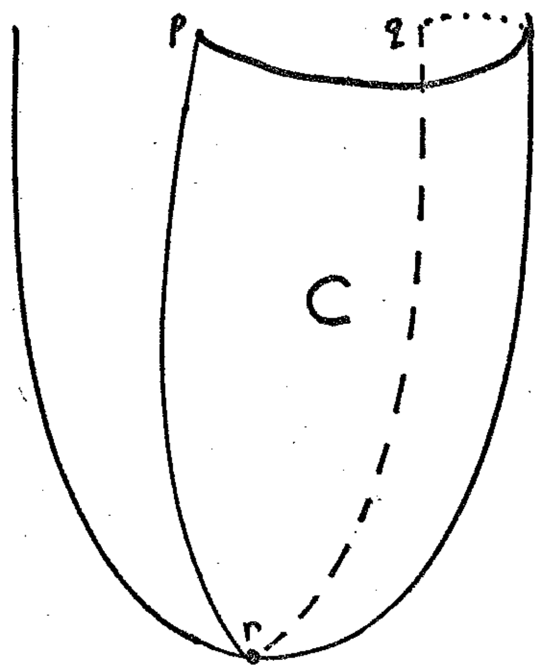

fig. 13

Unfortunately, the same example shows that the converse to Corollary IV.1 is not true either since the paraboloid is complete and not compact.

So the question still remains:

a) M a convex surface, $K > 0$ $(K \geq 0)$, $k \leq 0$ and

∂M contains conjugate points. What, if any, are the precise conditions guaranteeing the imbedding of M into a complete, non-compact surface of positive (non-negative) curvature?

It should be noted that the Gauss-Bonnet Theorem shows that if M is convex and $K \geq 0$, then $\int_M K \, dM \leq 2\pi$. So if $K > 0$ and ∂M is a (possibly reparametrized) geodesic, then it would be impossible to imbed M into a convex $\hat{M}$ of non-negative curvature, proper or not. But, the questions that still remain are:

b) M convex , $K \geq 0$, $\int_M K = 2\pi$. Can M be imbedded into a complete, non-compact surface of non-negative curvature?

c) M convex, $K \geq 0$, $\int_M K = 2\pi$. Can M be imbedded into a compact surface of non-negative curvature?

And, of course, many more questions still remain for dimensions higher than two.

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